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Quelques contributions aux équations différentielles d'ordre fractionnaire

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Introduction

Fractional calculus has its origin in the question of the extension of meaning. A well known example is the extension of meaning of real numbers to complex numbers, and another is the extension of meaning of factorials of integers to factorials of complex numbers. In generalized integration and differentiation the question of the extension of meaning is: Can the meaning of derivatives of integral order $\frac{d^n y}{dx^n}$ be extended to have meaning where n is any number irrational, fractional or complex ?

Leibnitz invented the above notation. Perhaps, it was naive play with symbols that prompted L'Hospital to ask Leibnitz about the possibility that n be a fraction. "What if n be $\frac{1}{2}$? " , asked L'Hospital. Leibnitz in 1695 replied, "It will lead to a paradox." But he added prophetically, "From this apparent paradox, one day useful consequences will be drawn." In 1697, Leibnitz, referring to Wallis's infinite product for $\frac{\pi}{2}$, used the notation $d^{\frac{1}{2}}y$ and stated that differential calculus might have been used to achieve the same result. In 1819 the first mention of a derivative of arbitrary order appears in a text. The French mathematician, S. F. Lacroix ,published a 700 page text on differential and integral calculus in which he devoted less than two pages to this topic. Starting with $y = x^n$, n a positive integer, he found the mth derivative to be

$$\frac{d^m y}{dx^m} = \frac{n!}{(n-m)!}x^{n-m}.$$

Using Legendre's symbol r which denotes the generalized factorial, and by replacing m by $\frac{1}{2}$ and n by any positive real number a, in the manner typical of the classical formalists of this period, Lacroix obtained the formula

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = \frac{\Gamma(a+1)}{\Gamma(a+\frac{1}{2})}x^{a-\frac{1}{2}}$$

which expresses the derivative of arbitrary order $\frac{1}{2}$ of the function x . He gives the example for $y = x$ and gets

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(x) = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

because $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$ and $\Gamma(2) = 1$. This result is the same yielded by the present day Riemann-Liouville definition of a fractional derivative. It has taken 279 years since L'Hospital first raised the question for a text to appear solely devoted to this topic5 . Euler and Fourier made mention of derivatives of arbitrary order but they gave no applications or examples. So the honor of making the first application belongs to Niels Henrik Abel in 1823. Abel applied the fractional calculus in the solution of an integral equation which arises in the formulation of the tautochrone problem. This problem, sometimes called the isochrone problem, is that of finding the shape of a frictionless wire lying in a vertical plane such that the time of slide of a bead placed on the wire slides to the lowest point of the wire in the same time regardless of where the bead is placed. The brachistochrone problem deals with the shortest time of slide.

Abel's solution was so elegant that it is my guess it attracted the attention of Liouville who made the first major attempt to give a logical definition of a fractional derivative. Hepublished three long memoirs in 1832 and several more through 1855.

Liouville's starting point is the known result for derivatives of integral order

$$D^m e^{ax} = a^m e^{ax}$$

which he extended in a natural way to derivatives of arbitrary order

$$D^\alpha e^{ax} = a^\alpha e^{ax}.$$

He expanded the function $f(x)$ in the series

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x},$$

and assumed the derivative of arbitrary order $f(x)$ to be

$$D^\alpha f(x) = \sum_{n=0}^{\infty} c_n a_n^\alpha e^{a_n x},$$

This formula is known as Liouville's first definition and has the obvious disadvantage that α must be restricted to values such that the series converges. Liouville's second method was applied to explicit functions

of the form $x^{-a}, a > 0$. He considered the integral

$$I = \int_0^\infty u^{a-1} e^{-xu} du.$$

The transformation $xu = t$ gives the result

$$x^{-a} = \frac{1}{\Gamma(1)} I.$$

Then after operating on both sides with D^α , the result

$$D^\alpha x^{-a} = \frac{(-1)^\alpha \Gamma(a+\alpha)}{\Gamma(a)} x^{-a-\alpha}.$$

Liouville was successful in applying these definitions to problems in potential theory. The first definition is restricted to certain values of α and the second method is not suitable to a wide class of functions. Between 1835 and 1850 there was a controversy which centered on two definitions of a fractional derivative. George Peacock favored Lacroix's generalization of a case of integral order. Other mathematicians favored Liouville's definition. Augustus De Morgan's judgement proved to be accurate when he stated that the two versions may very possibly be parts of a more general system. In 1850 William Center observed that the discrepancy between the two versions of a fractional derivative focused on the fractional derivative of a constant. According to the Peacock-Lacroix version the fractional derivative of a constant yields a result other than zero while according to Liouville's formula the fractional derivative of a constant equals zero because $\Gamma(0) = \infty$.

The state of affairs in the mid-nineteenth century is now cleared up. Harold Thayer Davis states, "The mathematicians at that time were aiming for a plausible definition of generalized differentiation but, in fairness to them, one should note they lacked the tools to examine the consequences of their definition in the complex plane."

Riemann in 1847 while a student wrote a paper published posthumously in which he gives a definition of a fractional operation. It is my guess that Riemann was influenced by one of Liouville's memoirs in which Liouville wrote, "The ordinary differential equation

$$\frac{d^n y}{dx^n} = 0.$$

has the complementary solution

$$y_c = c_0 + c_1 x + c_2 x^2 + ... + c_{n-1} x^{n-1}.$$

Thus

$$\frac{d^\alpha}{dx^\alpha} f(x) = 0.$$

Should have a corresponding complementary solution." So, I am inclined to believe Riemann saw fit to add a complementary function to his definition of a fractional integration:

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t) dt + \psi(x).$$

Cayley remarked in 1880 that Riemann's complementary function is of indeterminate nature. The development of mathematical ideas Ls not without error. Peacock made several errors in the topic of fractional calculus when he misapplied the Principle of the Permanence of Equivalent Forms which is stated for algebra and which did not always apply to the theory of operators. Liouville made an error when he failed to note in his discussion of a complementary function that the specialization of one of the parameters led to an absurdity. Riemann became hopelessly entangled with an indeterminate complementary function. Two different versions of a fractional derivative yielded different results when applied to a constant. Thus, I suggest that when Oliver Heaviside published his work in the last decade of the nineteenth century, he was met with haughty silence and disdain not only because of the hilarious jibes he made at mathematicians but also because of the distrust mathematicians had in the general concept of fractional operators. The subject of notation cannot be minimized. The succinctness of notation of fractional calculus adds to its elegance. In the papers that follow in this text, various notations are used. The notation Iprefer was invented by Harold T. Davis. All the information can be conveyed by the symbols

$${}_cI_x^\alpha f(x) \ , \ \alpha \geq 0.$$

Denoting integration of arbitrary order along the x-axis. The subscripts c and x denote the limits (terminals) of integration of a definite integral which defines fractional integration. The adjoining of these subscripts becomes a vital part of the operator symbol to avoid ambiguities in applications.

Probabilistic functional analysis is an important mathematical are a of research due to its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. Indeed,in many cases the mathematical models or equations used to describe phenomena in the biological, physical, engineering, and systems sciences contain certain parameters or coefficients which have specific interpretations, but whose values are unknown. Therefore, it is more realistic to consider such equations as random operator equations. These equations are much more difficult to handle mathematically than deterministic equations. Important contributions to the study of the mathematical aspects of such random equations have been undertaken in [25, 80, 95] among others.

In this thesis, we shall be concerned by the existence of solutions to the random system of fractional equations . Our results are based upon very recently fixed point theorems in vector metric space.
This thesis is structured in 4 chapters and each chapter contains more sections. It is arranged as follows:
In chapter 1 we give some basic concepts about Special functions (Euler's Gamma function , the Beta function and Mittag-Leffler function) and Fractional Calculus :notations, definitions, lemmas and theorems which are used throughout this thesis , and several approaches of Fractional Derivatives and Integrals , (Riemann-Liouville , Caputo and Hadamard).
We study the fixed point in Vector metric space , we mention that for generalized metric space, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces.

In chapter 2 , we prove the existence of solutions to the system of fractional discrete equation. More precisely, we will consider the following problem,

$$\begin{cases} \Delta^{\alpha}_{\alpha-1}x(k) = f_1(k+\alpha-1, x(k+\alpha-1), y(k+\alpha-1)), \ k \in \mathbb{N}_0(b) \\ \Delta^{\alpha}_{\alpha-1}y(k) = f_2(k+\alpha-1, x(k+\alpha-1), y(k+\alpha-1)), \ k \in \mathbb{N}_0(b), \\ \Delta^{\alpha-1}_{\alpha-1}x(0) = x_0 \\ \Delta^{\alpha-1}_{\alpha-1}y(0) = y_0, \end{cases} \tag{0.0.1}$$

where $k \in \mathbb{N}_0 = \{0, 1, ..., b+1\}$, $0 < \alpha \leq 1$, and $f_1, f_2 : \mathbb{N}_{\alpha-1}(b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

are given functions.

In chapter 3 , we prove the existence of solutions to the random system of fractional differential equations:

$$\begin{cases} D^{\alpha}x(t,\omega) = f(t,x(t,\omega),y(t,\omega),\omega), \ 0<\alpha<1, \ t\in[0,b], \\ D^{\beta}y(t,\omega) = g(t,x(t,\omega),y(t,\omega),\omega), \ 0<\beta<1, \ t\in[0,b], \\ x(0,\omega) = x_0(\omega), \ \omega\in\Omega \\ y(0,\omega) = y_0(\omega), \ \omega\in\Omega, \end{cases} \tag{0.0.2}$$

where $f,g:[0,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\Omega\to\mathbb{R}^m$, $(\Omega,\mathcal{A})$ is a measurable space and $x_0,y_0:\Omega\to\mathbb{R}^m$ are a random variable. $D^{\alpha}x$ is the Caputo fractional derivative of x with respect to the variable $t\in[0,b]$ with $b>0$.

In chapter 4 , we prove the existence of solutions to the random fractional differential equations via the Hadamard fractional derivative . We consider the system of Hadamard-type fractional differential equations:

$$\begin{cases} {}^{CH}D^{\alpha}x(t,\omega) = f(t,x(t,\omega),y(t,\omega),\omega), \ 0<\alpha<1, \ t\in[1,b], \\ {}^{CH}D^{\beta}y(t,\omega) = g(t,x(t,\omega),y(t,\omega),\omega), \ 0<\beta<1, \ t\in[1,b], \\ x(1,\omega) = x_0(\omega), \ \omega\in\Omega, \\ y(1,\omega) = y_0(\omega), \ \omega\in\Omega, \end{cases} \tag{0.0.3}$$

where $f,g:[1,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\Omega\to\mathbb{R}^m$, $(\Omega,\mathcal{F})$ is a measurable space and $x_0,y_0:\Omega\to\mathbb{R}^m$ are random variables. ${}^{CH}D^{\alpha}x$ is the Caputo-modification of the Hadamard fractional derivative.

Chapter 1

Fractional Calculus

In this chapter, we introduce notations, definitions, lemmas and theorems which are used throughout this thesis, and several approaches to the generalization of the notion of differentiation and integration are considered. The choice has been reduced to those definitions which are related to applications.

1.1 Special functions

1.1.1 Euler's Gamma function

In the study of special functions a fundamental cornerstone is given by Euler's Gamma function. The reason herein lies in the fact that this function can be encountered in nearly all parts of the subject and furthermore many special functions can be expressed in term of the Gamma functions directly or by contour integration. Before we give a formal definition of Euler's Gamma function we need an additional definition, which will be used in the proofs for some properties of the Gamma function.

Definition 1.1.1. *The Euler constant γ is given by*

$$\gamma = \lim_{n \longrightarrow \infty} (\sum_{k=1}^{n} \frac{1}{n} - \ln(n)) = 0.5772156649. \tag{1.1.1}$$

The Euler constant is also known as Euler-Mascheroni constant.
There are a number of ways, how Euler's Gamma function can be defined. We give the one, which will be most useful for our later considerations in fractional calculus.

Definition 1.1.2. *For $z \in \mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$ Euler's Gamma function $\Gamma(z)$ defined as*

$$\Gamma(z) = \begin{cases} \displaystyle\int_0^{\infty} t^{z-1} e^{-t} dt & \text{if } Re(z) > 0 \\ \dfrac{\Gamma(z+1)}{z} & \text{if } Re(z) \leq 0, \quad z \neq 0, -1, -2, -3, \ldots \end{cases} \tag{1.1.2}$$

Euler's Gamma function is defined in the whole complex plane except zero and negative integers, where Euler's Gamma function has poles; the values in $(-1, 0)$ are uniquely given by the ones from $(0,\ 1)$, the values in $(-2, 1)$ are uniquely defined by the ones in $(-1, 0)$ and so on. Next we state some properties of Euler's Gamma function, which will become useful in later chapters.

Theorem 1.1.1. *[65] , [67] Euler's Gamma function satisfies the following properties:*

(1) For $Re(z) > 0$, the first part of definition (1.1.2) is equivalent to

$$\Gamma(z) = \int_0^1 (\ln(\frac{1}{t}))^{z-1} dt.$$

(2) For $z \in \mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$

$$\Gamma(z+1) = z\Gamma(z).$$

(3) For $n \in \mathbb{N}$

$$\Gamma(n+1) = (n-1)!.$$

(4) For $z \in \mathbb{C} \setminus \{0, 1, 2, 3, \ldots\}$

$$\Gamma(1-z) = -z\Gamma(-z).$$

(5) (Limit representation) For $Re(z) > 0$ the following limit holds:

$$\Gamma(z) = \lim_{n \longrightarrow \infty} \frac{n! n^z}{z(z+1)(z+2)(z+3) \ldots\ (z+n)}. \tag{1.1.3}$$

The Limit representation is equivalent to Euler's infinite product, given by

$$\frac{1}{z} \prod_{n=1}^{\infty} \frac{(1+\frac{1}{n})^z}{1+\frac{z}{n}}.$$

(6) (Weierstrass definition) Let $z \in \mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$ *. Then Euler's Gamma function can be defined by*

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} (1 + \frac{z}{n}) e^{-\frac{z}{n}}.$$

where g is the Euler constant (1.1.1) .

(7) Euler's Gamma function is analytic for all $z \in \mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$ *.*

(8) Euler's Gamma function is never zero.

(9) (Reflection Theorem) For all non-integer $z \in \mathbb{C}$*,*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad and \quad \Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin(\pi z)}.$$

(9) For half-integer arguments , $\Gamma(\frac{n}{2}),\ n \in \mathbb{N}$ *has the special form*

$$\Gamma(\frac{n}{2}) = \frac{(n-2)!!\sqrt{\pi}}{2^{\frac{n-1}{2}}},$$

where n!! is the double factorial :

$$n!! = \begin{cases} n.(n-2)\ldots 5.3.1 & if \ \ n > 0 \ \ n \ odd \\ n.(n-2)\ldots 6.4.2 & if \ \ n > 0 \ \ n \ even \\ 1 \ \ if \ \ \ n = 0 \ , -1 \end{cases} .$$

1.1.2 The Beta function

A special function, which is connected to Euler's Gamma function in a direct way, is given by the Beta function, defined as follows :

Definition 1.1.3. *The Beta function* $B(a, b)$ *in two variables* $a, b \in \mathbb{C}$ *is defined by*

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{1.1.4}$$

Again we state some properties of this special function, which we will use later on. Especially the Beta integral in the following theorem will be used for examples in the chapter on fractional calculus.

Theorem 1.1.2. *[65] , [67] The Beta function possesses the following properties :*

(1) For $Re(z), Re(w) > 0$, the relationship (1.1.4) is equivalent to

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}dt. \tag{1.1.5}$$

$$B(a,b) = 2\int_0^{\frac{\pi}{2}} (\sin t)^{2a-1}(\cos t)^{2b-1}dt. \tag{1.1.6}$$

(2) $B(a+1, b+1)$ is the solution of the Beta Integral :

$$\int_0^1 t^a(1-t)^b dt = B(a+1, b+1).$$

(3) The following identities hold :

(a) $B(a,b) = B(b,a)$

(b) $B(a,b) = B(a+1,b) + B(a,b+1)$

(c) $B(a,b+1) = \frac{b}{a}B(a+1,b) = \frac{b}{a+b}B(a,b)$

1.1.3 Mittag-Leffler function

Definition 1.1.4. *For $z \in \mathbb{C}$ the Mittag-Leffler Function $E_\alpha(z)$ is defined by :*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \ , \quad \alpha > 0 \tag{1.1.7}$$

and the generalized Mittag-Leffler Function $E_{\alpha,\beta}(z)$by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \ , \quad \alpha, \beta > 0. \tag{1.1.8}$$

In the following theorem we state some of the properties of the Mittag-Leffler function, which will be of some use later on in the analysis of ordinary as well as partial differential equations of fractional order.

Theorem 1.1.3. *[65] , [67] The Mittag-Leffler function possesses the following properties:*

(1) *For $|z| < 1$ the generalized Mittag-Leffler function satisfies*

$$\int_0^\infty e^{-t} t^{\beta-1} E_{\alpha,\beta}(t^\alpha z) dt = \frac{1}{z-1}.$$

(2) *For $|z| < 1$, the Laplace transform of the Mittag-Leffler function $E_\alpha(z^\alpha)$ is given by*

$$\int_0^\infty e^{-zt} E_\alpha(z^\alpha) dt = \frac{1}{z - z^{1-\alpha}}.$$

(3) *The Mittag-Leffler function (1.1.7) converges for every $z \in \mathbb{C}$.*

(4) *For special values a the Mittag-Leffler function is given by :*

(a) $E_0(z) = \frac{1}{z-1}$

(b) $E_1(z) = e^z$

(c) $E_2(z^2) = cosh(z)$

(d) $E_2(-z^2) = cos(z)$

(5) *The generalized Mittag-Leffler function possesses the following properties:*

(i) $E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \dfrac{1}{\Gamma(\beta)}.$

(ii) $E_{\alpha,\beta}(z) = \beta E_{\alpha,\alpha+\beta}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z).$

(iii)

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_\Upsilon \frac{\lambda^{\alpha-\beta} e^\lambda}{\lambda^\alpha - z} d\lambda$$

where Υ is a contour which starts and ends at $-\infty$ and encircles the disc $|\lambda| \leq |z|^{\frac{1}{\alpha}}$ counterclockwise. If $0 < \lambda < 1$, $\beta > 0$, then the asymptotic expansion of $E_{\alpha,\beta}$ as $z \to \infty$ is given by

$$E_{\alpha,\beta}(z) = \begin{cases} \frac{1}{\alpha} z^{1-\beta} \exp\left(z^{\frac{1}{\alpha}}\right) + \mathcal{E}_{\alpha,\beta}(z), & \textit{for } |arg(z)| \leq \frac{1}{2}\alpha\pi, \\ \mathcal{E}_{\alpha,\beta}(z), & \textit{for } \ |arg(-z)| \leq (1-\frac{1}{2})\alpha\pi, \end{cases}$$

where

$$\mathcal{E}_{\alpha,\beta}(z) = -\sum_{k=1}^{n-1} \frac{z^{-k}}{\Gamma(\beta - \alpha n)} + O(|z|^{-n}) \ \textit{as } z \to \infty.$$

Set

1.2 Fractional Derivatives and Integrals

1.2.1 Riemann-Liouville Integrals

Definition 1.2.1. *Let $\alpha \in \mathbb{R}_+$ and let f be continuous function on $[a, b]$. The operator I_a^α, defined on $L^1[a, b]$ by*

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \tag{1.2.1}$$

for $a < x < b$, is called the Riemann-Liouville fractional integral operator of order α .

For $\alpha = 0$, we set $I_a^0 = I$, the identity operator.

The definition for $\alpha = 0$ is quite convenient for future manipulations. It is evident that the Riemann-Liouville fractional integral coincides with the classical definition of I_a^α in the case $\alpha \in \mathbb{N}$, except for the fact that we have extended the domain from Riemann integrable functions to Lebesgue integrable functions (which will not lead to any problems in our development).Moreover, in the case $\alpha \geq 1$ it is obvious that the integral $I_a^\alpha f(x)$ exists for every $x \in [a, b]$ because the integrand is the product of an integrable function f and the continuous function $(x-t)^{\alpha-1}$. In the case $0 < \alpha < 1$ though, the situation is less clear at first sight. However, the following result asserts that this definition is justified. All the results of this section, can be found in [1, 27, 32–34, 57, 67, 90].

Theorem 1.2.1. *Let $f \in L^1[a,b]$ and $\alpha > 0$. Then, the integral $I_a^\alpha f(x)$ exists for almost every $x \in [a,b]$. Moreover, the function $I_a^\alpha f$ itself is also an element of $L_1[a,b]$.*

Proof. We write the integral in question as

$$\int_a^x (x-t)^{\alpha-1} f(t)dt = \int_{-\infty}^{\infty} \phi_1(x-t)\phi_2(t)dt$$

where

$$\phi_1(u) = \begin{cases} u^{\alpha-1} & for\ 0 < u < b-a \\ 0 & else \end{cases} .$$

and

$$\phi_2(u) = \begin{cases} f(u) & for\ a < u < b \\ 0 & else \end{cases} .$$

By construction , $\phi_j \in L_1(\mathbb{R})\ for\ j = 1,2$, and thus by a classical result on lebesgue integration . □

Theorem 1.2.2. *Let $\alpha, \beta > 0$ and $\phi \in L^1[a,b]$. Then,*

$$I_a^\alpha I_a^\beta \phi = I_a^{\alpha+\beta}\phi. \tag{1.2.2}$$

holds almost everywhere on $[a,b]$. If additionally $\phi \in C[a,b]$ or $\alpha + \beta > 1$, then the identity holds everywhere on $[a,b]$.

Proof. The neutral element of semigroup is ascertained by definition 1.2.1 . Therefore we only need to prove this relation holds almost every where . By definition of the fractional integral have

$$I_a^\alpha I_a^\beta \phi(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha-1} \int_a^t (x-s)^{\beta-1} \phi(s) ds dt.$$

We may interchange the order of integration, obtaining

$$\begin{aligned} I_a^\alpha I_a^\beta \phi(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_s^x (x-t)^{\alpha-1}(t-s)^{\beta-1}\phi(s) dt ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \phi(s) \int_s^x (x-t)^{\alpha-1}(t-s)^{\beta-1} dt ds \end{aligned}$$

The substitution $t = s + \tau(x-s)$ yields

$$\begin{aligned} I_a^\alpha I_a^\beta \phi(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \phi(s) \int_0^1 [(x-t)(1-\tau)]^{\alpha-1} \times [\tau(t-s)]^{\beta-1}(x-s) d\tau ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \phi(s)(x-s)^{\alpha+\beta-1} \int_0^1 (1-\tau)^{\alpha-1}\tau^{\beta-1} d\tau ds \end{aligned}$$

The term $\int_0^1 (1-\tau)^{\alpha-1}\tau^{\beta-1} d\tau ds$ is the Beta function , and thus

$$I_a^\alpha I_a^\beta \phi(x) = \frac{1}{\Gamma(\alpha+\beta)} \int_a^x \phi(s)(x-s)^{\alpha+\beta-1} ds = I_a^{\alpha+\beta}\phi(x).$$

holds almost everywhere on $[a, b]$. □

Corollary 1.2.3. *Under the assumptions of theorem 1.2.12,*

$$I_a^\alpha I_a^\beta \phi = I_a^\beta I_a^\alpha \phi.$$

There is an algebraic way to state this result.

