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THESIS

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Existence of Solutions for Evolution Equations and Inclusions with Delay

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in front of the Jury made up of

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Dédicaces

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God created in man the ability to think ...

Man's ability to think may try to obtain

the desired information by combining

the universals with each other,

with the result that the mind obtains

a universal picture that conforms

to details outside. " ...

Ibn Khaldûn The Muqqaddimah.

The Science of Logic.

Abstract

In this work, we give a contribution to the study of existence and uniqueness results of mild solutions on a bounded interval for various classes of first order class and Caputo's fractional derivative order class of partial functional and neutral functional type, differential and integrodifferential, perturbed and nonperturbed *evolution* equations and inclusions with finite and infinite state-dependent delay.

To get the existence of these mild solutions ; sufficient conditions are considered in the study of different classes of *evolution* problems. Uniqueness results are also given for some classes of these problems.

The method used is to reduce the existence of these mild solutions to the search for the existence of fixed points of appropriate operators by applying different nonlinear alternatives in Fréchet and Banach spaces to entire the existence of fixed points of the above operator which are mild solutions of our problems. This method is based on famous and recent fixed point theorems and is combined with the semigroup theory.

Controllability of mild solutions is investigated for some classes of first order and fractional order of partial functional and neutral functional *evolution* equations in this work as applications.

Key words and phrases :

Evolution equations and inclusions with delay - neutral problems - Fractional derivative order - perturbed evolution problems - mild solution - state-dependent delay - existence - uniqueness - controllability - fixed point - nonlinear alternative - semigroup.

AMS Subject Classification : 26A33 - 93B05 - 34G20 - 34G25 - 34K26 - 34K37- 34K40 -35R11.

Résumé

Ce travail est consacré à l'étude de quelques résultats d'existence et d'unicité de la solution faible sur un intervalle borné pour quelques classes d'ordre un et d'ordre fractionnaire de type Caputo d'équations et d'inclusions d'évolution partielles fonctionelles et celles de type neutre, différentielles et intégrodifférentielles, perturbées et non perturbées, avec un retard fini et infini dépendant de l'état.

Pour obtenir l'existence de ces solutions faibles ; des conditions suffisantes seront considérées dans l'étude des différentes classes de problèmes d'*évolution*. Des résultats d'unicité sont également donnés pour quelques classes de ces problèmes.

La méthode utilisée est de ramener la recherche de l'existence de ces solutions faibles à la recherche de l'existence des points fixes d'opérateurs appropriés en appliquant différentes alternatives non linéaires dans les espaces de Fréchet et de Banach pour entirer l'existence des points fixes de cet opérateur qui sont les solutions faibles de nos problèmes. Cette méthode est basée sur des célèbres et récents théorèmes du point fixe et est combinée avec la théorie des semi-groupes.

La contrôlabilité des solutions faibles est donnée dans ce travail pour quelques classes d'ordre fractionnaire d'équations d'*évolution* et celles de type neutre à titre d'applications.

Mots et Phrases Clefs :

Equations et inclusions d'évolution à retard - problèmes de type neutre - Ordre fractionnaire - Problèmes d'évolution perturbées - solution faible - retard dépendant de l'état - existence - unicité - contrôlabilité - point fixe - alternative non linéaire - semi-groupes.

Classification AMS : 26A33 - 93B05 - 34G20 - 34G25 - 34K26 - 34K37- 34K40 -35R11.

Publications

1. **D. Aoued** and S. Baghli-Bendimerad, Mild solutions for perturbed evolution equations with infinite state-dependent delay, *Electronic Journal of Qualitative Theory of differential Equations*, **59**, 2013, 1-24.

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2. **D. Aoued** and S. Baghli-Bendimerad, Controllability of mild solutions for evolution equations with infinite state-dependent delay, *European Journal of Pure and Applied Mathematics*, **9** (4), 2016, 383-401.

   http://www.ejpam.com/index.php/ejpam/article/view/2456/485

3. **D. Aoued**, S. Baghli-Bendimerad and M. Benchohra, Fractional partial functional and neutral functional evolution equations with infinite state-dependent delay, submitted.

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5. **D. Aoued**, S. Baghli-Bendimerad and M. Benchohra, Controllability of mild solutions for fractional partial functional and neutral functional evolution equations with state-dependent delay, submitted.

6. **D. Aoued**, S. Baghli-Bendimerad and M. Benchohra, Multivalued fractional partial functional and neutral functional fractional evolution equations with state-dependent delay, submitted.

Contents

Introduction

Fractional calculus deals with the generalization of the integrals and derivatives of noninteger order. The idea of fractional calculus and fractional order differential equations has been a subject of interest not only among mathematicians, but also among physicists and engineers. Indeed, we can find numerous applications in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc... see the books of Baleanu *et al.* [28], Hilfer [71], Oldham and Spanier [88], Podlubny [91] and Tarasov [95].

There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Miller and Ross [87], Samko *et al.* [93], Podlubny [91], the papers of Abbas and Benchohra [2, 3], Benchohra *et al.* [29, 31], Kilbas and Marzan [78], Vityuk and Golushkov [97], Vityuk and Mykhailenko [98] and the references therein.

In fact, fractional differential equation is considered as an alternative model to nonlinear differential equations by Bonilla [35]. Though the concepts and the calculus of fractional derivative are few centuries old, it is realized only recently that these derivatives form an excellent framework for modeling real world problems. This in turn led to sustained study of the theory of fractional equations by Lakshmikantham *et al.* [82]. Numerical experiments for fractional models on population dynamics are discussed by Luchko and his collaborators in [86].

The first work, devoted exclusively to the subject of fractional calculus, is the book by Oldham and Spanier [88] published in 1974. One of the most recent works on the subject of fractional calculus is the book of Podlubny [91] published in 1999, which deals principally with fractional differential equations. Some of the lasted (but certainly not the last) works especially on fractional models of anomalous kinetics of complex process are the volumes edited by Carpinteri and Mainardi [39] in 1997 and by Hilfer [71] in 2000, and the book by Zaslavsky [102] published in 2005. Indeed, in the meantime, numerous other works (books, edited volumes, and conference proceeding) have also appeared. These include (for example) the monographs of Samko *et al.* [93], which was published in Russian in 1987 and in English in 1993, the book of Miller and Ross [87] in 1993, and recently the books of Baleanu and his collaborators [28], Diethelm [46], Kilbas *et al.* [79], Lakshmikantham and his collaborators [82] and Tarasov [95].

Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in last few decades. A good guide to the literature for neutral functional differential equations is in the books by Hale *et al.* [61, 62], Kolmanovskii and Myshkis [81] and the references therein.

Differential delay equations or functional differential equations, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books of Benchohra and his collaborators [30], Hale *et al.* [62, 60], Hino and his collaborators [74], Kolmanovskii and Myshkis [81], Lakshmikantham *et al.* [83, 41] and Wu in [100].

When the delay is infinite, the notation of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato (see [60]). For further applications see for instance the books of Hale [62], Hino [74], Lakshmikantham *et al.* [83] and their references.

However, complicated situations in which the delay depends on the unknown functions have been proposed in modeling in recent years (see for instance Arino *et al.* [17], Rezounenko and Wu [92], Willé and Baker [99] and the references therein). These equations are frequently called equations with state-dependent delay. Existence results, among other things, were derived recently for various classes of functional differential equations when the delay is depending on the solution. We refer the reader to the papers by Ait Dads and Ezzinbi [9], Gyri and Hartung [58], Hartung [63, 64, 65] and Hernandez *et al.* [69]. Darwish and Ntouyas considered in [43] a class of semilinear functional fractional order differential equations with state-dependent delay. By means of the Banach contraction principle and the nonlinear alternative of Leray-Schauder, Abada *et al.* in [4] present some existence as well as uniqueness results for each of our problems on bounded domain.

Baghli *et al.* in [5], [20]-[26] considered existence, uniqueness and controllability of mild solutions for first order classes of semilinear partial functional and neutral functional differential and integrodifferential perturbed and nonperturbed evolution equations and inclusions with finite and infinite delay. Then they look in [27] to the case where the delay is depending on the solution for evolution equations and in [19] she gives global mild solution for evolution inclusions with state-dependent delay.

We are going here to give the existence of mild solution for first order of the perturbed class of evolution equations with state-dependent delay in [10] and we are going to study the controllability of these mild solution in [11]. Then we extend the above results for Caputo derivative fractional order of different classes of evolution equations and inclusions with state-dependent delay in [12]-[15].

So in this thesis, we give existence and even uniqueness of mild solutions on a bounded interval for different classes of the first order and the Caputo's fractional derivative order for semilinear partial functional and neutral fractional, differential and integrodifferential, perturbed and nonperturbed of *evolution* equations and inclusions with finite and infinite state-dependent delay in Fréchet and Banach spaces. Our results are based upon fixed point techniques combined with semigroup theory.

Our thesis is organized as follows:

The first Chapter contains some notations, definitions, theorems and preliminary facts that will be used throughout this thesis.

In what follows each chapter concludes with examples applying the abstract theory and this thesis is clotured by the bibliography.

The second Chapter is devoted to give our main result, using a nonlinear alternative of Leary Schauder-type given by Frigon and Granas for contraction maps [52], combined with the semi-group theory. The existence of the unique mild solution is demonstrated in section 2.2 for the following class of fractional evolution equations with finite state-dependent delay

$$^cD_0^\alpha y(t) = A(t)y(t) + f(t, y_{\rho(t,y_t)}), \quad 0 < \alpha < 1 \text{ a.e. } t \in J := [0, b] \tag{1}$$

$$y(t) = \varphi(t), \quad t \in H := [-r, 0], \tag{2}$$

where $b > 0$, $0 < r < +\infty$, $^cD_0^\alpha$ is the standard Caputo's fractional derivative order of order $\alpha \in (0, 1)$, $f : J \times C(H; E) \to E, \rho : J \times C(H; E) \to \mathbb{R}$ and $\varphi \in C(H; E)$ are given functions and $\{A(t)\}_{0 \leq t}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t, s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem for the neutral type is given in section 2.3 for the following class of fractional neutral evolution equations with finite state-dependent delay

$$^cD_0^\alpha[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + f(t, y_{\rho(t,y_t)}), \quad 0 < \alpha < 1 \quad \text{a.e.} \quad t \in J \tag{3}$$

$$y(t) = \varphi(t), \quad t \in H, \tag{4}$$

where $A(\cdot)$, f and φ are as in problem $(2.1)-(2.2)$ and $g : J\times C(H; E) \to E$ is a given function.

In section 2.4 we study the fractional evolution equations with infinite state-dependent delay

$$^cD_0^\alpha y(t) = A(t)y(t) + f(t, y_{\rho(t,y_t)}), \quad 0 < \alpha < 1 \text{ a.e. } t \in J \tag{5}$$

$$y_0 = \phi \in \mathcal{B}, \tag{6}$$

where $\mathcal{B}$ is an abstract *phase space* to be specified later, $^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0, 1)$, $f : J \times \mathcal{B} \to E, \rho : J \times \mathcal{B} \to \mathbb{R}$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t, s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 2.5 for the following class of fractional neutral evolution equations with infinite state-dependent delay

$$^cD_0^\alpha[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + f(t, y_{\rho(t,y_t)}), \quad 0 < \alpha < 1 \quad \text{a.e.} \quad t \in J \tag{7}$$

$$y_0 = \phi \in \mathcal{B}, \tag{8}$$

where $A(\cdot)$, f and ϕ are as in problem $(2.5) - (2.6)$ and $g : J \times \mathcal{B} \to E$ is a given function.

The third Chapter is devoted to give our main result, using a nonlinear alternative of Frigon and Granas for contraction maps [52]. The existence of the unique mild solution is demonstrated in section 3.2 for the following class of fractional integrodifferential evolution equations with finite state-dependent delay

$$^cD_0^\alpha y(t) = A(t)y(t) + \int_0^t \mathcal{K}(t, s)f(s, y_{\rho(s,y_s)})ds, \quad 0 < \alpha < 1 \text{ a.e. } t \in J \tag{9}$$

$$y(t) = \varphi(t), \quad t \in H, \tag{10}$$

where $\mathcal{K} : J \times J \to E$, $0 < r < +\infty$, ${}^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0,1)$, $f : J \times C(H;E) \to E, \rho : J \times C(H;E) \to \mathbb{R}$ and $\varphi \in C(H;E)$ are given functions and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 3.3, we consider the following fractional neutral integrodifferential evolution equations with finite state-dependent delay

$$ {}^cD_0^\alpha[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + \int_0^t \mathcal{K}(t,s)f(s, y_{\rho(s,y_s)})ds, \quad 0 < \alpha < 1 \quad \text{a.e.} \quad t \in J \quad (11)$$

$$ y(t) = \varphi(t), \quad t \in H, \quad (12)$$

where $A(\cdot)$, f and φ are as in problem $(3.1)-(3.2)$ and $g : J \times C(H;E) \to E$ is a given function.

In section 3.4 we study the following fractional integrodifferential evolution equations with infinite state-dependent delay

$$ {}^cD_0^\alpha y(t) = A(t)y(t) + \int_0^t \mathcal{K}(t,s)f(s, y_{\rho(s,y_s)})ds, \quad 0 < \alpha < 1 \text{ a.e.} \quad t \in J \quad (13)$$

$$ y_0 = \phi \in \mathcal{B}, \quad (14)$$

where $\mathcal{K} : J \times J \to E$, $\mathcal{B}$ is an abstract *phase space* to be specified later, ${}^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0,1)$, $f : J \times \mathcal{B} \to E, \rho : J \times \mathcal{B} \to \mathbb{R}$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 3.5, we consider the following fractional neutral integrodifferential evolution equations with infinite state-dependent delay

$$ {}^cD_0^\alpha[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + \int_0^t \mathcal{K}(t,s)f(s, y_{\rho(s,y_s)})ds, \quad 0 < \alpha < 1 \quad \text{a.e.} \quad t \in J \quad (15)$$

$$ y_0 = \phi \in \mathcal{B}, \quad (16)$$

where $A(\cdot)$, $f, \mathcal{K}$ and ϕ are as in problem $(3.5) - (3.6)$ and $g : J \times \mathcal{B} \to E$ is a given function.

The fourth Chapter is devoted to study the perturbed classes using a nonlinear alternative of Avramescu for contractions maps in Fréchet spaces [18]. The existence of mild solutions on the positif real line is demonstrated in section 4.2 for the following class of first order perturbed evolution equations with infinite state-dependent delay

$$ y'(t) = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in \mathbb{R}^+, \quad (17)$$

$$ y_0 = \phi \in \mathcal{B} \quad (18)$$

where $\mathcal{B}$ is an abstract *phase space* to be specified later, $f, h : \mathbb{R}^+ \times \mathcal{B} \to E, \rho : \mathbb{R}^+ \times \mathcal{B} \to \mathbb{R}$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{t\geq 0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in\mathbb{R}^+\times\mathbb{R}^+}$ for $s \leq t$.

An extension of this problem is given in section 4.3, we consider the following first order neutral perturbed evolution equations with infinite state-dependent delay

$$\frac{d}{dt}[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in \mathbb{R}^+, \tag{19}$$

$$y_0 = \phi \in \mathcal{B} \tag{20}$$

where $A(\cdot)$, f and ϕ are as in problem $(4.1) - (4.2)$ and $g : \mathbb{R}^+ \times \mathcal{B} \to E$ is a given function.

In section 4.4, we study the following fractional perturbed evolution equations with finite state-dependent delay

$${}^cD_0^\alpha y(t) = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J = [0, b], \tag{21}$$

$$y(t) = \varphi(t), \quad t \in H = [-r, 0], \tag{22}$$

where $b > 0$, $0 < r < +\infty$, ${}^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0, 1)$, $f, h : J \times C(H; E) \to E, \rho : J \times C(H; E) \to \mathbb{R}$ and $\varphi \in C(H; E)$ are given functions and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t, s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 4.5, we consider the following fractional neutral perturbed evolution equations with finite state-dependent delay

$${}^cD_0^\alpha[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J, \tag{23}$$

$$y(t) = \varphi(t), \quad t \in H, \tag{24}$$

where $A(\cdot)$, f and ϕ are as in problem $(4.5) - (4.6)$ and $g : J \times \mathcal{B} \to E$ is a given function.

In section 4.6, we study the following fractional perturbed evolution equations with infinite state-dependent delay

$${}^cD_0^\alpha y(t) = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J, \tag{25}$$

$$y_0 = \phi \in \mathcal{B} \tag{26}$$

where $\mathcal{B}$ is an abstract *phase space* to be specified later, ${}^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0, 1)$, $f, h : J \times \mathcal{B} \to E, \rho : J \times \mathcal{B} \to \mathbb{R}$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t, s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 4.7, we consider the following fractional neutral perturbed evolution equations with infinite state-dependent delay

$${}^cD_0^\alpha[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J, \tag{27}$$

$$y_0 = \phi \in \mathcal{B} \tag{28}$$

where $A(\cdot)$, f and ϕ are as in problem $(4.9) - (4.10)$ and $g : J \times \mathcal{B} \to E$ is a given function.

The fifth Chapter is devoted to controllability problems, using a nonlinear alternative of Avramescu for contractions maps in Banach spaces [18]. The controllability of mild solutions over the real line is demonstrated in section 5.2 for the following class of first order evolution equations with infinite state-dependent delay

$$y'(t) = A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in \mathbb{R}^+, \tag{29}$$

$$y_0 = \phi \in \mathcal{B} \tag{30}$$

where $\mathcal{B}$ is an abstract *phase space* to be specified later, $f : \mathbb{R}^+ \times \mathcal{B} \to E, \rho : \mathbb{R}^+ \times \mathcal{B} \to \mathbb{R}$ and $\phi \in \mathcal{B}$ are given functions, the control function $u(.)$ is given in $L^2(\mathbb{R}^+ E)$, the Banach space of admissible control function with E is a real separable Banach space with the norm $|\cdot|$, C is a bounded linear operator from E into E and $\{A(t)\}_{t\in\mathbb{R}^+}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in\mathbb{R}^+\times\mathbb{R}^+}$ for $s \leq t$.

An extension of this problem is given in section 5.3, we consider the controllability of mild solutions over the real line for the following first order neutral evolution equations with infinite state-dependent delay

$$\frac{d}{dt}[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in \mathbb{R}^+, \tag{31}$$

$$y_0 = \phi \in \mathcal{B} \tag{32}$$

where $A(\cdot)$, f and ϕ are as in problem $(29) - (30)$ and $g : \mathbb{R}^+ \times \mathcal{B} \to E$ is a given function.

The controllability of mild solutions on a bounded interval is demonstrated in section 5.4 for the following class of fractional evolution equations with finite state-dependent delay

$$^cD_0^\alpha y(t) = A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J = [0, b] \tag{33}$$

$$y(t) = \varphi(t), \quad t \in H = [-r, 0], \tag{34}$$

where $0 < r < +\infty$, $^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0,1)$, $f : J \times C(H;E) \to E, \rho : J \times C(H;E) \to \mathbb{R}$ and $\varphi \in C(H;E)$ are given functions, the control function $u(.)$ is given in $L^2(J;E)$, the Banach space of admissible control function with E is a real separable Banach space with the norm $|\cdot|$, C is a bounded linear operator from E into E and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 5.5, we consider the controllability of mild solutions of the following fractional neutral evolution equations with finite state-dependent delay

$$^cD_0^\alpha[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J \tag{35}$$

$$y(t) = \varphi(t), \quad t \in H, \tag{36}$$

where $A(\cdot)$, f, u, C and φ are as in problem $(5.5) - (5.6)$ and $g : J \times C(H;E) \to E$ is a given function.

In section 5.6, we study the controllability of mild solutions of the following fractional evolution equations with infinite state-dependent delay

$$^cD_0^\alpha y(t) = A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J \tag{37}$$

$$y_0 = \phi \in \mathcal{B}, \tag{38}$$

where ${}^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0,1)$, $f : J \times \mathcal{B} \to E$ and $\phi \in \mathcal{B}$ are given functions, the control function $u(.)$ is given in $L^2(J;E)$, the Banach space of admissible control function with E is a real separable Banach space with the norm $|\cdot|$, C is a bounded linear operator from E into E and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 5.7, we consider the controllability of mild solutions of the following fractional neutral evolution equations with infinite state-dependent delay

$${}^cD_0^\alpha[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J \tag{39}$$

$$y_0 = \phi \in \mathcal{B}, \tag{40}$$

where $A(\cdot)$, f, u, C and ϕ are as in problem $(5.9)-(5.10)$ and $g : J\times\mathcal{B} \to E$ is a given function.

The sixth Chapter is devoted to give our main result, using a nonlinear alternative of Frigon contraction multivalued maps [50]. The existence of mild solutions on a bounded interval is demonstrated in section 6.2 for the following class of fractional evolution inclusion with finite state-dependent delay

$${}^cD_0^\alpha y(t) \in A(t)y(t) + F(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \;\; t \in J \tag{41}$$

$$y(t) = \varphi(t), \quad t \in H, \tag{42}$$

where ${}^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0,1)$, $F : J\times C(H;E) \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\mathcal{P}(E)$ is the family of all subsets of E, $\rho : J \times C(H;E) \to \mathbb{R}$ and $\varphi \in C(H;E)$ are given functions and $\{A(t)\}_{t\geq 0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 6.3, we consider the following fractional neutral evolution inclusion with finite state-dependent delay

$${}^cD_0^\alpha y(t)[y(t) - g(t, y_{\rho(t,y_t)})] \in A(t)y(t) + F(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J \tag{43}$$

$$y(t) = \varphi(t), \quad t \in H, \tag{44}$$

where $A(\cdot)$, F and φ are as in problem $(6.1) - (6.2)$ and $g : J \times C(H;E) \to E$ is a given function.

In section 6.4, we study the following fractional evolution inclusions with infinite state-dependent delay

$${}^cD_0^\alpha y(t) \in A(t)y(t) + F(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \;\; t \in J \tag{45}$$

$$y_0 = \phi \in \mathcal{B}, \tag{46}$$

where $\mathcal{B}$ is an abstract *phase space* to be specified later, $F : J \times C(H;E) \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\rho : J\times\mathcal{B} \to \mathbb{R}$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 6.5, we consider the following fractional neutral evolution inclusion with infinite state-dependent delay

$$^{c}D_{0}^{\alpha}y(t)[y(t)-g(t,y_{\rho(t,y_t)})]\in A(t)y(t)+F(t,y_{\rho(t,y_t)}),\quad \text{a.e.}\quad t\in J \tag{47}$$

$$y_0=\phi\in\mathcal{B}, \tag{48}$$

where $A(\cdot)$, F and φ are as in problem $(6.5)-(6.6)$ and $g : J\times\mathcal{B}\to E$ is a given function.

Finally, we give the conclusion of our results and the bibliography based on this work.

Chapter 1

Preliminaries

The aim of this Chapter is to introduce some basic concepts, notations and elementary results that used throughout this thesis.

1.1 Notations and Definitions

Let $J := [0, b]$ and $H := [-r, 0]$ be two closed and bounded intervals in $\mathbb{R}$ for the real numbers $b, r > 0$.

Let E be the Banach space of real numbers with the norm $\|.\|$

Consider $C(H; E)$ be the Banach space of continuous functions from H into E with the norm

$$\|y\| = \sup \{ \ |y(t)| : t \in H \ \}.$$

Let $B(E)$ be the space of all bounded linear operators from E into E, with the norm

$$\|N\|_{B(E)} = \sup \{ \ |N(y)| \ : \ |y| = 1 \ \}.$$

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the Bochner integral properties, see the classical monograph of Yosida [101]).

Let $L^1(J, E)$ denotes the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{+\infty} |y(t)| \ dt.$$

For any continuous function y defined on $[-r, b]$, and for all $t \in J$, note by y_t element of $\mathcal{C}(H; E)$ defined by

$$y_t(\theta) = y(t + \theta) \qquad \text{for } \theta \in H.$$

Here $y_t(\cdot)$ represent the history of the state from $t - r$ up to the present time t.

Definition 1.1.1. *A function $f : J \times E \to E$ is said to be an L^1 -Carathéodory function if it satisfies :*

(i) for each $t \in J$ the function $f(t, .) : E \to E$ is continuous ;

(ii) for each $y \in E$ the function $f(., y) : J \to E$ is measurable ;

(iii) for every positive integer k there exists $h_k \in L^1(J;\mathbb{R}^+)$ such that

$$|f(t,y)| \leq h_k(t) \quad \textit{for all} \ \ |y| \leq k \quad \textit{and almost every } t \in J.$$

1.2 Some Properties in Fréchet Spaces

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies :

$$\|x\|_1 \leq \|x\|_2 \leq \ \|x\|_3 \leq ... \quad \text{for every} \ \ x \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leq \overline{M}_n \qquad \text{for all} \ \ y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows : For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by : $x \sim_n y$ if and only if $\|x-y\|_n = 0$ for $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows : For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$. Respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n.

The appropriate concept of contraction in X is given in the following definition.

Definition 1.2.1. *[52] A function $f : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that*

$$\|f(x) - f(y)\|_n \leq k_n \ \|x-y\|_n \quad \textit{for all} \ \ x, y \in X.$$

1.3 Evolution System Generator

In what follows, we assume that $\{A(t)\}_{t\in J}$ is a family of closed densely defined linear unbounded operators on the Banach space E and with domain $D(A(t))$ independent of t. Additionally, we introduce the following hypothesis:

(P1) For $t \in [0,b]$, the domain $D(A(t)) = D$ is independent of t and is dense on X.

(P2) For $t \geq 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $Re(\lambda) \leq 0$, and there is a constant M independent of λ and t such that

$$\|R(t, A(t)\| \leq M(1+|\lambda|)^{-1} \qquad \text{for } Re(\lambda) \leq 0.$$

(P3) There exist constant $L > 0$ *and* $0 < \alpha \leq 1$ *such that*

$$\|(A(t) - A(\theta))A^{-1}(\tau)\| \leq L|t-\tau|^{\alpha} \quad t, \theta, \tau \in J.$$

(P4) The resolvent $R(t, A(t))$ is compact for $t \geq 0$.

Lemma 1.3.1. *([7], p. 159) Under the assumption $(P1)-(P4)$, the Cauchy problem*

$$\begin{cases} y'(t) - A(t)y(t) = 0 \quad t \in J \\ y(0) = y_0 \end{cases} \tag{1.1}$$

has a unique evolution system $\{U(t,s)\}_{0\le s\le t\le b}$ on E satisfying the following properties:

1. *$U(t,t) = I$ where I is the identity operator in E,*
2. *$U(t,s)\ U(s,\tau) = U(t,\tau)$ for $0 \le \tau \le s \le t \le b$,*
3. *$U(t,s) \in B(E)$ the space of bounded linear operators on E, where for every $(t,s) \in \Delta$ and for each $y \in E$, the mapping $(t,s) \to U(t,s)\ y$ is continuous.*
4. *For $0 \le s \le t \le b$, $U(t,s) : X \longrightarrow D$ and $t \longrightarrow U(t,s)$ is strongly differentiable on E. The derivative $\dfrac{\partial}{\partial t}U(t,s) \in B(E)$ and it is strongly continuous on $0 \le s \le t \le b$. Moreover,*
$$\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s) \quad \text{for} \quad 0 \le s \le t \le b,$$
$$\|\frac{\partial}{\partial t}U(t,s)\|_{B(E)} = \|A(t)U(t,s)\|_{B(E)} \le \frac{C}{t-s},$$
$$\|A(t)U(t,s)A^{-1}(s)\|_{B(E)} \le C \quad \text{for} \quad 0 \le s \le t \le b.$$
5. *For every $v \in D$ and $t \in (0,b]$, $U(t,s)v$ is differentiable with respect to s on $0 \le s \le t \le b$*
$$\frac{\partial}{\partial s}U(t,s)v = -U(t,s)A(s)v.$$
6. *$U(t,s)$ is a compact operator for $0 \le s \le t \le b$.*

And, for each $y_0 \in E$, the Cauchy problem (1.1) has a unique classical solution $y \in C^1(J,E)$ given by
$$y(t) = U(t,0)y_0, \quad t \in J$$

More details on evolution systems and their properties could be found on the books of Ahmed [7], Engel and Nagel [47] and Pazy [90].

1.4 Definition of Mild Solution

Definition 1.4.1. *We say that the function $y(\cdot) : H \cup J \to E$ is a mild solution for the following first order of partial functional differential evolution equations with delay*

$$\begin{cases} y'(t) = A(t)y(t) + f(t, y_t), \quad a.e. \ t \in J \\ y(t) = \varphi(t), \quad t \in H \end{cases} \tag{1.2}$$

If $y(t) = \varphi(t)$ for all $t \in H$ and y satisfies the following integral equation :

$$y(t) = U(t,0)\ \varphi(0) + \int_0^t U(t,s)f(s,y_s)\ ds \quad \text{for each} \quad t \in J. \tag{1.3}$$

Definition 1.4.2. *We say that the function $y(\cdot) : H \cup J \to E$ is a mild solution for the following Caputo's fractional derivative order of partial functional differential evolution equations with delay*

$$\begin{cases} {}^cD_0^\alpha y(t) = A(t)y(t) + f(t, y_t), \quad 0 < \alpha < 1 \ a.e. \ \ t \in J \\ \\ y(t) = \varphi(t), \quad t \in H \end{cases} \tag{1.4}$$

If $y(t) = \varphi(t)$ for all $t \in H$ and y satisfies the following integral equation :

$$y(t) = U(t,0)\ \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s)f(s, y_s)\ ds \quad \textit{for each} \ \ t \in J. \tag{1.5}$$

1.5 Controllability of Mild Solution

Controllability of differential equations is the origin of the study of differential inclusions. Let us consider the control problem of the following evolution equations with delay

$$\begin{cases} y'(t) &= A(t)y(t) + Cu(t) + f(t, y_t), \quad \text{a.e.} \ \ t \in J \\ y(t) &= \varphi(t), \quad t \in H, \end{cases} \tag{1.6}$$

where $f : J \times C(H; E) \to E$, $\rho : J \times C(H; E) \to \mathbb{R}$ and $\varphi \in C(H; E)$ are given functions and $\{A(t)\}_{0 \le t < b}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s) \in J \times J}$ for $0 \le s \le t$. The control function $u(.)$ is given in $L^2(J; E)$, the Banach space of admissible control function with E is a real separable Banach space with the norm $|\cdot|$, C is a bounded linear operator from E into E.

Definition 1.5.1. *The mild solution y of the problem evolution* (1.6) *is said to be controllable on the interval $H \cup J$ if for every initial function $\varphi \in H$ and $y_1 \in E$, there exists a control $u \in L^2(J; E)$ such that the mild solution $y(\cdot)$ of the problem* (1.6) *satisfies $y(b) = y_1$.*

1.6 Phase Space $\mathcal{B}$

The notation of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato (see [60]). For further applications see for instance the books of Hale *et al.* [62], Hino and his collaborators [74] and Lakshmikantham [83] and their references.

For any $t \in J$ denote $B_{+\infty} = \{y :]-\infty, b] \to E \ : \ y|_J \in \mathcal{C}(J; E), \ y_0 \in \mathcal{B}\}$ or $y|_J$ is the restriction of y on J. Consider that the space $(\mathcal{B}, \|(.,.)\|_{\mathcal{B}})$ is a seminormed linear space of functions that mapping $(-\infty, 0] \times (-\infty, 0]$ into E, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

(A_1) If $y : (-\infty, b) \to E, \ b > 0,$ is continuous on $[0, b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0, b)$ the following conditions hold :
(i) $y_t \in \mathcal{B}$;
(ii) There exists a positive constant H such that $|y(t)| \le H\|y_t\|_{\mathcal{B}}$;

(iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{ \ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$ valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

Denote $K_b = \sup\{K(t) : t \in [0, b]\}$ and $M_b = \sup\{M(t) : t \in [0, b]\}$.

Remark 1.6.1. 1. *(ii) is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.*

2. *Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.*

3. *From the previous equivalence in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$: We necessarily have that $\phi(0) = \psi(0)$.*

Now we indicate some examples of phase spaces. For other details we refer, for instance to the book by Hino *et al* [74].

Example 1.6.1. *The spaces BC, BUC, C_1 and C_0. Let:*

BC *the space of bounded continuous functions defined from $(-\infty, 0]$ to E;*

BUC *the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E;*

$$C^\infty := \left\{\phi \in BC : \lim_{\theta\to-\infty} \phi(\theta) \ \textit{exist in } E\right\};$$

$$C^0 := \left\{\phi \in BC : \lim_{\theta\to-\infty} \phi(\theta) = 0\right\}, \textit{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\{|\phi(\theta)| : \ \theta \leq 0\}.$$

We have that the spaces BUC, C^∞ and C^0 satisfy conditions $(A_1) - (A_3)$. However, BC satisfies $(A_1), (A_3)$ but (A_2) is not satisfied.

Remark 1.6.2. $C^0 \subset C^\infty$.

Example 1.6.2. *The spaces C_g, UC_g, C_g^∞ and C_g^0.*
Let g be a positive continuous function on $(-\infty, 0]$. We define

$$C_g := \left\{\phi \in C((-\infty, 0], E) : \frac{\phi(\theta)}{g(\theta)} \ \textit{is bounded on } (-\infty, 0]\right\};$$

$$C_g^0 := \left\{\phi \in C_g : \lim_{\theta\to-\infty} \frac{\phi(\theta)}{g(\theta)} = 0\right\}, \textit{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)} : \ \theta \leq 0\right\}.$$

Then we have that the spaces C_g and C_g^0 satisfy conditions (A_3). We consider the following condition on the function g.

(g_1) *For all $a > 0$,* $\sup\limits_{0\le t\le a} \sup \left\{ \dfrac{g(t+\theta)}{g(\theta)} : -\infty < \theta \le -t \right\} < \infty.$

They satisfy conditions (A_1) and (A_2) if (g_1) holds.

Remark 1.6.3. *For $g \equiv 1$, we have $C_1 = BC$ and $C_1^0 = C_0$*

Example 1.6.3. *The space C_γ.*
For any real constant γ, we define the functional space C_γ by

$$C_\gamma := \left\{ \phi \in C((-\infty, 0], E) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \ \textit{exists in } E \right\}$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \ \theta \le 0\}.$$

Then in the space C_γ the axioms $(A_1) - (A_3)$ are satisfied.

1.7 State-Dependent Delay

1. Finite delay : We always assume that $\rho : J \times C(H; E) \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

$$\mathcal{R}(\rho^-) = \{\rho(s, \varphi) : (s, \varphi) \in J \times C(\mathcal{H}; E), \ \rho(s, \varphi) \le 0\}.$$

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $C(H; E)$ and there exists a continuous and bounded function $\mathcal{L}^\varphi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\varphi_t\| \le \mathcal{L}^\varphi(t)\|\varphi\| \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 1.7.1. *The condition (H_φ), is frequently verified by functions continuous and bounded. For more details, see for instance [1, 70, 74].*

Lemma 1.7.1. *([70], Lemma 2.4) If $y : [-r, b] \to E$ is a function such that $y_0 = \varphi$, then*

$$\|y_s\| \le \mathcal{L}^\varphi\|\varphi\| + \sup\{|y(\theta)|; \theta \in [0, max\{0, s\}]\}, \ s \in \mathcal{R}(\rho^-) \cup J,$$

where $\mathcal{L}^\varphi = \sup\limits_{t\in\mathcal{R}(\rho^-)} \mathcal{L}^\varphi(t)$.

2. Infinite delay : We always assume that $\rho : J\times\mathrm{B} \longrightarrow \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

$$\mathcal{R}(\rho^-) = \{\rho(s, \phi) : (s, \phi) \in J \times \mathcal{B}, \rho(s, \phi) \le 0\}.$$

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \le \mathcal{L}^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 1.7.2. *The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see for instance [1, 70, 74].*

Lemma 1.7.2. *([70], Lemma 2.4) If $y : (-\infty, b] \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} + K_b \sup\{|y(\theta)|; \theta \in [0, max\{0, s\}]\}, \ s \in \mathcal{R}(\rho^-) \cup J,$$

where $\mathcal{L}^\phi = \sup\limits_{t \in \mathcal{R}(\rho^-)} \mathcal{L}^\phi(t)$. K_b and M_b are as defined in the previous assumption $(A1)$ in the condition (iii).

1.8 Some Properties of Fractional Calculs

In this section, we introduce the notations, definitions and preliminary lemmas concerning to partial fractional calculs theory.

Lemma 1.8.1. *Let $\upsilon : J \to [0, \infty)$ be a real function and $\omega(.)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < \alpha < 1$ such that*

$$\upsilon(t) \leq \omega(t) + c \int_0^t \frac{\upsilon(s)}{(t-s)^\alpha} ds, \tag{1.7}$$

then there exists a constant $\delta = \delta(\alpha)$ such that

$$\upsilon(t) \leq \omega(t) + \delta c \int_0^t \frac{\omega(s)}{(t-s)^\alpha} ds, \tag{1.8}$$

for every $t \in J$.

Lemma 1.8.2. *The system (1.4) is equivalent to the nonlinear integral equation*

$$y(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s) y(s)\ ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s)\ ds \qquad t \in J. \tag{1.9}$$

In other words, every solution of the integral equation (1.9) is also solution of the system (1.4) and vice versa.

Definition 1.8.1. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is defined as*

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \tag{1.10}$$

Provide the right hand side exists pointwise on $\mathbb{R}^+$. $\Gamma(.)$ is the Euler gamma function. For instance, $I^\alpha f$ exists for all $\alpha > 0$. When $h \in C^0(R^+) \cap L^1_{loc}(R^+)$. Note also that when $h \in C^0(R_0^+)$ then $I^\alpha f \in C^0(R_0^+)$ and moreover $I^\alpha f(0) = 0$.

Definition 1.8.2. *The fractional derivative of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ in the Caputo sense is given by*

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{m-\alpha-1} f(s) ds = \frac{d}{dt} I_0^{1-\alpha} f(t).$$

1.9 Definition of Multivalued Functions

Let (X,d) be a metric space. We use the following notations :

$$\mathcal{P}_{cl}(X)=\{Y\in\mathcal{P}(X):Y\ closed\},\quad \mathcal{P}_{b}(X)=\{Y\in\mathcal{P}(X):Y\ bounded\},$$

$$\mathcal{P}_{cv}(X)=\{Y\in\mathcal{P}(X):Y\ convexe\},\quad \mathcal{P}_{cp}(X)=\{Y\in\mathcal{P}(X):Y\ compact\}.$$

Consider $H_d:\mathcal{P}(X)\times\mathcal{P}(X)\longrightarrow\mathbb{R}_+\cup\{\infty\}$, given by

$$H_d(\mathcal{A},\mathcal{B})=\max\left\{\sup_{a\in\mathcal{A}}\ d(a,\mathcal{B})\ ,\ \sup_{b\in\mathcal{B}}\ d(\mathcal{A},b)\ \right\},$$

where $d(\mathcal{A},b)=\inf\limits_{a\in\mathcal{A}}\ d(a,b)$, $d(a,\mathcal{B})=\inf\limits_{b\in\mathcal{B}}\ d(a,b)$. Then $(\mathcal{P}_{b,cl}(X),H_d)$ is a metric space and $(\mathcal{P}_{cl}(X),H_d)$ is a generalized (complete) metric space (see [80]).

Definition 1.9.1. *A multivalued map $G:J\to\mathcal{P}_{cl}(X)$ is said to be measurable if for each $x\in E$, the function $Y:J\to X$ defined by*

$$Y(t)=d(x,G(t))=\inf\{|x-z|:z\in G(t)\}$$

is measurable where d is the metric induced by the normed Banach space X.

Definition 1.9.2. *A function $F:J\times C(H;E)\longrightarrow\mathcal{P}(X)$ is said to be an L^1_{loc}-Carathéodory multivalued map if it satisfies :*

(i) $x\mapsto F(t,y)$ is continuous (with respect to the metric H_d) for almost all $t\in J$;

(ii) $t\mapsto F(t,y)$ is measurable for each $y\in C(H;E)$;

(iii) for every positive constant k there exists $h_k\in L^1_{loc}(J;\mathbb{R}^+)$ such that

$$\|F(t,y)\|\leq h_k(t)\quad \text{for all } \|y\|\leq k \text{ and for almost all } t\in J.$$

Let $(X,\|\cdot\|)$ be a Banach space. A multivalued map $G:X\to\mathcal{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x\in X$. We say that G is *bounded* on bounded sets if $G(B)$ is bounded in X for each bounded set B of X, i.e.,

$$\sup_{x\in B}\{\sup\{\ \|y\|\ :\ y\in G(x)\}\}<\infty.$$

Finally, we say that G has a *fixed point* if there exists $x\in X$ such that $x\in G(x)$.

For each $y\in B_{+\infty}$ let the set $S_{F,y}$ known as *the set of selectors* from F defined by

$$S_{F,y}=\{v\in L^1(J;E):v(t)\in F(t,y_t)\ ,\ a.e.\ t\in J\}.$$

For more details on multivalued maps we refer to the books of Deimling [45], Górniewicz [56], Hu and Papageorgiou [75] and Tolstonogov [96].

Definition 1.9.3. *A multivalued map $F:X\to\mathcal{P}(X)$ is called an admissible contraction with constant $\{k_n\}_{n\in\mathbb{N}}$ if for each $n\in\mathbb{N}$ there exists $k_n\in(0,1)$ such that*

i) $H_d(F(x),F(y))\leq k_n\ \|x-y\|_n$ for all $x,y\in X$.

ii) for every $x\in X$ and every $\epsilon\in(0,\infty)^n$, there exists $y\in F(x)$ such that

$$\|x-y\|_n\leq\|x-F(x)\|_n+\epsilon_n \text{ for every } n\in\mathbb{N}$$

1.10 Fixed Point Theorems

In this section we give some fixed point theorems that will be used in the sequel. All the abstract results established in this thesis are based on these theorems.

1. Nonlinear Alternative in Fréchet space:

Theorem 1.10.1. *(Nonlinear Alternative of Frigon-Granas, [52]). Let X be a Fréchet space and $Y \subset X$ be a closed subset and let $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds :*

$(C1)$ N has a unique fixed point ;

$(C2)$ There exists $\lambda \in [0,1)$, $n \in \mathbb{N}$ and $x \in \partial_n Y^n$ such that $\|x - \lambda\ N\ (x)\|_n = 0$.

Theorem 1.10.2. *(Nonlinear Alternative of Avramescu [18]). Let X be a Fréchet space and $A, B : X \longrightarrow X$ two operators satisfying:*

(1) *A is a compact operator.*

(2) *B is a contraction.*
Then either

$(C1)$ *the operator $A + B$ has a fixed points.Or*

$(C2)$ *the set $\{x \in X, x = \lambda A(x) + \lambda B(\frac{x}{\lambda})\}$ is unbounded for $\lambda \in]0,1[$.*

Theorem 1.10.3. *(Nonlinear Alternative of Frigon, [50, 51]). Let X be a Fréchet space and U an open neighborhood of the origin in X and let $N : \overline{U} \to \mathcal{P}(X)$ be an admissible multivalued contraction. Assume that N is bounded. Then one of the following statements holds :*

$(C1)$ *N has a fixed point ;*

$(C2)$ *There exists $\lambda \in [0,1)$ and $x \in \partial U$ such that $x \in \lambda\ N\ (x)$.*

2. Nonlinear Alternative in Banach space:

Theorem 1.10.4. *(Nonlinear Alternative of Frigon-Granas, [52]). Let X be a Banach space and $Y \subset X$ be a closed subset and let $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds :*

$(C1)$ N has a unique fixed point ;

$(C2)$ There exists $\lambda \in [0,1)$ and $x \in \partial Y$ such that $\|x - \lambda\ N\ (x)\| = 0$.

Theorem 1.10.5. *(Nonlinear Alternative of Avramescu [18]). Let X be a Banach space and $A, B : X \longrightarrow X$ two operators satisfying:*

(1) *A is a compact operator.*

(2) *B is a contraction.*
Then either

$(C1)$ *the operator $A + B$ has a fixed points.Or*

$(C2)$ *the set $\{x \in X, x = \lambda A(x) + \lambda B(\frac{x}{\lambda})\}$ is unbounded for $\lambda \in]0,1[$.*

Theorem 1.10.6. *(Nonlinear Alternative of Frigon, [50, 51]). Let X be a Banach space and U an open neighborhood of the origin in X and let $N : \overline{U} \to \mathcal{P}(X)$ be an admissible multivalued contraction. Assume that N is bounded. Then one of the following statements holds :*

$(C1)$ *N has a fixed point ;*

$(C2)$ *There exists $\lambda \in [0, 1)$ and $x \in \partial U$ such that $x \in \lambda\ N\ (x)$.*

Chapter 2

Fractional Evolution Equations with State-Dependent Delay

2.1 Introduction

In this chapter, we shall establish sufficient conditions for the existence of the unique mild solution for some classes of fractional for partial functional and neutral functional differential evolution equations with finite and infinite state-dependent delay involving the Caputo's fractional derivative order[1].

Using the alternative of Leary-Schauder type for contraction maps given by Frigon and Granas in Banach space (see [52]), combined with the semi-group theory.

To our Knowledge, there are very few papers devoted to fractional differential equations with delay. By using a fractional version of Gronwall's inequality, we demonstrate the existence of the unique mild solution in section 2.2 for the following class of fractional evolution equations with finite state-dependent delay

$$^cD_0^\alpha y(t) = A(t)y(t) + f(t, y_{\rho(t,y_t)}), \quad 0 < \alpha < 1 \text{ a.e. } t \in J = [0, b] \tag{2.1}$$

$$y(t) = \varphi(t), \quad t \in H = [-r, 0], \tag{2.2}$$

where $b > 0$, $0 < r < +\infty$, $^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0, 1)$, $f : J \times C(H; E) \to E$, $\rho : J \times C(H; E) \to \mathbb{R}$ and $\varphi \in C(H; E)$ are given functions and $\{A(t)\}_{t \in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of bounded linear operators $\{U(t, s)\}_{(t,s) \in J \times J}$ for $s \leq t$.

An extension of this problem is given in section 2.3 for the following class of fractional neutral evolution equations with finite state-dependent delay

$$^cD_0^\alpha [y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + f(t, y_{\rho(t,y_t)}), \quad 0 < \alpha < 1 \quad \text{a.e.} \quad t \in J \tag{2.3}$$

$$y(t) = \varphi(t), \quad t \in H, \tag{2.4}$$

where $A(\cdot)$, f and φ are as in the above problem $(2.1) - (2.2)$ and $g : J \times C(H; E) \to E$ is a given function.

[1][12] **D. Aoued**, S. Baghli-Bendimerad and M. Benchohra, Fractional partial functional and neutral functional evolution equations with infinite state-dependent delay, submitted.

In section 2.4, we study the fractional evolution equations with infinite state-dependent delay

$$^{c}D_{0}^{\alpha}y(t) = A(t)y(t) + f(t, y_{\rho(t,y_t)}), \qquad 0 < \alpha < 1 \text{ a.e. } \ t \in J \tag{2.5}$$

$$y_0 = \phi \in \mathcal{B}, \tag{2.6}$$

where $\mathcal{B}$ is an abstract *phase space* to be specified later, $^{c}D_{0}^{\alpha}$ is the standard Caputo's fractional derivative of order $\alpha \in (0,1)$, $f : J \times \mathcal{B} \to E, \rho : J \times \mathcal{B} \to \mathrm{I\!R}$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 2.5 for the following class of fractional neutral evolution equations with infinite state-dependent delay

$$^{c}D_{0}^{\alpha}[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + f(t, y_{\rho(t,y_t)}), \qquad 0 < \alpha < 1 \quad \text{a.e.} \quad t \in J \tag{2.7}$$

$$y_0 = \phi \in \mathcal{B}, \tag{2.8}$$

where $A(\cdot)$, f and ϕ are as in problem $(2.5) - (2.6)$ and $g : J \times \mathcal{B} \to E$ is a given function. Finally, section 2.6 is devoted to examples illustrating the abstract theory considered in provious sections.

2.2 Partial Problem with Finite Delay

Before stating and proving the main result, we give first the definition of mild solution of the fractional problem $(2.1) - (2.2)$.

Lemma 2.2.1. *The system* $(2.1) - (2.2)$ *is equivalent to the nonlinear integral equation*

$$y(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A(s)y(s)\ ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_{\rho(s,y_s)})\ ds. \tag{2.9}$$

In other words, every solution of the integral equation (2.9) *is also mild solution of the system* $(2.1) - (2.2)$ *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system $(2.1) - (2.2)$, and using some classical results from fractional calculus to get (2.9).

Definition 2.2.1. *We say that the function* $y(\cdot) : [-r, b] \to E$ *is a mild solution of* $(2.1) - (2.2)$ *if* $y(t) = \varphi(t)$ *for all* $t \in [-r, 0]$ *and* y *satisfies the following integral equation*

$$y(t) = U(t,0)\ \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s, y_{\rho(s,y_s)})\ ds \qquad \textit{for each } t \in J. \tag{2.10}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times C(H;E), \rho(s,\varphi) \leq 0\}.$$

We always assume that $\rho : J \times C(H;E) \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $C(H;E)$ and there exists a continuous and bounded function $\mathcal{L}^\varphi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\varphi_t\| \leq \mathcal{L}^\varphi(t)\|\varphi\| \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

We will need to introduce the following hypothesis which are assumed thereafter :

$(H1)$ There exists a constant $\widehat{M} \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq \widehat{M} \quad \text{for every } (t,s) \in \Delta := \{(t,s) \in J \times J : 0 \leq s \leq t \leq b\}.$$

$(H2)$ There exist two functions $p, q \in L^1_{loc}(J; \mathbb{R}_+)$ such that

$$|f(t,u)| \leq p(t) + q(t)\ \|u\|_B \text{ for a.e. } t \in J \text{ and each } u \in C(H;E).$$

$(H3)$ For all $R > 0$, there exists $l_R \in L^1_{loc}(J; \mathbb{R}_+)$ such that

$$|f(t,u) - f(t,v)| \leq l_R(t)\ \|u - v\|$$

for all $u, v \in C(H;E)$ with $\|u\| \leq R$ and $\|v\| \leq R$.

Theorem 2.2.1. *Assume that the hypotheses (H_φ) and $(H1) - (H3)$ hold and moreover*

$$\frac{\widehat{M} l_n^* b^\alpha}{\Gamma(1+\alpha)} < 1 \tag{2.11}$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ and $l_n^* = \sup l_n(s)$. Then the fractional evolution problem $(2.1) - (2.2)$ has a unique mild solution on $[-r, b]$.

Proof. Transform the problem $(2.1) - (2.2)$ into a fixed-point problem. Consider the operator $N : C([-r,b];E) \to C([-r,b];E)$ defined by :

$$N(y)(t) = \begin{cases} \varphi(t), & \text{if } t \in H; \\ U(t,0)\ \varphi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s, y_{\rho(s,y_s)})\ ds, & \text{if } t \in J. \end{cases} \tag{2.12}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(2.1) - (2.2)$.

Let y be a possible solution of the problem $(2.1) - (2.2)$. Given $t \leq b$, then from $(H1)$, $(H2)$, (H_φ) and Lemma 5.2.1, we have for each $t \in [0,b]$

$$\begin{aligned} |y(t)| &\leq \|U(t,0)\|_{B(E)} |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \|U(t,s)\|_{B(E)}\ |f(s, y_{\rho(s,y_s)})|\ ds \\ &\leq \widehat{M}\|\varphi\| + \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s)\|y_{\rho(s,y_s)}\|\right] ds \end{aligned}$$

It follows that

$$|y(t)| + \mathcal{L}^\varphi\|\varphi\| \leq \widehat{M}\|\varphi\| + \mathcal{L}^\varphi\|\varphi\| + \frac{\widehat{M} p^* b^\alpha}{\Gamma(\alpha+1)} + \frac{\widehat{M} q^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \left[|y(s) + \mathcal{L}^\varphi\|\varphi\|\right] ds$$

Set

$$\delta_b := (\widehat{M} + \mathcal{L}^\varphi)\|\varphi\| + \frac{\widehat{M} p^* b^\alpha}{\Gamma(\alpha+1)}.$$

We consider the function μ defined by

$$\mu(t) := \sup \{ \ |y(s)| + \mathcal{L}^{\varphi}\|\varphi\| \ : \ 0 \leq s \leq t \ \}, \quad 0 \leq t \leq b.$$

Let $t^{\star} \in [-r, t]$ be such that $\mu(t) = |y(t^{\star})| + \mathcal{L}^{\varphi}\|\varphi\|$. If $t^{\star} \in [0, b]$, by the previous inequality, we get

$$\mu(t) \ \leq \ \delta_b + \frac{\widehat{M}q^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ \mu(s)ds \qquad \text{for } t \in [0, b].$$

If $t^{\star} \in [-r, 0]$, then $\mu(t) = \|\varphi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta_b = \delta_b(\alpha)$ such that

$$\mu(t) \ \leq \ \delta_b \times \left[1 + \frac{\widehat{M}q^* b^{\alpha}}{\Gamma(\alpha+1)}\right] := \Lambda_b$$

Since $\|y\| \leq \mu(t)$, we have $\|y\| \leq \max\{\|\varphi\|, \Lambda_b\} := \Theta_b$. Since for every $t \in [0, b]$, we have $\|y\| \leq \max\{\|\varphi\|, \Lambda_b\} := \Theta_b$.

Set

$$Y = \{ \ y \in C([-r, b]; E) \ : \ \sup\{|y(t)| : 0 \leq t \leq b\} \leq \Theta_b + 1\}.$$

Clearly, Y is a closed subset of $C([-r, b]; E)$.

