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Theme of Thesis

Blow-up results for Fractional Partial Differential Equations

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Resume

In the present thesis we study two models of damped wave equations with non linear memory. The first one is the Cauchy problem of weighted damped wave equation, it reads as follows

$$u_{tt} - \Delta u + g(\cdot)|u|^{m-1}u_t = \int_0^t (t-\tau)^{-\gamma}|u(\tau,\,\cdot)|^p\,d\tau \tag{1}$$

$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N \tag{2}$$

in the multi-dimensional real space $\mathbb{R}^N$, where the unknown function u is real valued, $m > 1,\ p > 1,\ 0 < \gamma < 1$ and Δ is the usual Laplace operator and g is a positive smooth function which will be specified later.
First, we prove the existence and uniqueness of local solution using the fixed point theorem where we search the solution as the fixed point of a suitable contracting mapping. For this reason we denote by X_T for the space of solutions that is

$$X_T = \mathscr{C}\big([0,\,T],\,H^s(\mathbb{R}^N)\big) \cap \mathscr{C}^1\big(\,[0,\,T]\,,\,H^{s-1}(\mathbb{R}^N)\big), \tag{3}$$

and we define the suitable supplementary space E_T as follows

$$E_T = L^\infty([0,T]\,, H^s(\mathbb{R}^N)) \cap W^{1,\infty}([0,T]\,,\,H^{s-1}(\mathbb{R}^N)) \tag{4}$$

with noting that $X_T \subset E_T$. Next, we take the centred ball in E_T of radius M (for some $M > 0$) which denoted $E_{T,M}$ and defined by

$$E_{T,M} = \Big\{u \in E_T;\, \sup_{t\in[0,T]}\big(\|u(t,\cdot)\|_{H^s(\mathbb{R}^N)} + \|u_t(t,\cdot)\|_{H^{s-1}(\mathbb{R}^N)}\big) \leq M\Big\} \tag{5}$$

Next, we rewrite the given equation (1) in the following equivalently form:

$$u_{tt} - \Delta u = -g(\cdot)|u|^{m-1}u_t \int_0^t (t-\tau)^{-\gamma}|u(\tau,\,\cdot)|^p\,d\tau := P_\alpha(g,\partial_t)u$$

with $\alpha = 1-\gamma$. After that, we define a mapping Φ from E_T to X_T which maps any $v \in E_T$ to $u \in X_T$ such that u is solution to the Cauchy problem of the following equation:

$$u_{tt} - \Delta u = P_\alpha(g,\partial_t)v \tag{6}$$

with initial data as in (2). In the first step we show that this map maps balls like (5) from E_T into balls from X_T which are defined by

$$X_{T,M} \;= X_T \cap E_{T,M}.$$

In the second step, we show that this mapping Φ is a contraction from $X_{T,M}$ into $E_{T,M}$ that is, it satisfies at

$$\sup_{0\leq t\leq T}\|\Phi(v_1)(t,\cdot) - \Phi(v_2)(t,\cdot)\|_{H^s} \leq k \sup_{0\leq t\leq T}\|v_1(t,\cdot) - v_2(t,\cdot)\|_{H^s},\ k \in]0,1[$$

for some all $v_1,\, v_2 \in E_{T,M}$.
Finally, by using recurrent sequences, we show that the fixed point of Φ is the desired solution to Problem (1)-(2) in X_T.
In the second part, we investigate the blow-up results of Problem (1)-(2) using the test function method of Pohozaev introduced by Pohozaev and Tesei [35], Mitidieri and Pohozaev ([11]-[10]), Fino [2], Berbiche and Hakem [22] and by Zhang [33]. To do so, we note that the right hand side of Equation (3.1) is, in fact, a fractional integral of order $1-\gamma$. For this reason, we choose a test function φ as

$$\varphi(t,x) = D^{\alpha}_{0|t}\psi(t,x) = \varphi_1(x) D^{\alpha}_{0|t}\varphi_2(t), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N$$

where $\alpha = 1-\gamma$, φ_1 is defined via a cut-off function and φ_2 is defined by

$$\varphi_2(t) = \big(1 - \frac{t}{T}\big)^{a}_{+}$$

for some $a > 1$. The result of this part is Theorem 4.1.2 (Theorem of non-existence of global solution) and it is demonstrated by absurd., so, we assumed that the solution is global non trivial and we showed that this leads to a contradiction, to do this we always show that

$$\int_0^{+\infty}\int_{\mathbb{R}^N} |u(t,x)|dtdx = 0$$

which implies necessary that $u = 0$.
The second model is the following Cauchy problem of fractional damped wave equations

$$u_{tt} - \Delta u + D^{\sigma}_{0|t}u_t = \int_0^t (t-\tau)^{-\gamma}|u(\tau,\,\cdot)|^p\, d\tau \tag{7}$$

where $\sigma, \gamma \in (0,1)$ and $D^{\sigma}_{0|t}$ is the right hand fractional derivative operator of order σ of Riemann-Liouville. In this model, the damping term $D^{\sigma}_{0|t}u_t$ is fractional. Indeed, by letting σ to 0 we find the model studied in [2]. In particular, we study, the blow-up phenomena of solutions of Cauchy problem for equation (7) and we managed to find the critical exponent in Fujita's sense of (7).

Keywords : weak solution, Fujita's exponent, non-linear memory, damped wave equations, test function, fractional derivative.

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Résumé en Francais

Sujet de these:
Résultats d'explosions pour des EDP fracrtionnaires

Dans cette thèse nous avons étudié deux modèles des équations des ondes. Le premier modèle est le problème de Cauchy suivant pour l'équation des ondes amorties à poids avec memoire non-linéaire

$$u_{tt} - \Delta u + g(\cdot)|u|^{m-1}u_t = \int_0^t (t-\tau)^{-\gamma}|u(\tau\ \cdot)|^p \, d\tau, \tag{8}$$

$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N, \tag{9}$$

dans l'espace multi-dimensionel réel $\mathbb{R}^N$, où la fonction inconnue est à valeurs réelles, $m > 1$, $p > 1$, $0 < \gamma < 1$, Δ est l'opérateur de Laplace classique et g est une fonction régulière positive à être spécifiée plus tard. D'abord, nous montrons le théorème d'existence et d'unicité d'une solution locale du problème posé en utilisant le théorème du point fixe où on cherche la solution sous forme de point fixe d'une application contractante et deuxièment le théorème de nonexistence des solutions globales sera établi, et là on a utilisé la méthode de fonction test de Pohozaev introduite par Pohozaev and Tesei [35], Mitidieri et Pohozaev ([11]-[10]), Fino [2], Berbiche et Hakem [22] et par Zhang [33], pour prouver le théorème de nonexistence globale (en temps).
Le seconde problème traité dans cette thèse est le problème de Cauchy suivant

$$u_{tt} - \Delta u + D^{\sigma}_{0|t}u_t = \int_0^t (t-\tau)^{-\gamma}|u(\tau,\ \cdot)|^p \, d\tau \tag{10}$$

$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N \tag{11}$$

où $\sigma, \gamma \in (0,1)$ et $D^{\sigma}_{0|t}$ est l'opérateur de dérivation fractionnaire d'ordre σ au sens Riemann-Liouville. Ce modèle est bien un problème de Cauchy pour l'équation des ondes amorties avec memoire non linéaire et de terme d'amortissement fractionaire. En particulier on a étudié le phénomène d'explosion de solutions du problème de Cauchy pour l'équation (10) où on a réussi de déterminer l'exposant d'explosion au sens Fujita. Bien enttendu que si σ tend vers 0 alors on obtient les résultats de [2].

Mots clés : Equation des ondes amorties, solution faible, fonction test, exposant de Fujita, dérivé fractionnaires, mémoire non linéaire.

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Sidi Bel Abbes, on *T. Hadj Kaddour*

September 20th, 2018

Dedicate

This modest work is dedicated to my family,

My supervisor Professeur Hakem Ali

My Colleagues of University of Sidi Belabbes

My Colleagues of University of Chlef.

T. HADJ KADDOUR

Contents

Chapter 1

Introduction

1.1 Brief History

1.1.1 History and Background of fractional derivation

On the topic of *Fractional Differentiation*, Most authors cite a particular date as the *birthday* of the so called "*Fractional Calculus.* In a letter dated September 30th, 1695 L'Hopital wrote to Leibniz asking him about a particular notation he had used in his publications for the n^{th}-derivative of the function $f(x) = x$. L'Hopital posed the question to Leibniz, what would the result be if $n = 1/2$. Leibniz's response: "*An apparent paradox, from which one day useful consequences will be drawn.*" In these words fractional calculus was born.

Following L'Hopital's and Leibniz's inquisition, fractional calculus was primarily a study reserved for the best minds in mathematics. Fourier, Euler, Laplace are among the many that dabbled with fractional calculus and the mathematical consequences [16]. Many found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most famous of these definitions that have been popularized in the world of fractional calculus (not yet the world as a whole) are the Riemann-Liouville and Grunwald-Letnikov, While the shear number of actual definitions are, no doubt, numerous (as men and women that study this field).

Most of mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the 20th century. However it is in the past 100 years that the most intriguing leaps in engineering and scientific application have been found. The mathematics has, in some cases, had to change to meet the requirements of physical reality. Caputo reformulated the more *classic* definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equation [16]. As recently as 1996, Kolowankar reformulate again, the Riemann-Liouville fractional derivative in order to differentiate fractal functions [20].

Leibniz's response based on studies over the intervening 300 years, has proven at least half right. It is clear that within the 20th century especially numerous applications and physical manifestations of fractional calculus have been found. However, these applications and the mathematical background surrounding fractional calculus are far from paradoxical. While the physical meaning is difficult (arguably impossible) to grasp, the definitions

themselves are no more rigorous than those of their integer order counterparts.
Understanding of definitions and use of fractional calculus will be made more clear by quickly discussing some necessary but relatively simple mathematical definitions that will arise in the study of these concepts .

1.1.2 Brief History on PDEs with non linear memory

In 2008, Cazenave and al [39] have generalized some results obtained by Fujita [14] in 1966 when they studied the following Cauchy problem:

$$u_t(t,x) - \Delta u(t,x) = \textstyle\int_0^t (t-\tau)^{-\gamma}|u|^{p-1}u(\tau,x)d\tau \quad t>0,\ x\in\mathbb{R}^N \tag{1.1}$$

$$u(0,x) = u_0(x), \quad x\in\mathbb{R}^N, \tag{1.2}$$

where $0 \leq \gamma < 1$, $p > 1$ and $u_0 \in \mathscr{C}_0(\mathbb{R}^N)$. This is the classical heat equation with non linear memory. Their results are the following:
Put

$$p_\gamma = 1 + \frac{2(2-\gamma)}{(N-2+2\gamma_+} \text{ and } p^* = \max\big(p_\gamma, \frac{1}{\gamma}\big)$$

with

$$(N-2+2\gamma)_+ := \max(N-2+2\gamma, 0). \tag{1.3}$$

Then

1. If $\gamma \neq 0, p \leq p^*$ and $u_0 > 0$, then the solution u of Cauchy problem (1.1)-(1.2) blows up in finite time.

2. If $\gamma \neq 0, p > p^*$ and $u_0 \in L_{q^*}(\mathbb{R}^N)$ (where $q^* = \frac{(p-1)N}{4-2\gamma}$) with $\|u_0\|_{L_{q^*}}$ small enough , then u exists globally (in time).

   In particular, they proved that the critical exponent in Fujita's sense p^* is not the one predicted by scaling. But this is not a surprising result since it is well known that scaling is efficient only for parabolic equations and not for pseudo-parabolic ones. To show this, it is sufficient to note that, formally, equation (1.1) is equivalent to

   $$D^\alpha_{0|t}u_t - D^\alpha_{0|t}\Delta u = \Gamma(\alpha)|u|^{p-1}u$$

   where $\alpha = 1-\gamma$ and $D^\alpha_{0|t}$ is the fractional derivative operator of order α ($\alpha \in]0,1[$) of Riemann-Liouville defined by

   $$D^\alpha_{0|t}u = \frac{d}{dt}J^{1-\alpha}_{0|t}u. \tag{1.4}$$

3. In the case of $\gamma = 0$, Souplet [32] has showed that non-zero positive solution blows up in finite time.

In 2010, Fino [2] studied the Cauchy problem of the following damped wave equation with non linear memory

$$u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = \int_0^t (t-\tau)^{-\gamma}|u|^{p-1}u(\tau,x)d\tau \quad t>0,\ x\in\mathbb{R}^N \tag{1.5}$$

with positive compact supported initial data

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N. \tag{1.6}$$

In his paper, he made some gaps according to the values of γ and p that is he didn't discuss all the possible values for γ and p. For this reason, M. D'Abbico in his paper [25] which published on 2014 tried to filling all the gaps and complete all the cases which were not investigated by Fino when he added an additional regularity on the intial data and so on the solution. For the existence of solution, Fino used the wighted energy method that is he work in weighted space while D'Abbicco has used an approach related to the so called Matsumura type estimates to prove the local existence of solutions. In particular he tried to estimate the nom in L^2 of u, ∇u and u_t and sometimes the norm of u in the homogeneous Sobolev spaces $\dot{H}^s$ $(s > 0)$. In particular they found the same critical exponent p^* in Fujita's sense founded by Cazenave and al [39]. In order to explain this, one can show by scaling argument that (1.1) is the corresponding heat equation of (1.5) roughly speaking, for a solution $u = u(t,x)$ of (1.5) setting

$$u(t,x) = w(\lambda t, \lambda^{1/2}x), \quad \lambda t = s, \ \lambda^{1/2}x = y$$

with a parameter $\lambda > 0$ we have

$$\lambda w_{ss}(s,y) - \Delta w(s,y) + w_s(s,y) = 0. \tag{1.7}$$

Thus, by letting $\lambda \to 0$ we obtain the heat equation

$$-\Delta w + w_s = 0. \tag{1.8}$$

It remains to note that $\lambda \to 0$ is corresponding to $t \to \infty$.
Concerning the blow-up results, M. D'Abbicco did not investigate the blow-up results in his paper [25], while Fino managed to prove the following results (See Theorem 2 in [2]): He take the data u_0 and u_1 from H^1 and L^2 respectively such that

$$\int_{\mathbb{R}^N} u_i(x)dx > 0, \quad i = 0,1.$$

Then he proved the following

- Let $1 < p < \frac{n}{n-2}$ for $n \geq 3$ and $p \in (1, +\infty)$ for $n = 1,2$. Assume that $\frac{n-2}{n} < \gamma < 1$. If $p \leq p_\gamma$, then the solution of Problem (1.5)-(1.6) blows up in finite time.
- Let $n \geq 3$ and $1 < p < \frac{n}{n-2}$. Assume that $0 < \gamma \leq \frac{n-2}{n}$, then the solution of Problem (1.5)-(1.6) blows up in finite time.