Theorem 1.2.4. *The operators $\{I_a^\alpha : L_1[a,b] \longrightarrow L^1[a,b]; \alpha > 0\}$ form a commutative semigroup with respect to concatenation. The identity operator $I_a^0$is the neutral element of this semigroup.*

Theorem 1.2.5. *Let $\alpha > 0$. Assume that $(f_k)_{k=1}^\infty$ is a uniformly convergent sequence of continuous functions on $[a, b]$. Then we may interchange the fractional integral operator and the limit process, i.e.*

$$(I_a^\alpha \lim_{k\longrightarrow\infty} f_k)(x) = (\lim_{k\longrightarrow\infty} I_a^\alpha f_k)(x).$$

In particular, the sequence of functions $(I_a^\alpha f_k)_{k=1}^\infty$ is uniformly convergent.

Proof. For the first statement we utilize the well known fact, that if f denotes the limit of the sequence (f_k), the function f is continuous. For $\alpha = 0$ the stated result follows directly from the uniform convergence and for $\alpha > 0$ we can deduce

$$\begin{aligned} \mid I_a^\alpha f_k(x) - I_a^\alpha f(x) \mid &\leq \frac{1}{\Gamma(\alpha)} \int_a^x \mid f_k(t) - f(t) \mid (x-t)^{\alpha-1} dt \\ &\leq \frac{1}{\Gamma(\alpha+1)} \parallel f_k - f \parallel_\infty (b-a)^\alpha \end{aligned}$$

The last term converges uniformly to zero as $k \longrightarrow \infty$ for all $x \in [a, b]$. □

1.2.2 Riemann-Liouville Derivatives

Having established these fundamental properties of Riemann -Liouville integral operators, we now come to the corresponding differential operators.

Definition 1.2.2. *Let $\alpha \in \mathbb{R}_+$, $n = \lceil \alpha \rceil + 1$ and let f be continuous function on $[a, b]$. The operator D_a^α, defined by*

$$D_a^\alpha f(x) = D^n I_a^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x (x-t)^{n-\alpha-1} f(t) dt \quad (1.2.3)$$

for $a \leq x \leq b$ is called the Riemann-Liouville differential operator of order α . For $\alpha = 0$, we set $D_a^0 = I$, the identity operator.

Lemma 1.2.6. *Let $\alpha \in \mathbb{R}_+$ and let $n \in \mathbb{N}$ such that $n > \alpha$. Then,*

$$D_a^\alpha = D^n I_a^{n-\alpha}.$$

Proof. The assumption on n implies that $n \geq \lceil \alpha \rceil$. Thus,

$$D^n I_a^{n-\alpha} = D^{\lceil \alpha \rceil} D^{n-\lceil \alpha \rceil} I_a^{n-\lceil \alpha \rceil} I_a^{\lceil \alpha \rceil - \alpha} = D^{\lceil \alpha \rceil} I_a^{\lceil \alpha \rceil - \alpha} = D_a^\alpha.$$

in view of the semigroup property of fractional integration and the fact that ordinary differentiation is left-inverse to integer integration. □

Definition 1.2.3. *By AC^n or $AC^n[a, b]$ we denote the set of functions with an absolutely continuous $(n-1)$st derivative, i.e. the functions f for which there exists (almost everywhere) a function $g \in L^1[a, b]$ such that*

$$f^{(n-1)}(x) = f^{(n-1)}(a) + \int_a^x g(t) dt.$$

In this case we call g the (generalized) nth derivative of f, and we simply write $g = f^{(n)}$.

Lemma 1.2.7. *Let $f \in AC^1[a, b]$ and $0 < a < 1$. Then D_a^α exists almost everywhere in $[a, b]$. Moreover, $D_a^\alpha \in L^p[a, b]$ for $1 \leq p < \frac{1}{n}$ and*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(x-a)^\alpha} + \int_a^x (x-t)^{-\alpha} f'(t) dt \right). \quad (1.2.4)$$

Classical differential operators $\{D^n : n \in \mathbb{N}_0\}$ exhibit a semigroup property, which follows immediately from their definition. Furthermore, we have proven in Theorem 1.2.12 that the Riemann-Liouville integral operators also form a semigroup. The following theorem yields a similar result for the Riemann-Liouville differential operator

Theorem 1.2.8. *Assume that* $\alpha, \beta \geq 0$. *Moreover, let* $g \in L^1[a,b]$ *and* $f = I_a^{\alpha+\beta} g$. *Then*

$$D_a^\alpha D_a^\beta f = D_a^{\alpha+\beta} f. \tag{1.2.5}$$

Proof. By our assumption on f and the definition of the Riemann-Liouville differential operator,

$$D_a^\alpha D_a^\beta f = D_a^\alpha D_a^\beta I_a^{\alpha+\beta} g = D^{\lceil\alpha\rceil} I_a^{\lceil\alpha\rceil-\alpha} D^{\lceil\beta\rceil} I_a^{\lceil\beta\rceil-\beta} I_a^{\alpha+\beta} g.$$

The semigroup property of the integral operators allows us to rewrite this expression as

$$\begin{aligned} D_a^\alpha D_a^\beta f &= D^{\lceil\alpha\rceil} I_a^{\lceil\alpha\rceil-\alpha} D^{\lceil\beta\rceil} I_a^{\lceil\beta\rceil+\alpha} g \\ &= D^{\lceil\alpha\rceil} I_a^{\lceil\alpha\rceil-\alpha} D^{\lceil\beta\rceil} I_a^{\lceil\beta\rceil} I_a^\alpha g \end{aligned}$$

By the fact that the classical differential operator is left inverse to integer integration and the fact that the orders of the integral and differential operators involved are natural numbers the expression is equivalent to

$$D_a^\alpha D_a^\beta f = D^{\lceil\alpha\rceil} I_a^{\lceil\alpha\rceil-\alpha} I_a^\alpha g = D^{\lceil\alpha\rceil} I_a^\alpha g.$$

where we have once again used the semigroup property of fractional integration. Again applying the integer differential operator as left inverse of the integral we find that

$$D_a^\alpha D_a^\beta f = g.$$

The proof that $D_a^{\alpha+\beta} f = g$. goes along similar lines. □

Theorem 1.2.9. *Let* $\alpha > 0$. *Assume that* $(f_k)_{k=1}^\infty$ *is a uniformly convergent sequence of continuous functions on* $[a,b]$, *and that* $D_a^\alpha f_k$ *exists for every* k. *Moreover, assume that* $(D_a^\alpha f_k)_{k=1}^\infty$ *converges uniformly on* $[a+\epsilon, b]$ *for every* $\epsilon > 0$. *Then, for every* $x \in (a,b]$, *we have*

$$(D_a^\alpha \lim_{k\longrightarrow\infty} f_k)(x) = (\lim_{k\longrightarrow\infty} D_a^\alpha f_k)(x).$$

Theorem 1.2.10. *Let f and g be two functions defined on $[a, b]$ such that $D_a^\alpha f$ and $D_a^\alpha g$ exist almost everywhere. Moreover, let $c_1, c_2 \in \mathbb{R}$. Then, $D_a^\alpha(c_1 f + c_2 g)$ exists almost everywhere, and*

$$D_a^\alpha(c_1 f + c_2 g) = c_1 D_a^\alpha f + c_2 D_a^\alpha g.$$

Proof. This linearity property of the fractional differential operator is an immediate consequence of the definition of D_a^α . □

Having defined both, the Riemann-Liouville integral and the differential operator, we can now state results on the interaction of both. A first result is concerned with the inverse property of the two operators:

Theorem 1.2.11. *Let $\alpha > 0$. Then , for every $f \in L^1[a, b]$,*

$$D_a^\alpha I_a^\alpha f = f. \tag{1.2.6}$$

almost everywhere . If furthermore there exists a function $g \in L^1[a, b]$ such that $f = I_a^\alpha g$ then

$$I_a^\alpha D_a^\alpha f = f$$

almost everywhere.

Proof. Let $n = \lceil \alpha \rceil + 1$. Then, by the definition of D_a^α and the semigroup property of fractional integration and the left inverse of the classical differential operator,

$$D_a^\alpha I_a^\alpha f(x) = D^n I_a^{n-\alpha} I_a^\alpha f(x) = D^n I_a^n f(x) = f(x).$$

The second statement is an immediate consequence of the previous result: We have,that

$$I_a^\alpha D_a^\alpha f = I_a^\alpha [D_a^\alpha I_a^\alpha g] = I_a^\alpha g = f.$$

□

Theorem 1.2.12. *Assume that $\alpha \geq 0$ and $n = \lceil \alpha \rceil + 1$ and $f \in AC^n[a, b]$. Then*

$$I_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \lim_{z \longrightarrow a^+} D^{n-k-1} I^{n-\alpha} f(z). \tag{1.2.7}$$

Specifically, for $0 < \alpha < 1$ we have

$$I_a^\alpha D_a^\alpha f(x) = f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \lim_{z \longrightarrow a^+} I^{1-\alpha} f(z).$$

Corollary 1.2.13. *(Taylor expansion for Riemann-Liouville derivatives) Let* $\alpha \geq 0$ *and* $n = \lceil \alpha \rceil + 1$ *, . Assume that* f *is such that* $I^{n-\alpha} f \in AC^n[a,b]$ *. Then*

$$f(x) = \frac{(x-a)^{\alpha-n}}{\Gamma(\alpha-n+1)} \lim_{z \longrightarrow a^+} I^{n-\alpha} f(z) + \sum_{k=0}^{n-1} \frac{(x-a)^{k+\alpha-n}}{\Gamma(k+\alpha-n+1)} \lim_{z \longrightarrow a^+} D_a^{k+\alpha-n} f(z) + I_a^\alpha D_a^\alpha f(x).$$

A more complex result in the classical case was given by Leibniz' formula as generalized product rule. For Riemann-Liouville derivatives a similar result can be obtained:

Theorem 1.2.14. *(Leibniz' formula for Riemann-Liouville operators) Let* $\alpha > 0$*, and assume that* f *and* g *are analytic on* $(a-h, a+h)$*. Then,*

$$D_a^\alpha (fg)(x) = \sum_{k=0}^{\lceil \alpha \rceil} \binom{\alpha}{k} \left(D^k f\right)(x) \left(D_a^{\alpha-k} g\right)(x) + \sum_{k=\lceil \alpha \rceil+1}^{\infty} \binom{\alpha}{k} \left(D^k f\right)(x) \left(I_a^{k-\alpha} g\right)(x) \tag{1.2.8}$$

for $a < x < a + \dfrac{h}{2}$.

1.2.3 Caputo operator

In 1967 Caputo was published, where a new definition of a fractional derivative was used. In this section we state the definition and some properties of this new operator, today called Caputo fractional derivative and most importantly show its connection to the fractional Riemann-Liouville integral and differential operators. We begin with a formal definition:

Definition 1.2.4. *Let* $\alpha \in \mathbb{R}_+$ *and* $n = \lceil \alpha \rceil + 1$*. The operator* ${}_C D_a^\alpha$*, defined by*

$${}_C D_a^\alpha f(x) = I_a^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} (\frac{d}{dx})^n f(t) dt \tag{1.2.9}$$

for $a \leq x \leq b$*, is called the Caputo differential operator of order* α*.*

Theorem 1.2.15. *Let* $\alpha \geq 0$ *and* $n = \lceil \alpha \rceil + 1$ *. Moreover, assume that* $D_a^\alpha f$ *exists and f possesses (n - 1) derivatives at a. Then,*

$${}_C D_a^\alpha f(x) = D_a^\alpha \left[f - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} D^k f(a) \right]$$

almost everywhere .

Another way to express the relation between both fractional differential operators is given by the following lemma:

Lemma 1.2.16. *Let $\alpha \geq 0$ and $n = \lceil \alpha \rceil + 1$. Assume that f is such that both $D_a^\alpha f$ and ${}_C D_a^\alpha f$ exist . Then,*

$$ {}_C D_a^\alpha f(x) = D_a^\alpha f(x) - \sum_{k=0}^{n-1} \frac{D^k f(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha}. $$

An immediate consequence of this Lemma is

Lemma 1.2.17. *Let $\alpha \geq 0$ and $n = \lceil \alpha \rceil + 1$. Assume that f is such that both $D_a^\alpha f$ and ${}_C D_a^\alpha f$ exist. Moreover, let $D^k f(x_0) = 0$ for $k = 0, 1, \ldots, n-1$ (i.e. we assume f to have an n-fold zero at x_0). Then,*

$$ D_a^\alpha f =_C D_a^\alpha f. $$

This is especially important in view of differential equations of fractional order. It basically states, that those equations formulated with Riemann-Liouville derivatives coincide with those formulated with Caputo derivatives, if the initial condition(s) are homogeneous.
Considering the interaction of Riemann-Liouville integrals and Caputo differential operators, we find that the Caputo derivative is also a left inverse of the Riemann-Liouville integral:

Theorem 1.2.18. *If f is continuous and $\alpha > 0$, then*

$$ {}_C D_a^\alpha I_a^\alpha f = f. $$

Theorem 1.2.19. *Assume that $\alpha \geq 0$ and $n = \lceil \alpha \rceil + 1$, and $f \in AC^n[a,b]$. Then*

$$ I_a^\alpha \left({}_C D_a^\alpha f(x) \right) = f(x) - \sum_{k=0}^{n-1} \frac{D^k f(a)}{k!} (x-a)^k. $$

Corollary 1.2.20. *(Taylor expansion for Caputo derivatives) Let $\alpha \geq 0$ and $n = \lceil \alpha \rceil$. Assume that f is such that $f \in AC^n[a,b]$. Then*

$$ f(x) = \sum_{k=0}^{n-1} \frac{D^k f(a)}{k!} (x-a)^k + I_a^\alpha \left({}_C D_a^\alpha f(x) \right). $$

A comparison of this result with Taylor's expansion in case of Riemann-Liouville differential operators given in (1.2.7) will - apart from the simpler structure in the Caputo case .
In terms of derivation rules for the Caputo derivative of composed functions, we can find similar, but not identical, results to those for the Riemann-Liouville derivative. We start with the linearity

Theorem 1.2.21. *Let f and g be two functions defined on $[a,b]$ such that ${}_CD_a^\alpha f$ and ${}_CD_a^\alpha g$ exist almost everywhere. Moreover, let $c_1, c_2 \in \mathbb{R}$. Then, ${}_CD_a^\alpha(c_1 f + c_2 g)$ exists almost everywhere, and*

$$
{}_CD_a^\alpha(c_1 f + c_2 g) = c_1\left({}_CD_a^\alpha f\right) + c_2\left({}_CD_a^\alpha g\right).
$$

Theorem 1.2.22. *(Leibniz' formula for Caputo operators) Let $0 < \alpha < 1$, and assume that f and g are analytic on $(a-h, a+h)$. Then,*

$$
{}_CD_a^\alpha\,(fg)\,(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} g(a)\left(f(x) - f(a)\right) + \left({}_CD_a^\alpha g(x)\right) f(x)
$$

$$
+ \sum_{k=1}^{\infty} \binom{\alpha}{k} \left({}_CD_a^k f\right)(x)\left(I_a^{k-\alpha} g(x)\right)
$$

The next two results on the Caputo differential operator establish another significant difference between Riemann-Liouville and Caputo derivatives.

Lemma 1.2.23. *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$ and $n = \lceil\alpha\rceil + 1$. Moreover, assume that $f \in C^n[a,b]$. Then, ${}_CD_a^\alpha f$ is continuous on [a, b] and ${}_CD_a^\alpha f(a) = 0$.*

We may relax the conditions on f slightly to obtain the following result:

Lemma 1.2.24. *$\alpha > 0$, $\alpha \notin \mathbb{N}$ and $n = \lceil\alpha\rceil + 1$. Moreover, let that $f \in A^n[a,b]$ and assume that ${}_CD_a^{\hat{\alpha}} f \in C[a,b]$ for some $\hat{\alpha} \in (\alpha, n)$. Then, ${}_CD_a^\alpha f$ is continuous on $[a,b]$ and ${}_CD_a^\alpha f(a) = 0$.*

1.2.4 Hadamard fractional calculus

In this section we introduce some notations and definitions from the fractional calculus. For the notation, definitions and lemmas of this section, we cite [39, 52, 56].

Definition 1.2.5. *The Hadamard fractional integral of order $\alpha \in \mathbb{R}_+$ of a function $f : [a,b] \to \mathbb{R}^m, 0 < a < b \leq \infty$, is defined by*

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}$$

where $\Gamma(\cdot)$ is the Euler-Gamma function.

Definition 1.2.6. *The Hadamard derivative of order $\alpha \in [n-1, n)$, of the function $f : [a,b] \to \mathbb{R}^m, 0 < a < b \leq \infty$, is given by*

$${}^H Df(t) = \delta^n (J^{n-\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} f(s) \frac{ds}{s}$$

where $\delta := t\frac{d}{dt}$, $\delta^0 f(t) = f(t)$, and $n = [\alpha] + 1$ with $[\alpha]$ denoting the smallest integer greater than or equal to α.

Definition 1.2.7 ([52])**.** For an $n-$times differentiable function and $c > 0$. The Caputo type Hadamard fractional derivative of order $\alpha > 0$ of a function $f : [a, \infty) \to \mathbb{R}^m$ is

$${}^{CH} D^\alpha_{c^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} \delta^n g(s) \frac{ds}{s} = J_a^{n-\alpha} \delta^n f(t),$$

where $\alpha < n \leq \alpha + 1$, i.e., $n = [\alpha] + 1$, provided that the right-hand side exists.

The Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral in the space $L^p[a,b]$, $1 \leq p \leq \infty$, that is ${}^H D^\alpha J^\alpha f = f$.

Also, define

$$AC^n_\delta([a,b]) = \{f : [a,b] \to \mathbb{R}^m : \delta^n f \in AC^n([a,b])\}.$$

The Caputo-type modification of the left-sided and right-sided Hadamard fractional derivatives are defined respectively by

$${}^{CH} D^\alpha f(t) = {}^H D^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{\delta^k(a)}{k!} \left(\ln \frac{t}{a}\right)^k \right]$$

and

$$ {}^{CH}D^{\alpha}f(t) = {}^{H}D^{\alpha}\left[f(t) - \sum_{k=0}^{n-1}\frac{\delta^k(b)}{k!}\left(\ln\frac{b}{t}\right)^k\right]. $$

In particular, if $0 < \alpha < 1$, then

$$ {}^{CH}D^{\alpha}f(t) = {}^{H}D^{\alpha}\left[y(t) - f(a)\right], $$

and

$$ {}^{CH}D^{\alpha}f(t) = {}^{H}D^{\alpha}\left[y(t) - f(b)\right]. $$

Lemma 1.2.25. *, Let $\alpha > 0$ and $\beta > 0$. Then, given $0 < a < b < \infty$ and $1 \leq p < \infty$, for every $f \in L^p(a,b)$,*

$$ D^{\alpha}J^{\alpha}f = J^{\alpha-\beta}f \quad \textit{and } J^{\alpha}J^{\beta}f = J^{\alpha+\beta}f. $$

Lemma 1.2.26. *Let $\alpha > 0,\ n = [\alpha] + 1$ and $f \in C[a,b]$. Then*

$$ {}^{CH}D^{\alpha}J^{\alpha}f(t) = f(t) \quad t \in [a,b]. $$

Lemma 1.2.27. *Let $\alpha > 0,\ n = [\alpha]+1$, and $f \in AC^n_{\delta}[a,b]$ or $f \in C^n([a,b])$. Then,*

$$ J^{\alpha}\ {}^{CH}Df(t) = f(t) - \sum_{k=0}^{n-1}\frac{\delta^k(a)}{k!}\left(\ln\frac{t}{a}\right)^k. $$

Properties 1.2.28. *[56, 101] If $\alpha > 0,\ \beta > 0$, and $0 < a < b < \infty$, then we have:*

(1)

$$ \left({}_{H}J^{\alpha}_{a^+}(\ln\frac{t}{a})^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\ln\frac{x}{a})^{\beta+\alpha-1}. $$

(2)

$$ \left({}_{H}D^{\alpha}_{a^+}(\ln\frac{t}{a})^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln\frac{x}{a})^{\beta-\alpha-1}. $$

1.3 Random variable on fractional calculus

Let $(\Omega, \mathcal{A})$ be a measurable space; that is, a set Ω with a σ-algebra of subsets of Ω. A probability measure $\mathbb{P}$ is a measure on with $\mathbb{P}(\Omega) = 1$. Then $(\Omega, \mathcal{A}, \mathbb{P})$ is called a probability space. In the following, assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is a complete probability space. Let X be a metric space, $B(X)$ will be the σ-algebra of all Borel subsets of X. A measurable function $x : \Omega \longrightarrow X$ is called a random element in X. A random element in X is called a random variable.
Let X, Y are two locally compact, metric spaces and $f : \Omega \times X \to Y$. By $C(X, Y)$ we denote the space of continuous functions from X into Y endowed with the compact-open topology.

Lemma 1.3.1. *[76] f is a Carathéodory function if and only if $\omega \to r(\omega)(.) = f(\omega, .)$ is a measurable function from $\Omega \to C(X, Y)$.*

Proof. First assume that$f(\cdot, \cdot)$ is Carathéodory. Let B be a basis element for $C(X, Y)$ with compact-open topology. Then $B = \{g(.) \in C(X, Y) : g(K) \subseteq V\}$ where $K \subseteq X$ is compact, $V \subset Y$ is open. We need to show that $r^{-1}(B) \in \Sigma$. Let $\{x_n\}_{n>1}$ be dense in K. Then we have

$$\begin{aligned} r^{-1}(B) &= \{\omega \in \Omega : r(\omega)(\cdot) \in B\} \\ &= \{\omega \in \Omega : r(\omega)(K) \subset V\} \\ &= \{w \in \Omega : f(\omega, K) \subset V\} \\ &= \textstyle\bigcap_{n\geq 1} \{\omega \in \Omega : f(\omega, x_n) \in V\} \in \Sigma. \end{aligned}$$

Since by hypothesis $f(\cdot, \cdot)$ is Carathéodory.

Now assume that $\omega \longrightarrow r(\omega)(\cdot) = f(\omega, \cdot)$ is measurable from Ω into $C(X, Y)$. Let $(r, Id) : \Omega \times X \longrightarrow C(X, Y) \times X$ be defined by $(r, Id)(\omega, x) = (r(\omega)(\cdot), x)$. Clearly this is measurable. Let e be the evaluation map on $C(X, Y) \longrightarrow X$. We know that $e(\cdot, \cdot)$ is continuous. Consider the map $u : \Omega \times X \longrightarrow Y$ defined by $u = e \circ [(r, Id)]$. Then $u(\omega, x) = e[(r(\omega)(x)(\cdot), x)] = r(\omega)(x) = f(w, x) \implies r(\cdot, \cdot) = f(\cdot, \cdot)$. But $u(\cdot, \cdot)$ is a Carathéodory function. Hence so is $f(\cdot, \cdot)$. $\square$

Let $([0, b], L, \lambda)$ be a Lebesgue-measure space, where $b > 0$ and let $x : [0, b] \times \Omega \longrightarrow \mathbb{R}^m$ be a product measurable function. We say that $x(., .)$ is

sample path Lebesgue integrable on $[0,b]$ if $x:[0,b] \longrightarrow \mathbb{R}^m$ is Legesgue integrable on $[0,b]$ for a.e. $w \in \Omega$.
Let $\alpha > 0$. If $x:[0,b]\times\Omega \longrightarrow \mathbb{R}^m$ is sample path Lebesgue integrable on $[0,b]$ then we can consider the fractional integral

$$I^\alpha x(t,\omega) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s,\omega)ds. \tag{1.3.1}$$

which will be called the sample path fractional integral of x, where Γ is the Euler's Gamma function.