We shall show that $N : Y \to C([-r, b]; E)$ is a contraction operator.
Indeed, consider $y, \overline{y} \in Y$, thus using $(H1)$ and $(H3)$ for each $t \in [0, b]$

$$\begin{aligned} |N(y)(t) - N(\overline{y})(t)| \ &\leq \ \frac{1}{\Gamma(\alpha)} \int_0^t \ (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \ \left|f(s, y_{\rho(s,y_s)}) - f(s, \overline{y}_{\rho(s,y_s)})\right| \ ds \\ &\leq \ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ l_n(s) \ \|y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)}\| \ ds. \end{aligned}$$

Using (H_{φ}) and Lemma 5.2.1, we obtain

$$\begin{aligned} |N(y)(t) - N(\overline{y})(t)| \ &\leq \ \frac{\widehat{M}l_n^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, |y(s) - \overline{y}(s)| \ ds \\ &\leq \ \frac{\widehat{M}l_n^* b^{\alpha}}{\Gamma(\alpha+1)} \|y - \overline{y}\| \end{aligned}$$

Therefore,

$$\|N(y) - N(\overline{y})\| \leq \frac{\widehat{M}l_n^* b^{\alpha}}{\Gamma(\alpha+1)} \|y - \overline{y}\|.$$

So by (6.16), the operator N is a contraction. From the choice of Y there is no $y \in \partial Y^n$ such that $y = \lambda \ N(y), \ \lambda \in (0, 1)$. Then the statement $(C2)$ in Theorem 1.10.4 does not hold. The nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. Thus, we deduce that the operator N has a unique fixed-point $y^{\star}$ which is the unique mild solution of the problem $(2.1) - (2.2)$.

2.3 Neutral Problem with Finite Delay

We give here an extension to previous results for the neutral case (2.3) − (2.4). Firstly, we define its mild solution.

Lemma 2.3.1. *The system* (2.3) − (2.4) *is equivalent to the nonlinear integral equation*

$$y(t) \;=\; [\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A(s)y(s)\,ds \tag{2.13}$$

$$+ \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_{\rho(s,y_s)})\,ds \qquad t \in J.$$

In other words, every solution of the integral equation (2.13) *is also solution of the system* (2.3) − (2.4) *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system (2.3) − (2.4), and using some classical results from fractional calculus to get (2.13).

Definition 2.3.1. *We say that the function* $y(\cdot) : [-r, b] \to E$ *is a mild solution of* (2.3) − (2.4) *if* $y(t) = \varphi(t)$ *for all* $t \in H$ *and* y *satisfies the following integral equation*

$$y(t) \;= U(t,0)[\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\; U(t,s) f(s, y_{\rho(s,y_s)})\,ds. \tag{2.14}$$

We consider the hypotheses (H_φ), $(H1) - (H3)$ and we need to introduce the following assumptions :

$(H4)$ There exists a constant $\overline{M}_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \le \overline{M}_0 \quad \text{for all } \; t \in J.$$

$(H5)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0}$ such that

$$|A(t)\; g(t,\varphi)| \le L\; (\|\varphi\| + 1) \text{ for all } t \in J \text{ and } \varphi \in C(H; E).$$

$(H6)$ There exists a constant $L_\star > 0$ such that

$$|A(s)g(s,\varphi) - A(\overline{s})g(\overline{s},\overline{\varphi})| \le L_\star\; (|s - \overline{s}| + \|\varphi - \overline{\varphi}\|)$$

for all $\varphi, \overline{\varphi} \in C(H; E)$.

Theorem 2.3.1. *Suppose that the hypotheses* $(H1) - (H5)$ *are satisfied and moreover*

$$\left[\overline{M}_0 L_\star + \frac{\widehat{M} L_n^* b^\alpha}{\Gamma(\alpha+1)}\right] < 1 \tag{2.15}$$

where $p^* = \sup p(s)$*,* $q^* = \sup q(s)$*and* $l_n^* = \sup l_n(s)$*. Then the problem* (2.3) − (2.4) *has a unique mild solution on* $[-r, b]$*.*

Proof. Transform as below the neutral problem (2.3) – (2.4) into a fixed point problem by considering the operator $\widetilde{N} : C([-r,b];E) \to C([-r,b];E)$ defined by :

$$\widetilde{N}(y)(t) = \begin{cases} \varphi(t), & \text{if } t \in H; \\ U(t,0)\ [\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) & \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s, y_{\rho(s,y_s)})\ ds, & \text{if } t \in J. \end{cases} \tag{2.16}$$

Clearly, the fixed points of the operator $\widetilde{N}$ are mild solutions of the problem (2.3) – (2.4).

Let y be a possible solution of the problem (2.3) – (2.4). Given $t \leq b$. Then, using $(H1)-(H2)$ and $(H4)-(H5)$, (H_φ) and Lemma 5.2.1, we have for each $t \in [0,b]$

$$\begin{aligned} |y(t)| &\leq |U(t,0)[\varphi(0) - g(0,\varphi)]| + |g(t, y_{\rho(t,y_t)})| \\ &+ \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s, y_{\rho(s,y_s)})\ ds \right| \\ &\leq \|A^{-1}(t)\|\ |A(t) g(t, y_{\rho(t,y_t)})| + \widehat{M}\|\varphi\| + \|U(t,0)\|_{B(E)}\ \|A^{-1}(0)\|\ |A(0) g(0,\varphi)| \\ &+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s)\|y_{\rho(s,y_s)}\|\right] ds \\ &\leq \widehat{M}\|\varphi\| + \overline{M}_0 L \left(\|y_{\rho(t,y_t)}\| + 1\right) + \widehat{M}\overline{M}_0 L\left(\|\varphi\| + 1\right) \\ &+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s)\|y_{\rho(s,y_s)}\|\right] ds \end{aligned}$$

Since $\|y_{\rho(t,y_t)}\| \leq |y(t)| + \mathcal{L}^\varphi\|\varphi\|$ we obtain

$$\begin{aligned} |y(t)| &\leq \overline{M}_0 L\left(|y(t)| + \mathcal{L}^\varphi\|\varphi\|\right) + \widehat{M}\|\varphi\|(1 + \overline{M}_0 L) + \widehat{M}\overline{M}_0 L + \frac{\widehat{M}p^* b^\alpha}{\Gamma(\alpha+1)} \\ &+ \frac{\widehat{M}q^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \left(|y(s)| + \mathcal{L}^\varphi\|\varphi\|\right)\ ds \end{aligned}$$

Then

$$\begin{aligned} (1 - \overline{M}_0 L)|y(t)| &\leq \left[\overline{M}_0 L \mathcal{L}^\varphi + \widehat{M}(1 + \overline{M}_0 L)\right] \|\varphi\| + \widehat{M}\overline{M}_0 L + \frac{\widehat{M}p^* b^\alpha}{\Gamma(\alpha+1)} \\ &+ \frac{\widehat{M}q^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \left(|y(s)| + \mathcal{L}^\varphi\|\varphi\|\right)\ ds \end{aligned}$$

Set

$$\beta_b := \frac{\left[\overline{M}_0 L \mathcal{L}^\varphi + \widehat{M}(1 + \overline{M}_0 L)\right]}{(1 - \overline{M}_0 L)}\|\varphi\| + \frac{\widehat{M}\overline{M}_0 L}{(1 - \overline{M}_0 L)} + \frac{\widehat{M}p^* b^\alpha}{(1 - \overline{M}_0 L)\Gamma(\alpha+1)} + \mathcal{L}^\varphi\|\varphi\|.$$

Thus

$$|y(t)| + \mathcal{L}^\varphi\|\varphi\| \leq \beta_b + \frac{\widehat{M}q^*}{(1 - \overline{M}_0 L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \left(|y(s)| + \mathcal{L}^\varphi\|\varphi\|\right)\ ds.$$

We consider the function μ defined by

$$\mu(t) := \sup \{ \ |y(s)| + \mathcal{L}^{\varphi}\|\varphi\| \ : \ 0 \leq s \leq t \ \}, \quad 0 \leq t \leq b.$$

Let $t^{\star} \in [-r, t]$ be such that $\mu(t) = |y(t^{\star})| + \mathcal{L}^{\varphi}\|\varphi\|$. If $t^{\star} \in [0, b]$, by the previous inequality, we get

$$\mu(t) \ \leq \ \beta_b + \frac{\widehat{M}q^*}{(1-\overline{M}_0 L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ \mu(s)ds \qquad \text{for } t \in [0, b].$$

If $t^{\star} \in [-r, 0]$, then $\mu(t) = \|\varphi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\beta_b = \beta_b(\alpha)$ such that

$$\mu(t) \ \leq \ \beta_b \times \left[1 + \frac{\widehat{M}q^* b^{\alpha}}{(1-\overline{M}_0 L)\Gamma(\alpha+1)} \right] := \Lambda_b$$

Since $\|y\| \leq \mu(t)$, we have $\|y\| \leq \max\{\|\varphi\|, \Lambda_b\} := \Theta_b$.

Now, we shall show that $\widetilde{N} : Y \to C([-r, b]; E)$ is a contraction operator.
Indeed, consider $y, \overline{y} \in Y$, thus using $(H1)$ and $(H3)-(H4)$ for each $t \in [0, b]$

$$\begin{aligned}
|\widetilde{N}(y)(t) - \widetilde{N}(\overline{y})(t)| \ &\leq \ |g(t, y_{\rho(t,y_t)}) - g(t, \overline{y}_{\rho(t,y_t)})| \\
&+ \ \frac{1}{\Gamma(\alpha)} \int_0^t \ (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \ \left| f(s, y_{\rho(s,y_s)}) - f(s, \overline{y}_{\rho(s,y_s)}) \right| \ ds \\
&\leq \ \|A^{-1}(t)\| |A(t)g(t, y_{\rho(t,y_t)}) - A(t)g(t, \overline{y}_{\rho(t,y_t)})| \\
&+ \ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ l_n(s) \ \|y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)}\| \ ds.
\end{aligned}$$

Using (H_{φ}) and Lemma 5.2.1, we obtain

$$\begin{aligned}
|\widetilde{N}(y)(t) - \widetilde{N}(\overline{y})(t)| \ &\leq \ \overline{M}_0 L_{\star} |y(t) - \overline{y}(t)| + \frac{\widehat{M}L_n^* b^{\alpha}}{\Gamma(\alpha+1)} \|y - \overline{y}\| \\
&\leq \ \left[\overline{M}_0 L_{\star} + \frac{\widehat{M}L_n^* b^{\alpha}}{\Gamma(\alpha+1)} \right] \|y - \overline{y}\|
\end{aligned}$$

Therefore,

$$\|\widetilde{N}(y) - \widetilde{N}(\overline{y})\| \leq \left[\overline{M}_0 L_{\star} + \frac{\widehat{M}L_n^* b^{\alpha}}{\Gamma(\alpha+1)} \right] \|y - \overline{y}\|.$$

So, for an appropriate choice of $\overline{M}_0 L_{\star}, L_n^*$ and b^{α} such that

$$\left[\overline{M}_0 L_{\star} + \frac{\widehat{M}L_n^* b^{\alpha}}{\Gamma(\alpha+1)} \right] < 1,$$

the operator $\widetilde{N}$ is a contraction . From the choice of Y there is no $y \in \partial Y^n$ such that $y = \lambda \, \widetilde{N}(y)$ for some $\lambda \in (0, 1)$. Then the statement $(C2)$ in Theorem 1.10.4 does not hold. A consequence of the nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. We deduce that the operator $\widetilde{N}$ has a unique fixed-point $y^{\star}$ which is the unique mild solution of the problem $(2.3)-(2.4)$.

2.4 Partial Problem with Infinite Delay

Before stating and proving the main result, we give first the definition of mild solution of the fractional problem $(2.5)-(2.6)$.

Lemma 2.4.1. *The system* $(2.5)-(2.6)$ *is equivalent to the nonlinear integral equation*

$$y(t)=\phi(0)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A(s)y(s)\ ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y_{\rho(s,y_s)})\ ds. \tag{2.17}$$

In other words, every solution of the integral equation (2.17) *is also solution of the system* $(2.5)-(2.6)$ *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system $(2.5)-(2.6)$, and using some classical results from fractional calculus to get (2.17).

Definition 2.4.1. *We say that the function* $y(\cdot):(-\infty,b]\to E$ *is a mild solution of* $(2.5)-(2.6)$ *if* $y(t)=\phi(t)$ *for all* $t\leq 0$ *and* y *satisfies the following integral equation*

$$y(t)=U(t,0)\ \phi(0)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ U(t,s)f(s,y_{\rho(s,y_s)})\ ds \qquad \text{for each } t\in J. \tag{2.18}$$

Set

$$\mathcal{R}(\rho^-)=\{\rho(s,\phi):(s,\phi)\in J\times\mathcal{B},\rho(s,\phi)\leq 0\}.$$

We always assume that $\rho:J\times\mathcal{B}\to\mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_ϕ) The function $t\to\phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi:\mathcal{R}(\rho^-)\to(0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}}\leq\mathcal{L}^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t\in\mathcal{R}(\rho^-).$$

We will need to introduce the following hypothesis which are assumed thereafter :

$(H01)$ There exists a constant $\widehat{M}\geq 1$ such that

$$\|U(t,s)\|_{B(E)}\leq\widehat{M} \quad \text{for every } (t,s)\in\Delta.$$

$(H02)$ There exist two functions $p,q\in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$|f(t,u)|\leq p(t)+q(t)\ \|u\|_B \text{ for a.e. } t\in J \text{ and each } u\in\mathcal{B}.$$

$(H03)$ For all $R>0$, there exists $l_R\in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$|f(t,u)-f(t,v)|\leq l_R(t)\ \|u-v\|$$

for all $u,v\in\mathcal{B}$ with $\|u\|\leq R$ and $\|v\|\leq R$.

We define in $C((-\infty, b]); E)$ the semi-norms by :

$$\|y\| := \sup \{ \; |y(t)| : t \in [0,b] \; \}$$

Consider the following space

$$\Omega = \left\{ y : (-\infty, b] \to E : y|_{(-\infty,0]} \; \in B \;\; \textit{and} \;\; y|_{[0,b]} \text{ is continuous } \right\},$$

Theorem 2.4.1. *Assume that the hypotheses (H_φ) and $(H01)-(H03)$ hold and moreover*

$$\frac{K_b \widehat{M} l_n^* b^\alpha}{\Gamma(\alpha+1)} < 1 \tag{2.19}$$

where $l_n^* = \sup l_n(s)$. Then the problem $(2.5)-(2.6)$ has a unique mild solution on $(-\infty, b]$.

Proof. We transform the problem $(2.5)-(2.6)$ into a fixed-point problem. Consider the operator $N : \Omega \to \Omega$ defined by :

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\ \phi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s, y_{\rho(s,y_s)})\ ds, & \text{if } t \in J. \end{cases} \tag{2.20}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(2.5)-(2.6)$.

For $\phi \in \mathcal{B}$, we will define the function $x(.) : (-\infty, b] \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\ \phi(0), & \text{if } \ t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in \Omega$, set

$$y(t) = z(t) + x(t)$$

It is obvious that y satisfies (2.18) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds \quad \text{ for } t \in J.$$

Let

$$\Omega^0 = \{ z \in \Omega : z_0 = 0 \}.$$

Define the operator $F : \Omega^0 \to \Omega^0$ by :

$$F(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds \quad \text{ for } t \in J. \tag{2.21}$$

Obviously the operator N has a fixed point is equivalent to F has one, so it turns to prove that F has a fixed point.

Let $z \in \Omega^0$ be be a possible fixed point of the operator. By the hypotheses $(H01)$ and $(H02)$, we have for each $t \in [0,b]$

$$
\begin{aligned}
|z(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \ |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \ ds \\
&\leq \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ \left[p(s) + q(s)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\right] \ ds
\end{aligned}
$$

From (H_ϕ), Lemma 1.7.2 and Assumption $(A1)$, we have for each $t \in [0,b]$

$$
\begin{aligned}
\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\leq \|z_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + \|x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \\
&\leq K_b|z(s)| + (M_b + \mathcal{L}^\phi)\|z_0\|_{\mathcal{B}} + K_b|x(s)| + (M_b + \mathcal{L}^\phi)\|x_0\|_{\mathcal{B}} \\
&\leq K_b|z(s)| + K_b\|U(s,0)\|_{B(E)}|\phi(0)| + (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} \\
&\leq K_b|z(s)| + K_b\widehat{M}|\phi(0)| + (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}
\end{aligned}
$$

Using (ii), we get

$$
\begin{aligned}
\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\leq K_b|z(s)| + K_b\widehat{M}H\|\phi\|_{\mathcal{B}} + (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} \\
&\leq K_b|z(s)| + (M_b + \mathcal{L}^\phi + K_b\widehat{M}H)\|\phi\|_{\mathcal{B}}
\end{aligned}
$$

Set $c_b := (M_b + \mathcal{L}^\phi + K_b\widehat{M}H)\|\phi\|_{\mathcal{B}}$. It follows that

$$
|z(t)| \leq \frac{\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)} + \frac{\widehat{M}q^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ \Big(K_b|z(s)| + (M_b + \mathcal{L}^\phi + K_b\widehat{M}H)\|\phi\|_{\mathcal{B}}\Big) \ ds
$$

Then, we have

$$
|z(t)| \leq \frac{\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)} + \frac{\widehat{M}q^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ (K_b|z(s)| + c_b) \ ds
$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ Then

$$
K_b|z(t)| + c_b \leq \frac{K_b\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)} + c_b + \frac{K_b\widehat{M}q^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ (K_b|z(s)| + c_b) \ ds.
$$

Set

$$
\delta_b := \frac{K_b\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)} + c_b.
$$

We consider the function μ defined by

$$
\mu(t) := \sup \{ \ K_b|z(s)| + c_b \ : \ 0 \leq s \leq t \ \}, \quad 0 \leq t \leq b.
$$

Let $t^\star \in [0,t]$ be such that $\mu(t) = K_b|z(t^\star)| + c_b$. By the previous inequality, we have

$$
\mu(t) \leq \delta_b + \frac{K_b\widehat{M}q^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ \mu(s)ds \qquad \text{for } t \in [0,b].
$$

If $t^\star \in (-\infty, 0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta = \delta(\alpha)$ such that

$$\mu(t) \quad \leq \quad \delta_b \times \left[1 + \frac{K_b \widehat{M} q^* b^\alpha}{\Gamma(\alpha+1)}\right] := \Lambda_b$$

Since $\|z\| \leq \mu(t)$, we have $\|z\| \leq \max\{\|\phi\|, \Lambda_b\} := \Theta_b$.

Set

$$Z = \{\ z \in \Omega^0\ :\ \sup_{0\leq t\leq b} |z(t)| \leq \Theta_b + 1\}.$$

Clearly, Z is a closed subset of Ω^0.

We shall show that $F : Z \to \Omega^0$ is a contraction operator.
Indeed, consider $z, \overline{z} \in Z$, thus using $(H01)$ and $(H03)$ for each $t \in [0, b]$

$$\begin{aligned} |F(z)(t) - F(\overline{z})(t)| \quad &\leq \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \ \big|f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \\ &\quad -f(s, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\big| \ ds \\ &\leq \quad \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ l_n(s)\ \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\ ds. \end{aligned}$$

Using (H_ϕ) and Lemma 1.7.2, we obtain

$$\begin{aligned} |F(z)(t) - F(\overline{z})(t)| \quad &\leq \quad \frac{\widehat{M} l_n^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_b\ \ |z(s) - \overline{z}(s)|\ \ ds \\ &\leq \quad \frac{K_b \widehat{M} l_n^* b^\alpha}{\Gamma(\alpha+1)} \|z(t) - \overline{z}(t)\|. \end{aligned}$$

Therefore,

$$\|F(z) - F(\overline{z})\| \leq \frac{K_b \widehat{M} l_n^* b^\alpha}{\Gamma(\alpha+1)} \|z - \overline{z}\|.$$

So, for $\dfrac{K_b \widehat{M} l_n^* b^\alpha}{\Gamma(\alpha+1)} < 1$ the operator F is a contraction. From the choice of Z there is no $z \in \partial Z^n$ such that $z = \lambda\ F(z),\ \ \lambda \in (0, 1)$. Then the statement $(C2)$ in Theorem 1.10.4 does not hold. The nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. Thus, we deduce that the operator F has a unique fixed-point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty, b]$ is a fixed point of the operator N, which is the unique mild solution of the problem $(2.5) - (2.6)$.

2.5 Neutral Problem with Infinite Delay

We give here an extension to previous results for the neutral case $(2.7) - (2.8)$. Firstly, we define its mild solution.

Lemma 2.5.1. *The system* $(2.7) - (2.8)$ *is equivalent to the nonlinear integral equation*

$$y(t) \quad = \quad [\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s) y(s)\ ds \qquad (2.22)$$

$$+ \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_{\rho(s,y_s)})\; ds.$$

In other words, every solution of the integral equation (2.22) *is also solution of the system* (2.7) − (2.8) .

Proof. It can be proved by applying the integral operator to both sides of the system (2.7) − (2.8), and using some classical results from fractional calculus to get (2.22).

Definition 2.5.1. *We say that the function* $y(\cdot) : (-\infty, b] \to E$ *is a mild solution of* (2.7)−(2.8) *if* $y(t) = \phi(t)$ *for all* $t \leq 0$ *and* y *satisfies the following integral equation*

$$y(t) \quad = U(t,0)[\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\; U(t,s) f(s, y_{\rho(s,y_s)})\; ds, \quad t \in J. \tag{2.23}$$

We consider the hypotheses (H_ϕ), $(H01) - (H03)$ and we need the following assumptions :

$(H04)$ There exists a constant $\overline{M}_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0 \quad \text{ for all } \; t \in J.$$

$(H05)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0 K_b}$ such that

$$|A(t)\; g(t,\varphi)| \leq L\; (\|\phi\| + 1) \text{ for all } t \in J \text{ and } \phi \in \mathcal{B}.$$

$(H06)$ There exists a constant $L_\star > 0$ such that

$$|A(s)g(s,\phi) - A(\overline{s})g(\overline{s},\overline{\phi})| \leq L_\star\; (|s-\overline{s}| + \|\phi - \overline{\phi}\|)$$

for all $\phi, \overline{\phi} \in \mathcal{B}$.

Theorem 2.5.1. *Suppose that the hypotheses* $(H01) - (H05)$ *are satisfied and moreover*

$$\left[\overline{M}_0 L_\star K_b + \frac{\widehat{M} K_b L_n^* b^\alpha}{\Gamma(\alpha+1)}\right] < 1 \tag{2.24}$$

where $l_n^* = \sup l_n(s)$. *Then the problem* (2.7) − (2.8) *has a unique mild solution on* $(-\infty, b]$.

Proof. Consider the operator $\widetilde{N} : \Omega \to \Omega$ defined by :

$$\widetilde{N}(y)(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\; [\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) & \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\; U(t,s) f(s, y_{\rho(s,y_s)})\; ds, & \text{if } t \in J. \end{cases} \tag{2.25}$$

Then, fixed points of the operator $\widetilde{N}$ are mild solutions of the problem (2.7) − (2.8).

For $\phi \in \mathcal{B}$, we consider the function $x(.) : (-\infty, b] \to E$ defined as below by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \leq 0; \\ U(t,0)\ \phi(0), & \text{if } \ t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in \Omega$, set

$$y(t) = z(t) + x(t)$$

It is obvious that y satisfies (2.23) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) &= g(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)}) - U(t,0)g(0,\phi) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) f(s, z_{\rho(s,y_s)} + x_{\rho(s,y_s)}) ds. \end{aligned}$$

Let

$$\Omega^0 = \{z \in \Omega : z_0 = 0\}.$$

Define the operator $\widetilde{F} : \Omega^0 \to \Omega^0$ by :

$$\begin{aligned} \widetilde{F}(z)(t) &= g(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)}) - U(t,0)g(0,\phi) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) f(s, z_{\rho(s,y_s)} + x_{\rho(s,y_s)}) ds. \end{aligned} \tag{2.26}$$

Obviously the operator $\widetilde{N}$ has a fixed point is equivalent to $\widetilde{F}$ has one, so it turns to prove that $\widetilde{F}$ has a fixed point.

Let $z \in \Omega^0$ be be a possible fixed point of the operator. Then, using $(H01) - (H06)$, we have for each $t \in [0, b]$

$$\begin{aligned} |z(t)| &\leq |g(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)})| + |U(t,0)g(0,\phi)| \\ &+ \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s, z_{\rho(s,y_s)} + x_{\rho(s,y_s)}) ds \right| \\ &\leq \|A^{-1}(t)\|\ |A(t)g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})| + \|U(t,0)\|_{B(E)}\ \|A^{-1}(0)\|\ |A(0)g(0,\phi)| \\ &+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s)\|z_{\rho(s,y_s)} + x_{\rho(s,y_s)}\|\right] ds \\ &\leq \overline{M}_0 L \left(\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} + 1\right) + \widehat{M}\overline{M}_0 L \left(\|\phi\|_{\mathcal{B}} + 1\right) \\ &+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s)\|z_{\rho(s,y_s)} + x_{\rho(s,y_s)}\|\right] ds \end{aligned}$$

Since $\|z_{\rho(t,y_t)} + x_{\rho(t,y_t)}\|_{\mathcal{B}} \leq K_b|z(t)| + c_b$ we obtain

$$\begin{aligned} |z(t)| &\leq \overline{M}_0 L(K_b|z(t)| + c_b + 1) + \widehat{M}\overline{M}_0 L(\|\phi\|_{\mathcal{B}} + 1) + \frac{\widehat{M} p^* b^\alpha}{\Gamma(\alpha+1)} \\ &+ \frac{\widehat{M} q^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ (K_b|z(s)| + c_b)\ ds \end{aligned}$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ Then

$$\begin{aligned}(1-\overline{M}_0LK_b)|z(t)| &\leq \overline{M}_0L(c_b+1)+\widehat{M}\overline{M}_0L(\|\phi\|_{\mathcal{B}}+1)+\frac{\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)}\\ &+ \frac{\widehat{M}q^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ (K_b|z(s)|+c_b)\ ds\end{aligned}$$

Set

$$\delta_b := c_b + \frac{K_b\overline{M}_0L(c_b+1)+K_b\widehat{M}\overline{M}_0L(\|\phi\|_{\mathcal{B}}+1)}{(1-\overline{M}_0LK_b)\Gamma(\alpha+1)}+\frac{K_b\widehat{M}p^*b^\alpha}{(1-\overline{M}_0LK_b)\Gamma(\alpha+1)}.$$

Thus

$$K_b|z(t)|+c_b \ \leq\ \delta_b+\frac{K_b\widehat{M}q^*}{(1-\overline{M}_0LK_b)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ (K_b|z(s)|+c_b)\ ds.$$

We consider the function μ defined by

$$\mu(t) := \sup\ \{\ K_b|z(s)|+c_b:\ 0\leq s\leq t\ \},\quad 0\leq t\leq b.$$

Let $t^\star\in(-\infty,t]$ be such that $\mu(t)=K_b|z(t^\star)|+c_b$. If $t^\star\in[0,b]$, by the previous inequality, we get

$$\mu(t)\ \leq\ \delta_b+\frac{K_b\widehat{M}q^*}{(1-\overline{M}_0LK_b)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ \mu(s)ds \qquad \text{for } t\in[0,b].$$

If $t^\star\in(-\infty,0]$, then $\mu(t)=\|\varphi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta_b=\delta_b(\alpha)$ such that

$$\mu(t)\ \leq\ \delta_b\times\left[1+\frac{K_b\widehat{M}q^*b^\alpha}{(1-\overline{M}_0LK_b)\Gamma(\alpha+1)}\right]:=\Lambda_b$$

Since $\|z\|\leq\mu(t)$, we have $\|z\|\leq\max\{\|\phi\|,\Lambda_b\}:=\Theta_b$.

Now, we shall show that $\widetilde{F}:Z\to\Omega^0$ is a contraction operator.
Indeed, consider $z,\overline{z}\in Z$, thus for each $t\in[0,b]$

$$\begin{aligned}|\widetilde{F}(y)(t)-\widetilde{F}(\overline{y})(t)| &\leq |g(t,z_{\rho(t,y_t)}+x_{\rho(t,y_t)})-g(t,\overline{z}_{\rho(t,y_t)}+x_{\rho(t,y_t)})|\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|_{B(E)}\ |f(s,z_{\rho(s,y_s)}\\ &+x_{\rho(s,y_s)})-f(s,\overline{z}_{\rho(s,y_s)}+x_{\rho(s,y_s)})|ds\\ &\leq \|A^{-1}(t)\||A(t)g(t,z_{\rho(t,y_t)}+x_{\rho(t,y_t)})-A(t)g(t,\overline{z}_{\rho(t,y_t)}+x_{\rho(t,y_t)})|\\ &+\frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ l_n(s)\ \|z_{\rho(s,y_s)}-\overline{z}_{\rho(s,y_s)}\|_{\mathcal{B}}\ ds\\ &\leq \overline{M}_0L_\star\|z_{\rho(t,y_t)}-\overline{z}_{\rho(t,y_t)}\|_{\mathcal{B}}\\ &+\frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ l_n(s)\ \|z_{\rho(s,y_s)}-\overline{z}_{\rho(s,y_s)}\|_{\mathcal{B}}\ ds.\end{aligned}$$

Since $\|z_{\rho(t,y_t)} - \overline{z}_{\rho(t,y_t)}\|_{\mathcal{B}} \leq K_b|z(t) - \overline{z}(t)|$ we obtain

$$\begin{aligned}
|\widetilde{N}(y)(t) - \widetilde{N}(\overline{y})(t)| &\leq \overline{M}_0 L_\star K_b \|z(t) - \overline{z}(t)\| + \frac{\widehat{M} K_b L_n^* b^\alpha}{\Gamma(\alpha+1)} \|z(t) - \overline{z}(t)\| \\
&\leq \left[\overline{M}_0 L_\star K_b + \frac{\widehat{M} K_b L_n^* b^\alpha}{\Gamma(\alpha+1)} \right] \|z(t) - \overline{z}(t)\|
\end{aligned}$$

Therefore,

$$\|\widetilde{F}(y) - \widetilde{F}(\overline{y})\| \leq \left[\overline{M}_0 L_\star K_b + \frac{\widehat{M} K_b L_n^* b^\alpha}{\Gamma(\alpha+1)} \|z - \overline{z}\| \right].$$

So, for an appropriate choice of $\overline{M}_0 L_\star$, L_n^* and b^α such that

$$\left[\overline{M}_0 L_\star K_b + \frac{\widehat{M} K_b L_b^* b^\alpha}{\Gamma(\alpha+1)} \right] < 1,$$

the operator $\widetilde{F}$ is a contraction. From the choice of Z there is no $z \in \partial Z^n$ such that $z = \lambda\, \widetilde{F}(z)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 1.10.4 does not hold. A consequence of the nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. We deduce that the operator $\widetilde{F}$ has a unique fixed-point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty, b]$ is a fixed point of the operator $\widetilde{N}$, which is the unique mild solution of the problem $(2.7) - (2.8)$.

2.6 Examples

We give in this section four examples to illustrate the previous results.

Example 1. Consider the partial differential equation

$$\begin{cases}
{}^cD_t^\alpha u(t,\xi) = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\
+ \displaystyle\int_{-r}^0 a_1(s-t) u \left[s - \rho_1(t)\rho_2 \left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta \right), \xi \right] ds, \\
\hfill 0 \leq t \leq b,\ \xi \in [0,\pi], \\
u(t,0) = u(t,\pi) = 0, \hfill 0 \leq t \leq b, \\
u(\theta,\xi) = u_0(\theta,\xi), \hfill -r < \theta \leq 0,\ \xi \in [0,\pi],
\end{cases} \tag{2.27}$$

where $a_1(t,\xi)$ is a continuous function and is uniformly Hölder continuous in t $0 < \alpha \leq 1$; $a_1 : [-r,0] \to \mathbb{R}$; $a_2 : [0,\pi] \to \mathbb{R}$; $\rho_i : [0,b] \to \mathbb{R}$ for $i = 1,2$ are continuous functions. To study this system, we consider the space $E = L^2([0,\pi],\mathbb{R})$ and the operator $A : D(A) \subset E \to E$ given by $Aw = w''$ with

$$D(A) := \{\ w \in E\ : w'' \in E,\ w(0) = w(\pi) = 0\ \}$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\in[0,b]}$ on E, with compact resolvent. On the domain $D(A)$, we define the operators $A(t) : D(A) \subset E \to E$ by

$$A(t)x(\xi) = Ax(\xi) + a_0(t,\xi)x(\xi).$$

By assuming that $a_0(.,.)$ is continuous and that $a_0(t,\xi) \le -\delta_0$ $(\delta_0 > 0)$ for every $t \in [0,b], \xi \in [0,\pi]$, and specific case $\alpha = 1$ it follows that the system

$$u'(t) = A(t)u(t) \quad t \ge s; \quad u(s) = x \in E,$$

has an associated evolution family given by

$$U(t,s)x(\xi) = \left[T(t-s)exp\left(\int_s^t a_0(\tau,\xi)d\tau\right)x\right](\xi).$$

From this expression, it follows that $U(t,s)$ is a compact linear operator and that

$$\|U(t,s)\| \le e^{-(1+\delta_0)(t-s)} \text{ for every } (t,s) \in [0,b] \times [0,b] \quad ; \quad s \le t.$$

Theorem 2.6.1. *Let $\varphi \in C(H;E)$. Assume that the condition (H_φ) holds, the functions $\rho_i : [0,b] \to \mathbb{R}$ for $i = 1,2$, $a_1 : [-r,0] \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a unique mild solution of* (2.27).

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-r}^0 a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (2.27) into the abstract system (2.1) − (2.2). Moreover, the function f is bounded linear operator. Now, the existence of the unique mild solution can be deduced from a direct application of Theorem 2.2.1. From Remark 1.7.1, we have the following result.

Corollary 2.6.1. *Let $\varphi \in C(H;E)$ be continuous and bounded. Then there exists a unique mild solution of* (2.27) *on* $[-r,b]$.

Example 2. Consider the partial differential equation

$$\begin{cases} {}^cD_t^\alpha\left[u(t,\xi) - \displaystyle\int_{-r}^0 a_3(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2d\theta\right),\xi\right)ds\right] \\ = \dfrac{\partial^2 u(t,\xi)}{\partial\xi^2} + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-r}^0 a_1(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2d\theta\right),\xi\right)ds, \\ \hfill 0 \le t \le b,\ \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \hfill 0 \le t \le b, \\ v(\theta,\xi) = v_0(\theta,\xi), \hfill -r < \theta \le 0,\ \xi \in [0,\pi], \end{cases} \tag{2.28}$$

where $a_3 : [-r,0] \to \mathbb{R}$ is a continuous function

Theorem 2.6.2. *Let $\varphi \in C(H;E)$. Assume that the condition (H_φ) holds, the functions $\rho_i : [0,b] \to \mathbb{R}$ for $i = 1,2$; $a_1, a_3 : [-r,0] \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a unique mild solution of* (2.28).

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-r}^{0} a_1(s)\psi(s,\xi)ds,$$

$$g(t,\psi)(\xi) = \int_{-r}^{0} a_3(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^{\pi} a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (2.28) into the abstract system (2.3) – (2.4). Moreover, the function f is bounded linear operator. Now, the existence of the unique mild solution can be deduced from a direct application of Theorem 2.3.1. From Remark 1.7.1, we have the following result.

Corollary 2.6.2. *Let $\varphi \in C(H;E)$ be continuous and bounded. Then there exists a unique mild solution of* (2.28) *on* $[-r,b]$.

Example 3. Consider the partial differential equation

$$\begin{cases} {}^cD_0^{\alpha}u(t,\xi) = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-\infty}^{0} a_1(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right]ds, \\ \qquad\qquad\qquad\qquad 0 \le t \le b,\ \xi \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \qquad\qquad 0 \le t \le b, \\ u(\theta,\xi) = u_0(\theta,\xi), \qquad\qquad -\infty < \theta \le 0,\ \xi \in [0,\pi], \end{cases} \tag{2.29}$$

where $a_1 : (-\infty,0] \to \mathbb{R}$ is a continuous functions.

Theorem 2.6.3. *Let $\mathcal{B} = BUC(\mathbb{R}_-;E)$ and $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds, the functions $\rho_i : [0,+\infty[\to \mathbb{R}$ for $i = 1,2$; $a_1 : (-\infty,0] \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a unique mild solution of* (2.29).

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^{\pi} a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (2.29) into the abstract system (2.5) – (2.6). Moreover, the function f is bounded linear operator. Now, the existence of the unique mild solution can be deduced from a direct application of Theorem 2.4.1. From Remark 1.7.2, we have the following result.

Corollary 2.6.3. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then there exists a unique mild solution of* (2.29) *on* $(-\infty,b]$.

Example 4. Consider the partial differential equation

$$\begin{cases} {}^cD_0^\alpha \left[u(t,\xi) - \displaystyle\int_{-\infty}^0 a_3(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)ds \right] \\ = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-\infty}^0 a_1(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)ds, \\ \hfill 0 \le t \le b,\ \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \hfill 0 \le t \le b, \\ v(\theta,\xi) = v_0(\theta,\xi), \hfill -\infty < \theta \le 0,\ \xi \in [0,\pi], \end{cases} \tag{2.30}$$

where $a_3 : (-\infty, 0] \to \mathbb{R}$ is a continuous function.

Theorem 2.6.4. *Let $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds, the functions $\rho_i : [0,b] \to \mathbb{R}$ for $i = 1,2$; $a_1, a_3 : (-\infty,0] \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a unique mild solution of* (2.30).

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-\infty}^0 a_1(s)\psi(s,\xi)ds,$$

$$g(t,\psi)(\xi) = \int_{-\infty}^0 a_3(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (2.30) into the abstract system (2.7) – (2.8). Moreover, the function f is bounded linear operator. Now, the existence of the unique mild solution can be deduced from a direct application of Theorem 2.5.1. From Remark 1.7.2, we have the following result.

Corollary 2.6.4. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then there exists a unique mild solution of* (2.30) *on* $(-\infty, b]$.

Chapter 3

Fractional Integrodifferential Evolution Equations with State-Dependent Delay

3.1 Introduction

Our attention in this chapter is to look for sufficient conditions for the existence of the unique mild solution for some classes of fractional order for partial functional and neutral functional integrodifferential evolution equations with finite and infinite state-dependent delay. Our analysis is based upon the nonlinear alternative of Frigon-Granas for contraction maps [52] combined with semigroup theory and the fractional version of Gronwall's inequality.

The existence of the unique mild solution is demonstrated in section 4.2 for the following class of fractional integrodifferential evolution equations with finite state-dependent delay

$$^cD_0^{\alpha}y(t) = A(t)y(t) + \int_0^t \mathcal{K}(t,s)f(s, y_{\rho(s,y_s)})ds, \quad 0<\alpha<1 \text{ a.e. } t\in J=[0,b] \tag{3.1}$$

$$y(t) = \varphi(t), \quad t\in H=[-r,0], \tag{3.2}$$

where $0<r<+\infty$, $^cD_0^{\alpha}$ is the standard Caputo's fractional derivative of order $\alpha\in(0,1)$, $\mathcal{K}: J\times J\to E$, $f: J\times C(H;E)\to E, \rho: J\times C(H;E)\to\mathbb{R}$ and $\varphi\in C(H;E)$ are given functions and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $s\le t$.

An extension of this problem is given in section 4.3, we consider the following fractional neutral integrodifferential evolution equations with finite state-dependent delay

$$^cD_0^{\alpha}[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + \int_0^t \mathcal{K}(t,s)f(s, y_{\rho(s,y_s)})ds, \quad 0<\alpha<1 \quad \text{a.e.} \quad t\in J \tag{3.3}$$

$$y(t) = \varphi(t), \quad t\in H, \tag{3.4}$$

where $A(\cdot)$, f and φ are as in problem $(3.1)-(3.2)$ and $g: J\times C(H;E)\to E$ is a given function.

In section 4.4 we study the following class of fractional integrodifferential evolution equations with infinite state-dependent delay

$$^cD_0^{\alpha}y(t) = A(t)y(t) + \int_0^t \mathcal{K}(t,s)f(s, y_{\rho(s,y_s)})ds, \quad 0<\alpha<1 \text{ a.e. } t\in J \tag{3.5}$$

$$y_0 = \phi \in \mathcal{B}, \tag{3.6}$$

where $\mathcal{K} : J \times J \to E$, $\mathcal{B}$ is an abstract *phase space* to be specified later, ${}^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0,1)$, $f : J \times \mathcal{B} \to E, \rho : J \times \mathcal{B} \to \mathrm{I\!R}$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 4.5, we consider the following fractional neutral integrodifferential evolution equations with infinite state-dependent delay

$${}^cD_0^\alpha[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + \int_0^t \mathcal{K}(t,s)f(s, y_{\rho(s,y_s)})ds, \quad 0 < \alpha < 1 \quad \text{a.e.} \quad t \in J \tag{3.7}$$

$$y_0 = \phi \in \mathcal{B}, \tag{3.8}$$

where $A(\cdot)$, $f, \mathcal{K}$ and ϕ are as in problem $(3.5) - (3.6)$ and $g : J \times \mathcal{B} \to E$ is a given function. Finally, section 4.6 is devoted to examples illustrating the abstract theory considered in previous sections.

3.2 Partial Problem with Finite Delay

Before stating and proving the main result, we give first the definition of mild solution of the semilinear evolution problem $(3.1) - (3.2)$.

Lemma 3.2.1. *The system* $(3.1) - (3.2)$ *is equivalent to the nonlinear integral equation*

$$y(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A(s)y(s)\ ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s \mathcal{K}(s,\tau)f(\tau, y_{\rho(\tau,y_\tau)})d\tau ds. \tag{3.9}$$

In other words, every solution of the integral equation (3.9) *is also solution of the system* $(3.1) - (3.2)$ *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system $(3.1) - (3.2)$, and using some classical results from fractional calculus to get (3.9).

Definition 3.2.1. *We say that the function $y(\cdot) : [-r, b] \to E$ is a mild solution of* $(3.1) - (3.2)$ *if $y(t) = \varphi(t)$ for all $t \in [-r, 0]$ and y satisfies the following integral equation*

$$y(t) = U(t,0)\ \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ U(t,s)\int_0^s \mathcal{K}(s,\tau)f(\tau, y_{\rho(\tau,y_\tau)})d\tau\ ds \qquad \textit{for each } t \in J. \tag{3.10}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times C(H; E), \rho(s,\varphi) \leq 0\}.$$

We always assume that $\rho : J \times C(H; E) \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $C(H; E)$ and there exists a continuous and bounded function $\mathcal{L}^\varphi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\varphi_t\| \leq \mathcal{L}^\varphi(t)\|\varphi\| \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

We will need to introduce the following hypothesis which are assumed thereafter :

$(H1)$ There exists a constant $\widehat{M} \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq \widehat{M} \quad \text{for every } (t,s) \in \Delta.$$

$(H2)$ There exist two functions $p, q \in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$|f(t,u)| \leq p(t) + q(t)\ \|u\|_B \text{ for a.e. } t \in J \text{ and each } u \in C(H;E).$$

$(H3)$ For all $R > 0$, there exists $l_R \in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$|f(t,u) - f(t,v)| \leq l_R(t)\ \|u - v\|$$

for all $u, v \in C(H;E)$ with $\|u\| \leq R$ and $\|v\| \leq R$.

$(H4)$ For each $t \in J$ $\mathcal{K}(t,s)$ is measurable on $[0,t]$ and

$$\mathcal{K}(t) = ess\sup\{|\mathcal{K}(t,s)|; 0 \leq s \leq t\}$$

is bonded on $[0,b]$; let $S_b := \sup \mathcal{K}(t)$

Theorem 3.2.1. *Assume that the hypotheses (H_φ) and $(H1) - (H4)$ hold and moreover*

$$\frac{\widehat{M} l_b^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} < 1 \tag{3.11}$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ and $l_b^* = \sup l_b(s)$. Then the problem $(3.1) - (3.2)$ has a unique mild solution on $[-r,b]$.

Proof. Transform the problem $(3.1) - (3.2)$ into a fixed-point problem. Consider the operator $N : C([-r,b];E) \to C([-r,b];E)$ defined by :

$$N(y)(t) = \begin{cases} \varphi(t), & \text{if } t \in H; \\ U(t,0)\ \varphi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) \int_0^s \mathcal{K}(s,\tau) f(\tau, y_{\rho(\tau,y_\tau)}) d\tau\ ds, & \text{if } t \in J. \end{cases} \tag{3.12}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(3.1) - (3.2)$.

Let y be a possible solution of the problem $(3.1)-(3.2)$. Given $t \leq b$, then from $(H1)-(H2)$, $(H4)$, (H_φ) and Lemma 5.2.1, we have for each $t \in [0,b]$

$$\begin{aligned} |y(t)| &\leq \|U(t,0)\|_{B(E)}|\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \|U(t,s)\|_{B(E)} \int_0^s |\mathcal{K}(s,\tau) f(\tau, y_{\rho(\tau,y_\tau)}) d\tau|\ ds \\ &\leq \widehat{M}\|\varphi\| + \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t \int_0^s (t-s)^{\alpha-1} |\mathcal{K}(s,\tau)|\ \left[p(s) + q(s)\|y_{\rho(s,y_s)}\|\right] ds \end{aligned}$$

It follows that

$$|y(t)| + \mathcal{L}^\varphi\|\varphi\| \leq \widehat{M}\|\varphi\| + \mathcal{L}^\varphi\|\varphi\| + \frac{\widehat{M} p^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} + \frac{\widehat{M} q^* b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \left[|y(s) + \mathcal{L}^\varphi\|\varphi\|\right] ds$$

Set

$$\delta_b := (\widehat{M} + \mathcal{L}^{\varphi})\|\varphi\| + \frac{\widehat{M} p^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)}.$$

We consider the function μ defined by

$$\mu(t) := \sup \{ \ |y(s)| + \mathcal{L}^{\varphi}\|\varphi\| \ : \ 0 \leq s \leq t \ \}, \quad 0 \leq t \leq b.$$

Let $t^\star \in [-r, t]$ be such that $\mu(t) = |y(t^\star)| + \mathcal{L}^{\varphi}\|\varphi\|$. If $t^\star \in [0, b]$, by the previous inequality, we get

$$\mu(t) \ \leq \ \delta_b + \frac{\widehat{M} q^* b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ \mu(s) ds \qquad \text{for } t \in [0, b].$$

If $t^\star \in [-r, 0]$, then $\mu(t) = \|\varphi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta_b = \delta_b(\alpha)$ such that

$$\begin{aligned} \mu(t) \ &\leq \ \delta_b \times \left[1 + \frac{\widehat{M} q^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} \right] \\ &= \ \Lambda_b \end{aligned}$$

Since $\|y\| \leq \mu(t)$, we have $\|y\| \leq \max\{\|\varphi\|, \Lambda_b\} := \Theta_b$. Since for every $t \in [0, b]$, we have $\|y\| \leq \max\{\|\varphi\|, \Lambda_b\} := \Theta_b$.

Set

$$Y = \{ \ y \in C([-r, b]; E) \ : \ \sup\{|y(t)| : 0 \leq t \leq b\} \leq \Theta_b + 1\}.$$

Clearly, Y is a closed subset of $C([-r, b]; E)$.

We shall show that $N : Y \to C([-r, b]; E)$ is a contraction operator.
Indeed, consider $y, \overline{y} \in Y$, thus using $(H1)$ and $(H3)$ for each $t \in [0, b]$

$$\begin{aligned} |N(y)(t) - N(\overline{y})(t)| \ &\leq \ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\| \times \\ &\quad \times \int_0^s |\mathcal{K}(s,\tau)| \left| f(\tau, y_{\rho(\tau, y_\tau)}) - f(\tau, \overline{y}_{\rho(\tau, y_\tau)}) \right| \ d\tau ds \\ &\leq \ \frac{\widehat{M} b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ l_b(s) \ \|y_{\rho(s, y_s)} - \overline{y}_{\rho(s, y_s)}\| \ ds. \end{aligned}$$

Using (H_φ) and Lemma 5.2.1, we obtain

$$\begin{aligned} |N(y)(t) - N(\overline{y})(t)| \ &\leq \ \frac{\widehat{M} l_b^* b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y(s) - \overline{y}(s)| \ ds \\ &\leq \ \frac{\widehat{M} l_b^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} \|y - \overline{y}\| \end{aligned}$$

Therefore,

$$\|N(y) - N(\overline{y})\| \leq \frac{\widehat{M} l_n^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} \|y - \overline{y}\|.$$

So, the operator N is a contraction. From the choice of Y there is no $y \in \partial Y^n$ such that $y = \lambda\ N(y),\ \lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 1.10.4 does not hold. The nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. Thus, we deduce that the operator N has a unique fixed-point $y^\star$ which is the unique mild solution of the problem (3.1) – (3.2).

3.3 Neutral Problem with Finite Delay

We give here an extension to previous results for the neutral case (3.3) – (3.4). Firstly, we define its mild solution.

Lemma 3.3.1. *The system* (3.3) – (3.4) *is equivalent to the nonlinear integral equation*

$$y(t) \quad = \quad [\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A(s)y(s)\ ds \tag{3.13}$$

$$+ \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s \mathcal{K}(s,\tau)f(\tau, y_{\rho(\tau,y_\tau)})d\tau ds.$$

In other words, every solution of the integral equation (3.22) *is also solution of the system* (3.3) – (3.4) *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system (3.3) – (3.4), and using some classical results from fractional calculus to get (3.22).

Definition 3.3.1. *We say that the function* $y(\cdot) : [-r,b] \to E$ *is a mild solution of* (3.3) – (3.4) *if* $y(t) = \varphi(t)$ *for all* $t \in H$ *and* y *satisfies the following integral equation*

$$\begin{aligned} y(t) \ &= U(t,0)[\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)\int_0^s \mathcal{K}(s,\tau)f(\tau, y_{\rho(\tau,y_\tau)})d\tau ds. \end{aligned} \tag{3.14}$$

We consider the hypotheses (H_φ), $(H1) - (H4)$ and we need to introduce the following assumptions :

$(H5)$ There exists a constant $L_\star > 0$ such that

$$|A(s)g(s,\varphi) - A(\overline{s})g(\overline{s},\overline{\varphi})| \le L_\star\ (|s-\overline{s}| + \|\varphi - \overline{\varphi}\|)$$

for all $\varphi, \overline{\varphi} \in C(H;E)$.

$(H6)$ There exists a constant $\overline{M}_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \le \overline{M}_0 \quad \text{for all} \ \ t \in J.$$

$(H7)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0 K_b}$ such that

$$|A(t)\ g(t,\varphi)| \le L\ (\|\varphi\| + 1) \text{ for all } t \in J \text{ and } \varphi \in C(H;E).$$

Theorem 3.3.1. *Suppose that the hypotheses* $(H1)-(H7)$ *are satisfied and moreover*

$$\left[\overline{M}_0 L_\star + \frac{\widehat{M} L_b^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)}\right] < 1 \tag{3.15}$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ *and* $l_b^* = \sup l_b(s)$. *Then the problem* $(3.3)-(3.4)$ *has a unique mild solution on* $[-r, b]$.

Proof. Transform as below the neutral problem $(3.3)-(3.4)$ into a fixed point problem by considering the operator $\widetilde{N} : C([-r, b]; E) \to C([-r, b]; E)$ defined by :

$$\widetilde{N}(y)(t) = \begin{cases} \varphi(t), & \text{if } t \in H; \\ U(t,0)\ [\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) & \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} U(t,s) \int_0^s \mathcal{K}(s,\tau) f(\tau, y_{\rho(\tau,y_\tau)}) d\tau ds, & \text{if } t \in J. \end{cases} \tag{3.16}$$

Clearly, the fixed points of the operator $\widetilde{N}$ are mild solutions of the problem $(3.3)-(3.4)$.

Let y be a possible solution of the problem $(3.3)-(3.4)$. Given $t \le b$. Then, using $(H1)-(H5)$, (H_φ) and Lemma 5.2.1, we have for each $t \in [0, b]$

$$\begin{aligned}
|y(t)| &\le |g(t, y_{\rho(t,y_t)})| + |U(t,0)[\varphi(0) - g(0,\varphi)]| \\
&+ \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s) \int_0^s \mathcal{K}(s,\tau) f(\tau, y_{\rho(\tau,y_\tau)}) d\tau ds \right| \\
&\le \|A^{-1}(t)\|\ |A(t) g(t, y_{\rho(t,y_t)})| + \|U(t,0)\|_{B(E)}\ \|A^{-1}(0)\|\ |A(0) g(0,\varphi)| \\
&+ \frac{\widehat{M} b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s)\|y_{\rho(s,y_s)}\|\right] ds \\
&\le \widehat{M}\|\varphi\| + \overline{M}_0 L \left(\|y_{\rho(t,y_t)}\| + 1\right) + \widehat{M}\overline{M}_0 L\,(\|\varphi\| + 1) \\
&+ \frac{\widehat{M} b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s)\|y_{\rho(s,y_s)}\|\right] ds
\end{aligned}$$

Since $\|y_{\rho(t,y_t)}\| \le |y(t)| + \mathcal{L}^\varphi \|\varphi\|$ we obtain

$$\begin{aligned}
|y(t)| &\le \overline{M}_0 L\,(|y(t)| + \mathcal{L}^\varphi\|\varphi\|) + \widehat{M}\|\varphi\|(1 + \overline{M}_0 L) + \widehat{M}\overline{M}_0 L + \frac{\widehat{M} p^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} \\
&+ \frac{\widehat{M} q^* b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \ (|y(s)| + \mathcal{L}^\varphi\|\varphi\|)\ ds
\end{aligned}$$

Then

$$\begin{aligned}
(1 - \overline{M}_0 L)|y(t)| &\le \left[\overline{M}_0 L \mathcal{L}^\varphi + \widehat{M}(1 + \overline{M}_0 L)\right] \|\varphi\| + \widehat{M}\overline{M}_0 L + \frac{\widehat{M} p^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} \\
&+ \frac{\widehat{M} q^* b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \ (|y(s)| + \mathcal{L}^\varphi\|\varphi\|)\ ds
\end{aligned}$$

Set

$$\delta_b := \frac{\Big[1-\overline{M}_0 L\mathcal{L}^{\varphi}+\widehat{M}(1-\overline{M}_0 L)\Big]}{(1-\overline{M}_0 L)}\|\varphi\| + \frac{\widehat{M}\overline{M}_0 L}{(1-\overline{M}_0 L)} + \frac{\widehat{M}p^{*}b^{\alpha+1}S_b}{(1-\overline{M}_0 L)\Gamma(\alpha+1)}.$$

Thus

$$|y(t)|+\mathcal{L}^{\varphi}\|\varphi\| \;\; \leq \;\; \beta_b + \frac{\widehat{M}q^{*}bS_b}{(1-\overline{M}_0 L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\;(|y(s)|+\mathcal{L}^{\varphi}\|\varphi\|)\;ds.$$

with $\beta_b=\mathcal{L}^{\varphi}\|\varphi\|+\delta_b$.