In particular, Fino proved that the exponent p^* is critical by proving the blow-up but D'Abbicco did not. For more details and faurther results one can check [2] and [25] respectively.
After that, and precisely in 2013, M. Berbiche and A. Hakem [22] have thought to generalize the above results for problems more general than the previous ones when they have addressed the study to the Cauchy problem of the following equation:

$$u_{tt}(t,x) - \Delta u(t,x) + |u|^{m-1}u_t(t,x) = \int_0^t (t-\tau)^{-\gamma}|u(\tau,x)|^p \, d\tau, \quad t > 0, \, x \in \mathbb{R}^n, \tag{1.9}$$

which describes a damped wave equation with nonlinear memory and the damping is also not linear. Exactly they have proved that if $p > m > 1$ and the initial data satisfies at

$$\int_{\mathbb{R}^N} u_0(x)dx > 0, \int_{\mathbb{R}^N} |u_0|^{m-1} u_0(x)dx > 0, \int_{\mathbb{R}^N} u_1(x)dx > 0$$

Then if

$$N \leq \Big(\frac{2(m+(1-\gamma)p)}{p-1+(1-\gamma)m-1)}, \frac{2\,(1+(2-\gamma)\,p)}{\big(\frac{(p-1)(2-\gamma)}{p-m}+\gamma-1)(p-1)}\Big)$$

or

$$p \leq \frac{1}{\gamma}$$

hence the solution of equation (1.9) with such initial data does not exist globally in time. This situation is relatively delicate, the non-linearity of the damping term does not allow the authors to find the critical exponent in Fujita sense, that is the upper bound of the set of admissible p can not be found explicitly. They found only sufficient conditions to the non global in time solutions, even with regular data. This means that the obtained condition for the parameters of the equation is not sharp!
In a recent paper, submitted and not published yet, T. Hadj kaddour and A. Hakem [41] have studied the following model

$$u_{tt} - \Delta u + D^{\sigma}_{0|t} u_t = \int_0^t (t-s)^{-\gamma} |u|^p ds$$

which is a fractional damped wave equation with non linear memory which is a generalization of (1.5), that is if we take $\sigma = 0$ then we get the model (1.5) and the results of [2] (See Chapter 6).

1.2 Motivation

The traditional integral and derivative are, to say the least, a staple for the technology professional essential as means of understanding and working with natural and artificial systems. Fractional calculus is field of mathematics study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integral value. Let us consider, for example, the physical meaning of the exponent, it is well known that exponents provide a short notation for what is essentially a repeated multiplication of a numerical value. However, this physical definition can clearly become confused when considering non integer exponents, While almost anyone can verify that $x^3 = \underbrace{x.x.x}_{3times}$ but how might one describe the physical meaning of, for example, $x^{4.4}$ or x^{π} . One can not conceive what it might be like to multiply a quantity by it self 4.4 times , and yet these expression have a definite value for any value x.

Now, in the same way consider the integral and the derivative. Although they are indeed concepts of higher complexity by nature, it is still fairly easy to physically represent their meaning. Once mastered, the idea of completing numerous of these operations, integrations or differentiations follows naturally. Given the satisfaction of a very few restrictions (e.g. function continuity) completing n integrations can become as methodical as multiplication. But the curious mind can not be restrained from asking the question what if n not were not restricted to an integer value? Again, at first glance, the physical meaning can become convoluted (pan intended), but as we want to show fractional calculus flows quite naturally from our traditional definitions. And just as fractional exponents, it will be become apparent that integrations of order 1/2 and beyond can find practical use in many modern problems. We can show this if we consider for example, the heat equation on the positive real axis, in particular the flux corresponding to a solution of a such PDE is given in the form of integration of order 1/2 as it will be explained hereafter

Let us consider the following problem at infinity of the heat equation on the real axis:

$$\begin{cases} u_t - ku_{xx} = f(t), \quad t > 0,\, x \geq 0, \\ u(t,0) = 0, \quad t > 0, \\ \frac{\partial u}{\partial x}(t,x) \to 0 \text{ as } x \to \infty. \end{cases} \tag{1.10}$$

where $k > 0$ and f is a given smooth function independent of x.
The additional condition can be interpreted by saying that the temperature is uniform for great distance. Let us interesting by the flux $\Phi(t)$ on the board $\{x = 0\}$ given by the Fourier law as follows:

$$\Phi(t) = -k\frac{\partial u}{\partial x}(t,0), \quad t > 0. \tag{1.11}$$

We can solve the problem (1.10) by using partial Fourier transform $\mathscr{F}_{t\to\tau}$ (in short $\mathscr{F}$) with respect to t defined by

$$\mathscr{F}u(\tau,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i\tau t}u(t,x)dt. \tag{1.12}$$

For the sake of brevity, we use sometimes the notation $\hat{u}$ instead of $\mathscr{F}u$. So it is well known that the solution is given by

$$u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\tau t}\hat{u}(\tau,x)d\tau. \tag{1.13}$$

Next, we derive both side of formula (1.13) with respect to t we find

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} i\tau e^{i\tau t}\hat{u}(\tau,x)d\tau \tag{1.14}$$

We know that the Fourier transform of the first derivative is obtained by multiplying the Fourier transform by $i\tau$, this leads to the well known classical identity

$$\widehat{\frac{\partial u}{\partial t}} = i\tau\hat{u}. \tag{1.15}$$

Then, denoting by $\hat{f}$ for the Fourier transform of f and applying Fourier transform on both sides of equation (1.10) we get

$$i\tau\hat{u} - k\hat{u}_{xx} = \hat{f}. \tag{1.16}$$

The general solution of the ordinary differential equation (1.16) is given by

$$\hat{u}(\tau, x) = \frac{1}{i\tau}\hat{f} + C_1 e^{\sqrt{\frac{i\tau}{k}}x} + C_2 e^{-\sqrt{\frac{i\tau}{k}}x}, \quad C_1, C_2 \in \mathbb{R}. \tag{1.17}$$

Introducing the auxiliary conditions we show that

$$\hat{u}(\tau, x) = \frac{1}{i\tau}\left(1 - e^{-\sqrt{\frac{i\tau}{k}}x}\right)\hat{f}. \tag{1.18}$$

Next, we derive the equation (1.18) with respect to x, we find

$$\frac{\partial \hat{u}}{\partial x}(\tau, x) = \frac{1}{\sqrt{ik\tau}}\left(1 - e^{-\sqrt{\frac{i\tau}{k}}x}\right)\hat{f}. \tag{1.19}$$

Hence, by setting $x = 0$ in (1.19) we get after using (1.11)

$$\Phi(t) = -\sqrt{\frac{k}{i\tau}}\hat{f}. \tag{1.20}$$

So, since the Fourier transform of the first derivative of f is obtained by multiplying the Fourier transform of f by $i\tau$ then, it seems that the quantity $\sqrt{\frac{1}{i\tau}}\hat{f}$ can be interpreted as the Fourier transform of integral of order *one half* of f. More precisely, if we introduce the function ρ defined by

$$\rho(t) = \frac{1}{\sqrt{t}}H(t)$$

where $H(t)$ is the Heaviside function then, one can verify easily that

$$\hat{\rho}(\tau) = \sqrt{\frac{\pi}{i\tau}}, \quad t > 0. \tag{1.21}$$

Taking in consideration (1.21), we get from (1.20)

$$\Phi(t) = -\sqrt{\frac{k}{\pi}}\hat{\rho}\hat{f}. \tag{1.22}$$

Using the proprieties of Fourier transform and convolution of functions, in particular the identity

$$\mathscr{F}(\rho * f) = \mathscr{F}(f)\mathscr{F}(\rho) = \hat{\rho}\hat{f}$$

we get from (1.22)

$$\begin{aligned} \Phi(t) &= -\sqrt{\frac{k}{\pi}}\mathscr{F}(\rho * f)(t) \\ &= -\sqrt{\frac{k}{\pi}}\int_0^t \frac{1}{\sqrt{t-s}}f(s)ds. \end{aligned} \tag{1.23}$$

The integral which appears in (1.23) is the integral of order $1/2$ of f.

1.3 Plan work

This thesis is composed of six principle chapters and an Annex. The first one is an *Introduction*, it contains in particular a brief history on the fractional calculus and partial differential equations with nonlinear memory. In order to motivate the requirement of the fractional calculus we have introduced a physical reason let the integration of order one half be created. This first chapter is ended by a third section devoted to the plane work of this thesis.

The second chapter is devoted to some notations and preliminaries, especially in the first section we remind some techniques and results which are necessary for the understanding of the fractional calculus's rules and several results as Sobolev embedding. Here, we restrict our selves to Riemann-Liouville and Caputo sense because there is a more compact theory. This chapter is finished by the notion of *blow-up* where we have introduce in particular what do authors mean by blow-up.

The third Chapter is consecrated to the existence and uniqueness of solution to Problem (3.1)-(3.2). Here we have used the famous fixed point theorem to show the existence and uniqueness of solution with high regular data, that is we assumed that $(u_0, u_1) \in H^s \times H^{s-1}$ with $s > \frac{N}{2}$ in order to apply Sobolev embedding.

The following chapter is number four and it is devoted to study the blow-up phenomena of solutions of Problem (3.1)-(3.2). Here we have tried to answer the question: under which conditions on the parameters γ, p and m, the solution does not exist globally in time? And the obtained results are proved by absurd using the test function method using a suitable test function.

The next chapter, number five, is devoted to study of the Blow-up of solutions for Problem (5.1)-(5.2) which concerns model of wave equations with fractional damping and non-linear memory, in particular, we have detreminated the critical exponent in the sens of Fujita for models like (5.1) and generalized some results obtained by Fino [2]. The last chapter is a conclusion which contains some remarks and proposal for open subjects. We have finished this thesis by an Annex, in which we have proved some results used to prove the main results in this thesis.

In the end of this thesis there is an alphabetic list of the references used to prepare this thesis under the title Bibliography.

Chapter 2

Notations and Preliminaries

This chapter contains some preliminaries used throughout the whole thesis. After presenting some notations and definitions, the operators of fractional integration and differentiation of Riemann-Liouville will be defined.

2.1 Some notations

Most notations used throughout this thesis are standard. So, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural, real and complex numbers, respectively, and $\mathbb{R}_+ := [0, \infty)$.
A multi-index is an element $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n)$ of $\mathbb{N}^n$.
The module (or length) of a multi-index α is denoted $|\alpha|$ and defined as

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

The factorial of α which is denoted $\alpha!$ is defined by

$$\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!.$$

For a multi-index $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$, ∂_x^α denotes the totality of all the α-*th* order derivatives with respect to $x \in \mathbb{R}^N$

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}.$$

For $0 < p < +\infty$, we denote by $L^p(\mathbb{R}^n)$ for the space of all (equivalent class) of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$, such that $|f|^p$ is integrable. The norm of a such f which is denoted $\|f\|_p$ instead of $\|f\|_{L^p}$ is defined as

$$\|f\|_p = \Big(\int_{\mathbb{R}^n} |f(x)|^p dx \Big)^{\frac{1}{p}}.$$

If $p = +\infty$ then we mean by $L^\infty(\mathbb{R}^n)$ the set of the functions essentially bounded on $\mathbb{R}^N$ with the norm

$$\|u\|_{L^\infty(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

The space $\mathscr{C}_0^\infty(\mathbb{R}^n)$ denotes the space of all infinitely differentiable functions with compact support.
Let $s \in \mathbb{R}$. Then $H^s(\mathbb{R}^n)$ denotes for the Sobolev space defined as follows

$$H^s(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n); \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\mathscr{F}u(\xi)|^s d\xi < \infty\}$$

equipped with the norm

$$\|u\|_{H^s} = \Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^s d\xi\Big)^{\frac{1}{2}}.$$

The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ of order s is defined by

$$\dot{H}^s(\mathbb{R}^n) = \left\{u \in \mathscr{S}'(\mathbb{R}^n); (-\Delta)^{s/2} u \in L^2(\mathbb{R}^n)\right\},$$

if $s \notin \mathbb{N}$, and by

$$\dot{H}^s(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n); (-\Delta)^{s/2} u \in L^2(\mathbb{R}^n)\right\},$$

if $s \in \mathbb{N}$, where $\mathscr{S}'(\mathbb{R}^n)$ is the space of Schwartz's (or tempered) distributions and $(-\Delta)^{s/2}$ is the fractional Laplacian operator defined as a pseudo-differential operator of symbol $p(x,\xi) = |\xi|^s$ that is (See [6])

$$\begin{aligned}(-\Delta)^{s/2} u(x) &= \mathscr{F}^{-1}(|\xi|^s \mathscr{F}(u)(\xi))(x) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\xi} |\xi|^s u(y) dy d\xi \qquad (2.1)\end{aligned}$$

for every $u \in D((-\Delta)^{s/2}) = \dot{H}^s(\mathbb{R}^n)$. Here $\mathscr{F}$ stands for the Fourier transform defined by

$$\mathscr{F}(f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} f(x) dx, \quad \xi \in \mathbb{R}^n.$$

and $\mathscr{F}^{-1}$ is its inverse.