Remark 1.3.1. *If $x(.,\omega) : [0,b] \longrightarrow \mathbb{R}^m$ is Lebesgue integrable on $[0,b]$ for each $w \in \Omega$, then $t \longmapsto I^\alpha x(t,w)$ is also Lebesgue integrable on $[0,b]$ for each $w \in \Omega$.*

Definition 1.3.1. *A function $x:[0,b]\times\Omega \longrightarrow \mathbb{R}^m$ is said to be a Carathéodory function if $t \longmapsto x(t,w)$ is continuous for a.e. $w \in \Omega$ and $w \longmapsto x(t,w)$ is measurable for each $t \in [0,b]$. We recall that a Carathéodory function is a product measurable function .*

Proposition 1.3.2. *If $x:[0,b]\times\Omega \longrightarrow \mathbb{R}^m$ is a Carathéodory function, then the function $(t,w) \longmapsto I^\alpha x(t,w)$ is also a Carathéodory function.*

Proof. Clear that $I^\alpha : C([0,b],\mathbb{R}^m) \to \mathbb{R}^m$ is a continuous operator, let $L : \Omega \to C([0,b],\mathbb{R}^m)$ defined by $L(\omega)(.) = x(.,\omega)$. From lemma 1.3.1, $L(.)$ is measurable. Then the operator $\omega \to (I^\alpha \circ L)(\omega)(.)$ is measurable. Since the function $t \to I^\alpha x(t,\omega)$ is continuous function. Hence $(t,\omega) \to I^\alpha x(t,\omega)$ is a Carathéory function. $\square$

Definition 1.3.2. *A function $x:[0,b]\times\Omega \longrightarrow \mathbb{R}^m$ is said to have a sample path derivative at $t \in [0,b]$ if the function $t \longmapsto x(t,w)$ has a derivative at t for a.e. $w \in \Omega$. We will denote by $\frac{d}{dt}x(t,w)$ or by $x'(t,w)$ the sample path derivative of $x(.,w)$ at t. We say that $x:[0,b]\times\Omega \longrightarrow \mathbb{R}^m$ is sample path differentiable on $[0,b]$ if $x(.,.)$ has a sample path derivative for each $t \in [0,b]$ and possesses a one-sided sample path derivative at the end points 0 and b.*

Proposition 1.3.3. *If $x:[0,b]\times\Omega \longrightarrow \mathbb{R}^m$ is said to have a sample path derivative at $t \in [0,b]$ is a sample path absolutely continuous on $[0,b]$ (that is, $t \longmapsto x(t,w)$ is absolutely continuous on $[0,b]$ for a.e. $w \in \Omega$) , then the sample path derivative $x'(t,\omega)$ exists for λ-a.e. $t \in [0,b]$.*

Definition 1.3.3. *Let $x : [0,b] \times \Omega \longrightarrow \mathbb{R}^m$ be a sample path absolutely continuous on $[0,b]$ and let $\alpha \in (0,1]$. Then, for λ-a.e.$t \in [0,b]$ and for a.e. $w \in \Omega$, we define the Caputo sample path fractional derivative of x by:*

$$D^{\alpha}x(t,w) = I^{1-\alpha}x(t,w) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x'(s,w)ds. \tag{1.3.2}$$

Proposition 1.3.4. *If $x : [0,b] \times \Omega \longrightarrow \mathbb{R}^m$ is sample path differentiable on $[0,b]$ and $t \longmapsto x'(t,w)$ is continuous on $[0,b]$; then $D^{\alpha}x(t,w)$ exists for every $t \in [0,b]$ and $t \longmapsto D^{\alpha}x(t,w)$is continuous on $[0,b]$.*

Proposition 1.3.5. *If $x : [0,b] \times \Omega \longrightarrow \mathbb{R}^m$ is a Carathéodory function then :*

$$D^{\alpha}I^{\alpha}x(t,w) = x(t,w) \tag{1.3.3}$$

for all $t \in [0,b]$ and a.e. $w \in \Omega$.

Proposition 1.3.6. *If $x : [0,b] \times \Omega \longrightarrow \mathbb{R}^m$ is sample path absolutely continuous on $[0,b]$ then:*

$$I^{\alpha}D^{\alpha}x(t,w) = x(t,w) - x(0,w) \tag{1.3.4}$$

for all $t \in [0,b]$ and a.e. $w \in \Omega$.

Proposition 1.3.7. *If $x : [0,b] \times \Omega \longrightarrow \mathbb{R}^m$ is sample path absolutely continuous on $[0,b]$ then :*

$$t \longmapsto h(t,w) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x(s,w)ds.$$

is also sample path absolutely continuous on $[0,b]$. Moreover, for λ-a.e. $t \in [0,b]$ and a.e. $w \in \Omega$, we have that

$$h'(t,w) = \frac{d}{dt}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x(s,w)ds = D^{\alpha}x(t,w) + \frac{x(0,w)}{\Gamma(1-\alpha)}t^{-\alpha}. \tag{1.3.5}$$

For the definitions and propositions of this section we see Diethelm [32], Kilbas *et al* [57], Samko *et al* [90] and Podlubny [82].

1.4 Discrete Fractional Calculus

In this section, we recall from the literature some notations, definitions, and auxiliary results that will be used throughout this paper. All the results of this section, can be found in [3, 4, 14, 15, 17, 18].
Notations

$$\mathbb{N}_a = \{a, a+1, \dots\}, a \in \mathbb{R}, \tag{1.4.1}$$

$$\mathbb{N}_a(b) = \{a, a+1, \dots a+b+1\}, \quad a \in \mathbb{R}, b-a \geq 0, \; a-b \in \mathbb{Z}. \tag{1.4.2}$$

and

$$\mathbb{N}_0(b) = \{0, 1, \dots b+1\}, \quad b \in \mathbb{N}. \tag{1.4.3}$$

Definition 1.4.1. *(The Falling Function) Let $t \in \mathbb{R}$, $\alpha > 0$ we define the Falling function by :*

$$t^{\alpha} = t(t-1)(t-2)(t-3)(t-3) \; ... \; (t-\alpha+1). \tag{1.4.4}$$

Lemma 1.4.1. *For $\alpha > 0$*

$$t^{\alpha} = \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)} \tag{1.4.5}$$

Proof we have :

$$t^{\alpha} = t(t-1)(t-2)(t-3)(t-3) \; ... \; (t-\alpha+1)$$

we find that

$$t^{\alpha} = \frac{t(t-1)(t-2)(t-3)(t-4) \; ... \; (t-\alpha+1)\Gamma(t-\alpha+1)}{\Gamma(t-\alpha+1)}$$

Then :

$$t^{\alpha} = \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}.$$

Theorem 1.4.2. *(Power Rule). The following formula holds :*

$$\Delta t^{\alpha} = \alpha t^{\alpha-1}.$$

Lemma 1.4.3. *For $k \in \mathbb{N}_0^+$, we have :*

$$\sum_{s=0}^{s=k} (k-s+\alpha-1)^{\alpha-1} = \frac{\Gamma(k+\alpha+1)}{\alpha\Gamma(k+1)}. \tag{1.4.6}$$

Proof we have :

$$\sum_{s=0}^{s=k}(k-s+\alpha-1)^{\alpha-1}=\sum_{s=0}^{s=k}\frac{\Gamma(k-s+\alpha)}{\Gamma(k-s+1)}$$

and

$$\sum_{s=0}^{s=k}(k-s+\alpha-1)^{\alpha-1}=\sum_{s=0}^{s=k-1}\frac{\Gamma(k-s+\alpha)}{\Gamma(k-s+1)}+\Gamma(\alpha)$$

we find that

$$\sum_{s=0}^{s=k}(k-s+\alpha-1)^{\alpha-1}=\sum_{s=0}^{s=k-1}\frac{1}{\alpha}[\frac{\Gamma(k-s+\alpha+1)}{\Gamma(k-s+1)}-\frac{\Gamma(k-s+\alpha)}{\Gamma(k-s)}]+\Gamma(\alpha)$$

then

$$\sum_{s=0}^{s=k}(k-s+\alpha-1)^{\alpha-1}=\frac{1}{\alpha}[\frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}-\frac{\Gamma(\alpha+1)}{\Gamma(1)}]+\Gamma(\alpha)$$

then

$$\sum_{s=0}^{s=k}(k-s+\alpha-1)^{\alpha-1}=\frac{\Gamma(k+\alpha+1)}{\alpha\Gamma(k+1)}.$$

Lemma 1.4.4. *For $k\in\mathbb{N}_0^+$, we have :*

$$\sum_{s=0}^{s=k}\frac{\Gamma(k-s+\alpha)}{\Gamma(\alpha)\Gamma(k+s)}=\frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)\Gamma(k+1)}. \tag{1.4.7}$$

Proof we have :

$$\sum_{s=0}^{s=k}(k-s+\alpha-1)^{\alpha-1}=\frac{\Gamma(k+\alpha+1)}{\alpha\Gamma(k+1)}$$

and

$$(k-s+\alpha-1)^{\alpha-1}=\frac{\Gamma(k-s+\alpha)}{\Gamma(k-s+1)}$$

then

$$\sum_{s=0}^{s=k} \frac{\Gamma(k-s+\alpha)}{\Gamma(k-s+1)} = \frac{\Gamma(k+\alpha+1)}{\alpha\Gamma(k+1)}$$

we get that : $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$, then

$$\sum_{s=0}^{s=k} \frac{\Gamma(k-s+\alpha)}{\Gamma(k-s+1)} = \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \times \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}$$

then

$$\sum_{s=0}^{s=t} \frac{\Gamma(k-s+\alpha)}{\Gamma(\alpha)\Gamma(k-s)} = \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)\Gamma(k+1)}.$$

We define the forward difference operator by

$$(\Delta\varphi)(t) = \varphi(t+1) - \varphi(t),\ t \in \mathbb{N}_a,\ a \in \mathbb{R}.$$

Definition 1.4.2. *Let $\varphi : \mathbb{N}_a \longrightarrow \mathbb{R}$ and $\alpha > 0$. Then the $\alpha - th$ order fractional sum of φ started at a is defined by*

$$(\Delta_a^{-\alpha}\varphi)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)}\varphi(s). \tag{1.4.8}$$

where $\Delta_a^{-\alpha}\varphi$ is defined for $t \in \mathbb{N}_{a+\alpha}$. Moreover, we additionally define $(\Delta_a^{-0}\varphi)(t) = \varphi(t)$ for $t \in \mathbb{N}_a$.

For $\alpha = 1$, formula (1.3.5) takes the form

$$(\Delta_a^{-1}\varphi)(t) = \sum_{s=0}^{t-1} \varphi(s) = \int_a^t \varphi(s)ds$$

which is the delta integral of φ on the set $[a,t] \cap \mathbb{N}_0$.

Definition 1.4.3. *Let $\alpha \in (0,1]$. Then the difference operator is defined as*

$$(\Delta_a^{-\alpha}\varphi)(t) = (\Delta(\Delta_a^{-(1-\alpha)}\varphi))(t), t \in \mathbb{N}_{a+1-\alpha}. \tag{1.4.9}$$

where $(\Delta\varphi)(t) = \varphi(t+1) - \varphi(t)$ and $\varphi : \mathbb{N}_a \longrightarrow \mathbb{R}$.

Theorem 1.4.5. *Let φ be a real- valued function defined on $\mathbb{N}_a$ and let $\alpha, \beta > 0$. Then the following equalities hold :*

$$(\Delta_{\alpha+\beta}^{-\alpha}(\Delta_a^{-\beta}\varphi))(t) = (\Delta_a^{-(\alpha+\beta)}\varphi)(t) = (\Delta_{a+\alpha}^{-\beta}(\Delta_a^{-\alpha}\varphi))(t).$$

Theorem 1.4.6. *For any $\alpha > 0$ the following holds :*

$$(\Delta_a^{-\alpha}(\Delta\varphi))(t) = (\Delta(\Delta_a^{-\alpha)}\varphi))(t) - \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)}\varphi(a) \tag{1.4.10}$$

where φ is defined on $\mathbb{N}_a$.

Lemma 1.4.7. *Let $a \in \mathbb{R}$ and $p > 0$. Then*

$$\Delta(t-a)^{(p)} = p(t-a)^{(p-1)} \tag{1.4.11}$$

for any t for which both sides are well-defined . Furthermore , for $\alpha > 0$

$$\Delta_{a+p}^{-\alpha}(t-a)^{(p)} = p^{(-\alpha)}(t-a)^{(p+\alpha)}, t \in \mathbb{N}_{a+p+\alpha}. \tag{1.4.12}$$

and

$$\Delta_{a+p}^{\alpha}(t-a)^{(p)} = p^{(\alpha)}(t-a)^{(p-\alpha)}, t \in \mathbb{N}_{a+p+1-\alpha}.$$

Equation (1.4.12) can be also transformed as follows let $\varphi(s) = (s-a+p)^{(p)}$, then for $s \in \mathbb{N}_a$, $(\Delta_a^{-\alpha}\varphi)(s+\alpha) = \dfrac{\Gamma(p+1)}{\Gamma(p+\alpha+1)}(k+p+\alpha)^{(p+\alpha)}, s = a+k$.

Theorem 1.4.8. *Let $\alpha \in (0,1]$. Then for $t \in \mathbb{N}_a$ and $U : \mathbb{N}_{a-1} \longrightarrow \mathbb{R}$ the following formula holds :*

$$(\Delta_0^{-\alpha}(\Delta_{\alpha-1}^{\alpha}U))(t) = U(t) - \frac{(t)^{(\alpha-1)}}{\Gamma(\alpha)}U(\alpha-1), t \in \mathbb{N}_a \tag{1.4.13}$$

Theorem 1.4.9. *For any $\alpha > 0$ and $U : \mathbb{N}_a \longrightarrow \mathbb{R}$, the following equality holds :*

$$\Delta_a^{-\alpha}\Delta U(t) = \Delta\Delta_a^{-\alpha}U(t) - \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)}U(a), t \in \mathbb{N}_a. \tag{1.4.14}$$

Theorem 1.4.10. *For any real number α and any positive integer p and $U : \mathbb{N}_a \longrightarrow \mathbb{R}$, following equality :*

$$\Delta_a^{-\alpha}\Delta^p U(t) = \Delta^p\Delta_a^{-\alpha}U(t) - \sum_{k=0}^{p-1}\frac{(t-a)^{(\alpha-p+k)}}{\Gamma(\alpha+k-p+1)}\Delta^k U(a), t \in \mathbb{N}_a.$$

Lemma 1.4.11. *Let $\beta \neq -1$ and assume $\alpha + \beta + 1$ is not a non positive integer. Then :*

$$\Delta_a^{-\alpha} t^{(\beta)} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} t^{(\beta+\alpha)}.$$

Lemma 1.4.12. *: Let $0 \leq N - 1 < \alpha \leq N$. Then*

$$\Delta^{-\alpha} \Delta^{\alpha} x(t) = x(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_N t^{\alpha-N}$$

for some $C_i \in \mathbb{R}$ with $1 \leq i \leq N$.

1.5 Some inequalities

Theorem 1.5.1. *(Hölder's Inequality) : Let $p > 1$ and p and q be conjugate exponents. If $x \in l^p$ and $y \in l^q$, then*

$$\sum_{n=1}^{\infty} \mid x_n y_n \mid \leq \left(\sum_{n-1}^{\infty} \mid x_n \mid^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \mid y_n \mid^q \right)^{\frac{1}{q}}$$

where $x = (x_n)$, $y = (y_n)$. and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.5.2. *(Jensen's Inequality) : Let $f(x)$ be a convex function defined on an interval I. If $x_1, x_2, x_3, ..., x_N \in I$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N \in (0, 1)$ with $\sum_{i=1}^{N} \lambda_i = 1$, then*

$$f \left(\sum_{i=1}^{N} \lambda_i x_i \right) \leq \sum_{i=1}^{N} \lambda_i f(x_i).$$

Alternatively, if $f(x)$ is a convex function and $X \in \{x_i : 1, 2, ..., N\}$ is a random variable with probabiliites $P(x_i)$ where $\sum P(x_i) = 1$, then

$$f(E\{X\}) \leq E\{f(X)\}$$

$$f \left(\sum_{i=1}^{N} x_i P(x_i) \right) \leq \sum_{i=1}^{N} f(x_i) P(x_i).$$

Theorem 1.5.3. *(Markov's Inequality) : For a nonnegative random variable, $X : \Omega \longrightarrow \mathbb{R}$ where $X(s) \geq 0$ for all $s \in \Omega$, for any positive real number $a > 0$:*

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

Lemma 1.5.4. *[3] Let $p, q, f, u : \mathbb{N}_a \to \mathbb{R}_+$ are nonnegative functions such that*

$$u(k) \leq p(k) + q(k) \sum_{l=a}^{l=k-1} f(l)u(l), \quad for\ all \quad k \in \mathbb{N}_a.$$

Then

$$u(k) \leq p(k) + q(k) \sum_{l=a}^{l=k-1} p(l)f(l) \prod_{\tau=l+1}^{\tau=k-1} (1 + q(\tau)f(\tau)).$$

Lemma 1.5.5. *Let $v : [0, b] \to [0, \infty)$ be a real function and $w(\cdot)$ is a non-negative, locally integrable function on $[0, b]$. Assume that there are constants $a > 0$ and $0 < \gamma < 1$ such that*

$$v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t-s)^\gamma} ds,$$

then,there exists a constant $K = K(\beta)$ such that

$$v(t) \leq w(t) + Ka \int_0^t \frac{w(s)}{(t-s)^\gamma} ds,$$

for every $t \in [0, b]$.

We recall Gronwall's lemma for singular kernels, whose proof can be found in Lemma 7.1.1 of [61].

Lemma 1.5.6. *Let $v, a, \bar{a} : [1, b] \to [0, \infty)$ be continuous functions. If, for any $t \in [1, b]$,*

$$v(t) \leq a(t) + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds,$$

then there exists a constant $K = K(\beta)$ such that

$$v(t) \leq a(t) + \bar{a}(t) \int_1^t \left[\sum_{k=1}^{\infty} \frac{(\bar{a}(t)\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log \frac{t}{s}\right)^{\alpha-1} a(s) \right] \frac{ds}{s},$$

for every $t \in [1, b]$.

1.6 Vector metric space

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this section .We refer the reader to the monographs [92]
If, $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \dots, n$. Also $|x| = (|x_1|, \dots, |x_n|)$, $\max(x, y) = (\max(x_1, y_1), \dots, \max(x_n, y_n))$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \quad x_i > 0\}$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \dots, n$.

Definition 1.6.1. *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:*

(i) $d(u, v) \geq 0$ for all $u, v \in X$; if $d(u, v) = 0$ then $u = v$;

(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

We call the pair (X, d) a generalized metric space with $d(x, y) := \begin{pmatrix} d_1(x, y) \\ \vdots \\ d_n(x, y) \end{pmatrix}$.

Notice that d is a generalized metric space on X if and only if $d_i,\ i = 1, \dots, n$ are metrics on X.
For $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\} = \{x \in X : d_i(x_0, x) < r_i, i = 1, \dots, n\}$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\} = \{x \in X : d_i(x_0, x) \leq r_i, i = 1, \dots, n\}$$

the closed ball centered in x_0 with radius r. We mention that for generalized metric space, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces.
Let (X, d) be a generalized metric space we define the following metric spaces: Let $X_i = X,\ i = 1, \dots, n$. Consider $\prod_{i=1}^{n} X_i$ with $\bar{d}$:

$$\bar{d}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{i=1}^{n} d_i(x_i, y_i).$$

The diagonal space of $\prod_{i=1}^{n} X_i$ defined by

$$\widetilde{X} = \left\{ (x, \ldots, x) \in \prod_{i=1}^{n} X_i : x \in X, \; i = 1, \ldots, n \right\}.$$

Thus it is a metric space with the following distance

$$d_*((x, \ldots, x), (y, \ldots, y)) = \sum_{i=1}^{n} d_i(x, y), \text{ for each } x, y \in X.$$

It is clear that $\widetilde{X}$ is closed set in $\prod_{i=1}^{n} X_i$.

This is showed in the following result.

Lemma 1.6.1. *[92] Let (X, d) be a generalized metric space. Then there exists $h : X \to \widetilde{X}$ homeomorphism map.*

Definition 1.6.2. *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$.*

A classical result in matric analysis is the following theorem (see [9, 87, 98]).

Theorem 1.6.2. *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) *M is convergent towards zero;*

(ii) *$M^k \to 0$ as $k \to \infty$;*

(iii) *The matrix $(I - M)$ is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots,$$

(iv) *The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Definition 1.6.3. *Let (X,d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that*

$$d(N(x),N(y)) \leq M d(x,y) \text{ for all } x,y \in X.$$

For $n = 1$ we recover the classical Banach's contraction fixed point result.

Definition 1.6.4. *We say that a non-singular matrix $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ has the absolute value property if*

$$A^{-1}|A| \leq I,$$

where

$$|A| = (|a_{ij}|)_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+).$$

Some examples of matrices convergent to zero $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, which also satisfies the property $(I-A)^{-1}|I-A| \leq I$ are:

1) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a,b \in \mathbb{R}_+$ and $\max(a,b) < 1$

2) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a,b,c \in \mathbb{R}_+$ and $a+b<1,\ c<1$

3) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a,b,c \in \mathbb{R}_+$ and $|a-b|<1,\ a>1, b>0$.

Definition 1.6.5. *Let $Q \in \mathcal{M}_{2\times 2}(\mathbb{R})$ is said to be order preserving (or positive) if $p_1 \geq p_0$, $q_1 \geq q_0$ imply*

$$Q\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \leq Q\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}.$$

in the sense of components.

Lemma 1.6.3. *Let*

$$Q = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

where $a,b,c,d \geq 0$ and $detQ > 0$. Then Q^{-1} is order preserving.

1.7 Fixed point theorems

Theorem 1.7.1. *[81], [45] Let (X,d) be a complete generalized metric space and $N : X \to X$ a contractive operator with Lipschitz matrix M. Then N has a unique fixed point x_* and for each $x_0 \in X$ we have*

$$d(N^k(x_0), x_*) \leq M^k(I-M)^{-1}d(x_0, N(x_0)) \quad \text{for all } k \in \mathbb{N}.$$

Theorem 1.7.2. *[45] Let $(E, \|\cdot\|)$ be a generalized Banach space and $N : E \to E$ is a continuous compact mapping. Moreover assume that the set*

$$\mathcal{A} = \{x \in E : x = \lambda N(x) \quad \text{for some } \lambda \in (0,1)\}$$

is bounded. Then N has a fixed point.

Denote by $\mathcal{P}(X) = \{Y \subset X\colon\ Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ bounded}\}$. Let (X,d) and (Y,ρ) be two metric spaces and $F : X \to \mathcal{P}(Y)$ be a multi-valued mapping. Then F is said to be *lower semi-continuous* (*l.s.c.*) if the inverse image of V by F

$$F^{-1}(V) = \{x \in X\ :\ F(x) \cap V \neq \emptyset\}$$

is open for any open set V in Y. Equivalently, F is *l.s.c.* if the core of V by F

$$F^{+1}(V) = \{x \in X\ :\ F(x) \subset V\}$$

is closed for any closed set V in Y.

Likewise, the map F is called *upper semi-continuous* (*u.s.c.*) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $F(x_0)$, there exists an open neighborhood M of x_0 such that $F(M) \subseteq Y$. That is, if the set $F^{-1}(V)$ is closed for any closed set V in Y. Equivalently, F is *u.s.c.* if the set $F^{+1}(V)$ is open for any open set V in Y.

The mapping F is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq X$, $F(A)$ is relatively compact, i.e., there exists a relatively compact set $K = K(A) \subset X$ such that

$$F(A) = \bigcup\{F(x) : x \in A\} \subset K.$$

Also, F is *compact* if $F(X)$ is relatively compact, and it is called *locally compact* if for each $x \in X$, there exists an open set U containing x such that $F(U)$ is relatively compact.