We consider the function μ defined by

$$\mu(t) := \sup\;\{\;|y(s)|+\mathcal{L}^{\varphi}\|\varphi\|\;:\;0\leq s\leq t\;\},\quad 0\leq t\leq b.$$

Let $t^{\star}\in[-r,t]$ be such that $\mu(t)=|y(t^{\star})|+\mathcal{L}^{\varphi}\|\varphi\|$. If $t^{\star}\in[0,b]$, by the previous inequality, we get

$$\mu(t) \;\; \leq \;\; \beta_b + \frac{\widehat{M}q^{*}bS_b}{(1-\overline{M}_0 L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\;\mu(s)ds \qquad \text{for } t\in[0,b].$$

If $t^{\star}\in[-r,0]$, then $\mu(t)=\|\varphi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta_b=\delta_b(\alpha)$ such that

$$\begin{aligned}\mu(t) \;\; &\leq \;\; \beta_b\times\left[1+\frac{\widehat{M}q^{*}b^{\alpha+1}S_b}{(1-\overline{M}_0 L)\Gamma(\alpha+1)}\right]\\ &= \;\; \Lambda_b\end{aligned}$$

Since $\|y\|\leq\mu(t)$, we have $\|y\|\leq\max\{\|\varphi\|,\Lambda_b\}:=\Theta_b$.

Now, we shall show that $\widetilde{N}:Y\to C([-r,b];E)$ is a contraction operator. Indeed, consider $y,\overline{y}\in Y$, thus for each $t\in[0,b]$

$$\begin{aligned}|\widetilde{N}(y)(t)-\widetilde{N}(\overline{y})(t)| \;\; &\leq \;\; |g(t,y_{\rho(t,y_t)})-g(t,\overline{y}_{\rho(t,y_t)})| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|\times\\ &\quad\times\int_0^s|\mathcal{K}(s,\tau)|\left|f(\tau,y_{\rho(\tau,y_\tau)})-f(\tau,\overline{y}_{\rho(\tau,y_\tau)})\right|\;d\tau ds\\ &\leq \;\; \|A^{-1}(t)\||A(t)g(t,y_{\rho(t,y_t)})-A(t)g(t,\overline{y}_{\rho(t,y_t)})|\\ &+ \;\; \frac{\widehat{M}nS_b}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\;l_b(s)\;\|y_{\rho(s,y_s)}-\overline{y}_{\rho(s,y_s)}\|\;ds.\end{aligned}$$

Using (H_{φ}) and Lemma 5.2.1, we obtain

$$\begin{aligned}|\widetilde{N}(y)(t)-\widetilde{N}(\overline{y})(t)| \;\; &\leq \;\; \overline{M}_0 L_{\star}|y(t)-\overline{y}(t)| + \frac{\widehat{M}L_b^{*}b^{\alpha+1}S_b}{\Gamma(\alpha+1)}\|y-\overline{y}\|\\ &\leq \;\; \left[\overline{M}_0 L_{\star}+\frac{\widehat{M}L_b^{*}b^{\alpha+1}S_b}{\Gamma(\alpha+1)}\right]\|y-\overline{y}\|\end{aligned}$$

Therefore,

$$\|\widetilde{N}(y)-\widetilde{N}(\overline{y})\| \leq \left[\overline{M}_0 L_\star + \frac{\widehat{M} L_b^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)}\right] \|y-\overline{y}\|.$$

So, for an appropriate choice of $\overline{M}_0 L_\star$, L_b^* and b^α such that

$$\left[\overline{M}_0 L_\star + \frac{\widehat{M} L_b^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)}\right] < 1,$$

the operator $\widetilde{N}$ is a contraction. From the choice of Y there is no $y \in \partial Y^n$ such that $y = \lambda\, \widetilde{N}(y)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 1.10.4 does not hold. A consequence of the nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. We deduce that the operator $\widetilde{N}$ has a unique fixed-point $y^\star$ which is the unique mild solution of the problem $(3.3)-(3.4)$.

3.4 Partial Problem with Infinite Delay

Before stating and proving the main result, we give first the definition of mild solution of the semilinear evolution problem $(3.5)-(3.6)$.

Lemma 3.4.1. *The system* $(3.5)-(3.6)$ *is equivalent to the nonlinear integral equation*

$$y(t) = \phi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} A(s)y(s)\; ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s \mathcal{K}(s,\tau) f(\tau, y_{\rho(\tau,y_\tau)}) d\tau ds. \tag{3.17}$$

In other words, every solution of the integral equation (3.17) *is also solution of the system* $(3.5)-(3.6)$ *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system $(3.5)-(3.6)$, and using some classical results from fractional calculus to get (3.17).

Definition 3.4.1. *We say that the function* $y(\cdot) : (-\infty, b] \to E$ *is a mild solution of* $(3.5)-(3.6)$ *if* $y(t) = \phi(t)$ *for all* $t \leq 0$ *and* y *satisfies the following integral equation*

$$y(t) = U(t,0)\; \phi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\; U(t,s)\int_0^s \mathcal{K}(s,\tau) f(\tau, y_{\rho(\tau,y_\tau)}) d\tau\; ds \qquad \text{for each } t \in J. \tag{3.18}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\phi) : (s,\phi) \in J \times \mathcal{B}, \rho(s,\phi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq \mathcal{L}^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

We will need to introduce the following hypothesis which are assumed thereafter :

$(H01)$ There exists a constant $\widehat{M} \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq \widehat{M} \quad \text{for every } (t,s) \in \Delta.$$

$(H02)$ There exist two functions $p, q \in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$|f(t,u)| \leq p(t) + q(t)\ \|u\|_B \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$

$(H03)$ For all $R > 0$, there exists $l_R \in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$|f(t,u) - f(t,v)| \leq l_R(t)\ \|u - v\|$$

for all $u, v \in \mathcal{B}$ with $\|u\| \leq R$ and $\|v\| \leq R$.

$(H04)$ For each $t \in J$ $\mathcal{K}(t,s)$ is measurable on $[0,t]$ and

$$\mathcal{K}(t) = ess\sup\{|\mathcal{K}(t,s)|; 0 \leq s \leq t\}$$

is bonded on $[0,b]$; let $S_b := \sup \mathcal{K}(t)$

Consider the following space

$$\Omega = \left\{ y : (-\infty, b] \to E : y|_{(-\infty,0]} \ \in B \ \ and \ \ y|_{[0,b]} \text{is continuous } \right\},$$

Theorem 3.4.1. *Assume that the hypothesis* (H_φ) *and* $(H01) - (H03)$ *hold and moreover*

$$\frac{K_b \widehat{M} l_b^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} < 1 \tag{3.19}$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ *and* $l_b^* = \sup l_b(s)$. *Then the problem* $(3.5) - (3.6)$ *has a unique mild solution on* $(-\infty, b]$.

Proof. We transform the problem $(3.5) - (3.6)$ into a fixed-point problem. Consider the operator $N : \Omega \to \Omega$ defined by :

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\ \phi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) \int_0^s \mathcal{K}(s,\tau) f(\tau, y_{\rho(\tau,y_\tau)}) d\tau\ ds, & \text{if } t \in J. \end{cases} \tag{3.20}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(3.5) - (3.6)$.

For $\phi \in \mathcal{B}$, we will define the function $x(.) : (-\infty, b] \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \in [0,b]; \\ U(t,0)\ \phi(0), & \text{if } \ t \in (-\infty, 0]. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in \Omega$, set

$$y(t) = z(t) + x(t)$$

It is obvious that y satisfies (3.18) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s) \int_0^s \mathcal{K}(s,\tau)\ f(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})d\tau ds \quad \text{for } t \in J.$$

Let

$$\Omega^0 = \{z \in \Omega : z_0 = 0\}.$$

Define the operator $F : \Omega^0 \to \Omega^0$ by :

$$F(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s) \int_0^s \mathcal{K}(s,\tau)\ f(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})d\tau ds \quad \text{for } t \in J. \tag{3.21}$$

Obviously the operator N has a fixed point is equivalent to F has one, so it turns to prove that F has a fixed point.

Let $z \in \Omega^0$ be be a possible fixed point of the operator. By the hypotheses $(H01)$ and $(H02)$, we have for each $t \in [0,b]$

$$\begin{aligned}
|z(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \left| \int_0^s \mathcal{K}(s,\tau)\ f(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}) \right| d\tau ds \\
&\leq \frac{\widehat{M} b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \left[p(s) + q(s)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\right]\ ds
\end{aligned}$$

From (H_ϕ), Lemma 1.7.2 and Assumption $(A1)$, we have for each $t \in [0,b]$

$$\begin{aligned}
\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\leq \|z_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + \|x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \\
&\leq K_b|z(s)| + (M_b + \mathcal{L}^\phi)\|z_0\|_{\mathcal{B}} + K_b|x(s)| + (M_b + \mathcal{L}^\phi)\|x_0\|_{\mathcal{B}} \\
&\leq K_b|z(s)| + K_b\|U(s,0)\|_{B(E)}|\phi(0)| + (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} \\
&\leq K_b|z(s)| + K_b\widehat{M}|\phi(0)| + (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}
\end{aligned}$$

Using (ii), we get

$$\begin{aligned}
\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\leq K_b|z(s)| + K_b\widehat{M}H\|\phi\|_{\mathcal{B}} + (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} \\
&\leq K_b|z(s)| + (M_b + \mathcal{L}^\phi + K_b\widehat{M}H)\|\phi\|_{\mathcal{B}}
\end{aligned}$$

It follows that

$$|z(t)| \leq \frac{\widehat{M}p^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} + \frac{\widehat{M}q^* b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \Big(K_b|z(s)| + (M_b + \mathcal{L}^\phi + K_b\widehat{M}H)\|\phi\|_{\mathcal{B}}\Big)\ ds$$

Set $c_b := (M_b + \mathcal{L}^\phi + K_b\widehat{M}H)\|\phi\|_{\mathcal{B}}$. Then, we have

$$|z(t)| \leq \frac{\widehat{M}p^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} + \frac{\widehat{M}q^* b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ (K_b|z(s)| + c_b)\ ds$$

Then

$$K_b|z(t)| + c_b \leq \frac{K_b\widehat{M}p^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} + c_b + \frac{K_b\widehat{M}q^* b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ (K_b|z(s)| + c_b)\ ds.$$

Set

$$\delta_b := \frac{K_b\widehat{M}p^* b^{\alpha+1}S_b}{\Gamma(\alpha+1)} + c_b.$$

We consider the function μ defined by

$$\mu(t) := \sup\ \{\ K_b|z(s)| + c_b\ :\ 0 \le s \le t\ \}, \quad 0 \le t \le b.$$

Let $t^\star \in [0,t]$ be such that $\mu(t) = K_b|z(t^\star)| + c_b$. By the previous inequality, we have

$$\mu(t) \ \le\ \delta_b + \frac{K_b\widehat{M}q^* bS_b}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ \mu(s)ds \qquad \text{for } t \in [0,b].$$

If $t^\star \in (-\infty,0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta_b = \delta_b(\alpha)$ such that

$$\begin{aligned}\mu(t) \ &\le\ \delta_b \times \left[1 + \frac{K_b\widehat{M}q^* b^{\alpha+1}S_b}{\Gamma(\alpha+1)}\right] \\ &=\ \Lambda_b\end{aligned}$$

Since $\|y\| \le \mu(t)$, we have $\|y\| \le \max\{\|\phi\|,\Lambda_b\} := \Theta_b$. Since for every $t \in [0,b]$, we have $\|y\| \le \max\{\|\varphi\|,\Lambda_b\} := \Theta_b$.

Set

$$Z = \{\ z \in \Omega^0\ :\ \sup_{0\le t\le b}|z(t)| \le \Theta_b + 1\}.$$

Clearly, Z is a closed subset of Ω^0.

We shall show that $F : Z \to \Omega^0$ is a contraction operator.
Indeed, consider $z,\overline{z} \in Z$, thus using $(H01)$ and $(H03)-(H04)$ for each $t \in [0,b]$

$$\begin{aligned}|F(z)(t) - F(\overline{z})(t)| \ &\le\ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|\int_0^s |\mathcal{K}(s,\tau)| \times \\ &\quad \times \left|f(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}) - f(\tau,\overline{z}_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})\right|\ d\tau ds \\ &\le\ \frac{\widehat{M}bS_b}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ l_b(s)\ \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\ ds.\end{aligned}$$

Using (H_ϕ) and Lemma 1.7.2, we obtain

$$\begin{aligned}|F(z)(t) - F(\overline{z})(t)| \ &\le\ \frac{\widehat{M}bS_b l_b^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K_b\ |z(s) - \overline{z}(s)|\ ds \\ &\le\ \frac{K_b\widehat{M}l_b^* b^{\alpha+1}S_b}{\Gamma(\alpha+1)}\|z(t) - \overline{z}(t)\|.\end{aligned}$$

Therefore,

$$\|F(z) - F(\overline{z})\| \le \frac{K_b\widehat{M}l_b^* b^{\alpha+1}S_b}{\Gamma(\alpha+1)}\|z - \overline{z}\|.$$

So, the operator F is a contraction. From the choice of Z there is no $z \in \partial Z^n$ such that $z = \lambda\ F(z)$, $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 1.10.4 does not hold. The nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. Thus, we deduce that the operator F has a unique fixed-point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty,b]$ is a fixed point of the operator N, which is the unique mild solution of the problem $(3.5)-(3.6)$.

3.5 Neutral Problem with Infinite Delay

We give here an extension to previous results for the neutral case (3.7) – (3.8). Firstly, we define its mild solution.

Lemma 3.5.1. *The system* (3.7) – (3.8) *is equivalent to the nonlinear integral equation*

$$y(t) = [\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s) y(s)\ ds \tag{3.22}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^s \mathcal{K}(s,\tau) f(\tau, y_{\rho(\tau,y_\tau)}) d\tau\ ds.$$

In other words, every solution of the integral equation (3.22) *is also solution of the system* (3.7) – (3.8) *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system (3.7) – (3.8), and using some classical results from fractional calculus to get (3.22).

Definition 3.5.1. *We say that the function* $y(\cdot) : (-\infty, b] \to E$ *is a mild solution of* (3.7)–(3.8) *if* $y(t) = \phi(t)$ *for all* $t \leq 0$ *and* y *satisfies the following integral equation*

$$\begin{aligned} y(t) &= U(t,0)[\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) \int_0^s \mathcal{K}(s,\tau) f(\tau, y_{\rho(\tau,y_\tau)}) d\tau\ ds. \end{aligned} \tag{3.23}$$

We consider the hypotheses (H_ϕ), $(H01) - (H04)$ and we need the following assumptions

$(H05)$ There exists a constant $L_\star > 0$ such that

$$|A(s)g(s,\varphi) - A(\overline{s})g(\overline{s},\overline{\varphi})| \leq L_\star \left(|s - \overline{s}| + \|\varphi - \overline{\varphi}\| \right)$$

for all $\varphi, \overline{\varphi} \in C(H;E)$.

$(H06)$ There exists a constant $\overline{M}_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0 \quad \text{for all} \quad t \in J.$$

$(H07)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0 K_b}$ such that

$$|A(t)\ g(t,\varphi)| \leq L\ (\|\varphi\| + 1) \text{ for all } t \in J \text{ and } \varphi \in C(H;E).$$

Theorem 3.5.1. *Suppose that the hypotheses* $(H01) - (H07)$ *are satisfied and moreover*

$$\left[\overline{M}_0 L_\star K_b + \frac{\widehat{M} K_b L_b^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} \right] < 1 \tag{3.24}$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ *and* $l_b^* = \sup l_b(s)$. *Then the problem* (3.7) – (3.8) *has a unique mild solution on* $(-\infty, b]$.

Proof. Consider the operator $\widetilde{N} : \Omega \to \Omega$ defined by :

$$\widetilde{N}(y)(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\ [\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) \int_0^s \mathcal{K}(s,\tau) f(\tau, y_{\rho(\tau,y_\tau)}) d\tau ds, & \text{if } t \in J. \end{cases} \tag{3.25}$$

Then, fixed points of the operator $\widetilde{N}$ are mild solutions of the problem (3.7) – (3.8).

For $\phi \in \mathcal{B}$, we consider the function $x(.) : (-\infty, b] \to E$ defined as below by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\ \phi(0), & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in \Omega$, set

$$y(t) = z(t) + x(t)$$

It is obvious that y satisfies (3.23) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) &= g(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)}) - U(t,0) g(0,\phi) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) \int_0^s \mathcal{K}(s,\tau)\ f(\tau, z_{\rho(\tau, z_\tau + x_\tau)} + x_{\rho(\tau, z_\tau + x_\tau)}) d\tau ds. \end{aligned}$$

Let

$$\Omega^0 = \{ z \in \Omega : z_0 = 0 \}.$$

Define the operator $\widetilde{F} : \Omega^0 \to \Omega^0$ by :

$$\begin{aligned} \widetilde{F}(z)(t) &= g(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)}) - U(t,0) g(0,\phi) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) \int_0^s \mathcal{K}(s,\tau)\ f(\tau, z_{\rho(\tau, z_\tau + x_\tau)} + x_{\rho(\tau, z_\tau + x_\tau)}) d\tau ds. \end{aligned} \tag{3.26}$$

Obviously the operator $\widetilde{N}$ has a fixed point is equivalent to $\widetilde{F}$ has one, so it turns to prove that $\widetilde{F}$ has a fixed point.

Let $z \in \Omega^0$ be be a possible fixed point of the operator. Then, using $(H01) - (H04)$, we have for each $t \in [0, b]$

$$\begin{aligned} |z(t)| &\leq |g(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)})| + |U(t,0) g(0,\phi)| \\ &+ \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s) \int_0^s \mathcal{K}(s,\tau) f(\tau, z_{\rho(\tau, z_\tau + x_\tau)} + x_{\rho(\tau, z_\tau + x_\tau)}) d\tau ds \right| \\ &\leq \|A^{-1}(t)\|\ |A(t) g(t, z_{\rho(t, z_t + x_t)} + x_{\rho(t, z_t + x_t)})| + \|U(t,0)\|_{B(E)}\ \|A^{-1}(0)\|\ |A(0) g(0,\phi)| \\ &+ \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s) \int_0^s \mathcal{K}(s,\tau) f(\tau, z_{\rho(\tau, z_\tau + x_\tau)} + x_{\rho(\tau, z_\tau + x_\tau)}) d\tau ds \right| \\ &\leq \overline{M}_0 L \left(\|z_{\rho(t, z_t + x_t)} + x_{\rho(t, z_t + x_t)}\|_{\mathcal{B}} + 1 \right) + \widehat{M} \overline{M}_0 L \left(\|\phi\|_{\mathcal{B}} + 1 \right) \\ &+ \frac{\widehat{M} b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s) \|z_{\rho(s,y_s)} + x_{\rho(s,y_s)}\| \right] ds \end{aligned}$$

Since $\|z_{\rho(t,y_t)} + x_{\rho(t,y_t)}\|_{\mathcal{B}} \leq K_b|z(t)| + c_b$ we obtain

$$\begin{aligned} |z(t)| &\leq \overline{M}_0 L(K_n|z(t)| + c_b + 1) + \widehat{M}\overline{M}_0 L(\|\phi\|_{\mathcal{B}} + 1) + \frac{\widehat{M}p^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} \\ &+ \frac{\widehat{M}q^* b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ (K_b|z(s)| + c_b) \ ds \end{aligned}$$

Then

$$\begin{aligned} (1 - \overline{M}_0 L K_b)|z(t)| &\leq \overline{M}_0 L(c_b + 1) + \widehat{M}\overline{M}_0 L(\|\phi\|_{\mathcal{B}} + 1) + \frac{\widehat{M}p^* b^{\alpha+1} S_b}{\Gamma(\alpha+1)} \\ &+ \frac{\widehat{M}q^* b S_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ (K_b|z(s)| + c_b) \ ds \end{aligned}$$

Set

$$\delta_b := c_b + \frac{K_b\overline{M}_0 L(c_b+1) + K_b\widehat{M}\overline{M}_0 L(\|\phi\|_{\mathcal{B}}+1)}{(1-\overline{M}_0 L K_b)\Gamma(\alpha+1)} + \frac{K_b\widehat{M}p^* b^{\alpha+1} S_b}{(1-\overline{M}_0 L K_b)\Gamma(\alpha+1)}.$$

Thus

$$K_b|z(t)| + c_b \leq \delta_b + \frac{K_b\widehat{M}q^* b S_b}{(1-\overline{M}_0 L K_b)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ (K_b|z(s)| + c_b) \ ds.$$

We consider the function μ defined by

$$\mu(t) := \sup \{ \ K_b|z(s)| + c_b : \ 0 \leq s \leq t \ \}, \quad 0 \leq t \leq b.$$

Let $t^\star \in (-\infty, t]$ be such that $\mu(t) = K_b|z(t^\star)| + c_b$. If $t^\star \in [0, b]$, by the previous inequality, we get

$$\mu(t) \leq \delta_b + \frac{K_b\widehat{M}q^* b S_b}{(1-\overline{M}_0 L K_b)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ \mu(s) ds \qquad \text{for } t \in [0, b].$$

If $t^\star \in (-\infty, 0]$, then $\mu(t) = \|\varphi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta_b = \delta_b(\alpha)$ such that

$$\mu(t) \leq \delta_b \times \left[1 + \frac{K_b\widehat{M}q^* b^{\alpha+1} S_b}{(1-\overline{M}_0 L K_b)\Gamma(\alpha+1)} \right] := \Lambda_b$$

Since $\|z\| \leq \mu(t)$, we have $\|z\| \leq \max\{\|\phi\|, \Lambda_b\} := \Theta_b$.

Now, we shall show that $\widetilde{F} : Z \to \Omega^0$ is a contraction operator.
Indeed, consider $z, \overline{z} \in Z$, thus for each $t \in [0, b]$

$$\begin{aligned}
|\widetilde{F}(y)(t)-\widetilde{F}(\overline{y})(t)| &\leq |g(t,z_{\rho(t,y_t)}+x_{\rho(t,y_t)})-g(t,\overline{z}_{\rho(t,y_t)}+x_{\rho(t,y_t)})| \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|\int_0^s |\mathcal{K}(s,\tau)|\left|f(\tau,z_{\rho(\tau,z_\tau+x_\tau)}+x_{\rho(\tau,z_\tau+x_\tau)})\right. \\
&\left. -f(\tau,\overline{z}_{\rho(\tau,z_\tau+x_\tau)}+x_{\rho(\tau,z_\tau+x_\tau)})\right| \, d\tau ds \\
&\leq \|A^{-1}(t)\||A(t)g(t,z_{\rho(t,y_t)}+x_{\rho(t,y_t)})-A(t)g(t,\overline{z}_{\rho(t,y_t)}+x_{\rho(t,y_t)})| \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|\int_0^s |\mathcal{K}(s,\tau)|\left|f(\tau,z_{\rho(\tau,z_\tau+x_\tau)}+x_{\rho(\tau,z_\tau+x_\tau)})\right. \\
&\left. -f(\tau,\overline{z}_{\rho(\tau,z_\tau+x_\tau)}+x_{\rho(\tau,z_\tau+x_\tau)})\right| \, d\tau ds \\
&\leq \overline{M}_0 L_\star \|z_{\rho(t,y_t)}-\overline{z}_{\rho(t,y_t)}\|_{\mathcal{B}} \\
&+ \frac{\widehat{M} b S_b}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\; l_b(s)\; \|z_{\rho(s,y_s)}-\overline{z}_{\rho(s,y_s)}\|_{\mathcal{B}}\, ds.
\end{aligned}$$

Since $\|z_{\rho(t,y_t)}\|_{\mathcal{B}} \leq K_b|z(t)|+c_b$ we obtain

$$\begin{aligned}
|\widetilde{F}(y)(t)-\widetilde{F}(\overline{y})(t)| &\leq \overline{M}_0 L_\star k_b\|z(t)-\overline{z}(t)\| + \frac{\widehat{M}K_b L_b^* b^{\alpha+1}S_b}{\Gamma(\alpha+1)}\|z(t)-\overline{z}(t)\| \\
&\leq \left[\overline{M}_0 L_\star K_b + \frac{\widehat{M}k_b L_b^* b^{\alpha+1}S_b}{\Gamma(\alpha+1)}\right]\|z(t)-\overline{z}(t)\|
\end{aligned}$$

Therefore,

$$\|\widetilde{F}(y)-\widetilde{F}(\overline{y})\| \leq \left[\overline{M}_0 L_\star K_b + \frac{\widehat{M}K_b L_b^* b^{\alpha+1}S_b}{\Gamma(\alpha+1)}\right]\|z-\overline{z}\|.$$

So, for an appropriate choice of $\overline{M}_0 L_\star$, L_b^* and b^α such that

$$\left[\overline{M}_0 L_\star K_b + \frac{\widehat{M}K_b L_b^* b^{\alpha+1}S_b}{\Gamma(\alpha+1)}\right] < 1,$$

the operator $\widetilde{F}$ is a contraction. From the choice of Z there is no $z \in \partial Z^n$ such that $z = \lambda\, \widetilde{F}(z)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 1.10.4 does not hold. A consequence of the nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. We deduce that the operator $\widetilde{F}$ has an unique fixed-point $z^\star$. Then $y^\star(t) = z^\star(t)+x(t)$, $t \in (-\infty, b]$ is a fixed point of the operator $\widetilde{N}$, which is the unique mild solution of the problem $(3.7)-(3.8)$.

3.6 Examples

We give in this section four examples to illustrate the previous results.
Example 1. Consider the partial differential equation

$$\begin{cases} {}^cD_t^\alpha u(t,\xi) = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-r}^t \eta(t,\tau) \int_{-r}^0 a_1(s-t)u\left[s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right), \xi\right] d\tau ds, \\ \qquad\qquad 0 \leq t \leq b,\ \xi \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \qquad 0 \leq t \leq b, \\ u(\theta,\xi) = u_0(\theta,\xi), \qquad -r < \theta \leq 0,\ \xi \in [0,\pi], \end{cases} \tag{3.27}$$

where $a_1(t,\xi)$ is a continuous function and is uniformly Hölder continuous in t $0 < \alpha \leq 1$; $a_1 : [-r,0] \to \mathbb{R}$; $a_2 : [0,\pi] \to \mathbb{R}$; $\rho_i : [0,b] \to \mathbb{R}$ for $i = 1,2$ are continuous functions.

To study this system, we consider the space $E = L^2([0,\pi],\mathbb{R})$ and the operator $A : D(A) \subset E \to E$ given by $Aw = w''$ with

$$D(A) := \{\ w \in E\ : w'' \in E,\ w(0) = w(\pi) = 0\ \}$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\in[0,b]}$ on E, with compact resolvent. On the domain $D(A)$, we define the operators $A(t) : D(A) \subset E \to E$ by $A(t)x(\xi) = Ax(\xi) + a_0(t,\xi)x(\xi)$.

By assuming that $a_0(.,.)$ is continuous and that $a_0(t,\xi) \leq -\delta_0$ ($\delta_0 > 0$) for every $t \in [0,b], \xi \in [0,\pi]$, and specific case $\alpha = 1$ it follows that the system

$$u'(t) = A(t)u(t) \quad t \geq s; \quad u(s) = x \in E,$$

has an associated evolution family given by

$$U(t,s)x(\xi) = \left[T(t-s)exp\left(\int_s^t a_0(\tau,\xi)d\tau\right)x\right](\xi).$$

From this expression, it follows that $U(t,s)$ is a compact linear operator and that

$$\|U(t,s)\| \leq e^{-(1+\delta_0)(t-s)} \text{ for every } (t,s) \in [0,b]\times[0,b] \quad ; \quad s \leq t.$$

Theorem 3.6.1. *Let $\varphi \in C(H;E)$. Assume that the condition (H_φ) holds, the functions $\rho_i : [0,b] \to \mathbb{R}$ for $i = 1,2$; $a_1 : [-r,0] \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a unique mild solution of* (3.27).

Proof. From the assumptions, we have that

$$\mathcal{K}(t,s) = \eta(t,s)$$

$$f(t,\psi)(\xi) = \int_{-r}^0 a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (3.27) into the abstract system (3.1) − (3.2). Moreover, the function f is bounded linear operator. Now, the existence of the unique mild solution can be deduced from a direct application of Theorem 3.2.1. From Remark 1.7.1, we have the following result.

Corollary 3.6.1. *Let $\varphi \in C(H;E)$ be continuous and bounded. Then there exists a unique mild solution of* (3.27) *on* $[-r,b]$.

Example 2. Consider the partial differential equation

$$\begin{cases} {}^cD_t^\alpha\left[u(t,\xi) - \int_{-r}^0 a_3(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)ds\right] \\ = \dfrac{\partial^2 u(t,\xi)}{\partial\xi^2} + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-r}^t \eta(t,\tau)\int_{-r}^0 a_1(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)d\tau ds, \\ \hfill 0\le t\le b,\ \xi\in[0,\pi], \\ v(t,0) = v(t,\pi) = 0, \hfill 0\le t\le b, \\ v(\theta,\xi) = v_0(\theta,\xi), \hfill -r<\theta\le 0,\ \xi\in[0,\pi], \end{cases} \tag{3.28}$$

where $a_3 : [-r,0] \to \mathbb{R}$ is a continuous function.

Theorem 3.6.2. *Let $\varphi \in C(H;E)$. Assume that the condition (H_φ) holds, the functions $\rho_i : [0,b] \to \mathbb{R}$ for $i = 1,2$; $a_i : [-r,0] \to \mathbb{R}$ for $i = 1,3$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a unique mild solution of* (3.28).

Proof. From the assumptions, we have that

$$\mathcal{K}(t,s) = \eta(t,s)$$

$$f(t,\psi)(\xi) = \int_{-r}^0 a_1(s)\psi(s,\xi)ds,$$

$$g(t,\psi)(\xi) = \int_{-r}^0 a_3(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (3.28) into the abstract system (3.3) − (3.4). Moreover, the function f is bounded linear operator. Now, the existence of the unique mild solution can be deduced from a direct application of Theorem 3.3.1. From Remark 1.7.1, we have the following result.

Corollary 3.6.2. *Let $\varphi \in C(H;E)$ be continuous and bounded. Then there exists a unique mild solution of* (3.28) *on* $[-r,b]$.

Example 3. Consider the partial differential equation

$$\begin{cases} {}^cD_t^\alpha u(t,\xi) = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-\infty}^t \eta(t,\tau)\int_{-\infty}^0 a_1(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right] d\tau ds, \\ \hfill 0\le t\le b,\ \xi\in[0,\pi], \\ u(t,0) = u(t,\pi) = 0, \hfill 0\le t\le b, \\ u(\theta,\xi) = u_0(\theta,\xi), \hfill -\infty<\theta\le 0,\ \xi\in[0,\pi], \end{cases} \tag{3.29}$$

where $a_1 : (-\infty, 0] \to \mathbb{R}$ is continuous function.

Theorem 3.6.3. *Let $\mathcal{B} = BUC(\mathbb{R}_-; E)$ and $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds, the functions $\rho_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$ for $i = 1, 2$; $a_1 : \mathbb{R}_- \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a unique mild solution of* (3.29) *on* $]-\infty, b]$.

Proof. From the assumptions, we have that

$$\mathcal{K}(t,s) = \eta(t,s)$$

$$f(t,\psi)(\xi) = \int_{-\infty}^0 a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (3.29) into the abstract system (3.5) – (3.6). Moreover, the function f is bounded linear operator. Now, the existence of the unique mild solution can be deduced from a direct application of Theorem 3.4.1. From Remark 1.7.2, we have the following result.

Corollary 3.6.3. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then there exists a mild solution of* (3.29) *on* $]-\infty, b]$.

Example 4. Consider the partial differential equation

$$\begin{cases} {}^cD_0^\alpha \left[u(t,\xi) - \displaystyle\int_{-\infty}^0 a_3(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right) ds\right] \\ = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-\infty}^t \eta(t,\tau)\int_{-\infty}^0 a_1(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right) d\tau ds, \\ \hfill 0\le t\le b,\ \xi\in[0,\pi], \\ v(t,0) = v(t,\pi) = 0, \hfill 0\le t\le b, \\ v(\theta,\xi) = v_0(\theta,\xi), \hfill -\infty<\theta\le 0,\ \xi\in[0,\pi], \end{cases} \tag{3.30}$$

where $a_3 : (-\infty, 0] \to \mathbb{R}$ is a continuous function.

Theorem 3.6.4. *Let $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds, the functions $\rho_i : [0,b] \to \mathbb{R}$ for $i = 1,2$; $a_i : (-\infty, 0] \to \mathbb{R}$ for $i = 1,3$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a unique mild solution of* (3.30).

Proof. From the assumptions, we have that

$$\mathcal{K}(t,s) = \eta(t,s)$$

$$f(t,\psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s,\xi)ds,$$

$$g(t,\psi)(\xi) = \int_{-\infty}^{0} a_3(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (3.30) into the abstract system (3.7) – (3.8). Moreover, the function f is bounded linear operator. Now, the existence of the unique mild solution can be deduced from a direct application of Theorem 3.5.1. From Remark 1.7.2, we have the following result.

Corollary 3.6.4. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then there exists a unique mild solution of* (3.30) *on* $(-\infty, b]$.

Chapter 4

Perturbed Evolution Equations with State-Dependent Delay

4.1 Introduction

In this chapter, we will demonstrate the existence of mild solutions for some classes of first order of partial functional and neutral functional differential perturbed evolution equations with infinite state-dependent delay on Fréchet spaces[1], then we will show the existence of mild solutions for some classes of Caputo's fractional derivative order for partial functional and neutral functional perturbed evolution equations with finite and infinite state-dependent delay[2] by using the nonlinear alternative of Avramescu for the sum of contraction and completely continuous operators on Banach spaces.

The existence of mild solutions on the real positif interval is demonstrated in section 3.2 for the following class of perturbed evolution equations with infinite state-dependent delay

$$y'(t) = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e. } \ t \geq 0, \tag{4.1}$$

$$y_0 = \phi \in \mathcal{B} \tag{4.2}$$

where $\mathcal{B}$ is an abstract *phase space* to be specified later, $f, h : \mathbb{R}^+ \times \mathcal{B} \to E, \rho : \mathbb{R}^+ \times \mathcal{B} \to \mathbb{R}$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{0 \leq t < +\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+}$ for $s \leq t$.

An extension of this problem is given in section 3.3, we consider the following class of neutral perturbed evolution equations with infinite state-dependent delay

$$\frac{d}{dt}[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e. } \ t \geq 0, \tag{4.3}$$

$$y_0 = \phi \in \mathcal{B} \tag{4.4}$$

where $A(\cdot)$, f, h and ϕ are as in problem (4.1) – (4.2) and $g : \mathbb{R}^+ \times \mathcal{B} \to E$ is a given function.

[1][10] **D. Aoued** and S. Baghli-Bendimerad, Mild solutions for perturbed evolution equations with infinite state-dependent delay, *Electronic Journal of Qualitative Theory of Differential Equations* 2013, No. 59, 1–24.

[2][13] **D. Aoued**, S. Baghli-Bendimerad and M. Benchohra, Fractional perturbed partial functional and neutral functional evolution equations with infinite state-dependent delay, submitted.

In section 3.4, we give the existence of the unique mild solution of the following class of fractional perturbed evolution equations with finite state-dependent delay

$$^cD_0^\alpha y(t) = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J = [0, b], \tag{4.5}$$

$$y(t) = \varphi(t), \quad t \in H = [-r, 0], \tag{4.6}$$

where $0 < r < +\infty$, $^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0, 1)$, $f, h : J \times C(H; E) \to E, \rho : J \times C(H; E) \to \mathbb{R}$ and $\varphi \in C(H; E)$ are given functions and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t, s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 3.5, we consider the following class of fractional neutral perturbed evolution equations with finite state-dependent delay

$$^cD_0^\alpha [y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J, \tag{4.7}$$

$$y(t) = \varphi(t), \quad t \in H, \tag{4.8}$$

where $A(\cdot)$, f and ϕ are as in problem $(4.5)-(4.6)$ and $g : J \times C(H; E) \to E$ is a given function.

In section 3.6, we investigate the existence of mild solutions of the following class of fractional perturbed evolution equations with infinite state-dependent delay

$$^cD_0^\alpha y(t) = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J, \tag{4.9}$$

$$y_0 = \phi \in \mathcal{B} \tag{4.10}$$

where $\mathcal{B}$ is an abstract *phase space* to be specified later, $^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0, 1)$, $f, h : J \times \mathcal{B} \to E, \rho : J \times \mathcal{B} \to \mathrm{I\!R}$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t, s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 3.7, we consider the following class of fractional neutral perturbed evolution equations with infinite state-dependent delay

$$^cD_0^\alpha [y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J, \tag{4.11}$$

$$y_0 = \phi \in \mathcal{B} \tag{4.12}$$

where $A(\cdot)$, f and ϕ are as in problem $(4.9) - (4.10)$ and $g : J \times \mathcal{B} \to E$ is a given function. Finally, in section 3.8 we give examples to illustrate the abstract theory presented in previous sections.

4.2 Partial Problem with Infinite Delay

Before stating and proving the main result, we give first the definition of a mild solution of the perturbed semilinear evolution problem $(4.1) - (4.2)$.

Definition 4.2.1. *We say that the function $y : \mathbb{R} \to E$ is a mild solution of* $(4.1) - (4.2)$ *if $y(t) = \phi(t)$ for all $t \leq 0$ and y satisfies the following integral equation*

$$y(t) = U(t, 0)\ \phi(0) + \int_0^t U(t, s)\ [f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})]\ ds \qquad a.e.\ t \geq 0. \tag{4.13}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\phi) : (s,\phi) \in \mathbb{R}^+ \times \mathcal{B}, \rho(s,\phi) \leq 0\}.$$

We always assume that $\rho : \mathbb{R}^+ \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq \mathcal{L}^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

We will need to introduce the following hypothesis which are assumed thereafter

$(H0)$ $U(t,s)$ is compact for $t - s > 0$.

$(H1)$ There exists a constant $\widehat{M} \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq \widehat{M} \quad \text{for every } 0 \leq s \leq t < +\infty.$$

$(H2)$ There exists a function $p \in L^1_{loc}(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to (0,\infty)$ and such that

$$|f(t,u)| \leq p(t)\ \psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t \geq 0 \text{ and each } u \in \mathcal{B}.$$

$(H3)$ For all $R > 0$, there exists $l_R \in L^1_{loc}(\mathbb{R}^+;\mathbb{R}_+)$ such that

$$|f(t,u) - f(t,v)| \leq l_R(t)\ \|u - v\|_{\mathcal{B}}$$

for all $u, v \in \mathcal{B}$ with $\|u\|_{\mathcal{B}} \leq R$ and $\|v\|_{\mathcal{B}} \leq R$.

$(H4)$ There exists a function $\eta \in L^1(J,\mathbb{R}_+)$ such that

$$|h(t,u) - h(t,v)| \leq \eta(t)\ \ \|u - v\|_{\mathcal{B}} \quad a.e.\ t \in J \ \ et\ \ \forall u,v \in \mathcal{B}.$$

Consider the following space

$$B_{+\infty} = \left\{y : \mathbb{R} \to E : y|_{[0,T]} \text{ continuous for } T > 0 \text{ and } y_0 \in \mathcal{B}\right\},$$

where $y|_{[0,T]}$ is the restriction of y to the real compact interval $[0,T]$.

Let us fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms by

$$\|y\|_n := \sup\ \{\ e^{-\tau\ L^*_n(t)}\ |y(t)| : t \in [0,n]\ \}$$

where $L^*_n(t) = \displaystyle\int_0^t \bar{l}_n(s)\ ds$, $\bar{l}_n(t) = K_n\widehat{M}l_n(t)$ and l_n is the function from $(H3)$.

Then $B_{+\infty}$ is a Fréchet space with the family of semi-norms $\|\cdot\|_{n\in\mathbb{N}}$.

Proposition 4.2.1. *By (H_ϕ), and Lemma 1.7.2 and the assumption (A_1), we have for each $t \in [0,n]$ and $n \in \mathbb{N}$*

$$\|y_{\rho(t,y_t)}\| \leq K_n|y(t)| + (M_n + \mathcal{L}^\varphi)\ \|y_0\|_{\mathcal{B}}$$

Theorem 4.2.1. *Assume that the hypotheses* (H_ϕ), $(H0)-(H2)$ *and* $(H4)$ *hold and moreover for all* $n \in \mathbb{N}$, *we have*

$$\int_{\sigma_n}^{+\infty} \frac{ds}{s+\psi(s)} > K_n \widehat{M} \int_0^n \max(p(s);\eta(s))\ ds. \tag{4.14}$$

with $\sigma_n = (M_n + \mathcal{L}^\phi + K_n\widehat{M}H)\|\phi\|_{\mathcal{B}} + K_n\widehat{M}\int_0^n |h(s,0)|\ ds$. *Then the problem* $(4.1)-(4.2)$ *has a mild solution on* $(-\infty,+\infty)$.

Proof. We transform the problem $(4.1)-(4.2)$ into a fixed-point problem. Consider the operator $N : B_{+\infty} \to B_{+\infty}$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in \mathbb{R}^-; \\ U(t,0)\ \phi(0) + \int_0^t U(t,s)\ f(s, y_{\rho(s,y_s)})\ ds, & \\ + \int_0^t U(t,s)\ h(s, y_{\rho(s,y_s)})\ ds, & \text{if } t \geq 0. \end{cases}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(4.1)-(4.2)$.

For $\phi \in \mathcal{B}$, we will define the function $x(.) : \mathbb{R} \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \leq 0; \\ U(t,0)\ \phi(0), & \text{if } \ t \geq 0. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$, set

$$y(t) = z(t) + x(t)$$

It is obvious that y satisfies (4.13) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) &= \int_0^t U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds \\ &+ \int_0^t U(t,s)\ h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds. \end{aligned}$$

Let

$$B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0\}.$$

Define the operators $F, G : B^0_{+\infty} \to B^0_{+\infty}$ by

$$F(z)(t) = \int_0^t U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds$$

and

$$G(z)(t) = \int_0^t U(t,s)\ h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds.$$

Obviously the operator N has a fixed point is equivalent to $F+G$ has one, so it turns to prove that $F+G$ has a fixed point.

First, show that F is continuous and compact.

Step 1 : F continuous. Let $(z_n)_{n\in\mathbb{N}}$ a sequence in $B^0_{+\infty}$ such that $z_n \to z$ in $B^0_{+\infty}$. By the hypothesis $(H1)$, we have

$$\begin{aligned}
&|F(z_n)(t) - F(z)(t)| \leq \\
&\leq \int_0^t \|U(t,s)\|_{B(E)} |f(s, z_{n\rho(s,z_{ns}+x_s)} + x_{\rho(s,z_{ns}+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds \\
&\leq \widehat{M} \int_0^t \left| f(s, z_{n\rho(s,z_{ns}+x_s)} + x_{\rho(s,z_{ns}+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \right| \, ds.
\end{aligned}$$

Since f is continuous, by dominated convergence theorem of Lebesgue, we get

$$|F(z_n)(t) - F(z)(t)| \longrightarrow 0 \ \text{ when } \ n \longrightarrow +\infty$$

So F is continuous.

Step 2 : Show that F transforms any bounded of $B^0_{+\infty}$ in a bounded set. For each $d > 0$, there exists a positive constant ξ such that for all $z \in B_d = \{z \in B^0_{+\infty} \ : \|z\|_n \leq d\}$ we get $\|F(z)\|_n \leq \xi$. Soit $z \in B_d$, from assumption $(H1)$ and $(H2)$, we have for each $t \in [0,n]$

$$\begin{aligned}
|F(z)(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)} \, |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \, ds \\
&\leq \widehat{M} \int_0^t p(s) \, \psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}) \, ds.
\end{aligned}$$

From (H_ϕ), Lemma 1.7.2 and Assumption $(A1)$, we have for each $t \in [0,n]$

$$\begin{aligned}
\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\leq \|z_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + \|x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \\
&\leq K_n|z(s)| + (M_n + \mathcal{L}^\phi)\|z_0\|_{\mathcal{B}} + K_n|x(s)| + (M_n + \mathcal{L}^\phi)\|x_0\|_{\mathcal{B}} \\
&\leq K_n|z(s)| + K_n\|U(s,0)\|_{B(E)}|\phi(0)| + (M_n + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} \\
&\leq K_n|z(s)| + K_n\widehat{M}|\phi(0)| + (M_n + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}
\end{aligned}$$

Using (ii), we get

$$\begin{aligned}
\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\leq K_n|z(s)| + K_n\widehat{M}H\|\phi\|_{\mathcal{B}} + (M_n + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} \\
&\leq K_n|z(s)| + (M_n + \mathcal{L}^\phi + K_n\widehat{M}H)\|\phi\|_{\mathcal{B}}
\end{aligned}$$

Set $c_n := (M_n + \mathcal{L}^\phi + K_n\widehat{M}H)\|\phi\|_{\mathcal{B}}$ and $\delta_n := K_n d + c_n$. Then

$$\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \leq K_n|z(s)| + c_n \leq \delta_n. \tag{4.15}$$

Using the nondecreasing character of ψ, we get for each $t \in [0,n]$

$$|F(z)(t)| \leq \widehat{M}\psi(\delta_n)\|p\|_{L^1} := \varrho.$$

So there is a positive constant ϱ such that $\|F(z)\|_n \leq \varrho$. Then $F(B_d) \subset B_\varrho$.

Step 3 : F maps bounded sets into equi-continuous sets of $B^0_{+\infty}$. We consider B_d as in Step 2 and we show that $F(B_d)$ is equi-continuous. Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $z \in B_d$.

$$\begin{aligned}|F(z)(\tau_2) - F(z)(\tau_1)| &\leq \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} \, |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \, ds \\ &+ \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)} \, |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \, ds.\end{aligned}$$

Then by (5.41) and the nondecreasing character of ψ we get

$$\begin{aligned}|F(z)(\tau_2) - F(z)(\tau_1)| &\leq \psi(\delta_n) \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} \, p(s) \, ds \\ &+ \widehat{M}\psi(\delta_n) \int_{\tau_1}^{\tau_2} p(s) \, ds.\end{aligned}$$

Noting that $|F(z)(\tau_2) - F(z)(\tau_1)| \longrightarrow 0$ tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$ independently of $z \in B_d$. The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$, since $U(t, s)$ is a strongly continuous operator and the compactness of $U(t, s)$ for $t > s$, implies the continuity in the uniform operator topology (voir [7, 90]). As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that the operator F maps B_d into a precompact set in E.

Let $t \in J$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $z \in B_d$ we define

$$F_\varepsilon(z)(t) = U(t, t - \varepsilon) \int_0^{t-\varepsilon} U(t - \varepsilon, s) C u_{z+x}(s) \, ds.$$

Since $U(t, s)$ is a compact operator, the set $Z_\varepsilon(t) = \{F_\varepsilon(z)(t) \ : z \in B_d\}$ is pre-compact in E for every ε, $0 < \varepsilon < t$. Moreover, using the definition of w, we get

$$|F_(z)(t) - F_\varepsilon(z)(t)| \leq \int_{t-\varepsilon}^{t} \|U(t, s)\|_{B(E)} \, |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \, ds$$

Therefore the set $Z(t) = \{F(z)(t) \ : z \in B_d\}$ is totally bounded.

Hence the set $\{F(z)(t) \ : z \in B_d\}$. So, we deduce from Steps 1, 2 and 3 that F is a compact operator.

Step 4 : G is a contraction. Let $z, \overline{z} \in B^0_{+\infty}$. By the hypotheses $(H1)$ and $(H4)$, we get for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$\begin{aligned}&|G(z)(t) - G(\overline{z})(t)| \leq \\ &\leq \int_0^t \|U(t, s)\|_{B(E)} \mid h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - h(s, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \mid ds \\ &\leq \widehat{M} \int_0^t \eta(s) \, \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)} - x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \, ds \\ &\leq \widehat{M} \int_0^t \eta(s) \, \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \, ds.\end{aligned}$$

Use the inequality (4.15), to get

$$\begin{aligned} |G(z)(t)-G(\overline{z})(t)| &\le \int_0^t \widehat{M}\ K_n\ \eta(s)\ |z(s)-(\overline{z})(s)|\ ds \\ &\le \int_0^t \left[\overline{L}_n(s)\ e^{\tau L_n^*(s)}\right]\ \left[e^{-\tau L_n^*(s)}\ |z(s)-\overline{z}(s)|\right]\ ds \\ &\le \int_0^t \left[\frac{e^{\tau\ L_n^*(s)}}{\tau}\right]'\ ds\ \|z-\overline{z}\|_n \\ &\le \frac{1}{\tau}\ e^{\tau\ L_n^*(t)}\ \|z-\overline{z}\|_n. \end{aligned}$$

Therefore

$$\|G(z)-G(\overline{z})\|_n \le \frac{1}{\tau}\ \|z-\overline{z}\|_n.$$

Then the operator G is a contraction for all $n\in\mathbb{N}$.

Step 5 : For applying Theorem (1.10.2), we must check $(Av2)$: i.e. it remains to show that the set

$$\Sigma = \left\{ z\in B^0_{+\infty}\ : z=\lambda F(z)+\lambda G\left(\frac{z}{\lambda}\right) \quad \text{for some } \lambda\in]0,1[\right\}.$$

is bounded.

Let $z\in\Sigma$. By $(H1)-(H2)$ and $(H4)$, we have for each $t\in[0,n]$

$$\begin{aligned} \frac{1}{\lambda}|z(t)| &\le \int_0^t \|U(t,s)\|_{B(E)}\ \left|f(s, z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})\right|\ ds \\ &+ \int_0^t \|U(t,s)\|_{B(E)}\ \left|h\left(s, \frac{z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}}{\lambda}\right)-h(s,0)+h(s,0)\right|\ ds \\ &\le \widehat{M}\int_0^t p(s)\ \psi(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})\|_{\mathcal{B}}\ ds) \\ &+ \widehat{M}\int_0^t \eta(s)\ \left\|\frac{z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}}{\lambda}\right\|_{\mathcal{B}}\ ds+\widehat{M}\int_0^t\ |h(s,0)|\ ds. \end{aligned}$$

Use Proposition 4.2.1 and inequality (4.15)

$$\begin{aligned} \frac{1}{\lambda}|z(t)| &\le \widehat{M}\int_0^t p(s)\ \psi(K_n|z(s)|+c_n)\ ds \\ &+ \widehat{M}\int_0^t \eta(s)\ \left(\frac{K_n}{\lambda}|z(s)|+c_n\right)\ ds+\widehat{M}\int_0^t\ |h(s,0)|\ ds. \end{aligned}$$

We consider the function $u(t) := \sup_{\theta\in[0,t]}|z(\theta)|$. The use of nondecreasing character of ψ gives with the fact that $0<\lambda<1$

$$\begin{aligned} \frac{K_n}{\lambda}u(t)+c_n &\le c_n+K_n\widehat{M}\int_0^t p(s)\ \psi\left(\frac{K_n}{\lambda}u(s)+c_n\right)\ ds \\ &+ K_n\widehat{M}\int_0^t \eta(s)\ \left(\frac{K_n}{\lambda}u(s)+c_n\right)\ ds+\widehat{M}\int_0^t\ |h(s,0)|\ ds. \end{aligned}$$

Set $\sigma_n := c_n + K_n\widehat{M}\int_0^n |h(s,0)|\ ds$. Then, we have

$$\begin{aligned} \frac{K_n}{\lambda}u(t) + c_n &\le \sigma_n + K_n\widehat{M}\int_0^t p(s)\ \psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)\ ds \\ &+ K_n\widehat{M}\int_0^t \eta(s)\ \left(\frac{K_n}{\lambda}u(s) + c_n\right)\ ds. \end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\ \{K_n u(s) + c_n\ :\ 0 \le s \le t\},\quad 0 \le t \le +\infty$$

Let $t^\star \in [0,t]$ such that $\mu(t) = K_n u(t^\star) + c_n$. From the previous inequality, we have for all $t \in [0,n]$

$$\mu(t) \le \sigma_n + K_n\widehat{M}\int_0^t p(s)\psi(\mu(s))ds + K_n\widehat{M}\int_0^t \eta(s)\mu(s)ds.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have

$$\mu(t) \le v(t) \quad \forall t \in [0,n].$$

From the definition of v, we have

$$v(0) = \sigma_n \quad \text{and} \quad v'(t) = K_n\widehat{M}\left[p(t)\psi(\mu(t)) + \eta(t)\mu(t)\right] \quad \text{a.e.} \quad t \in [0,n].$$

Using the nondecreasing character of ψ we get

$$v'(t) \le K_n\widehat{M}[p(t)\psi(v(t)) + \eta(t)v(t)] \quad a.e.\ t \in [0,n]$$

This implies that for each $t \in [0,n]$ and using (4.14), we get

$$\begin{aligned} \int_{\sigma_n}^{v(t)} \frac{ds}{s+\psi(s)} &\le K_n\widehat{M}\int_0^t \max(p(s);\eta(s))\ ds \\ &\le K_n\widehat{M}\int_0^n \max(p(s);\eta(s))\ ds \\ &< \int_{c_n}^{+\infty} \frac{ds}{s+\psi(s)}. \end{aligned}$$

Thus, for every $t \in [0,n]$, there exists a constant Λ_n such that $v(t) \le \Lambda_n$ and hence $\mu(t) \le \Lambda_n$. Since $\|z\|_n \le \mu(t)$, we have $\|z\|_n \le \Lambda_n$. This shows that the set Σ is bounded. Then the statement $(Av2)$ in Theorem 1.10.2 does not hold. The nonlinear alternative of Avramescu implies that $(Av1)$ is satisfied, we deduce that the operator $F+G$ has a fixed point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in]-\infty,+\infty[$ is a fixedpoint of the operator N which is a mild solution of the problem $(4.1)-(4.2)$.