2.2 Fractional integration and differentiation

Let $\alpha > 0$, $m = [\alpha]$ and $I = (0,T)$ for some $T > 0$. For the sake of brevity we introduce, for $\beta > 0$, the following function:

$$g_\beta(t) := \begin{cases} \frac{1}{\Gamma(\beta)} t^{\beta-1} & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases} \qquad (2.2)$$

where Γ is the Gamma function defined by (See e.g [16])

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \qquad (2.3)$$

One can note that $g_0(t) = 0$ since $\Gamma(0) = +\infty$ (divergent positive integral). This family of functions satisfy, in particular, the following semi group property

$$g_\alpha * g_\beta = g_{\alpha+\beta}, \quad \alpha,\, \beta > 0. \qquad (2.4)$$

The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined, for all $f \in L^1(I)$, as follows:

$$J_t^\alpha f(t) := (g_\alpha * f)(t), \; t > 0, \tag{2.5}$$

with $J_t^0 f(t) := f(t)$.
Thanks to (2.4) and the associativity of the convolution, one can show that the operators of fractional integration obey the following semi group property

$$J_t^\alpha J_t^\beta = J_t^{\alpha+\beta}, \tag{2.6}$$

for all $\alpha, \beta \geq 0$.
The Riemann-Liouville fractional derivative of order α is defined for all f satisfying

$$f \in L^1(I) \text{ and } g_{m-\alpha} * f \in W^{1,m}(I) \tag{2.7}$$

by

$$D_t^\alpha f(t) := D_t^m (g_{m-\alpha} * f)(t) = D_t^m J_t^{m-\alpha} f(t), \tag{2.8}$$

where

$$D_t^m := \frac{d^m}{dt^m}, m \in \mathbb{N}.$$

As in the case of differentiation and integration of integer order, D_t^α is the left inverse of J_t^α, but in general it is not right inverse. More precisely, we have the following theorem [37]:

Theorem 2.2.1. *Let $\alpha > 0$ and $m = [\alpha]$. Then for any $f \in L^1(I)$ it holds*

$$D_t^\alpha J_t^\alpha f = f.$$

If moreover (2.5) holds then

$$J_t^\alpha D_t^\alpha f = f(t) - \sum_{k=0}^{m-1} (g_{m-\alpha} * f)^{(k)}(0) g_{\alpha+k+1-m}(t). \tag{2.9}$$

In the particular case $g_{m-\alpha} * f \in W_0^{1,m}(I)$, we have $J_t^\alpha D_t^\alpha f = f$.

In particular, if $\alpha \in (0, 1)$, and if $g_{1-\alpha} * f \in W^{1,1}(I)$ then (2.9) reads

$$J_t^\alpha D_t^\alpha f = f(t) - (g_{1-\alpha} * f)(0) g_\alpha(t).$$

If $f \in W^{m,1}(I)$ (which implies (2.5)), then $D_t^\alpha f$ may be represented in the form

$$D_t^\alpha f = \sum_{k=0}^{m-1} f^{(k)}(0) g_{k-\alpha+1}(t) + J_t^{m-\alpha} D_t^\alpha f(t). \tag{2.10}$$

It follows from the representation (2.10) of the elements of $W^{m,1}(I)$ and the definition of D_t^α. In many cases it is more convenient to use the second term in the right-hand side of (2.10) as a definition of fractional derivative of order α. The usefulness of such a definition in the mathematical analysis is demonstrated in [27]. Later, this alternative definition of fractional derivative was introduced by Caputo [23], and adopted by Caputo and Mainardi [24] in the framework of the theory of linear viscoelasticity. So the Caputo fractional derivative of order α ($\alpha > 0$) is defined by

$$\mathbf{D}_t^\alpha f(t) := J_t^{m-\alpha} D_t^m f(t). \tag{2.11}$$

Some simple but relevant results valid for $\alpha, \beta, t > 0$ are:

$$J_t^\alpha g_\beta = g_{\alpha+\beta}, \quad D_t^\alpha g_\beta = g_{\beta-\alpha}, \quad \beta \geq \alpha. \tag{2.12}$$

In particular, $D_t^\alpha g_\alpha = 0$. We also note $D_t^\alpha 1 = g_{1-\alpha},\ \alpha \leq 1$, while $\mathbf{D}_t^\alpha 1 = 0$ for all $\alpha > 0$. If instead of $f \in W^{m,1}(I)$ we have only (2.7) and $f \in \mathscr{C}^{m-1}(I)$, then we can use the following equivalent representation, which follows immediately from (2.10), (2.11) and (2.12):

$$\mathbf{D}_t^\alpha f(t) = D_t^\alpha \Big(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) g_{k+1}(t) \Big). \tag{2.13}$$

The Caputo derivative $\mathbf{D}_t^\alpha$ is again a left inverse of J_t^α but in general not right inverse, that is

$$\mathbf{D}_t^\alpha J_t^\alpha f = f, \ J_t^\alpha \mathbf{D}_t^\alpha f = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) g_{k+1}(t). \tag{2.14}$$

The first identity is valid for all $f \in L^1(I)$, the second one, for $f \in \mathscr{C}^{m-1}(I)$, such that (2.7) is satisfied. In particular, if $\alpha \in (0,1)$, $g_{1-\alpha} * f \in W^{1,1}(I)$ and $f \in \mathscr{C}^{m-1}(I)$, then

$$J_t^\alpha \mathbf{D}_t^\alpha f = f(t) - f(0). \tag{2.15}$$

Since $\mathscr{L}(g_{\alpha)})(\lambda) = \lambda^{-\alpha}$, we obtain after applying the properties of the Laplace transform

$$\mathscr{L}(D_t^\alpha f)(\lambda) = \lambda^\alpha \mathscr{L}(f)(\lambda) - \sum_{k=0}^{m-1} (g_{m-\alpha} * f)^{(k)}(0) \lambda^{m-1-k}, \tag{2.16}$$

and

$$\mathscr{L}(\mathbf{D}_t^\alpha f)(\lambda) = \lambda^\alpha \mathscr{L}(f)(\lambda) - \sum_{k=0}^{m-1} f^{(k)}(0) \lambda^{\alpha-1-k}. \tag{2.17}$$

The left-handed derivative and the right-handed derivative in the Riemann-Liouville sense are defined, for $\Psi \in L^1(0,T)$ and $0 < \alpha < 1$ as follows:

$$D_{0|t}^\alpha \Psi(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{\Psi(\sigma)}{(t-\sigma)^\alpha} d\sigma, \tag{2.18}$$

and

$$D_{t|T}^\alpha \Psi(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{\Psi(\sigma)}{(\sigma-t)^\alpha} d\sigma, \tag{2.19}$$

respectively, where the symbol Γ stands for the usual Euler's gamma function defined by (2.3).

Relationship between Caputo and Riemann-Liouville fractional derivative

If one try to compare the Caputo fractional derivative with the Riemann-Liouville one, one can find that the Caputo fractional derivative

$$\mathbf{D}^{\alpha}_{0|t}\Psi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\Psi'(\sigma)}{(t-\sigma)^{\alpha}}d\sigma,$$

requires $\Psi' \in L^1(0,T)$. Clearly we have

$$D^{\alpha}_{0|t}g(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{g(0)}{t^{\alpha}} + \int_0^t \frac{g'(\sigma)}{(t-\sigma)^{\alpha}}d\sigma\right]$$

and

$$D^{\alpha}_{t|T}f(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(T)}{(T-t)^{\alpha}} - \int_t^T \frac{f'(\sigma)}{(t-\sigma)^{\alpha}}d\sigma\right].$$

Therefore the Caputo derivative is related to the Riemann-Liouville derivative by

$$\mathbf{D}^{\alpha}_{0|t}\Psi(t) = D^{\alpha}_{0|t}\left[\Psi(t) - \Psi(0)\right]$$

We end this section by the fractional version of integration by parts

Proposition 2.2.2. *(See ([16]) For all $f, g \in L^1[0,T]$ such that $D^{\alpha}_{t|T}f$ and $D^{\alpha}_{0|t}g$ exist and are continuous, it holds the following formula of integration by parts*

$$\int_0^T f(t)(D^{\alpha}_{0|t}g)(t)dt = \int_0^T g(t)(D^{\alpha}_{t|T}f)(t)dt.$$

2.2.3 Fundamental examples

Example 2.2.4. *Let $a > 1$ and φ_a the function defined by*

$$\varphi_a(t) = \left(1 - \frac{t^2}{T^2}\right)^{2a}, \quad t \in [0,T].$$

Then, we have

$$|D^{\alpha}_{t|T}\varphi_a(t)| \leq CT^{-\alpha}, \quad |D^{\alpha-1}_{t|T}\varphi_a(t)| \leq C'T^{1-\alpha}$$

for some positive constants C, C'.

Proof. For the first term, we compute $D^{\alpha}_{t|T}\varphi_a$

$$\begin{aligned}\Gamma(2-\alpha)D^{\alpha}_{t|T}\varphi_a &= \int_t^T \frac{\varphi_a''(\sigma)}{(\sigma-t)^{\alpha-1}}d\sigma \\ &= \frac{-4a}{T^2}\int_t^T \left(\left(1-\frac{\sigma^2}{T^2}\right)^{2a-1} - 2\frac{\sigma^2}{T^2}(2a-1)\left(1-\frac{\sigma^2}{T^2}\right)^{2a-2}\right)(\sigma-t)^{1-\alpha}d\sigma, \quad (2.20)\end{aligned}$$

therefore

$$\Gamma(2-\alpha)D^{\alpha}_{t|T}\varphi_a = -4aT^{-4a}\int_t^T (T^2-\sigma^2)^{2a-1}(\sigma-t)^{1-\alpha}d\sigma$$
$$+8a(2a-1)T^{-4a}\int_t^T \sigma^2(T^2-\sigma^2)^{2a-2}(\sigma-t)^{1-\alpha}d\sigma = I+J. \quad (2.21)$$

Using the Euler's change of variable

$$y = \frac{\sigma-t}{T-t} \quad (2.22)$$

we see that

$$\sigma - t = (T-t)y, \quad y = \frac{\sigma-t}{T-t}, \quad 1-y = \frac{T-\sigma}{T-t},$$
$$1-y^2 = \frac{T^2-\sigma^2}{(T-t)^2} - 2t\frac{1-y}{T-t},$$

and

$$T^2-\sigma^2 = (1-y^2)(T-t)^2 + 2t(1-y)(T-t),$$

Then

$$\begin{aligned} I &= -4aT^{-4a}\int_t^T (T^2-\sigma^2)^{2a-1}(\sigma-t)^{1-\alpha}d\sigma \\ &= -4aT^{-4a}(T-t)^{1-\alpha+2a}\int_0^1 (1-y)^{2a-1}\left((T-t)(1+y)+2t\right)^{2a-1} y^{1-\alpha}dy. \quad (2.23)\end{aligned}$$

Since we have

$$(T-t)(1+y)+2t = (T+t)+y(T-t),$$

and as

$$y(T-t) < (T-t) \leq (T+t), \text{ for } y<1,$$

then one can apply the Binomial formula for non integer power to

$$I = -4aT^{-4a}(T-t)^{1-\alpha+2a}\int_0^1 (1-y)^{2a-1}\left((T+t)+y(T-t)\right)^{2a-1} y^{1-\alpha}dy$$

In this way, we find

$$I = -4aT^{-4a}\sum_{k=0}^{\infty} C_k^{2a-1}(T-t)^{1-\alpha+2a+k}(T+t)^{2a-k-1}\int_0^1 (1-y)^{2a-1}y^{1-\alpha+k}dy, \quad (2.24)$$

where

$$C_k^{2a-1} = \frac{(2a-1)!}{k!(2a-1-k)!}.$$

Using the following Beta formula

$$\int_0^1 (1-\tau)^{u-1}\tau^{v-1}d\tau = \mathscr{B}(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad u,v>0, \quad (2.25)$$

we obtain

$$I = -4aT^{-4a}\sum_{k=0}^{\infty} C_k^{2a-1}(T-t)^{1-\alpha+2a+k}(T+t)^{2a-k-1}\frac{\Gamma(2a)\Gamma(2-\alpha+k)}{\Gamma(2a+2-\alpha+k)}.$$