Theorem 1.7.3. *Let* $F : X \to \mathcal{P}_{cp}(Y)$ *be a closed locally compact multifunction. Then* F *is u.s.c. (See , [45, 58]).*

Chapter 2

Fractional difference equations

In this chapter, we prove the existence of solutions and the compactness of solution sets of a system of fractional discrete equation.
More precisely , we consider the system of Caputo-type fractional difference equations:

$$\begin{cases} \Delta^{\alpha}_{\alpha-1}x(k) = f_1(k+\alpha-1, x(k+\alpha-1), y(k+\alpha-1)), \ k \in \mathbb{N}_0(b) \\ \Delta^{\alpha}_{\alpha-1}y(k) = f_2(k+\alpha-1, x(k+\alpha-1), y(k+\alpha-1)), \ k \in \mathbb{N}_0(b), \\ \Delta^{\alpha-1}_{\alpha-1}x(0) = x_0 \\ \Delta^{\alpha-1}_{\alpha-1}y(0) = y_0, \end{cases} \tag{2.0.1}$$

where $k \in \mathbb{N}_0(b) = \{0, 1, ..., b+1\}$, $0 < \alpha \leq 1$, and $f_1, f_2 : \mathbb{N}_{\alpha-1}(b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions.

The chapter is organized as follows . In Section 2.1 , we prove the existence and uniqueness and continuous dependence of solution to problem (2.0.1). The existence and compactness of solutions set to the problem (2.0.1) is investigated in Section 2.2.

2.1 Existence and Uniqueness

Before stating the results of this section we consider the following spaces.

$$C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) = \Big\{ x : \mathbb{N}_{\alpha-1}(b) \to \mathbb{R} \mid x \text{ continuous} \Big\}.$$

It is clear that $C(\mathbb{N}_\alpha(b), \mathbb{R})$ is a Banach space with norm

$$\|x\|_\infty = \sup_{t \in \mathbb{N}_{\alpha-1}(b)} |x(t)|.$$

Let $f_1, f_2 : \mathbb{N}_\alpha(b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are a continuous functions which satisfies the following assumptions:

(H_1) There exist $L_1,\ L_2,\ K_1,\ K_2 > 0$ such that

$$| f_1(k, x, y) - f_1(k, \overline{x}, \overline{y}) | \leq L_1 \mid x - \overline{x} \mid + L_2 \mid y - \overline{y} \mid,$$

and

$$| f_2(k, x, y) - f_2(k, \overline{x}, \overline{y}) | \leq K_1 \mid x - \overline{x} \mid + K_2 \mid y - \overline{y} \mid$$

for any $x, y, \overline{x}, \overline{y} \in \mathbb{R}$.

From (1.4.14) , we can easily prove the following auxiliary lemma.

Lemma 2.1.1. *A function $x : \mathbb{N}_{\alpha-1}(b) \to \mathbb{R}$ is a solution of the following problem*

$$\begin{cases} \Delta^{\alpha}_{\alpha-1} x(k) = f(k + \alpha - 1), \quad k \in \mathbb{N}_0(b) \\ \Delta^{\alpha-1}_{\alpha-1} x(0) = x_0, \end{cases} \tag{2.1.1}$$

if and only if x is solution of the following discrete equation

$$x(k + \alpha - 1) = \frac{(k + \alpha - 1)^{(\alpha-1)}}{\Gamma(\alpha)} x_0 + \sum_{s=0}^{k} \frac{(k + \alpha - s - 2)^{(\alpha-1)}}{\Gamma(\alpha)} f(s + \alpha - 1).$$

For our main consideration of Problem (2.0.1), a Preov fixed point is used to investigate the existence and uniqueness of solutions for system of nonlinear fractional discrete equations.

Theorem 2.1.2. *Assume that hypotheses (H_1) holds. If*

$$\begin{pmatrix} \sum_{s=0}^{b} \Gamma_\infty(s) L_1 & \sum_{s=0}^{b} \Gamma_\infty(s) L_2, \\ \sum_{s=0}^{b} \Gamma_\infty(s) K_1 & \sum_{s=0}^{b} \Gamma_\infty(s) K_2 \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$$

where

$$\Gamma_\infty(s) = \max_{k \in \mathbb{N}_0(b)} \frac{\Gamma(k - s + \alpha - 1)}{\Gamma(\alpha)\Gamma(k - s)}$$

be a matrix converge to zero. Then the Problem (2.0.1) has a unique solution.

Proof. Let $N : C(\mathbb{N}_{\alpha-1}(b),\mathbb{R})\times C(\mathbb{N}_{\alpha-1}(b),\mathbb{R}) \longrightarrow C(\mathbb{N}_{\alpha-1}(b),\mathbb{R})\times C(\mathbb{N}_{\alpha-1}(b),\mathbb{R})$ be defined by,

$$N(x,y) = (N_1(x,y), N_2(x,y)), \quad (x,y) \in C(\mathbb{N}_{\alpha-1}(b),\mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b),\mathbb{R}),$$

where

$$\begin{aligned} N_1(x(t),y(t)) &= \sum_{s=0}^{t-\alpha} \frac{(t-s-1)^{(\alpha-1)}}{\Gamma(\alpha)} f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \\ &+ \frac{t^{(\alpha-1)}}{\Gamma(\alpha)} x_0, \quad t \in \mathbb{N}_{\alpha-1}(b), \end{aligned}$$

and

$$\begin{aligned} N_2(x(t),y(t)) &= \sum_{s=0}^{t-\alpha} \frac{(t-s-1)^{(\alpha-1)}}{\Gamma(\alpha)} f_2(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \\ &+ \frac{t^{(\alpha-1)}}{\Gamma(\alpha)} y_0, \ t \in \mathbb{N}_{\alpha-1}(b). \end{aligned}$$

Let $(x,y), (\overline{x},\overline{y}) \in C(\mathbb{N}_{\alpha-1}(b),\mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b),\mathbb{R})$, then for every $t \in \mathbb{N}_{\alpha-1}(b), t = k+\alpha-1, \ k \in \mathbb{N}_0(b)$ we have

$$\begin{aligned} |N_1(x(t),y(t)) - N_1(\overline{x}(t),\overline{y}(t))| &\leq \sum_{s=0}^{t-\alpha} \frac{(t-s-1)^{(\alpha-1)}}{\Gamma(\alpha)} | \\ & f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \\ & -f_1((s+\alpha-1), \overline{x}(s+\alpha-1), \overline{y}(s+\alpha-1))|, \end{aligned}$$

From the definition $t^{(\alpha-1)}$, we get

$$\begin{aligned} |N_1(x(t),y(t)) - N_1(\overline{x}(t),\overline{y}(t))| &\leq \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} | \\ & f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \\ &- f_1(s+\alpha-1, \overline{x}(s+\alpha-1), \overline{y}(s+\alpha-1))|. \end{aligned}$$

Hence

$$\| N_1(x,y) - N_1(\overline{x},\overline{y}) \|_\infty \leq \sum_{s=0}^{b} \Gamma_\infty(s)[L_1 \| x - \overline{x} \|_\infty + L_2 \| y - \overline{y} \|_\infty].$$

Similarly,

$$\| N_2(x,y) - N_2(\overline{x},\overline{y}) \|_\infty \leq \sum_{s=0}^{b} \Gamma_\infty[K_1 \| x - \overline{x} \|_\infty + K_2 \| y - \overline{y} \|_\infty].$$

Thus

$$\| N(x,y) - N(\overline{x},\overline{y}) \|_\infty \leq \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \| x - \overline{x} \|_\infty \\ \| y - \overline{y} \|_\infty \end{pmatrix},$$

where

$$\Gamma_\infty(s) = \max_{k \in \mathbb{N}_0(b)} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)}, \; a_i = \sum_{s=0}^{b} \Gamma_\infty(s) L_i, \text{ and } a_{i+2} = \sum_{s=0}^{b} \Gamma_\infty(s) K_i, \quad i = 1, 2.$$

From theorem 1.7.1, the operator N has a unique fixed $(x,y) \in C(\mathbb{N}_{\alpha-1}(b),\mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b),\mathbb{R})$ which is unique solution of problem (2.0.1). □

Theorem 2.1.3. *Assume the following conditions*

(H_2) *There exist nonnegative functions* $\lambda_i, \gamma_i : \mathbb{N}_0(b) \to \mathbb{R}_+$ *for each* $i = 1, 2$

$$\begin{cases} |f_1(k,x,y) - f_1(k,\overline{x},\overline{y})| \leq \lambda_1(k)|x - \overline{x}| + \lambda_2(k)|y - \overline{y}| \\ |f_2(k,x,y) - f_2(k,\overline{x},\overline{y})| \leq \gamma_1(k|x - \overline{x}| + \gamma_2(k)|y - \overline{y}| \end{cases}$$

for all x, y, $\overline{x}$, $\overline{y} \in \mathbb{R}$.

(H_3) $h_1, h_2 : N_a \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *be functions such that*

$$|h_i(k,x,y)| \leq \mu_i(k), \;\; i = 1, 2,$$

where μ_i *are a nonnegative functions defined on* $\mathbb{N}_a$.

Then, for the solutions $(x(k,x_0), y(k,y_0))$ *and* $(u(k,u_0), v(k,v_0))$ *on* $\mathbb{N}_0(b)$ *of the initial value problem (2.0.1) and*

$$\begin{cases} \Delta^{\alpha}_{\alpha-1}u(k) &= h_1(k+\alpha-1, u(k+\alpha-1), v(k+\alpha-1)) \\ & \quad +f_1(k+\alpha-1, u(k+\alpha-1), v(k+\alpha-1)), \\ \Delta^{\alpha}_{\alpha-1}v(k) &= h_2(k+\alpha-1, u(k+\alpha-1), v(k+\alpha-1)) \\ & \quad +f_2(k+\alpha-1, u(k+\alpha-1), v(k+\alpha-1)), \\ \Delta^{\alpha-1}_{\alpha-1}u(0) &= u_0, \\ \Delta^{\alpha-1}_{\alpha-1}v(0) &= v_0, \end{cases} \tag{2.1.2}$$

where $k \in \mathbb{N}_0(b)$, $0 < \alpha \leq 1$, *and* $k+\alpha \in \mathbb{N}_\alpha(b)$.

the following inequality holds

(H_4)

$$|x(k,x_0) - u(k,u_0)| \leq L\left(|x_0-u_0| + |y_0-v_0| + \sum_{l=0}^{l=k}\mu(l)\right)\prod_{l=0}^{l=k}(1+\lambda(l)).$$

and

$$|y(k,x_0) - v(k,v_0)| \leq L\left(|y_0-v_0| + |x_0-u_0| + \sum_{l=0}^{l=k}\mu(l)\right)\prod_{l=0}^{l=k}(1+\lambda(l)).$$

where

$$\lambda(k) = \lambda_1(k) + \lambda_2(k) + \gamma_1(k) + \gamma_2(k), \quad \mu(k) = \mu_1(k) + \mu_2(k), \quad k \in \mathbb{N}(0).$$

Proof. The solutions of problems (2.0.1) and (2.1.2) are equivalent to

$$\begin{cases} x(k+\alpha-1) &= \displaystyle\sum_{s=0}^{k}\frac{(k+\alpha-s-2)^{(\alpha-1)}}{\Gamma(\alpha)}f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \\ & \quad +\displaystyle\frac{(k+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)}x_0 \\ \\ y(k+\alpha-1) &= \displaystyle\sum_{s=0}^{k}\frac{(k+\alpha-s-2)^{(\alpha-1)}}{\Gamma(\alpha)}f_2(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \\ & \quad +\displaystyle\frac{(k+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)}y_0 \end{cases}$$

and

$$\begin{cases} u(k+\alpha-1) &= \displaystyle\sum_{s=0}^{k} \frac{(k+\alpha-s-2)^{(\alpha-1)}}{\Gamma(\alpha)} \\ & \quad \times [h_1((s+\alpha-1), u(s+\alpha-1), v(s+\alpha-1)) \\ & \quad + f_1(s+\alpha-1, u(s+\alpha-1), v(s+\alpha-1))] + \dfrac{(k+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)} u_0 \\ v(k+\alpha-1) &= \displaystyle\sum_{s=0}^{k} \frac{(k+\alpha-s-2)^{(\alpha-1)}}{\Gamma(\alpha)} \\ & \quad \times [h_2((s+\alpha-1), u(s+\alpha-1), v(s+\alpha-1)) \\ & \quad + f_2(s+\alpha-1, u(s+\alpha-1), v(s+\alpha-1))] + \dfrac{(k+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)} v_0. \end{cases}$$

Hence

$$\begin{cases} x(k+\alpha-1) &= \dfrac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} x_0 + \\ & \quad \displaystyle\sum_{s=0}^{k} \frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)} f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)), \\ y(k+\alpha-1) &= \dfrac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} y_0 \\ & \quad + \displaystyle\sum_{s=0}^{k} \frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)} f_2(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \end{cases}$$

and

$$\begin{cases} u(k+\alpha-1) &= \displaystyle\sum_{s=0}^{k} \frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)} \times [h_1((s+\alpha-1), u(s+\alpha-1), v(s+\alpha-1)) \\ & \quad + f_1(s+\alpha-1, u(s+\alpha-1), v(s+\alpha-1))] + \dfrac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} u_0, \\ v(k+\alpha-1) &= \displaystyle\sum_{s=0}^{k} \frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)} \times [h_2((s+\alpha-1), u(s+\alpha-1), v(s+\alpha-1)) \\ & \quad + f_2(s+\alpha-1, u(s+\alpha-1), v(s+\alpha-1))] + \dfrac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} v_0. \end{cases}$$

Then, we get

$$x(k+\alpha-1)-u(k+\alpha-1)=\frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)}(x_0-u_0)+\sum_{s=0}^{k}\frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)}$$
$$\times[f_1(s+\alpha-1,x(s+\alpha-1),y(s+\alpha-1))-f_1(s+\alpha-1,u(s+\alpha-1),v(s+\alpha-1))]$$
$$-\sum_{s=0}^{k}\frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)}h_1(s+\alpha-1,u(s+\alpha-1),v(s+\alpha-1)),$$

and

$$y(k+\alpha-1)-v(k+\alpha-1)=\frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)}(y_0-v_0)+\sum_{s=0}^{k}\frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)}$$
$$\times[f_2(s+\alpha-1,x(s+\alpha-1),y(s+\alpha-1))-f_2(s+\alpha-1,u(s+\alpha-1),v(s+\alpha-1))]$$
$$-\sum_{s=0}^{k}\frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)}h_2(s+\alpha-1,u(s+\alpha-1),v(s+\alpha-1)).$$

This implies that

$$\begin{aligned}\mid x(k+\alpha-1)-u(k+\alpha-1)\mid \quad &\leq \quad \sum_{s=0}^{k}\frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)}\times[\lambda_1(s+\alpha-1)\mid x-u\mid \\ &+\lambda_2(s+\alpha-1)\mid y-v\mid]+\frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)}\mid x_0-u_0\mid \\ &+\sum_{s=0}^{k}\frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)}\mu_1(s+\alpha-1),\end{aligned}$$

and

$$\begin{aligned}\mid y(k+\alpha-1)-v(k+\alpha-1)\mid \quad &\leq \quad \sum_{s=0}^{k}\frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)}\times[\gamma_1(s+\alpha-1)\mid x-u\mid \\ &+\gamma_2(s+\alpha-1)\mid y-v\mid]+\frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)}\mid y_0-v_0\mid \\ &+\sum_{s=0}^{k}\frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)}\mu_2(s+\alpha-1).\end{aligned}$$

Then

$$\begin{cases} \mid x(k+\alpha-1) - u(k+\alpha-1) \mid & \leq & L_2 \sum_{s=0}^{k} [\lambda_1(s+\alpha-1) \mid x-u \mid + \\ & & \lambda_2(s+\alpha-1) \mid y-v \mid] \\ & & +L_2 \sum_{s=0}^{k} \mu_1(s+\alpha-1) + L_1 \mid x_0 - u_0 \mid \\ & & \\ \mid y(k+\alpha-1) - v(k+\alpha-1) \mid & \leq & L_2 \sum_{s=0}^{k} [\gamma_1(s+\alpha-1) \mid x-u \mid \\ & & +\gamma_2(s+\alpha-1) \mid y-v \mid] \\ & & +L_2 \sum_{s=0}^{k} \mu_2(s+\alpha-1) + L_1 \mid y_0 - v_0 \mid, \end{cases}$$

where

$$L_1 = \max_{k \in \mathbb{N}_0(b)} \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)}, \quad \text{and} \quad L_2 = \max_{k \in \mathbb{N}(b)} \max_{s \in \mathbb{N}_0(b)} \frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)}.$$

Set

$w(.) = |x(.) - u(.)| + |y(.) - v(.)|$, $\lambda(.) = \lambda_1(.) + \lambda_2(.) + \gamma_1(.) + \gamma_2(.)$ and $L = \max(L_1, L_2)$.

From above inequality, we obtain

$$w(k+\alpha-1) \leq L \left[|x_0 - u_0| + |y_0 - v_0| + \sum_{l=\alpha-1}^{k+\alpha-1} \lambda(l) w(l) + \sum_{l=\alpha-1}^{k+\alpha-1} \mu(l) \right].$$

Hence

$$w(k+\alpha-1) \leq L \left(|x_0 - u_0| + |y_0 - v_0| + \sum_{l=\alpha-1}^{k+\alpha-1} \mu(l) \right) \prod_{l=\alpha-1}^{k+\alpha-1} (1+\lambda(l)).$$

So

$$|x(k+\alpha-1, x_0) - u(k+\alpha-1, u_0)|$$
$$\leq L \left(|x_0 - u_0| + |y_0 - v_0| + \sum_{l=\alpha-1}^{l=k+\alpha-1} \mu(l) \right) \prod_{l=\alpha-1}^{l=k+\alpha-1} (1+\lambda(l)).$$

and

$$\begin{aligned}&|y(k+\alpha-1, x_0) - v(k+\alpha-1, v_0)|\\ &\leq L\left(|y_0 - v_0| + |x_0 - u_0| + \sum_{l=\alpha-1}^{l=k+\alpha-1} \mu(l)\right) \prod_{l=\alpha-1}^{l=k+\alpha-1} (1+\lambda(l)).\end{aligned}$$

□

2.2 Existence and Compactness results

Let $(E, \ |\cdot|)$ be a Banach space, we denote the space of continuous functions on $\mathbb{N}_{\alpha-1}(b)$ by

$$C(\mathbb{N}_{\alpha-1}(b), E) = \{y : \mathbb{N}_{\alpha-1}(b) \to E, \text{ is continuous}\}$$

with norm

$$\|y\|_\infty = \sup_{t\in\mathbb{N}_{\alpha-1}(b)} |y(t)|$$

is Banach space. Now we set the discrete Arzela-Ascoli Theorem.

Theorem 2.2.1. *[30] Let F be a closed subset of $C(\mathbb{N}_{\alpha-1}(b), E)$. If F is uniformly bounded and the set*

$$\{y(k+\alpha-1) : y \in F\}$$

is relatively compact for each $k \in \mathbb{N}_0(b)$, then F is compact.

Theorem 2.2.2. *Let $f_1, f_2 : \mathbb{N}_{\alpha-1}(b)\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions. Assume that condition :*

(H5) There exist $p_1, p_2, \bar{p}_1, \bar{p}_2 \in C(\mathbb{N}(0, b-1), \mathbb{R}_+)$ such that for any $(x, y) \in \mathbb{R}\times\mathbb{R}$ and $k \in \mathbb{N}_{\alpha-1}(b)$, we have

$$|f_1(k+\alpha-1, x, y)| \leq p_1(k+\alpha-1)(|x|+|y|) + \bar{p}_1(k+\alpha-1),$$

and

$$|f_2(k+\alpha-1, x, y)| \leq p_2(k+\alpha-1)(|x|+|y|) + \bar{p}_2(k+\alpha-1).$$

holds. Then the problem (2.0.1) has at least one solution. Moreover, the solution set $S(x_0, y_0)$ is compact and the multivalued map $S : (x_0, y_0) \multimap S(x_0, y_0)$ is u.s.c.

Proof. Clearly, the fixe point of N are solutions to (2.0.1), we first show that N is completely continuous. The proof will be given in several steps.

- **Step 1.** N is continuous.
  Let (x_m, y_m) be a sequence such that $(x_m, y_m) \to (x, y) \in C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b), \mathbb{R})$ as $m \to \infty$. Then

$$\begin{aligned}
|N_1(x_m(t), y_m(t)) - N_1(x(t), y(t))| &= \Bigg|\sum_{s=0}^{t-\alpha} \frac{\Gamma(t-s)}{\Gamma(\alpha)\Gamma(t-s-\alpha+1)}[\\
& \quad f_1(s+\alpha-1, x_m(s+\alpha-1), y_m(s+\alpha-1)) \\
&- \quad f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1))]\Bigg| \\
&\leq \sum_{s=0}^{k} \frac{\Gamma(t-s)}{\Gamma(\alpha)\Gamma(t-s-\alpha+1)} \; | \\
& \quad f_1(s+\alpha-1, x_m(s+\alpha-1), y_m(s+\alpha-1)) \\
&- \quad f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \; | \; .
\end{aligned}$$

  Then

$$\begin{aligned}
\|N_1(x_m, y_m) - N_1(x, y)\|_\infty &\leq \sum_{s=0}^{b} \Gamma_\infty(s) \; | \; [\\
& \quad f_1(k+\alpha-1, x_m(s+\alpha-1), y_m(s+\alpha-1)) \\
&- \quad f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \; |].
\end{aligned}$$

  Similarly

$$\begin{aligned}
\|N_2(x_m, y_m) - N_2(x, y)\|_\infty &\leq \sum_{s=0}^{b} \Gamma_\infty(s)[| \\
& \quad f_2(s+\alpha-1, x_m(s+\alpha-1), y_m(s+\alpha-1)) \\
& \quad -f_2(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \; |],
\end{aligned}$$

where

$$\Gamma_\infty(s) = \max_{t\in\mathbb{N}_{\alpha-1}} \frac{\Gamma(t-s)}{\Gamma(\alpha)\Gamma(t-s-\alpha+1)}$$

Since f_1, f_2 are continuous functions, we get

$$\| N_1(x_m, y_m) - N_1(x,y) \|_\infty \longrightarrow 0 \text{ as } m \longrightarrow \infty$$

and

$$\| N_2(x_m, y_m) - N_2(x,y) \|_\infty \longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

- **Step 2.** N maps bounded sets into bounded sets in $C(\mathbb{N}_{\alpha-1}(b),\mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b),\mathbb{R})$. Indeed, it is enough to show that for any $q > 0$ there exists a positive constant l such that for each $(x,y) \in B_q = \{(x,y) \in C(\mathbb{N}_{\alpha-1}(b),\mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b),\mathbb{R}) : \|x\|_\infty \leq q,\ \|y\|_\infty \leq q\}$, we have

$$\|N(x,y)\|_\infty \leq l = (l_1, l_2).$$

Then for each $k \in \mathbb{N}_0$ and $t = k+\alpha-1$, we get

$$\begin{aligned}
|N_1(x(t),y(t))| &\leq \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} \mid f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \mid \\
&\quad + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)}|x_0| \\
&\leq \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)}[|x_0| + \sum_{s=0}^{k} \mid f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \mid]
\end{aligned}$$

Therefore

$$\|N_1(x,y)\|_\infty \leq L_1[|x_0| + \sum_{l=\alpha-1}^{k+\alpha-1} 2(qp_1(l) + \bar{p}_1(l))] := l_1.$$

Similarly,

$$\|N_2(x,y)\|_\infty \leq L_2[|y_0| + \sum_{l=\alpha-1}^{k+\alpha-1} (2qp_2(l) + \bar{p}_2(l))] := l_2.$$

where

$$L_1 = \max_{k\in\mathbb{N}_{\alpha-1}(b)} \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)}, \quad \text{and} \quad L_2 = \max_{k\in\mathbb{N}(b)} \max_{s\in\mathbb{N}_{\alpha-1}(b)} \frac{\Gamma(k+\alpha-s-1)}{\Gamma(k-s)\Gamma(\alpha)}.$$

Moreover, for each $k \in \mathbb{N}_0(b)$, we have

$$\{N_1(x(k+\alpha-1), y(k+\alpha-1)) : (x,y) \in B_q\}, \quad \{N_2(x(k+\alpha-1), y(k+\alpha-1)) : (x,y) \in B_q\}$$

are relatively compact in $\mathbb{R}$. Then from theorem 2.2.1 we conclude that $\overline{N(B_q \times B_q)}$ is compact in $C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b), \mathbb{R})$. As a consequence of Steps 1 to 2 , $N : C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) \to C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b), \mathbb{R})$ is completely continuous.