4.3 Neutral Problem with Infinite Delay

In this section, we give an existence result of mild solution for the problem $(4.3)-(4.4)$. Firstly, we define the concept of the mild solution for that problem.

Definition 4.3.1. *We say that the function $y(\cdot):\mathbb{R}\to E$ is a mild solution of* $(4.3)-(4.4)$ *if* $y(t)=\phi(t)$ *for all* $t\leq 0$ *and* y *satisfies the following integral equation*

$$\begin{aligned} y(t) &= U(t,0)[\phi(0)-g(0,\phi)]+g(t,y_{\rho(t,y_t)})+\int_0^t U(t,s)A(s)\ g(s,y_{\rho(s,y_s)})\ ds \\ &+\int_0^t U(t,s)\ [f(s,y_{\rho(s,y_s)})+h(s,y_{\rho(s,y_s)})]\ ds \qquad \forall t\geq 0. \end{aligned} \tag{4.16}$$

We consider the hypotheses (H_ϕ), $(H0)-(H4)$ and we need the following assumptions

$(H5)$ There exists a constant $\overline{M}_0>0$ such that

$$\|A^{-1}(t)\|_{B(E)}\leq \overline{M}_0 \qquad \text{for all } t\geq 0.$$

$(H6)$ There exists a constant $0<L<\dfrac{1}{\overline{M}_0K_n}$, such that

$$|A(t)\ g(t,\phi)|\leq L\ (\|\phi\|_{\mathcal{B}}+1) \text{ for all } t\geq 0 \text{ and } \phi\in\mathcal{B}.$$

$(H7)$ There exists a constant $L_*>0$ such that

$$|A(s)\ g(s,\phi)-A(\overline{s})\ g(\overline{s},\overline{\phi})|\leq L_*\ (|s-\overline{s}|+\|\phi-\overline{\phi}\|_{\mathcal{B}})$$

for all $s,\overline{s}\geq 0$ and $\phi,\overline{\phi}\in\mathcal{B}$.

$(H8)$ The function g is completely continuous and for any bounded set $Q\subset\mathcal{B}$ the set $\{t\longrightarrow g(t,x_{\rho(t,y_t)})\}$ is equi-continuous in $\mathcal{B}$.

Theorem 4.3.1. *Suppose that the hypotheses* (H_ϕ), $(H0)-(H8)$ *are satisfied and moreover for all* $n\in\mathbb{N}$, *we have*

$$\int_{\xi_n}^{+\infty}\frac{ds}{s+\psi(s)}>\frac{K_n\widehat{M}}{1-\overline{M}_0LK_n}\int_0^n \max(L+\eta(s),p(s))\ ds \tag{4.17}$$

with $c_n=(M_n+\mathcal{L}^\phi+K_n\widehat{M}H)\|\phi\|_{\mathcal{B}}$ *and*

$$\xi_n=c_n+K_n\frac{\left(\widehat{M}+1\right)\overline{M}_0L+\widehat{M}Ln+\overline{M}_0L\left(c_n+\widehat{M}\right)\|\phi\|_{\mathcal{B}}+\widehat{M}\displaystyle\int_0^n |h(s,0)|\ ds}{1-\overline{M}_0LK_n}.$$

Then the problem $(4.3)-(4.4)$ *has a mild solution.*

Proof. Consider the operator $\widetilde{N}:B_{+\infty}\to B_{+\infty}$ defined by

$$\widetilde{N}(y)(t)=\begin{cases} \phi(t), & \text{if } t\in\mathbb{R}^-; \\ U(t,0)\ [\phi(0)-g(0,\phi)]+g(t,y_{\rho(t,y_t)}) & \\ +\displaystyle\int_0^t U(t,s)A(s)\ g(s,y_{\rho(s,y_s)})\ ds, & \\ +\displaystyle\int_0^t U(t,s)\ \left[f(s,y_{\rho(s,y_s)})+h(s,y_{\rho(s,y_s)})\right]\ ds & \text{if } t\geq 0. \end{cases}$$

Then, fixed points of the operator $\widetilde{N}$ are mild solutions of the problem (4.3) – (4.4).

For $\phi \in \mathcal{B}$, we consider the function $x(.) : \mathbb{R} \to E$ defined as below by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \leq 0; \\ U(t,0)\ \phi(0), & \text{if } \ t \geq 0. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$, set

$$y(t) = z(t) + x(t)$$

It is obvious that y satisfies (4.16) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) \ &= g(t, z_{\rho(s,z_t+x_t)} + x_{\rho(s,z_t+x_t)}) - U(t,0)g(0,\phi) \\ &+ \int_0^t U(t,s)A(s)\ g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds \\ &+ \int_0^t U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds \\ &+ \int_0^t U(t,s)\ h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds. \end{aligned}$$

Let

$$B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0\}.$$

Define the operators $F, \widetilde{G} : B^0_{+\infty} \to B^0_{+\infty}$ by

$$F(z)(t) = \int_0^t U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds$$

And

$$\begin{aligned} \widetilde{G}(z)(t) \ &= g(t, z_{\rho(s,z_t+x_t)} + x_{\rho(s,z_t+x_t)}) - U(t,0)g(0,\phi) \\ &+ \int_0^t U(t,s)A(s)\ g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds \\ &+ \int_0^t U(t,s)\ h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds. \end{aligned}$$

Obviously the operator $\widetilde{N}$ has a fixed point is equivalent to $F + \widetilde{G}$ has one, so it turns to prove that $F + \widetilde{G}$ has a fixed point.

We have shown that the operator F is continuous and compact as in Section 3.2 Remains to show that the operator $\widetilde{G}$ is a contraction.

Let $z,\overline{z}\in B^0_{+\infty}$. By $(H1)$, $(H4)$, $(H5)$ and $(H7)$, we have for each $t\in[0,n]$ and $n\in\mathbb{N}$

$$
\begin{aligned}
&|\widetilde{G}(z)(t)-\widetilde{G}(\overline{z})(t)|\leq\\
&\leq|g(t,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})-g(t,\overline{z}_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})|\\
&+\int_0^t\|U(t,s)\|_{B(E)}\ |A(s)[g(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})-g(s,\overline{z}_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})]|\ ds\\
&+\int_0^t\|U(t,s)\|_{B(E)}\ |h(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})-h(s,\overline{z}_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})|\ ds\\
&\leq\|A^{-1}(t)\|_{B(E)}\ |A(t)g(t,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})-A(t)g(t,\overline{z}_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})|\\
&+\int_0^t\widehat{M}\ |A(s)g(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})-A(s)g(s,\overline{z}_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})|\ ds\\
&+\int_0^t\widehat{M}\ |h(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})-h(s,\overline{z}_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})|\ ds\\
&\leq\overline{M_0}L_*\|z_{\rho(t,z_t+x_t)}-\overline{z}_{\rho(t,z_t+x_t)}\|_{\mathcal{B}}+\int_0^t\widehat{M}L_*\|z_{\rho(s,z_s+x_s)}-\overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\ ds\\
&+\int_0^t\widehat{M}\eta(s)\|z_{\rho(s,z_s+x_s)}-\overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\ ds.
\end{aligned}
$$

Use the inequality (4.15), we get

$$
\begin{aligned}
|\widetilde{G}(z)(t)-\widetilde{G}(\overline{z})(t)| &\leq \overline{M_0}L_*K_n|z(t)-\overline{z}(t)|+\int_0^t\widehat{M}L_*K_n|z(s)-\overline{z}(s)|\ ds\\
&+\int_0^t\widehat{M}K_n\ \eta(s)|z(s)-\overline{z}(s)|\ ds\\
&\leq \overline{M_0}L_*K_n|z(t)-\overline{z}(t)|+\int_0^t\widehat{M}K_n[L_*+\eta(s)]|z(s)-\overline{z}(s)|\ ds.
\end{aligned}
$$

Set $\bar{l}_n(t)=\widehat{M}K_n[L_*+\eta(t)]$ for the family of semi-norms $\{\|\cdot\|_{n\in\mathbb{N}}\}$, then

$$
\begin{aligned}
|\widetilde{G}(z)(t)-\widetilde{G}(\overline{z})(t)| &\leq \overline{M}_0L_*K_n\ |z(t)-\overline{z}(t)|+\int_0^t\bar{l}_n(s)\ |z(s)-\overline{z}(s)|ds\\
&\leq \left[\overline{M}_0L_*K_n\ e^{\tau\ L_n^*(t)}\right]\ \left[e^{-\tau\ L_n^*(t)}\ |z(t)-\overline{z}(t)|\right]\\
&+\int_0^t\left[\bar{l}_n(s)\ e^{\tau\ L_n^*(s)}\right]\ \left[e^{-\tau\ L_n^*(s)}\ |z(s)-\overline{z}(s)|\right]ds\\
&\leq \overline{M}_0L_*K_n\ e^{\tau\ L_n^*(t)}\ \|z-\overline{z}\|_n+\int_0^t\left[\frac{e^{\tau\ L_n^*(s)}}{\tau}\right]'\ ds\ \|z-\overline{z}\|_n\\
&\leq \left[\overline{M}_0L_*K_n+\frac{1}{\tau}\right]\ e^{\tau\ L_n^*(t)}\ \|z-\overline{z}\|_n.
\end{aligned}
$$

Therefore

$$\|\widetilde{G}(z)-\widetilde{G}(\overline{z})\|_n\leq\left[\overline{M_0}L_*K_n+\frac{1}{\tau}\right]\|z-\overline{z}\|_n$$

Let us fix $\tau>0$ and assume that

$$\overline{M}_0L_*K_n+\frac{1}{\tau}<1,$$

then the operator $\widetilde{G}$ is a contraction for all $n \in \mathbb{N}$.

For applying Theorem (1.10.2), we must check $(Av2)$ i.e. it remains to show that the set

$$\widetilde{\Sigma} = \left\{ z \in B^0_{+\infty} \quad : \ z = \lambda F(z) + \lambda \widetilde{G}\left(\frac{z}{\lambda}\right) \quad pour \ 0 < \lambda < 1 \right\}.$$

is bounded.

Let $z \in \widetilde{\Sigma}$. By $(H1) - (H2)$, we have for each $t \in [0, n]$

$$\begin{aligned}
\frac{|z(t)|}{\lambda} &\le \|A^{-1}(t)\| |A(t) g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})| + \widehat{M} \|A^{-1}(0)\| |A(0) g(0, \phi)| \\
&+ \widehat{M} \int_0^t |A(s) g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \ ds \\
&+ \widehat{M} \int_0^t p(s) \ \psi\left(\left\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right\|\right) \ ds \\
&+ \widehat{M} \int_0^t \left| h\left(s, \frac{z_{\rho\left(s, \frac{z_s}{\lambda} + x_s\right)}}{\lambda} + x_{\rho\left(s, \frac{z_s}{\lambda} + x_s\right)}\right) - h(s, 0) \right| \ ds + \widehat{M} \int_0^n |h(s, 0)| \ ds.
\end{aligned}$$

Using assumptions $(H5) - (H6)$ and $(H4)$

$$\begin{aligned}
\frac{|z(t)|}{\lambda} &\le \overline{M}_0 L(\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} + 1) + \widehat{M}\overline{M}_0 L(\|\phi\|_{\mathcal{B}} + 1) \\
&+ \widehat{M} L \int_0^t (\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + 1) \ ds \\
&+ \widehat{M} \int_0^t \ p(s) \ \psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}) \ ds \\
&+ \widehat{M} \int_0^t \ \eta(s) \ \left\| \frac{z_{\rho\left(s, \frac{z_s}{\lambda} + x_s\right)}}{\lambda} + x_{\rho\left(s, \frac{z_s}{\lambda} + x_s\right)} \right\|_{\mathcal{B}} \ ds + \widehat{M} \int_0^t \ |h(s, 0)| \ ds \\
&\le \left(\widehat{M} + 1\right) \overline{M}_0 L + \widehat{M} L n + \widehat{M}\overline{M}_0 L \|\phi\|_{\mathcal{B}} + \widehat{M} \int_0^n \ |h(s, 0)| \ ds \\
&+ \overline{M}_0 L \|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} + \widehat{M} L \int_0^t \ \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \ ds \\
&+ \widehat{M} \int_0^t \ p(s) \ \psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}) \ ds \\
&+ \widehat{M} \int_0^t \ \eta(s) \ \left\| \frac{z_{\rho\left(s, \frac{z_s}{\lambda} + x_s\right)}}{\lambda} + x_{\rho\left(s, \frac{z_s}{\lambda} + x_s\right)} \right\|_{\mathcal{B}} \ ds.
\end{aligned}$$

Use Proposition (4.2.1) and inequality (4.15) for get

$$\begin{aligned}
\left\| \frac{z_{\rho\left(s, \frac{z_s}{\lambda} + x_s\right)}}{\lambda} + x_{\rho\left(s, \frac{z_s}{\lambda} + x_s\right)} \right\|_{\mathcal{B}} &\le \frac{1}{\lambda} \left\| z_{\rho\left(s, \frac{z_s}{\lambda} + x_s\right)} \right\|_{\mathcal{B}} + \left\| x_{\rho\left(s, \frac{z_s}{\lambda} + x_s\right)} \right\|_{\mathcal{B}} \\
&\le \frac{K_n}{\lambda} |z(s)| + \frac{M_n + \mathcal{L}^\phi}{\lambda} \|z_0\|_{\mathcal{B}} + K_n |x(s)| + \left(M_n + \mathcal{L}^\phi\right) \|x_0\|_{\mathcal{B}} \\
&\le \frac{K_n}{\lambda} |z(s)| + K_n \|U(s, 0)\|_{B(E)} |\phi(0)| + \left(M_n + \mathcal{L}^\phi\right) \|\phi\|_{\mathcal{B}} \\
&\le \frac{K_n}{\lambda} |z(s)| + K_n \widehat{M} H \|\phi\|_{\mathcal{B}} + \left(M_n + \mathcal{L}^\phi\right) \|\phi\|_{\mathcal{B}} \\
&\le \frac{K_n}{\lambda} |z(s)| + \left(M_n + \mathcal{L}^\phi + K_n \widehat{M} H\right) \|\phi\|_{\mathcal{B}}.
\end{aligned}$$

Hence

$$\left\| \frac{z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}}{\lambda} + x_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)} \right\|_{\mathcal{B}} \le \frac{K_n}{\lambda}|z(s)| + c_n. \tag{4.18}$$

Use the function $u(\cdot)$ and the nondecreasing character of ψ to get

$$\begin{aligned}\frac{u(t)}{\lambda} \;&\le\; \left(\widehat{M}+1\right)\overline{M}_0 L + \widehat{M}Ln + \widehat{M}\overline{M}_0 L\|\phi\|_{\mathcal{B}} + \widehat{M}\int_0^n \; |h(s,0)| \; ds \\ &+\; \overline{M}_0 L \;\left(K_n u(t) + c_n\right) + \widehat{M}L\int_0^t \;\left(K_n u(s) + c_n\right)\; ds \\ &+\; \widehat{M}\int_0^t \; p(s)\; \psi\left(K_n u(s) + c_n\right)\; ds + \widehat{M}\int_0^t \; \eta(s)\; \left(\frac{K_n}{\lambda}u(s) + c_n\right)\; ds.\end{aligned}$$

Then

$$\begin{aligned}\frac{u(t)}{\lambda} \;&\le\; \left(\widehat{M}+1\right)\overline{M}_0 L + \widehat{M}Ln + \overline{M}_0 L\left[M_n + \mathcal{L}^{\phi} + \widehat{M}(1+K_nH)\right]\|\phi\|_{\mathcal{B}} \\ &+\; \widehat{M}\int_0^n \; |h(s,0)|\; ds + \overline{M}_0 L\;\frac{K_n}{\lambda}u(t) + \widehat{M}L\int_0^t \;\left(\frac{K_n}{\lambda}u(s) + c_n\right)\; ds \\ &+\; \widehat{M}\int_0^t \; p(s)\; \psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)\; ds + \widehat{M}\int_0^t \; \eta(s)\; \left(\frac{K_n}{\lambda}u(s) + c_n\right)\; ds.\end{aligned}$$

Set

$$\zeta_n := \left(\widehat{M}+1\right)\overline{M}_0 L + \widehat{M}Ln + \overline{M}_0 L\left[M_n + \mathcal{L}^{\phi} + \widehat{M}(1+K_nH)\right]\|\phi\|_{\mathcal{B}} + \widehat{M}\int_0^n \; |h(s,0)|\; ds.$$

So

$$\begin{aligned}\frac{K_n}{\lambda}\left(1 - \overline{M}_0 L K_n\right)u(t) \;&\le\; K_n\zeta_n + K_n\widehat{M}\int_0^t \;[L+\eta(s)]\;\left(\frac{K_n}{\lambda}u(s) + c_n\right)\; ds \\ &+\; K_n\widehat{M}\int_0^t \; p(s)\; \psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)\; ds.\end{aligned}$$

Set $\xi_n := c_n + \dfrac{K_n\zeta_n}{1-\overline{M}_0 L K_n}$. Then

$$\begin{aligned}\frac{K_n}{\lambda}u(t) + c_n \;&\le\; \xi_n + \frac{K_n\widehat{M}}{1-\overline{M}_0 L K_n}\int_0^t \;[L+\eta(s)]\;\left(\frac{K_n}{\lambda}u(s) + c_n\right)\; ds \\ &+\; \frac{K_n\widehat{M}}{1-\overline{M}_0 L K_n}\int_0^t \; p(s)\; \psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)\; ds.\end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\left\{\frac{K_n}{\lambda}u(s) + c_n \quad : \; 0 \le s \le t\right\}, \quad 0 \le t \le +\infty$$

Let $t^{\star} \in [0,t]$ be such that $\mu(t) = \dfrac{K_n}{\lambda}u(t^{\star}) + c_n$. By the previous inequality, we have for $t \in [0,n]$

$$\mu(t) \le \xi_n + \frac{K_n\widehat{M}}{1-\overline{M}_0 L K_n}\int_0^t \;[L+\eta(s)]\;\mu(s)\; ds + \frac{K_n\widehat{M}}{1-\overline{M}_0 L K_n}\int_0^t \; p(s)\; \psi\left(\mu(s)\right)\; ds.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\mu(t) \leq v(t) \quad \forall t \in [0, n].$$

From the definition of v, we get $v(0) = \xi_n$ and

$$v'(t) = \frac{K_n \widehat{M}}{1 - \overline{M}_0 L K_n} \; [L + \eta(t)] \; \mu(t) + \frac{K_n \widehat{M}}{1 - \overline{M}_0 L K_n} \; p(t) \; \psi\left(\mu(t)\right) \quad \text{a.e. } \; t \in [0, n].$$

Using the nondecreasing character of ψ we have

$$v'(t) \leq \frac{K_n \widehat{M}}{1 - \overline{M}_0 L K_n} \; [L + \eta(t)] \; v(t) + \frac{K_n \widehat{M}}{1 - \overline{M}_0 L K_n} \; p(t) \; \psi\left(v(t)\right) \quad \text{a.e. } \; t \in [0, n].$$

This implies that for each $t \in [0, n]$ and using (4.17) we get

$$\begin{aligned} \int_{\xi_n}^{v(t)} \frac{ds}{s + \psi(s)} &\leq \frac{K_n \widehat{M}}{1 - \overline{M}_0 L K_n} \int_0^t \max(L + \eta(s), p(s)) \; ds \\ &\leq \frac{\widehat{M} K_n}{1 - \overline{M}_0 L K_n} \int_0^n \max(L + \eta(s), p(s)) \; ds \\ &< \int_{\xi_n}^{+\infty} \frac{ds}{s + \psi(s)}. \end{aligned}$$

Thus, for every $t \in [0, n]$, there exists a constant Λ_n such that $v(t) \leq \Lambda_n$ and hence $\mu(t) \leq \Lambda_n$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq \Lambda_n$. This shows that the set $\widetilde{\Sigma}$ is bounded. Then the statement $(Av2)$ in Theorem 1.10.2 does not hold. The nonlinear alternative of Avramescu implies that $(Av1)$ is satisfied. we deduce that the operator $F + \widetilde{G}$ has a fixed point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in]-\infty, +\infty[$ is a fixed point of the operator N which is a mild solution of the problem $(4.3) - (4.4)$.

4.4 Fractional Partial Problem with Finite Delay

Before stating and proving the main result, we give first the definition of the unique mild solution of the perturbed fractional problem $(4.5) - (4.6)$.

Lemma 4.4.1. *The system* $(4.5) - (4.6)$ *is equivalent to the nonlinear integral equation*

$$y(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s) y(s) \; ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})] \; ds. \tag{4.19}$$

In other words, every solution of the integral equation (4.19) *is also solution of the system* $(4.5) - (4.6)$ *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system $(4.5) - (4.6)$, and using some classical results from fractional calculus to get (4.19).

Definition 4.4.1. *We say that the function* $y(\cdot) : [-r, b] \to E$ *is a mild solution of* $(4.5) - (4.6)$ *if* $y(t) = \varphi(t)$ *for all* $t \in H$ *and* y *satisfies the following integral equation*

$$y(t) = U(t, 0) \; \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \; U(t, s) [f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})] \; ds \qquad \text{for each } t \in J. \tag{4.20}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times C(H;E), \rho(s,\varphi) \leq 0\}.$$

We always assume that $\rho : J \times C(H;E) \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $C(H;E)$ and there exists a continuous and bounded function $\mathcal{L}^\varphi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\varphi_t\| \leq \mathcal{L}^\varphi(t)\|\varphi\| \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

We will need to introduce the following hypothesis which are assumed thereafter

$(H'0)$ $U(t,s)$ is compact for $t-s>0$.

$(H'1)$ There exists a constant $\widehat{M} \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq \widehat{M} \quad \text{for every } s \leq t.$$

$(H'2)$ There exist two functions $p,q \in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$|f(t,u)| \leq p(t) + q(t)\ \|u\|_B \text{ for a.e. } t \in J \text{ and each } u \in C(H;E).$$

$(H'3)$ For all $R>0$, there exists $l_R \in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$|f(t,u) - f(t,v)| \leq l_R(t)\ \|u-v\|$$

for all $u,v \in C(H;E)$ with $\|u\| \leq R$ and $\|v\| \leq R$.

$(H'4)$ There exists a function $\eta \in L^1(J,\mathbb{R}_+)$ such that

$$|h(t,u) - h(t,v)| \leq \eta(t)\ \ \|u-v\| \quad a.e.\ t \in J \ \ et \ \ \forall u,v \in C(H;E).$$

Theorem 4.4.1. *Assume that the hypotheses (H_φ), $(H'0)-(H'4)$ hold and moreover for all $n \in \mathbb{N}$,*

$$\frac{\widehat{M}(l_b^* + \eta^*)b^\alpha}{\Gamma(\alpha+1)} < 1 \tag{4.21}$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ and $\eta^* = \sup \eta(s)$. Then the problem (4.5) – (4.6) has a mild solution on $[-r,b]$.

Proof. We transform the problem (4.5) – (4.6) into a fixed-point problem. Consider the operator $N : C([-r,b];E) \to C([-r,b];E)$ defined by

$$N(y)(t) = \begin{cases} \varphi(t), & \text{if } t \in H; \\ U(t,0)\ \varphi(0) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s, y_{\rho(s,y_s)})\ ds, & \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) h(s, y_{\rho(s,y_s)})\ ds, & \text{if } t \in J. \end{cases}$$

Clearly, fixed points of the operator N are mild solutions of the problem (4.1) – (4.2).

Let y be a possible solution of the problem (4.5)–(4.6). Given $t \leq b$, then from $(H'1)$–$(H'4)$, (H_φ) and Lemma 5.2.1, we have for each $t \in [0,b]$

$$\begin{aligned} |y(t)| &\leq \|U(t,0)\|_{B(E)}|\varphi(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ \|U(t,s)\|_{B(E)}\ |f(s,y_{\rho(s,y_s)})|\ ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|_{B(E)}\ |\ h(s,y_{\rho(s,y_s)})\ |\ ds. \\ &\leq \widehat{M}\|\varphi\| + \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[p(s)+q(s)\|y_{\rho(s,y_s)}\|\right] ds \\ &+ \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)\|y_{\rho(s,y_s)}\|ds. \end{aligned}$$

It follows that

$$\begin{aligned} |y(t)| + \mathcal{L}^\varphi\|\varphi\| &\leq (\widehat{M}+\mathcal{L}^\varphi)\|\varphi\| + \frac{\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)} + \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ q(s)\left[|y(s)|+\mathcal{L}^\varphi\|\varphi\|\right] ds \\ &+ \int_0^t (t-s)^{\alpha-1}\ \eta(s)(|y(s)|+\mathcal{L}^\varphi\|\varphi\|)ds \end{aligned}$$

Set

$$\delta_b := (\widehat{M}+\mathcal{L}^\varphi)\|\varphi\| + \frac{\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)}.$$

We consider the function μ defined by

$$\mu(t) := \sup\ \{\ |y(s)| + \mathcal{L}^\varphi\|\varphi\|\ :\ 0 \leq s \leq t\ \},\quad 0 \leq t \leq b.$$

Let $t^\star \in [-r,t]$ be such that $\mu(t) = |y(t^\star)| + \mathcal{L}^\varphi\|\varphi\|$. If $t^\star \in [0,b]$, by the previous inequality, we get

$$\mu(t) \ \leq\ \delta_b + \frac{\widehat{M}(q^*+\eta^*)}{\Gamma(\alpha)}\ \int_0^t (t-s)^{\alpha-1}\ \mu(s)ds \qquad \text{for } t \in [0,b].$$

If $t^\star \in [-r,0]$, then $\mu(t) = \|\varphi\|$ and the previous inequality holds. And Lemma 1.7.2 implies that there exists a positive constant $\delta_b = \delta_b(\alpha)$ such that

$$\begin{aligned} \mu(t) &\leq \delta_b \times \left[1 + \frac{\widehat{M}(q^*+\eta^*)b^\alpha}{\Gamma(\alpha+1)}\right] \\ &= \Lambda_b \end{aligned}$$

Since $\|y\|_n \leq \mu(t)$, we have $\|y\|_n \leq \max\{\|\varphi\|, \Lambda_b\} := \Theta_b$. Since for every $t \in [0,b]$, we have $\|y\|_n \leq \max\{\|\varphi\|, \Lambda_b\} := \Theta_b$.

Set

$$Y = \{\ y \in C([-r,b];E)\ :\ \sup\{|y(t)| : 0 \leq t \leq b\} \leq \Theta_b + 1\}.$$

Clearly, Y is a closed subset of $C([-r,b];E)$.

We shall show that $N : Y \to C([-r,b];E)$ is a contraction operator.
Indeed, consider $y, \overline{y} \in Y$, thus using $(H^{'}1)$ and $(H^{'}3)-(H^{'}4)$ for each $t \in [0,b]$

$$\begin{aligned}
|N(y)(t) - N(\overline{y})(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \left|f(s, y_{\rho(s,y_s)}) - f(s, \overline{y}_{\rho(s,y_s)})\right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \left|h(s, y_{\rho(s,y_s)}) - h(s, \overline{y}_{\rho(s,y_s)})\right| ds \\
&\leq \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, l_b(s)\, \|y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)}\|\, ds. \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, \eta(s)\, \|y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)}\|\, ds.
\end{aligned}$$

Using (H_{φ}) and Lemma 5.2.1, we obtain

$$\begin{aligned}
|N(y)(t) - N(\overline{y})(t)| &\leq \frac{\widehat{M}(l_b^* + \eta^*)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y(s) - \overline{y}(s)|\, ds \\
&\leq \frac{\widehat{M}(l_b^* + \eta^*)b^{\alpha}}{\Gamma(\alpha+1)} \|y - \overline{y}\|_n
\end{aligned}$$

Therefore,

$$\|N(y) - N(\overline{y})\|_n \leq \frac{\widehat{M}(l_b^* + \eta^*)b^{\alpha}}{\Gamma(\alpha+1)} \|y - \overline{y}\|_n.$$

So, the operator N is a contraction. From the choice of Y there is no $y \in \partial Y^n$ such that $y = \lambda\, N(y)$, $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 1.10.4 does not hold. The nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. Thus, we deduce that the operator N has a unique fixed-point $y^{\star}$ which is the unique mild solution of the problem $(4.5)-(4.6)$.

4.5 Fractional Neutral Problem with Finite Delay

Before stating and proving the main result, we give first the definition of a mild solution of the perturbed semilinear evolution problem $(4.7)-(4.8)$.

Lemma 4.5.1. *The system* $(4.7)-(4.8)$ *is equivalent to the nonlinear integral equation*

$$y(t) = [\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s) y(s)\, ds \tag{4.22}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})]\, ds.$$

In other words, every solution of the integral equation (4.22) *is also solution of the system* $(4.7)-(4.8)$ *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system (4.7) – (4.8), and using some classical results from fractional calculus to get (4.22).

Definition 4.5.1. *We say that the function $y : [-r, b] \to E$ is a mild solution of* (4.7) – (4.8) *if $y(t) = \varphi(t)$ for all $t \in H$ and y satisfies the following integral equation*

$$\begin{array}{rcl} y(t) & = & U(t,0)[\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) \\ & + & \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} \, U(t,s)[f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})] \, ds \end{array} \tag{4.23}$$

We consider the hypotheses (H_φ), $(H^{'}0) - (H^{'}4)$ and we need the following assumptions

$(H^{'}5)$ There exists a constant $L_\star > 0$ such that

$$|A(s)g(s,\varphi) - A(\overline{s})g(\overline{s},\overline{\varphi})| \leq L_\star \left(|s - \overline{s}| + \|\varphi - \overline{\varphi}\|\right)$$

for all $\varphi, \overline{\varphi} \in C(H; E)$.

$(H^{'}6)$ There exists a constant $\overline{M}_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0 \quad \text{for all} \;\; t \in J.$$

$(H^{'}7)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0}$ such that

$$|A(t)\; g(t,\varphi)| \leq L \; (\|\varphi\| + 1) \text{ for all } t \in J \text{ and } \varphi \in C(H; E).$$

Theorem 4.5.1. *Suppose that the hypotheses (H_ϕ), $(H^{'}0) - (H^{'}7)$ are satisfied and moreover*

$$\left[\overline{M}_0 L_\star + \frac{\widehat{M}(L_b^* + \eta^*)b^\alpha}{\Gamma(\alpha+1)}\right] < 1 \tag{4.24}$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ and $\eta^* = \sup \eta(s)$. Then the problem (4.7) – (4.8) *has a unique mild solution on $[-r, b]$.*

Proof. Transform as below the neutral problem (4.7) – (4.8) into a fixed point problem by considering the operator $\widetilde{N} : C([-r,b]; E) \to C([-r,b]; E)$ defined by :

$$\widetilde{N}(y)(t) = \begin{cases} \varphi(t), & \text{if } t \in H; \\ U(t,0) \;\; [\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) & \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} U(t,s) \;\; \left[f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})\right] \, ds & \text{if } t \in J. \end{cases}$$

Clearly, the fixed points of the operator $\widetilde{N}$ are mild solutions of the problem (4.7) – (4.8).

Let y be a possible solution of the problem (4.7) – (4.8). Given $t \leq b$. Using $(H^{'}1) - (H^{'}2)$, $(H^{'}6) - (H^{'}7)$, (H_{φ}) and Lemma 5.2.1, we have for each $t \in [0, b]$

$$\begin{aligned}
|y(t)| &\leq |g(t, y_{\rho(t,y_t)})| + |U(t,0)[\varphi(0) - g(0,\varphi)]| \\
&+ \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, U(t,s) f(s, y_{\rho(s,y_s)})\, ds \right| \\
&+ \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, U(t,s) h(s, y_{\rho(s,y_s)})\, ds \right| \\
&\leq \|A^{-1}(t)\|\, |A(t) g(t, y_{\rho(t,y_t)})| + \|U(t,0)\|_{B(E)}\, \|A^{-1}(0)\|\, |A(0) g(0,\varphi)| \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s) \|y_{\rho(s,y_s)}\| \right] ds \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \|y_{\rho(s,y_s)}\| ds.
\end{aligned}$$

Since $\|y_{\rho(t,y_t)}\| \leq |y(t)| + \mathcal{L}^{\varphi}\|\varphi\|$, we obtain

$$\begin{aligned}
|y(t)| &\leq \overline{M}_0 L \left(|y(t)| + \mathcal{L}^{\varphi}\|\varphi\|\right) + \widehat{M}\|\varphi\|(1 + \overline{M}_0 L) + \frac{\widehat{M} p^* b^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s)\ \left(|y(s)| + \mathcal{L}^{\varphi}\|\varphi\|\right)\ ds \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s)\ \left(|y(s)| + \mathcal{L}^{\varphi}\|\varphi\|\right)\ ds
\end{aligned}$$

Then

$$\begin{aligned}
(1 - \overline{M}_0 L)|y(t)| &\leq \left[\overline{M}_0 L \mathcal{L}^{\varphi} + \widehat{M}(1 + \overline{M}_0 L)\right] \|\varphi\| + \frac{\widehat{M} p^* b^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \frac{\widehat{M}(q^* + \eta^*)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \left(|y(s)| + \mathcal{L}^{\varphi}\|\varphi\|\right)\ ds
\end{aligned}$$

Set

$$\delta_b := \mathcal{L}^{\varphi}\|\varphi\| + \frac{\left[\overline{M}_0 L \mathcal{L}^{\varphi} + \widehat{M}(1 + \overline{M}_0 L)\right]}{(1 - \overline{M}_0 L)} \|\varphi\| + \frac{\widehat{M} p^* b^{\alpha}}{(1 - \overline{M}_0 L)\Gamma(\alpha+1)}.$$

Thus

$$|y(t)| + \mathcal{L}^{\varphi}\|\varphi\| \leq \delta_b + \frac{\widehat{M}(q^* + \eta^*)}{(1 - \overline{M}_0 L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \left(|y(s)| + \mathcal{L}^{\varphi}\|\varphi\|\right)\ ds.$$

We consider the function μ defined by

$$\mu(t) := \sup\ \{\ |y(s)| + \mathcal{L}^{\varphi}\|\varphi\|\ :\ 0 \leq s \leq t\ \}, \quad 0 \leq t \leq b.$$

Let $t^{\star} \in [-r, t]$ be such that $\mu(t) = |y(t^{\star})| + \mathcal{L}^{\varphi}\|\varphi\|$. If $t^{\star} \in [0, b]$, by the previous inequality, we get

$$\mu(t) \leq \delta_b + \frac{\widehat{M}(q^* + \eta^*)}{(1 - \overline{M}_0 L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ \mu(s) ds \qquad \text{for } t \in [0, b].$$

If $t^\star \in [-r, 0]$, then $\mu(t) = \|\varphi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta_b = \delta_b(\alpha)$ such that

$$\begin{aligned} \mu(t) &\leq \delta_b \times \left[1 + \frac{\widehat{M}(q^* + \eta^*)b^\alpha}{(1 - \overline{M}_0 L)\Gamma(\alpha+1)}\right] \\ &= \Lambda_b \end{aligned}$$

Since $\|y\| \leq \mu(t)$, we have $\|y\| \leq \max\{\|\varphi\|, \Lambda_b\} := \Theta_b$.

Now, we shall show that $\widetilde{N} : Y \to C([-r, b]; E)$ is a contraction operator. Indeed, consider $y, \overline{y} \in Y$, thus for each $t \in [0, b]$

$$\begin{aligned} |\widetilde{N}(y)(t) - \widetilde{N}(\overline{y})(t)| &\leq |g(t, y_{\rho(t,y_t)}) - g(t, \overline{y}_{\rho(t,y_t)})| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \left| f(s, y_{\rho(s,y_s)}) - f(s, \overline{y}_{\rho(s,y_s)}) \right| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \left| h(s, y_{\rho(s,y_s)}) - h(s, \overline{y}_{\rho(s,y_s)}) \right| ds \\ &\leq \|A^{-1}(t)\| |A(t)g(t, y_{\rho(t,y_t)}) - A(t)g(t, \overline{y}_{\rho(t,y_t)})| \\ &+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, l_b(s)\, \|y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)}\|\, ds \\ &+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, \eta(s)\, \|y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)}\|\, ds \end{aligned}$$

Using (H_φ) and Lemma 5.2.1, we obtain

$$\begin{aligned} |\widetilde{N}(y)(t) - \widetilde{N}(\overline{y})(t)| &\leq \overline{M}_0 L_\star \|y - \overline{y}\| + \frac{\widehat{M}(L_b^* + \eta^*)b^\alpha}{\Gamma(\alpha+1)} \|y - \overline{y}\| \\ &\leq \left[\overline{M}_0 L_\star + \frac{\widehat{M}(L_b^* + \eta^*)b^\alpha}{\Gamma(\alpha+1)}\right] \|y - \overline{y}\| \end{aligned}$$

Therefore,

$$\|\widetilde{N}(y) - \widetilde{N}(\overline{y})\|_n \leq \left[\overline{M}_0 L_\star + \frac{\widehat{M}(L_b^* + \eta^*)b^\alpha}{\Gamma(\alpha+1)}\right] \|y - \overline{y}\|.$$

So, for an appropriate choice of $\overline{M}_0 L_\star, L_b^*, \eta^*$ and b^α such that

$$\left[\overline{M}_0 L_\star + \frac{\widehat{M}(L_b^* + \eta^*)b^\alpha}{\Gamma(\alpha+1)}\right] < 1,$$

the operator $\widetilde{N}$ is a contraction. From the choice of Y there is no $y \in \partial Y^n$ such that $y = \lambda\, \widetilde{N}(y)$ for some $\lambda \in (0, 1)$. Then the statement $(C2)$ in Theorem 1.10.4 does not hold. A consequence of the nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. We deduce that the operator $\widetilde{N}$ has a unique fixed-point $y^\star$ which is the unique mild solution of the problem $(4.7) - (4.8)$.

4.6 Fractional Partial Problem with Infinite Delay

Before stating and proving the main result, we give first the definition of a mild solution of the perturbed semilinear evolution problem (4.9) – (4.10).

Lemma 4.6.1. *The system* (4.9) – (4.10) *is equivalent to the nonlinear integral equation*

$$y(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s) y(s)\ ds \tag{4.25}$$

$$+\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})]\ ds.$$

In other words, every solution of the integral equation (4.25) *is also solution of the system* (4.9) – (4.10) *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system (4.9) – (4.10), and using some classical results from fractional calculus to get (4.25).

Definition 4.6.1. *We say that the function* $y(\cdot) : (-\infty, b] \to E$ *is a mild solution of* (4.9) – (4.10) *if* $y(t) = \phi(t)$ *for all* $t \leq 0$ *and* y *satisfies the following integral equation*

$$y(t) = U(t,0)\ \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s)[f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})]\ ds \quad \text{for each } t \in J. \tag{4.26}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\phi) : (s,\phi) \in J \times \mathcal{B}, \rho(s,\phi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq \mathcal{L}^\phi(t) \|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

We will need to introduce the following hypothesis which are assumed thereafter

($H'0$) $U(t,s)$ is compact for $t - s > 0$.

($H'01$) There exists a constant $\widehat{M} \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq \widehat{M} \quad \text{for every } (t,s) \in \Delta.$$

($H'02$) There exist two functions $p, q \in L^1_{loc}(J; \mathbb{R}_+)$ such that

$$|f(t,u)| \leq p(t) + q(t)\ \|u\|_B \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$

($H'03$) For all $R > 0$, there exists $l_R \in L^1_{loc}(J; \mathbb{R}_+)$ such that

$$|f(t,u) - f(t,v)| \leq l_R(t)\ \|u - v\|$$

for all $u, v \in \mathcal{B}$ with $\|u\| \leq R$ and $\|v\| \leq R$.

$(H'04)$ There exists a function $\eta \in L^1(J, \mathbb{R}_+)$ such that

$$|h(t,u) - h(t,v)| \leq \eta(t) \quad \|u-v\|_{\mathcal{B}} \quad a.e.\ t \in J \quad et \quad \forall u, v \in \mathcal{B}.$$

Consider the following space

$$\Omega = \left\{ y : (-\infty, b] \to E : y|_{(-\infty,0]} \ \in B \quad and \quad y|_{[0,b]} \text{is continuous } \right\},$$

Theorem 4.6.1. *Assume that the hypotheses* (H_ϕ), $(H'0) - (H'04)$ *hold and moreover for all* $n \in \mathbb{N}$,

$$\frac{K_b \widehat{M} \eta^* b^\alpha}{\Gamma(\alpha+1)} < 1 \tag{4.27}$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ *and* $\eta^* = \sup \eta(s)$. *Then the problem* $(4.9) - (4.10)$ *has a unique mild solution on* $(-\infty, b]$.

Proof. We transform the problem $(4.9) - (4.10)$ into a fixed-point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in \mathbb{R}^-; \\ U(t,0)\ \phi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s, y_{\rho(s,y_s)})\ ds, & \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) h(s, y_{\rho(s,y_s)})\ ds, & \text{if } t \in J. \end{cases}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(4.9) - (4.10)$.

For $\phi \in \mathcal{B}$, we will define the function $x(.) : (-\infty, b] \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \in (-\infty, 0]; \\ U(t,0)\ \phi(0), & \text{if } \ t \in [0, b]. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in \Omega$, set

$$y(t) = z(t) + x(t)$$

It is obvious that y satisfies (4.26) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s) h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds. \end{aligned}$$

Let

$$\Omega^0 = \{ z \in \Omega : z_0 = 0 \}.$$

Define the operators $F, G : \Omega^0 \to \Omega^0$ by

$$F(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds.$$

and

$$G(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, U(t,s)h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\, ds.$$

Obviously the operator N has a fixed point is equivalent to $F+G$ has one, so it turns to prove that $F+G$ has a fixed point.

First, show that F is continuous and compact.

Step 1 : F continuous. Let $(z_n)_{n\in\mathbb{N}}$ a sequence in Ω^0 such that $z_n \to z$ in Ω^0. By the hypothesis $(H1)$, we have

$$\begin{aligned}
|F(z_n)(t) - F(z)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t |(t-s)^{\alpha-1}| \|U(t,s)\|_{B(E)} \\
&\times |f(s, z_{n\rho(s,z_{ns}+x_s)} + x_{\rho(s,z_{ns}+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds \\
&\le \frac{\widehat{M} b^\alpha \|f(., z_{n\rho(.,.)} + x_{\rho(.,.)}) - f(., z_{\rho(.,.)} + x_{\rho(.,.)})\|_\infty}{\Gamma(\alpha+1)}.
\end{aligned}$$

$$\begin{aligned}
|F(z)(t) - F(\overline{z})(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)}\ \big|f(s, z_{n\rho(s,z_{ns}+x_s)} + x_{\rho(s,z_{ns}+x_s)}) \\
&\quad - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\big|\ ds \\
&\le \frac{\widehat{M} b^\alpha \|f(., z_{n\rho(.,.)} + x_{\rho(.,.)}) - f(., z_{\rho(.,.)} + x_{\rho(.,.)})\|_\infty}{\Gamma(\alpha+1)}.
\end{aligned}$$

Since f is continuous, by dominated convergence theorem of Lebesgue, we get

$$|F(z_n)(t) - F(z)(t)| \longrightarrow 0 \ \text{ when } \ n \longrightarrow +\infty$$

So F is continuous.

Step 2 : Show that F transforms any bounded of Ω^0 in a bounded set. For each $d > 0$, there exists a positive constant ξ such that for all $z \in B_d = \{z \in \Omega^0 \ : \|z\|_n \le d\}$ we get $\|F(z)\|_n \le \xi$. Soit $z \in B_d$, from assumption $(H'01)$ and $(H'02)$, we have for each $t \in [0,b]$

$$\begin{aligned}
|F(z)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)}\ |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|\ ds \\
&\le \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t p(s)\ \left[p(s) + q(s)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\right]\ ds
\end{aligned}$$

From (H_ϕ), Lemma 1.7.2 and Assumption $(A1)$, we have for each $t \in [0,b]$

$$\begin{aligned}
\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\le \|z_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + \|x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \\
&\le K_b|z(s)| + (M_b + \mathcal{L}^\phi)\|z_0\|_{\mathcal{B}} + K_b|x(s)| + (M_b + \mathcal{L}^\phi)\|x_0\|_{\mathcal{B}} \\
&\le K_b|z(s)| + K_b\|U(s,0)\|_{B(E)}|\phi(0)| + (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} \\
&\le K_b|z(s)| + K_b\widehat{M}|\phi(0)| + (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}
\end{aligned}$$

Using (ii), we get

$$\begin{aligned}
\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\le K_b|z(s)| + K_b\widehat{M}H\|\phi\|_{\mathcal{B}} + (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} \\
&\le K_b|z(s)| + (M_b + \mathcal{L}^\phi + K_b\widehat{M}H)\|\phi\|_{\mathcal{B}}
\end{aligned}$$

Set $c_b := (M_b + \mathcal{L}^\phi + K_b\widehat{M}H)\|\phi\|_{\mathcal{B}}$ and $\delta_b := K_b d + c_b$. Then

$$\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \leq K_b|z(s)| + c_b \leq \delta_b. \tag{4.28}$$

For each $t \in [0,b]$, it follows that

$$\begin{aligned} |F(z)(t)| &\leq \frac{\widehat{M}p^* b^\alpha}{\Gamma(\alpha+1)} + \frac{\widehat{M}q^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \ (K_b|z(s)| + c_b) \ ds \\ &\leq \frac{\widehat{M}b^\alpha(p^* + q^*\delta_b)}{\Gamma(\alpha+1)} \end{aligned}$$

$$|F(z)(t)| \leq \frac{\widehat{M}b^\alpha(p^* + q^*\delta_b)}{\Gamma(\alpha+1)} := \varrho.$$

So there is a positive constant ϱ such that $\|F(z)\|_n \leq \varrho$. Then $F(B_d) \subset B_\varrho$.

Step 3 : F maps bounded sets into equi-continuous sets of Ω^0. We consider B_d as in Step 2 and we show that $F(B_d)$ is equi-continuous. Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $z \in B_d$.

$$\begin{aligned} |F(z)(\tau_2) - F(z)(\tau_1)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}| \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} \\ &\quad |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \\ &+ \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{\alpha-1}| \|U(\tau_2, s)\|_{B(E)} \ |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \ ds. \end{aligned}$$

Then by (5.41) we get

$$\begin{aligned} |F(z)(\tau_2) - F(z)(\tau_1)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}| \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} \\ &\quad [p(s) + q(s)\delta_b] ds. \\ &+ \frac{\widehat{M}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{\alpha-1}| \ [p(s) + q(s)\delta_b] \ ds. \end{aligned}$$

Noting that $|F(z)(\tau_2) - F(z)(\tau_1)| \longrightarrow 0$ tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$ independently of $z \in B_d$. The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$, since $U(t,s)$ is a strongly continuous operator and the compactness of $U(t,s)$ for $t > s$, implies the continuity in the uniform operator topology (voir [7, 90]). As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that the operator F maps B_d into a precompact set in E.

Let $t \in J$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $z \in B_d$ we define

$$F_\varepsilon(z)(t) = \frac{1}{\Gamma(\alpha)}U(t, t-\varepsilon)\int_0^{t-\varepsilon} (t-s)^{\alpha-1} \ U(t-\varepsilon, s)Cu_{z+x}(s) \ ds.$$

Since $U(t,s)$ is a compact operator, the set $Z_\varepsilon(t) = \ \{F_\varepsilon(z)(t) \ : z \in B_d\}$ is pre-compact in E for every ε, $0 < \varepsilon < t$. Moreover, using the definition of w, we get

$$|F(z)(t) - F_\varepsilon(z)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_{t-\varepsilon}^t \ (t-s)^{\alpha-1}\|U(t,s)\|_{B(E)} \ |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \ ds$$

Therefore the set $Z(t) = \{F(z)(t) \ : z \in B_d\}$ is totally bounded. Hence the set $\{F(z)(t) \ : z \in B_d\}$. So we deduce from Steps 1, 2 and 3 that F is a compact operator.

Step 4 : G is a contraction. Let $z, \overline{z} \in \Omega^0$. By the hypotheses $(H^{'}01)$ and $(H^{'}04)$, we get for each $t \in [0, b]$ and $n \in \mathbb{N}$

$$\begin{aligned}
&|G(z)(t) - G(\overline{z})(t)| \leq \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \mid h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \\
&-h(s, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \mid ds \\
&\leq \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \ \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)} - x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \ ds \\
&\leq \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \ \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \ ds.
\end{aligned}$$

Use the inequality (4.28), for get

$$\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| &\leq \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_b \eta(s) \ |z(s) - \overline{z}(s)| \ ds \\
&\leq \frac{\widehat{M} K_b \eta^* b^\alpha}{\Gamma(\alpha+1)} \|z(t) - \overline{z}(t)\|
\end{aligned}$$

Therefore

$$\|G(z) - G(\overline{z})\|_n \leq \frac{\widehat{M} K_b \eta^* b^\alpha}{\Gamma(\alpha+1)} \|z(t) - \overline{z}(t)\|.$$

Then the operator G is a contraction.

Step 5 : For applying Theorem (1.10.2), we must check $(Av2)$: i.e. it remains to show that the set

$$\Sigma = \left\{ z \in \Omega^0 \ : z = \lambda F(z) + \lambda G\left(\frac{z}{\lambda}\right) \quad \text{for some } \lambda \in]0,1[\right\}.$$

is bounded.

Let $z \in \Sigma$. By $(H^{'}01) - (H^{'}02)$ and $(H^{'}04)$, we have for each $t \in [0, b]$

$$\begin{aligned}
\frac{1}{\lambda}|z(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \ \left| f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \right| \ ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \ \left| h\left(s, \frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}\right) - h(s,0) + h(s,0) \right| \ ds \\
&\leq \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t \ \left[p(s) + q(s) \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \right] \ ds \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \ \left\| \frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} \right\|_{\mathcal{B}} \ ds + \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, |h(s,0)| \ ds.
\end{aligned}$$

Use Proposition (4.2.1) and inequality (4.15) to get

$$
\begin{aligned}
\frac{1}{\lambda}|z(t)| &\leq \frac{\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)} + \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}q(s)\left(\frac{K_b}{\lambda}|z(s)|+c_b\right)\,ds \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)\,\left(\frac{K_b}{\lambda}|z(s)|+c_b\right)\,ds + \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|h(s,0)|\,\,ds.
\end{aligned}
$$

We consider the function $u(t) := \sup\limits_{\theta\in[0,t]} |z(\theta)|$ and use the fact that $0<\lambda<1$

$$
\begin{aligned}
\frac{K_b}{\lambda}u(t)+c_b &\leq c_b + \frac{K_b\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)} + \frac{K_b\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}q(s)\left(\frac{K_b}{\lambda}|z(s)|+c_b\right)\,ds \\
&+ \frac{K_b\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)\,\left(\frac{K_b}{\lambda}|z(s)|+c_b\right)\,ds + \frac{K_b\widehat{M}}{\Gamma(\alpha)}\int_0^b (t-s)^{\alpha-1}\,|h(s,0)|\,\,ds.
\end{aligned}
$$

Set $\sigma_b := c_b + \dfrac{K_b\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)} + \dfrac{K_b\widehat{M}}{\Gamma(\alpha)}\displaystyle\int_0^b (t-s)^{\alpha-1}\,|h(s,0)|\,\,ds$. Then, we have

$$
\begin{aligned}
\frac{K_b}{\lambda}u(t)+c_b &\leq \sigma_b + \frac{K_b\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}q(s)\left(\frac{K_b}{\lambda}|z(s)|+c_b\right)\,ds \\
&+ \frac{K_b\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)\,\left(\frac{K_b}{\lambda}|z(s)|+c_b\right)\,ds.
\end{aligned}
$$

We consider the function μ defined by

$$\mu(t) = \sup\,\{\frac{K_b}{\lambda}u(s)+c_b\,\,:\,\,0\leq s\leq t\},\quad 0\leq t\leq b$$

Let $t^\star\in[0,t]$ such that $\mu(t) = \dfrac{K_b}{\lambda}u(t^\star)+c_b$. From the previous inequality, we have

$$\mu(t)\leq\sigma_b + \frac{K_b\widehat{M}q^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mu(s)ds + \frac{K_b\widehat{M}\eta^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mu(s)ds.$$

If $t^\star\in(-\infty,0]$, then $\mu(t)=\|\phi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta_b=\delta_b(\alpha)$ such that

$$
\begin{aligned}
\mu(t) &\leq \delta_b\times\left[1+\frac{K_b\widehat{M}(q^*+\eta^*)b^\alpha}{\Gamma(\alpha+1)}\right] \\
&= \Lambda_b
\end{aligned}
$$

Since $\|z\|\leq\mu(t)$, we have $\|z\|\leq\max\{\|\phi\|,\Lambda_b\}:=\Theta_b$. Since for every $t\in[0,b]$, we have $\|z\|\leq\max\{\|\phi\|,\Lambda_b\}:=\Theta_b$.

Since $\|z\|\leq\mu(t)$, we have $\|z\|\leq\Lambda_b$. This shows that the set Σ is bounded. Then the statement $(Av2)$ in Theorem 1.10.5 does not hold. The nonlinear alternative of Avramescu implies that $(Av1)$ is satisfied, we deduce that the operator $F+G$ has a fixed point $z^\star$. Then $y^\star(t)=z^\star(t)+x(t)$, $t\in]-\infty,b]$ is a fixed point of the operator N which is a mild solution of the problem $(4.9)-(4.10)$.