Analogously for J, we get

$$\begin{aligned} J &= 8a(2a-1)T^{-4a}\int_t^T \sigma^2(T^2-\sigma^2)^{2a-2}(\sigma-t)^{1-\alpha}d\sigma \\ &= 8a(2a-1)T^{-4a}\int_0^1 (t+(T-t)y)^2(1-y)^{2a-2}((T+t)+y(T-t))^{2a-2}(T-t)^{-\alpha+2a}y^{1-\alpha}dy. \end{aligned}$$

By Newton formula, we obtain

$$\begin{aligned} J &= 8a(2a-1)T^{-4a}\sum_{k=0}^{\infty} C_k^{2a-2} \\ &\times \int_0^1 (t+(T-t)y)^2(T+t)^{2a-2-k}(T-t)^{k-\alpha+2a}(1-y)^{2a-2}y^{k+1-\alpha}dy \\ &= 8a(2a-1)T^{-4a}\sum_{k=0}^{+\infty} C_k^{2a-2}\Bigg(t^2(T+t)^{2a-2-k}(T-t)^{k-\alpha+2a}\int_0^1(1-y)^{2a-2}y^{k+1-\alpha}dy \\ &+2t(T+t)^{2a-2-k}(T-t)^{k-\alpha+2a+1}\int_0^1(1-y)^{2a-2}y^{k+2-\alpha}dy \\ &+(T+t)^{2a-2-k}(T-t)^{k-\alpha+2a+2}\int_0^1(1-y)^{2a-2}y^{k+3-\alpha}dy\Bigg). \end{aligned} \tag{2.26}$$

Using the Beta formula (2.25), we have

$$\begin{aligned} J &= 8a(2a-1)T^{-4a}\sum_{k=0}^{\infty} C_k^{2a-2}\Bigg(t^2(T+t)^{2a-2-k}(T-t)^{k-\alpha+2a}\frac{\Gamma(2a-1)\Gamma(2-\alpha+k)}{\Gamma(2a+1-\alpha+k)} \\ &+2t(T+t)^{2a-2-k}(T-t)^{k-\alpha+2a+1}\frac{\Gamma(2a-1)\Gamma(3-\alpha+k)}{\Gamma(2a+2-\alpha+k)} \\ &+(T+t)^{2a-2-k}(T-t)^{k-\alpha+2a+2}\frac{\Gamma(2a-1)\Gamma(4-\alpha+k)}{\Gamma(2a+3-\alpha+k)}\Bigg). \end{aligned} \tag{2.27}$$

Hence

$$\begin{aligned} D^{\alpha}_{t|T}\big(1-\frac{t^2}{T^2}\big)^{2a} &= \frac{-4aT^{-4a}}{\Gamma(2-\alpha)}\Bigg(\sum_{k=0}^{\infty} C_k^{2a-1}(T-t)^{1-\alpha+2a+k}(T+t)^{2a-k-1}\frac{\Gamma(l)\Gamma(2-\alpha+r)}{\Gamma(l+2-\alpha+r)}\Bigg) \\ &+\frac{8a(2a-1)}{\Gamma(2-\alpha)}T^{-4a}\sum_{k=0}^{\infty} C_k^{2a-2}\Bigg(t^2(T+t)^{2a-2-k}(T-t)^{k-\alpha+2a}\frac{\Gamma(2a-1)\Gamma(2-\alpha+k)}{\Gamma(2a+1-\alpha+k)} \\ &+2t(T+t)^{2a-2-k}(T-t)^{k-\alpha+2a+1}\frac{\Gamma(2a-1)\Gamma(3-\alpha+k)}{\Gamma(2a+2-\alpha+k)} \\ &+(T+t)^{2a-2-k}(T-t)^{k-\alpha+2a+2}\frac{\Gamma(2a-1)\Gamma(4-\alpha+k)}{\Gamma(2a+3-\alpha+k)}\Bigg). \end{aligned} \tag{2.28}$$

Now, if we set $t=\tau T$, we find

$$D^{\alpha}_{t|T}(1-\frac{t^2}{T^2})^{2a}=\frac{-4aT^{-\alpha}}{\Gamma(2-\alpha)}\left(\sum_{k=0}^{\infty}C_k^{2a-1}(1-\tau)^{1-\alpha+2a+k}(1+\tau)^{2a-k-1}\frac{\Gamma(l)\Gamma(2-\alpha+r)}{\Gamma(l+2-\alpha+r)}\right)$$
$$+\frac{8a(2a-1)}{\Gamma(2-\alpha)}\sum_{k=0}^{\infty}C_k^{2a-2}\left(\tau^2(1+\tau)^{2a-2-k}(1-\tau)^{k-\alpha+2a}\frac{\Gamma(2a-1)\Gamma(2-\alpha+k)}{\Gamma(2a+1-\alpha+k)}\right.$$
$$+2\tau(1+\tau)^{2a-2-k}(1-\tau)^{k-\alpha+2a+1}\frac{\Gamma(2a-1)\Gamma(3-\alpha+k)}{\Gamma(2a+2-\alpha+k)}$$
$$\left.+(1+\tau)^{2a-2-k}(1-\tau)^{k-\alpha+2a+2}\frac{\Gamma(2a-1)\Gamma(4-\alpha+k)}{\Gamma(2a+3-\alpha+k)}\right). \tag{2.29}$$

In the same way we compute $D^{\alpha-1}_{t|T}\big(1-\frac{t^2}{T^2}\big)^{2a}$. We have

$$\begin{aligned}\Gamma(2-\alpha)D^{\alpha-1}_{t|T}\big(1-\frac{t^2}{T^2}\big)^{2a} &= -\int_t^T(\sigma-t)^{1-\alpha}\Big(\frac{d}{d\sigma}\big(\frac{T^2-\sigma^2}{T^2}\big)^{2a}\Big)d\sigma\\ &= \frac{4a}{T^2}\int_t^T\sigma(T^2-\sigma^2)^{2a-1}(\sigma-t)^{1-\alpha}d\sigma. \end{aligned}\tag{2.30}$$

Next we calculate the integral I where

$$I(t,T):=\int_t^T\sigma(T^2-\sigma^2)^{2a-1}(\sigma-t)^{1-\alpha}d\sigma,$$

by using Euler's change of variable (2.22), we find

$$I(t,T)=(T-t)^{2a-\alpha+1}\int_t^T((T-t)y+t)(1-y)^{2a-1}\left((T+t)+y(T-t)\right)^{2a-1}y^{1-\alpha}dy.$$

Then by using the generalized binomial formula, we may write

$$I(t,T)=$$
$$(T-t)^{2a-\alpha+1}\int_0^1((T-t)y+t)(1-y)^{2a-1}\sum_{k=0}^{\infty}C^k_{2a-1}(T+t)^{2a-1-k}(T-t)^ky^{k+1-\alpha}dy.$$

Using the Beta integral formula (2.25) we obtain

$$I(t,T)=(T-t)^{2a-\alpha+2}\sum_{k=0}^{\infty}C^k_{2a-1}(T+t)^{2a-1-k}(T-t)^k\int_0^1y^{k+2-\alpha}(1-y)^{2a-1}dy$$
$$+(T-t)^{2a-\alpha+1}t\sum_{k=0}^{\infty}C^k_{2a-1}(T+t)^{2a-1-k}(T-t)^k\int_0^1y^{k+1-\alpha}(1-y)^{2a-1}dy.$$

Hence

$$D^{\alpha-1}_{t|T}\big(1-\frac{t^2}{T^2}\big)^{2a}=$$
$$\frac{4aT^{-4a}}{\Gamma(2-\alpha)}\left((T-t)^{2a-\alpha+2}\sum_{k=0}^{\infty}C^k_{2a-1}(T+t)^{2a-1-k}(T-t)^k\frac{\Gamma(k+3-\alpha)\Gamma(2a)}{\Gamma(k+3-\alpha+2a)}\right.$$
$$\left.+(T-t)^{2a-\alpha+1}t\sum_{k=0}^{\infty}C^k_{2a-1}(T+t)^{2a-1-k}(T-t)^k\frac{\Gamma(k+2-\alpha)\Gamma(2a)}{\Gamma(k+2-\alpha+2a)}\right).$$

In particular we have

$$D_{t|T}^{\alpha-1}\varphi_a(0) = \frac{4a}{\Gamma(2-\alpha)} \sum_{k=0}^{\infty} C_k^{2a-1} \frac{\Gamma(k+3-\alpha)\Gamma(2a)}{\Gamma(k+3-\alpha+2a)} T^{-\alpha+1}.$$

□

2.3 Furthermore results and Sobolev embedding

We need also to some results as Sobolev embedding theorems. We need the following Lemmas:

Lemma 2.3.1. *(Sobolev embedding)(See [26])*
If $s > N/2$ then one has

$$H^s(\mathbb{R}^N) \subset \mathscr{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),$$

where the inclusion is continuous, indeed there exists a constant $C > 0$ such that

$$\text{for all } u \in H^s(\mathbb{R}^N): \quad \|u\|_{L^\infty(\mathbb{R}^N)} \leq C\,\|u\|_{H^s(\mathbb{R}^N)}$$

Lemma 2.3.2. *(See [22]) Assume that $s_1, s_2 \geq s > N/2$, then for all $u \in H^{s_1}(\mathbb{R}^N)$ and $v \in H^{s_2}(\mathbb{R}^N)$ there exists a positive constant C independent of u and v such that*

$$\|uv\|_{H^s(\mathbb{R}^N)} \leq C\,\|u\|_{H^{s_1}(\mathbb{R}^N)}\,\|v\|_{H^{s_2}(\mathbb{R}^N)}$$

Lemma 2.3.3 (*Fractional powers*)**.** *(See [22]) Let $s,\ p \geq 1$ such that $N \geq \max\{s-1,\, 1\}$, then for all nonnegative function u belonging to $L^\infty(\mathbb{R}^N) \cap H^{s-1}(\mathbb{R}^N)$ and $n \in \mathbb{N}^*$ it holds the following inequality:*

$$\|u^p\|_{H^{s-1}(\mathbb{R}^N)} \leq C\|u\|_{L^\infty(\mathbb{R}^N)}^{p-1}\|u\|_{H^{s-1}(\mathbb{R}^N)}.$$

for some constant $C > 0$.

Remark 2.3.4. *In fact, the result of Lemma 2.3.3 requires the embedding $L^\infty \hookrightarrow H^{s-1}$ but the condition $u \in L^\infty(\mathbb{R}^N)$ is sufficient.*

The next lemma is an immediate consequence of Proposition 3.7 p.11 in [26], the fact that f is bounded with, together, all its derivatives as it is mentioned in Lemma 2.3.5 and Leibniz formula (see formula 3.23 p.11 in [26])

$$D^\alpha(uv) = \sum_{\beta+\gamma=\alpha} C_\alpha^\beta (D^\beta u)(D^\gamma v), \text{ where } C_\alpha^\beta = \frac{\alpha!}{\beta!(\alpha-\beta)!}. \tag{2.31}$$

Lemma 2.3.5. *Let $s > 1$, $u \in H^{s-1}(\mathbb{R}^N)$ and f be a real valued bounded function with all its derivatives then one has*

- $fu \in H^{s-1}(\mathbb{R}^N)$.

- *There exists some constant $C_{s,f}$ depending only on f and s such that*
$$\|fu\|_{H^{s-1}(\mathbb{R}^N)} \leq C_{s,f} \|u\|_{H^{s-1}(\mathbb{R}^N)}.$$
More precisely $C_{s,f}$ is given by
$$C_{s,f} = \sqrt{\max_{|\alpha|\leq s-1} \sup_{x\in\mathbb{R}^N} |\partial^\alpha f(x)|}.$$

(See Annex for a proof of Lemma 2.3.5)

Change of Order of Integration

Change of order of integration is a trick which we will use e.g. during the calculation of the fractional integral of the power function especially in our case we will use it to treat the non-linear memory term which is in some sense a fractional integral of order γ . We point out that this process does not impose any new condition for the integrated function, it is only a different view at the area we integrate over. There are known more general versions (e.g. Fubini theorem), but for us the case of triangular areas is sufficient. The formula of change of order of integration is given in the following theorem:

Theorem 2.3.6. *Let $f = f(t, \tau, s)$ be a function integrable with respect to τ and s. Then it holds*
$$\int_a^t \int_a^\tau f(t, \tau, s) ds d\tau = \int_a^t \int_s^t f(t, \tau, s) d\tau ds \tag{2.32}$$
The geometrical idea of this formula will become clear. That is matter

2.4 Notion of Well Posedness

It is important to discuss what "*solving a problem of a PDE* " means. Ideally, for a given PDE with an extra conditions (as boundary or (and) initial conditions), there is three main questions are waiting for a reply:

- Are there solutions?(Existence).
- The solution is unique? (Uniqueness).
- Does the solution (which is unique) depend continuously on various parameters, such as forcing term f, initial or (and) boundary conditions? (Stability).

Definition 2.4.1. *(See [8])*
We say that a given problem for a PDE is well posed in the sens of Hadamard if

- *There exists a solution.*
- *This solution is unique.*
- *The solution (which is unique) depends continuously on various parameters, such as forcing term f, initial or (and) boundary conditions.*

Remark 2.4.2. *One should specify in what space the solution exists, is unique and depends continuously on the data.*

2.5 Notion of Blow-up and exponent of Fujita

Sometimes, we are interested by the behaviour of solutions of a given problem for an evolution PDE, especially, if this PDE describes a concrete phenomena e.g. propagation of pollutant in the air, if we denote by $u(t,x)$ for the concentration of this pollutant in the point x at the time t then it is reasonable that one has $\lim_{x\to\infty} u(t,x) = 0$ since there will be no pollutant in the great distance . From this point of view we start, and we have the following definition

Definition 2.5.1. *Let $A \subset \mathbb{R}^N$ and $u = u(t,x)$ be a solution of a given evolution PDE on the set $\Omega := [0,T] \times A$. We say that u blows up in finite time T if such that*

$$\lim_{t\to T^-} |u(t,x)| = +\infty$$

In this case one has

$$\sup_{x\in\Omega} |u(t,x)| = +\infty$$

and T is called the time of Blow-up.