Step 3. It remains to show that

$$\mathcal{A} = \{(x,y) \in C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) : \ (x,y) = \lambda N(x,y), \lambda \in (0,1)\}$$

is bounded.
Let $(x,y) \in \mathcal{A}$. Then $x = \lambda N_1(x,y)$ and $y = \lambda N_2(x,y)$ for some $0 < \lambda < 1$. Thus, for $k \in \mathbb{N}_0(b)$, we have

$$\begin{aligned} |x(k+\alpha-1)| &\le L[|x_0| + \sum_{l=\alpha-1}^{k+\alpha-1} |f(l, x(l), y(l))|] \\ &\le L[|x_0| + \sum_{l=\alpha-1}^{k+\alpha-1} \bar{p}_1(l) + \sum_{l=\alpha-1}^{k+\alpha-1} p_1(l)(|x(l)| + |y(l)|)], \end{aligned}$$

and

$$|y(k+\alpha-1)| \le L[|y_0| + \sum_{l=\alpha-1}^{k+\alpha-1} \bar{p}_2(l) + \sum_{l=\alpha-1}^{k+\alpha-1} p_2(l)(|x(l)| + |y(l)|)].$$

Therefore

$$\begin{aligned} |x(k+\alpha-1)| + |y(k+\alpha-1)| &\le L[|x_0| + |y_0| + \sum_{l=\alpha-1}^{k+\alpha-1} (\bar{p}_1(l) + \bar{p}_2(l)) \\ &\quad + \sum_{l=\alpha-1}^{k+\alpha-1} p(l)(|x(l)| + |y(l)|)], \end{aligned}$$

where

$$L = \max(L_1, L_2),\ p(k) = p_1(k)+p_2(k),\ \bar{p}(k) = \sum_{l=\alpha-1}^{k+\alpha-1} (\bar{p}_1(l)+\bar{p}_2(l)) \quad k \in \mathbb{N}_{\alpha-1}(b).$$

By theorem 2.1.3, we have

$$\begin{aligned} |x(k+\alpha-1)| + |y(k+\alpha-1)| \quad &\leq \quad L\left(|x_0| + |y_0| + \sum_{l=\alpha-1}^{b+\alpha-1} \bar{p}(l)\right) \times \\ &\left(1 + \sum_{l=\alpha-1}^{k+\alpha-1} p(k) \prod_{l+1}^{k+\alpha-1} (1+p(\tau))\right). \end{aligned}$$

Hence

$$\|x\|_\infty + \|y\|_\infty \leq L\left(|x_0| + |y_0| + \sum_{l=\alpha-1}^{b+\alpha-1} \bar{p}(l)\right)\left(1 + \sum_{l=\alpha-1}^{k+\alpha-1} p(k) \prod_{l+1}^{k+\alpha-1} (1+p(\tau))\right).$$

This shows that $\mathcal{A}$ is bounded. As a consequence of Theorem 1.7.2 we deduce that N has a fixed point (x, y) which is a solution to the problem (2.0.1).

Step 4: Compactness of the solution set. For each $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$, let

$$S(x_0, y_0) = \{(x, y) \in C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) : (x, y) \text{ is a solution of } (2.0.1)\}.$$

From Step 3, there exists $\widetilde{M}$ such that for every $(x, y) \in S((x_0, y_0))$, we have

$$\|x\|_\infty \leq \widetilde{M}; \|y\|_\infty \leq \widetilde{M}.$$

Since N is completely continuous, $N(S(x_0, y_0))$ is relatively compact in $(x, y) \in C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b), \mathbb{R})$. Let $(x, y) \in S(x_0, y_0))$; then $(x, y) = N(x, y)$ hence $S(x_0, y_0) \subset \overline{N(S(x_0, y_0))}$. It remains to prove that $S(x_0, y_0)$ is a closed subset in $(x, y) \in C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b), \mathbb{R})$. Let $\{(x_m, y_m) : m \in \mathbb{N}\} \subset S(x_0, y_0)$ be such that $(x_m, y_m)_{m\in\mathbb{N}}$ converges to (x, y). For every $m \in \mathbb{N}$, and $k \in \mathbb{N}_0(b)$, we have

$$\begin{aligned} x_m(k+\alpha-1) \quad &= \quad \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x_m(s+\alpha-1), y_m(s+\alpha-1)) \\ &+ \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} x_0 \end{aligned}$$

and

$$y_m(k+\alpha-1) = \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x_m(s+\alpha-1), y_m(s+\alpha-1)) + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} y_0.$$

Set

$$z_1(k+\alpha-1) = \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} x_0$$

and

$$z_2(k+\alpha-1) = \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} y_0.$$

Since f_1 and f_2 are continuous functions, we can prove that

$$x_m \to z_1, \quad y_m \to z_2, \quad \text{as } m \to \infty.$$

Thus, we concluded,

$$x(k+\alpha-1) = \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} x_0, \quad k \in \mathbb{N}_{\alpha-1}(b),$$

and

$$y(k+\alpha-1) = \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} y_0, \quad k \in \mathbb{N}_{\alpha-1}(b).$$

Therefore $(x, y) \in S(x_0, y_0)$ which yields that $S(x_0, y_0)$ is closed, hence compact subset in $C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b), \mathbb{R})$. Finally, we prove that $S(.)$ is $u.s.c.$ by proving that the graph of S

$$\Gamma_S := \{(\bar{x}, \bar{y}, x, y) : \ (x, y) \in S(\bar{x}, \bar{y})\}$$

is closed. Let $(\bar{x}_m, \bar{y}_m, x_m, y_m) \in \Gamma_S$ be such that $(\bar{x}_m, \bar{y}_m, x_m, y_m) \to (\bar{x}, \bar{y}, x, y)$ as $m \to \infty$. Since $(x_m, y_m) \in S(\bar{x}_m, \bar{y}_m)$, then

$$\begin{aligned} x_m(k+\alpha-1) &= \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x_m(s+\alpha-1), y_m(s+\alpha-1)) \\ &\quad + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} x_m \end{aligned}$$

and

$$\begin{aligned} y_m(k+\alpha-1) &= \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x_m(s+\alpha-1), y_m(s+\alpha-1)) \\ &\quad + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} y_m. \end{aligned}$$

Arguing as in Step 2, we can prove that

$$\begin{aligned} x(k+\alpha-1) &= \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \\ &\quad + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} \bar{x}, \end{aligned}$$

and

$$\begin{aligned} y(k+\alpha-1) &= \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \\ &\quad + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} \bar{y}. \end{aligned}$$

Thus, $(x, y) \in S(\bar{x}, \bar{y})$. Now, we show that S maps bounded sets into relatively compact sets of $C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b), \mathbb{R})$. Let B be a bounded set in $\mathbb{R} \times \mathbb{R}$ and let $\{(x_m, y_m)\} \subset S(B)$. Then there exists $\{(\bar{x}_m, \bar{y}_m)\} \subset B$ such that

$$\begin{aligned} x_m(k+\alpha-1) &= \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x_m(s+\alpha-1), y_m(s+\alpha-1)) \\ &\quad + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} \bar{x}_m \end{aligned}$$

and

$$\begin{aligned} y_m(k+\alpha-1) &= \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x_m(s+\alpha-1), y_m(s+\alpha-1)) \\ &\quad + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} \bar{y}_m. \end{aligned}$$

Since $\{(\bar{x}_m, \bar{y}_m)\}$ is a bounded sequence, there exists a subsequence of $\{(\bar{x}_m, \bar{y}_m)\}$ converging to $(\bar{x}, \bar{y})$. As in Step 2, we can show that $\{(x_m, y_m) : m \in \mathbb{N}\}$ is uniformly bounded. Then there exists a subsequence of $\{(x_m, y_m)\}$ converging to (x, y) in $C(\mathbb{N}_{\alpha-1}(b), \mathbb{R}) \times C(\mathbb{N}_{\alpha-1}(b), \mathbb{R})$. By the continuity of f_1 and f_2, we can prove that

$$\begin{aligned} x(k+\alpha-1) &= \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \\ &\quad + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} \bar{x} \end{aligned}$$

and

$$\begin{aligned} y(k+\alpha-1) &= \sum_{s=0}^{k} \frac{\Gamma(k-s+\alpha-1)}{\Gamma(\alpha)\Gamma(k-s)} f_1(s+\alpha-1, x(s+\alpha-1), y(s+\alpha-1)) \\ &\quad + \frac{\Gamma(k+\alpha)}{\Gamma(k)\Gamma(\alpha)} \bar{y}. \end{aligned}$$

Thus, $(x, y) \in \overline{S(B)}$. Then $S(.)$ is *u.s.c.*

$\square$

Chapter 3

Random fractional differential equations

In this chapter, we prove the existence of solutions to the random system of fractional differential equations:

$$\begin{cases} D^{\alpha}x(t,\omega) = f(t,x(t,\omega),y(t,\omega),\omega),\ 0<\alpha<1,\ t\in[0,b], \\ D^{\beta}y(t,\omega) = g(t,x(t,\omega),y(t,\omega),\omega),\ 0<\beta<1,\ t\in[0,b], \\ x(0,\omega) = x_0(\omega),\ \omega\in\Omega \\ y(0,\omega) = y_0(\omega),\ \omega\in\Omega, \end{cases} \tag{3.0.1}$$

where $f, g : [0,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\Omega\to\mathbb{R}^m$, $(\Omega,\mathcal{A})$ is a measurable space and $x_0, y_0 : \Omega\to\mathbb{R}^m$ are random variable. $D^{\alpha}x$ is the Caputo fractional derivative of x with respect to the variable $t\in[0,b]$ with $b>0$.

The chapter is organized as follows. In Section 3.1, we prove the existence and uniqueness and compactness of solutions set for a system of fractional random differential equations with initial condition. In Section 3.2, we give an example .

3.1 Existence and Uniqueness

Definition 3.1.1. *A random operator $T:\Omega\times X\to X$ is said to be continuous at $x_0\in X$*
if $\lim_{n\to\infty}\|x_n-x_0\|=0$ *implies* $\lim_{n\to\infty}\|T(\omega,x_n)-T(\omega,x)\|=0$ *a.s.*

Theorem 3.1.1. *[92] Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, X be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a continuous random operator, and let $M(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matric such that $M(\omega)$ converge to 0 a.s. and*

$$d(F(\omega, x_1), F(\omega, x_2)) \leq M(\omega)d(x_1, x_2) \text{ for each } x_1, x_2 \in X,\ \omega \in \Omega.$$

then there exists any random variable $x : \Omega \to X$ which is the unique random fixed point of F.

Theorem 3.1.2. *[92] Let $(\Omega, \mathcal{F})$ be a measurable space, X be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a continuous random operator, and let $M(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matric such that for every $\omega \in \Omega$ the matrix, $M(\omega)$ converge to 0 and*

$$d(F(\omega, x_1), F(\omega, x_2)) \leq M(\omega)d(x_1, x_2) \text{ for each } x_1, x_2 \in X,\ \omega \in \Omega.$$

then there exists any random variable $x : \Omega \to X$ which is the unique random fixed point of F.

Theorem 3.1.3. *[92] Let X be a separable generalized Banach space and let $F : \Omega \times X \to X$ be a completely continuous random operator. Then, either*

(i) the random equation $F(\omega, x) = x$ has a random solution, i.e., there is a measurable function $x : \Omega \to X$ such that $F(\omega, x(\omega)) = x(\omega)$ for all $\omega \in \Omega$, or

(ii) the set $\mathcal{M} = \{x : \Omega \to X \text{ is measurable } / \ \lambda(\omega)F(\omega, x) = x\}$ is unbounded for some measurable $\lambda : \Omega \to X$ with $0 < \lambda(\omega) < 1$ on Ω.

Definition 3.1.2. *A function $f : [0, b] \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ is called random Carathéodory if the following conditions are satisfied:*

(i) the map $(t, \omega) \to f(t, x, \omega)$ is jointly measurable for all $x \in \mathbb{R}^m$,

(ii) the map $x \to f(t, x, \omega)$ is continuous for all $t \in [0, b]$ and $\omega \in \Omega$.

Definition 3.1.3. *A Carathéodory function $f : [0, b]\times\mathbb{R}^m\times\Omega \to \mathbb{R}^m$ is called random L^1-Carathéodory if for each real number $r > 0$ there is a measurable and bounded function $h_r \in L^1([0, b], \mathbb{R}_+)$ such that*

$$\|f(t, x, \omega)\| \leq h_r(t, \omega), \quad a.e. \ \ t \in [0, b]$$

for all $\omega \in \Omega$ and $x \in \mathbb{R}$ with $\|x\| \leq r$.

Lemma 3.1.4. *Concerning the problem :*

$$\begin{cases} D^\alpha x(t,\omega) = f(t, x(t,\omega), y(t,\omega), \omega), \;\; 0 < \alpha < 1 \\ x(0,\omega) = x_0(\omega), \;\; \omega \in \Omega \end{cases} \tag{3.1.1}$$

If $(t,w) \longmapsto f(t, x(t,\omega), y(t,\omega), \omega)$ *is product measurable and* $t \longmapsto f(t, x(t,\omega), y(t,\omega), \omega)$ *is Lebesgue integrable on* $[0,b]$ *for a.e.* $w \in \Omega$ *then, the function* $x : [0,b] \times \Omega \longrightarrow \mathbb{R}^m$ *is a solution for (3.1.1) if and only if*

$$x(t,w) = x_0(w) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s,\omega), y(s,\omega), \omega) ds$$

for all $t \in [0,b]$ *and for a.e.* $w \in \Omega$, $0 < \alpha < 1$.

Proof. We have:

$$D^\alpha x(t,\omega) = f(t, x(t,\omega), y(t,\omega), \omega).$$

Then

$$I^\alpha D^\alpha x(t,\omega) = I^\alpha f(t, x(t,\omega), y(t,\omega), \omega)$$

From proposition 1.3.5, we get

$$I^\alpha D^\alpha x(t,\omega) = x(t,w) - x(0,w).$$

Thus

$$x(t,w) - x(0,w) = I^\alpha f(t, x(t,\omega), y(t,\omega), \omega).$$

Next, the definition of I^α, we have:

$$x(t,w) - x(0,w) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s,w) ds.$$

Hence

$$x(t,w) = x(0,w) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s,w) ds.$$

We can state the solution for the problem (3.1.1) :

$$x(t,w) = x_0(w) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s,\omega), y(s,\omega), \omega) ds.$$

Similarly we have

$$y(t,w) = y_0(w) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s, x(s,\omega), y(s,\omega), \omega)ds.$$

$\square$

Our main first result is the existence and uniqueness of random solution of the problem (3.0.1).

Theorem 3.1.5. *Let $f, g : [0,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\Omega\to\mathbb{R}^m$ are two Carathéodory functions. Assume that the following condition*

($\mathcal{L}_1$) *There exist $p_1, p_2, p_3, p_4 : \Omega\to\mathbb{R}_+$ are random variable such that*

$$\|f(t,x,y,\omega)-f(t,\widetilde{x},\widetilde{y},\omega)\| \leq p_1(\omega)\|x-\widetilde{x}\|+p_2(\omega)\|y-\widetilde{y}\|,\ \forall\ x, y, \widetilde{x}, \widetilde{y},\ \mathbb{R}^m$$

and

$$\|g(t,x,y,\omega)-g(t,\widetilde{x},\widetilde{y},\omega)\| \leq p_3(\omega)\|x-\widetilde{x}\|+p_4(\omega)\|y-\widetilde{y}\|,\ \forall\ x, y, \widetilde{x}, \widetilde{y},\ \mathbb{R}^m$$

holds. If for every $\omega\in\Omega$, $\widetilde{M}(\omega)$ converge to 0, where

$$\widetilde{M}(\omega) = \begin{pmatrix} \frac{b^\alpha p_1(\omega)}{\Gamma(\alpha+1)} & \frac{b^\alpha p_2(\omega)}{\Gamma(\alpha+1)} \\ \frac{b^\beta p_3(\omega)}{\Gamma(\beta+1)} & \frac{b^\beta p_4(\omega)}{\Gamma(\beta+1)} \end{pmatrix}.$$

Then the problem (3.0.1) has unique random solution.

Proof. Consider the operator $N : C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)\times\Omega\to C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)$,

$$(x(.,\omega), y(.,\omega), w) \mapsto (N_1(t, x(t,\omega), y(t,\omega), \omega), N_2(t, x(t,\omega), y(t,\omega), \omega))$$

where

$$N_1(x(t,\omega), y(t,\omega), \omega) \ = \ x_0(w) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s,\omega), y(s,\omega), \omega)ds$$

and

$$N_2(x(t,\omega), y(t,\omega), \omega) \ = \ y_0(w) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s, x(s,\omega), y(s,\omega), \omega)ds.$$

First we show that N is a random operator on $C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)$. Since f and g are Carathéodory functions, then $\omega\to f(t,x,y,\omega)$ and $\omega\to g(t,x,y,\omega)$ are measurable maps in view of proposition 1.3.2 we concluded that, the maps

$$\omega\to N_1(x(t,\omega),y(t,\omega),\omega),\ \ \omega\to N_2(x(t,\omega),y(t,\omega),\omega)$$

are measurable. As a result, N is a random operator on $C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)\times\Omega$ into $C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)$.

We show that N satisfies all the conditions of theorem 3.1.1 on $C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)$. Let $(x,y),(\widetilde{x},\widetilde{y})\in C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)$, then

$$\begin{gathered}\|N_1(t,x(t),y(t),\omega)-N_1(t,\widetilde{x}(t),\widetilde{y}(t),\omega)\|\\ \leq\\ \tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|f(s,x(s,x),y(s,\omega),\omega)-f(s,\widetilde{x}(s,\omega),\widetilde{y}(s,\omega),\omega)\|ds\\ \leq\\ \tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p_1(\omega)\|x(s,\omega)-\widetilde{x}(s,\omega)\|ds+\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p_2(\omega)\|y(s,\omega)-\widetilde{y}(s,\omega)\|ds\\ \leq\\ \tfrac{p_1(\omega)t^\alpha}{\Gamma(\alpha+1)}\|x(.,\omega)-\widetilde{x}(.,\omega)\|_\infty+\tfrac{t^\alpha p_2(\omega)}{\Gamma(\alpha+1)}\|y(.,\omega)-\widetilde{y}(.,\omega)\|.\end{gathered}$$

Then

$$\|N_1(.,x,y,\omega)-N_1(t,\widetilde{x},\widetilde{y},\omega))\|_\infty\leq\|x-\widetilde{x}\|_\infty p_1(\omega)\frac{b^\alpha}{\Gamma(\alpha+1)}+\|y-\widetilde{y}\|_\infty p_2(\omega)\frac{b^\alpha}{\Gamma(\alpha+1)}.$$

Similarly, we obtains

$$\|N_2(x,y,\omega)-N_2(\widetilde{x},\widetilde{y},\omega)\|_\infty\leq\|x-\widetilde{x}\|_\infty p_3(\omega)\frac{b^\beta}{\Gamma(\beta+1)}+\|y-\widetilde{y}\|_\infty p_4(\omega)\frac{b^\beta}{\Gamma(\beta+1)}.$$

Hence

$$d(N(x(.,\omega),y(.,\omega),\omega),N(\widetilde{x}(.,\omega),\widetilde{y}(.,\omega),\omega))\leq\widetilde{M}(\omega)d((x(.,\omega),y(.,\omega)),(\widetilde{x}(.,\omega),\widetilde{y}(.,\omega))),$$

where

$$d(x,y)=\begin{pmatrix}\|x(.,\omega)-y(.,\omega)\|_\infty\\ \|x(.,\omega)-y(.,\omega)\|_\infty\end{pmatrix}$$

and

$$\widetilde{M}(\omega) = \begin{pmatrix} \dfrac{b^{\alpha}}{\Gamma(\alpha+1)}p_1(\omega) & \dfrac{b^{\alpha}}{\Gamma(\alpha+1)}p_2(\omega) \\ \dfrac{b^{\beta}}{\Gamma(\beta+1)}p_3(\omega) & \dfrac{b^{\beta}}{\Gamma(\beta+1)}p_4(\omega) \end{pmatrix}.$$

Since for every $\omega \in \Omega$, $\widetilde{M}(\omega) \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$ converge to zero, then from theorem 3.1.2 there exists unique random solution of problem (3.0.1). $\square$

Next, we present existence result without Lipschitz conditions. We consider the following hypotheses:

($\mathcal{L}_2$) For every $\omega \in \Omega$, the functions $f(.,.,.,\omega)$ and $g(.,.,.,\omega)$ are continuous and $\omega \to f(.,.,.,\omega)$, $\omega \to g(.,.,.,\omega)$ are measurable.

($\mathcal{L}_3$) There exist a measurable and bounded functions $\gamma_1, \gamma_2 : \Omega \to \mathbb{R}_+$ such that

$$\|f(t,x,y,\omega)\| \leq \gamma_1(\omega)(\|x\| + \|y\|), \ \|g(t,x,y,\omega)\| \leq \gamma_2(\omega)(\|x\| + \|y\|),$$

for all $t \in [0,b]$, $\omega \in \Omega$ and $x, y \in \mathbb{R}^m$.

Now,we give prove of the existence result of problem (3.0.1) by using Leary-Schauder random fixed point theorem type in generalized Banach space.

Theorem 3.1.6. *Assume that the hypotheses* ($\mathcal{L}_2$) *and* ($\mathcal{L}_3$) *hold. Then the problem (3.0.1) has a random solution defined on* $[0,b]$. *Moreover, the solution set*

$$S(x_0,y_0) = \{(x,y) : \Omega \to C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m) : (x(.,\omega), y(.,\omega)), \omega \in \Omega$$

$$\textit{is solution of } (3.0.1)\}$$

is compact.

Proof. Let $N : C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)\times\Omega \to C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)$ be a random operator defined in Theorem 3.1.5

In order to apply theorem 3.1.3, we first show that N is completely continuous. The proof will be given in several steps.