4.7 Fractional Neutral Problem with Infinite Delay

Before stating and proving the main result, we give first the definition of a mild solution of the perturbed semilinear evolution problem (4.11) – (4.12).

Lemma 4.7.1. *The system* (4.11) – (4.12) *is equivalent to the nonlinear integral equation*

$$y(t) = [\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s) y(s)\ ds \tag{4.29}$$

$$+\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})]\ ds.$$

In other words, every solution of the integral equation (4.29) *is also solution of the system* (4.11) – (4.12) *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system (4.11) – (4.12), and using some classical results from fractional calculus to get (4.29).

Definition 4.7.1. *We say that the function* $y(\cdot) : (-\infty, b] \to E$ *is a mild solution of* (4.11) – (4.12) *if* $y(t) = \phi(t)$ *for all* $t \le 0$ *and* y *satisfies the following integral equation*

$$y(t) = U(t,0)[\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) \tag{4.30}$$

$$+\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s)[f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})]\ ds$$

We consider the hypotheses (H_ϕ), $(H^{'}0) - (H^{'}04)$ and we need the following assumptions

$(H^{'}05)$ There exists a constant $L_\star > 0$ such that

$$|A(s)g(s,\phi) - A(\overline{s})g(\overline{s},\overline{\phi})| \le L_\star\ (|s-\overline{s}| + \|\phi - \overline{\phi}\|)$$

for all $\phi, \overline{\phi} \in \mathcal{B}$.

$(H^{'}06)$ There exists a constant $\overline{M}_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \le \overline{M}_0 \quad \text{for all} \ \ t \in J.$$

$(H^{'}07)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0 K_b}$ such that

$$|A(t)\ g(t,\varphi)| \le L\ (\|\phi\| + 1) \text{ for all } t \in J \text{ and } \phi \in \mathcal{B}.$$

Theorem 4.7.1. *Suppose that the hypotheses* (H_ϕ), $(H^{'}0) - (H^{'}02)$ *and* $(H^{'}04) - (H^{'}07)$ *are satisfied and moreover*

$$\left[\overline{M}_0 L_\star K_b + \frac{\widehat{M} k_b \eta^* b^\alpha}{\Gamma(\alpha+1)}\right] < 1 \tag{4.31}$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ *and* $\eta^* = \sup \eta(s)$. *Then the problem* (4.11) – (4.12) *has a unique mild solution on* $(-\infty, b]$.

Proof. Consider the operator $\widetilde{N}:\Omega\to\Omega$ defined by

$$\widetilde{N}(y)(t)=\begin{cases}\phi(t), & \text{if } t\in\mathbb{R}^-;\\ U(t,0)\ \ [\phi(0)-g(0,\phi)]+g(t,y_{\rho(t,y_t)}) & \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}U(t,s)\ \ \left[f(s,y_{\rho(s,y_s)})+h(s,y_{\rho(s,y_s)})\right]\ ds & \text{if } t\in J.\end{cases}$$

Then, fixed points of the operator $\widetilde{N}$ are mild solutions of the problem (4.11) – (4.12).

For $\phi\in\mathcal{B}$, we consider the function $x(.):(-\infty,b]\to E$ defined as below by

$$x(t)=\begin{cases}\phi(t), & \text{if } \ t\leq 0;\\ U(t,0)\ \phi(0), & \text{if } \ t\in J.\end{cases}$$

Then $x_0=\phi$. For each function $z\in\Omega$, set

$$y(t)=z(t)+x(t)$$

It is obvious that y satisfies (4.30) if and only if z satisfies $z_0=0$ and

$$\begin{aligned} z(t) &= g(t,z_{\rho(s,z_t+x_t)}+x_{\rho(s,z_t+x_t)})-U(t,0)g(0,\phi)\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)\ f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})\ ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)\ h(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})\ ds.\end{aligned}$$

Let

$$\Omega^0=\{z\in\Omega: z_0=0\}.$$

Define the operators $F,\widetilde{G}:\Omega^0\to\Omega^0$ by

$$F(z)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)\ f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})\ ds$$

And

$$\begin{aligned}\widetilde{G}(z)(t) &= g(t,z_{\rho(s,z_t+x_t)}+x_{\rho(s,z_t+x_t)})-U(t,0)g(0,\phi)\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)\ h(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})\ ds.\end{aligned}$$

Obviously the operator $\widetilde{N}$ has a fixed point is equivalent to $F+\widetilde{G}$ has one, so it turns to prove that $F+\widetilde{G}$ has a fixed point.

We have shown that the operator F is continuous and compact as in Section 3.6 Remains to show that the operator $\widetilde{G}$ is a contraction.

Let $z,\overline{z}\in\Omega^0$. By $(H'01)$ and $(H'04)$ we have for each $t\in[0,b]$

$$\begin{aligned}|\widetilde{G}(z)(t)-\widetilde{G}(\overline{z})(t)| &\leq |g(t,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})-g(t,\overline{z}_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})|\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|_{B(E)}\ |h(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})-\\ &\quad h(s,\overline{z}_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})|\ ds\\ &\leq \|A^{-1}(t)\||A(t)g(t,z_{\rho(t,y_t)}+x_{\rho(t,y_t)})-A(t)g(t,\overline{z}_{\rho(t,y_t)}+x_{\rho(t,y_t)})|\\ &+\frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)\|z_{\rho(s,z_s+x_s)}-\overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\ ds.\end{aligned}$$

Use the inequality (4.15), and $(H'05)-(H'06)$, we get

$$\begin{aligned}
|\widetilde{G}(z)(t)-\widetilde{G}(\overline{z})(t)| &\leq \overline{M}_0 L_\star K_b\|z(t)-\overline{z}(t)\| + \frac{\widehat{M}K_b\eta^* b^\alpha}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}|z(s)-\overline{z}(s)|\,ds \\
&\leq \overline{M}_0 L_\star K_b\|z(t)-\overline{z}(t)\| + \frac{\widehat{M}K_b\eta^* b^\alpha}{\Gamma(\alpha+1)}\|z(t)-\overline{z}(t)\| \\
&\leq \left[\overline{M}_0 L_\star K_b + \frac{\widehat{M}k_b\eta^* b^\alpha}{\Gamma(\alpha+1)}\right]\|z(t)-\overline{z}(t)\|.
\end{aligned}$$

Therefore

$$\|\widetilde{G}(z)-\widetilde{G}(\overline{z})\|_n \leq \left[\overline{M}_0 L_\star K_b + \frac{\widehat{M}k_b\eta^* b^\alpha}{\Gamma(\alpha+1)}\right]\|z(t)-\overline{z}(t)\|$$

So, for an appropriate choice of $\overline{M}_0 L_\star$, η^* K_b and b^α such that

$$\left[\overline{M}_0 L_\star K_b + \frac{\widehat{M}k_b\eta^* b^\alpha}{\Gamma(\alpha+1)}\right] < 1,$$

operator $\widetilde{G}$ is a contraction.

For applying Theorem (1.10.2), we must check $(Av2)$ i.e. it remains to show that the set

$$\widetilde{\Sigma} = \left\{ z\in B^0_{+\infty} \quad : \quad z = \lambda F(z) + \lambda\widetilde{G}\left(\frac{z}{\lambda}\right) \quad pour\ 0<\lambda<1 \right\}.$$

is bounded.

Let $z\in\widetilde{\Sigma}$. By $(H'01)-(H'02)$, we have for each $t\in[0,b]$

$$\begin{aligned}
\frac{|z(t)|}{\lambda} &\leq |g(t, z_{\rho(t,y_t)}+x_{\rho(t,y_t)})| + |U(t,0)g(0,\phi)| \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|_{B(E)}\ \left|f(s, z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})\right|\ ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|_{B(E)}\ \left|h\left(s, \frac{z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}}{\lambda}\right) - h(s,0) + h(s,0)\right| \\
&\leq \|A^{-1}(t)\|\ |A(t)g(t, z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})| + \|U(t,0)\|_{B(E)}\ \|A^{-1}(0)\|\ |A(0)g(0,\phi)| \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[p(s)+q(s)\|z_{\rho(s,y_s)}+x_{\rho(s,y_s)}\|\right] ds \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)\ \left\|\frac{z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}}{\lambda} + x_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}\right\|_{\mathcal{B}} ds + \widehat{M}\int_0^t\ |h(s,0)|\ ds.ds
\end{aligned}$$

Using assumptions $(H'04)$, $(H'06)$ and $(H'07)$

$$\begin{aligned}
\frac{|z(t)|}{\lambda} &\leq \overline{M}_0 L(\|z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}}+1) + \widehat{M}\overline{M}_0 L(\|\phi\|_{\mathcal{B}}+1) \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[p(s)+q(s)\|z_{\rho(s,y_s)}+x_{\rho(s,y_s)}\|\right] ds \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)\ \left\|\frac{z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}}{\lambda} + x_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}\right\|_{\mathcal{B}} ds + \widehat{M}\int_0^t\ |h(s,0)|\ ds.
\end{aligned}$$

Use Proposition (4.2.1) and inequality (4.15) for get

$$
\begin{aligned}
\left\| \frac{z_{\rho\left(s, \frac{z_s}{\lambda}+x_s\right)}}{\lambda} + x_{\rho\left(s, \frac{z_s}{\lambda}+x_s\right)} \right\|_{\mathcal{B}} &\leq \frac{1}{\lambda} \left\| z_{\rho\left(s, \frac{z_s}{\lambda}+x_s\right)} \right\|_{\mathcal{B}} + \left\| x_{\rho\left(s, \frac{z_s}{\lambda}+x_s\right)} \right\|_{\mathcal{B}} \\
&\leq \frac{K_b}{\lambda}|z(s)| + \frac{M_b + \mathcal{L}^{\phi}}{\lambda} \|z_0\|_{\mathcal{B}} + K_b|x(s)| + \left(M_b + \mathcal{L}^{\phi}\right) \|x_0\|_{\mathcal{B}} \\
&\leq \frac{K_b}{\lambda}|z(s)| + K_b \|U(s,0)\|_{B(E)} |\phi(0)| + \left(M_b + \mathcal{L}^{\phi}\right) \|\phi\|_{\mathcal{B}} \\
&\leq \frac{K_b}{\lambda}|z(s)| + K_b \widehat{M} H \|\phi\|_{\mathcal{B}} + \left(M_b + \mathcal{L}^{\phi}\right) \|\phi\|_{\mathcal{B}} \\
&\leq \frac{K_b}{\lambda}|z(s)| + \left(M_b + \mathcal{L}^{\phi} + K_b \widehat{M} H\right) \|\phi\|_{\mathcal{B}}.
\end{aligned}
$$

Hence

$$
\left\| \frac{z_{\rho\left(s, \frac{z_s}{\lambda}+x_s\right)}}{\lambda} + x_{\rho\left(s, \frac{z_s}{\lambda}+x_s\right)} \right\|_{\mathcal{B}} \leq \frac{K_b}{\lambda}|z(s)| + c_b. \tag{4.32}
$$

Use the function $u(\cdot)$ to get

$$
\begin{aligned}
\frac{u(t)}{\lambda} &\leq \overline{M}_0 L (K_b |u(t)| + c_b) + \overline{M}_0 L \left(1 + \widehat{M}(\|\phi\| + 1)\right) \\
&+ \frac{\widehat{M} p^* b^{\alpha}}{\Gamma(\alpha+1)} + \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |h(s,0)| \, ds + \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s) \left(K_b u(s) + c_b\right) \, ds \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \left(\frac{K_b}{\lambda} u(s) + c_b\right) \, ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{u(t)}{\lambda} &\leq \overline{M}_0 L (\frac{K_b}{\lambda} u(t) + c_b) + \overline{M}_0 L \left(1 + \widehat{M}(\|\phi\| + 1)\right) + \frac{\widehat{M} p^* b^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} |h(s,0)| \, ds + \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s) \left(\frac{K_b}{\lambda} u(s) + c_b\right) \, ds \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \left(\frac{K_b}{\lambda} u(s) + c_b\right) \, ds.
\end{aligned}
$$

Set

$$
\zeta_b := \overline{M}_0 L c_b + \overline{M}_0 L \left(1 + \widehat{M}(\|\phi\| + 1)\right) + \frac{\widehat{M} p^* b^{\alpha}}{\Gamma(\alpha+1)} + \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} |h(s,0)| \, ds.
$$

So

$$
\begin{aligned}
\frac{K_b}{\lambda} \left(1 - \overline{M}_0 L K_b\right) u(t) &\leq K_b \zeta_b + \frac{K_b \widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s) \left(\frac{K_b}{\lambda} u(s) + c_b\right) \, ds \\
&+ \frac{K_b \widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \left(\frac{K_b}{\lambda} u(s) + c_b\right) \, ds.
\end{aligned}
$$

Set $\xi_b := c_b + \dfrac{K_b\zeta_b}{(1-\overline{M}_0LK_b)}$. Then

$$\begin{aligned}\frac{K_b}{\lambda}u(t)+c_b &\leq \xi_b + \frac{K_b\widehat{M}}{(1-\overline{M}_0LK_b)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}q(s)\left(\frac{K_b}{\lambda}u(s)+c_b\right)\,ds\\ &+ \frac{K_b\widehat{M}}{(1-\overline{M}_0LK_b)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)\left(\frac{K_b}{\lambda}u(s)+c_b\right)\,ds.\end{aligned}$$

We consider the function μ defined by

$$\mu(t)=\sup\left\{\frac{K_b}{\lambda}u(s)+c_b \ : \ 0\leq s\leq t\right\},\quad 0\leq t\leq b$$

Let $t^\star\in[0,t]$ be such that $\mu(t)=\dfrac{K_b}{\lambda}u(t^\star)+c_b$. By the previous inequality, we have for $t\in[0,b]$

$$\mu(t) \ \leq \ \xi_b + \frac{K_b\widehat{M}(q^*+\eta^*)}{(1-\overline{M}_0LK_b)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\mu(s)ds \qquad \text{for } t\in[0,b].$$

If $t^\star\in(-\infty,0]$, then $\mu(t)=\|\phi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta_b=\delta_b(\alpha)$ such that

$$\begin{aligned}\mu(t) &\leq \delta_b\times\left[1+\frac{K_b\widehat{M}(q^*+\eta^*)b^\alpha}{\Gamma(\alpha+1)}\right]\\ &= \Lambda_b\end{aligned}$$

Since $\|z\|\leq\mu(t)$, we have $\|z\|\leq\max\{\|\phi\|,\Lambda_b\}:=\Theta_b$. Since for every $t\in[0,b]$, we have $\|z\|\leq\max\{\|\phi\|,\Lambda_b\}:=\Theta_b$.

Since $\|z\|\leq\mu(t)$, we have $\|z\|\leq\Lambda_b$. This shows that the set Σ is bounded. Then the statement $(Av2)$ in Theorem 1.10.5 does not hold. The nonlinear alternative of Avramescu implies that $(Av1)$ is satisfied, we deduce that the operator $F+G$ has a fixed point $z^\star$. Then $y^\star(t)=z^\star(t)+x(t)$, $t\in]-\infty,b]$ is a fixed point of the operator N which is a mild solution of the problem $(4.11)-(4.12)$.

4.8 Examples

To illustrate the previous results, we give in this section six examples.

Example 1. Consider the partial differential equation

$$\begin{cases}\dfrac{\partial u}{\partial t}(t,\xi)=\dfrac{\partial^2u(t,\xi)}{\partial\xi^2}+a_0(t,\xi)u(t,\xi)\\ +\displaystyle\int_{-\infty}^0 a_1(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2d\theta\right),\xi\right]ds\\ +\displaystyle\int_{-\infty}^0 a_3(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2d\theta\right),\xi\right]ds,\\ \hfill t\geq 0,\ \xi\in[0,\pi],\\ u(t,0)=\ u(t,\pi)\ =0, \hfill t\geq 0,\\ u(\theta,\xi)=\ u_0(\theta,\xi), \hfill -\infty<\theta\leq 0,\ \xi\in[0,\pi].\end{cases}\tag{4.33}$$

where $a_0(t,\xi)$ is a continuous function and is uniformly Hölder continuous in t ; $a_1, a_3 : \mathbb{R}_- \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$, $\rho_i : [0,+\infty[\to \mathbb{R}$ are continuous functions $i = 1,2$.

To study this system, we consider the space $E = L^2([0,\pi],\mathbb{R})$ and the operator $A : D(A) \subset E \to E$ given by $Aw = w''$ with

$$D(A) := \{ \ w \in E \ \ : w'' \in E, \ w(0) = w(\pi) = 0 \ \}$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on E, with compact resolvent. On the domain $D(A)$, we define the operators $A(t) : D(A) \subset E \to E$ by

$$A(t)x(\xi) = Ax(\xi) + a_0(t,\xi)x(\xi).$$

By assuming that $a_0(.,.)$ is continuous and that $a_0(t,\xi) \leq -\delta_0$ ($\delta_0 > 0$) for every $t \in \mathbb{R}, \xi \in [0,\pi]$, and specific case $\alpha = 1$ it follows that the system

$$u'(t) = A(t)u(t) \quad t \geq s; \quad u(s) = x \in E,$$

has an associated evolution family given by

$$U(t,s)x(\xi) = \left[T(t-s)exp\left(\int_s^t a_0(\tau,\xi)d\tau \right) x \right](\xi).$$

From this expression, it follows that $U(t,s)$ is a compact linear operator and that

$$\|U(t,s)\| \leq e^{-(1+\delta_0)(t-s)} \text{ for every } (t,s) \in \mathbb{R}^+ \times \mathbb{R}^+ \quad ; \quad s \leq t.$$

Theorem 4.8.1. *Let $\mathcal{B} = BUC(\mathbb{R}_-;E)$ and $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds, $\rho_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $i = 1,2$, are continuous and the functions $a_1, a_3 : \mathbb{R}_- \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a mild solution of* (4.33) *on* $]-\infty,+\infty[$.

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-r}^{0} a_1(s)\psi(s,\xi)ds,$$

$$h(t,\psi)(\xi) = \int_{-r}^{0} a_3(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta \right),$$

are well defined functions, which permit to transform system (4.33) into the abstract system (4.1) – (4.2). Moreover, the function f and h is bounded linear operator. Now, the existence of mild solutions can be deduced from a direct application of Theorem 4.2.1.
From Remark 1.7.2, we have the following result

Corollary 4.8.1. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then there exists a mild solution of* (4.33) *on* $]-\infty,+\infty[$.

Example 2. Consider the partial differential equation

$$
\begin{cases}
\dfrac{\partial}{\partial t}\left[u(t,\xi)-\displaystyle\int_{-\infty}^{0}a_4(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^{\pi}a_2(\theta)|u(t,\theta)|^2d\theta\right),\xi\right)ds\right] \\
=\dfrac{\partial^2 u(t,\xi)}{\partial \xi^2}+a_0(t,\xi)u(t,\xi) \\
+\displaystyle\int_{-\infty}^{0}a_1(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^{\pi}a_2(\theta)|u(t,\theta)|^2d\theta\right),\xi\right)ds \\
+\displaystyle\int_{-\infty}^{0}a_3(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^{\pi}a_2(\theta)|u(t,\theta)|^2d\theta\right),\xi\right)ds, \\
\hfill t\geq 0,\ \xi\in[0,\pi], \\
v(t,0)=\ v(t,\pi)\ =0, \hfill t\geq 0, \\
v(\theta,\xi)=\ v_0(\theta,\xi), \hfill -\infty<\theta\leq 0,\ \xi\in[0,\pi].
\end{cases}
\tag{4.34}
$$

$a_4:\mathbb{R}_-\to\mathbb{R}$ is a continuous function.

Theorem 4.8.2. *Let $\mathcal{B}=BUC(\mathbb{R}_-;E)$ and $\phi\in\mathcal{B}$. Assume that the condition (H_ϕ) holds, $d:[0,\pi]\to E$, $\rho_i:\mathbb{R}_+\longrightarrow\mathbb{R}$ for $i=1,2$, $a_1,a_3,a_4:\mathbb{R}_-\to\mathbb{R}$ and $a_2:[0,\pi]\to\mathbb{R}$ are continuous. Then there exists a mild solution of* (4.34) *on* $]-\infty,+\infty[$.

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi)=\int_{-\infty}^{0}a_1(s)\psi(s,\xi)ds,$$

$$g(t,\psi)(\xi)=\int_{-\infty}^{0}a_3(s)\psi(s,\xi)ds,$$

$$h(t,\psi)(\xi)=\int_{-r}^{0}a_4(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi)=s-\rho_1(s)\rho_2\left(\int_0^{\pi}a_2(\theta)|\psi(0,\xi)|^2d\theta\right),$$

are well defined functions, which permit to transform system (4.34) into the abstract system $(4.3)-(4.4)$. Moreover, the function f, g and h is bounded linear operator. Now, the existence of mild solutions can be deduced from a direct application of Theorem 4.3.1.

From Remark 1.7.2, we have the following result

Corollary 4.8.2. *Let $\phi\in\mathcal{B}$ be continuous and bounded. Then there exists a mild solution of* (4.34) *on* $]-\infty,+\infty[$.

Example 3. Consider the partial differential equation

$$\begin{cases} {}^cD_0^\alpha u(t,\xi) = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-r}^{0} a_1(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right] ds \\ + \displaystyle\int_{-r}^{0} a_3(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right] ds, \\ \hfill \xi \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \hfill 0 \le t \le b, \\ u(\theta,\xi) = u_0(\theta,\xi), \hfill -r < \theta \le 0,\ \xi \in [0,\pi], \end{cases} \tag{4.35}$$

where $a_1, a_3 : [-r,0] \to \mathbb{R}$ is a continuous function.

Theorem 4.8.3. *Let $\varphi \in C(H,E)$. Assume that the condition (H_φ) holds, $\rho_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $i = 1,2$, are continuous and the functions $a_1, a_3 : [-r,0] \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a mild solution of* (4.35) *on* $]-r,b]$.

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-r}^{0} a_1(s)\psi(s,\xi)ds,$$

$$h(t,\psi)(\xi) = \int_{-r}^{0} a_3(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (4.35) into the abstract system (4.1) – (4.2). Moreover, the function f and h is bounded linear operator. Now, the existence of mild solutions can be deduced from a direct application of Theorem 4.4.1.
From Remark 1.7.1, we have the following result

Corollary 4.8.3. *Let $\varphi \in C(H,E)$ be continuous and bounded. Then there exists a mild solution of* (4.35) *on* $]-r,b]$.

Example 4. Consider the partial differential equation

$$\begin{cases} {}^cD_0^\alpha \left[u(t,\xi) - \displaystyle\int_{-r}^{0} a_4(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right) ds\right] \\ = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-r}^{0} a_1(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right) ds \\ + \displaystyle\int_{-r}^{0} a_3(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right) ds, \\ \hfill ,\ \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \hfill 0 \le t \le b, \\ v(\theta,\xi) = v_0(\theta,\xi), \hfill -r < \theta \le 0,\ \xi \in [0,\pi]. \end{cases} \tag{4.36}$$

where $a_4 : [-r, 0] \to \mathbb{R}$ is a continuous function.

Theorem 4.8.4. *Let $\varphi \in C(H, E)$. Assume that the condition (H_φ) holds, $d : [0, \pi] \to E$, $\rho_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$ for $i = 1, 2$, $a_1, a_3, a_4 : [-r, 0] \to \mathbb{R}$ and $a_2 : [0, \pi] \to \mathbb{R}$ are continuous. Then there exists a mild solution of* (4.36) *on* $] - r, b]$.

Proof. From the assumptions, we have that

$$f(t, \psi)(\xi) = \int_{-r}^{0} a_1(s)\psi(s, \xi)ds,$$

$$g(t, \psi)(\xi) = \int_{-r}^{0} a_3(s)\psi(s, \xi)ds,$$

$$h(t, \psi)(\xi) = \int_{-r}^{0} a_4(s)\psi(s, \xi)ds,$$

$$\rho(s, \psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0, \xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (4.36) into the abstract system $(4.3) - (4.4)$. Moreover, the function f, g and h is bounded linear operator. Now, the existence of mild solutions can be deduced from a direct application of Theorem 4.5.1.

From Remark 1.7.1, we have the following result

Corollary 4.8.4. *Let $\varphi \in C(H, E)$ be continuous and bounded. Then there exists a mild solution of* (4.36) *on* $] - r, b]$.

Example 5. Consider the partial differential equation

$$\begin{cases} {}^cD_0^\alpha u(t, \xi) = \dfrac{\partial^2 u(t, \xi)}{\partial \xi^2} + a_0(t, \xi)u(t, \xi) \\ + \displaystyle\int_{-\infty}^{0} a_1(s - t)u\left[s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t, \theta)|^2 d\theta\right), \xi\right] ds \\ + \displaystyle\int_{-\infty}^{0} a_3(s - t)u\left[s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t, \theta)|^2 d\theta\right), \xi\right] ds, \\ \hfill \xi \in [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, \hfill 0 \le t \le b, \\ u(\theta, \xi) = u_0(\theta, \xi), \hfill -\infty < \theta \le 0, \ \xi \in [0, \pi], \end{cases} \tag{4.37}$$

where $a_1, a_3 : (-\infty, 0] \to \mathbb{R}$ are continuous functions.

Theorem 4.8.5. *Let $\mathcal{B} = BUC(\mathbb{R}_-; E)$ and $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds, $\rho_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $i = 1, 2$, are continuous and the functions $a_1, a_3 : \mathbb{R}_- \to \mathbb{R}$ and $a_2 : [0, \pi] \to \mathbb{R}$ are continuous. Then there exists a mild solution of* (4.37) *on* $] - \infty, b]$.

Proof. From the assumptions, we have that

$$f(t, \psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s, \xi)ds,$$

$$h(t,\psi)(\xi) = \int_{-\infty}^{0} a_3(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (4.37) into the abstract system (4.1) – (4.2). Moreover, the function f and h is bounded linear operator. Now, the existence of mild solutions can be deduced from a direct application of Theorem 4.6.1.
From Remark 1.7.2, we have the following result

Corollary 4.8.5. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then there exists a mild solution of* (4.37) *on* $]-\infty, b]$.

Example 6. Consider the partial differential equation

$$\begin{cases} {}^cD_0^\alpha\left[u(t,\xi) - \int_{-\infty}^{0} a_4(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)ds\right] \\ = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \int_{-\infty}^{0} a_1(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)ds \\ + \int_{-\infty}^{0} a_3(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad, \ \xi \in [0,\pi], \\ v(t,0) = \ v(t,\pi) \ = 0, \qquad\qquad\qquad\qquad\qquad 0 \le t \le b, \\ v(\theta,\xi) = \ v_0(\theta,\xi), \qquad\qquad\qquad -\infty < \theta \le 0, \ \xi \in [0,\pi]. \end{cases} \tag{4.38}$$

where $a_4 : (-\infty, 0] \to \mathbb{R}$ is a continuous function.

Theorem 4.8.6. *Let $\mathcal{B} = BUC(\mathbb{R}_-; E)$ and $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds, $d : [0,\pi] \to E$, $\rho_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$ for $i = 1, 2$, $a_1, a_3, a_4 : \mathbb{R}_- \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a mild solution of* (4.38) *on* $]-\infty, b]$.

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s,\xi)ds,$$

$$g(t,\psi)(\xi) = \int_{-\infty}^{0} a_3(s)\psi(s,\xi)ds,$$

$$h(t,\psi)(\xi) = \int_{-\infty}^{0} a_4(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (4.38) into the abstract system (4.3) – (4.4). Moreover, the function f, g and h is bounded linear operator. Now, the existence of mild solutions can be deduced from a direct application of Theorem 4.7.1.

From Remark 1.7.2, we have the following result

Corollary 4.8.6. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then there exists a mild solution of* (4.38) *on* $]-\infty, b]$.

Chapter 5

Controllability Results for Evolution Equations with State-Dependent Delay

5.1 Introduction

In this chapter, we will prove the controllability of mild solutions for some classes of first order of partial functional and neutral functional differential evolution equations with infinite state-dependent delay in Fréchet spaces[1], then we will show the controllability of mild solutions for some classes of Caputo's fractional derivative order of partial functional and neutral functional differential evolution equations with finite and infinite state-dependent delay[2] by using the nonlinear alternative of Avramescu for the sum of contraction and completely continuous operators on Banach spaces.

The controllability of mild solutions on the real positif interval is demonstrated in section 5.2 for the following class of evolution equations with infinite state-dependent delay :

$$y'(t) = A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \geq 0 \tag{5.1}$$

$$y_0 = \phi \in \mathcal{B}, \tag{5.2}$$

where $\mathcal{B}$ is an abstract *phase space* to be specified later, $f : \mathbb{R}^+ \times \mathcal{B} \to E, \rho : \mathbb{R}^+ \times \mathcal{B} \to \mathbb{R}$ and $\phi \in \mathcal{B}$ are given functions, the control function $u(.)$ is given in $L^2(J; E)$, the Banach space of admissible control function with E is a real separable Banach space with the norm $|\cdot|$, C is a bounded linear operator from E into E and $\{A(t)\}_{0 \leq t < +\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t, s)\}_{(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+}$ for $s \leq t$.

An extension of this problem is given in section 5.3, we consider the following class of neutral evolution equations with infinite state-dependent delay

$$\frac{d}{dt}[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \geq 0 \tag{5.3}$$

$$y_0 = \phi \in \mathcal{B}, \tag{5.4}$$

[1][11] **D. Aoued** and S. Baghli-Bendimerad, Controllability of mild solutions for evolution equations with infinite state-dependent delay, *Europian Journal of Pure and Applied Mathematics*, **9** (4), 2016, 383-401.

[2][14] **D. Aoued**, S. Baghli-Bendimerad and M. Benchohra, Controllability of mild solutions for fractional partial functional and neutral functional evolution equations with state-dependent delay, submitted.

where $A(\cdot)$, f, ϕ and C are as in problem (5.1) – (5.2) and $g : \mathbb{R}^+ \times \mathcal{B} \to E$ is a given function.

In section 5.4, we give the controllability of the unique mild solution of the following class of fractional evolution equations with finite state-dependent delay

$$ {}^cD_0^\alpha y(t) = A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J = [0, b] \tag{5.5} $$

$$ y(t) = \varphi(t), \quad t \in H = [-r, 0], \tag{5.6} $$

where $0 < r < +\infty$, ${}^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0, 1]$, $f : J \times C(H; E) \to E, \rho : J \times C(H; E) \to \mathbb{R}$ and $\varphi \in C(H; E)$ are given functions, the control function $u(.)$ is given in $L^2(J; E)$, the Banach space of admissible control function with E is a real separable Banach space with the norm $|\cdot|$, C is a bounded linear operator from E into E and $\{A(t)\}_{0\le t\le b}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t, s)\}_{(t,s)\in J\times J}$ for $0 \le s \le t \le b$.

An extension of this problem is given in section 5.5, we consider the following class of fractional neutral evolution equations with finite state-dependent delay

$$ {}^cD_0^\alpha [y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J \tag{5.7} $$

$$ y(t) = \varphi(t), \quad t \in H, \tag{5.8} $$

where $A(\cdot)$, f, u, C and φ are as in problem (5.5) – (5.6) and $g : J \times C(H; E) \to E$ is a given function.

In section 5.6, we investigate the controllability of mild solutions of the following class of fractional evolution equations with infinite state-dependent delay

$$ {}^cD_0^\alpha y(t) = A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J \tag{5.9} $$

$$ y_0 = \phi \in \mathcal{B}, \tag{5.10} $$

where ${}^cD_t^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0, 1]$, $f : J \times \mathcal{B} \to E$ and $\phi \in \mathcal{B}$ are given functions, the control function $u(.)$ is given in $L^2(J; E)$, the Banach space of admissible control function with E is a real separable Banach space with the norm $|\cdot|$, C is a bounded linear operator from E into E and $\{A(t)\}_{0\le t\le b}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t, s)\}_{(t,s)\in J\times J}$ for $0 \le s \le t \le b$.

An extension of this problem is given in section 5.7, we consider the following class of fractional neutral evolution equations with infinite state-dependent delay

$$ {}^cD_0^\alpha [y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J \tag{5.11} $$

$$ y_0 = \phi \in \mathcal{B}, \tag{5.12} $$

where $A(\cdot)$, f, u, C and ϕ are as in problem (5.11) – (5.12) and $g : J \times \mathcal{B} \to E$ is a given function. Finally, section 5.8 is devoted to examples illustrating the abstract theory considered in previous sections.

5.2 Partial Problem with Infinite Delay

Before stating and proving our first main result, we define firstly the corresponding mild solution of the semilinear evolution problem (5.1)−(5.2) then we define the concept of controllability for that problem and finally we expose the properties of state-dependent delay.

Definition 5.2.1. *We say that the function $y : \mathbb{R} \to E$ is a mild solution of* $(5.1) - (5.2)$ *if $y(t) = \phi(t)$ for all $t \leq 0$ and y satisfies for each $t \geq 0$ the following integral equation*

$$y(t) = U(t,0)\phi(0) + \int_0^t U(t,s)Cu(s)ds + \int_0^t U(t,s)\ f(s, y_{\rho(s,y_s)})ds. \tag{5.13}$$

Definition 5.2.2. *The evolution problem* $(5.1) - (5.2)$ *is said to be controllable if for every initial function $\phi \in \mathcal{B}$, $y^* \in E$ and for some $n \in \mathbb{N}$, there is some control $u \in L^2([0,n]; E)$ such that the mild solution $y(\cdot)$ of* $(5.1) - (5.2)$ *satisfies the terminal condition $y(n) = y^*$.*

Set $\mathcal{R}(\rho^-) = \{\rho(s,\phi) : (s,\phi) \in J \times \mathcal{B}, \rho(s,\phi) \leq 0\}$. We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0, +\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Lemma 5.2.1. *([70]) If $y : (-\infty, b] \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b \sup\{|y(\theta)|; \theta \in [0, max\{0,s\}]\},\ \ s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup\limits_{t \in \mathcal{R}(\rho^-)} L^\phi(t)$.

Proposition 5.2.1. *From* (H_ϕ), (A_1) *and Lemma* (5.2.1), *for all $t \in [0,n]$ and $n \in \mathbb{N}$ we have*

$$\|y_{\rho(t,y_t)}\|_{\mathcal{B}} \leq K_n|y(t)| + (M_n + L^\phi)\|\phi\|_{\mathcal{B}}.$$

We will need to introduce the following hypothesis which are assumed thereafter :

($H0$) $U(t,s)$ is compact for $t - s > 0$.

($H1$) There exists a constant $\widehat{M} \geq 1$ such that $\|U(t,s)\|_{B(E)} \leq \widehat{M}$ for every $(t,s) \in \Delta$.

($H2$) There exists a function $p \in L^1_{loc}(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to (0,+\infty)$ and such that

$$|f(t,u)| \leq p(t)\ \psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$

($H3$) For all $R > 0$, there exists $l_R \in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$|f(t,u) - f(t,v)| \leq l_R(t)\ \|u - v\|_{\mathcal{B}}$$

for all $u, v \in \mathcal{B}$ with $\|u\|_{\mathcal{B}} \leq R$ and $\|v\|_{\mathcal{B}} \leq R$.

$(H4)$ For each $n \in \mathbb{N}$, the linear operator $W : L^2([0,n];E) \to E$ is defined by

$$Wu = \int_0^n U(n,s)Cu(s)ds,$$

has a pseudo invertible operator $\tilde{W}^{-1}$ which takes values in $L^2([0,n];E)/\ker W$ and there exists positive constants $\widetilde{M}$ and $\widetilde{M_1}$ such that $\|C\| \leq \widetilde{M}$ and $\|\tilde{W}^{-1}\| \leq \widetilde{M_1}$.

For the construction of $\tilde{W}^{-1}$ see the paper of Carmichael *et al.* [38].

Consider the following space $B_{+\infty} = \left\{ y : \mathbb{R} \to E : y|_{[0,b]} \text{ continuous for } b > 0 \text{ and } y_0 \in \mathcal{B} \right\}$, where $y|_{[0,b]}$ is the restriction of y to the real compact interval $[0,b]$.

Let us fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms by :

$$\|y\|_n := \sup_{t \in [0,n]} e^{-\tau \, L_n^*(t)} \, |y(t)|$$

where $L_n^*(t) = \displaystyle\int_0^t \bar{l}_n(s)\, ds$, $\bar{l}_n(t) = K_n \widehat{M} l_n(t)$ and l_n is the function from $(H3)$.

Then $B_{+\infty}$ is a Fréchet space with the family of semi-norms $\| \cdot \|_{n \in \mathbb{N}}$.

Theorem 5.2.1. *Assume that the hypotheses* (H_ϕ) *and* $(H0) - (H4)$ *hold and moreover for each* $n \in \mathbb{N}$*, there exists a constant* $M_\star^n > 0$ *such that*

$$\frac{M_\star^n}{\alpha_n + K_n \widehat{M} (\widehat{M} \widetilde{M} \widetilde{M_1} n + 1) \; \psi(M_\star^n) \; \|p\|_{L^1}} > 1, \tag{5.14}$$

with $\alpha_n = K_n \widehat{M} \widetilde{M} \widetilde{M_1} n |y^*| + \left[M_n + L^\phi + K_n \widehat{M} H \left(\widehat{M} \widetilde{M} \widetilde{M_1} n + 1 \right) \right] \|\phi\|_{\mathcal{B}}$. *Then the evolution problem* $(5.1) - (5.2)$ *is controllable on* $\mathbb{R}$.

Proof. We transform the problem $(5.1) - (5.2)$ into a fixed-point problem. Consider the operator $N : B_{+\infty} \to B_{+\infty}$ defined by :

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\; \phi(0) + \displaystyle\int_0^t U(t,s)\; C\; u_y(s) ds + \int_0^t U(t,s)\; f(s, y_{\rho(s,y_s)}) ds, & \text{if } t \in J. \end{cases}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(5.1) - (5.2)$.

Using assumption $(H4)$, for arbitrary function $y(\cdot)$, we define the control

$$u_y(t) = \tilde{W}^{-1} \left[y^* - U(n,0)\; \phi(0) - \int_0^n U(n,\tau)\; f(\tau, y_{\rho(\tau, y_\tau)})\; d\tau \right](t).$$

Applying $(H2)$, we get

$$|u_y(t)| \;\; \leq \;\; \widetilde{M_1} \left[|y^*| + \widehat{M} H \|\phi\|_{\mathcal{B}} + \widehat{M} \int_0^n p(\tau)\; \psi(\|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}}) d\tau \right]. \tag{5.15}$$

We shall show that using this control the operator N has a fixed point $y(\cdot)$. Then $y(\cdot)$ is a mild solution of the evolution system (2.1).

For $\phi \in \mathcal{B}$, we will define the function $x(.) : \mathbb{R} \to E$ by $x(t) = \phi(t)$ for $t \leq 0$ and $x(t) = U(t,0)\ \phi(0)$ for $t \in J$. Then $x_0 = \phi$. For each function $z \in B_{+\infty}$ with $z(0) = 0$, we denote by $\overline{z}$ the function defined by $\overline{z}(t) = 0$ for $t \leq 0$ and $\overline{z}(t) = z(t)$ for $t \in J$.

If $y(\cdot)$ satisfies (5.13), we can decompose it as $y(t) = z(t) + x(t)$, $t \geq 0$, which implies $y_t = z_t + x_t$, for every $t \in J$ and the function $z(\cdot)$ satisfies for $t \in J$

$$z(t) = \int_0^t U(t,s)\ C\ u_{z+x}(s)ds + \int_0^t\ U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds.$$

Let $B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0 \in \mathcal{B}\}$. For any $z \in B^0_{+\infty}$ we have $\|z\|_{+\infty} = \sup\limits_{s\geq 0} |z(s)|$. Thus $(B^0_{+\infty}, \|\cdot\|_{+\infty})$ is a Banach space. We define the operators $F, G : B^0_{+\infty} \to B^0_{+\infty}$ by $F(z)(t) = \displaystyle\int_0^t U(t,s)\ C\ u_{z+x}(s)ds$ and $G(z)(t) = \displaystyle\int_0^t\ U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds$.

Obviously the operator N has a fixed point is equivalent to $F+G$ has one, so it turns to prove that $F+G$ has a fixed point. The proof will be given in several steps. First we show that F is continuous and compact.

Step 1 : F is continuous. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $B^0_{+\infty}$ such that $z_n \to z$ in $B^0_{+\infty}$. By $(H1)$, $(H4)$ and (5.15), we get for every $t \in [0,n]$

$$\begin{aligned}
|F(z_n)(t) - F(z)(t)| &\leq \widehat{M}\widetilde{M}\int_0^t\ |u_{z_n+x}(s) - u_{z+x}(s)|ds \\
&\leq \widehat{M}^2\widetilde{M}\widetilde{M}_1\int_0^t\int_0^n |f(\tau, z_{n\rho(\tau,z_{n\tau}+x_\tau)} + x_{\rho(\tau,z_{n\tau}+x_\tau)}) \\
&\qquad\qquad -f(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})|d\tau ds \\
&\leq \widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n\ |f(s, z_{n\rho(s,z_{ns}+x_s)} + x_{\rho(s,z_{ns}+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds.
\end{aligned}$$

Since f is continuous, we obtain by the Lebesgue Dominated Convergence theorem

$$|F(z_n)(t) - F(z)(t)| \to 0 \quad \text{as} \ \ n \to +\infty.$$

Thus F is continuous.

Step 2 : F maps bounded sets of $B^0_{+\infty}$ into bounded sets. For any $d > 0$, there exists a positive constant ℓ such that for each $z \in B_d = \{z \in B^0_{+\infty} : \|z\|_n \leq d\}$ one has $\|F(z)\|_n \leq \ell$. Let $z \in B_d$. By $(H1)$, $(H2)$ and (5.15), we have for each $t \in [0,n]$

$$\begin{aligned}
|F(z)(t)| &\leq \widehat{M}\widetilde{M}\int_0^t \widetilde{M}_1\Big[|\widehat{y}| + \widehat{M}H\|\phi\|_{\mathcal{B}} \\
&\qquad + \widehat{M}\int_0^n p(\tau)\ \psi(\|z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathcal{B}})\ d\tau\Big]\,ds \\
&\leq \widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|\widehat{y}| + \widehat{M}H\|\phi\|_{\mathcal{B}} + \widehat{M}\int_0^n p(s)\ \psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})ds\right].
\end{aligned}$$

Using Proposition (5.2.1), we get

$$\begin{aligned}
\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\leq K_n|z(s)| + (M_n + \mathcal{L}^\phi)\|z_0\|_{\mathcal{B}} + K_n|x(s)| + (M_n + \mathcal{L}^\phi)\|x_0\|_{\mathcal{B}} \\
&\leq K_n|z(s)| + K_n\|U(s,0)\|_{B(E)}|\phi(0)| + (M_n + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} \\
&\leq K_n|z(s)| + (M_n + \mathcal{L}^\phi + K_n\widehat{M}H)\|\phi\|_{\mathcal{B}}.
\end{aligned}$$

Set $c_n := (M_n + \mathcal{L}^\phi + K_n\widehat{M}H)\|\phi\|_{\mathcal{B}}$, then we obtain

$$\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \le K_n|z(s)| + c_n. \tag{5.16}$$

Since $z \in B_d$, then we have for $\delta_n := K_n d + c_n$

$$\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \le K_n|z(s)| + c_n \le \delta_n. \tag{5.17}$$

Using the nondecreasing character of ψ, we get for each $t \in [0, n]$

$$|F(z)(t)| \le \widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|\widehat{y}| + \widehat{M}H\|\phi\|_{\mathcal{B}} + \widehat{M}\psi(\delta_n)\|p\|_{L^1}\right] := \varrho.$$

Thus there exists a positive number ϱ such that $\|F(z)\|_n \le \varrho$. Hence $F(B_d) \subset B_\varrho$.
Step 3 : F maps bounded sets into equi-continuous sets of $B^0_{+\infty}$. We consider B_d as in Step 2 and we show that $F(B_d)$ is equi-continuous. Let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$ and $z \in B_d$. Then

$$\begin{aligned} |F(z)(\tau_2) - F(z)(\tau_1)| &\le \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)}\ \|C\||u_{z+x}(s)|\ ds \\ &+ \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)}\ \|C\||u_{z+x}(s)|\ ds. \end{aligned}$$

By the inequalities (5.15) and (5.16) and using the nondecreasing character of ψ, we get

$$|u_{z+x}(t)| \le \widetilde{M}_1\left[|y^*| + \widehat{M}H\|\phi\|_{\mathcal{B}} + \widehat{M}\ \psi(\delta_n)\ \|p\|_{L^1}\right] := \omega. \tag{5.18}$$

Then

$$\begin{aligned} |F(z)(\tau_2) - F(z)(\tau_1)| &\le \|C\|_{B(E)}\ \omega \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)}\ ds \\ &+ \|C\|_{B(E)}\ \omega \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)}\ ds. \end{aligned}$$

Noting that $|F(z)(\tau_2) - F(z)(\tau_1)|$ tends to zero as $\tau_2 - \tau_1 \to 0$ independently of $z \in B_d$. The right-hand side of the above inequality tends to zero as $\tau_2 - \tau_1 \to 0$. Since $U(t, s)$ is a strongly continuous operator and the compactness of $U(t, s)$ for $t > s$ implies the continuity in the uniform operator topology (see [7, 90]). As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that the operator F maps B_d into a precompact set in E.

Let $t \in J$ be fixed and let ϵ be such that $0 < \epsilon < t$. For $z \in B_d$ we define

$$F_\epsilon(z)(t) = U(t, t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon, s)\ C\ u_{z+x}(s)\ ds.$$

Since $U(t, s)$ is a compact operator, the set $Z_\epsilon(t) = \{F_\epsilon(z)(t) :\ z \in B_d\}$ is pre-compact in E for every ϵ sufficiently small, $0 < \epsilon < t$. Moreover using (5.18), we have

$$\begin{aligned} |F(z)(t) - F_\epsilon(z)(t)| &\le \int_{t-\epsilon}^{t} \|U(t, s)\|_{B(E)}\ \|C\|\ |u_{z+x}(s)|\ ds \\ &\le \|C\|_{B(E)}\ \omega \int_{t-\epsilon}^{t} \|U(t, s)\|_{B(E)}\ ds. \end{aligned}$$

Therefore there are precompact sets arbitrary close to the set $\{F(z)(t) : z \in B_d\}$. Hence the set $\{F(z)(t) : z \in B_d\}$ is precompact in E. So we deduce from Steps 1, 2 and 3 that F is a continuous compact operator.
Step 4 : We shall show now that the operator G is a contraction. Indeed, consider $z, \overline{z} \in B^0_{+\infty}$. By $(H1)$, $(H3)$ and (5.17), we get for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| &\leq \int_0^t \widehat{M}\ l_n(s)\ \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} ds \\
&\leq \int_0^t \widehat{M}\ K_n\ l_n(s)\ |z(s) - \overline{z}(s)| ds \\
&\leq \int_0^t \left[\overline{l}_n(s)\ e^{\tau L_n^*(s)}\right]\ \left[e^{-\tau L_n^*(s)}\ |z(s) - \overline{z}(s)|\right] ds \\
&\leq \int_0^t \left[\frac{e^{\tau\ L_n^*(s)}}{\tau}\right]' ds\ \|z - \overline{z}\|_n \\
&\leq \frac{1}{\tau}\ e^{\tau\ L_n^*(t)}\ \|z - \overline{z}\|_n.
\end{aligned}$$

Therefore,

$$\|G(z) - G(\overline{z})\|_n \leq \frac{1}{\tau}\ \|z - \overline{z}\|_n.$$

So, the operator G is a contraction for all $n \in \mathbb{N}$.
Step 5 : To apply Theorem (1.10.2), we must check $(C2)$: i.e. it remains to show that the following set is bounded $\mathcal{E} = \left\{z \in B^0_{+\infty} :\ z = \lambda\ F(z) + \lambda\ G\left(\frac{z}{\lambda}\right) \quad \text{for some } 0 < \lambda < 1\right\}$.

Let $z \in \mathcal{E}$. By (5.15), we have for each $t \in [0, n]$

$$\begin{aligned}
\frac{|z(t)|}{\lambda} &\leq \widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|\widehat{y}| + \widehat{M}H\|\phi\|_{\mathcal{B}}\right] \\
&+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 n \int_0^n p(s)\ \psi\left(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\right) ds \\
&+ \widehat{M}\int_0^t p(s)\ \psi\left(\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathcal{B}}\right) ds.
\end{aligned}$$

Using the first inequality in (5.16), we get

$$\begin{aligned}
\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathcal{B}} &\leq \frac{K_n|z(s)|}{\lambda} + \frac{M_n + \mathcal{L}^\phi}{\lambda}\|z_0\|_{\mathcal{B}} \\
&+ K_n|x(s)| + \left(M_n + \mathcal{L}^\phi\right)\|x_0\|_{\mathcal{B}} \\
&\leq \frac{K_n|z(s)|}{\lambda} + K_n\|U(s,0)\|_{B(E)}|\phi(0)| + \left(M_n + \mathcal{L}^\phi\right)\|\phi\|_{\mathcal{B}} \\
&\leq \frac{K_n|z(s)|}{\lambda} + \left(K_n\widehat{M}H + M_n + \mathcal{L}^\phi\right)\|\phi\|_{\mathcal{B}}.
\end{aligned}$$

Then, we get

$$\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathcal{B}} \leq \frac{K_n|z(s)|}{\lambda} + c_n. \tag{5.19}$$

By the previous inequality and the nondecreasing character of ψ, we obtain

$$\begin{aligned}\frac{|z(t)|}{\lambda} &\leq \widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|\widehat{y}| + \widehat{M}H\|\phi\|_{\mathcal{B}}\right] + \widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\ \psi(K_n|z(s)| + c_n)ds \\ &+ \widehat{M}\int_0^t p(s)\ \psi\left(\frac{K_n|z(s)|}{\lambda} + c_n\right)ds.\end{aligned}$$

Consider the function $\widetilde{u}(t) := \sup_{\theta\in[0,t]}|z(\theta)|$. Then by the nondecreasing character of ψ, we get for $\lambda < 1$ and for $t \in [0,n]$

$$\frac{\widetilde{u}(t)}{\lambda} \leq \widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|\widehat{y}| + \widehat{M}H\|\phi\|_{\mathcal{B}}\right] + \widehat{M}\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right)\int_0^n p(s)\ \psi\left(\frac{K_n\widetilde{u}(s)}{\lambda} + c_n\right)ds.$$

We consider the function μ defined by $\mu(t) = \sup\limits_{s\in[0,t]}\frac{K_n\widetilde{u}(s)}{\lambda} + c_n$ for $t \in J$. Let $t^\star \in [0,t]$ be such that $\mu(t) = \frac{K_n u(t^\star)}{\lambda} + c_n$. If $t^\star \in [0,n]$, by the previous inequality and the nondecreasing character of ψ, we have for $\alpha_n := c_n + K_n\widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|\widehat{y}| + \widehat{M}H\|\phi\|_{\mathcal{B}}\right]$

$$\mu(t) \leq \alpha_n + K_n\widehat{M}\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right)\int_0^n p(s)\ \psi(\mu(s))\ ds.$$

Consequently,

$$\frac{\|z\|_n}{\alpha_n + K_n\widehat{M}\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right)\psi(\|z\|_n)\|p\|_{L^1}} \leq 1.$$

Then by the condition (5.14), there exists a constant $M_\star^n$ such that $\mu(t) \leq M_\star^n$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq M_\star^n$. This shows that the set $\mathcal{E}$ is bounded, i.e. the statement $(C2)$ in Theorem (1.10.2) does not hold. Then the Avramescu's nonlinear alternative ([18]) implies that $(C1)$ holds : i.e. the operator $F + G$ has a fixed-point $z^\star$. Then, there exists at least $y^\star(t) = z^\star(t) + x(t)$, $t \in \mathbb{R}$ which is a fixed point of the operator N, which is a mild solution of the problem $(5.1) - (5.2)$. Thus the evolution system $(5.1) - (5.2)$ is controllable on $\mathbb{R}$. Then, the proof is complete.

5.3 Neutral Problem with Infinite Delay

Before stating and proving our second main result, we define firstly the corresponding mild solution then we define the concept of controllability for that problem.

Definition 5.3.1. *We say that the function $y(\cdot) : \mathbb{R} \to E$ is a mild solution of $(5.3) - (5.4)$ if $y(t) = \phi(t)$ for all $t \leq 0$ and y satisfies the following integral equation*

$$\begin{aligned}y(t) &= U(t,0)[\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) + \int_0^t U(t,s)A(s)g(s, y_{\rho(s,y_s)})ds \\ &+ \int_0^t U(t,s)Cu(s)ds + \int_0^t U(t,s)f(s, y_{\rho(s,y_s)})ds \quad \text{for each } t \geq 0.\end{aligned} \tag{5.20}$$

Definition 5.3.2. *The neutral evolution problem $(5.3) - (5.4)$ is said to be controllable if for every initial function $\phi \in \mathcal{B}$, $y^* \in E$ and $n \in \mathbb{N}$, there is some control $u \in L^2([0,n];E)$ such that the mild solution $y(\cdot)$ of $(5.3) - (5.4)$ satisfies $y(n) = y^*$.*

We consider the function $\rho : J \times \mathcal{B} \longrightarrow \mathbb{R}$ satisfies the hypothesis (H_ϕ) and the Lemma (5.2.1). We assume here that the hypotheses $(H0)-(H4)$ hold and we need the following assumptions :

$(H5)$ There exists a constant $\overline{M}_0 > 0$ such that $\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0$ for all $t \in J$.