Referential examples

Case of ODE

The simplest example to show the blow-up phenomena in the case of ordinary differential equations (ODE) is the following (non-linear) Cauchy problem

$$x'(t) = x^2(t), \quad t > 0, \quad x(0) = x_0.$$

One can show immediately that if $x_0 = \frac{1}{T}$ for some $T > 0$ then, this Cauchy problem admits in the interval $]0,T[$ the unique solution $x(t) = \frac{1}{T-t}$. This solution is a smooth function on $]0,T[$ and satisfies in particular at $\lim_{t\to T^-} x(t) = +\infty$.
This means that, according to the previous definition, the solution blows up in finite time. One can thought to generalize this remark as a main phenomena of ODEs and PDEs.

Case of PDE

The Blow-up's phenomena appears in particular when the unknown function in the considered problem depends not only on time, but also on the spacial variable, especially in the reaction-diffusion problems, propagation evolution problems, the famous example is the Fujita's equation which we recall in the following section.

Critical exponent of Fujita

Consider the following Cauchy problem of Fujita's equation

$$\begin{cases} u_t = \Delta u + u^p \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{2.33}$$

Where the unknown function $u = u(t,x)$ is real valued, $t > 0$, $p > 1$ and Δ is the classical Laplace operator.
This equation is studied by Fujita in 1966 in his pioneering works [14], in particular, he showed that if $1 < p < 1 + \frac{2}{N}$ then all solutions in a given class blow-up in finite time.

Definition 2.5.2. *The upper bound $1 + \frac{2}{N}$, of the parameter p, is called the exponent of Fujita, it is denoted p^* or sometimes p_{Fuj}. That is $p^* = 1 + \frac{2}{N}$.*
It is characterized by the following:

- *if $1 < p < p^*$ then all solution blows up in finite time.*
- *If $p > p^*$ all solution is global in time, that is defined on $(0, \infty) \times \Omega$, $(\Omega \subset \mathbb{R}^N)$.*
- *if $p = p^*$ this is a critical case!*

Remark 2.5.3. *Throughout this thesis, the constants will be denoted C and are, in general, different from line to line and even in the same line from one place to another one.*

Chapter 3

Well Posedness of Problem (3.1)-(3.2)

3.1 Introduction and statement of problem

In this section, we will use the fixed point theorem to prove the theorem of existence and uniqueness of solutions for the following Cauchy problem which describes a weighted damped wave equation with nonlinear memory

$$u_{tt} - \Delta u + g(\cdot)|u|^{m-1}u_t = \int_0^t (t-\tau)^{-\gamma}|u(\tau,\cdot)|^p\,d\tau \tag{3.1}$$

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N. \tag{3.2}$$

where $m > 1,\ p > 1,\ 0 < \gamma < 1,\ \Delta$ is the classical Laplace operator defined, for all $v \in \mathscr{C}^2(\mathbb{R}^n)$ by

$$\Delta v = \frac{\partial^2 v}{\partial x_1^2} + ... + \frac{\partial^2 v}{\partial x_n^2} \tag{3.3}$$

and the weight $g(\cdot)$ is stationary (that is independent of time).

3.2 Main result

Our result concerning local existence and uniqueness of solution to Problem (3.1)-(3.2) is given in the following theorem

Theorem 3.2.1. *Let* $N \geq 1$, $s > \frac{N}{2}$, $m, p \in (1, \infty)$ *such that* $m, p > s-1$ *and let* $g \in \mathscr{C}^{[s]+1}(\mathbb{R}^N)$ *such that* g *is positive and satisfies for all* $\beta \in \mathbb{N}^N$ *such that* $|\beta| \leq [s]+1$

$$\partial_x^\beta g = \mathcal{O}(1) \quad \text{uniformly with respect to } x \in \mathbb{R}^N,$$

then for any $(u_0, u_1) \in H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)$, *Problem (3.1)-(3.2) admits a unique solution*

$$u \in \mathscr{C}([0,T], H^s(\mathbb{R}^N)) \cap \mathscr{C}^1([0,T], H^{s-1}(\mathbb{R}^N))$$

for some positive T *which depends only on*

$$\|u_0\|_{H^s(\mathbb{R}^N)} + \|u_1\|_{H^{s-1}(\mathbb{R}^N)}.$$

3.3 Proof of Theorem 3.2.1

Proof of Theorem 3.2.1. The main tool used to prove Theorem 3.2.1 is the fixed point theorem as it is mentioned here above, for this reason, we need a suitable functional space and a contraction mapping. To do so, we denote by X_T (for some $T > 0$) for the space of solutions of Problem (3.1)-(3.2), that is

$$X_T = \mathscr{C}\big([0,\, T],\, H^s(\mathbb{R}^N)\big) \cap \mathscr{C}^1\big(\,[0,\, T]\,,\, H^{s-1}(\mathbb{R}^N)\big)$$

for some $T > 0$ and we define for $T > 0$ and $M > 0$ the following auxiliary functional spaces:

$$\begin{aligned} E_T &= L^\infty([0,T], H^s(\mathbb{R}^N)) \cap W^{1,\infty}([0,T],\, H^{s-1}(\mathbb{R}^N)) \\ E_{T,M} &= \big\{u \in E_T;\ \sup_{t\in[0,T]} \big(\|u\|_{H^s(\mathbb{R}^N)} + \|u_t\|_{H^{s-1}(\mathbb{R}^N)}\big) \leq M\big\}, \end{aligned}$$

and we put by definition

$$X_{T,M} := X_T \cap E_{T,M}.$$

Remark 3.3.1. *One should remark that* $X_T \subset E_T$ *and* $X_{T,M} \subset E_{T,M}$ *for all* $T > 0$ *and* $M > 0$.

The proof of Theorem 3.2.1 will be concluded from the following sections. So, we begin by

3.4 Some interesting estimates

Let us define the operator $P_\alpha(g, \partial_t)$ on E_T which acts on the elements of E_T as follows:

$$\begin{aligned} P_\alpha(g,\partial_t)u(t,\cdot) &:= -g(\cdot)|u|^{m-1}u_t + \int_0^t (t-\tau)^{-\gamma}|u(\tau,\,\cdot)|^p\, d\tau \\ &= -g(\cdot)|u|^{m-1}u_t + \Gamma(\alpha) I^\alpha_{0|t}(|u|^p). \end{aligned} \tag{3.4}$$

for all $u \in E_T$, where $\alpha = 1-\gamma$ and $I^{\alpha}_{0|t}$ is the Riemann-Liouville fractional integral of order α ($\alpha \in]0,1[$) defined by (See [37])

$$I^{\alpha}_{0|t} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau \tag{3.5}$$

and Γ is the usual Euler's Gamma function defined by (2.3).
First of all, we state some results to prove Theorem 3.2.1.

Lemma 3.4.1. *Let $P_\alpha(g,\partial_t)$ be the operator defined by (3.4), then for all $T>0$, $t \in [0,T]$ and $u \in E_T$ we have*

$$\begin{aligned}\int_0^t P_\alpha(g,\partial_t) u(\tau,\cdot)\, d\tau &= \frac{1}{m} g\big(|u_0|^{m-1}u_0 - |u(t,\cdot)|^{m-1}u(t,\cdot)\big)\\ &+ \frac{1}{\alpha}\int_0^t (t-\tau)^{\alpha}|u(\tau,\cdot)|^p\, d\tau.\end{aligned}$$

Proof. For all $u \in E_T$, we have

$$\begin{aligned}\int_0^t P_\alpha(g,\partial_t) u(\tau,\cdot)\, d\tau &= -g(\cdot)\int_0^t |u(\tau,\cdot)|^{m-1}u_t(\tau,\cdot)\, d\tau \qquad (3.6)\\ &+ \int_0^t\int_0^\tau (\tau-s)^{\alpha-1}|u(s,\cdot)|^p\, dsd\tau.\\ &=: -g(\cdot)I + J \qquad (3.7)\end{aligned}$$

In order to calculate I, we note that for all $u \in E_T$, we have

$$\partial_t(|u(t,\cdot)|^{m-1}u(t,\cdot)) = m|u(t,\cdot)|^{m-1}u_t(t,\cdot), \tag{3.8}$$

therefore

$$\begin{aligned} I &= \frac{1}{m}|u(\tau,\cdot)|^{m-1}u(\tau,\cdot)\Big|_{\tau=0}^{\tau=t}\\ &= \frac{1}{m}\big(|u(t,\cdot)|^{m-1}u(t,\cdot) - |u(0,\cdot)|^{m-1}u(0,\cdot)\big)\\ &= \frac{1}{m}\big(|u(t,\cdot)|^{m-1}u(t,\cdot) - |u_0|^{m-1}u_0\big). \qquad (3.9)\end{aligned}$$

For J we apply the theorem of change of order of integration (2.3.6) to calculate the interior integral with respect to s. In this way, we find

$$\begin{aligned} J &= \int_0^t \Big(\int_s^t (\tau-s)^{\alpha-1}d\tau\Big)|u(s,\cdot)|^p ds\\ &= \frac{1}{\alpha}\int_0^t (t-\tau)^{\alpha}|u(\tau,\cdot)|^p\, d\tau. \qquad (3.10)\end{aligned}$$

Combining (3.9) and (3.10) into (3.6) we get

$$\begin{aligned}\int_0^t P_\alpha(g,\partial_t) u(\tau,\cdot)\, d\tau &= -\frac{1}{m} g\big(|u(t,\cdot)|^{m-1}u(t,\cdot) - |u_0|^{m-1}u_0\big)\\ &+\frac{1}{\alpha}\int_0^t (t-\tau)^{\alpha}|u(\tau,\cdot)|^p\, d\tau.\end{aligned}$$

This completes the proof of Lemma 3.4.1. □

One of consequences of Lemma 3.4.1 is to estimate the norm in $H^{s-1}(\mathbb{R}^N)$ of $P_\alpha(g, \partial_t)u$ for all $u \in E_{T,M}$ which we need hereafter.

Proposition 3.4.2. *Let $P_\alpha(g, \partial_t)$ be the operator defined by (3.4) and*

$$M = C\big(\|u_0\|_{H^s(\mathbb{R}^N)} + \|u_1\|_{H^{s-1}(\mathbb{R}^N)}\big)$$

for some positive constant C independent of $\|u_0\|_{H^s}$ and $\|u_1\|_{H^{s-1}}$, then there exists some positive constants C_1 and C_2 depending only on T and M such that for all $u \in E_{T,M}$ one has

$$\int_0^t \|P_\alpha(g, \partial_t)u(\tau, \cdot)\|_{H^{s-1}(\mathbb{R}^N)}\, d\tau \leq C_1 M^m + C_2 T^{\alpha+1} M^p.$$

Proof. Using Lemma 3.4.1 we get for all $\beta \in \mathbb{N}^N$ such that $|\beta| \leq [s]$ and $x \in \mathbb{R}^N$

$$\begin{aligned} \partial_x^\beta \int_0^t P_\alpha(g,\, \partial_t)u(\tau,\, x)\, d\tau \;&=\; \frac{1}{m}\partial_x^\beta\big(g(x)|u_0(x)|^{m-1}u_0(x)\big) \\ &\quad -\frac{1}{m}\partial_x^\beta\big(g(x)|u(t,x)|^{m-1}u(t,x)\big) \\ &\quad +\frac{1}{\alpha}\partial_x^\beta \int_0^t (t-\tau)^\alpha |u(\tau,\, x)|^p\, d\tau. \end{aligned} \tag{3.11}$$

Next, by using Lebesgue dominated convergence theorem (*Theorem 1.1.4, p. 3 in [42]*) we get

$$\begin{aligned} \int_0^t \|P_\alpha(g,\, \partial_t)u(\tau,\cdot)\|_{H^{s-1}(\mathbb{R}^N)} d\tau \;&\leq\; \|g|u(0,\, \cdot)|^{m-1}u(0,\, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} \\ &\quad +\|g|u(t,\, \cdot)|^{m-1}u(t,\, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} \\ &\quad +\int_0^t (t-\tau)^\alpha \||u(\tau,\, \cdot)|^p\|_{H^{s-1}(\mathbb{R}^N)} d\tau. \end{aligned} \tag{3.12}$$

By Sobolev's embedding theorem (Lemma 2.3.1), Lemma 2.3.3 and Lemma 2.3.5 we find

$$\begin{aligned} \|g|u(t,\, \cdot)|^{m-1}u(t,\, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} \;&\leq\; C_{s,g}\||u(t,\, \cdot)|^{m-1}u(t,\, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} \\ &\leq\; C\|u(t,\, \cdot)\|^{m-1}_{L^\infty(\mathbb{R}^N)}\|u(t,\, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} \\ &\leq\; C \sup \|u(t,\, \cdot)\|^m_{H^{s-1}(\mathbb{R}^N)} \end{aligned} \tag{3.13}$$

where $C_{s,g}$ is positive constant depending only on s and g. In the same way, by noting that $u(0,\, \cdot)$ is independent of t, we have

$$\|g|u(0,\, \cdot)|^{m-1}u(0,\, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} \leq C\|u_0\|^m_{H^{s-1}(\mathbb{R}^N)}. \tag{3.14}$$