- **Step 1.** $N(.,.,\omega) = (N_1(.,.,\omega), N_2(.,.,\omega))$ is continuous.
Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (x, y) \in C([0,b],\mathbb{R}^m) \times C([0,b],\mathbb{R}^m)$ as $n \to \infty$. Then

$$\begin{aligned} &\|N_1(x_n(.,\omega), y_n(.,\omega),\omega) - N_1(x(.,\omega), y(.,\omega),\omega)\|_\infty \\ \leq\ & \frac{b^\alpha}{\Gamma(\alpha+1)}\|f(., x_n(.,\omega), y_n(.,\omega),\omega) - f(., x(.,\omega), y(.,\omega),\omega)\|. \end{aligned}$$

Since f is a continuous function. Thus

$\|N_1(x_n(.,\omega), y_n(.,\omega),\omega) - N_1(x(.,\omega), y(.,\omega),\omega)\|_\infty \to 0$ as $n \to \infty$.

$$\begin{aligned} &\|N_2(x_n(.,\omega), y_n(.,\omega),\omega) - N_2(x(.,\omega), y(.,\omega),\omega)\|_\infty \\ \leq\ & \frac{b^\beta}{\Gamma(\beta+1)}\|g(., x_n(.,\omega), y_n(.,\omega),\omega) - g(., x(.,\omega), y(.,\omega),\omega)\|_\infty. \end{aligned}$$

Then

$\|N_2(x_n(.,\omega), y_n(.,\omega),\omega) - N_2(x(.,\omega), y(.,\omega),\omega)\|_\infty \to 0$ as $n \to \infty$.

Thus N is continuous.

- **Step 2.** N maps bounded sets into bounded sets in $C([0,b],\mathbb{R}^m) \times C([0,b],\mathbb{R}^m)$. Indeed, it is enough to show that for any $q > 0$ there exists a positive constant l such that for each $(x,y) \in B_q = \{(x,y) \in C([0,b],\mathbb{R}) \times C([0,b],\mathbb{R}) : \|x\|_\infty \leq q,\ \|y\|_\infty \leq q\}$, we have

$$\|N(x,y,\omega)\|_\infty \leq l = (l_1, l_2).$$

Then for each $t \in [0,b]$, we get

$$\begin{aligned} \|N_1(x(t,\omega), y(t,\omega),\omega)\| &= \left\| x_0(\omega) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s,\omega), y(s,\omega),\omega)ds \right\| \\ &\leq \|x_0(\omega)\| + \frac{\gamma_1(\omega)}{\Gamma(\alpha)}\int_0^b \|f(s, x(s,\omega), y(s,\omega),\omega)\|ds. \end{aligned}$$

From $(\mathcal{L}_3)$ we get

$$\|N_1(x(.,\omega), y(.,\omega),\omega)\|_\infty \leq \|x_0(\omega)\| + \frac{2b^\alpha q}{\Gamma(\alpha+1)}\gamma_1(\omega) := l_1.$$

Similarly, we have

$$\|N_2(x(.,\omega),y(.,\omega),\omega)\|_\infty \leq \|y_0(\omega)\| + \frac{2b^\beta q}{\Gamma(\beta+1)}\gamma_2(\omega) := l_2.$$

- **Step 3.** N maps bounded sets into equicontinuous sets of $C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)$.
  Let B_q be a bounded set in $C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)$ as in Step 2. Let $r_1, r_2 \in J, r_1 < r_2$ and $u \in B_q$. Thus we have

$$\|N_1(x(r_2,\omega),y(r_2,\omega),\omega) - N_1(x(r_1,\omega),y(r_1,\omega),\omega)\| \leq \frac{2q\gamma_1(\omega)}{\Gamma(\alpha)}\left[\int_{r_1}^{r_2}(r_2-s)^{\alpha-1}ds + \int_0^{r_1}\Big((r_1-s)^{\alpha-1}-(r_2-s)^{\alpha-1}\Big)ds\right].$$

  Hence

$$\|N_1(x(r_2,\omega),y(r_2,\omega),\omega) - N_1(x(r_1,\omega),y(r_1,\omega),\omega)\| \leq \frac{4q\gamma_1(\omega)}{\Gamma(\alpha+1)}(r_2-r_1)^\alpha.$$

  and

$$\|N_2(x(r_2,\omega),y(r_2,\omega),\omega) - N_2(x(r_1,\omega),y(r_1,\omega),\omega)\| \leq \frac{4q\gamma_2(\omega)}{\Gamma(\beta+1)}(r_2-r_1)^\beta.$$

  The right-hand term tends to zero as $|r_2-r_1| \to 0$. As a consequence of Steps 1 to 3 together with the Arzela-Ascoli, we conclude that N maps B_q into a precompact set in $C([0,b],\mathbb{R})\times C([0,b],\mathbb{R})$.

- **Step 4.** It remains to show that

$$\mathcal{A}(\omega) = \{(x(.,\omega),y(.,\omega)) \in C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m) : (x(.,\omega),y(.,\omega)) = \lambda(\omega)N(x(.,\omega),y(.,\omega),\omega),\ \lambda(\omega)\in(0,1)\}$$

  is bounded.
  Let $(x,y)\in\mathcal{A}$. Then $x=\lambda(\omega)N_1(x,y)$ and $y=\lambda(\omega)N_2(x,y)$ for some $0<\lambda<1$. Thus, for $t\in[0,b]$, we have

$$\begin{aligned}\|x(t,\omega)\| &\leq \|x_0(\omega)\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s,\omega),y(s,\omega),\omega)\|ds\\ &\leq \|x_0(\omega)\| + \frac{1}{\Gamma(\alpha)}\int_0^t \gamma_1(\omega)(t-s)^{\alpha-1}(\|x(s,\omega)\|+\|y(s,\omega)\|)ds.\end{aligned}$$

Hence

$$\|x(t,\omega)\| \leq \|x_0(\omega)\| + \frac{\gamma_1(\omega)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\|x(s,\omega)\| + \|y(s,\omega)\|)ds$$

and

$$\|y(t,\omega)\| \leq \|y_0(\omega)\| + \frac{\gamma_2(\omega)}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(\|x(s,\omega)\| + \|y(s,\omega)\|)ds.$$

Therefore

$$\|x(t,\omega)\| + \|y(t,\omega)\| \leq c + \left[\frac{\gamma_1(\omega)}{\Gamma(\alpha)} + \frac{\gamma_2(\omega)}{\Gamma(\beta)}\right] \times \int_0^t (t-s)^{\min(\alpha,\beta)-1}(\|x(s,\omega)\| + \|y(s,\omega)\|)ds,$$

where

$$c = \|x_0(\omega)\| + \|y_0(\omega)\|.$$

By lemma 1.5.5, there exists $K(\min(\alpha,\beta)) > 0$ such that

$$\|x(t,\omega)\| + \|y(t,\omega)\| \leq c + c\left(\frac{\gamma_1(\omega)}{\Gamma(\alpha)} + \frac{\gamma_2(\omega)}{\Gamma(\beta)}\right)\int_0^t (t-s)^{\min(\alpha,\beta)-1}ds.$$

Hence

$$\|x(.,\omega)\|_\infty + \|y(.,\omega)\|_\infty \leq c + cb^{\min(\alpha,\beta)}\left(\frac{\gamma_1(\omega)}{\Gamma(\alpha)} + \frac{\gamma_2(\omega)}{\Gamma(\beta)}\right) := K_*.$$

Consequently

$$\|x\|_\infty \leq K_* \text{ and } \|y\|_\infty \leq K_*.$$

This shows that $\mathcal{A}$ is bounded. As a consequence of Theorem 3.1.3 we deduce that N has a random fixed point $\omega \to (x(.,\omega), y(.,\omega))$ which is a solution to the problem (3.0.1).

- **Step 5:** Compactness of the solution set. Let $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset S(x_0, y_0)$ be a sequence. For every $n \in \mathbb{N}$ and for fixe $\omega \in \Omega$, we get

$$x_n(t,\omega) = x_0(w) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x_n(s,\omega), y_n(s,\omega), \omega)ds$$

and

$$y_n(t,\omega) \quad = \quad y_0(w) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s, x_n(s,\omega), y_n(s,\omega),\omega)ds.$$

As in Steps 3, 4, we can prove that subsequence $\{(x_{n_k}, y_{n_k})\}_{k\in\mathbb{N}}$ of $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converge to some $(x(.,\omega), y(.,\omega)) \in C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)$, such that

$$\omega \to x(t,\omega), \quad \omega \to y(t,\omega)$$

are measurable functions. Since $f(.,.,.,\omega)$ and $g(.,.,.,\omega)$ are continuous functions, then

$$x(t,\omega) \quad = \quad x_0(w) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s,\omega), y(s,\omega),\omega)ds, \quad t\in[0,b],$$

and

$$y(t,\omega) \quad = \quad y_0(w) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s, x(s,\omega), y(s,\omega),\omega)ds, \quad t\in[0,b].$$

So $S(x_0, y_0)$ is compact.

□

3.2 An example

Let $\Omega = \mathbb{R}$ be equipped with the usual $\sigma-$ algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$ and $J := [0,1]$.
Consider the following random differential equation system.

$$\begin{cases} D^\alpha x(t,\omega) = \frac{t\omega^2 x^2(t,\omega)}{(1+\omega^2)(1+x^2(t,\omega)+y^2(t,\omega))}, \ \alpha\in(0,1) \\ D^\beta y(t,\omega) = \frac{t\omega^2 y^2(t,\omega)}{(1+\omega^2)(1+x^2(t,\omega)+y^2(t,\omega))}, \ \beta\in(0,1), \\ x(0,\omega) = \sin\omega, \ \omega\in\Omega \\ y(0,\omega) = \cos\omega, \ \omega\in\Omega. \end{cases} \tag{3.2.1}$$

Here

$$f(t,x,y,\omega) = \frac{t\omega^2 x^2}{2(1+\omega^2)(1+x^2+y^2)}$$

$$g(t,x,y,\omega) = \frac{t\omega^2 y^2}{2(1+\omega^2)(1+x^2+y^2)}$$

Clearly, the map $(t,\omega) \mapsto f(t,x,y,\omega)$ is jointly continuous for all $x, y \in [1,\infty)$. The same for the map g. Also the maps $x \mapsto f(t,x,y,\omega)$ and $y \mapsto f(t,x,y,\omega)$ are continuous for all $t \in J$ and $\omega \in \Omega$. Similarly for the maps corresponding to function g. Thus the functions f and g are Carathéodory on $J \times [1,\infty) \times [1,\infty) \times \Omega$. Firstly, we show that f and g are Lipschitz functions. Indeed, let $x, y \in \mathbb{R}$, then

$$\begin{aligned} |f(t,x,y,\omega) - f(t,\widetilde{x},\widetilde{y},\omega)| &= \left| \frac{t\omega^2 x^2}{2(1+\omega^2)(1+x^2+y^2)} - \frac{t\omega^2 x^2}{2(1+\omega^2)(1+\widetilde{x}^2+\widetilde{y}^2)} \right| \\ &= \left| \frac{t\omega^2[(1+\widetilde{x}^2+\widetilde{y}^2)x^2 - (1+x^2+y^2)\widetilde{x}^2}{2(1+\omega^2)(1+x^2+y^2)(1+\widetilde{x}^2+\widetilde{y}^2)} \right| \\ &\leq \frac{\omega^2}{2(1+\omega^2)}|x-\widetilde{x}| + \frac{\omega^2}{2(1+\omega^2)}|y-\widetilde{y}| \end{aligned}$$

Then

$$|f(t,x,y,\omega) - f(t,\widetilde{x},\widetilde{y},\omega)| \leq \frac{\omega^2}{2(1+\omega^2)}|x-\widetilde{x}| + \frac{\omega^2}{2(1+\omega^2)}|y-\widetilde{y}|.$$

Analogously for the function g, we get

$$|g(t,x,y,\omega) - g(t,\widetilde{x},\widetilde{y},\omega)| \leq \frac{\omega^2}{2(1+\omega^2)}|x-\widetilde{x}| + \frac{\omega^2}{2(1+\omega^2)}|y-\widetilde{y}|.$$

We take,

$$p_1(\omega) = p_2(\omega) = p_3(\omega) = p_4(\omega) = \frac{\omega^2}{2(1+\omega^2)}$$

and

$$M(\omega) = \begin{pmatrix} \frac{\omega^2}{2(1+\omega^2)} & \frac{\omega^2}{2(1+\omega^2)} \\ \frac{\omega^2}{2(1+\omega^2)} & \frac{\omega^2}{2(1+\omega^2)} \end{pmatrix}.$$

We remark that

$$|\rho(M(\omega))| = \frac{\omega^2}{2(1+\omega^2)} < 1,$$

then

$$M(\omega), \text{ converge to } 0.$$

Therefore, all the conditions of theorem 3.1.5 are satisfied. Hence the problem (3.2.1) has a unique random solution.

Chapter 4

Random Hadamard Fractional Differential Equations

In this chapter, we prove the existence of solutions and the compactness of solution sets of a random system of fractional differential equations via the Hadamard-type derivative. The existence, modification and stochastically continuity of an M^2-solution are also proved.

We consider the system of Hadamard-type fractional differential equations:

$$\begin{cases} {}^{CH}D^{\alpha}x(t,\omega) = f(t,x(t,\omega),y(t,\omega),\omega), \ 0<\alpha<1, \ t\in[1,b], \\ {}^{CH}D^{\beta}y(t,\omega) = g(t,x(t,\omega),y(t,\omega),\omega), \ 0<\beta<1, \ t\in[1,b], \\ x(1,\omega) = x_0(\omega), \ \omega\in\Omega, \\ y(1,\omega) = y_0(\omega), \ \omega\in\Omega, \end{cases} \tag{4.0.1}$$

where $f, g : [1,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\Omega\to \ \mathbb{R}^m$, $(\Omega,\mathcal{A}))$ is a measurable space and $x_0, y_0 : \Omega\to\mathbb{R}^m$ are random variables. ${}^{CH}D^{\alpha}x$ is the Caputo-modification of the Hadamard fractional derivative.

We say that $x(\cdot,\cdot) : [1,b]\times\Omega\to\mathbb{R}^m$ is sample path Lebesgue integrable on $[1,b]$ if $x(\cdot,\omega) : [1,b] \longrightarrow \mathbb{R}^m$ is Legesgue integrable on $[1,b]$ for a.e. $\omega\in\Omega$.

Let $\alpha>0$. If $x : [1,b]\times\Omega \longrightarrow \mathbb{R}^m$ is sample path Lebesgue integrable on $[1,b]$, then we can consider the fractional integral

$$J^{\alpha}x(t,\omega) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} x(s,\omega)\frac{ds}{s}. \tag{4.0.2}$$

which will be called the sample path fractional integral of x, where Γ is the Euler's Gamma function.

4.1 Existence and Uniqueness

In this section, we prove some existences results and the compactness of the solution set.

Lemma 4.1.1. *Consider the problem,*

$$\begin{cases} D^{\alpha}x(t,\omega) = f(t,x(t,\omega),y(t,\omega),\omega), \ 0<\alpha<1, \\ x(1,\omega) = x_0(\omega), \ \omega\in\Omega, \\ \end{cases} \tag{4.1.1}$$

If $(t,\omega) \mapsto f(t,x(t,\omega),y(t,\omega),\omega)$ is product measurable and $t \mapsto f(t,x(t,\omega),y(t,\omega),\omega)$ is Lebesgue integrable on $[1,b]$ for a.e. $\omega\in\Omega$, then, the function $x:[1,b]\times\Omega \longrightarrow \mathbb{R}^m$ is a solution of (4.1.1) if and only if

$$x(t,\omega) = x_0(\omega) + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s,x(s,\omega),y(s,\omega),\omega)\frac{ds}{s}$$

for all $t\in[1,b]$ and for a.e. $\omega\in\Omega$.

Proof. We have:

$${}^{CH}D^{\alpha}x(t,\omega) = f(t,x(t,\omega),y(t,\omega),\omega).$$

Then

$$J^{\alpha}D^{\alpha}x(t,\omega) = J^{\alpha}f(t,x(t,\omega),y(t,\omega),\omega).$$

From Lemma 1.2.27, we get

$$J^{\alpha\,CH}D^{\alpha}x(t,\omega) = x(t,\omega) - x(1,\omega).$$

Thus

$$x(t,\omega) - x(1,\omega) = J^{\alpha}f(t,x(t,\omega),y(t,\omega),\omega).$$

Next, from the definition of J^{α}, we have:

$$x(t,\omega) - x(1,\omega) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s,x(s,\omega),y(s,\omega),\omega)\frac{ds}{s}.$$

Hence

$$x(t,\omega)=x_0(\omega)+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}f(s,x(s,\omega),y(s,\omega),\omega)\frac{ds}{s}.$$

The converse is straightforward.

□

Our first main result is the existence and uniqueness of a random solution of the problem (4.0.1).

Theorem 4.1.2. *Let $f,g:[1,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\Omega\to\ \mathbb{R}^m$ be two Carathéodory functions. Assume that the following conditions holds*

(h_1) *There exist $p_1,p_2,p_3,p_4:\Omega\to\mathbb{R}_+$ random variables such that*

$$\|f(t,x,y,\omega)-f(t,\widetilde{x},\widetilde{y},\omega)\|\leq p_1(\omega)\|x-\widetilde{x}\|+p_2(\omega)\|y-\widetilde{y}\|,\ \forall\, x,y,\widetilde{x},\widetilde{y}\in\mathbb{R}^m,\ \omega\in\Omega,$$

and

$$\|g(t,x,y,\omega)-g(t,\widetilde{x},\widetilde{y},\omega)\|\leq p_3(\omega)\|x-\widetilde{x}\|+p_4(\omega)\|y-\widetilde{y}\|,\ \forall\, x,y,\widetilde{x},\widetilde{y}\in\mathbb{R}^m,\ \omega\in\Omega.$$

If for every $\omega\in\Omega$, $\widetilde{M}(\omega)$ converges to 0, where

$$\widetilde{M}(\omega)=\begin{pmatrix}\frac{(\ln b)^{\alpha}p_1(\omega)}{\Gamma(\alpha+1)} & \frac{(\ln b)^{\alpha}p_2(\omega)}{\Gamma(\alpha+1)}\\ \frac{(\ln b)^{\beta}p_3(\omega)}{\Gamma(\beta+1)} & \frac{(\ln b)^{\beta}p_4(\omega)}{\Gamma(\beta+1)}\end{pmatrix},$$

then problem (4.0.1) has unique random solution.

Proof. Consider the operator $N:C([1,b],\mathbb{R}^m)\times C([1,b],\mathbb{R}^m)\times\Omega\to C([1,b],\mathbb{R}^m)\times C([1,b],\mathbb{R}^m)$ defined by

$$(x(\cdot,\omega),y(\cdot,\omega),\omega)\mapsto(N_1(x(t,\omega),y(t,\omega),\omega),N_2(x(t,\omega),y(t,\omega),\omega))$$

where

$$N_1(x(t,\omega),y(t,\omega),\omega)\ =\ x_0(\omega)+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}f(s,x(s,\omega),y(s,\omega),\omega)\frac{ds}{s}$$

and

$$N_2(x(t,\omega),y(t,\omega),\omega)\ =\ y_0(\omega)+\frac{1}{\Gamma(\beta)}\int_1^t\left(\ln\frac{t}{s}\right)^{\beta-1}g(s,x(s,\omega),y(s,\omega),\omega)\frac{ds}{s}.$$

First we show that N is a random operator on $C([1,b],\mathbb{R}^m)\times C([1,b],\mathbb{R}^m)$. Since f and g are Carathéodory functions, then $\omega\to f(t,x,y,\omega)$ and $\omega\to g(t,x,y,\omega)$ are measurable maps in view of Proposition 1.3.2, and we conclude that the maps

$$\omega\to N_1(x(t,\omega),y(t,\omega),\omega),\ \ \omega\to N_2(x(t,\omega),y(t,\omega),\omega)$$

are measurable. As a result, N is a random operator on $C([1,b],\mathbb{R}^m)\times C([1,b],\mathbb{R}^m)\times\Omega$ into $C([1,b],\mathbb{R}^m)\times C([1,b],\mathbb{R}^m)$.

We now show that N satisfies all the conditions of Theorem 3.1.2 on $C([1,b],\mathbb{R}^m)\times C([1,b],\mathbb{R}^m)$. Let $(x(\cdot,\omega),y(\cdot,\omega)),(\widetilde{x}(\cdot,\omega),\widetilde{y}(\cdot,\omega))\in C([1,b],\mathbb{R}^m)\times C([1,b],\mathbb{R}^m)$, ; then

$$\begin{aligned}
&\|N_1(t,x(t),y(t),\omega)-N_1(t,\widetilde{x}(t),\widetilde{y}(t),\omega)\|\\
&=\left\|\tfrac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\tfrac{t}{s}\right)^{\alpha-1}\left(f(s,x(s,\omega),y(s,\omega),\omega)-f(s,\widetilde{x}(s,\omega),\widetilde{y}(s,\omega),\omega)\right)\tfrac{ds}{s}\right\|\\
&\le\tfrac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\tfrac{t}{s}\right)^{\alpha-1}\|f(s,x(s,\omega),y(s,\omega),\omega)-f(s,\widetilde{x}(s,\omega),\widetilde{y}(s,\omega),\omega)\|\tfrac{ds}{s}\\
&\le\tfrac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\tfrac{t}{s}\right)^{\alpha-1}p_1(\omega)\|x(s,\omega)-\widetilde{x}(s,\omega)\|\tfrac{ds}{s}\\
&\quad+\tfrac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\tfrac{t}{s}\right)^{\alpha-1}p_2(\omega)\|y(s,\omega)-\widetilde{y}(s,\omega)\|\tfrac{ds}{s}\\
&\le\tfrac{p_1(\omega)}{\Gamma(\alpha)}\int_1^t\left(\ln\tfrac{t}{s}\right)^{\alpha-1}\tfrac{ds}{s}\|x(.,\omega)-\widetilde{x}(.,\omega)\|_\infty\quad+\tfrac{p_2(\omega)}{\Gamma(\alpha)}\int_1^t\left(\ln\tfrac{t}{s}\right)^{\alpha-1}\tfrac{ds}{s}\|y(.,\omega)-\widetilde{y}(.,\omega)\|_\infty\\
&=\tfrac{p_1(\omega)}{\Gamma(\alpha)}\int_0^{\ln t}(\ln t-s)^{\alpha-1}ds\|x(\cdot,\omega)-\widetilde{x}(\cdot,\omega)\|_\infty\\
&\quad+\tfrac{p_2(\omega)}{\Gamma(\alpha)}\int_0^{\ln t}(\ln t-s)^{\alpha-1}ds\|y(.,\omega)-\widetilde{y}(.,\omega)\|_\infty\\
&\le\tfrac{p_1(\omega)(\ln t)^\alpha}{\Gamma(\alpha+1)}\|x(\cdot,\omega)-\widetilde{x}(\cdot,\omega)\|_\infty+\tfrac{p_2(\omega)(\ln t)^\alpha}{\Gamma(\alpha+1)}\|y(\cdot,\omega)-\widetilde{y}(\cdot,\omega)\|_\infty.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|N_1(x(\cdot,\omega),y(\cdot,\omega),\omega)-N_1(\widetilde{x}(\cdot,\omega),\widetilde{y}(\cdot,\omega),\omega))\|_\infty\\
&\le\|x(\cdot,\omega)-\widetilde{x}(\cdot,\omega)\|_\infty p_1(\omega)\tfrac{(\ln b)^\alpha}{\Gamma(\alpha+1)}+\|y(\cdot,\omega)-\widetilde{y}(\cdot,\omega)\|_\infty p_2(\omega)\tfrac{(\ln b)^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\|N_2(x(\cdot,\omega),y(\cdot,\omega),\omega)-N_2(\widetilde{x}(\cdot,\omega),\widetilde{y}(\cdot,\omega),\omega)\|_\infty\\
&\le\|x(\cdot,\omega)-\widetilde{x}(\cdot,\omega)\|_\infty p_3(\omega)\tfrac{(\ln b)^\beta}{\Gamma(\beta+1)}+\|y(\cdot,\omega)-\widetilde{y}(\cdot,\omega)\|_\infty p_4(\omega)\tfrac{(\ln b)^\beta}{\Gamma(\beta+1)}.
\end{aligned}$$

Hence

$$d(N(x(\cdot,\omega),y(\cdot,\omega),\omega),N(\widetilde{x}(\cdot,\omega),\widetilde{y}(\cdot,\omega),\omega))\le\widetilde{M}(\omega)d((x(\cdot,\omega),y(\cdot,\omega)),(\widetilde{x}(\cdot,\omega),\widetilde{y}(\cdot,\omega))),$$

where

$$d(x(\cdot,\omega),y(\cdot,\omega)) = \begin{pmatrix} \|x(\cdot,\omega)-y(\cdot,\omega)\|_\infty \\ \|x(\cdot,\omega)-y(\cdot,\omega)\|_\infty \end{pmatrix}.$$

Since for every $\omega\in\Omega$, $\widetilde{M}(\omega)\in\mathcal{M}_{2\times 2}(\mathbb{R}_+)$ converges to zero, then from Theorem 3.1.2, there exists a unique random fixed point of N which is a solution of problem (4.0.1). □

Next, we present an existence result without Lipschitz conditions. We consider the following hypotheses:

(h_2) For every $\omega\in\Omega$, the functions $f(\cdot,\cdot,\cdot,\omega)$ and $g(\cdot,\cdot,\cdot,\omega)$ are continuous, and $\omega\to f(\cdot,\cdot,\cdot,\omega),\ \omega\to g(\cdot,\cdot,\cdot,\omega)$ are measurable.