$(H6)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0 K_n}$, such that

$$|A(t)\ g(t,\phi)| \leq L\ (\|\phi\|_{\mathcal{B}} + 1) \text{ for all } t \in J \text{ and } \phi \in \mathcal{B}.$$

$(H7)$ There exists a constant $L_* > 0$ such that

$$|A(s)\ g(s,\phi) - A(\overline{s})\ g(\overline{s},\overline{\phi})| \leq L_*\ (|s-\overline{s}| + \|\phi - \overline{\phi}\|_{\mathcal{B}})$$

for all $s, \overline{s} \in J$ and $\phi, \overline{\phi} \in \mathcal{B}$.

$(H8)$ The function g is completely continuous and for each bounded sub-set $Q \subset \mathcal{B}$, the mapping $\{t \longrightarrow g(t, x_{\rho(s,y_s)})\}$ is equicontinous in $C(J,E)$.

Theorem 5.3.1. *Suppose that the hypotheses* $(H0)-(H8)$ *are satisfied and moreover*

$$\frac{M^{\star\star}}{\gamma_n + \dfrac{K_n\widehat{M}}{1-\overline{M}_0 L K_n}\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right)[M^{\star\star} + \psi(M^{\star\star})]\ \ \psi(\|z\|_n)\|\zeta\|_{L^1}} > 1, \tag{5.21}$$

where $\zeta(t) = \max(L; p(t))$ *and* $\gamma_n = (M_n + \mathcal{L}^\phi + K_n\widehat{M}H)\|\phi\|_{\mathcal{B}} + \dfrac{K_n\beta_n}{1-\overline{M}_0 L K_n}$ *with*

$$\begin{aligned}\beta_n &= \left[(\widehat{M}+1)\overline{M}_0 L + \widehat{M}Ln\right]\left[\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right] + \widehat{M}\widetilde{M}\widetilde{M}_1 n\left(1 + K_n\overline{M}_0 L\right)|\widehat{y}| \\ &+ \left[\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right)\overline{M}_0 L\left[\widehat{M} + M_n + \mathcal{L}^\phi\right] + \widehat{M}H\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n + \overline{M}_0 L K_n\right)\right]\|\phi\|_{\mathcal{B}}.\end{aligned}$$

Then the neutral evolution problem $(5.3)-(5.4)$ *is controllable on* $\mathbb{R}$.

Proof. Consider the operator $\widetilde{N} : B_{+\infty} \to B_{+\infty}$ defined by :

$$\widetilde{N}(y)(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\ [\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) + \displaystyle\int_0^t U(t,s)A(s)g(s,y_{\rho(s,y_s)})ds & \\ + \displaystyle\int_0^t U(t,s)Cu(s)ds + \int_0^t U(t,s)f(s,y_{\rho(s,y_s)})ds, & \text{if } t \in J. \end{cases}$$

Then, fixed points of the operator $\widetilde{N}$ are mild solutions of the problem $(5.3)-(5.4)$. Using assumption $(H4)$, for arbitrary function $y(\cdot)$, we define the control

$$\begin{aligned} u_y(t) &= \tilde{W}^{-1}\ \left[y^* - U(n,0)\left(\phi(0) - g(0,\phi)\right) - g(n, y_{\rho(n,y_n)})\right. \\ &\left. - \int_0^n U(n,\tau)A(\tau)g(\tau, y_{\rho(\tau,y_\tau)})d\tau - \int_0^n U(n,\tau)f(\tau, y_{\rho(\tau,y_\tau)})d\tau\right](t). \end{aligned}$$

Noting that by $(H1)$, $(H2)$, $(H4)$, $(H5)$ and $(H7)$ we get

$$
\begin{aligned}
|u_y(t)| &\leq \widetilde{M}_1\left[|y^*| + \widehat{M}\left(H + \overline{M}_0 L\right)\|\phi\|_{\mathcal{B}} + \left(\widehat{M}+1\right)\overline{M}_0 L + \widehat{M} L n\right] \\
&+ \widetilde{M}_1\overline{M}_0 L\|y_{\rho(n,y_n)}\|_{\mathcal{B}} + \widetilde{M}_1\widehat{M} L\int_0^n \|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}} d\tau \\
&+ \widetilde{M}_1\widehat{M}\int_0^n p(\tau)\psi(\|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}})d\tau.
\end{aligned} \tag{5.22}
$$

Using this control the operator $\widetilde{N}$ has a fixed point $y(\cdot)$. Then $y(\cdot)$ is a mild solution of the neutral evolution system (5.3) – (5.4).

For $\phi \in \mathcal{B}$, we will define the function $x(.) : \mathbb{R} \to E$ by $x(t) = \phi(t)$ for $t \leq 0$ and $x(t) = U(t,0)\ \phi(0)$ for $t \in J$. Then $x_0 = \phi$. For each function $z \in B_{+\infty}$ with $z(0) = 0$, we denote by $\overline{z}$ the function defined by $\overline{z}(t) = 0$ for $t \leq 0$ and $\overline{z}(t) = z(t)$ for $t \in J$.

If $y(\cdot)$ satisfies (5.20), we decompose it as $y(t) = z(t)+x(t)$, $t \geq 0$, which implies $y_t = z_t + x_t$, for every $t \in J$ and the function $z(\cdot)$ satisfies $z_0 = 0$ and for $t \in J$, we get

$$
\begin{aligned}
z(t) &= g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - U(t,0)g(0,\phi) \\
&+ \int_0^t U(t,s)A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds + \int_0^t U(t,s)Cu_{z+x}(s)ds \\
&+ \int_0^t U(t,s)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds.
\end{aligned}
$$

Let us define the operators $\widetilde{F}, G : B^0_{+\infty} \longrightarrow B^0_{+\infty}$ by

$$
\begin{aligned}
\widetilde{F}(z)(t) &= g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - U(t,0)g(0,\phi) \\
&+ \int_0^t U(t,s)A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds + \int_0^t U(t,s)Cu_{z+x}(s)ds
\end{aligned}
$$

and $$G(z)(t) = \int_0^t U(t,s)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds.$$

Obviously the operator $\widetilde{N}$ has a fixed point is equivalent to $\widetilde{F} + G$ has one, so it turns to prove that $\widetilde{F} + G$ has a fixed point.

We can show as in Section 3 that the operator $\widetilde{F}$ is continuous and compact and the operator G is a contraction. For applying Avramescu's nonlinear alternative, we must check $(C2)$ in Theorem (1.10.2): i.e. it remains to show that the following set

$$\widetilde{\mathcal{E}} = \left\{z \in B^0_{+\infty} :\ z = \lambda\widetilde{F}(z) + \lambda G\left(\frac{z}{\lambda}\right) \quad \text{for some } 0 < \lambda < 1\right\}$$

is bounded.

Let $z \in \widetilde{\mathcal{E}}$. Then, using $(H1)-(H6)$ and (5.22), we have for each $t \in [0, n]$

$$\begin{aligned}
\frac{|z(t)|}{\lambda} &\leq \left[(\widehat{M}+1)\overline{M}_0 L + \widehat{M}Ln\right]\left[\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right] + \widehat{M}\widetilde{M}\widetilde{M}_1 n|\widehat{y}| \\
&+ \widehat{M}\left[\overline{M}_0 L\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right) + \widehat{M}\widetilde{M}\widetilde{M}_1 nH\right]\|\phi\|_{\mathcal{B}} \\
&+ \widehat{M}\widetilde{M}\widetilde{M}_1\overline{M}_0 Ln\|z_{\rho(n,z_n+x_n)} + x_{\rho(n,z_n+x_n)}\|_{\mathcal{B}} \\
&+ \overline{M}_0 L\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} + \widehat{M}L\int_0^t \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\, ds \\
&+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 Ln\int_0^n \|\left(z_{\rho(\tau,z_\tau+x_\tau)}\right) + x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathcal{B}}\, d\tau \\
&+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(\tau)\ \psi(\|\left(z_{\rho(\tau,z_\tau+x_\tau)}\right) + x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathcal{B}})\, d\tau \\
&+ \widehat{M}\int_0^t p(s)\ \psi\left(\left\|\frac{z_{\rho(s,\frac{z_\tau}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathcal{B}}\right) ds.
\end{aligned}$$

By Proposition (5.2.1), we obtain $\|z_{\rho(n,z_n+x_n)} + x_{\rho(n,z_n+x_n)}\|_{\mathcal{B}} \leq K_n|\widehat{y}| + (M_n + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}$. Using the inequalities (5.16) and (5.19), we have

$$\begin{aligned}
\frac{|z(t)|}{\lambda} &\leq \left[(\widehat{M}+1)\overline{M}_0 L + \widehat{M}Ln\right]\left[\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right] + \widehat{M}\widetilde{M}\widetilde{M}_1 n|\widehat{y}| \\
&+ \widehat{M}\left[\overline{M}_0 L\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right) + \widehat{M}\widetilde{M}\widetilde{M}_1 nH\right]\|\phi\|_{\mathcal{B}} \\
&+ \widehat{M}\widetilde{M}\widetilde{M}_1\overline{M}_0 Ln\left(K_n|\widehat{y}| + (M_n + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}\right) \\
&+ \overline{M}_0 L\left(K_n|z(t)| + (M_n + \mathcal{L}^\phi + K_n\widehat{M}H)\|\phi\|_{\mathcal{B}}\right) \\
&+ \widehat{M}L\int_0^t (K_n|z(s)| + c_n)\, ds + \widehat{M}^2\widetilde{M}\widetilde{M}_1 Ln\int_0^n (K_n|z(\tau)| + c_n)\, d\tau \\
&+ \widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(\tau)\ \psi\left(K_n|z(\tau)| + c_n\right) d\tau \\
&+ \widehat{M}\int_0^t p(s)\ \psi\left(\frac{K_n|z(s)|}{\lambda} + c_n\right) ds.
\end{aligned}$$

We consider the function $\widetilde{u}(t) := \sup_{\theta\in[0,t]} |z(\theta)|$ then by the nondecreasing character of ψ, we obtain for $\beta_n := \left[(\widehat{M}+1)\overline{M}_0 L + \widehat{M}Ln\right]\left[\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right] + \widehat{M}\widetilde{M}\widetilde{M}_1 n\left(1 + K_n\overline{M}_0 L\right)|\widehat{y}| + \left[\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right)\overline{M}_0 L\left[\widehat{M} + M_n + \mathcal{L}^\phi\right] + \widehat{M}H\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n\overline{M}_0 LK_n\right)\right]\|\phi\|_{\mathcal{B}}$ and for $\lambda < 1$,

$$\begin{aligned}
\frac{\widetilde{u}(t)}{\lambda}\left(1 - \overline{M}_0 LK_n\right) &\leq \beta_n + \widehat{M}L\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right)\int_0^n \left(\frac{K_n\widetilde{u}(s)}{\lambda} + c_n\right) ds \\
&+ \widehat{M}\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n + 1\right)\int_0^n p(s)\ \psi\left(\frac{K_n\widetilde{u}(s)}{\lambda} + c_n\right) ds.
\end{aligned}$$

We consider the function μ defined by $\mu(t) = \sup_{s\in[0,t]} \dfrac{K_n\widetilde{u}(s)}{\lambda} + c_n$ for $t \in J$.

Let $t^\star \in [0, t]$ be such that $\mu(t) = \frac{K_n u(t^\star)}{\lambda} + c_n$. By the previous inequality, we have for $\gamma_n :=$

$c_n + \dfrac{K_n\beta_n}{1-\overline{M}_0LK_n}$, then we obtain for $t \in [0,n]$

$$\mu(t) \leq \gamma_n + \frac{K_n\widehat{M}}{1-\overline{M}_0LK_n}\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)\int_0^n [L\mu(s)+p(s)\ \psi(\mu(s))]\,ds$$

Set $\zeta(t) := \max(L; p(t))$ for $t \in [0,n]$. Consequently, we get

$$\frac{\|z\|_n}{\gamma_n + \dfrac{K_n\widehat{M}}{1-\overline{M}_0LK_n}\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)[\|z\|_n+\psi(\|z\|_n)]\ \ \psi(\|z\|_n)\|\zeta\|_{L^1}} \leq 1.$$

Then by the condition (5.21), there exists a constant $M^n_{\star\star}$ such that $\mu(t) \leq M^n_{\star\star}$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq M^n_{\star\star}$. This shows that the set $\widetilde{\mathcal{E}}$ is bounded, i.e. the statement $(C2)$ in Theorem (1.10.2) does not hold. Then the nonlinear alternative due to Avramescu ([18]) implies that $(C1)$ holds : i.e. the operator $\widetilde{F}+G$ has a fixed-point $z^{\star\star}$. Then, there exists at least $y^{\star\star}(t) = z^{\star\star}(t) + x(t)$, $t \in \mathbb{R}$ which is a fixed point of the operator $\widetilde{N}$, which is a mild solution of the problem $(5.3)-(5.4)$. Thus the neutral evolution system $(5.3)-(5.4)$ is controllable on $\mathbb{R}$. Then, the proof is complete.

5.4 Fractional Partial Problem with Finite Delay

Before stating and proving the main result, we give first the definition of mild solution of the semilinear evolution problem $(5.5)-(5.6)$.

Lemma 5.4.1. *The system* $(5.5)-(5.6)$ *is equivalent to the nonlinear integral equation*

$$\begin{aligned} y(t) \ &= \ \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A(s)y(s)\ ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Cu(s)ds \\ &\qquad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y_{\rho(s,y_s)})\ ds. \end{aligned} \tag{5.23}$$

In other words, every solution of the integral equation (5.33) *is also solution of the system* $(5.5)-(5.6)$ *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system $(5.5)-(5.6)$, and using some classical results from fractional calculus to get (5.33).

Definition 5.4.1. *We say that the function* $y(\cdot) : [-r,b] \to E$ *is a mild solution of* $(5.5)-(5.6)$ *if* $y(t) = \varphi(t)$ *for all* $t \in H$ *and* y *satisfies the following integral equation*

$$\begin{aligned} y(t) \ &= \ U(t,0)\ \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)Cu(s)ds \\ &\qquad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ U(t,s)f(s,y_{\rho(s,y_s)})\ ds\ \ t \in J. \end{aligned} \tag{5.24}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times C(H;E), \rho(s,\varphi) \leq 0\}.$$

We always assume that $\rho : J \times C(H;E) \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $C(H;E)$ and there exists a continuous and bounded function $\mathcal{L}^\varphi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\varphi_t\| \le \mathcal{L}^\varphi(t)\|\varphi\| \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

We will need to introduce the following hypothesis which are assumed thereafter :

$(H'1)$ There exists a constant $\widehat{M} \ge 1$ such that

$$\|U(t,s)\|_{B(E)} \le \widehat{M} \quad \text{for every } (t,s) \in \Delta.$$

$(H'2)$ There exist two functions $p, q \in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$|f(t,u)| \le p(t) + q(t)\ \|u\|_B \text{ for a.e. } t \in J \text{ and each } u \in C(H;E).$$

$(H'3)$ For all $R > 0$, there exists $l_R \in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$|f(t,u) - f(t,v)| \le l_R(t)\ \|u - v\|$$

for all $u, v \in C(H;E)$ with $\|u\| \le R$ and $\|v\| \le R$.

$(H'4)$ The linear operator $W : L^2(J;E) \to E$ is defined by

$$Wu = \int_0^b U(b,s)Cu(s)ds,$$

has an induced invertible operator $\tilde{W}^{-1}$ which takes values in $L^2(J;E)/\ker W$ and there exists positive constants $\widetilde{M}$ and $\widetilde{M}_1$ such that

$$\|C\| \le \widetilde{M} \quad \text{and} \quad \|\tilde{W}^{-1}\| \le \widetilde{M}_1.$$

Theorem 5.4.1. *Suppose that the hypotheses $(H'1) - (H'4)$ are satisfied and moreover there exists a constant $M_\star > 0$ such that*

$$\frac{M_\star}{\delta_b + \dfrac{\widehat{M}q^*[\widehat{M}\widetilde{M}\widetilde{M}_1 + \Gamma(\alpha)]}{\alpha\Gamma(\alpha)^2}M_\star\ b^\alpha} > 1 \tag{5.25}$$

with

$$\delta_b := (\widehat{M} + \mathcal{L}^\varphi)\|\varphi\| + \frac{K_b\widehat{M}\widetilde{M}\widetilde{M}_1 b}{\Gamma(\alpha)}\Big[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}}\Big] + \frac{K_b\widehat{M}p^*b^\alpha[\alpha\widehat{M}\widetilde{M}\widetilde{M}_1 b + \Gamma(\alpha)]}{\alpha\Gamma(\alpha)^2}.$$

Then the problem $(5.5) - (5.6)$ has a unique controllable mild solution on $[-r,b]$.

Proof. Transform the problem $(5.5) - (5.6)$ into a fixed-point problem. Consider the operator $N : B_b \to B_b$ defined by :

$$N(y)(t) = \begin{cases} \varphi(t), & \text{if } t \in [-r,0]; \\ U(t,0)\ \varphi(0) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}U(t,s)\ C\ u_y(s)\ ds & \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}U(t,s)\ f(s, y_{\rho(s,y_s)})\ ds, & \text{if } t \in J. \end{cases} \tag{5.26}$$

Using assumption $(H'3)$, for arbitrary function $y(\cdot)$, we define the control

$$u_y(t) = \tilde{W}^{-1}\left[y_1 - U(b,0)\ \varphi(0) - \frac{1}{\Gamma(\alpha)}\int_0^b (t-s)^{\alpha-1}\ U(b,s)\ f(s, y_{\rho(s,y_s)})\ ds\right](t).$$

Noting that, we have

$$\begin{aligned} |u_y(t)| &\leq \|\tilde{W}^{-1}\|\ \Big[|y_1| + \|U(t,0)\|_{B(E)}|\varphi(0)| \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^b (t-\tau)^{\alpha-1}\|U(b,\tau)\|_{B(E)}|f(\tau, y_{\rho(\tau,y_\tau)})|d\tau\Big]. \end{aligned}$$

From $(H'2)$, we get

$$|u_y(t)| \leq \widetilde{M}_1\left[|y_1| + \widehat{M}H\|\varphi\|_{\mathcal{B}} + \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^b (t-\tau)^{\alpha-1}\left[p(\tau) + q(\tau)\|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}}\right]\ ds\right] \tag{5.27}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(5.5) - (5.6)$.

Let y be a possible solution of the problem $(5.5) - (5.6)$. Given $t \leq b$, then from $(H'1)$, $(H'2)$, (H_φ) and Lemma 5.2.1, we have for each $t \in [0,b]$

$$\begin{aligned} |y(t)| &\leq \|U(t,0)\|_{B(E)}|\varphi(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ \|U(t,s)\|_{B(E)}\ |f(s, y_{\rho(s,y_s)})|\ ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|_{B(E)}\ \|C\|\ |u_y(s)|\ ds \\ &\leq \widehat{M}\|\varphi\| + \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[p(s) + q(s)\|y_{\rho(s,y_s)}\|\right] ds \\ &+ \frac{\widehat{M}\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|u_y(s)|\ ds \\ &\leq \widehat{M}\|\varphi\| + \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[p(s) + q(s)\|y_{\rho(s,y_s)}\|\right] ds \\ &+ \frac{\widehat{M}\widetilde{M}}{\Gamma(\alpha)}\int_0^t \widetilde{M}_1\Big[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} \\ &\quad + \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^b (t-s)^{\alpha-1}\left[p(\tau) + q(\tau)\|y_{\rho(\tau,y_\tau)}\|d\tau\right]\Big]\ ds \end{aligned}$$

It follows that

$$\begin{aligned} |y(t)| + \mathcal{L}^\varphi\|\varphi\| &\leq \widehat{M}\|\varphi\| + \mathcal{L}^\varphi\|\varphi\| + \frac{\widehat{M}\widetilde{M}\widetilde{M}_1 b}{\Gamma(\alpha)}\left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}}\right] \\ &+ \frac{\widehat{M}b^\alpha p^*(\Gamma(\alpha) + \widehat{M}\widetilde{M}\widetilde{M}_1 b)}{\alpha\Gamma^2(\alpha)} \\ &+ \frac{\widehat{M}q^*(\Gamma(\alpha) + \widehat{M}\widetilde{M}\widetilde{M}_1 b)}{\Gamma^2(\alpha)}\int_0^b (t-s)^{\alpha-1}\ \left[|y(s) + \mathcal{L}^\varphi\|\varphi\|\right] ds \end{aligned}$$

Set

$$\delta_b := (\widehat{M} + \mathcal{L}^{\varphi})\|\varphi\| + \frac{\widehat{M}\widetilde{M}\widetilde{M}_1 b}{\Gamma(\alpha)}\Big[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}}\Big] + \frac{\widehat{M}b^{\alpha}p^*(\Gamma(\alpha) + \widehat{M}\widetilde{M}\widetilde{M}_1 b)}{\alpha\Gamma^2(\alpha)}.$$

We consider the function μ defined by

$$\mu(t) := \sup\ \{\ |y(s)| + \mathcal{L}^{\varphi}\|\varphi\|\ :\ 0 \le s \le t\ \}, \quad 0 \le t \le b.$$

Let $t^{\star} \in [-r, b]$ be such that $\mu(t) = |y(t^{\star})| + \mathcal{L}^{\varphi}\|\varphi\|$. If $t^{\star} \in [0, b]$, by the previous inequality, we get

$$\mu(t) \ \le\ \delta_b + \frac{\widehat{M}q^*(\Gamma(\alpha) + \widehat{M}\widetilde{M}\widetilde{M}_1 b)}{\Gamma^2(\alpha)} \int_0^b (t-s)^{\alpha-1}\ \mu(s)ds \qquad \text{for } t \in [0, b].$$

If $t^{\star} \in [-r, 0]$, then $\mu(t) = \|\varphi\|$ and the previous inequality holds. Consequently,

$$\frac{\|z\|_n}{\delta_b + \dfrac{\widehat{M}q^*[\widehat{M}\widetilde{M}\widetilde{M}_1 + \Gamma(\alpha)]}{\alpha\Gamma(\alpha)^2}\|z\|_n\ b^{\alpha}} \le 1.$$

Then by (5.25), there exists a constant $M^{\star}$ such that $\mu(t) \le M^{\star}$ Since $\|y\| \le \mu(t)$,Since for every $t \in [0, b]$ we have $\|y\| \le \max\{\|\varphi\|, M^{\star}\} := \Theta_b$.

Set

$$Y = \{\ y \in C([-r, b]; E)\ :\ \sup\{|y(t)| : 0 \le t \le b\} \le \Theta_b + 1\}.$$

Clearly, Y is a closed subset of $C([-r, b]; E)$.

We shall show that $N : Y \to C([-r, b]; E)$ is a contraction operator.
Indeed, consider $y, \overline{y} \in Y$, thus using $(H^{'}1)$ and $(H^{'}3)$ for each $t \in [0, b]$

$$\begin{aligned}
|N(y)(t) - N(\overline{y})(t)| \ &\le\ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|_{B(E)}\ \left|f(s, y_{\rho(s,y_s)}) - f(s, \overline{y}_{\rho(s,y_s)})\right|\ ds \\
&+\ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|_{B(E)}\ \|C\|\ |u_y(s) - u_{\overline{y}}|\ ds \\
&\le\ \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ l_b(s)\ \|y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)}\|\ ds \\
&+\ \frac{\widehat{M}\widetilde{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\widetilde{M}_1\widehat{M}\int_0^b |f(\tau, y_{\rho(\tau,y_\tau)}) \\
&-\ f(\tau, \overline{y}_{\rho(\tau,y_\tau)})|\ d\tau\ ds
\end{aligned}$$

Using (H_{φ}) and Lemma 5.2.1, we obtain

$$\begin{aligned}
|N(y)(t) - N(\overline{y})(t)| \ &\le\ \frac{\widehat{M}l_b^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ |y(s) - \overline{y}(s)|\ ds \\
&+\ \frac{\widehat{M^2}\widetilde{M}\widetilde{M}_1 l_b^* b^{\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ |y(s) - \overline{y}(s)|\ ds \\
&\le\ \frac{\widehat{M}l_b^* b^{\alpha}}{\Gamma(\alpha+1)}\|y - \overline{y}\| + \frac{\widehat{M^2}\widetilde{M}\widetilde{M}_1 l_b^* b^{\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha)}\|y - \overline{y}\|
\end{aligned}$$

Therefore,

$$\|N(y)-N(\overline{y})\|_n \le \frac{\widehat{M}l_b^* b^\alpha(\Gamma(\alpha)+\widehat{M}\widetilde{M}\widetilde{M}_1)}{\Gamma(\alpha+1)\Gamma(\alpha)}\|y-\overline{y}\|.$$

So, for $\dfrac{\widehat{M}l_b^* b^\alpha(\Gamma(\alpha)+\widehat{M}\widetilde{M}\widetilde{M}_1)}{\Gamma(\alpha+1)\Gamma(\alpha)} < 1$ the operator N is a contraction . From the choice of Y there is no $y \in \partial Y^n$ such that $y = \lambda\ N(y),\ \lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 1.10.4 does not hold. The nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. Thus, we deduce that the operator N has a unique fixed-point $y^\star$ which is the unique controllable mild solution of the problem $(5.5)-(5.6)$.

5.5 Fractional Neutral Problem with Finite Delay

In this section, we give controllability result for the neutral functional fractional differential evolution problem with infinite delay $(5.7)-(5.8)$. Firstly, we define its mild solution.

Lemma 5.5.1. *The system* $(5.7)-(5.8)$ *is equivalent to the nonlinear integral equation*

$$\begin{aligned} y(t) \quad &= \quad [\varphi(0)-g(0,\varphi)]+g(t,y_{\rho(t,y_t)})+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A(s)y(s)\ ds \qquad (5.28)\\ &+ \quad \frac{1}{\Gamma(\alpha)}\int_0^t U(t,s)Cu(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y_{\rho(s,y_s)})\ ds. \end{aligned}$$

In other words, every solution of the integral equation (5.28) *is also solution of the system* $(5.7)-(5.8)$ *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system $(5.7)-(5.8)$, and using some classical results from fractional calculus to get (5.28).

Definition 5.5.1. *We say that the function* $y(\cdot):[-r,b]\to E$ *is a mild solution of* $(5.7)-(5.8)$ *if* $y(t)=\varphi(t)$ *for all* $t\in H$ *and* y *satisfies the following integral equation*

$$\begin{aligned} y(t) \quad &= U(t,0)[\varphi(0)-g(0,\varphi)]+g(t,y_{\rho(t,y_t)})+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ U(t,s)f(s,y_{\rho(s,y_s)})\ ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)Cu(s)ds \qquad \textit{for each } t\in J. \end{aligned} \tag{5.29}$$

Definition 5.5.2. *The neutral evolution problem* $(5.7)-(5.8)$ *is said to be controllable on* $[-r,b]$ *if for every initial function* $\varphi\in C(H;E)$ *and* $y_1\in E$ *there exists a control* $u\in L^2(J;E)$ *such that the mild solution* $y(\cdot)$ *of* $(5.7)-(5.8)$ *satisfies* $y(b)=y_1$.

We consider the hypotheses (H_φ), $(H1)-(H3)$ and we need the following assumptions

$(H'5)$ There exists a constant $L_\star>0$ such that

$$|A(s)g(s,\varphi)-A(\overline{s})g(\overline{s},\overline{\varphi})|\le L_\star\ (|s-\overline{s}|+\|\varphi-\overline{\varphi}\|)$$

for all $\varphi,\overline{\varphi}\in C(H;E)$.

$(H'6)$ There exists a constant $\overline{M}_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0 \quad \text{for all } \ t \in J.$$

$(H'7)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0 K_b}$ such that

$$|A(t)\ g(t,\varphi)| \leq L\ (\|\varphi\| + 1) \text{ for all } t \in J \text{ and } \varphi \in C(H;E).$$

$(H'8)$ The function g is completely continuous and for any bounded set $Q \subseteq \mathcal{B}$ the set $\{t \to g(t,x_t) : x \in Q\}$ is equi-continuous in $C(J;E)$.

Theorem 5.5.1. *Suppose that the hypotheses $(H'1) - (H'8)$ are satisfied and moreover there exists a constant $M^{\star\star} > 0$ with*

$$\frac{M^{\star\star}}{\beta_b + \dfrac{\widehat{M} q^* [\widehat{M}\widetilde{M}\widetilde{M}_1 b + \Gamma(\alpha)]}{\alpha \Gamma(\alpha)^2} M^{\star\star}\ b^{\alpha}} >\ 1, \tag{5.30}$$

where $\beta_b = \mathcal{L}^{\varphi}\|\varphi\| + \dfrac{\delta_b}{(1 - \overline{M}_0 L)}$ *and*

$$\begin{aligned} \delta_b :\ &= \widetilde{M}_1 \overline{M}_0 L |y_1| + \widehat{M}\|\varphi\| + \overline{M}_0 L\,(\mathcal{L}^{\varphi}\|\varphi\| + 1) + \widehat{M}\overline{M}_0 L\,(\|\varphi\| + 1) \\ &+ \frac{\widehat{M}\widetilde{M}\widetilde{M}_1 b}{\Gamma(\alpha)} \Big[|y_1| + \widehat{M} H \|\varphi\|_{\mathcal{B}}\Big] + \frac{\widehat{M} b^{\alpha} p^* (\Gamma(\alpha) + \widehat{M}\widetilde{M}\widetilde{M}_1 b)}{\alpha \Gamma^2(\alpha)}. \end{aligned}$$

Then the neutral evolution problem (5.7) – (5.8) *has a unique controllable mild solution on* $[-r,b]$.

Proof. Consider the operator $\widetilde{N} : B_b \to B_b$ defined by :

$$\widetilde{N}(y)(t) = \begin{cases} \varphi(t), & \text{if } t \in [-r,0]; \\ U(t,0)\ [\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) & \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} U(t,s)\ C\ u_y(s)\ ds & \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} U(t,s)\ f(s, y_{\rho(s,y_s)})\ ds, & \text{if } t \in J. \end{cases} \tag{5.31}$$

Using assumption $(H'4)$, for arbitrary function $y(\cdot)$, we define the control

$$\begin{aligned} u_y(t) &= \tilde{W}^{-1}\Big[y_1 - U(b,0)\,(\varphi(0) - g(0,\varphi)) - g(b, y_{\rho(b,y_b)}) \\ &- \frac{1}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} U(b,s) f(s, y_{\rho(s,y_s)}) ds\Big](t). \end{aligned}$$

Noting that

$$\begin{aligned}
|u_y(t)| &\leq \|\tilde{W}^{-1}\| \Big[\ |y_1| + \|U(t,0)\|_{B(E)} \big(|\varphi(0)| + \|A^{-1}(0)\| |A(0)g(0,\varphi)|\big) \\
&\quad + \|A^{-1}(b)\| |A(b)g(b, y_{\rho(b,y_b)})| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} U(b,\tau) f(\tau, y_{\rho(\tau,y_\tau)}) d\tau \Big] \\
&\leq \widetilde{M}_1 \Big[\ |y_1| + \widehat{M} H \|\varphi\| + \widehat{M}\overline{M}_0 L(\|\varphi\|+1) + \overline{M}_0 L(\|y_{\rho(b,y_b)}\|+1) \Big] \\
&+ \frac{\widetilde{M}_1 \widehat{M}}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} \left[p(\tau) + q(\tau) \|y_{\rho(\tau,y_\tau)}\| \right].
\end{aligned}$$

we get

$$\begin{aligned}
|u_y(t)| &\leq \widetilde{M}_1 \Big[\ |y_1| + \widehat{M} \left(H + \overline{M}_0 L\right) \|\varphi\| + \left(\widehat{M}+1\right) \overline{M}_0 L \Big] + \widetilde{M}_1 \overline{M}_0 L \|y_{\rho(b,y_b)}\| \\
&+ \frac{\widehat{M}\widetilde{M}_1 p^* b^\alpha}{\Gamma(\alpha+1)} + \frac{\widehat{M}\widetilde{M}_1 q^*}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} \ \|y_{\rho(s,y_s)}\| \ ds
\end{aligned} \tag{5.32}$$

We shall show that using this control the operator $\widetilde{N}$ has a fixed point $y(\cdot)$, which is a mild solution of the neutral evolution system $(5.7)-(5.8)$.

Let y be a possible solution of the problem $(5.7)-(5.8)$. Given $t \leq b$, then from $(H'1)-(H'2)$, $(H'4)-(H'7)$, (H_φ) and Lemma 5.2.1, we have for each $t \in [0,b]$

$$\begin{aligned}
|y(t)| &\leq |g(t, y_{\rho(t,y_t)})| + |U(t,0)[\varphi(0) - g(0,\varphi)]| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ \|U(t,s)\|_{B(E)} \ |f(s, y_{\rho(s,y_s)})| \ ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \ \|C\| \ |u_y(s)| \ ds \\
&\leq \|A^{-1}(t)\| \ |A(t)g(t, y_{\rho(t,y_s)})| + \|U(t,0)\|_{B(E)} \ \|A^{-1}(0)\| \ |A(0)g(0,\varphi)| \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s) \|y_{\rho(s,y_s)}\| \right] ds \\
&+ \frac{\widehat{M}\widetilde{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u_y(s)| \ ds
\end{aligned}$$

$$\begin{aligned}
|y(t)| &\leq \widehat{M}\|\varphi\| + \overline{M}_0 L \left(\|y_{\rho(t,y_t)}\| + 1\right) + \widehat{M}\overline{M}_0 L \left(\|\varphi\| + 1\right) \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s) \|y_{\rho(s,y_s)}\| \right] ds \\
&+ \frac{\widehat{M}\widetilde{M}}{\Gamma(\alpha)} \int_0^t \widetilde{M}_1 \Big[|y_1| + \widehat{M} \left(H + \overline{M}_0 L\right) \|\varphi\| + \left(\widehat{M}+1\right) \overline{M}_0 L \\
&\quad + \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} \left[p(\tau) + q(\tau) \|y_{\rho(\tau,y_\tau)}\| d\tau \right] \Big] ds \\
&+ \widetilde{M}_1 \overline{M}_0 L \|y_{\rho(b,y_b)}\|
\end{aligned}$$

Since $\|y_{\rho(t,y_t)}\| \leq |y(t)| + \mathcal{L}^\varphi \|\varphi\|$, we have

$$\begin{aligned}
(1-\overline{M}_0L)|y(t)| &\leq \widetilde{M}_1\overline{M}_0L|y_1| + \widehat{M}\|\varphi\| + \overline{M}_0L\left(\mathcal{L}^{\varphi}\|\varphi\|+1\right) + \widehat{M}\overline{M}_0L\left(\|\varphi\|+1\right)\\
&+ \frac{\widehat{M}\widetilde{M}\widetilde{M}_1b}{\Gamma(\alpha)}\left[|y_1| + \widehat{M}H\|\varphi\|\right] + \frac{\widehat{M}b^{\alpha}p^*(\Gamma(\alpha)+\widehat{M}\widetilde{M}\widetilde{M}_1b)}{\alpha\Gamma^2(\alpha)}\\
&+ \frac{\widehat{M}q^*(\Gamma(\alpha)+\widehat{M}\widetilde{M}\widetilde{M}_1b)}{\Gamma^2(\alpha)}\int_0^b (t-s)^{\alpha-1}\ \left[|y(s)+\mathcal{L}^{\varphi}\|\varphi\|\right]ds
\end{aligned}$$

Set

$$\begin{aligned}
\delta_b: &= \widetilde{M}_1\overline{M}_0L|y_1| + \widehat{M}\|\varphi\| + \overline{M}_0L\left(\mathcal{L}^{\varphi}\|\varphi\|+1\right) + \widehat{M}\overline{M}_0L\left(\|\varphi\|+1\right)\\
&+ \frac{\widehat{M}\widetilde{M}\widetilde{M}_1b}{\Gamma(\alpha)}\left[|y_1| + \widehat{M}H\|\varphi\|\right] + \frac{\widehat{M}b^{\alpha}p^*(\Gamma(\alpha)+\widehat{M}\widetilde{M}\widetilde{M}_1b)}{\alpha\Gamma^2(\alpha)}.
\end{aligned}$$

Then

$$(1-\overline{M}_0L)|y(t)| \leq \delta_b + \frac{\widehat{M}q^*(\Gamma(\alpha)+\widehat{M}\widetilde{M}\widetilde{M}_1b)}{\Gamma^2(\alpha)}\int_0^b (t-s)^{\alpha-1}\ \left[|y(s)+\mathcal{L}^{\varphi}\|\varphi\|\right]ds,$$

with $\beta_b = \mathcal{L}^{\varphi}\|\varphi\| + \dfrac{\delta_b}{(1-\overline{M}_0L)}$.

Then

$$|y(t)| + \mathcal{L}^{\varphi}\|\varphi\| \leq \beta_b + \frac{\widehat{M}q^*\left(\Gamma(\alpha)+\widehat{M}\widetilde{M}\widetilde{M}_1b\right)}{(1-\overline{M}_0L)\Gamma^2(\alpha)}\int_0^b (t-s)^{\alpha-1}\ \left[|y(s)+\mathcal{L}^{\varphi}\|\varphi\|\right]ds$$

We consider the function μ defined by

$$\mu(t) := \sup\ \{\ |y(s)| + \mathcal{L}^{\varphi}\|\varphi\|\ :\ 0\leq s\leq t\ \},\quad 0\leq t\leq b.$$

Let $t^{\star}\in[-r,b]$ be such that $\mu(t) = |y(t^{\star})| + \mathcal{L}^{\varphi}\|\varphi\|$. If $t^{\star}\in[0,b]$, by the previous inequality, we get

$$\mu(t) \leq \beta_b + \frac{\widehat{M}q^*(\Gamma(\alpha)+\widehat{M}\widetilde{M}\widetilde{M}_1b)}{(1-\overline{M}_0L)\Gamma^2(\alpha)}\int_0^b (t-s)^{\alpha-1}\ \mu(s)ds \qquad \text{for } t\in[0,b].$$

If $t^{\star}\in[-r,0]$, then $\mu(t) = \|\varphi\|$ and the previous inequality holds. Consequently,

$$\frac{\|y\|}{\beta_b + \dfrac{\widehat{M}q^*[\widehat{M}\widetilde{M}\widetilde{M}_1b+\Gamma(\alpha)]}{\alpha\Gamma(\alpha)^2}\|y\|\ b^{\alpha}} \leq 1.$$

Then by (5.30), there exists a constant $M^{\star\star}$ such that $\mu(t)\leq M^{\star\star}$, thus $\|y\|\leq\mu(t)$. Since for every $t\in[0,b]$, we have $\|y\|\leq\max\{\|\varphi\|, M^{\star\star}\} := \Theta_b$.

Set

$$Y = \{\ y\in C([-r,b];E)\ :\ \sup\{|y(t)| : 0\leq t\leq b\}\leq\Theta_b+1\}.$$

Clearly, Y is a closed subset of $C([-r,b];E)$.

We shall show that $N : Y \to C([-r, b]; E)$ is a contraction operator.
Indeed, consider $y, \overline{y} \in Y$, thus using $(H^{'}1)$ and $(H^{'}3)$ for each $t \in [0, b]$, we get

$$\begin{aligned}
|N(y)(t) - N(\overline{y})(t)| &\leq |g(t, y_{\rho(t,y_t)}) - g(t, \overline{y}_{\rho(t,y_t)})| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \left| f(s, y_{\rho(s,y_s)}) - f(s, \overline{y}_{\rho(s,y_s)}) \right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \|C\| \, |u_y(s) - u_{\overline{y}}| \, ds \\
&\leq |g(t, y_{\rho(t,y_t)}) - g(t, \overline{y}_{\rho(t,y_t)})| \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, l_b(s) \, \|y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)}\| \, ds \\
&+ \frac{\widehat{M}\widetilde{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \widetilde{M}_1 \widehat{M} \int_0^b |f(\tau, y_{\rho(\tau,y_\tau)}) - f(\tau, \overline{y}_{\rho(\tau,y_\tau)})| \, d\tau \, ds
\end{aligned}$$

Using (H_φ) and Lemma 5.2.1, we obtain

$$\begin{aligned}
|N(y)(t) - N(\overline{y})(t)| &\leq \|A^{-1}(t)\| |A(t) g(t, y_{\rho(t,y_t)}) - A(t) g(t, \overline{y}_{\rho(t,y_t)})| \\
&+ \frac{\widehat{M} l_b^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, |y(s) - \overline{y}(s)| \, ds \\
&+ \frac{\widehat{M^2} \widetilde{M} \widetilde{M}_1 l_b^* b^\alpha}{\Gamma(\alpha+1)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, |y(s) - \overline{y}(s)| \, ds \\
&\leq \overline{M}_0 L_\star |y(t) - \overline{y}(t)| + \frac{\widehat{M} l_b^* b^\alpha}{\Gamma(\alpha+1)} \|y - \overline{y}\|_n + \frac{\widehat{M^2} \widetilde{M} \widetilde{M}_1 l_b^* b^\alpha}{\Gamma(\alpha+1)\Gamma(\alpha)} \|y - \overline{y}\|
\end{aligned}$$

Therefore,

$$\|N(y) - N(\overline{y})\|_n \leq \left[\overline{M}_0 L_\star + \frac{\widehat{M} l_b^* b^\alpha \left(\Gamma(\alpha) + \widehat{M}\widetilde{M}\widetilde{M}_1 \right)}{\Gamma(\alpha+1)\Gamma(\alpha)} \right] \|y - \overline{y}\|.$$

So, for $\left[\overline{M}_0 L_\star + \dfrac{\widehat{M} l_b^* b^\alpha (\Gamma(\alpha) + \widehat{M}\widetilde{M}\widetilde{M}_1)}{\Gamma(\alpha+1)\Gamma(\alpha)} \right] < 1$, the operator N is a contraction. From the choice of Y there is no $y \in \partial Y^n$ such that $y = \lambda \, N(y)$, $\lambda \in (0, 1)$. Then the statement $(C2)$ in Theorem 1.10.4 does not hold. The nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. Thus, we deduce that the operator N has a unique fixed-point $y^\star$ which is the unique controllable mild solution of the problem $(5.7) - (5.8)$.

5.6 Fractional Partial Problem with Infinite Delay

Before stating and proving the main result, we give first the definition of mild solution of the semilinear evolution problem $(5.9) - (5.10)$.

Lemma 5.6.1. *The system* $(5.9) - (5.10)$ *is equivalent to the nonlinear integral equation*

$$y(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s) y(s) \, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} C u(s) ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_{\rho(s,y_s)})\ ds. \tag{5.33}$$

In other words, every solution of the integral equation (5.33) *is also solution of the system* (5.9) – (5.10) *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system (5.9) – (5.10), and using some classical results from fractional calculus to get (5.33).

Definition 5.6.1. *We say that the function $y(\cdot) : (-\infty, b] \to E$ is a mild solution of* (5.9) – (5.10) *if $y(t) = \phi(t)$ for all $t \leq 0$ and y satisfies the following integral equation*

$$\begin{aligned} y(t) &= U(t,0)\ \phi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} U(t,s) Cu(s) ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s, y_{\rho(s,y_s)})\ ds \quad t \in J. \end{aligned} \tag{5.34}$$

Definition 5.6.2. *The neutral evolution problem* (5.9) – (5.10) *is said to be controllable on $[-\infty, b]$ if for every initial function $\phi \in \mathcal{B}$ and $y_1 \in E$ there exists a control $u \in L^2(J; E)$ such that the mild solution $y(\cdot)$ of* (5.9) – (5.10) *satisfies $y(b) = y_1$.*

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\phi) : (s,\phi) \in J \times \mathcal{B}, \rho(s,\phi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq \mathcal{L}^\phi(t) \|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

We will need to introduce the following hypothesis which are assumed thereafter

$(H01)$ There exists a constant $\widehat{M} \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq \widehat{M} \quad \text{for every } (t,s) \in \Delta.$$

$(H02)$ There exist two functions $p, q \in L^1_{loc}(J; \mathbb{R}_+)$ such that

$$|f(t,u)| \leq p(t) + q(t)\ \|u\|_B \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$

$(H03)$ For all $R > 0$, there exists $l_R \in L^1_{loc}(J; \mathbb{R}_+)$ such that

$$|f(t,u) - f(t,v)| \leq l_R(t)\ \|u - v\|$$

for all $u, v \in \mathcal{B}$ with $\|u\| \leq R$ and $\|v\| \leq R$.

Consider the following space

$$\Omega = \left\{ y : (-\infty, b] \to E : y|_{(-\infty,0]} \in B \ \ and \ \ y|_{[0,b]} \text{is continuous } \right\},$$

$(H04)$ The linear operator $W : L^2(J;E) \to E$ is defined by

$$Wu = \int_0^b U(b,s)Cu(s)ds,$$

has an induced invertible operator $\tilde{W}^{-1}$ which takes values in $L^2(J;E)/\ker W$ and there exists positive constants $\widetilde{M}$ and $\widetilde{M_1}$ such that

$$\|C\| \leq \widetilde{M} \quad \text{and} \quad \|\tilde{W}^{-1}\| \leq \widetilde{M_1}.$$

Remark 5.6.1. *For the construction of W and $\tilde{W}^{-1}$ see the paper by Carmichael and Quinn [38].*

Theorem 5.6.1. *Suppose that the hypotheses $(H01)-(H04)$ are satisfied and moreover there exists a constant $M_\star > 0$ such that*

$$\frac{M_\star}{\sigma_b + \dfrac{K_b\widehat{M}q^*[\widehat{M}\widetilde{M}\widetilde{M_1} + \Gamma(\alpha)]}{\alpha\Gamma(\alpha)^2} M_\star\ b^\alpha} > 1 \tag{5.35}$$

with $\sigma_b := c_b + \dfrac{K_b\widehat{M}\widetilde{M}\widetilde{M_1}b}{\Gamma(\alpha)}\left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}}\right] + \dfrac{K_b\widehat{M}p^*b^\alpha[\alpha\widehat{M}\widetilde{M}\widetilde{M_1}b + \Gamma(\alpha)]}{\alpha\Gamma(\alpha)^2}$. *Then the problem $(5.9)-(5.10)$ is controllable on $(-\infty, b]$.*

Proof. Transform the problem $(5.9)-(5.10)$ into a fixed-point problem. Consider the operator $N : B_b \to B_b$ defined by :

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ U(t,0)\ \phi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} U(t,s)\ C\ u_y(s)\ ds & \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} U(t,s)\ f(s, y_{\rho(s,y_s)})\ ds, & \text{if } t \in J. \end{cases} \tag{5.36}$$

Using assumption $(H03)$, for arbitrary function $y(\cdot)$, we define the control

$$u_y(t) = \tilde{W}^{-1}\left[y_1 - U(b,0)\ \phi(0) - \frac{1}{\Gamma(\alpha)}\int_0^b (t-s)^{\alpha-1}\ U(b,s)\ f(s, y_{\rho(s,y_s)})\ ds\right](t).$$

Noting that, we have

$$\begin{aligned} |u_y(t)| \ \leq\ & \|\tilde{W}^{-1}\| \left[|y_1| + \|U(t,0)\|_{B(E)}|\phi(0)|\right. \\ & \left. +\frac{1}{\Gamma(\alpha)} \int_0^b (t-\tau)^{\alpha-1}\|U(b,\tau)\|_{B(E)}|f(\tau, y_{\rho(\tau,y_\tau)})|d\tau\right]. \end{aligned}$$

From $(H02)$, we get

$$|u_y(t)| \ \leq\ \widetilde{M_1}\left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} + \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^b (t-\tau)^{\alpha-1}\left[p(\tau) + q(\tau)\|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}}\right]\ ds\right] \tag{5.37}$$

Clearly, fixed points of the operator N are mild solutions of the problem (5.9) – (5.10).

For $\phi \in \mathcal{B}$, we will define the function $x(.) : (-\infty, b] \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ U(t,0)\ \phi(0), & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_b$, set

$$y(t) = z(t) + x(t). \tag{5.38}$$

It is obvious that y satisfies (5.34) if and only if z satisfies $z_0 = 0$ and for $t \in J$, we have

$$\begin{aligned} z(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s)\ C\ u_{z+x}(s)\ ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds \end{aligned}$$

Let

$$B_b^0 = \{z \in B_b : z_0 = 0\}.$$

For any $z \in B_b^0$ we have

$$\|z\|_b = \sup\{\ |z(t)|\ :\ t \in J\ \} + \|z_0\|_{\mathcal{B}} = \sup\{\ |z(t)|\ :\ t \in J\ \}.$$

Thus $(B_b^0, \|\cdot\|_b)$ is a Banach space.

Define the operators $F, G : B_b^0 \to B_b^0$ by :

$$F(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s)\ C\ u_{z+x}(s)\ ds \tag{5.39}$$

and

$$G(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds \quad \text{for } t \in J. \tag{5.40}$$

Obviously the operator N has a fixed point is equivalent to $F + G$ has one, so it turns to prove that $F + G$ has a fixed point. The proof will be given in several steps.

Let us first show that the operator F is continuous and compact.

Step 1 : F is continuous.

Let $(z_n)_n$ be a sequence in B_b^0 such that $z_n \to z$ in B_b^0. Then using (5.37), we get

$$\begin{aligned} |F(z_n)(t) - F(z)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)}\ \|C\|\ |u_{z_n}(s) - u_z(s)|\ ds \\ &\leq \frac{\widehat{M}\widetilde{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \widetilde{M}_1 \widehat{M} \times \\ &\quad \times \int_0^b |f(\tau, z_{n\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}) - f(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(s,z_\tau+x_\tau)})|\ d\tau\ ds \\ &\leq \frac{\widehat{M}^2 \widetilde{M}\widetilde{M}_1 b^{\alpha+1}}{\Gamma(\alpha+1)} \times \\ &\quad \times \int_0^b |f(s, z_{n\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|\ ds \end{aligned}$$

Since f is L^1-Carathéodory, we obtain by the Lebesgue dominated convergence theorem

$$|F(z_n)(t) - F(z)(t)| \to 0 \quad \text{as } \ n \to +\infty.$$

Thus F is continuous.

Step 2 : F maps bounded sets of B_b^0 into bounded sets. For any $d > 0$, there exists a positive constant ℓ such that for each $z \in B_d = \{z \in B_b^0 : \|z\|_n \leq d\}$ one has $\|F(z)\|_n \leq \ell$.

Let $z \in B_d$. By (5.37), we have for each $t \in J$

$$\begin{aligned}
|F(z)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \; \|C\| \; |u_{z+x}(s)| \; ds \\
&\leq \frac{\widehat{M}\widetilde{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u_{z+x}(s)| \; ds \\
&\leq \frac{\widehat{M}\widetilde{M}}{\Gamma(\alpha)} \int_0^t \widetilde{M}_1 \Bigg[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} \\
&\quad + \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} \left[p(\tau) + q(\tau) \|z_{\rho(\tau, z_\tau + x_\tau)} + x_{\rho(\tau, z_\tau + x_\tau)}\| d\tau \right] \Bigg] ds \\
&\leq \frac{\widehat{M}\widetilde{M}\widetilde{M}_1 b}{\Gamma(\alpha)} \Bigg[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} + \frac{\widehat{M}p^* b^\alpha}{\Gamma(\alpha+1)} \times \\
&\quad \times + \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} q(s) \|z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\| ds \Bigg]
\end{aligned}$$

From (H_ϕ), Lemma 1.7.2 and Assumption $(A1)$, we have for each $t \in [0, b]$

$$\begin{aligned}
\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\leq \|z_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + \|x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \\
&\leq K_b|z(s)| + (M_b + \mathcal{L}^\phi)\|z_0\|_{\mathcal{B}} + K_b|x(s)| + (M_b + \mathcal{L}^\phi)\|x_0\|_{\mathcal{B}} \\
&\leq K_b|z(s)| + K_b\|U(s,0)\|_{B(E)}|\phi(0)| + (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} \\
&\leq K_b|z(s)| + K_b\widehat{M}|\phi(0)| + (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}
\end{aligned}$$

Using (ii), we get

$$\begin{aligned}
\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\leq K_b|z(s)| + K_b\widehat{M}H\|\phi\|_{\mathcal{B}} + (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} \\
&\leq K_b|z(s)| + (M_b + \mathcal{L}^\phi + K_b\widehat{M}H)\|\phi\|_{\mathcal{B}}
\end{aligned}$$

Set $C_b := (M_b + \mathcal{L}^\phi + K_b\widehat{M}H)\|\phi\|_{\mathcal{B}}$ and $\delta_b := K_b d + C_b$. Then

$$\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \leq K_b|z(s)| + C_b \leq \delta_b. \tag{5.41}$$

For each $t \in [0, b]$, it follows that

$$|F(z)(t)| \leq \frac{\widehat{M}\widetilde{M}\widetilde{M}_1 b}{\Gamma(\alpha)} \left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}} + \frac{\widehat{M}b^\alpha(p^* + q^*\delta_b)}{\Gamma(\alpha+1)} \right] := \ell.$$

Thus there exists a positive number ℓ such that

$$\|F(z)\|_n \leq \ell.$$

Hence $F(B_d) \subset B_d$.

Step 3 : F maps bounded sets into equi-continuous sets of B_b^0. We consider B_d as in Step 2 and we show that $F(B_d)$ is equi-continuous.