Including the estimates (3.13) and (3.14) into (3.12) we find

$$\begin{aligned} \int_0^t \|P_\alpha(g,\, \partial_t)u(\tau,\, \cdot.)\|_{H^{s-1}(\mathbb{R}^N)}\, d\tau \;&\leq\; C\|u_0\|^m_{H^s(\mathbb{R}^N)} + C \sup_{0\leq t\leq T} \|u(t,\, \cdot)\|^m_{H^s(\mathbb{R}^N)} \\ &\quad + \sup_{0\leq t\leq T} \|u(t,\, \cdot)\|^p_{H^s(\mathbb{R}^N)} \sup_{0\leq t\leq T} \int_0^t (t-\tau)^\alpha\, d\tau \\ &\leq\; C_1 M^m + C_2 T^{\alpha+1} M^p \end{aligned}$$

Hence the proof is completed. □

3.5 The contracting mapping

As it is well known, to apply the fixed point theorem, we need to a contracting mapping, so we introduce the following:

Definition 3.5.1. *Let $T > 0$ and $P_\alpha(g, \partial_t)$ be the operator defind by (3.4). For all $v \in E_T$ we define a mapping Φ by $\Phi(v) = u$ where u is solution in X_T to the following Cauchy problem:*

$$(P2) \quad \begin{cases} u_{tt}(t,x) - \Delta u(t,x) = P_\alpha(g, \partial_t)v(t,x), \quad t > 0, x \in \mathbb{R}^N \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N \\ u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N. \end{cases}$$

Remark 3.5.2. *Let $T > 0$. Suppose that $u \in X_T$ is a solution of the Cauchy Problem (P2) then for all $v \in E_T$ the mapping Φ defined by $\Phi(v) = u$ is well defined.*

Proof. The fact that $v \in E_T$ implies that $u(\cdot, x)$, $u_t(\cdot, x) \in L^\infty([0,T])$ for all $x \in \mathbb{R}^N$, then we have

$$\begin{aligned} \sup_{0\le t\le T} |P_\alpha(g,\, \partial_t)v(t,x)| &\le \sup_{0\le t\le T} |g(x)|u(t,x)|^{m-1}u_t(t,x)| + \sup_{0\le t\le T} \left| \int_0^t (t-s)^{-\gamma}|u(t,x)|^p ds \right| \\ &\le g(x) \sup_{0\le t\le T} |u(t,x)|^{m-1} \sup_{0\le t\le T} |u_t(t,x)| \\ &\quad + \sup_{0\le t\le T} |u(t,x)|^p \sup_{0\le t\le T} \int_0^t (t-s)^{-\gamma} ds \\ &\le H(x)M^m(x) + T^{1-\gamma}M^p(x) \end{aligned} \tag{3.15}$$

where

$$H(x) = g(x) \sup_{0\le t\le T} |u_t(t,x)|, \text{ and } M(x) = \sup_{0\le t\le T} |u(t,x)|$$

Since g is bounded and $u(\cdot, x)$, $u_t(\cdot, x) \in L^\infty([0,T])$ then H and M are bounded functions, as a consequence we get

$$P_\alpha(g,\, \partial_t)v(\cdot, x) \in L^\infty([0,T])$$

for all $x \in \mathbb{R}^N$.
Next by Sobolev's embedding theorem we have $P_\alpha(g,\, \partial_t)v \in L^\infty([0,T];\, H^s(\mathbb{R}^N))$ which implies the existence and uniqueness of such u in X_T.
In fact, by using Duhamel's principle we show that $\Phi(v)$ can be expressed in the form

$$\Phi(v) = u^{lin} + u^{nl} \tag{3.16}$$

where u^{lin} is solution of the corresponding linear Cauchy problem

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) = 0, \qquad t > 0, x \in \mathbb{R}^N \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N \\ u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N. \end{cases}$$

In particular u^{lin} is given by

$$u^{lin}(t,x) = E_0(t,0,x) *_x u_0 + E_1(t,0,x) *_x u_1, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n \tag{3.17}$$

where the notation $*_x$ stands for the convolution with respect to x, and E_0 and E_1 are the fundamental solutions to the (above) linear problem, that is the solutions corresponding to $(u_0, u_1) = (\delta,\ 0)$ and $(u_0, u_1) = (0,\ \delta)$ respectively where δ is the Dirac distribution, and u^{nl} is given by

$$u^{nl}(t, x) = \int_0^t w(t, s, x)ds, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \tag{3.18}$$

where $w(t, s, x)$ is solution to the corresponding parameter depending problem

$$\begin{cases} w_{tt}(t, s, x) - \Delta w(t, s, x) = 0, & t > s, x \in \mathbb{R}^N \\ w(s, s, x) = 0, \quad x \in \mathbb{R}^N \\ w_t(s, s, x) = P_\alpha(g, \partial_t)v(s, x), & x \in \mathbb{R}^N. \end{cases}$$

In particular $w(t, s, x)$ is given by

$$w(t, s, x) = E_1(t, s, x) *_x P_\alpha(g, \partial_t)v(s, x). \tag{3.19}$$

Finally, $\Phi(v)$ is defined by

$$\Phi(v) = E_0(t, 0, x) *_x u_0 + E_1(t, 0, x) *_x u_1 + \int_0^t E_1(t, s, x) *_x P_\alpha(g, \partial_t)v(s, x)ds. \tag{3.20}$$

□

The following proposition is one of consequences of Proposition 3.4.2.
Noting that, in fact, $E_{T,M}$ and $X_{T,M}$ are the centred balls of radius M in the spaces E_T and X_T respectively, then Proposition 3.5.3 shows us that the mapping Φ maps $E_{T,M}$ into $X_{T,M}$. In other words, Φ maps centred balls from E_T to centred ones in X_T, this result allows us to deduce using the fixed point theorem.

Proposition 3.5.3. *Let Φ be the mapping defined in Definition 3.5.1, then for all $v \in E_{T,M}$ one has $\Phi(v) = u \in X_{T,M}$.*

Proof. Assume that $v \in E_{T,M}$ with $M = M(u_0, u_1)$ is chosen in the following way

$$M = C_0\big(\|u_0\|_{H^s(\mathbb{R}^N)} + \|u_1\|_{H^{s-1}(\mathbb{R}^N)}\big),$$

for some constant $C_0 > 0$. Then, we have by Lemma 3.4.2 and the theory of linear wave equations (See E.g. [38])

$$\begin{aligned} \sup_{0\le t\le T} &\big(\|u(t, \cdot)\|_{H^s} + \|u_t(t, \cdot)\|_{H^{s-1}}\big) \\ &\le C(1 + T)\Big(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \int_0^t \|P_\alpha(g,\ \partial_t)v(\tau,\ \cdot)\|_{H^{s-1}}\, d\tau\Big) \\ &\le C(1 + T)(C_0^{-1}M + C_1M^m + C_2T^{\alpha+1}M^p) \\ &\le C(T, M)M, \end{aligned}$$

where

$$C(T, M) = C(1 + T)(C_0^{-1} + CM^{m-1} + CT^{\alpha+1}M^{p-1}).$$

since we can find $T_1 > 0$ such that

$$\textit{for all } T \in [0, T_1] : C(T, M) \leq 1,$$

we deduce that

$$\sup_{0 \leq t \leq T} \left(\|u(t, \cdot)\|_{H^s(\mathbb{R}^N)} + \|u_t(t, \cdot)\|_{H^{s-1}(\mathbb{R}^N)} \right) \leq M.$$

□

Proposition 3.5.4. *Let $T > 0$ and Φ be the mapping defined in Definition 3.5.1, then Φ is a contraction from $X_{T,M}$ into $X_{T,M}$.*

Proof. Since the function $(x, y) \mapsto |x|^{m-1} y$ is not Lipschitz continuous with respect to $(x, y) \in \mathbb{R}^2$ for $1 < m < 2$ then, we can not apply directly the mean value theorem. To overcome this obstacle, we will use the linearity and we modify the technique as it has been done by Lions and Strauss [17], Berbiche and Hakem [22], Katayam [36] and MD. Abu Naim [21].
Let v_1, $v_2 \in E_{T,M}$ such that $v_1(0, x) = v_2(0, x) = u_0(x), x \in \mathbb{R}^N$ and let w, ν be solutions for the following problems respectively

$$(P.1.1) \quad \begin{cases} w_{tt}(t, x) - \Delta w(t, x) = \int_0^t (t - s)^{\alpha-1} |v_1(s, x)|^p \, ds, \quad t > 0, x \in \mathbb{R}^n \\ w(0, x) = u_0(x), \quad x \in \mathbb{R}^n \\ w_t(0, x) = u_1(x) + \frac{g(x)}{m} |u_0|^{m-1} u_0, \quad x \in \mathbb{R}^n \end{cases}$$

and

$$(P.1.2) \quad \begin{cases} \nu_{tt}(t, x) - \Delta \nu(t, x) = \int_0^t (t - s)^{\alpha-1} |v_2(s, x)|^p \, ds, \quad t > 0, x \in \mathbb{R}^N \\ \nu(0, x) = u_0(x), \quad x \in \mathbb{R}^N \\ \nu_t(0, x) = u_1(x) + \frac{g(x)}{m} |u_0|^{m-1} u_0, \quad x \in \mathbb{R}^N. \end{cases}$$

Let also $\tilde{w}$ and $\tilde{\nu}$ be solutions for the following problems respectively

$$(P.2.1) \quad \begin{cases} \tilde{w}_{tt}(t, x) - \Delta \tilde{w}(t, x) = -\frac{g(x)}{m} |v_1(t, x)|^{m-1} v_1(t, x), \quad t > 0, x \in \mathbb{R}^n \\ \tilde{w}(0, x) = 0, \quad x \in \mathbb{R}^N \\ \tilde{w}_t(0, x) = 0, \quad x \in \mathbb{R}^N \end{cases}$$

and

$$(P.2.2) \quad \begin{cases} \tilde{\nu}_{tt}(t, x) - \Delta \tilde{\nu}(t, x) = -\frac{g(x)}{m} |v_2(t, x)|^{m-1} v_2(t, x), \quad t > 0, x \in \mathbb{R}^n \\ \tilde{\nu}(0, x) = 0, \quad x \in \mathbb{R}^N \\ \tilde{\nu}_t(0, x) = 0, \quad x \in \mathbb{R}^N. \end{cases}$$

Remark 3.5.5. *The facts that g is bounded with all its derivatives and $v_i \in E_{T,M}$ imply that for $i = 1, 2$ we have*

$$\int_0^t (t - s)^{\alpha-1} |v_i|^p ds; \ g|v_i|^{m-1} v_i; \ g|v_i|^{m-1} \partial_t v_i \in L^\infty\big([0, T]; \, H^{s-1}(\mathbb{R}^N)\big),$$

consequently, by Sobolev embedding theorem, we deduce that $w, \nu \in E_{T,M}$ and

$$\tilde{w}, \tilde{\nu} \in \mathscr{C}([0, T]; \, H^{s+1}(\mathbb{R}^N)) \cap \mathscr{C}^1([0, T]; \, H^s(\mathbb{R}^N)) \cap \mathscr{C}^2([0, T]; \, H^{s-1}(\mathbb{R}^N))$$

Then we have the following results:

Proposition 3.5.6. *Denoting* $\bar{w} := w + \tilde{w}_t$*, then* $\bar{w}$ *is solution for the following Cauchy problem*

$$(\bar{P}_1) \quad \begin{cases} \bar{w}_{tt}(t,x) - \Delta\bar{w}(t,x) = P_\alpha(g,\partial_t)v_1(t,x), \quad t>0, x\in\mathbb{R}^N \\ \bar{w}(0,x) = u_0(x), \quad x\in\mathbb{R}^N \\ \bar{w}_t(0,x) = u_1(x), \quad x\in\mathbb{R}^N. \end{cases}$$

Proof. One can remark that for all $v_1 \in E_T$ and $t\in[0,T]$ we have

$$\partial_t(|v_1(t,\,\cdot)|^{m-1}v_1(t,\,\cdot)) = m|v_1(t,\,\cdot)|^{m-1}\partial_t v_1(t,\,\cdot),$$

hence, this drives us to arrive at

$$\begin{aligned} \bar{w}_{tt} - \Delta\bar{w} &= \partial_t^2(w+\tilde{w}_t) - \Delta(w+\tilde{w}_t) \\ &= w_{tt}(t,x) - \Delta w(t,x) - \partial_t(\tilde{w}_{tt}(t,x) - \Delta\tilde{w}(t,x)) \\ &= \int_0^t (t-s)^{\alpha-1}|v_1(t,x)|^p ds - g(x)|v_1(t,x)|^{m-1}v_1(t,x) \\ &= P_\alpha(g,\partial_t)v_1(t,x). \end{aligned}$$

In the other hand, it is easy to show that the initial conditions are satisfied too as follows: For the first condition we have

$$\bar{w}(0,x) = w(0,x) + \tilde{w}_t(0,x) = u_0(x) \quad \textit{for all } x\in\mathbb{R}^N$$

For the second, we have

$$\tilde{w}_{tt}(t,x) - \Delta\tilde{w}(t,x) = -\frac{g(x)}{m}|v_1(t,x)|^{m-1}v_1(t,x).$$

Then for $t=0$ we get

$$\tilde{w}_{tt}(0,x) - \Delta\tilde{w}(0,x) = -\frac{g(x)}{m}|v_1(0,x)|^{m-1}v_1(0,x). \tag{3.21}$$

Since $\Delta\tilde{w}(0,x)=0$ because $\tilde{w}(0,x)=0$, and $v_1(0,x)=u_0(x)$ we get from (3.21)

$$\tilde{w}_{tt}(0,x) = -\frac{g(x)}{m}|u_0(x)|^{m-1}u_0(x). \tag{3.22}$$

Finally, by using formula (3.22) we get

$$\begin{aligned} \bar{w}_t(0,x) &= w_t(0,x) + \tilde{w}_{tt}(0,x) \\ &= u_1(x) + \frac{g(x)}{m}|u_0(x)|^{m-1}u_0(x) - \frac{g(x)}{m}|u_0(x)|^{m-1}u_0(x) \\ &= u_1(x) \end{aligned}$$

This completes the proof of Proposition 3.5.6. □

Proposition 3.5.7. *Denoting $\bar{\nu} := \nu + \tilde{\nu}_t$, then $\bar{\nu}$ is solution for the following problem*

$$(\bar{P}_2) \quad \begin{cases} \bar{\nu}_{tt}(t,x) - \Delta\bar{\nu}(t,x) = P_\alpha(g,\partial_t)v_2(t,x), \quad t>0, x\in\mathbb{R}^N \\ \bar{\nu}(0,x) = u_0(x), \quad x\in\mathbb{R}^n \\ \bar{\nu}_t(0,x) = u_1(x), \quad x\in\mathbb{R}^n. \end{cases}$$

Proof. The proof is similar to the proof of Proposition(3.5.6), so omit it. □

Although the following corollary is very simple since it is an immediate consequence of Propositions 3.5.6, and 3.5.7 and the definition of the mapping Φ, it is useful and very interesting.