(h_3) There exist measurable and bounded functions $\gamma_1,\gamma_2:\Omega\to\mathbb{R}_+$ such that

$$\|f(t,x,y,\omega)\|\le\gamma_1(\omega)(\|x\|+\|y\|),\ \ \|g(t,x,y,\omega)\|\le\gamma_2(\omega)(\|x\|+\|y\|),$$

for all $t\in[1,b],\ \omega\in\Omega$ and $x,y\in\mathbb{R}^m$.

Now,we prove an existence result for problem (4.0.1) by using a Leary-Schauder type random fixed point theorem in generalized Banach spaces.

Theorem 4.1.3. *Assume that the hypotheses (h_2) and (h_3) hold. Then the problem (4.0.1) has a random solution defined on $[1,b]$. Moreover, the solution set*

$$\begin{aligned} S(x_0,y_0) &= \{(x,y):\Omega\to C([1,b],\mathbb{R}^m)\times C([1,b],\mathbb{R}^m):(x(\cdot,\omega),y(\cdot,\omega)),\\ &\qquad \omega\in\Omega \textit{ is a solution of } (4.0.1)\} \end{aligned}$$

is compact (i.e. for every $(x_n,y_n)_{n\in\mathbb{N}}\subset S(x_0,y_0)$ there exists a subsequence of $(x_n,y_n)_{n\in\mathbb{N}}$ converging to some element $(x,y)\in S(x_0,y_0)$).

Proof. Let $N: C([1,b],\mathbb{R}^m)\times C([1,b],\mathbb{R}^m)\times\Omega\to C([1,b],\mathbb{R}^m)\times C([1,b],\mathbb{R}^m)$ be a random operator defined in Theorem 4.1.2.

In order to apply Theorem 3.1.3, we first show that N is completely continuous. The proof will be given in several steps.

- **Step 1** $N(\cdot,\cdot,\omega) = (N_1(\cdot,\cdot,\omega), N_2(\cdot,\cdot,\omega))$ is continuous.
  Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (x, y) \in C([1,b],\mathbb{R}^m) \times C([1,b],\mathbb{R}^m)$ as $n \to \infty$. Then
  $$\begin{aligned} &\|N_1(x_n(\cdot,\omega), y_n(\cdot,\omega),\omega) - N_1(x(\cdot,\omega), y(\cdot,\omega),\omega)\|_\infty \\ \leq\ & \tfrac{(\ln b)^\alpha}{\Gamma(\alpha+1)} \|f(\cdot, x_n(\cdot,\omega), y_n(\cdot,\omega),\omega) - f(\cdot, x(\cdot,\omega), y(\cdot,\omega),\omega)\|_\infty. \end{aligned}$$
  Since f is a continuous function, we get
  $$\|N_1(x_n(\cdot,\omega), y_n(\cdot,\omega),\omega) - N_1(x(\cdot,\omega), y(\cdot,\omega),\omega)\|_\infty \to 0 \text{ as } n \to \infty.$$
  Similarly
  $$\begin{aligned} &\|N_2(x_n(\cdot,\omega), y_n(\cdot,\omega),\omega) - N_2(x(\cdot,\omega), y(\cdot,\omega),\omega)\|_\infty \\ \leq\ & \tfrac{(\ln b)^\beta}{\Gamma(\beta+1)} \|g(\cdot, x_n(\cdot,\omega), y_n(\cdot,\omega),\omega) - g(\cdot, x(\cdot,\omega), y(\cdot,\omega),\omega)\|_\infty. \end{aligned}$$
  Then
  $$\|N_2(x_n(\cdot,\omega), y_n(\cdot,\omega),\omega) - N_2(x(\cdot,\omega), y(\cdot,\omega),\omega)\|_\infty \to 0 \text{ as } n \to \infty.$$
  Thus N is continuous.

- **Step 2.** N maps bounded sets into bounded sets in $C([1,b],\mathbb{R}^m) \times C([1,b],\mathbb{R}^m)$. Indeed, it is enough to show that for any $q > 0$ there exists a positive constant l such that for each $(x, y) \in B_q = \{(x, y) \in C([1,b],\mathbb{R}) \times C([1,b],\mathbb{R}) : \|x\|_\infty \leq q,\ \|y\|_\infty \leq q\}$, we have
  $$\|N(x, y,\omega)\|_\infty \leq l = (l_1, l_2).$$
  Then for each $t \in [1, b]$, we get
  $$\begin{aligned} &\|N_1(x(t,\omega), y(t,\omega),\omega)\| \\ =\ & \left\| x_0(\omega) + \tfrac{1}{\Gamma(\alpha)} \int_1^t (\ln \tfrac{t}{s})^{\alpha-1} f(s, x(s,\omega), y(s,\omega),\omega) \tfrac{ds}{s} \right\| \\ & \leq \|x_0(\omega)\| + \tfrac{\gamma_1(\omega)}{\Gamma(\alpha)} \int_1^b \|f(s, x(s,\omega), y(s,\omega),\omega)\| \tfrac{ds}{s}. \end{aligned}$$
  From (h_3), we get
  $$\|N_1(x(\cdot,\omega), y(\cdot,\omega),\omega)\|_\infty \leq \|x_0(\omega)\| + \frac{2(\ln b)^\alpha q}{\Gamma(\alpha+1)} \gamma_1(\omega) := l_1.$$
  Similarly, we have
  $$\|N_2(x(\cdot,\omega), y(\cdot,\omega),\omega)\|_\infty \leq \|y_0(\omega)\| + \frac{2(\ln b)^\beta q}{\Gamma(\beta+1)} \gamma_2(\omega) := l_2.$$

- **Step 3.** N maps bounded sets into equicontinuous sets of $C([1,b],\mathbb{R}^m)\times C([1,b],\mathbb{R}^m)$.
  Let B_q be a bounded set in $C([1,b],\mathbb{R}^m)\times C([1,b],\mathbb{R}^m)$ as in Step 2. Let $r_1,r_2\in J, r_1<r_2$ and $u\in B_q$. Thus we have

$$\begin{aligned}&\|N_1(x(r_2,\omega),y(r_2,\omega),\omega)-N_1(x(r_1,\omega),y(r_1,\omega),\omega)\|\\ &\leq \tfrac{2q\gamma_1(\omega)}{\Gamma(\alpha)}\Big[\int_{r_1}^{r_2}\left(\ln\tfrac{r_2}{s}\right)^{\alpha-1}\tfrac{ds}{s}+\int_1^{r_1}\Big(\left(\ln\tfrac{r_2}{s}\right)^{\alpha-1}-\left(\ln\tfrac{r_2}{s}\right)^{\alpha-1}\Big)\tfrac{ds}{s}\Big].\end{aligned}$$

  Hence

$$\begin{aligned}&\|N_1(x(r_2,\omega),y(r_2,\omega),\omega)-N_1(x(r_1,\omega),y(r_1,\omega),\omega)\|\\ &\leq \frac{2q\gamma_1(\omega)}{\Gamma(\alpha+1)}(\ln r_2-\ln r_1)^{\alpha}+\frac{2q\gamma_1(\omega)}{\Gamma(\alpha+1)}\Big[(\ln r_2)^{\alpha}-(\ln r_1)^{\alpha}\Big],\end{aligned}$$

  and

$$\begin{aligned}&\|N_2(x(r_2,\omega),y(r_2,\omega),\omega)-N_2(x(r_1,\omega),y(r_1,\omega),\omega)\|\\ &\leq \frac{2q\gamma_1(\omega)}{\Gamma(\beta+1)}(\ln r_2-\ln r_1)^{\beta}+\frac{2q\gamma_1(\omega)}{\Gamma(\beta+1)}\Big[(\ln r_2)^{\beta}-(\ln r_1)^{\beta}\Big].\end{aligned}$$

  The right-hand term tends to zero as $|r_2-r_1|\to 0$. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we conclude that N maps B_q into a precompact set in $C([1,b],\mathbb{R})\times C([1,b],\mathbb{R})$.

- **Step 4.** It remains to show that

$$\begin{aligned}\mathcal{A}(\omega)=\{&(x(\cdot,\omega),y(\cdot,\omega))\in C([1,b],\mathbb{R}^m)\times C([1,b],\mathbb{R}^m):\\ &(x(\cdot,\omega),y(\cdot,\omega))=\lambda(\omega)N(x(\cdot,\omega),y(\cdot,\omega),\omega),\ \lambda(\omega)\in(0,1)\}\end{aligned}$$

  is bounded.
  Let $(x,y)\in\mathcal{A}$. Then $x=\lambda(\omega)N_1(x,y)$ and $y=\lambda(\omega)N_2(x,y)$ for some $0<\lambda<1$. Thus, for $t\in[1,b]$, we have

$$\begin{aligned}\|x(t,\omega)\| &\leq \|x_0(\omega)\|+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\|f(s,x(s,\omega),y(s,\omega),\omega)\|\frac{ds}{s}\\ &\leq \|x_0(\omega)\|+\frac{1}{\Gamma(\alpha)}\int_1^t\gamma_1(\omega)\left(\ln\frac{t}{s}\right)^{\alpha-1}(\|x(s,\omega)\|+\|y(s,\omega)\|)\frac{ds}{s}.\end{aligned}$$

  Hence

$$\|x(t,\omega)\| \leq \|x_0(\omega)\|+\frac{\gamma_1(\omega)}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}(\|x(s,\omega)\|+\|y(s,\omega)\|)\frac{ds}{s}$$

and

$$\|y(t,\omega)\| \leq \|y_0(\omega)\| + \frac{\gamma_2(\omega)}{\Gamma(\beta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\beta-1}(\|x(s,\omega)\|+\|y(s,\omega)\|)\frac{ds}{s}.$$

Therefore

$$\|x(t,\omega)\|+\|y(t,\omega)\| \leq c + c\int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1}(\|x(s,\omega)\|+\|y(s,\omega)\|)\frac{ds}{s},$$

where

$$c = \|x_0(\omega)\|+\|y_0(\omega)\| + \frac{\gamma_1(\omega)}{\Gamma(\alpha)} + \frac{\gamma_2(\omega)}{\Gamma(\beta)}, \quad \gamma = \min(\alpha,\beta).$$

By Lemma 1.5.6, we have

$$\begin{aligned}
\|x(t,\omega)\|+\|y(t,\omega)\| &\leq c + c\int_1^t \left[\sum_{k=1}^{\infty}\frac{(c\Gamma(\gamma))^k}{\Gamma(k\gamma)}\left(\ln\frac{t}{s}\right)^{k\gamma-1}\right]\frac{ds}{s} \\
&\leq c + c\sum_{k=1}^{\infty}\frac{(c\Gamma(\gamma))^k}{\Gamma(k\gamma+1)}(\ln t)^{k\gamma} \\
&\leq c\left[1+\sum_{k=1}^{\infty}\frac{(c\Gamma(\gamma)(\ln t)^{\gamma})^k}{\Gamma(k\gamma+1)}\right] \\
&\leq c\left[1+\sum_{k=1}^{\infty}\frac{(c\Gamma(\gamma)(\ln b)^{\gamma})^k}{\Gamma(k\gamma+1)}\right] \\
&= cE_{\gamma}\left(c\Gamma(\gamma)(\ln b)^{\gamma}\right).
\end{aligned}$$

Hence

$$\|x(\cdot,\omega)\|_{\infty}+\|y(\cdot,\omega)\|_{\infty} \leq cE_{\gamma}\left(c\Gamma(\gamma)(\ln b)^{\gamma}\right) := K_*.$$

Consequently

$$\|x\|_{\infty} \leq K_* \text{ and } \|y\|_{\infty} \leq K_*.$$

This shows that $\mathcal{A}$ is bounded.

As a consequence of Theorem 3.1.3 we deduce that N has a random fixed point $\omega \to (x(\cdot,\omega), y(\cdot,\omega))$ which is a solution to the problem (4.0.1).

- **Step 5:** Compactness of the solution set. Let $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset S(x_0, y_0)$ be a sequence. For every $n \in \mathbb{N}$ and for fixed $\omega \in \Omega$, we get

$$x_n(t,\omega) \;=\; x_0(\omega) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, x_n(s,\omega), y_n(s,\omega), \omega) \frac{ds}{s}$$

and

$$y_n(t,\omega) \;=\; y_0(\omega) + \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} g(s, x_n(s,\omega), y_n(s,\omega), \omega) \frac{ds}{s}.$$

As in Steps 3 and 4, we can prove that the subsequence $\{(x_{n_k}, y_{n_k})\}_{k\in\mathbb{N}}$ of $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to some $(x(\cdot,\omega), y(\cdot,\omega)) \in C([1,b],\mathbb{R}^m) \times C([1,b],\mathbb{R}^m)$, such that

$$\omega \to x(t,\omega), \quad \omega \to y(t,\omega)$$

are measurable functions. Since $f(\cdot,\cdot,\cdot,\omega)$ and $g(\cdot,\cdot,\cdot,\omega)$ are continuous functions,

$$x(t,\omega) \;=\; x_0(\omega) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, x(s,\omega), y(s,\omega), \omega) \frac{ds}{s}, \quad t \in [1,b],$$

and

$$y(t,\omega) \;=\; y_0(\omega) + \frac{1}{\Gamma(\beta)} \int_1^t (\ln \frac{t}{s})^{\beta-1} g(s, x(s,\omega), y(s,\omega), \omega) \frac{ds}{s}, \quad t \in [1,b].$$

So $S(x_0, y_0)$ is compact.

$\square$

4.2 M^2-Solution

Our objective in this section is to apply the new concept of M^2- solution to Problem (4.0.1); for this we need some preliminary results which will be used throughout this section.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F} = \mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all

$\mathbb{P}$-null sets). For a stochastic process $x : [1, b] \times \Omega \to \mathbb{R}^m$ we will write $x(t)$ (or simply x when no confusion is possible) instead of $x_t(\omega) = x(t, \omega)$. We say $x(\cdot, \cdot)$ is jointly measurable if the map $(t, \omega) \to x(t, \omega)$ is measurable as a map $B([1, b] \bigotimes \mathcal{F}) \to \mathcal{B}(\mathbb{R}^m)$. For $\omega \in \Omega$, the path $t \to x(t, \omega)$ is called left-continuous if for each $t \in [1, b]$

$$x_s(\omega) \to x_t(\omega) \quad s \uparrow t.$$

A process $x(t, \omega)$ is stochastically continuous at a point $s \in [1, b]$ if for each $\epsilon > 0$

$$\lim_{t \to s} \mathbb{P}\{\omega \in \Omega : \|x(t, \omega) - x(s, \omega)\| > \epsilon\} = 0.$$

Theorem 4.2.1. *[77] If x is a stochastic process with state space $\mathbb{R}^m$ and all the paths of x are left-continuous (or right-continuous), then x is jointly measurable.*

If x and y are stochastic processes, we say that x is a modification of y if for each $t \in [1, b]$

$$\mathbb{P}(\{\omega \in \Omega : x_t(\omega) = y_t(\omega)\} = 1.$$

Theorem 4.2.2. *(Kolmogorov continuity theorem) [77] Suppose that $(\Omega, \mathcal{F}, \mathbb{P}, (x_t)_{t \geq 0})$, is a stochastic process with state space $\mathbb{R}^m$. If there are $\bar{\alpha}, \bar{\beta}, \sigma > 0$ such that*

$$\mathbb{E}\|x_t - x_s\|^{\bar{\alpha}} \leq \sigma |t - s|^{1+\bar{\beta}}, \quad t, s \in \mathbb{R}_+,$$

then the stochastic process has a continuous modification.

We introduce the notations:

- Denote by $L^p(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}^m)$, $p > 0$, the linear space of random variables (equivalence classes) $x : \Omega \to \mathbb{R}^m$ such that

$$\mathbb{E}\|x\|^p < \infty.$$

- $M^p(1, b)$: the space of (equivalence classes of) progressively measurable processes $x : [1, b] \times \Omega \to \mathbb{R}^m$ such that

$$\int_1^b \|x_t\|^2 dt < \infty, \ \mathbb{P}, \ \text{p.s } \omega \in \Omega, \ \text{if } p = 0$$

and

$$\mathbb{E}\left(\int_1^b \|x_t\|^2 dt\right)^{\frac{p}{2}} < \infty, \quad \text{if } p > 0.$$

Note that the property of progressive measurability is independent of the choice of an element in an equivalence class x, and for every $p \geq 0$;

$$M^p(1,b) \subset L^p(\Omega, \mathcal{F}, \mathbb{P}, L^p(1,b,\mathbb{R}^m)),$$

as a closed linear subspace. Hence, for each $p \in [1,\infty)$ the space $M^p(1,b)$ is a Banach space with respect to the norm

$$\|x\|_{M^p} = \left(\mathbb{E} \left(\int_1^b \|x_t\|^2 dt \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}.$$

Moreover the space $M^2(1,b)$ is a Hilbert space.

Definition 4.2.1. *A pair $x, y \in M^2$ is called an M^2-solution of problem (4.0.1) if*

$$x(t,\omega) = x_0(\omega) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} f(s, x(s,\omega), y(s,\omega), \omega) \frac{ds}{s}, \mathbb{P}, p.s\ \omega \in \Omega, t \in [1,b],$$

and

$$y(t,\omega) = y_0(\omega) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\ln \frac{t}{s} \right)^{\beta-1} g(s, x(s,\omega), y(s,\omega), \omega) \frac{ds}{s}, \mathbb{P}, p.s\ \omega \in \Omega, t \in [1,b].$$

4.2.1 Existence and uniqueness of M^2-solutions

In this part we investigate the existence, uniqueness, modification continuity and stochastically continuity of M^2-solution. Let us now introduce the following hypotheses which will be basic tools in the treatment of M^2-solutions.

(h_4) Let $f, g : [1,b] \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ be two functions such that, $\omega \to f(\cdot,\cdot,\cdot,\omega)$, $g(\cdot,\cdot,\cdot,\omega)$ are measurable, $t \to f(t,\cdot,\cdot,\cdot)$, $g(t,\cdot,\cdot,\cdot)$ are continuous and

$$\|f(\cdot,0,0,\cdot)\|_{M^2} < \infty, \quad \|g(\cdot,0,0,\cdot)\|_{M^2} < \infty.$$

(h_5) There exist positive real numbers c_1, c_2, c_3, c_4 such that

$$\|f(t,x,y,\omega) - f(t,\widetilde{x},\widetilde{y},\omega)\| \leq c_1\|x-\widetilde{x}\| + c_2\|y-\widetilde{y}\|, \ \forall\, x, y, \widetilde{x}, \widetilde{y} \in \mathbb{R}^m,\ \omega \in \Omega,$$

and

$$\|g(t,x,y,\omega) - g(t,\widetilde{x},\widetilde{y},\omega)\| \leq c_3\|x-\widetilde{x}\| + c_4\|y-\widetilde{y}\|, \ \forall\, x, y, \widetilde{x}, \widetilde{y} \in \mathbb{R}^m,\ \omega \in \Omega.$$

In this section we assume that $\alpha, \beta \in (\frac{1}{2}, 1)$ and $\mathbb{E}\|x_0\|^2 < \infty,\ \mathbb{E}\|y_0\|^2 < \infty$. Now we are in the position to present the first result of this section.

Theorem 4.2.3. *Assume that the conditions* (h_4) *and* (h_5) *hold and the matrix* $\widetilde{M}_* \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$ *defined by*

$$\widetilde{M}_* = \begin{pmatrix} \frac{(\ln b)^{\alpha}\sqrt{2}bc_1}{\sqrt{2\alpha-1}\Gamma(\alpha)} & \frac{(\ln b)^{\alpha}\sqrt{2}bc_2}{\sqrt{2\alpha-1}\Gamma(\alpha)} \\ \frac{(\ln b)^{\beta}\sqrt{2}bc_3}{\sqrt{2\beta-1}\Gamma(\beta)} & \frac{(\ln b)^{\beta}\sqrt{2}bc_4}{\sqrt{2\beta-1}\Gamma(\beta)} \end{pmatrix}$$

converges to zero. Then problem (4.0.1) has unique solution in $M^2(1,b)$.

Proof. Consider the operator $N : M^2(1,b) \times M^2(1,b) \to M^2(1,b) \times M^2(1,b)$, defined by

$$(x(\cdot,\omega), y(\cdot,\omega)) \mapsto (N_1(x(t,\omega), y(t,\omega)), N_2(x(t,\omega), y(t,\omega)))$$

where

$$N_1(x(t,\omega), y(t,\omega)) \;=\; x_0(\omega) + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s, x(s,\omega), y(s,\omega), w)\frac{ds}{s}$$

and

$$N_2(x(t,\omega), y(t,\omega)) \;=\; y_0(\omega) + \frac{1}{\Gamma(\beta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\beta-1} g(s, x(s,\omega), y(s,\omega), w)\frac{ds}{s}.$$

First we show that N is a random operator on $M^2(1,b) \times M^2(1,b)$. Since f and g are Carathéodory functions, then $\omega \to f(t,x,y,\omega)$ and $\omega \to g(t,x,y,\omega)$ are measurable maps in view of Proposition 1.3.2, and we conclude that the maps

$$\omega \to N_1(x(t,\omega), y(t,\omega)),\ \ \omega \to N_2(x(t,\omega), y(t,\omega))$$

are measurable.