Let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$ and $z \in B_d$. Then

$$\begin{aligned}|F(z)(\tau_2) - F(z)(\tau_1)| \leq \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}| \|U(\tau_2, s) - U(\tau_1, s)\| \times \\ \times \|C\| \; |u_{z+x}(s)| \; ds + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{\alpha-1}| \|U(\tau_2, s)\|_{B(E)} \; \|C\| \; |u_{z+x}(s)| \; ds.\end{aligned}$$

By the inequalities (5.37) and (5.41) we get

$$|u_{z+x}(t)| \leq \widetilde{M}_1 \left[|y_1| + \widehat{M} H \|\phi\|_{\mathcal{B}} + \frac{\widehat{M} b^\alpha (p^* + q^* \delta_b)}{\Gamma(\alpha+1)} \right] := \omega. \tag{5.42}$$

Then

$$\begin{aligned}|F(z)(\tau_2) - F(z)(\tau_1)| \;\; &\leq \;\; \frac{\|C\|_{B(E)} \; \omega}{\Gamma(\alpha)} \int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}| \|U(\tau_2, s) - U(\tau_1, s)\| ds \\ &+ \;\; \frac{\|C\|_{B(E)} \; \omega}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{\alpha-1}| \|U(\tau_2, s)\|_{B(E)} ds.\end{aligned}$$

Noting that $|F(z)(\tau_2) - F(z)(\tau_1)|$ tends to zero as $\tau_2 - \tau_1 \to 0$ independently of $z \in B_d$. The right-hand side of the above inequality tends to zero as $\tau_2 - \tau_1 \to 0$. Since $U(t, s)$ is a strongly continuous operator and the compactness of $U(t, s)$ for $t > s$ implies the continuity in the uniform operator topology (see [7, 90]). As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that the operator F maps B_d into a precompact set in E.
Let $t \in J$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_d$ we define

$$F_\epsilon(z)(t) \;\; = \;\; \frac{1}{\Gamma(\alpha)} U(t, t-\epsilon) \int_0^{t-\epsilon} (t - s)^{\alpha-1} U(t-\epsilon, s) \; C \; u_{z+x}(s) \; ds$$

Since $U(t, s)$ is a compact operator, the set $Z_\epsilon(t) = \{F_\epsilon(z)(t) : \;\; z \in B_d\}$ is pre-compact in E for every ϵ sufficiently small, $0 < \epsilon < t$. Moreover using (5.42), we have

$$\begin{aligned}|F(z)(t) - F_\epsilon(z)(t)| \;\; &\leq \;\; \frac{1}{\Gamma(\alpha)} \int_{t-\epsilon}^{t} (t - s)^{\alpha-1} \|U(t, s)\|_{B(E)} \; \|C\| \; |u_{z+x}(s)| \; ds \\ &\leq \;\; \frac{\|C\|_{B(E)} \; \omega}{\Gamma(\alpha)} \int_{t-\epsilon}^{t} (t - s)^{\alpha-1} \|U(t, s)\|_{B(E)} \; ds.\end{aligned}$$

Therefore there are precompact sets arbitrary close to the set $\{F(z)(t) : z \in B_d\}$. Hence the set $\{F(z)(t) : z \in B_d\}$ is precompact in E. So we deduce from Steps 1, 2 and 3 that F is a compact operator.

Step 4 : G is a contraction. Let $z, \overline{z} \in \Omega^0$. By the hypotheses $(H01)$ and $(H03)$, we get for each $t \in [0, b]$

$$\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \times \\
&\times \mid f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - f(s, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \mid ds \\
&\leq \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l_b(s) \; \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)} - x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \; ds \\
&\leq \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l_b(s) \; \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \; ds.
\end{aligned}$$

Use the inequality (5.41), for get

$$\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| &\leq \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_b l_b(s) \; |z(s) - \overline{z}(s)| \; ds \\
&\leq \frac{\widehat{M} K_b l_b^* b^\alpha}{\Gamma(\alpha+1)} \|z(t) - \overline{z}(t)\|
\end{aligned}$$

Therefore

$$\|G(z) - G(\overline{z})\|_n \leq \frac{\widehat{M} K_b l_b^* b^\alpha}{\Gamma(\alpha+1)} \|z(t) - \overline{z}(t)\|.$$

Then the operator G is a contraction.

Step 5 : For applying Theorem (1.10.2), we must check $(Av2)$: i.e. it remains to show that the set

$$\Sigma = \left\{ z \in \Omega^0 \;: z = \lambda F(z) + \lambda G\left(\frac{z}{\lambda}\right) \quad \text{for some } \lambda \in]0,1[\right\}.$$

is bounded.

Let $z \in \Sigma$. By $(H01) - (H02)$ and $(H04)$, we have for each $t \in [0, b]$

$$\begin{aligned}
\frac{1}{\lambda}|z(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \|C\| \; |u_{z+x}(s)| \; ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \; \left| f(s, z_{\rho\left(s, \frac{z_s}{\lambda}+x_s\right)} + x_{\rho\left(s, \frac{z_s}{\lambda}+x_s\right)}) \right| \; ds \\
&\leq \frac{\widehat{M}\widetilde{M}\widetilde{M}_1 b}{\Gamma(\alpha)} \Big[|y_1| + \widehat{M} H \|\phi\|_{\mathcal{B}} \\
&\qquad \times + \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} \left[p(\tau) + q(\tau) \|z_{\rho(\tau,.)} + x_{\rho(\tau,.)}\| d\tau \right] ds \Big] \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \; \left[p(s) + q(s) \|z_{\rho\left(s, \frac{z_s}{\lambda}+x_s\right)} + x_{\rho\left(s, \frac{z_s}{\lambda}+x_s\right)}\|_{\mathcal{B}} \right] \; ds
\end{aligned}$$

Lemma 1.7.2 and inequality (5.41)

$$\begin{aligned}
\frac{1}{\lambda}|z(t)| &\leq \frac{\widehat{M}\widetilde{M}\widetilde{M_1}b}{\Gamma(\alpha)}\left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}}\right] \\
&+ \frac{\widehat{M^2}\widetilde{M}\widetilde{M_1}b}{\Gamma(\alpha)^2}\int_0^b (t-s)^{\alpha-1}\ \left[p(s) + q(s)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\right]\ ds \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ \left[p(s) + q(s)\|z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)} + x_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}\|_{\mathcal{B}}\right]\ ds
\end{aligned}$$

We consider the function $u(t) := \sup_{\theta\in[0,b]} |z(\theta)|$ and use the fact that $0 < \lambda < 1$

$$\begin{aligned}
\frac{K_b}{\lambda}u(t) + C_b &\leq C_b + \frac{K_b\widehat{M}\widetilde{M}\widetilde{M_1}b}{\Gamma(\alpha)}\left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}}\right] + \frac{K_b\widehat{M}p^* b^\alpha[\alpha\widehat{M}\widetilde{M}\widetilde{M_1}b + \Gamma(\alpha)]}{\alpha\Gamma(\alpha)^2} \\
&+ \frac{K_b\widehat{M^2}\widetilde{M}\widetilde{M_1}}{\Gamma(\alpha)^2}\int_0^b (t-s)^{\alpha-1}q(s)\left(\frac{K_b}{\lambda}|z(s)| + C_b\right)\ ds \\
&+ \frac{K_b\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}q(s)\ \left(\frac{K_b}{\lambda}|z(s)| + C_b\right)\ ds
\end{aligned}$$

Set $\sigma_b := C_b + \dfrac{K_b\widehat{M}\widetilde{M}\widetilde{M_1}b}{\Gamma(\alpha)}\left[|y_1| + \widehat{M}H\|\phi\|_{\mathcal{B}}\right] + \dfrac{K_b\widehat{M}p^* b^\alpha[\alpha\widehat{M}\widetilde{M}\widetilde{M_1}b + \Gamma(\alpha)]}{\alpha\Gamma(\alpha)^2}$. Then, we have

$$\begin{aligned}
\frac{K_b}{\lambda}u(t) + C_b &\leq \sigma_b + \frac{K_b\widehat{M^2}\widetilde{M}\widetilde{M_1}}{\Gamma(\alpha)^2}\int_0^b (t-s)^{\alpha-1}q(s)\left(\frac{K_b}{\lambda}|z(s)| + c_b\right)\ ds \\
&+ \frac{K_b\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}q(s)\ \left(\frac{K_b}{\lambda}|z(s)| + C_b\right)\ ds.
\end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\ \{\frac{K_b}{\lambda}u(s) + C_b\ :\ 0 \leq s \leq t\},\quad 0 \leq t \leq b$$

Let $t^\star \in [0,t]$ such that $\mu(t) = \dfrac{K_b}{\lambda}u(t^\star) + C_b$. From the previous inequality, we have for all $t \in [0,b]$

$$\mu(t) \leq \sigma_b + \frac{K_b\widehat{M^2}\widetilde{M}\widetilde{M_1}q^*}{\Gamma(\alpha)^2}\int_0^b (t-s)^{\alpha-1}\mu(s)ds + \frac{K_b\widehat{M}q^*}{\Gamma(\alpha)}\int_0^b (t-s)^{\alpha-1}\mu(s)ds.$$

Then, we have

$$\mu(t) \leq \sigma_b + \frac{K_b\widehat{M}q^*[\widehat{M}\widetilde{M}\widetilde{M_1} + \Gamma(\alpha)]}{\Gamma(\alpha)^2}\int_0^b (t-s)^{\alpha-1}\mu(s)ds.$$

Consequently,

$$\frac{\|z\|_n}{\sigma_b + \dfrac{K_b\widehat{M}q^*[\widehat{M}\widetilde{M}\widetilde{M_1} + \Gamma(\alpha)]}{\alpha\Gamma(\alpha)^2}\|z\|_n\ b^\alpha} \leq 1.$$

Then by (5.35), there exists a constant $M^\star$ such that $\|z\|_n \neq M^\star$. Set

$$\mathcal{U} = \{\ z \in B_b^0\ :\ \|z\|_n\ \leq\ M^\star + 1\ \}.$$

Clearly, $\mathcal{U}$ is a closed subset of B_b^0. From the choice of $\mathcal{U}$ there is no $z \in \partial\mathcal{U}$ such that $z = \lambda F(z) + \lambda G\left(\frac{z}{\lambda}\right)$ some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 1.10.5 does not hold. As a consequence of the nonlinear alternative of Avramescu ([18]), we deduce that $(C1)$ holds : i.e. the operator $F+G$ as a fixed-point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty, b]$ is a fixed point of the operator N, which is a mild solution of the problem $(5.9) - (5.10)$. Thus the evolution system $(5.9) - (5.10)$ is controllable on $(-\infty, b]$.

5.7 Fractional Neutral Problem with Infinite Delay

In this section, we give controllability result for the neutral functional fractional differential evolution problem with infinite state-dependent delay $(5.11) - (5.12)$. Firstly, we define its mild solution.

Lemma 5.7.1. *The system* $(5.11) - (5.12)$ *is equivalent to the nonlinear integral equation*

$$\begin{aligned} y(t) &= [\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} A(s) y(s)\ ds \qquad (5.43) \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t U(t,s) C u(s) ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_{\rho(s,y_s)})\ ds. \end{aligned}$$

In other words, every solution of the integral equation (5.43) *is also solution of the system* $(5.11) - (5.12)$ *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system $(5.11) - (5.12)$, and using some classical results from fractional calculus to get (5.43).

Definition 5.7.1. *We say that the function* $y(\cdot) : (-\infty, b] \to E$ *is a mild solution of* $(5.11) - (5.12)$ *if* $y(t) = \phi(t)$ *for all* $t \leq 0$ *and* y *satisfies the following integral equation*

$$\begin{aligned} y(t) &= U(t,0)[\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s, y_{\rho(s,y_s)})\ ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} U(t,s) C u(s) ds \qquad \text{for each } t \in J. \end{aligned} \qquad (5.44)$$

Definition 5.7.2. *The neutral evolution problem* $(5.11) - (5.12)$ *is said to be controllable on the* $(-\infty, b]$ *if for every initial function* $\phi \in \mathcal{B}$ *and* $y_1 \in E$ *there exists a control* $u \in L^2(J;E)$ *such that the mild solution* $y(\cdot)$ *of* $(5.11) - (5.12)$ *satisfies* $y(b) = y_1$.

We consider the hypotheses (H_ϕ), $(H01) - (H03)$ and we need the following assumptions

$(H05)$ There exists a constant $L_\star > 0$ such that

$$|A(s)g(s,\phi) - A(\overline{s})g(\overline{s},\overline{\phi})| \leq L_\star\ (|s - \overline{s}| + \|\phi - \overline{\phi}\|)$$

for all $\phi, \overline{\phi} \in \mathcal{B}$.

$(H06)$ There exists a constant $\overline{M}_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0 \quad \text{for all} \;\; t \in J.$$

$(H07)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0 K_b}$ such that

$$|A(t)\; g(t,\phi)| \leq L\; (\|\phi\|_{\mathcal{B}} + 1) \text{ for all } t \in J \text{ and } \phi \in \mathcal{B}.$$

$(H08)$ The function g is completely continuous and for any bounded set $Q \subseteq \mathcal{B}$ the set $\{t \to g(t, x_t) : x \in Q\}$ is equi-continuous in $C(J; E)$.

Theorem 5.7.1. *Suppose that the hypotheses* $(H01) - (H08)$ *are satisfied and moreover there exists a constant* $M^{\star\star} > 0$ *with*

$$\frac{M^{\star\star}}{\delta_b + \dfrac{K_b\widehat{M}(\widehat{M}\widetilde{M}\widetilde{M}_1 b^\alpha + \Gamma(\alpha+1))}{(1-\overline{M}_0 L K_b)\Gamma^2(\alpha+1)}\;[q^* b^\alpha M^{\star\star}]} \;>\; 1, \tag{5.45}$$

where $\delta_b := C_b + \frac{K_b\sigma_b}{(1-\overline{M}_0 L K_b)}$. *and*

$$\begin{aligned}
\sigma_b &= K_b\overline{M}_0 L(C_b + 1 + \widehat{M}(\|\phi\|_{\mathcal{B}} + 1)) \\
&+ \frac{K_b\widehat{M}\widetilde{M}\widetilde{M}_1 b^\alpha}{\Gamma(\alpha+1)}\left[\,|y_1| + \widehat{M}\left(H + \overline{M}_0 L\right)\|\phi\|_{\mathcal{B}} + \left(\widehat{M}+1\right)\overline{M}_0 L\right] \\
&+ \frac{K_b\widehat{M}\widetilde{M}\widetilde{M}_1 b^\alpha \overline{M}_0 L}{\Gamma(\alpha+1)}(K_b|y_1| + (M_b + L^\phi)\|\phi\|_{\mathcal{B}}) \\
&+ \frac{K_b\widehat{M}p^* b^\alpha[\Gamma(\alpha+1) + \widehat{M}\widetilde{M}\widetilde{M}_1\widetilde{M}_1 b^\alpha]}{\Gamma^2(\alpha+1)}.
\end{aligned}$$

Then the neutral evolution problem $(5.11) - (5.12)$ *is controllable on* $(-\infty, b]$.

Proof. Consider the operator $\widetilde{N} : B_b \to B_b$ defined by :

$$\widetilde{N}(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ U(t,0)\;[\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) & \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} U(t,s)\; C\; u_y(s)\; ds & \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} U(t,s)\; f(s, y_{\rho(s,y_s)})\; ds, & \text{if } t \in J. \end{cases} \tag{5.46}$$

Using assumption $(H3)$, for arbitrary function $y(\cdot)$, we define the control

$$\begin{aligned}
u_y(t) &= \tilde{W}^{-1}\left[y_1 - U(b,0)\,(\phi(0) - g(0,\phi)) - g(b, y_{\rho(b,y_b)})\right. \\
&- \left.\frac{1}{\Gamma(\alpha)}\int_0^b (t-s)^{\alpha-1} U(b,s) f(s, y_{\rho(s,y_s)}) ds\right](t).
\end{aligned}$$

Noting that

$$\begin{aligned}
|u_y(t)| &\leq \|\tilde{W}^{-1}\| \Big[\ |y_1| + \|U(t,0)\|_{B(E)} \left(|\phi(0)| + \|A^{-1}(0)\| |A(0)g(0,\phi)|\right) \\
& \quad + \|A^{-1}(b)\| |A(b)g(b, y_{\rho(b,y_b)})| \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} U(b,\tau) f(\tau, y_{\rho(\tau,y_\tau)}) d\tau \Big] \\
&\leq \widetilde{M_1} \Big[\ |y_1| + \widehat{M} H \|\phi\|_{\mathcal{B}} + \widehat{M}\overline{M}_0 L(\|\phi\|_{\mathcal{B}} + 1) + \overline{M}_0 L(\|y_{\rho(b,y_b)}\|_{\mathcal{B}} + 1) \Big] \\
&+ \frac{\widetilde{M_1}\widehat{M}}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} \left[p(\tau) + q(\tau) \|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}}\right].
\end{aligned}$$

we get

$$\begin{aligned}
|u_y(t)| &\leq \widetilde{M_1} \Big[\ |y_1| + \widehat{M} \left(H + \overline{M}_0 L\right) \|\phi\|_{\mathcal{B}} + \left(\widehat{M} + 1\right) \overline{M}_0 L \Big] \\
&+ \widetilde{M_1} \overline{M}_0 L \|y_{\rho(b,y_b)}\|_{\mathcal{B}} + \frac{\widehat{M}\widetilde{M_1} p^* b^\alpha}{\Gamma(\alpha+1)} + \frac{\widehat{M}\widetilde{M_1} q^*}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} \ \|y_{\rho(s,y_s)}\|_{\mathcal{B}} \ ds
\end{aligned} \tag{5.47}$$

We shall show that using this control the operator $\widetilde{N}$ has a fixed point $y(\cdot)$, which is a mild solution of the neutral evolution system (5.11) – (5.12).

For $\phi \in \mathcal{B}$, we will define the function $x(.) : (-\infty, b] \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \in (-\infty, 0]; \\ U(t,0) \ \phi(0), & \text{if } \ t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_b$, set

$$y(t) = z(t) + x(t). \tag{5.48}$$

It is obvious that y satisfies (5.44) if and only if z satisfies $z_0 = 0$ and for $t \in J$, we get

$$\begin{aligned}
z(t) &= g(t, z_t + x_t) - U(t,0)g(0,\phi) + \int_0^t U(t,s)A(s)g(s, z_s + x_s)ds \\
&+ \int_0^t U(t,s)Cu_z(s)ds + \int_0^t U(t,s)f(s, z_s + x_s)ds.
\end{aligned}$$

Define the operators $\widetilde{F}, \widetilde{G} : B_b^0 \to B_b^0$ by :

$$\begin{aligned}
\widetilde{F}(z)(t) &= g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - U(t,0)g(0,\phi) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) \ C \ u_{z+x}(s) \ ds
\end{aligned}$$

$$\widetilde{G}(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) \ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \ ds.$$

Obviously the operator $\widetilde{N}$ has a fixed point is equivalent to $\widetilde{F} + \widetilde{G}$ has one, so it turns to prove that $\widetilde{F} + \widetilde{G}$ has a fixed point. We can show as in Section 3 that the operator $\widetilde{F}$ is continuous and compact and $\widetilde{G}$ is a contraction. To apply Theorem 1.10.5, we must check $(C2)$, i.e., it remains to show that the set

$$\Sigma = \left\{ z \in \Omega^0 \ : z = \lambda \widetilde{F}(z) + \lambda \widetilde{G}\left(\frac{z}{\lambda}\right) \quad \text{for some } \lambda \in]0,1[\right\}.$$

is bounded.

Let $z \in \Sigma$. By $(H01)$ to $(H05)$ and (5.47), we have for each $t \in J$

$$\begin{aligned}
\frac{1}{\lambda}|z(t)| &\leq \|A^{-1}(t)\| \ |A(t)g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})| + \|U(t,0)\|_{B(E)} \ \|A^{-1}(0)\| \ |A(0)g(0,\phi)| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \|C\| \ |u_{z+x}(s)| \ ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \ \left| f(s, z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)} + x_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}) \right| \ ds \\
&\leq \overline{M}_0 L \left(\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} + 1 \right) + \widehat{M}\overline{M}_0 L \left(\|\phi\|_{\mathcal{B}} + 1 \right) \\
&+ \frac{\widehat{M}\widetilde{M}\widetilde{M}_1 b^\alpha}{\Gamma(\alpha+1)} \Big[\ |y_1| + \widehat{M}\left(H + \overline{M}_0 L\right) \|\phi\|_{\mathcal{B}} + \left(\widehat{M}+1\right) \overline{M}_0 L \Big] \\
&+ \frac{\widehat{M}\widetilde{M}\widetilde{M}_1 b^\alpha \overline{M}_0 L}{\Gamma(\alpha+1)} \|z_{\rho(b,y_b)} + x_{\rho(b,y_b)}\|_{\mathcal{B}} + \frac{\widehat{M} p^* b^\alpha [\Gamma(\alpha+1) + \widehat{M}\widetilde{M}\widetilde{M}_1 \widetilde{M}_1 b^\alpha]}{\Gamma^2(\alpha+1)} \\
&+ \frac{\widehat{M^2}\widetilde{M}\widetilde{M}_1 b^\alpha}{\Gamma(\alpha)\Gamma(\alpha+1)} \int_0^b (t-s)^{\alpha-1} \ q(s) \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \ ds \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \ \int_0^t (t-s)^{\alpha-1} \ q(s) \|z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)} + x_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}\|_{\mathcal{B}} \ ds
\end{aligned}$$

Noting that we have $\|z_{\rho(b,y_b)} + x_{\rho(b,y_b)}\|_{\mathcal{B}} \leq K_b|y_1| + (M_b + L^\phi)\|\phi\|_{\mathcal{B}}$ and using the first inequality $\||z_{\rho(t,y_t)} + x_{\rho(t,y_t)}\|_{\mathcal{B}} \leq K_b|z(t)| + C_b$ we obtain

$$\begin{aligned}
\frac{K_b}{\lambda}|z(t)| &\leq K_b \overline{M}_0 L (K_b|z(t)| + C_b + 1) + K_b \widehat{M}\overline{M}_0 L (\|\phi\|_{\mathcal{B}} + 1) \\
&+ \frac{K_b \widehat{M}\widetilde{M}\widetilde{M}_1 b^\alpha}{\Gamma(\alpha+1)} \Big[\ |y_1| + \widehat{M}\left(H + \overline{M}_0 L\right) \|\phi\|_{\mathcal{B}} + \left(\widehat{M}+1\right) \overline{M}_0 L \Big] \\
&+ \frac{K_b \widehat{M}\widetilde{M}\widetilde{M}_1 b^\alpha \overline{M}_0 L}{\Gamma(\alpha+1)} (K_b|y_1| + (M_b + L^\phi)\|\phi\|_{\mathcal{B}}) \\
&+ \frac{K_b \widehat{M} p^* b^\alpha [\Gamma(\alpha+1) + \widehat{M}\widetilde{M}\widetilde{M}_1 \widetilde{M}_1 b^\alpha]}{\Gamma^2(\alpha+1)} \\
&+ \frac{K_b \widehat{M^2}\widetilde{M}\widetilde{M}_1 b^\alpha}{\Gamma(\alpha)\Gamma(\alpha+1)} \int_0^b (t-s)^{\alpha-1} \ q(s)(K_b|z(s)| + C_b) ds \\
&+ \frac{K_b \widehat{M}}{\Gamma(\alpha)} \ \int_0^t (t-s)^{\alpha-1} \ q(s)\left(\frac{K_b}{\lambda}|z(s)| + C_b\right) ds
\end{aligned}$$

We consider the function $u(t) := \sup\limits_{\theta\in[0,b]} |z(\theta)|$ and use the fact that $0 < \lambda < 1$ Then, we have

$$\begin{aligned}\frac{K_b}{\lambda}u(t) &\leq \sigma_b + K_b\overline{M}_0 L\frac{|u(t)|}{\lambda} + \frac{K_b\widehat{M^2}\widetilde{M}\widetilde{M}_1 b^\alpha}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_0^b (t-s)^{\alpha-1}q(s)\left(\frac{K_b}{\lambda}|z(s)|+c_b\right)ds \\ &+ \frac{K_b\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}q(s)\left(\frac{K_b}{\lambda}|z(s)|+C_b\right)ds.\end{aligned}$$

Set:

$$\begin{aligned}\sigma_b &= K_b\overline{M}_0 L(C_b + 1 + \widehat{M}(\|\phi\|_{\mathcal{B}}+1)) \\ &+ \frac{K_b\widehat{M}\widetilde{M}\widetilde{M}_1 b^\alpha}{\Gamma(\alpha+1)}\left[\,|y_1| + \widehat{M}\left(H+\overline{M}_0 L\right)\|\phi\|_{\mathcal{B}} + \left(\widehat{M}+1\right)\overline{M}_0 L\right] \\ &+ \frac{K_b\widehat{M}\widetilde{M}\widetilde{M}_1 b^\alpha\overline{M}_0 L}{\Gamma(\alpha+1)}(K_b|y_1| + (M_b+L^\phi)\|\phi\|_{\mathcal{B}}) + \frac{K_b\widehat{M}p^* b^\alpha[\Gamma(\alpha+1)+\widehat{M}\widetilde{M}\widetilde{M}_1\widetilde{M}_1 b^\alpha]}{\Gamma^2(\alpha+1)}\end{aligned}$$

Then

$$\begin{aligned}\frac{K_b}{\lambda}(1-\overline{M}_0 L)u(t) &\leq K_b\sigma_b + \frac{K_b\widehat{M^2}\widetilde{M}\widetilde{M}_1 b^\alpha}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_0^b (t-s)^{\alpha-1}\, q(s)(K_b u(s)+C_b)ds \\ &+ \frac{K_b\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, q(s)(\frac{K_b}{\lambda}u(s)+C_b)ds\end{aligned}$$

with $\delta_b := C_b + \frac{K_b\sigma_b}{(1-\overline{M}_0 L)}$, thus

$$\begin{aligned}\frac{K_b}{\lambda}u(t)+C_b &\leq \delta_b + \frac{K_b\widehat{M^2}\widetilde{M}\widetilde{M}_1 b^\alpha}{(1-\overline{M}_0 L)\Gamma(\alpha)\Gamma(\alpha+1)}\int_0^b (t-s)^{\alpha-1}\, q(s)(K_b u(s)+C_b)ds \\ &+ \frac{K_b\widehat{M}}{(1-\overline{M}_0 L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, q(s)(\frac{K_b}{\lambda}u(s)+C_b)ds\end{aligned}$$

We consider the function μ defined by

$$\mu(t) := \sup\left\{\ \frac{K_b}{\lambda}|u(s)| + C_b\ :\ 0\leq s\leq t\ \right\},\quad 0\leq t\leq b.$$

Let $t^\star\in[0,t]$ be such that $\mu(t) = \dfrac{K_b}{\lambda}|u(t^\star)| + C_b$. If $t\in J$, by the previous inequality, we have for $t\in J$

$$\mu(t) \leq \delta_b + \frac{K_b\widehat{M}(\widehat{M}\widetilde{M}\widetilde{M}_1 b^\alpha + \Gamma(\alpha+1))}{(1-\overline{M}_0 L)\Gamma(\alpha)\Gamma(\alpha+1)}\int_0^b (t-s)^{\alpha-1}\, q(s)\mu(s)ds$$

Consequently,

$$\frac{\|z\|_b}{\delta_b + \dfrac{K_b\widehat{M}(\widehat{M}\widetilde{M}\widetilde{M}_1 b^\alpha + \Gamma(\alpha+1))}{(1-\overline{M}_0 L)\Gamma^2(\alpha+1)}\ [q^* b^\alpha\|z\|_b]} \leq 1.$$

Then by (5.45), there exists a constant $M^{\star\star}$ such that $\|z\|_b \neq M^{\star\star}$. Set

$$\widetilde{\mathcal{U}} = \{ \ z \in B_b^0 \ : \ \|z\|_b \ \leq \ M^{\star\star} + 1 \ \}.$$

Clearly, $\widetilde{\mathcal{U}}$ is a closed subset of B_b^0. From the choice of $\widetilde{\mathcal{U}}$ there is no $z \in \partial\widetilde{\mathcal{U}}$ such that $z = \lambda \ \widetilde{F}(z)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 1.10.5 does not hold. As a consequence of the nonlinear alternative of Avramescu ([18]), we deduce that $(C1)$ holds : i.e. the operator $\widetilde{F}$ has a fixed-point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty, b]$ is a fixed point of the operator $\widetilde{N}$, which is a mild solution of the problem (5.11) – (5.12). Thus the fractional neutral evolution system (5.11) – (5.12) is controllable on $(-\infty, b]$.

5.8 Examples

To illustrate the previous results, we give in this section six examples.

Example 1. Consider the partial differential equation

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,\xi) = \dfrac{\partial^2 z(t,\xi)}{\partial \xi^2} + d(\xi)u(t) + a_0(t,\xi)z(t,\xi) \\ + \displaystyle\int_{-\infty}^{0} a_1(s-t)z\left[s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|z(t,\theta)|^2 d\theta\right), \xi\right] ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad for \ t \geq 0, \ \xi \in [0,\pi], \\ z(t,0) = \ z(t,\pi) \ = 0, \qquad\qquad\qquad\qquad \text{for } t \geq 0, \\ z(\theta,\xi) = \ z_0(\theta,\xi), \qquad\qquad \text{for } -\infty < \theta \leq 0, \ \xi \in [0,\pi], \end{cases} \tag{5.49}$$

where $a : \mathbb{R}^+ \times [0,\pi] \to \mathbb{R}$ is a continuous function and is uniformly Hölder continuous in t ; $a_0 : \mathbb{R}^+ \times [0,\pi] \to \mathbb{R}$; $a_1 : \mathbb{R}^- \to \mathbb{R}$; $a_2 : [0,\pi] \to \mathbb{R}$; $\rho_i : \mathbb{R}^+ \to \mathbb{R}$ for $i = 1,2$; $z_0 : \mathbb{R}_- \times [0,\pi] \to \mathbb{R}$ and $d : \mathbb{R}_+ \to E$ are continuous functions. $u(\cdot) : \mathbb{R}_+ \to E$ is a given control.

To study this system, we consider the space $E = L^2([0,\pi],\mathbb{R})$ and the operator $A : D(A) \subset E \to E$ given by $Aw = w''$ with $D(A) := \{ \ w \in E \ : w'' \in E, \ w(0) = w(\pi) = 0 \ \}$. It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on E. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(\xi) = \frac{\sin(n\xi)\sqrt{2}}{\sqrt{\pi}}$. In addition, $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E and $T(t)x = \sum_{n=1}^{+\infty} e^{-n^2 t}(x, z_n)z_n$ for $x \in E$ and $t \geq 0$. It follows from this representation that $T(t)$ is compact for every $t > 0$ and that $\|T(t)\| \leq e^{-t}$ for every $t \geq 0$. On the domain $D(A)$, we define the operators $A(t) : D(A) \subset E \to E$ by

$$A(t)x(\xi) = Ax(\xi) + a_0(t,\xi)x(\xi).$$

By assuming that $a_0(.,.)$ is continuous and that $a_0(t,\xi) \leq -\delta_0$ ($\delta_0 > 0$) for every $t \in \mathbb{R}, \xi \in [0,\pi]$, it follows that the system

$$u'(t) = A(t)u(t) \quad t \geq s; \quad u(s) = x \in E,$$

has an associated evolution family given by $U(t,s)x(\xi) = \left[T(t-s)exp\left(\int_s^t a_0(\tau,\xi)d\tau\right)x\right](\xi)$.

From this expression, it follows that $U(t,s)$ is a compact linear operator and that

$$\|U(t,s)\| \leq e^{-(1+\delta_0)(t-s)} \text{ for every } (t,s) \in \Delta.$$

Let $\mathcal{B} = BUC(\mathbb{R}_-; E)$ the space of bounded uniformly continuous functions defined from $\mathbb{R}_-$ to E endowed with the uniform norm $\|\phi\| = \sup\limits_{\theta\in\mathbb{R}_-} |\phi(\theta)|$.

Theorem 5.8.1. *Let $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds and the functions $d : \mathbb{R}_+ \to E$, $\rho_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, 2$, $a_1 : \mathbb{R}^- \to \mathbb{R}$ and $a_2 : [0, \pi] \to \mathbb{R}$ are continuous. Then the evolution system* (5.49) *is controllable on* $(-\infty, +\infty)$.

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions and let $C \in L(\mathbb{R}; E)$ be defined as :
$Cu(t)(\xi) = d(\xi)u(t),\ t \geq 0,\ \xi \geq 0,\ u \in \mathbb{R},\ d(\xi) \in E,$
which permit to transform system (5.49) into the abstract system (5.1) $-$ (5.2). Moreover, the function f is a bounded linear operator. Now, the controllability of mild solutions can be deduced from a direct application of Theorem (1.10.2). Thus, the conclusion of our theorem hold.

From Remark (1.7.2), we have the following result.

Corollary 5.8.1. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then the evolution problem* (5.49) *is controllable on* $\mathbb{R}$.

Example 2. Consider the partial differential equation

$$\left\{\begin{array}{ll} \dfrac{\partial}{\partial t}\left[u(t,\xi) - \displaystyle\int_{-\infty}^{0} a_3(s-t)u\left(s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right), \xi\right) ds\right] & \\ = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) & \\ + \displaystyle\int_{-\infty}^{0} a_1(s-t)u\left(s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right), \xi\right) ds, & \\ \hfill for\ t \geq 0,\ \xi \in [0,\pi], & \\ v(t,0) = v(t,\pi) = 0, \hfill \text{for } t \geq 0, & \\ v(\theta,\xi) = v_0(\theta,\xi), \hfill \text{for } -\infty < \theta \leq 0,\ \xi \in [0,\pi], & \end{array}\right. \tag{5.50}$$

where $a_3 : \mathbb{R}^- \to \mathbb{R}$ is a continuous function and a, a_i for $i = 0, 1, 2$, ρ_i for $i = 1, 2$, z_0, d and $u(\cdot)$ are defined as in (5.49).

Theorem 5.8.2. *Let $\mathcal{B} = BUC(\mathbb{R}_-; E)$ and $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds and the functions $d : \mathbb{R}_+ \to E$, $\rho_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, 2$, $a_1, a_3 : \mathbb{R}^- \to \mathbb{R}$ and $a_2 : [0, \pi] \to \mathbb{R}$ are continuous. Then the evolution system* (5.50) *is controllable on* $(-\infty, +\infty)$.

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s,\xi)ds,$$

$$g(t,\psi)(\xi) = \int_{-\infty}^{0} a_3(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^{\pi} a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

$$Cu(t)(\xi) = d(\xi)u(t),\ t \in [0,b],\ \xi \in [0,\pi],\ u \in \mathbb{R},\ d(\xi) \in E.$$

which permit to transform system (5.50) into the abstract system (5.3) – (5.4). Moreover, the function f is a bounded linear operator. Now, the controllability of mild solutions can be deduced from a direct application of Theorem (5.3.1). Thus, the conclusion of our theorem hold.

From Remark (1.7.2), we have the following result.

Corollary 5.8.2. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then there exists a unique mild solution of* (5.50) *on* $\mathbb{R}$.

Example 3. Consider the partial differential equation

$$\begin{cases} {}^cD_0^{\alpha}u(t,\xi) = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + d(\xi)u(t) + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-r}^{0} a_1(s-t)u\left[s - \rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\theta)|u(t,\theta)|^2 d\theta\right), \xi\right] ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le t \le b,\ \xi \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \qquad\qquad\qquad 0 \le t \le b, \\ u(\theta,\xi) = u_0(\theta,\xi), \qquad\qquad -r < \theta \le 0,\ \xi \in [0,\pi], \end{cases} \tag{5.51}$$

where $a_1 : (-\infty, 0] \to \mathbb{R}$ is a continuous function.

Theorem 5.8.3. *Let $\varphi \in C(H;E)$. Assume that the condition (H_φ) holds, $\rho_i : [0,b] \to \mathbb{R}$ for $i = 1,2$ are continuous and the functions $a_1 : [-r,0] \to \mathbb{R}$; $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then the problem* (5.51) *is controllable on* $[-r,b]$.

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-r}^{0} a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^{\pi} a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

Finally let $C \in B(\mathbb{R};E)$ be defined as

$$Cu(t)(\xi) = d(\xi)u(t),\ t \in [0,b],\ \xi \in [0,\pi],\ u \in \mathbb{R},\ d(\xi) \in E.$$

are well defined functions, which permit to transform system (5.51) into the abstract system (5.5) – (5.6). Moreover, the function f is bounded linear operator. Now, the controllability of mild solutions can be deduced from a direct application of Theorem 5.4.1. From Remark 1.7.1, we have the following result.

Corollary 5.8.3. *Let $\varphi \in C(H;E)$ be continuous and bounded. Then the problem* (5.51) *is controllable on* $[-r,b]$.

Example 4. Consider the partial differential equation

$$\begin{cases} {}^cD_0^\alpha\left[u(t,\xi) - \displaystyle\int_{-r}^{0} a_3(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)ds\right] \\ = \dfrac{\partial^2 u(t,\xi)}{\partial\xi^2} + d(\xi)u(t) + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-r}^{0} a_1(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)ds, \\ \qquad\qquad 0 \le t \le b,\ \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \qquad 0 \le t \le b, \\ v(\theta,\xi) = v_0(\theta,\xi), \qquad -r < \theta \le 0,\ \xi \in [0,\pi], \end{cases} \tag{5.52}$$

where $a_3 : [-r,0] \to \mathbb{R}$ is a continuous function

Theorem 5.8.4. *Let $\varphi \in C(H;E)$. Assume that the condition (H_φ) holds, $\rho_i : [0,b] \to \mathbb{R}$ for $i = 1,2$ are continuous and the functions $a_1, a_3 : [-r,0] \to \mathbb{R}$; $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then the problem* (5.52) *is controllable on* $[-r,b]$.

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-r}^{0} a_1(s)\psi(s,\xi)ds,$$

$$g(t,\psi)(\xi) = \int_{-r}^{0} a_3(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

$$Cu(t)(\xi) = d(\xi)u(t),\ t \in [0,b],\ \xi \in [0,\pi],\ u \in \mathbb{R},\ d(\xi) \in E.$$

are well defined functions, which permit to transform system (5.52) into the abstract system (5.7) – (5.8). Moreover, the function f is bounded linear operator. Now, the controllability of mild solutions can be deduced from a direct application of Theorem 5.5.1. From Remark 1.7.1, we have the following result.

Corollary 5.8.4. *Let $\varphi \in C(H;E)$ be continuous and bounded. Then the problem* (5.52) *is conntrollable on* $[-r,b]$.

Example 5. Consider the partial differential equation

$$\begin{cases} {}^cD_0^\alpha u(t,\xi) = \dfrac{\partial^2 u(t,\xi)}{\partial\xi^2} + d(\xi)u(t) + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-\infty}^{0} a_1(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right]ds, \\ \qquad\qquad 0 \le t \le b,\ \xi \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \qquad 0 \le t \le b, \\ u(\theta,\xi) = u_0(\theta,\xi), \qquad -\infty < \theta \le 0,\ \xi \in [0,\pi], \end{cases} \tag{5.53}$$

where $a_1 : (-\infty, 0] \to \mathbb{R}$ is a continuous function.

Theorem 5.8.5. *Let $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds, $\rho_i : [0, b] \to \mathbb{R}$ for $i = 1, 2$ are continuous and the functions $a_1 : (-\infty, 0] \to \mathbb{R}$; $a_2 : [0, \pi] \to \mathbb{R}$ are continuous. Then the problem* (5.53) *is controllable on* $[-r, b]$.

Proof. From the assumptions, we have that

$$f(t, \psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s, \xi)ds,$$

$$\rho(s, \psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0, \xi)|^2 d\theta\right),$$

Finally let $C \in B(\mathbb{R}; E)$ be defined as

$$Cu(t)(\xi) = d(\xi)u(t),\ t \in [0, b],\ \xi \in [0, \pi],\ u \in \mathbb{R},\ d(\xi) \in E.$$

are well defined functions, which permit to transform system (5.53) into the abstract system (5.9) – (5.10). Moreover, the function f is bounded linear operator. Now, the controllability of mild solutions can be deduced from a direct application of Theorem 5.6.1. From Remark 1.7.2, we have the following result.

Corollary 5.8.5. *Let $\phi \in \mathcal{B}$ be continuous and bounded. The problem* (5.53) *is controllable on* $(-\infty, b]$.

Example 6. Consider the partial differential equation

$$\begin{cases} {}^cD_0^\alpha\left[u(t,\xi) - \displaystyle\int_{-\infty}^{0} a_3(s-t)u\left(s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right), \xi\right) ds\right] \\ = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + d(\xi)u(t) + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-\infty}^{0} a_1(s-t)u\left(s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right), \xi\right) ds, \\ \hfill 0 \le t \le b,\ \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \hfill 0 \le t \le b, \\ v(\theta,\xi) = v_0(\theta,\xi), \hfill -\infty < \theta \le 0,\ \xi \in [0,\pi], \end{cases} \tag{5.54}$$

where $a_3 : (-\infty, 0] \to \mathbb{R}$ is a continuous function.

Theorem 5.8.6. *Let $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds, $\rho_i : [0, b] \to \mathbb{R}$ for $i = 1, 2$ are continuous and the functions $a_1, a_3 : \mathbb{R}_- \to \mathbb{R}$ and $a_2 : [0, \pi] \to \mathbb{R}$ are continuous. Then the problem* (5.54) *is controllable on* $(-\infty, b]$.

Proof. From the assumptions, we have that

$$f(t, \psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s, \xi)ds,$$

$$g(t,\psi)(\xi) = \int_{-\infty}^{0} a_3(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^{\pi} a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

$$Cu(t)(\xi) = d(\xi)u(t),\ t \in [0,b],\ \xi \in [0,\pi],\ u \in \mathbb{R},\ d(\xi) \in E.$$

are well defined functions, which permit to transform system (5.54) into the abstract system (5.11) – (5.12). Moreover, the function f is bounded linear operator. Now, the controllability of mild solutions can be deduced from a direct application of Theorem 5.7.1. From Remark 1.7.2, we have the following result.

Corollary 5.8.6. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then the problem* (5.54) *is controllable on* $(-\infty, b]$.

Chapter 6

Fractional Evolution Inclusions with State-Dependent Delay

6.1 Introduction

In this chapter, we will give sufficient conditions for the existence of mild solutions for some classes of Caputo's fractional derivative order of partial functional and neutral functional differential evolution inclusions with finite and infinite state-dependent delay[1].

Using the alternative of Frigon for multivalued contraction maps in Banach space (see [50]), combined with the semi-group theory.

The existence of the mild solutions is demonstrated in section 6.2 for the following class of fractional evolution inclusion with finite state-dependent delay

$$^cD_0^\alpha y(t) \in A(t)y(t) + F(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J = [0, b] \tag{6.1}$$

$$y(t) = \varphi(t), \quad t \in H = [-r, 0], \tag{6.2}$$

where $0 < r < +\infty$, $^cD_0^\alpha$ is the standard Caputo's fractional derivative of order $\alpha \in (0, 1)$, $F : J \times C(H; E) \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\mathcal{P}(E)$ is the family of all subsets of E, $\rho : J \times C(H; E) \to \mathbb{R}$ and $\varphi \in C(H; E)$ are given functions and $\{A(t)\}_{t\geq 0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t, s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 6.3, we consider the following class of fractional neutral evolution inclusion with finite state-dependent delay

$$^cD_0^\alpha y(t)[y(t) - g(t, y_{\rho(t,y_t)})] \in A(t)y(t) + F(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J \tag{6.3}$$

$$y(t) = \varphi(t), \quad t \in H, \tag{6.4}$$

where $A(\cdot)$, F and φ are as in problem $(6.1) - (6.2)$ and $g : J \times C(H; E) \to E$ is a given function.

In section 6.4, we study the following class of fractional evolution inclusion with infinite state-dependent delay

$$^cD_0^\alpha y(t) \in A(t)y(t) + F(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J \tag{6.5}$$

[1][15] **D. Aoued**, S. Baghli-Bendimerad and M. Benchohra, Multivalued fractional partial functional and neutral functional fractional evolution equations with state-dependent delay, submitted.

$$y_0 = \phi \in \mathcal{B}, \tag{6.6}$$

where $\mathcal{B}$ is an abstract *phase space* to be specified later, $F : J \times \mathcal{B} \to E, \rho : J \times \mathcal{B} \to \mathbb{R}$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{t\in J}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $s \leq t$.

An extension of this problem is given in section 6.5, we consider the following class of fractional neutral evolution inclusion with infinite state-dependent delay

$${}^cD_0^\alpha y(t)[y(t) - g(t, y_{\rho(t,y_t)})] \in A(t)y(t) + F(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J \tag{6.7}$$

$$y_0 = \phi \in \mathcal{B}, \tag{6.8}$$

where $A(\cdot)$, F and φ are as in problem $(6.5) - (6.6)$ and $g : J \times \mathcal{B} \to E$ is a given function. Finally, section 6.6 is devoted to examples illustrating the abstract theory considered in previous sections.

6.2 Partial Multivalued Problem with Finite Delay

Before stating and proving the main result, we give first the definition of mild solution of the semilinear evolution problem $(6.1) - (6.2)$.

Lemma 6.2.1. *The system* $(6.1) - (6.2)$ *is equivalent to the nonlinear integral equation*

$$y(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s)y(s)\; ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\; ds. \tag{6.9}$$

In other words, every solution of the integral equation (6.9) *is also solution of the system* $(6.1) - (6.2)$ *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system $(6.1) - (6.2)$, and using some classical results from fractional calculus to get (6.9).

Definition 6.2.1. *We say that the function $y(\cdot) : [-r, b] \to E$ is a mild solution of* $(6.1) - (6.2)$ *if $y(t) = \varphi(t)$ for all $t \in [-r, 0]$ and y satisfies the following integral equation*

Definition 6.2.2. *We say that the function $y(\cdot) : [-r, b] \to E$ is a mild solution of the evolution system* $(6.1) - (6.2)$ *if $y(t) = \varphi(t)$ for all $t \in H$ and the restriction of $y(\cdot)$ to the interval J is continuous and there exists $f(\cdot) \in L^1(J; E) : f(t) \in F(t, y_{\rho(t,y_t)})$ a.e. in J such that y satisfies the following integral equation :*

$$y(t) = U(t,0)\; \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\; U(t,s)f(s)\; ds \qquad \textit{for each } t \in J. \tag{6.10}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times C(H;E), \rho(s,\varphi) \leq 0\}.$$

We always assume that $\rho : J \times C(H;E) \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $C(H;E)$ and there exists a continuous and bounded function $\mathcal{L}^\varphi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\varphi_t\| \leq \mathcal{L}^\varphi(t)\|\varphi\| \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

We need to introduce the following hypothesis

$(H1)$ There exists a constant $\widehat{M} \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq \widehat{M} \quad \text{for every } (t,s) \in \Delta.$$

$(H2)$ The multifunction $F : J \times C(H;E) \longrightarrow \mathcal{P}(E)$ is L^1_{loc}-Carathéodory with compact and convex values for each $u \in C(H;E)$ and there exist a function $p \in L^1_{loc}(J;\mathbb{R}_+)$ and such that

$$\|F(t,u)\|_{\mathcal{P}(E)} \leq p(t) + q(t)\ \|u\|_B \text{ for a.e. } t \in J \text{ and each } u \in C(H;E).$$

$(H3)$ For all $R > 0$, there exists $l_R \in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$H_d(F(t,u) - F(t,v)) \leq l_R(t)\ \|u - v\|$$

for each $t \in J$ and for all $u, v \in C(H;E)$ with $\|u\| \leq R$ and $\|v\| \leq R$ and

$$d(0, F(t,0)) \leq l_R(t) \quad \text{a.e.} \quad t \in J.$$

Theorem 6.2.1. *Assume that the hypotheses* (H_φ) *and* $(H1) - (H3)$ *hold and moreover*

$$\frac{\widehat{M} l_b^* b^\alpha}{\Gamma(1+\alpha)} < 1 \tag{6.11}$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ *and* $l_b^* = \sup l_b(s)$. *Then the problem* (6.1) – (6.2) *has at least mild solution on* $[-r, b]$.

Proof. Transform the problem (6.1) – (6.2) into a fixed-point problem. Consider the multivalued operator $N : C([-r,b];E) \to \mathcal{P}(C([-r,b];E))$ defined by :

$$N(y) = \left\{ h \in C([-r,b];E) : h(t) = \left\{ \begin{array}{ll} \varphi(t), & \text{if } t \in H; \\ U(t,0)\ \varphi(0) & \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s)\ ds, & \text{if } t \in J. \end{array} \right. \right\}$$

where $f \in S_{F,y} = \{v \in L^1(J;E) : v(t) \in F(t, y_{\rho(t,y_t)}) \text{ for a.e. } t \in J\}$.

Clearly, fixed points of the operator N are mild solutions of the problem (6.1) – (6.2). We remark also that, for each $y \in C([-r,b];E)$, the set $S_{F,y}$ is nonempty since, by $(H2)$, F has a measurable selection (see [40], Theorem III.6).

Let y be a possible fixed point of the operator N. Given $t \leq b$, then y should be solution of the inclusion $y \in \lambda\ N(y)$ for some $\lambda \in (0,1)$ and there exists $f \in S_{F,y} \Leftrightarrow f(t) \in F(t, y_{\rho(t,y_t)})$

such that, for each $t \in J$. So, from $(H1)$, $(H2)$, (H_φ) and Lemma 5.2.1, we have for each $t \in [0,b]$

$$\begin{aligned} |y(t)| &\leq \|U(t,0)\|_{B(E)}\ |\varphi(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|_{B(E)}\ |f(s)|\ ds \\ &\leq \widehat{M}\ \|\varphi\| + \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[p(s) + q(s)\|y_{\rho(s,y_s)}\|\right] ds \end{aligned}$$

It follows that

$$|y(t)| + \mathcal{L}^\varphi\|\varphi\| \leq \widehat{M}\|\varphi\| + \mathcal{L}^\varphi\|\varphi\| + \frac{\widehat{M}p^* b^\alpha}{\Gamma(\alpha+1)} + \frac{\widehat{M}q^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ [|y(s) + \mathcal{L}^\varphi\|\varphi\|]\, ds$$

Set

$$\delta_b := (\widehat{M} + \mathcal{L}^\varphi)\|\varphi\| + \frac{\widehat{M}p^* b^\alpha}{\Gamma(\alpha+1)}.$$

We consider the function μ defined by

$$\mu(t) := \sup\ \{\ |y(s)| + \mathcal{L}^\varphi\|\varphi\|\ :\ 0 \leq s \leq t\ \}, \quad 0 \leq t \leq b.$$

Let $t^\star \in [-r,t]$ be such that $\mu(t) = |y(t^\star)| + \mathcal{L}^\varphi\|\varphi\|$. If $t^\star \in [0,b]$, by the previous inequality, we get

$$\mu(t) \ \leq\ \delta_b + \frac{\widehat{M}q^*}{\Gamma(\alpha)}\ \int_0^t (t-s)^{\alpha-1}\ \mu(s)ds \qquad \text{for } t \in [0,b].$$

If $t^\star \in [-r,0]$, then $\mu(t) = \|\varphi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta_b = \delta_b(\alpha)$ such that

$$\begin{aligned} \mu(t) &\leq \delta_b \times \left[1 + \frac{\widehat{M}q^* b^\alpha}{\Gamma(\alpha+1)}\right] \\ &= \Lambda_b \end{aligned}$$

Since $\|y\| \leq \mu(t)$, we have $\|y\| \leq \max\{\|\varphi\|, \Lambda_b\} := \Theta_b$. Since for every $t \in [0,b]$, we have $\|y\| \leq \max\{\|\varphi\|, \Lambda_b\} := \Theta_b$.

Set $Y = \{\ y \in C([-r,b];E)\ :\ \sup\{|y(t)| : 0 \leq t \leq b\} \leq \Theta_b + 1\}$. Clearly, Y is a closed subset of $C([-r,b]);E)$.

We shall show that $N : \overline{Y} \to \mathcal{P}(C([-r,b];E))$ is a contraction and an admissible operator. First, we prove that N is a contraction ; Let $y, \overline{y} \in C([-r,b];E)$ and $h \in N(y)$. Then there exists $f(t) \in F(t, y_{\rho(t,y_t)})$ such that for each $t \in [0,b]$

$$h(t) = U(t,0)\ \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ U(t,s)\ f(s)\ ds.$$

From $(H3)$ it follows that

$$H_d(F(t, y_{\rho(t,y_t)}), F(t, \overline{y}_{\rho(t,y_t)})) \leq l_b(t)\ \|y_{\rho(t,y_t)} - \overline{y}_{\rho(t,y_t)}\|.$$

Hence, there is $\varrho \in F(t, \overline{y}_{\rho(t,y_t)})$ such that

$$|f(t) - \varrho| \leq l_b(t)\ \|y_{\rho(t,y_t)} - \overline{y}_{\rho(t,y_t)}\|, \quad t \in [0,b].$$

Consider $\mathcal{U}_\star : [0,b] \to \mathcal{P}(E)$, given by

$$\mathcal{U}_\star = \{\rho \in E : |f(t) - \rho| \leq l_b(t)\ \|y_{\rho(t,y_t)} - \overline{y}_{\rho(t,y_t)}\|\}.$$

Since the multivalued operator $\mathcal{V}(t) = \mathcal{U}_\star(t) \cap F(t, \overline{y}_{\rho(t,y_t)})$ is measurable (in [40], see Proposition III.4), there exists a function $\overline{f}(t)$, which is a measurable selection for $\mathcal{V}$. So, $\overline{f}(t) \in F(t, \overline{y}_{\rho(t,y_t)})$ and we obtain for each $t \in [0,b]$

$$|f(t) - \overline{f}(t)| \leq l_b(t)\ \|y_{\rho(t,y_t)} - \overline{y}_{\rho(t,y_t)}\|$$

Using (H_φ) and Lemma 5.2.1, we obtain

$$\begin{aligned} |f(t) - \overline{f}(t)| &\leq l_b(t)\ |y(t) - \overline{y}(t)| \\ &\leq l_b(t)\|y - \overline{y}\| \end{aligned}$$

Let us define, for each $t \in [0,b]$

$$\overline{h}(t) = U(t,0)\ \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} U(t,s)\ \overline{f}(s)\ ds.$$

Then, we can show as that we have for each $t \in [0,b]$

$$\begin{aligned} |h(t) - \overline{h}(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|_{B(E)}\ \left|f(s) - \overline{f}(s)\right|\ ds \\ &\leq \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ l_b(s)\|y - \overline{y}\|\ ds \\ &\leq \frac{\widehat{M} l_b^* b^\alpha}{\Gamma(\alpha+1)}\|y - \overline{y}\| \end{aligned}$$

Therefore, we have

$$\|h - \overline{h}\| \leq \frac{\widehat{M} l_b^* b^\alpha}{\Gamma(\alpha+1)}\|y - \overline{y}\|.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \leq \frac{\widehat{M} l_b^* b^\alpha}{\Gamma(\alpha+1)}\|y - \overline{y}\|.$$

So, for $\dfrac{\widehat{M} l_b^* b^\alpha}{\Gamma(\alpha+1)} < 1$, N is a contraction.