Corollary 3.5.8. *One has $\Phi(v_1) = w + \tilde{w}_t$ and $\Phi(v_2) = \nu + \tilde{\nu}_t$*

Proof. An immediate consequence of definition of Φ, Propositions 3.5.6, and 3.5.7 and the uniqueness of solution to linear Cauchy problem for wave equations. □

In order to prove that Φ is a contraction mapping into $X_{T,M}$, we also need to the following:

Proposition 3.5.9. *Denoting $w^* := w - \nu$ then there exists a constant $C>0$ such that, for all $t\in[0,\,T]$:*

$$\|w^*(t,\,\cdot)\|_{H^s} + \|w_t^*(t,\,\cdot)\|_{H^{s-1}} \le C(1+T)T^{\alpha+1}M^{p-1}\sup_{0\le t\le T}\|v_1(t,\,\cdot) - v_2(t,\,\cdot)\|_{H^s}$$

Proof. First of all, we show that w^* is a solution for the following homogeneous Cauchy problem:

$$\begin{cases} \partial_t^2 w^* - \Delta w^* = \int_0^t (t-s)^{-\gamma}(|v_1(s,\,\cdot)|^p - |v_2(s,\,\cdot)|^p)\,ds \\ w^*(0,x) = w_t^*(0,x) = 0, \quad x\in\mathbb{R}^N \end{cases}.$$

To do so, it is enough to note, at first, that

$$\begin{aligned} \partial_t^2 w^* - \Delta w^* &= w_{tt} - \Delta w - (\nu_{tt} - \Delta\nu) \\ &= \int_0^t (t-s)^{-\gamma}|v_1(s,\,\cdot)|^p\,ds - \int_0^t (t-s)^{-\gamma}|v_2(s,\cdot)|^p\,ds \\ &= \int_0^t (t-s)^{-\gamma}\big(|v_1(s,\,\cdot)|^p - |v_2(s,\cdot)|^p\big)\,ds, \end{aligned}$$

Secondly, for all $x\in\mathbb{R}^N$: we have

$$w^*(0,x) = w(0,x) - \nu(0,x) = u_0(x) - u_0(x) = 0.$$

and

$$\begin{aligned} w_t^*(0,x) &= w_t(0,x) - \nu_t(0,x) \\ &= u_1(x) + \frac{g(x)}{m}|u_0|^{m-1}u_0 - u_1(x) + \frac{g(x)}{m}|u_0|^{m-1}u_0 = 0. \end{aligned}$$

Then, by using the theory of linear wave equations we get, for all $t\in[0,T]$:

$$\|w^*(t,\cdot)\|_{H^s} + \|w_t^*(t,\cdot)\|_{H^{s-1}} \le C(1+T)\int_0^t\int_0^\tau (\tau-\nu)^{-\gamma}\||v_1(\nu,\cdot)|^p - |v_2(\nu,\cdot)|^p\|_{H^{s-1}}\,d\nu d\tau.$$

The application of the theorem of change of order of integration (2.3.6), allows us to calculate the integral with respect to τ. This leads to

$$\begin{aligned}&\|w^*(t,\cdot)\|_{H^s}+\|w_t^*(t,\cdot)\|_{H^{s-1}}\\&\quad\le C(1+T)\int_0^t\Big(\int_s^t(\tau-s)^{-\gamma}\,d\tau\Big)\||v_1(s,\cdot)|^p-|v_2(s,\cdot)|^p\|_{H^{s-1}}\,ds\\&\quad\le C(1-\gamma)^{-1}(1+T)\int_0^t(t-\tau)^{1-\gamma}\||v_1(\tau,\cdot)|^p-|v_2(\tau,\cdot)|^p\|_{H^{s-1}}\,d\tau.\end{aligned}$$

Taking into consideration the estimate

$$\big||u|^p-|v|^p\big|\le C|u-v|\big(|u|^{p-1}+|v|^{p-1}\big)\text{ for some }C>0,$$

we have after application of Hölder's inequality the estimates

$$\begin{aligned}\||u|^p-|v|^p\|_{L^2}&\le C\||u-v|(|u|^{p-1}+|v|^{p-1})\|_{L^2}\\&\le C\|u-v\|_{L^{2p}}\big(\||u|^{p-1}+|v|^{p-1}\|_{L^{2p}}\big)\\&\le C\|u-v\|_{L^{2p}}\left(\|u\|^{p-1}_{L^{2p}}+\|v\|^{p-1}_{L^{2p}}\right),\end{aligned}\tag{3.23}$$

for some constant $C>0$.
So, by using the estimate (3.23), we find, for all $s>1$, the following estimate

$$\begin{aligned}\||u|^p-|v|^p\|_{H^{s-1}}&\le C\||u-v|(|u|^{p-1}+|v|^{p-1})\|_{H^{s-1}}\\&\le C\|u-v\|_{H^{s-1}}\left(\||u|^{p-1}+|v|^{p-1}\|_{H^{s-1}}\right)\\&\le C\|u-v\|_{H^{s-1}}\left(\|u\|^{p-1}_{H^{s-1}}+\|v\|^{p-1}_{H^{s-1}}\right)\end{aligned}\tag{3.24}$$

Then, by using Sobolev's embedding theorem we obtain

$$\begin{aligned}\|w^*(t,\cdot)\|_{H^s}+\|w_t^*(t,\cdot)\|_{H^{s-1}}&\le C(1-\gamma)^{-1}(1+T)\sup_{0\le t\le T}(\|v_1(t,\cdot)\|^{p-1}_{H^s}+\|v_2(t,\cdot)\|^{p-1}_{H^s})\\&\quad\times\sup_{0\le t\le T}\|v_1(t,\cdot)-v_2(t,\cdot)\|_{H^s}\sup_{0\le t\le T}\int_0^t(t-s)^{1-\gamma}ds\\&\le C(1+T)T^{2-\gamma}M^{p-1}\sup_{0\le t\le T}\|v_1(t,\cdot)-v_2(t,\cdot)\|_{H^s}.\end{aligned}$$

It remains, only, to replace $\alpha=1-\gamma$. This ends the proof. □

Proposition 3.5.10. *Denoting $\tilde{w}^*:=\tilde{w}-\tilde{\nu}$ then there exists a constant $C>0$ such that, for all $t\in[0,T]$ we have*

$$\|\tilde{w}^*(t,\cdot)\|_{H^s}+\|\tilde{w}^*_t(t,\cdot)\|_{H^{s-1}}\le C(1+T)TM^{m-1}\sup_{0\le t\le T}\|v_1(t,\cdot)-v_2(t,\cdot)\|_{H^s}.$$

Proof. The proof is similar to the proof of the previous proposition. So, we begin by showing that $\tilde{w}^*$ is solution to the following homogeneous Cauchy problem

$$\begin{cases}\partial_t^2\tilde{w}^*-\Delta\tilde{w}^*=-\frac{g}{m}(|v_1(t,\cdot)|^{m-1}v_1-|v_2(t,\cdot)|^{m-1}v_2)\\\tilde{w}^*(0,x)=\tilde{w}^*_t(0,x)=0,\quad x\in\mathbb{R}^N.\end{cases}$$

Hence, by the theory of linear wave equations we find

$$\|\tilde{w}^*(t,\cdot)\|_{H^s} + \|\tilde{w}_t^*(t,\cdot)\|_{H^{s-1}} \leq C(1+T)\int_0^t \left\| \frac{g}{m}|v_1(s,\cdot)|^{m-1}v_1 - \frac{g}{m}|v_2(s,\cdot)|^{m-1}v_2 \right\|_{H^{s-1}} ds$$

Since g is assumed to be bounded with all its derivatives up to the $[s]^{th}$ one, then we arrive by Lemma 2.3.5 at

$$\|\tilde{w}^*(t,\cdot)\|_{H^s} + \|\tilde{w}_t^*(t,\cdot)\|_{H^{s-1}} \leq C(1+T)\int_0^t \left\| |v_1(s,\cdot)|^{m-1}v_1 - |v_2(s,\cdot)|^{m-1}v_2 \right\|_{H^{s-1}} ds.$$

By using of Hölder's inequality (3.24) we get

$$\begin{aligned}
&\|\tilde{w}^*(t,\cdot)\|_{H^s} + \|\tilde{w}_t^*(t,\cdot)\|_{H^{s-1}} \\
&\quad \leq C(1+T)\int_0^t (\|v_1(s,\cdot)\|_{H^s}^{m-1} + \|v_2(s,\cdot)\|_{H^s}^{m-1})\|v_1(s,\cdot) - v_2(s,\cdot)\|_{H^s} ds \\
&\quad \leq C(1+T)\sup_{0\leq t\leq T}(\|v_1(t,\cdot)\|_{H^s}^{m-1} + \|v_2(t,\cdot)\|_{H^s}^{m-1}) \\
&\qquad \times \sup_{0\leq t\leq T}\|v_1(t,\cdot) - v_2(t,\cdot)\|_{H^s}\sup_{0\leq t\leq T}\int_0^t ds \\
&\quad \leq C(1+T)TM^{m-1}\sup_{0\leq t\leq T}\|v_1(t,\cdot) - v_2(t,\cdot)\|_{H^s}.
\end{aligned}$$

□

Remind ourselves that the aim is to show that Φ is a contraction mapping in $X_{T,M}$. For this reason, we show that, there exists some $k \in]0,1[$ such that

$$\sup_{0\leq t\leq T}\|\Phi(v_1) - \Phi(v_2)\|_{H^s} \leq k\sup_{0\leq t\leq T}\|v_1(t,\cdot) - v_2(t,\cdot)\|_{H^s}, \tag{3.25}$$

for all $v_1,\, v_2 \in E_{T,M}$.
So, let $v_1,\, v_2 \in E_{T,M}$. We have

$$\begin{aligned}
&\|\Phi(v_1)(t,\cdot) - \Phi(v_2)(t,\cdot)\|_{H^s} = \|w(t,\cdot) + \tilde{w}_t(t,\cdot) - \nu(t,\cdot) - \tilde{\nu}_t(t,\cdot)\|_{H^s} \\
&\quad \leq \|w - \nu\|_{H^s} + \|\tilde{w}_t - \tilde{\nu}_t\|_{H^s} \\
&\quad \leq \|w^*(t,\cdot)\|_{H^s} + \|w_t^*(t,\cdot)\|_{H^{s-1}} + \|\tilde{w}^*(t,\cdot)\|_{H^s} + \|\tilde{w}_t^*(t,\cdot)\|_{H^{s-1}} \\
&\quad \leq C(1+T)T^{\alpha+1}M^{p-1}\sup_{0\leq t\leq T}\|v_1(t,\cdot) - v_2(t,\cdot)\|_{H^s} \\
&\qquad + C(1+T)TM^{m-1}\sup_{0\leq t\leq T}\|v_1(t,\cdot) - v_2(t,\cdot)\|_{H^s} \\
&\quad \leq C(1+T)\big(T^{\alpha+1}M^{p-1} + TM^{m-1}\big)\sup_{0\leq t\leq T}\|v_1(t,\cdot) - v_2(t,\cdot)\|_{H^s}.
\end{aligned} \tag{3.26}$$

As the inequality (3.26) holds for all $t \in [0,T]$ then we deduce that

$$\begin{aligned}
&\sup_{0\leq t\leq T}\|\Phi(v_1)(t,\cdot) - \Phi(v_2)(t,\cdot)\|_{H^s} \\
&\quad \leq C(1+T)\big(T^{\alpha+1}M^{p-1} + TM^{m-1}\big)\sup_{0\leq t\leq T}\|v_1(t,\cdot) - v_2(t,\cdot)\|_{H^s}
\end{aligned} \tag{3.27}$$