Also,

$$\begin{aligned} &\int_1^b \|N_1(x(s,\omega), y(s,\omega))\|^2 ds \\ &= \int_1^b \left[\left\| x_0(\omega) + \tfrac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\tfrac{t}{s}\right)^{\alpha-1} f(s, x(s,\omega), y(s,\omega), \omega)\tfrac{ds}{s}\right\|^2\right] dt \\ &\leq \tfrac{2}{\Gamma^2(\alpha)}\int_1^b \left[\left\| \int_1^t \left(\ln\tfrac{t}{s}\right)^{\alpha-1} f(s, x(s,\omega), y(s,\omega), \omega)\tfrac{ds}{s}\right\|^2\right] dt + 2\int_1^b \|x_0(\omega)\|^2 dt \\ &\leq 2(b-1)\|x_0(\omega)\|^2 + \tfrac{b(\ln b)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\int_1^b \|f(s, x(s,\omega), y(s,\omega), \omega)\|^2 ds \\ &\leq 2(b-1)\|x_0(\omega)\|^2 + \tfrac{2b(\ln b)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\int_1^b \|f(s,0,0;\omega)\|^2 ds \\ &\quad + \tfrac{2b(\ln b)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\int_1^b [c_1^2\|x(s,\omega)\|^2 + c_2^2\|y(s,\omega)\|^2] ds. \end{aligned}$$

Therefore

$$\begin{aligned}
&\int_1^b \|N_1(x(s,\omega),y(s,\omega))\|^2 ds\\
&\leq 2(b-1)\|x_0(\omega)\|^2 + \tfrac{2b(\ln b)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)} \int_1^b \|f(s,0,0;\omega)\|^2 ds\\
&+\tfrac{2b(\ln b)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)} \int_1^b [c_1^2\|x(s,\omega)\|^2 + c_2^2\|y(s,\omega)\|^2] ds < \infty.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\int_1^b \|N_1(x(s,\omega),y(s,\omega))\|^2 ds\\
&\leq 2(b-1)\mathbb{E}\|x_0(\omega)\|^2 + \tfrac{2b(\ln b)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E} \int_1^b \|f(s,0,0;\omega)\|^2 ds\\
&+\tfrac{2b(\ln b)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E} \int_1^b [c_1^2\|x(s,\omega)\|^2 + c_2^2\|y(s,\omega)\|^2] ds < \infty.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\int_1^b \|N_2(x(s,\omega),y(s,\omega))\|^2 ds\\
&\leq 2(b-1)\|y_0(\omega)\|^2 + \tfrac{2b(\ln b)^{2\beta}}{(2\beta-1)\Gamma^2(\beta)} \int_1^b \|g(s,0,0,\omega)\|^2 ds\\
&+\tfrac{2b(\ln b)^{2\beta}}{(2\beta-1)\Gamma^2(\beta)} \int_1^b [c_3^2\|x(s,\omega)\|^2 + c_4^2\|y(s,\omega)\|^2] ds < \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\int_1^b \|N_2(x(s,\omega),y(s,\omega))\|^2 ds\\
&\leq 2(b-1)\mathbb{E}\|y_0(\omega)\|^2 + \tfrac{2b(\ln b)^{2\beta}}{(2\beta-1)\Gamma^2(\beta)}\mathbb{E} \int_1^b \|g(s,0,0,\omega)\|^2 ds\\
&+\tfrac{2b(\ln b)^{2\beta}}{(2\beta-1)\Gamma^2(\beta)}\mathbb{E} \int_1^b [c_3^2\|x(s,\omega)\|^2 + c_4^2\|y(s,\omega)\|^2] ds < \infty.
\end{aligned}$$

So, N is a random operator on $M^2(1,b)\times M^2(1,b)$ into $M^2(1,b)\times M^2(1,b)$.

We show that N satisfies all the conditions of Theorem 1.7.1 on $M^2(1,b)\times M^2(1,b)$. Let $(x(\cdot,\omega),y(\cdot,\omega)),(\widetilde{x}(\cdot,\omega),\widetilde{y}(\cdot,\omega))\in M^2(1,b)\times M^2(1,b)$, then

$$\begin{aligned}
&\|N_1(t,x(t),y(t,\omega)) - N_1(t,\widetilde{x}(t),\widetilde{y}(t,\omega))\|^2\\
&= \left\|\tfrac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\tfrac{t}{s}\right)^{\alpha-1}\left[f(s,x(s,\omega),y(s,\omega),w) - f(s,\widetilde{x}(s,\omega),\widetilde{y}(s,\omega),w)\right]\tfrac{ds}{s}\right\|^2\\
&\leq \tfrac{2c_1^2}{\Gamma^2(\alpha)}\int_1^t\left(\ln\tfrac{t}{s}\right)^{2\alpha-2}\tfrac{ds}{s^2}ds\int_1^t\|x(s,\omega)-\widetilde{x}(s,\omega)\|^2 ds\\
&+\tfrac{2c_2^2}{\Gamma^2(\alpha)}\int_1^t\left(\ln\tfrac{t}{s}\right)^{2\alpha-2}\tfrac{ds}{s^2}\int_1^t\|y(s,\omega)-\widetilde{y}(s,\omega)\|^2 ds\\
&= \tfrac{2c_1^2}{\Gamma^2(\alpha)}\int_0^{\ln t}(\ln t - s)^{2\alpha-2}\tfrac{ds}{e^s}\int_1^t\|x(s,\omega)-\widetilde{x}(s,\omega)\|^2 ds\\
&+\tfrac{2c_2^2}{\Gamma^2(\alpha)}\int_0^{\ln t}(\ln t - s)^{2\alpha-2}\tfrac{ds}{e^s}\int_1^t\|y(s,\omega)-\widetilde{y}(s,\omega)\|^2 ds\\
&\leq \tfrac{2c_1^2(\ln t)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\int_1^t\|x(s,\omega)-\widetilde{x}(s,\omega)\|^2 ds + \tfrac{2c_2^2(\ln t)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\int_1^t\|y(s,\omega)-\widetilde{y}(s,\omega)\|^2 ds.
\end{aligned}$$

Then

$$\begin{aligned}&\int_1^b \|N_1(x(s,\omega),y(s,\omega))-N_1(\widetilde{x}(s,\omega),\widetilde{y}(s,\omega))\|^2 ds\\&\quad\leq \tfrac{2bc_1^2(\ln b)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\int_1^b \|x(s,\omega)-\widetilde{x}(s,\omega)\|^2 ds\\&\quad+\tfrac{2bc_2^2(\ln b)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\int_1^b \|y(s,\omega)-\widetilde{y}(s,\omega)\|^2 ds.\end{aligned}$$

Consequently,

$$\begin{aligned}&\mathbb{E}\left(\int_1^b \|N_1(x(s,\omega),y(s,\omega))-N_1(\widetilde{x}(s,\omega),\widetilde{y}(s,\omega))\|^2 ds\right)\\&\quad\leq \tfrac{2bc_1^2(\ln b)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\left(\int_1^b \|x(s,\omega)-\widetilde{x}(s,\omega)\|^2 ds\right)\\&\quad+\tfrac{2bc_2^2(\ln b)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\left(\int_1^b \|y(s,\omega)-\widetilde{y}(s,\omega)\|^2 ds\right).\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&\mathbb{E}\left(\int_1^b \|N_1(x(s,\omega),y(s,\omega))-N_1(\widetilde{x}(s,\omega),\widetilde{y}(s,\omega))\|^2 ds\right)\\&\quad\leq \tfrac{2bc_3^2(\ln b)^{2\beta}}{(2\beta-1)\Gamma^2(\alpha)}\mathbb{E}\left(\int_1^b \|x(s,\omega)-\widetilde{x}(s,\omega)\|^2 ds\right)\\&\quad+\tfrac{2bc_4^2(\ln b)^{2\beta}}{(2\beta-1)\Gamma^2(\beta)}\mathbb{E}\left(\int_1^b \|y(s,\omega)-\widetilde{y}(s,\omega)\|^2 ds\right).\end{aligned}$$

Hence

$$\begin{aligned}&\left(\mathbb{E}\left(\int_1^b \|N_1(x(s,\omega),y(s,\omega))-N_1(\widetilde{x}(s,\omega),\widetilde{y}(s,\omega))\|^2 ds\right)\right)^{\frac{1}{2}}\\&\quad\leq \tfrac{\sqrt{2}bc_1(\ln b)^{\alpha}}{\sqrt{(2\alpha-1)}\Gamma(\alpha)}\left(\mathbb{E}\left(\int_1^b \|x(s,\omega)-\widetilde{x}(s,\omega)\|^2 ds\right)\right)^{\frac{1}{2}}\\&\quad+\tfrac{\sqrt{2}bc_2(\ln b)^{\alpha}}{\sqrt{(2\alpha-1)}\Gamma(\alpha)}\left(\mathbb{E}\left(\int_1^b \|y(s,\omega)-\widetilde{y}(s,\omega)\|^2 ds\right)\right)^{\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}&\left(\mathbb{E}\left(\int_1^b \|N_1(x(s,\omega),y(s,\omega))-N_1(\widetilde{x}(s,\omega),\widetilde{y}(s,\omega))\|^2 ds\right)\right)^{\frac{1}{2}}\\&\quad\leq \tfrac{\sqrt{2}bc_3(\ln b)^{\beta}}{\sqrt{(2\beta-1)}\Gamma(\alpha)}\left(\mathbb{E}\left(\int_1^b \|x(s,\omega)-\widetilde{x}(s,\omega)\|^2 ds\right)\right)^{\frac{1}{2}}\\&\quad+\tfrac{\sqrt{2}bc_4(\ln b)^{\beta}}{\sqrt{(2\beta-1)}\Gamma(\beta)}\left(\mathbb{E}\left(\int_1^b \|y(s,\omega)-\widetilde{y}(s,\omega)\|^2 ds\right)\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore
$$d(N(x,y), N(\widetilde{x},\widetilde{y}) \leq \widetilde{M}_* d((x,y),(\widetilde{x},\widetilde{y}),$$
where
$$d(x,y) = \begin{pmatrix} \|x-y\|_{M^2} \\ \|x-y\|_{M^2} \end{pmatrix}.$$
Since $\widetilde{M}_* \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$ converges to zero, then from Theorem 1.7.1 there exists a unique M^2-solution of problem (4.0.1). $\square$

Theorem 4.2.4. *Assume that* (h_4) *and* (h_5) *hold. Then for each* $x, y \in M^2(1,b)$*, the processes*

$$z_t(\omega) = x_0(\omega) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, x(s,\omega), y(s,\omega), w)\frac{ds}{s}, \quad \frac{1}{2} < \alpha < 1,$$

and

$$\bar{z}_t(\omega) \;=\; y_0(\omega) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\beta-1} g(s, x(s,\omega), y(s,\omega), w)\frac{ds}{s}, \quad \frac{1}{2} < \beta < 1,$$

have continuous modifications. Moreover $t \to z_t, \bar{z}_t$ *are stochastically continuous.*

Proof. Let $x, y \in M^2(1,b)$ and $t, r \in [1,b], \quad r < t$. By the Jensen and Hölder inequalities, we have

$$\begin{aligned}
\sqrt{\|z_t(\omega) - z_r(\omega)\|} \;&\leq\; \sqrt{\frac{1}{\Gamma(\alpha)} \int_1^r \Big[\left(\ln \frac{t}{s}\right)^{\alpha-1} - \left(\ln \frac{r}{s}\right)^{\alpha-1} \Big] \|f(s, x(s,\omega), y(s,\omega), \omega)\| \frac{ds}{s}} \\
&\quad + \sqrt{\frac{1}{\Gamma(\alpha)} \int_r^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \|f(s, x(s,\omega), y(s,\omega), \omega)\| \frac{ds}{s}} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^r \Big[\left(\ln \frac{t}{s}\right)^{\alpha-1} - \left(\ln \frac{r}{s}\right)^{\alpha-1} \Big] \frac{ds}{s} \\
&\quad \int_1^r \|f(s, x(s,\omega), y(s,\omega), w)\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_r^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \int_r^t \|f(s, x(s,\omega), y(s,\omega), \omega)\| ds.
\end{aligned}$$

Then

$$\begin{aligned}\sqrt{\|z_t(\omega)-z_r(\omega)\|} \;\leq\;& \frac{2}{\Gamma(\alpha+1)}\,|(\ln t-\ln r)^{\alpha}|\int_1^b \Big[c_1\|x(s,\omega)\|+c_2\|y(s,\omega)\|\Big]ds\\ &+\frac{2}{\Gamma(\alpha+1)}\,|(\ln t-\ln r)^{\alpha}|\int_1^b \|f(s,0,0,\omega)\|ds\\ &+\frac{1}{\Gamma(\alpha+1)}\,|(\ln t)^{\alpha}-(\ln r)^{\alpha}|\int_1^b \Big[c_1\|x(s,\omega)\|+c_2\|y(s,\omega)\|\Big]ds\\ &+\frac{1}{\Gamma(\alpha+1)}\,|(\ln t)^{\alpha}-(\ln r)^{\alpha}|\int_1^b \|f(s,0,0,\omega)\|ds\end{aligned}$$

By Hölder's inequality, we obtain that

$$\begin{aligned}\sqrt{\|z_t(\omega)-z_r(\omega)\|} \;\leq\;& \frac{2b}{\Gamma(\alpha+1)}\,|(\ln t-\ln r)^{\alpha}|\int_1^b \Big[c_1^2\|x(s,\omega)\|^2+c_2^2\|y(s,\omega)\|^2\Big]ds\\ &+\frac{2b}{\Gamma(\alpha+1)}\,|(\ln t-\ln r)^{\alpha}|\int_1^b \|f(s,0,0,\omega)\|^2ds\\ &+\frac{b}{\Gamma(\alpha+1)}\,|(\ln t)^{\alpha}-(\ln r)^{\alpha}|\int_1^b \Big[c_1^2\|x(s,\omega)\|^2+c_2^2\|y(s,\omega)\|^2\Big]ds\\ &+\frac{b}{\Gamma(\alpha+1)}\,|(\ln t)^{\alpha}-(\ln r)^{\alpha}|\int_1^b \|f(s,0,0,\omega)\|^2ds\end{aligned}$$

and

$$\begin{aligned}\sqrt{\|\bar{z}_t(\omega)-\bar{z}_r(\omega)\|} \;\leq\;& \frac{2b}{\Gamma(\beta+1)}\left|(\ln t-\ln r)^{\beta}\right|\int_1^b \Big[c_3^2\|x(s,\omega)\|^2+c_4^2\|y(s,\omega)\|^2\Big]ds\\ &+\frac{2b}{\Gamma(\beta+1)}\left|(\ln t-\ln r)^{\beta}\right|\int_1^b \|g(s,0,0,\omega)\|^2ds\\ &+\frac{b}{\Gamma(\beta+1)}\left|(\ln t)^{\beta}-(\ln r)^{\beta}\right|\int_1^b \Big[c_3^2\|x(s,\omega)\|^2+c_4^2\|y(s,\omega)\|^2\Big]ds\\ &+\frac{b}{\Gamma(\beta+1)}\left|(\ln t)^{\beta}-(\ln r)^{\beta}\right|\int_1^b \|g(s,0,0,\omega)\|^2ds.\end{aligned}$$

Consequently

$$\begin{aligned}
\mathbb{E}\sqrt{\|z_t(\omega)-z_r(\omega)\|} &\leq \frac{2b}{\Gamma(\alpha+1)}\left|(\ln t-\ln r)^{\alpha}\right|\left[c_1^2\|x\|_{M^2}^2+c_2^2\|y\|_{M^2}^2\right]\\
&\quad+\frac{2b}{\Gamma(\alpha+1)}\left|(\ln t-\ln r)^{\alpha}\right|\|f(\cdot,0,0,\cdot)\|_{M^2}^2\\
&\quad+\frac{b}{\Gamma(\alpha+1)}\left|(\ln t)^{\alpha}-(\ln r)^{\alpha}\right|\left[c_1^2\|x\|_{M^2}^2+c_2^2\|y\|_{M^2}^2\right]\\
&\quad+\frac{b}{\Gamma(\alpha+1)}\left|(\ln t)^{\alpha}-(\ln r)^{\alpha}\right|\|f(\cdot,0,0,\cdot)\|_{M^2}^2
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\sqrt{\|\bar{z}_t(\omega)-\bar{z}_r(\omega)\|} &\leq \frac{2b}{\Gamma(\beta+1)}\left|(\ln t-\ln r)^{\beta}\right|\left[c_3^2\|x\|_{M^2}^2+c_4^2\|y\|_{M^2}^2\right]\\
&\quad+\frac{2b}{\Gamma(\beta+1)}\left|(\ln t-\ln r)^{\beta}\right|\|g(\cdot,0,0,\cdot)\|_{M^2}^2\\
&\quad+\frac{b}{\Gamma(\beta+1)}\left|(\ln t)^{\beta}-(\ln r)^{\beta}\right|\left[c_3^2\|x\|_{M^2}^2+c_4^2\|y\|_{M^2}^2\right]\\
&\quad+\frac{b}{\Gamma(\beta+1)}\left|(\ln t)^{\beta}-(\ln r)^{\beta}\right|\|g(\cdot,0,0,\cdot)\|_{M^2}^2.
\end{aligned}$$

Since $r<t$, this implies that

$$\begin{aligned}
\left|(\ln t)^{\alpha}-(\ln r)^{\alpha}\right| &= (\ln t)^{\alpha}-(\ln r)^{\alpha}\\
&= \left(\frac{\ln t-\ln r}{2}+\frac{\ln t+\ln r}{2}\right)^{\alpha}-(\ln r)^{\alpha}
\end{aligned}$$

Using the fact that ξ^{α} is a convex function on $(0,\ln b]$, we obtain

$$\begin{aligned}
(\ln t)^{\alpha}-(\ln r)^{\alpha} &\leq \frac{1}{2}(\ln t-\ln r)^{\alpha}+\frac{1}{2}(\ln t+\ln r)^{\alpha}-(\ln r)^{\alpha}\\
&= \frac{1}{2}(\ln t-\ln r)^{\alpha}+\frac{2^{\alpha}}{2}\left(\frac{\ln t}{2}+\frac{\ln r}{2}\right)^{\alpha}-(\ln r)^{\alpha}\\
&\leq \frac{(\ln t-\ln r)^{\alpha}}{2}+\frac{(\ln t)^{\alpha}}{2^{2-\alpha}}+\frac{(\ln r)^{\alpha}}{2^{2-\alpha}}-(\ln r)^{\alpha}\\
&\leq \frac{(\ln t-\ln r)^{\alpha}}{2}+\frac{(\ln t)^{\alpha}}{2}-\frac{(\ln r)^{\alpha}}{2}.
\end{aligned}$$

Hence

$$(\ln t)^\alpha - (\ln r)^\alpha \quad \le \quad (\ln t - \ln r)^\alpha.$$

Applying, the Finite Increment Theorem to $\ln t - \ln r,\ r, t \in [1, b]$, we get

$$\begin{aligned}
\mathbb{E}\sqrt{\|z_t(\omega) - z_r(\omega)\|} \quad \le \quad & \frac{2b}{\Gamma(\alpha+1)}\,|t-r|^\alpha \left[c_1^2\|x\|_{M^2}^2 + c_2^2\|y\|_{M^2}^2\right] \\
& + \frac{2b}{\Gamma(\alpha+1)}\,|t-r|^\alpha \,\|f(\cdot,0,0,\cdot)\|_{M^2}^2 \\
& + \frac{b}{\Gamma(\alpha+1)}\,|t-r|^\alpha \left[c_1^2\|x\|_{M^2}^2 + c_2^2\|y\|_{M^2}^2\right] \\
& + \frac{b}{\Gamma(\alpha+1)}\,|t-r|^\alpha \,\|f(\cdot,0,0,\cdot)\|_{M^2}^2
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\sqrt{\|\bar{z}_t(\omega) - \bar{z}_r(\omega)\|} \quad \le \quad & \frac{2b}{\Gamma(\beta+1)}\,|t-r|^\beta \left[c_3^2\|x\|_{M^2}^2 + c_4^2\|y\|_{M^2}^2\right] \\
& + \frac{2b}{\Gamma(\beta+1)}\,|t-r|^\beta \,\|g(\cdot,0,0,\cdot)\|_{M^2}^2 \\
& + \frac{b}{\Gamma(\beta+1)}\,|t-r|^\beta \left[c_3^2\|x\|_{M^2}^2 + c_4^2\|y\|_{M^2}^2\right] \\
& + \frac{b}{\Gamma(\beta+1)}\,|t-r|^\beta \,\|g(\cdot,0,0,\cdot)\|_{M^2}^2.
\end{aligned}$$

Hence there exist $C, \bar{C} > 0$ such that

$$\mathbb{E}\sqrt{\|z_t(\omega) - z_r(\omega)\|} \le C\,|t-r|^{1+\alpha} \tag{4.2.1}$$

and

$$\mathbb{E}\sqrt{\|\bar{z}_t(\omega) - \bar{z}_r(\omega)\|} \le C\,|t-r|^{1+\beta}\,. \tag{4.2.2}$$

So, from Kolmogrov's Theorem 4.2.2, z_t and $\bar{z}_t$ have continuous modification.

Now we show that z and $\bar{z}$ are stochastically continuous. Indeed, let $t, s \in [1, b]$, and $\epsilon > 0$, then

$$\mathbb{P}(\{\omega \in \Omega :\ \|z_t(\omega) - z_r(\omega)\| > \epsilon\}) = \mathbb{P}(\{\omega \in \Omega :\ \sqrt{\|z_t(\omega) - z_r(\omega)\|} > \sqrt{\epsilon}\}).$$

Using Markov's inequality, we obtain that

$$\mathbb{P}(\{\omega \in \Omega : \ \|z_t(\omega) - z_r(\omega)\| > \epsilon\}) \le \frac{1}{\sqrt{\epsilon}}\mathbb{E}(\sqrt{\ \|z_t(\omega) - z_r(\omega)\|})$$

and

$$\mathbb{P}(\{\omega \in \Omega : \ \|\bar{z}_t(\omega) - \bar{z}_r(\omega)\| > \epsilon\}) \le \frac{1}{\sqrt{\epsilon}}\mathbb{E}(\sqrt{\|\bar{z}_t(\omega) - \bar{z}_r(\omega)\|}).$$

Hence, (4.2.1) and (4.2.2), implies that

$$\mathbb{P}(\{\omega \in \Omega : \ \|z_t(\omega) - z_r(\omega)\| > \epsilon\}) \le \frac{C}{\sqrt{\epsilon}}|t-r|^{1+\alpha} \to 0 \text{ as } t \to r$$

and

$$\mathbb{P}(\{\omega \in \Omega : \ \|\bar{z}_t(\omega) - \bar{z}_r(\omega)\| > \epsilon\}) \le \frac{1}{\sqrt{\epsilon}}|t-r|^{1+\beta} \to 0 \text{ as } t \to r.$$

□

As a consequence of above theorem we can easily prove the following result.

Corollary 4.2.5. *Under the conditions of Theorem 4.2.3, every M^2-solution of problem (4.0.1) has a continuous modification and is stochastically continuous..*

For the existence of modification of an M^2-solution of the problem (4.1.1), we assume in addition to the Lipschitz condition (H_5) the hypothesis:

(h_6) There exist positive constants $\bar{c}_i > 0, \ i = 1, \dots, 6$ such that

$$\|f(t,x,y,\omega)\|^2 \le \bar{c}_1\|x\|^2 + \bar{c}_2\|y\|^2 + \bar{c}_3, \quad \|g(t,x,y,\omega)\|^2 \le \bar{c}_4\|x\|^2 + \bar{c}_5\|y\|^2 + \bar{c}_6,$$

for each $x, y \in \mathbb{R}^m, \ t \in [1,b], \quad \mathbb{P}$ a.e. $\omega \in \Omega$.

By some simple modifications of the proof of Theorem 4.2.3, we present the following result.

Theorem 4.2.6. *Suppose that (h_5) and (h_6) hold. Then every M^2-solution of problem (4.0.1) has a continuous modification and is stochastically continuous.*

Conclusion and Perspectives

In this thesis, we prove some random fixed point theorems in generalized Banach spaces. We establish a random version of a Krasnosel'skii type fixed point theorem for the sum of a contraction random operator and a compact operator. The results are used to prove to the existence of solutions to the random system of fractional differential equations , then we shall be concerned with the existence and uniqueness of solutions for some classes of system of fractional discrete equation.

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