It's remains to show that N is an admissible operator. Let $y \in C([-r,b];E)$ and consider the operator $N : C([-r,b];E) \to \mathcal{P}(C([-r,b];E)$, given by

$$N(y) = \left\{ h \in C([-r,b],E) : h(t) = \begin{cases} \varphi(t), & \text{if } t \in H; \\ U(t,0)\ \varphi(0) & \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} U(t,s)\ f(s)\ ds, & \text{if } t \in [0,b], \end{cases} \right\}$$

where $f \in S^n_{F,y} = \{v \in L^1([0,b];E) : v(t) \in F(t, y_{\rho(t,y_t)}) \text{ for a.e. } t \in [0,b]\}$.

From $(H1)-(H3)$ and since F is a multivalued map with compact values, we can prove that for every $y \in C([-r,b];E)$, $N(y) \in \mathcal{P}_{cp}(C([-r,b];E))$ and there exists $y_\star \in C([-r,b];E)$ such that $y_\star \in N(y_\star)$. Let $h \in C([-r,b];E)$, $\overline{y} \in \overline{Y}$ and $\epsilon > 0$. Assume that $y_\star \in N(\overline{y})$, then we have

$$\begin{aligned} \|\overline{y}(t) - y_\star(t)\| &\leq \|\overline{y}(t) - h(t)\| + \|y_\star(t) - h(t)\| \\ &\leq \|\overline{y} - N(\overline{y})\| + \|y_\star(t) - h(t)\|. \end{aligned}$$

Since h is arbitrary, we may suppose that

$$h \in B(y_\star, \epsilon) = \{h \in C([-r,b];E) : \|h - y_\star\| \leq \epsilon\}.$$

Therefore,

$$\|\overline{y} - y_\star\| \leq \|\overline{y} - N(\overline{y})\| + \epsilon.$$

If y is not in $N(\overline{y})$, then $\|y_\star - N(\overline{y})\| \neq 0$. Since $N(\overline{y})$ is compact, there exists $x \in N(\overline{y})$ such that $\|y_\star - N(\overline{y})\| = \|y_\star - x\|$. Then we have

$$\begin{aligned} \|\overline{y}(t) - x(t)\| &\leq \|\overline{y}(t) - h(t)\| + \|x(t) - h(t)\| \\ &\leq \|\overline{y} - N(\overline{y})\| + \|x(t) - h(t)\|. \end{aligned}$$

Thus,

$$\|\overline{y} - x\| \leq \|\overline{y} - N(\overline{y})\| + \epsilon.$$

So, N is an admissible operator contraction. From the choice of Y there is no $y \in \partial Y^n$ such that $y = \lambda\ N(y)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 1.10.6 does not hold. A consequence of the nonlinear alternative of Frigon that $(C1)$ holds, we deduce that the operator N has a fixed point $y^\star$ which is a mild solution of the fractional evolution inclusion problem $(6.1)-(6.2)$.

6.3 Neutral Multivalued Problem with Finite Delay

We give here an extension to previous results for the neutral case $(6.3)-(6.4)$. Firstly, we define its mild solution.

Lemma 6.3.1. *The system* $(6.3)-(6.4)$ *is equivalent to the nonlinear integral equation*

$$y(t) \;=\; [\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s) y(s)\ ds \qquad (6.12)$$

$$+ \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\ ds.$$

In other words, every solution of the integral equation (6.12) *is also solution of the system* (6.3) – (6.4) *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system (6.3) – (6.4), and using some classical results from fractional calculus to get (6.12).

Definition 6.3.1. *We say that the function $y(\cdot) : [-r, b] \to E$ is a mild solution of the neutral functional evolution system* (6.3) – (6.4) *if $y(t) = \varphi(t)$ for all $t \in H$ and the restriction of $y(\cdot)$ to the interval J is continuous and there exists $f(\cdot) \in L^1(J; E)$: $f(t) \in F(t, y_{\rho(t,y_t)})$ a.e. in J such that y satisfies the following integral equation*

$$y(t) \ = U(t,0)[\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s)\ ds.$$

We consider the hypotheses (H_φ), $(H1) - (H3)$ and we need to introduce the following assumptions :

$(H4)$ There exists a constant $L_\star > 0$ such that

$$|A(s)g(s,\varphi) - A(\overline{s})g(\overline{s},\overline{\varphi})| \leq L_\star\ (|s-\overline{s}| + \|\varphi - \overline{\varphi}\|)$$

for all $\varphi, \overline{\varphi} \in C(H; E)$.

$(H5)$ There exists a constant $\overline{M}_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0 \quad \text{for all } \ t \in J.$$

$(H6)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0}$ such that

$$|A(t)\ g(t,\varphi)| \leq L\ (\|\varphi\| + 1) \text{ for all } t \in J \text{ and } \varphi \in C(H; E).$$

Theorem 6.3.1. *Suppose that the hypotheses* $(H1) - (H6)$ *are satisfied and moreover*

$$\left[\overline{M}_0 L_\star + \frac{\widehat{M} L_b^* b^\alpha}{\Gamma(\alpha+1)}\right] < 1 \tag{6.13}$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ and $l_b^* = \sup l_b(s)$. Then the problem (6.3) – (6.4) *has a least mild solution on* $[-r, b]$.

Proof. Transform as below the neutral problem (6.3) – (6.4) into a fixed point problem by considering the multivalued operator $\widetilde{N} : C([-r, b]; E) \to \mathcal{P}(C([-r, b]; E))$ defined by :

$$\widetilde{N}(y) = \left\{ h \in C([-r,b];E) : h(t) = \left\{ \begin{array}{ll} \varphi(t), & \text{if } t \in H; \\ U(t,0)\ [\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) & \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s)\ ds, & \text{if } t \in J. \end{array} \right. \right\}$$

where $f \in S_{F,y} = \{v \in L^1(J;E) : v(t) \in F(t, y_{\rho(t,y_t)}) \text{ for a.e. } t \in J\}$.

Clearly, the fixed points of the operator $\widetilde{N}$ are mild solutions of the problem $(6.3) - (6.4)$. We remark also that, for each $y \in C([-r,b];E)$, the set $S_{F,y}$ is nonempty since, by $(H2)$, F has a measurable selection (see [40], Theorem III.6).

Let y be a possible fixed point of the operator $\widetilde{N}$. Given $t \leq b$, then y should be solution of the inclusion $y \in \lambda\, \widetilde{N}(y)$ for some $\lambda \in (0,1)$ and there exists $f \in S_{F,y} \Leftrightarrow f(t) \in F(t, y_{\rho(t,y_t)})$ such that, for each $t \in J$, by using $(H1) - (H2)$, and $(H5) - (H6)$ we have for each $t \in [0,b]$

$$\begin{aligned}
|y(t)| &\leq |g(t, y_{\rho(t,y_t)})| + |U(t,0)[\varphi(0) - g(0,\varphi)]| \\
&+ \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, U(t,s) f(s, y_{\rho(s,y_s)})\, ds \right| \\
&\leq \|A^{-1}(t)\|\ |A(t) g(t, y_{\rho(t,y_t)})| + \widehat{M}\|\varphi\| + \|U(t,0)\|_{B(E)}\ \|A^{-1}(0)\|\ |A(0) g(0,\varphi)| \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s) \|y_{\rho(s,y_s)}\| \right] ds \\
&\leq \widehat{M}\|\varphi\| + \overline{M}_0 L \left(\|y_{\rho(t,y_t)}\| + 1 \right) + \widehat{M}\overline{M}_0 L\, (\|\varphi\| + 1) \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p(s) + q(s) \|y_{\rho(s,y_s)}\| \right] ds
\end{aligned}$$

Since $\|y_{\rho(t,y_t)}\| \leq |y(t)| + \mathcal{L}^\varphi \|\varphi\|$ we obtain

$$\begin{aligned}
|y(t)| &\leq \overline{M}_0 L\, (|y(t)| + \mathcal{L}^\varphi \|\varphi\| + 1) + \widehat{M}\|\varphi\|(1 + \overline{M}_0 L) + \frac{\widehat{M} p^* b^\alpha}{\Gamma(\alpha+1)} \\
&+ \frac{\widehat{M} q^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ (|y(s)| + \mathcal{L}^\varphi \|\varphi\|)\ ds
\end{aligned}$$

Then

$$\begin{aligned}
(1 - \overline{M}_0 L)|y(t)| &\leq \left[\overline{M}_0 L \mathcal{L}^\varphi + \widehat{M}(1 + \overline{M}_0 L) \right] \|\varphi\| + \overline{M}_0 L + \frac{\widehat{M} p^* b^\alpha}{\Gamma(\alpha+1)} \\
&+ \frac{\widehat{M} q^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ (|y(s)| + \mathcal{L}^\varphi \|\varphi\|)\ ds
\end{aligned}$$

Set

$$\delta_b := \mathcal{L}^\varphi \|\varphi\| + \frac{\left[\overline{M}_0 L \mathcal{L}^\varphi + \widehat{M}(1 + \overline{M}_0 L) \right]}{(1 - \overline{M}_0 L)} \|\varphi\| + \overline{M}_0 L + \frac{\widehat{M} p^* b^\alpha}{(1 - \overline{M}_0 L)\Gamma(\alpha+1)}.$$

Thus

$$|y(t)| + \mathcal{L}^\varphi \|\varphi\| \leq \delta_b + \frac{\widehat{M} q^*}{(1 - \overline{M}_0 L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ (|y(s)| + \mathcal{L}^\varphi \|\varphi\|)\ ds.$$

We consider the function μ defined by

$$\mu(t) := \sup\ \{\ |y(s)| + \mathcal{L}^\varphi \|\varphi\|\ :\ 0 \leq s \leq t\ \}, \quad 0 \leq t \leq b.$$

Let $t^\star \in [-r,t]$ be such that $\mu(t) = |y(t^\star)| + \mathcal{L}^\varphi\|\varphi\|$. If $t^\star \in [0,b]$, by the previous inequality, we get

$$\mu(t) \quad \leq \quad \delta_b + \frac{\widehat{M}q^*}{(1-\overline{M}_0L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\mu(s)ds \qquad \text{for } t \in [0,b].$$

If $t^\star \in [-r,0]$, then $\mu(t) = \|\varphi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta_b = \delta_b(\alpha)$ such that

$$\begin{aligned}\mu(t) &\leq \delta_b \times \left[1 + \frac{\widehat{M}q^*b^\alpha}{(1-\overline{M}_0L)\Gamma(\alpha+1)}\right]\\ &= \Lambda_b\end{aligned}$$

Since $\|y\| \leq \mu(t)$, we have $\|y\| \leq \max\{\|\varphi\|, \Lambda_b\} := \Theta_b$.

We can show as in the previous section that $\widetilde{N}$ is an admissible operator and we shall prove now that $\widetilde{N} : \overline{Y} \to \mathcal{P}(C([-r,b];E))$ is a contraction.

Let $y, \overline{y} \in C([-r,b];E)$ and $h \in \widetilde{N}(y)$. Then there exists $f(t) \in F(t, y_{\rho(t,y_t)})$ such that for each $t \in [0,b]$, we have

$$h(t) = U(t,0)[\varphi(0) - g(0,\varphi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,U(t,s)f(s)\,ds$$

From $(H3)$ it follows that

$$H_d(F(t, y_{\rho(t,y_t)}), F(t, \overline{y}_{\rho(t,y_t)})) \leq l_b(t)\;\|y_{\rho(t,y_t)} - \overline{y}_{\rho(t,y_t)}\|.$$

Hence, there is $\varrho \in F(t, \overline{y}_{\rho(t,y_t)})$ such that

$$|f(t) - \varrho| \leq l_b(t)\;\|y_{\rho(t,y_t)} - \overline{y}_{\rho(t,y_t)}\| \quad t \in [0,b].$$

Consider $\mathcal{U}_\star : [0,b] \to \mathcal{P}(E)$, given by

$$\mathcal{U}_\star = \{\varrho \in E : |f(t) - \varrho| \leq l_b(t)\;\|y_{\rho(t,y_t)} - \overline{y}_{\rho(t,y_t)}\|\}.$$

Since the multivalued operator $\mathcal{V}(t) = \mathcal{U}_\star(t) \cap F(t, \overline{y}_{\rho(t,y_t)})$ is measurable (in [40], see Proposition III.4), there exists a function $\overline{f}(t)$, which is a measurable selection for $\mathcal{V}$. So, $\overline{f}(t) \in F(t, \overline{y}_{\rho(t,y_t)})$, and we obtain for each $t \in [0,b]$

$$|f(t) - \overline{f}(t)| \leq l_b(t)\;\|y_{\rho(t,y_t)} - \overline{y}_{\rho(t,y_t)}\|$$

Using $(H3)$, (H_φ) and Lemma 5.2.1, we obtain

$$|f(t) - \overline{f}(t)| \leq l_b(t)\;\;|y(t) - \overline{y}(t)|$$

Let us define, for each $t \in [0,b]$

$$\overline{h}(t) = U(t,0)[\varphi(0) - g(0,\varphi)] + g(t, \overline{y}_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}U(t,s)\overline{f}(s)ds$$

Then, by $(H1)$, $(H3)$ and $(H4)$, we have for each $t \in [0, b]$

$$\begin{aligned} |h(t) - \overline{h}(t)| &\leq |g(t, y_{\rho(t,y_t)}) - g(t, \overline{y}_{\rho(t,y_t)})| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)} \left| f(s, y_{\rho(s,y_s)}) - f(s, \overline{y}_{\rho(s,y_s)}) \right| \, ds \\ &\leq \|A^{-1}(t)\| |A(t) g(t, y_{\rho(t,y_t)}) - A(t) g(t, \overline{y}_{\rho(t,y_t)})| \\ &+ \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, l_b(s) \, \|y_{\rho(s,y_s)} - \overline{y}_{\rho(s,y_s)}\| \, ds. \end{aligned}$$

Using (H_φ) and Lemma 5.2.1, we obtain

$$\begin{aligned} |h(t) - \overline{h}(t)| &\leq \overline{M}_0 L_\star |y(t) - \overline{y}(t)| + \frac{\widehat{M} L_b^* b^\alpha}{\Gamma(\alpha+1)} \|y - \overline{y}\| \\ &\leq \left[\overline{M}_0 L_\star + \frac{\widehat{M} L_b^* b^\alpha}{\Gamma(\alpha+1)} \right] \|y - \overline{y}\| \end{aligned}$$

Therefore,

$$|h(t) - \overline{h}(t)| \leq \left[\overline{M}_0 L_\star + \frac{\widehat{M} L_b^* b^\alpha}{\Gamma(\alpha+1)} \right] \|y - \overline{y}\|.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(\widetilde{N}(y), \widetilde{N}(\overline{y})) \leq \left[\overline{M}_0 L_\star + \frac{\widehat{M} L_b^* b^\alpha}{\Gamma(\alpha+1)} \right] \|y - \overline{y}\|.$$

So, for an appropriate choice of C_b, L_b^* and b^α such that

$$\left[\overline{M}_0 L_\star + \frac{\widehat{M} L_b^* b^\alpha}{\Gamma(\alpha+1)} \right] < 1,$$

The operator $\widetilde{N}$ is a contraction and an admissible operator. From the choice of Y there is no $y \in \partial Y^n$ such that $y = \lambda \ \widetilde{N}(y)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 1.10.6 does not hold. By the nonlinear alternative due to Frigon we get that $(C1)$ holds, we deduce that the operator $\widetilde{N}$ has a fixed point $y^\star$ which is a mild solution of the fractional neutral evolution inclusion problem (6.3) – (6.4).

6.4 Partial Multivalued Problem with Infinite Delay

Before stating and proving the main result, we give first the definition of mild solution of the semilinear fractional evolution problem (6.5) – (6.6).

Lemma 6.4.1. *The system* (6.5) – (6.6) *is equivalent to the nonlinear integral equation*

$$y(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s) y(s) \, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds. \tag{6.14}$$

In other words, every solution of the integral equation (6.14) *is also solution of the system* (6.5) – (6.6) *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system (6.5) – (6.6), and using some classical results from fractional calculus to get (6.14).

Definition 6.4.1. *We say that the function* $y(\cdot) : (-\infty, b] \to E$ *is a mild solution of* (6.5)–(6.6) *if* $y(t) = \phi(t)$ *for all* $t \in (-\infty, 0]$ *and* y *satisfies the following integral equation*

Definition 6.4.2. *We say that the function* $y(\cdot) : (-\infty, b] \to E$ *is a mild solution of the evolution system* (6.5) – (6.6) *if* $y(t) = \phi(t)$ *for all* $t \in (-\infty, 0]$ *and the restriction of* $y(\cdot)$ *to the interval* J *is continuous and there exists* $f(\cdot) \in L^1(J; E) : f(t) \in F(t, y_{\rho(t,y_t)})$ *a.e. in* J *such that* y *satisfies the following integral equation :*

$$y(t) = U(t,0)\ \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s)\ ds \qquad \textit{for each } t \in J. \tag{6.15}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\phi) : (s,\phi) \in J \times \mathcal{B}, \rho(s,\phi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq \mathcal{L}^\phi(t) \|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

We will need to introduce the following hypothesis which are assumed thereafter :

$(H01)$ There exists a constant $\widehat{M} \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq \widehat{M} \quad \text{for every } (t,s) \in \Delta.$$

$(H02)$ The multifunction $F : J \times \mathcal{B} \longrightarrow \mathcal{P}(E)$ is L^1_{loc}-Carathéodory with compact and convex values for each $u \in \mathcal{B}$ and there exist a function $p \in L^1_{loc}(J; \mathbb{R}_+)$ and such that

$$\|F(t,u)\|_{\mathcal{P}(E)} \leq p(t) + q(t)\ \|u\|_B \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$

$(H03)$ For all $R > 0$, there exists $l_R \in L^1_{loc}(J; \mathbb{R}_+)$ such that

$$H_d(F(t,u) - F(t,v)) \leq l_R(t)\ \|u - v\|$$

for each $t \in J$ and for all $u, v \in \mathcal{B}$ with $\|u\| \leq R$ and $\|v\| \leq R$ and

$$d(0, F(t,0)) \leq l_R(t) \quad \text{a.e.} \quad t \in J.$$

Consider the following space

$$\Omega = \left\{ y : (-\infty, b] \to E : y|_{(-\infty,0]}\ \in B \ \ \textit{and} \ \ y|_{[0,b]} \text{is continuous } \right\},$$

Theorem 6.4.1. *Assume that the hypotheses (H_φ) and $(H01)-(H03)$ hold and moreover*

$$\frac{\widehat{M} l_b^* b^\alpha}{\Gamma(1+\alpha)} < 1 \tag{6.16}$$

where $p^* = \sup p(s)$, $q^* = \sup q(s)$ and $l_b^* = \sup l_b(s)$. Then the problem $(6.5)-(6.6)$ has at least mild solution on $[-r, b]$.

Proof. Transform the problem $(6.5)-(6.6)$ into a fixed-point problem. Consider the multivalued operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by :

$$N(y) = \left\{ h \in \Omega : h(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\ \phi(0) & \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s)\ ds, & \text{if } t \in J. \end{array} \right. \right\}$$

where $f \in S_{F,y} = \{v \in L^1(J;E) : v(t) \in F(t, y_{\rho(t,y_t)}) \text{ for a.e. } t \in J\}$.

Clearly, fixed points of the operator N are mild solutions of the problem $(6.5)-(6.6)$. We remark also that, for each $y \in \Omega$, the set $S_{F,y}$ is nonempty since, by $(H2)$, F has a measurable selection (see [40], Theorem III.6).

For $\phi \in \mathcal{B}$, we will define the function $x(.) : (-\infty, b] \to E$ by

$$x(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if } \ t \in [0,b]; \\ U(t,0)\ \phi(0), & \text{if } \ t \in (-\infty, 0]. \end{array} \right.$$

Then $x_0 = \phi$. For each function $z \in \Omega$, set

$$y(t) = z(t) + x(t)$$

It is obvious that y satisfies (6.15) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s)\ f(s)\ ds \quad \text{ for } t \in J.$$

where $f(t) \in F(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)})$ *a.e* $t \in J$

Let

$$\Omega^0 = \{ z \in \Omega : z_0 = 0 \}.$$

Define the operator $F : \Omega^0 \to \mathcal{P}(\Omega^0)$ by :

$$F(z) = \left\{ h \in \Omega : h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s)\ f(s)\ ds \quad \text{ for } t \in J. \right\} \tag{6.17}$$

Obviously the operator N has a fixed point is equivalent to F has one, so it turns to prove that F has a fixed point.

Let $z \in \Omega^0$ be a possible fixed point of the operator. By the hypotheses $(H01)$ and $(H02)$, we have for each $t \in [0,b]$

$$\begin{aligned}|z(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|_{B(E)}\ |f(s)|\ ds\\ &\leq \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ \left[p(s)+q(s)\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\right]\ ds\end{aligned}$$

From (H_ϕ), Lemma 1.7.2 and Assumption $(A1)$, we have for each $t \in [0,b]$

$$\begin{aligned}\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\leq \|z_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}+\|x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\\ &\leq K_b|z(s)|+(M_b+\mathcal{L}^\phi)\|z_0\|_{\mathcal{B}}+K_b|x(s)|+(M_b+\mathcal{L}^\phi)\|x_0\|_{\mathcal{B}}\\ &\leq K_b|z(s)|+K_b\|U(s,0)\|_{B(E)}|\phi(0)|+(M_b+\mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}\\ &\leq K_b|z(s)|+K_b\widehat{M}|\phi(0)|+(M_b+\mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}\end{aligned}$$

Using (ii), we get

$$\begin{aligned}\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\leq K_b|z(s)|+K_b\widehat{M}H\|\phi\|_{\mathcal{B}}+(M_b+\mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}\\ &\leq K_b|z(s)|+(M_b+\mathcal{L}^\phi+K_b\widehat{M}H)\|\phi\|_{\mathcal{B}}\end{aligned}$$

It follows that

$$|z(t)| \leq \frac{\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)}+\frac{\widehat{M}q^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ \Big(K_b|z(s)|+(M_b+\mathcal{L}^\phi+K_b\widehat{M}H)\|\phi\|_{\mathcal{B}}\Big)\ ds$$

Set $c_b := (M_b+\mathcal{L}^\phi+K_b\widehat{M}H)\|\phi\|_{\mathcal{B}}$. Then, we have

$$|z(t)| \leq \frac{\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)}+\frac{\widehat{M}q^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ (K_b|z(s)|+c_b)\ ds$$

Then

$$K_b|z(t)|+c_b \leq \frac{K_b\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)}+c_b+\frac{K_b\widehat{M}q^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ (K_b|z(s)|+c_b)\ ds.$$

Set

$$\delta_b := \frac{K_b\widehat{M}p^*b^\alpha}{\Gamma(\alpha+1)}+c_b.$$

We consider the function μ defined by

$$\mu(t) := \sup\ \{\ K_b|z(s)|+c_b\ :\ 0\leq s\leq t\ \},\quad 0\leq t\leq b.$$

Let $t^\star \in [0,t]$ be such that $\mu(t)=K_b|z(t^\star)|+c_b$. By the previous inequality, we have

$$\mu(t) \leq \delta_b+\frac{K_b\widehat{M}q^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ \mu(s)ds \qquad \text{for } t\in[0,b].$$

If $t^\star \in (-\infty, 0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta_b = \delta_b(\alpha)$ such that

$$\begin{aligned} \mu(t) &\leq \delta_b \times \left[1 + \frac{K_b \widehat{M} q^* b^\alpha}{\Gamma(\alpha+1)} \right] \\ &= \Lambda_b \end{aligned}$$

Since $\|z\| \leq \mu(t)$, we have $\|z\| \leq \max\{\|\phi\|, \Lambda_b\} := \Theta_b$. Since for every $t \in [0, b]$, we have $\|z\| \leq \max\{\|\phi\|, \Lambda_b\} := \Theta_b$.

Set

$$Z = \{\ z \in \Omega^0 \ :\ \sup_{0 \leq t \leq b} |z(t)| \leq \Theta_b + 1\}.$$

Clearly, Z is a closed subset of Ω^0.

We shall show that $N : \overline{Z} \to \mathcal{P}(\Omega^0)$ is a contraction and an admissible operator. First, we prove that F is a contraction ; Let $z, \overline{z} \in \Omega^0$ and $h \in F(z)$. Then there exists $f(t) \in F(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)})$ such that for each $t \in [0, b]$

$$h(t) = U(t, 0)\ \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t, s)\ f(s)\ ds.$$

From $(H03)$, it follows that

$$H_d(F(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)}), F(t, \overline{z}_{\rho(t,y_t)} + x_{\rho(t,y_t)}) \leq l_b(t)\ \|z_{\rho(t,y_t)} - \overline{z}_{\rho(t,y_t)}\|.$$

Hence, there is $\varrho \in F(t, \overline{y}_{\rho(t,y_t)})$ such that

$$|f(t) - \varrho| \leq l_b(t) \|z_{\rho(t,y_t)} - \overline{z}_{\rho(t,y_t)}\|, \quad t \in [0, b].$$

Consider $\mathcal{U}_\star : [0, b] \to \mathcal{P}(E)$, given by

$$\mathcal{U}_\star = \{\rho \in E : |f(t) - \rho| \leq l_b(t)\ \|z_{\rho(t,y_t)} - \overline{z}_{\rho(t,y_t)}\|\}.$$

Since the multivalued operator $\mathcal{V}(t) = \mathcal{U}_\star(t) \cap F(t, \overline{z}_{\rho(t,y_t)})$ is measurable (in [40], see Proposition III.4), there exists a function $\overline{f}(t)$, which is a measurable selection for $\mathcal{V}$. So, $\overline{f}(t) \in F(t, \overline{z}_{\rho(t,y_t)})$ and we obtain for each $t \in [0, b]$

$$|f(t) - \overline{f}(t)| \leq l_b(t)\ \|z_{\rho(t,y_t)} - \overline{z}_{\rho(t,y_t)}\|$$

Using (H_ϕ) and Lemma 1.7.2, we obtain

$$\begin{aligned} |f(t) - \overline{f}(t)| &\leq l_b(t)\ |z(t) - \overline{z}(t)| \\ &\leq l_b(t) \|z - \overline{z}\| \end{aligned}$$

Let us define, for each $t \in [0, b]$

$$\overline{h}(t) = U(t, 0)\ \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t, s)\ \overline{f}(s)\ ds.$$

Then, we can show that, we have, for each $t \in [0,b]$

$$\begin{aligned} |h(t)-\overline{h}(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|U(t,s)\|_{B(E)}\ \left|f(s)-\overline{f}(s)\right|\ ds \\ &\leq \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ l_b(s)\|z-\overline{z}\|\ ds \\ &\leq \frac{\widehat{M}l_b^*b^\alpha}{\Gamma(\alpha+1)}\|y-\overline{y}\| \end{aligned}$$

Therefore, we have

$$\|h-\overline{h}\| \leq \frac{\widehat{M}l_b^*b^\alpha}{\Gamma(\alpha+1)}\|z-\overline{z}\|.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y),N(\overline{y})) \leq \frac{\widehat{M}l_b^*b^\alpha}{\Gamma(\alpha+1)}\|z-\overline{z}\|.$$

So, for $\dfrac{\widehat{M}l_b^*b^\alpha}{\Gamma(\alpha+1)} < 1$, N is a contraction.

Now we show that F is an admissible operator. Let $z \in \Omega^0$. Set, the space

$$\Omega^0 = \{z \in \Omega : z_0 = 0\}$$

and let us consider the multivalued operator $F : \Omega^0 \to \mathcal{P}(\Omega^0)$

$$F(z) = \left\{h \in \Omega : h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ U(t,s)\ f(s)\ ds \quad \text{for } t \in J.\right\} \tag{6.18}$$

where $f \in S^n_{F,y} = \{v \in L^1([0,b];E) : v(t) \in F(t, y_{\rho(t,y_t)}) \text{ for a.e. } t \in [0,b]\}$.

From $(H01)-(H03)$ and since F is a multivalued map with compact values, we can prove that for every $z \in \Omega^0$, $F(z) \in \mathcal{P}_{cp}(\Omega^0)$ and there exists $z_\star \in \Omega^0$ such that $z_\star \in F(z_\star)$. Let $h \in \Omega^0$, $\overline{y} \in \overline{Y}$ and $\epsilon > 0$. Assume that $z_\star \in F(\overline{z})$, then we have

$$\begin{aligned} \|\overline{z}(t)-z_\star(t)\| &\leq \|\overline{z}(t)-h(t)\| + \|z_\star(t)-h(t)\| \\ &\leq \|\overline{z}-N(\overline{z})\| + \|z_\star(t)-h(t)\|. \end{aligned}$$

Since h is arbitrary, we may suppose that

$$h \in B(z_\star,\epsilon) = \{h \in C([-r,b];E) : \|h-z_\star\| \leq \epsilon\}.$$

Therefore,

$$\|\overline{z}-z_\star\| \leq \|\overline{z}-N(\overline{z})\| + \epsilon.$$

If z is not in $F(\overline{z})$, then $\|z_\star - F(\overline{z})\| \neq 0$. Since $F(\overline{z})$ is compact, there exists $x \in F(\overline{z})$ such that $\|z_\star - F(\overline{z})\| = \|z_\star - x\|$. Then we have

$$\begin{aligned} \|\overline{z}(t)-x(t)\| &\leq \|\overline{z}(t)-h(t)\| + \|x(t)-h(t)\| \\ &\leq \|\overline{z}-F(\overline{z})\| + \|x(t)-h(t)\|. \end{aligned}$$

Thus,

$$\|\overline{z} - x\| \leq \|\overline{z} - F(\overline{z})\| + \epsilon.$$

So, F is an admissible operator contraction. From the choice of z there is no $z \in \partial Z^n$ such that $z = \lambda\ F(z)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 1.10.6 does not hold. A consequence of the nonlinear alternative of Frigon that $(C1)$ holds, we deduce that the operator F has a fixed point $y^\star$ which is a mild solution of the fractional evolution inclusion problem $(6.5) - (6.6)$.

6.5 Neutral Multivalued Problem with Infinite Delay

We give here an extension to previous results for the neutral case $(6.7) - (6.8)$. Firstly, we define its mild solution.

Lemma 6.5.1. *The system* $(6.7) - (6.8)$ *is equivalent to the nonlinear integral equation*

$$\begin{aligned} y(t) &= [\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(s) y(s)\ ds \qquad (6.19) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\ ds. \end{aligned}$$

In other words, every solution of the integral equation (6.19) *is also solution of the system* $(6.7) - (6.8)$ *and vice versa.*

Proof. It can be proved by applying the integral operator to both sides of the system $(6.7) - (6.8)$, and using some classical results from fractional calculus to get (6.19).

Definition 6.5.1. *We say that the function* $y(\cdot) : (-\infty, b] \to E$ *is a mild solution of the fractional neutral functional evolution system* $(6.7) - (6.8)$ *if* $y(t) = \phi(t)$ *for all* $t \leq 0$ *and the restriction of* $y(\cdot)$ *to the interval* J *is continuous and there exists* $f(\cdot) \in L^1(J; E) : f(t) \in F(t, y_{\rho(t,y_t)})$ *a.e. in* J *such that* y *satisfies the following integral equation*

$$y(t) = U(t,0)[\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s)\ ds \qquad (6.20)$$

We consider the hypotheses (H_ϕ), $(H01) - (H03)$ and we need to introduce the following assumptions :

$(H04)$ There exists a constant $L_\star > 0$ such that

$$|A(s)g(s,\phi) - A(\overline{s})g(\overline{s},\overline{\phi})| \leq L_\star\ (|s - \overline{s}| + \|\phi - \overline{\phi}\|)$$

for all $s, \overline{s} \in J$, $\phi, \overline{\phi} \in \mathcal{B}$.

$(H05)$ There exists a constant $\overline{M}_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0 \quad \text{for all } \ t \in J.$$

$(H06)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0 K_b}$ such that

$$|A(t)\ g(t,\varphi)| \leq L\ (\|\phi\| + 1) \text{ for all } t \in J \text{ and } \phi \in \mathcal{B}.$$

Theorem 6.5.1. *Suppose that the hypotheses* $(H01) - (H06)$ *are satisfied and moreover*

$$\left[\overline{M}_0 L_\star + \frac{\widehat{M} L_b^* b^\alpha}{\Gamma(\alpha+1)}\right] < 1 \tag{6.21}$$

where $p^* = \sup p(s)$*,* $q^* = \sup q(s)$*and* $l_b^* = \sup l_b(s)$*. Then the problem* $(6.7) - (6.8)$ *has at least mild solution on* $(-\infty, b]$*.*

Proof. Transform as below the neutral problem $(6.7) - (6.8)$ into a fixed point problem by considering the multivalued operator $\widetilde{N} : \Omega \to \mathcal{P}(\Omega)$ defined by :

$$\widetilde{N}(y) = \left\{ h \in \Omega : h(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\ [\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) & \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\ U(t,s) f(s)\ ds, & \text{if } t \in J. \end{array} \right. \right\}$$

where $f \in S_{F,y} = \{v \in L^1(J;E) : v(t) \in F(t, y_{\rho(t,y_t)}) \text{ for a.e. } t \in J\}$.

Clearly, the fixed points of the operator $\widetilde{N}$ are mild solutions of the problem $(6.7)-(6.8)$. We remark also that, for each $y \in \Omega$, the set $S_{F,y}$ is nonempty since, by $(H02)$, F has a measurable selection (see [40], Theorem III.6).

For $\phi \in \mathcal{B}$, we consider the function $x(.) : (-\infty, b] \to E$ defined as below by

$$x(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if } \ t \leq 0; \\ U(t,0)\ \phi(0), & \text{if } \ t \in J. \end{array} \right.$$

Then $x_0 = \phi$. For each function $z \in \Omega$, set

$$y(t) = z(t) + x(t)$$

It is obvious that y satisfies (6.20) if and only if z satisfies $z_0 = 0$ and

$$z(t) \ = \ g(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)}) - U(t,0) g(0,\phi) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) f(s) ds.$$

where $f(t) \in F(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)})$ $\ a.e \ \ t \in J$
Let

$$\Omega^0 = \{z \in \Omega : z_0 = 0\}.$$

Define the operator $\widetilde{F} : \Omega^0 \to \mathcal{P}(\Omega^0)$ by :

$$\begin{aligned} \widetilde{F}(z) \ &= \{h \in \Omega : h(t) = g(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)}) - U(t,0) g(0,\phi) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) f(s) ds.\} \end{aligned} \tag{6.22}$$

Obviously the operator $\widetilde{N}$ has a fixed point is equivalent to $\widetilde{F}$ has one, so it turns to prove that $\widetilde{F}$ has a fixed point.

Let $z \in \Omega^0$ be be a possible fixed point of the operator. Then, using $(H01)-(H02)$, $(H05)-(H06)$ we have for each $t \in [0,b]$

$$\begin{aligned}
|z(t)| &\leq |g(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)})| + |U(t,0)g(0,\phi)| \\
&+ \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, U(t,s)f(s, z_{\rho(s,y_s)} + x_{\rho(s,y_s)})ds\ ds\right| \\
&\leq \|A^{-1}(t)\|\ |A(t)g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})| + \|U(t,0)\|_{B(E)}\ \|A^{-1}(0)\|\ |A(0)g(0,\phi)| \\
&+ \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, U(t,s)f(s, z_{\rho(s,y_s)} + x_{\rho(s,y_s)})ds\ ds\right| \\
&\leq \overline{M}_0 L(\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} + 1) + \widehat{M}\overline{M}_0 L(\|\phi\|_{\mathcal{B}} + 1) \\
&+ \frac{\widehat{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[p(s) + q(s)\|z_{\rho(s,y_s)} + x_{\rho(s,y_s)}\|\right] ds
\end{aligned}$$

Since $\|z_{\rho(t,y_t)} + x_{\rho(t,y_t)}\|_{\mathcal{B}} \leq K_b|z(t)| + c_b$ we obtain

$$\begin{aligned}
|z(t)| &\leq \overline{M}_0 L\left(K_b|z(t)| + c_b + 1\right) + \overline{M}_0 L\widehat{M}(\|\phi\| + 1) + \frac{\widehat{M}p^* b^\alpha}{\Gamma(\alpha+1)} \\
&+ \frac{K_b\widehat{M}q^*}{\Gamma(\alpha)}\ \int_0^t (t-s)^{\alpha-1}\ \left(|z(s)| + c_b\right)\ ds
\end{aligned}$$

Then

$$\begin{aligned}
(1-\overline{M}_0 L K_b)|z(t)| &\leq \overline{M}_0 L(c_b+1) + \overline{M}_0 L\widehat{M}(\|\phi\| + 1) + \frac{\widehat{M}p^* b^\alpha}{\Gamma(\alpha+1)} \\
&+ \frac{K_b\widehat{M}q^*}{\Gamma(\alpha)}\ \int_0^t (t-s)^{\alpha-1}\ \left(|z(s)| + c_b\right)\ ds
\end{aligned}$$

Set

$$\delta_b := c_b + \frac{K_b\overline{M}_0 L\Big((c_b+1) + \widehat{M}(\|\phi\|+1)\Big)}{(1-\overline{M}_0 L K_b)\Gamma(\alpha+1)} + \frac{K_b\widehat{M}p^* b^\alpha}{(1-\overline{M}_0 L K_b)\Gamma(\alpha+1)}.$$

Thus

$$K_b|z(t)| + c_b \leq \delta_b + \frac{K_b\widehat{M}q^*}{\Gamma(\alpha)}\ \int_0^t (t-s)^{\alpha-1}\ \left(K_b|z(s)| + c_b\right)\ ds.$$

We consider the function μ defined by

$$\mu(t) := \sup\ \{\ K_b|z(s)| + c_b:\ 0 \leq s \leq t\ \},\quad 0 \leq t \leq b.$$

Let $t^\star \in (-\infty, t]$ be such that $\mu(t) = K_b|z(t^\star)| + c_b$. If $t^\star \in [0,b]$, by the previous inequality, we get

$$\mu(t) \leq \delta_b + \frac{K_b\widehat{M}q^*}{(1-\overline{M}_0 L K_b)\Gamma(\alpha)}\ \int_0^t (t-s)^{\alpha-1}\ \mu(s)ds \qquad \text{for } t \in [0,b].$$

If $t^\star \in (-\infty, 0]$, then $\mu(t) = \|\varphi\|$ and the previous inequality holds. And Lemma 1.8.1 implies that there exists a positive constant $\delta_b = \delta_b(\alpha)$ such that

$$\begin{aligned}\mu(t) &\leq \delta_b \times \left[1 + \frac{K_b \widehat{M} q^* b^\alpha}{(1 - \overline{M}_0 L K_b)\Gamma(\alpha+1)}\right] \\ &= \Lambda_b\end{aligned}$$

Since $\|z\| \leq \mu(t)$, we have $\|z\| \leq \max\{\|\phi\|, \Lambda_b\} := \Theta_b$.

We can show as in the previous section that $\widetilde{N}$ is an admissible operator and we shall prove now that $\widetilde{N} : \overline{Z} \to \mathcal{P}(\Omega^0)$ is a contraction.

Let $z, \overline{z} \in \Omega^0$ and $h \in \widetilde{F}(z)$. Then there exists $f(t) \in F(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)})$ such that for each $t \in [0, b]$

$$h(t) = U(t,0)[\phi(0) - g(0,\phi)] + g(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, U(t,s) f(s)\, ds$$

From $(H03)$ it follows that

$$H_d(F(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)}), F(t, \overline{z}_{\rho(t,y_t)} + x_{\rho(t,y_t)}) \leq l_b(t)\ \|z_{\rho(t,y_t)} - \overline{z}_{\rho(t,y_t)}\|.$$

Hence, there is $\varrho \in F(t, \overline{z}_{\rho(t,y_t)} + x_{\rho(t,y_t)})$ such that

$$|f(t) - \varrho| \leq l_b(t)\ \|z_{\rho(t,y_t)} - \overline{z}_{\rho(t,y_t)}\| \quad t \in [0, b].$$

Consider $\mathcal{U}_\star : [0, b] \to \mathcal{P}(E)$, given by

$$\mathcal{U}_\star = \{\varrho \in E : |f(t) - \varrho| \leq l_b(t)\ \|z_{\rho(t,y_t)} - \overline{z}_{\rho(t,y_t)}\|\}.$$

Since the multivalued operator $\mathcal{V}(t) = \mathcal{U}_\star(t) \cap F(t, \overline{z}_{\rho(t,y_t)} + x_{\rho(t,y_t)})$ is measurable (in [40], see Proposition III.4), there exists a function $\overline{f}(t)$, which is a measurable selection for $\mathcal{V}$. So, $\overline{f}(t) \in F(t, \overline{z}_{\rho(t,y_t)} + x_{\rho(t,y_t)})$, and we obtain for each $t \in [0, b]$

$$|f(t) - \overline{f}(t)| \leq l_b(t)\ \|z_{\rho(t,y_t)} - \overline{z}_{\rho(t,y_t)}\|$$

Using $(H03)$, (H_φ) and Lemma 1.7.2, we obtain

$$|f(t) - \overline{f}(t)| \leq l_b(t)\ \ |z(t) - \overline{z}(t)|$$

Let us define, for each $t \in [0, b]$

$$\overline{h}(t) = U(t,0)[\phi(0) - g(0,\phi)] + g(t, \overline{z}_{\rho(t,y_t)} + x_{\rho(t,y_t)}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) \overline{f}(s) ds$$

Then, by $(H01)$, $(H03)$ and $(H04)$, we have for each $t \in [0, b]$

$$\begin{aligned}|h(t) - \overline{h}(t)| &\leq |g(t, z_{\rho(s,y_s)} + x_{\rho(s,y_s)}) - g(t, \overline{z}_{\rho(t,y_t)} + x_{\rho(t,y_t)})| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)}\ \left|f(s) - \overline{f}(s)\right|\ ds \\ &\leq \|A^{-1}(t)\| |A(t) g(t, z_{\rho(t,y_t)} + x_{\rho(t,y_t)}) - A(t) g(t, \overline{z}_{\rho(t,y_t)} + x_{\rho(t,y_t)})| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|U(t,s)\|_{B(E)}\ \left|f(s) - \overline{f}(s)\right|\ ds \\ &\leq \overline{M}_0 L_\star \|z_{\rho(t,y_t)} - \overline{z}_{\rho(t,y_t)}\| + \frac{\widehat{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ l_b(s)\ \|z_{\rho(s,y_s)} - \overline{z}_{\rho(s,y_s)}\|\ ds.\end{aligned}$$

Using (H_ϕ) and Lemma 1.7.2, we obtain

$$\begin{aligned} |h(t)-\overline{h}(t)| &\leq \overline{M}_0 L_\star \|z-\overline{z}\| + \frac{\widehat{M} L_b^* b^\alpha}{\Gamma(\alpha+1)} \|z-\overline{z}\| \\ &\leq \left[\overline{M}_0 L_\star + \frac{\widehat{M} L_b^* b^\alpha}{\Gamma(\alpha+1)} \right] \|z-\overline{z}\| \end{aligned}$$

Therefore,

$$|h(t)-\overline{h}(t)| \leq \left[\overline{M}_0 L_\star + \frac{\widehat{M} L_b^* b^\alpha}{\Gamma(\alpha+1)} \right] \|z-\overline{z}\|.$$

By an analogous relation, obtained by interchanging the roles of z and $\overline{z}$, it follows that

$$H_d(\widetilde{N}(z), \widetilde{N}(\overline{z})) \leq \left[\overline{M}_0 L_\star + \frac{\widehat{M} L_b^* b^\alpha}{\Gamma(\alpha+1)} \right] \|z-\overline{z}\|.$$

So, for an appropriate choice of $\overline{M}_0 L_\star$, L_b^* and b^α such that

$$\left[\overline{M}_0 L_\star + \frac{\widehat{M} L_b^* b^\alpha}{\Gamma(\alpha+1)} \right] < 1,$$

The operator $\widetilde{N}$ is a contraction and an admissible operator. From the choice of Z there is no $z \in \partial Z^n$ such that $z = \lambda\, \widetilde{N}(z)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 1.10.6 does not hold. By the nonlinear alternative due to Frigon we get that $(C1)$ holds, we deduce that the operator $\widetilde{N}$ has a fixed point $z^\star$ which is a mild solution of the fractional neutral evolution inclusion (6.7) – (6.8).

6.6 Examples

To illustrate the previous results, we give in this section four applications:
Example 1. Consider the partial differential inclusion

$$\begin{cases} {}^cD_0^\alpha u(t,\xi) \in \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-r}^0 a_1(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right), \xi\right] ds, \\ \qquad\qquad\qquad\qquad -r \leq t \leq b,\ \xi \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \qquad -r \leq t \leq b, \\ u(\theta,\xi) = u_0(\theta,\xi), \qquad -r < \theta \leq 0,\ \xi \in [0,\pi], \end{cases} \tag{6.23}$$

where $a_1(t,\xi)$ is a continuous function and is uniformly Hölder continuous in t
$0 < \alpha \leq 1$; $a_1 : [-r,0] \to \mathbb{R}$; $a_2 : [0,\pi] \to \mathbb{R}$; $u_0 \in C(H;E)$; $\rho_1 : [0,+\infty) \to \mathbb{R}$ are continuous functions ; $\rho_2 : [0,+\infty) \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact convex values.

To study this system, we consider the space $E = L^2([0,\pi],\mathbb{R})$ and the operator $A : D(A) \subset E \to E$ given by $Aw = w''$ with

$$D(A) := \{\ w \in E\ : w'' \in E,\ w(0) = w(\pi) = 0\ \}$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\in[0,b]}$ on E, with compact resolvent. On the domain $D(A)$, we define the operators $A(t) : D(A) \subset E \to E$ by

$$A(t)x(\xi) = Ax(\xi) + a_0(t,\xi)x(\xi).$$

By assuming that $a_0(.,.)$ is continuous and that $a_0(t,\xi) \le -\delta_0$ ($\delta_0 > 0$) for every $t \in \mathbb{R}, \xi \in [0,\pi]$, and specific case $\alpha = 1$ it follows that the system

$$u'(t) = A(t)u(t) \quad t \ge s,$$

$$u(s) = x \in E,$$

has an associated evolution family given by

$$U(t,s)x(\xi) = \left[T(t-s)exp\left(\int_s^t a_0(\tau,\xi)d\tau\right)x\right](\xi).$$

From this expression, it follows that $U(t,s)$ is a compact linear operator and that

$$\|U(t,s)\| \le e^{-(1+\delta_0)(t-s)} \text{ for every } (t,s) \in [0,b]\times[0,b] \quad ; \quad s \le t.$$

Theorem 6.6.1. *Let $\varphi \in C(H;E)$. Assume that the condition (H_φ) holds, $\rho_1 : [0,+\infty) \to \mathbb{R}$ is a continuous function, $\rho_2 : [0,+\infty) \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact convex values and the functions $a_1 : [-r,0] \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a mild solution of* (6.23).

Proof. From the assumptions, we have that

$$F(t,\psi)(\xi) = \int_{-r}^0 a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (6.23) into the abstract system (6.1) – (6.2). Moreover, the function F is bounded linear operator. Now, the existence of mild solutions can be deduced from a direct application of Theorem 6.2.1.

From Remark 1.7.1, we have the following result.

Corollary 6.6.1. *Let $\varphi \in C(H;E)$ be continuous and bounded. Then there exists a mild solution of* (6.23) *on* $[-r,b]$.

Example 2. Consider the partial differential inclusion

$$
\begin{cases}
{}^cD_0^\alpha \left[u(t,\xi) - \displaystyle\int_{-r}^{0} a_3(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right) ds \right] \\
\in \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\
+\displaystyle\int_{-r}^{0} a_1(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right) ds, \\
\hfill -r \le t \le 0,\ \xi \in [0,\pi], \\
v(t,0) = \ v(t,\pi) \ = 0, \hfill -r \le t \le 0, \\
v(\theta,\xi) = \ v_0(\theta,\xi), \hfill -r < \theta \le 0,\ \xi \in [0,\pi],
\end{cases}
\tag{6.24}
$$

where $a_3 : [-r,0] \to \mathbb{R}$ is a continuous function.

Theorem 6.6.2. *Let $\varphi \in C(H;E)$. Assume that the condition (H_φ) holds, $\rho_1 : [0,+\infty) \to \mathbb{R}$ is a continuous function, $\rho_2 : [0,+\infty) \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact convex values and the functions $a_1, a_3 : [-r,0] \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a mild solution of* (6.24).

Proof. From the assumptions, we have that

$$F(t,\psi)(\xi) = \int_{-r}^{0} a_1(s)\psi(s,\xi)ds,$$

$$g(t,\psi)(\xi) = \int_{-r}^{0} a_3(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (6.24) into the abstract system (6.3) – (6.4). Moreover, the function F is bounded linear operator. Now, the existence of mild solutions can be deduced from a direct application of Theorem 6.3.1.

From Remark 1.7.1, we have the following result.

Corollary 6.6.2. *Let $\varphi \in C(H;E)$ be continuous and bounded. Then there exists a mild solution of* (6.24) *on* $[-r,b]$.

Example 3. Consider the partial differential inclusion

$$
\begin{cases}
{}^cD_0^\alpha u(t,\xi) \in \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\
+\displaystyle\int_{-\infty}^{0} a_1(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right] ds, \\
\hfill 0 \le t \le b,\ \xi \in [0,\pi], \\
u(t,0) = \ u(t,\pi) \ = 0, \hfill 0 \le t \le b, \\
u(\theta,\xi) = \ u_0(\theta,\xi), \hfill -\infty < \theta \le 0,\ \xi \in [0,\pi],
\end{cases}
\tag{6.25}
$$

where $a_1 : (-\infty,0] \to \mathbb{R}$ is a continuous function.

Theorem 6.6.3. *Let $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds, $\rho_1 : [0,+\infty) \to \mathbb{R}$ is a continuous function, $\rho_2 : [0,+\infty) \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact convex values and the functions $a_1 : (-\infty,0] \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a mild solution of* (6.25).

Proof. From the assumptions, we have that

$$F(t,\psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (6.25) into the abstract system (6.5) – (6.6). Moreover, the function F is bounded linear operator. Now, the existence of mild solutions can be deduced from a direct application of Theorem 6.4.1.

From Remark 1.7.2, we have the following result.

Corollary 6.6.3. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then there exists a mild solution of* (6.25) *on* $(-\infty,b]$.

Example 4. Consider the partial differential inclusion

$$\begin{cases} {}^cD_0^\alpha\left[u(t,\xi) - \displaystyle\int_{-\infty}^{0} a_3(s-t)u\left(s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)ds\right] \\ \in \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-\infty}^{0} a_1(s-t)u\left(s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right)ds, \\ \hfill 0 \le t \le b,\ \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \hfill 0 \le t \le b, \\ v(\theta,\xi) = v_0(\theta,\xi), \hfill -\infty < \theta \le 0,\ \xi \in [0,\pi], \end{cases} \tag{6.26}$$

where $a_3 : (-\infty,0] \to \mathbb{R}$ is a continuous functions.

Theorem 6.6.4. *Let $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds, $\rho_1 : [0,+\infty) \to \mathbb{R}$ is a continuous function, $\rho_2 : [0,+\infty) \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact convex values and the functions $a_1, a_3 : (-\infty,0] \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then there exists a mild solution of* (6.26).

Proof. From the assumptions, we have that

$$F(t,\psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s,\xi)ds,$$

$$g(t,\psi)(\xi) = \int_{-\infty}^{0} a_3(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta\right),$$

are well defined functions, which permit to transform system (6.26) into the abstract system (6.7) – (6.8). Moreover, the function F is bounded linear operator. Now, the existence of mild solutions can be deduced from a direct application of Theorem 6.5.1.

From Remark 1.7.2, we have the following result.

Corollary 6.6.4. *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then there exists a mild solution of* (6.26) *on* $(-\infty, b]$.

Conclusion

In this thesis, we have present some results of existence, even uniqueness and controllability of mild solutions on a bounded and unbounded interval for different classes of firt order and Caputo's fractional derivative order of differential and integrodifferential, perturbed and nonperturbed,partial functional and neutral functional *evolution* equations and inclusions with finite and infinite state-dependent delay using various nonlinear alternatives on Fréchet and Banach spaces depending on the fixed point argument and combined with semigroups theory.

We have generalized the various evolution problems in the thesis of Baghli [20] for the Caputo's fractional derivative order and we have considered the case when the delay is depending on the solution following the paper of Baghli-Benchohra-Nieto [**?**] and Baghli [19].

Our first work, in this thesis was proving the existence of mild solutions on unbounded interval for the first order perturbed evolution equations with infinite state-dependent delay with the Theorem 3.5, page 8 of our first paper [10] and we have extended the result to the first order neutral perturbed evolution equations with infinite state-dependent delay with the Theorem 4.2, page 14 of our first paper [10].

Then in *our second work*, we have give in this thesis the controllability of mild solutions on unbounded interval for the first order evolution equations with infinite state-dependent delay on the Theorem 2, page 388 of our second paper [11] and we have extended the result to the first order neutral evolution equations with infinite state-dependent delay with the Theorem 3, page 394 of our first paper [11].

The others results about fractional evolution equations and inclusions are submitted or in preparation.

Perspectives

Our perspectives are to study the existence, even uniqueness and the controllability of mild solutions on the whole positif real line for the different classes of fractional evolution equations studied in this thesis when the delay is finite and infinite and also is depending on the solution.

We look also for the existence and the controllability of mild solutions for fractional perturbed and nonperturbed evolution inclusions with finite and infinite state-dependent delay.

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