Since it is possible to find $T_1 > 0$ such that

$$C(1+T)\big(T^{\alpha+1}M^{p-1} + TM^{m-1}\big) < 1, \, \forall T \in [0, T_1]$$

we deduce from (3.27) that

$$\sup_{0\le t\le T} \|\Phi(v_1)(t,\cdot) - \Phi(v_2)(t,\cdot)\|_{H^s} \le k \sup_{0\le t\le T} \|v_1(t,\cdot) - v_2(t,\cdot)\|_{H^s}, \tag{3.28}$$

for some $k \in]0,1[$. Using the fact that $X_{T,M} \subset E_{T,M}$ (result in Remark 3.3.1) and Proposition 3.5.3 we show that

$$\Phi(X_{T,M}) \subset X_{T,M}.$$

By this step, the proof of Proposition 3.5.4 is achieved. □

3.6 Fixed point and existence of solution

In this section, we turn ourselves to sequence language and some related notions.
Let $T > 0$. Define the recurrent functional sequence $(u^{(n)})_n$ on $[0,T] \times \mathbb{R}^N$ as follows:

$$\begin{cases} u^{(0)}(t,x) := u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \\ u^{(n)}(t,x) = \Phi(u^{(n-1)})(t,x), \;\; n \ge 1, \;\; 0 < t \le T, \;\; x \in \mathbb{R}^N. \end{cases} \tag{3.29}$$

Then, $(u^{(n)})_n$ satisfies at the following propriety:

Proposition 3.6.1. *Let $(u^{(n)})_n$ be the sequence defined by (3.29), then there exists some $\bar{u} \in \mathscr{C}\big([0,T];\, H^s\big)$ such that $u^{(n)}$ converges to $\bar{u}$ in $\mathscr{C}\big([0,T];\, H^s\big)$ as $n \to \infty$ with respect to the topology of $L^\infty\big([0,T];\, H^s\big)$.*

Proof. First, let us proof that $u^{(n)}$ is a Cauchy functional sequence.
For all $n \in \mathbb{N}$ and $t \in [0,T]$ we have

$$\begin{aligned} \|u^{(n+1)}(t,\cdot) - u^{(n)}(t,\cdot)\|_{H^s} &= \|\Phi(u^{(n+1)})(t,\cdot) - \Phi(u^{(n)})(t,\cdot)\|_{H^s} \\ &\le k\|u^{(n)}(t,\cdot) - u^{(n-1)}(t,\cdot)\|_{H^s}. \end{aligned} \tag{3.30}$$

We iterate inequality (3.30) n times we find

$$\|u^{(n+1)}(t,\cdot) - u^{(n)}(t,\cdot)\|_{H^s} \le \; k^n\|u^{(1)}(t,\cdot) - u^{(0)}\|_{H^s}. \tag{3.31}$$

Next, for all $p \in \mathbb{N}$ we get after using (3.31)

$$\begin{aligned} \|u^{(n+p)}(t,\cdot) - u^{(n)}(t,\cdot)\|_{H^s} &\le \|u^{(n+p)}(t,\cdot) - u^{(n+p-1)}(t,\cdot)\|_{H^s} + \cdots + \|u^{(n+1)}(t,\cdot) - u^{(n)}(t,\cdot)\|_{H^s} \\ &\le (k^{n+p-1} + \cdots + k^n)\|u^{(1)}(t,\cdot) - u^{(0)}\|_{H^s} \\ &\le \frac{k^n(1-k^p)}{1-k}\|u^{(1)}(t,\cdot) - u^{(0)}\|_{H^s} \\ &\le \frac{k^n}{1-k}\|u^{(1)}(t,\cdot) - u^{(0)}\|_{H^s}, \quad \text{since } \; k < 1. \end{aligned} \tag{3.32}$$

The inequality (3.32) holds for all $t \in [0,T]$ therefore, we get from (3.32)

$$\sup_{t\in[0,T]} \|u^{(n+p)}(t,\cdot) - u^{(n)}(t,\cdot)\|_{H^s} \leq \frac{k^n}{1-k} \sup_{t\in[0,T]} \|u^{(1)}(t,\cdot) - u^{(0)}\|_{H^s} \tag{3.33}$$

Since the right hand side of (3.33) is independent of p and goes to 0 as $n \to +\infty$ then $(u^{(n)})_n$ is a Cauchy sequence in $\mathscr{C}\big([0,T];\, H^s\big)$.
Finally, since the space $\mathscr{C}\big([0,T];\, H^s\big)$ is a complete (and then closed) subspace of $L^\infty\big([0,T];\, H^s\big)$ then $(u^{(n)})_n$ converges to some $\bar{u}$ in $\mathscr{C}\big([0,T];\, H^s\big)$. □

The aim, now, is to show that this $\bar{u}$ belongs to X_T and is solution to Problem (3.1)-(3.2). This is the aim of the following proposition.

Proposition 3.6.2. *Let $(u^{(n)})_n$ be the sequence defined by (3.29) and let $\bar{u}$ be its limit in $\mathscr{C}\big([0,T];\, H^s\big)$. Then $\bar{u}$ belongs to X_T and satisfies at*

$$\Phi(\bar{u}) = \bar{u}.$$

In particular $\bar{u}$ is solution to Problem (3.1)-(3.2).

Proof. Since $u^{(n)} \in X_{T,M}$, then $\big(u^{(n)}\big)_n$ and $\big(u_t^{(n)}\big)_n$ has a weak convergent subsequence $\big(u^{(n_k)}\big)_k$ and $(u_t^{(n_k)})_k)$ in $L^\infty\big([0,T];\, H^s\big)$ and $L^\infty\big([0,T];\, H^{s-1}\big))$ respectively. By Proposition 3.6.1, $(u^{(n)})_n$ converges to $\bar{u}$ in $\mathscr{C}\big([0,T];\, H^s\big)$, then, the above subsequence converge weakly to $\bar{u}$ and $\bar{u}_t$ in $L^\infty\big([0,T];\, H^s)$ and $L^\infty\big([0,T];\, H^{s-1}\big))$ respectively, as a consequence, we see that $\bar{u} \in L^\infty\big([0,T]; H^s\big)$ and $\bar{u}_t \in L^\infty\big([0,T]; H^{s-1}\big)$. This means that $\bar{u} \in E_{T,M}$. Then, Proposition 3.5.3 shows us that

$$\Phi(\bar{u}) \in X_{T,M}$$

immediately. Next, applying (3.28) we get

$$\sup_{0\leq t\leq T} \|\Phi(u^{(n)}) - \Phi(\bar{u})\|_{H^s} \leq k \sup_{0\leq t\leq T} \|u^{(n)}(t,\cdot) - \bar{u}(t,\cdot)\|_{H^s}, \quad k \in]0,1[\tag{3.34}$$

Since the right hand side of (3.34) goes to 0 as $n \to +\infty$, then $\Phi(u^{(n)})$ converges to $\Phi(\bar{u})$ in $\mathscr{C}\big([0,T];\, H^s\big)$. Passing to the limit in $u^{(n)} = \Phi(u^{(n-1)})$ as $n \to +\infty$, we get after using the fact that $u_n^{(n)} \to \bar{u}$ in $\mathscr{C}\big([0,T];\, H^s\big)$

$$\Phi(\bar{u}) = \bar{u} \in X_{T,M}$$

Finally, by definition of Φ, this $\bar{u}$ is apparently, the desired solution of Problem (3.1)-(3.2). □

3.7 Uniqueness of the solution

The uniqueness of a such solution in $X_{T,M}$ follows immediately from formula (3.28). Assume that Problem (3.1)-(3.2) admits two solutions u_1 and u_2. Then one has

$$\Phi(u_1) = u_1 \text{ and } \Phi(u_2) = u_2. \tag{3.35}$$

So, by using formula (3.28) we get

$$\sup_{0\le t\le T} \|\Phi(u_1)(t,\cdot) - \Phi(u_2)(t,\cdot)\|_{H^s} \le k \sup_{0\le t\le T} \|u_1(t,\cdot) - u_2(t,\cdot)\|_{H^s}. \tag{3.36}$$

Thanks to (3.35), we get from (3.36) the following inequality

$$(1-k) \sup_{0\le t\le T} \|u_1(t,\cdot) - u_2(t,\cdot)\|_{H^s} \le 0. \tag{3.37}$$

Using the fact that $0 < k < 1$ we get from (3.37)

$$\sup_{0\le t\le T} \|u_1(t,\cdot) - u_2(t,\cdot)\|_{H^s} = 0. \tag{3.38}$$

This implies that

$$u_1(t,x) - u_2(t,x) = 0, \quad \textit{for all } t > 0 \quad \textit{and } x \in \mathbb{R}^N.$$

that is

$$u_1 = u_2.$$

This achieves the proof of Theorem 3.2.1. □

Chapter 4

Blow-up results for Problem (3.1)-(3.2)

This chapter is devoted to the Blow-up results for Problem (3.1)-(3.2). The method which we will use to prove our results is the test function method considered by Pohozaev and Tesei [35], Mitidieri and Pohozaev ([11]-[10]), Fino [2], Berbiche and Hakem [22] and by Zhang [33].
Important to note that the non-linearity of the damping term in Problem (3.1)-(3.2) turned the situation more delicate! In particular, we can not find explicitly the upper bound of the parameter p and then we don't manage to determine the critical exponent of Fujita of Problem (3.1)-(3.2). As a consequence, the conditions obtained in Theorem 4.1.2 on the parameters of the considered equation (3.1) are not sharp!

4.1 The main result

Let us introduce what do we mean by *weak solution* to Problem (3.1)-(3.2). Noting that the right hand side of equation (3.1) is, in some sense, the fractional integral of order $1-\gamma$ of $|u|^p$, this is helpful! So, we have the following definition:

Definition 4.1.1 (Weak solution).
Let $u_0 \in L^1_{loc}(\mathbb{R}^N) \cap L^m_{loc}(\mathbb{R}^N)$, $u_1 \in L^1_{loc}(\mathbb{R}^N)$ and $T > 0$. A weak solution of problem (3.1)-(3.2) is a locally integrable function

$$u \in L^p\big([0,T];\, L^p_{loc}(\mathbb{R}^N)\big) \cap L^m\big((0,T);\, L^m_{loc}(\mathbb{R}^N)\big)$$

satisfies the following formula:

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)\varphi(t,x)dtdx + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)dx - \int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)dx \\
&+\frac{1}{m}\int_{\mathbb{R}^n} g(x)|u_0|^{m-1}(x)u_0(x)\varphi(0,x)dx = \int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_{tt}(t,x)dtdx \\
&-\frac{1}{m}\int_0^T\int_{\mathbb{R}^n} g(x)(|u|^{m-1}u)(t,x)\varphi_t(t,x)dtdx - \int_0^T\int_{\mathbb{R}^n} u(t,x)\Delta\varphi(t,x)dtdx \qquad (4.1)
\end{aligned}$$

for all non-negative test function $\varphi \in \mathscr{C}^2\big([0,T]\times\mathbb{R}^N\big)$ *such that* $\varphi_t(T,\cdot)=\varphi(T,\cdot)=0$ *and* $\alpha = 1-\gamma$.

The main result of this chapter is the following theorem:

Theorem 4.1.2. *Let* $0<\gamma<1$, $p,m\in\mathbb{R}$ *such that* $p>m>1$ *and* g *be a positive and bounded smooth function. Assume that*

$$\int_{\mathbb{R}^n} u_i(x)dx > 0, \quad i=0,1. \tag{4.2}$$

Then, if one of the following conditions:

i) $N \le \min\Big(\frac{2(1-\gamma)p+2m}{(1-\gamma)(m-1)+(p-1)}, -\frac{2p(2-\gamma)+2}{(\frac{(2-\gamma)(p-1)}{m-p}+(1-\gamma))(p-1)}\Big)$,

ii) $p<\frac{1}{\gamma}$ *or* $p=\frac{1}{\gamma}$ *and moreover* $\frac{N-2}{N}<\gamma<1$ *if* $N\ge 3$.

is fulfilled ,then the solution of the Cauchy problem (3.1)-(3.2) does not exist globally in time.

Proof of Theorem 4.1.2. The Theorem (4.1.2) will be demonstrated by absurd. So we suppose that u is a global (in time) non trivial weak solution to Problem (3.1)-(3.2). The aim is to show that this assumption leads to a contradiction. For this reason we need also to some results and tools, we give them in the following section.

4.2 Preliminary results

Since the principle of the method is the right choice of the test function, let us choose a test function φ as follows:

$$\varphi(t,x) = D^\alpha_{t|T}\psi(t,x) = \varphi_1^r(x)D^\alpha_{t|T}\varphi_2(t), \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^N. \tag{4.3}$$

where $r>1$, $\alpha\in]0,1[$ (In fact $\alpha=1-\gamma$) and $D^\alpha_{t|T}$ is the right fractional derivative operator of Riemann-Liouville defined by (4.4) below (See [37]).

$$D^\alpha_{t|T}v(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{v(s)}{(s-t)^\alpha}\,ds. \tag{4.4}$$

For the sake of brevity, we have used here the notation $D^\alpha_{t|T}v(t)$ instead of ${}_{RL}D^\alpha_{t|T}v(t)$ since there is no possibility of confusion.
The functions φ_1 and φ_2 are defined, respectively, by

$$\varphi_1(x) = \phi\big(\frac{x^2}{T^\theta}\big) \quad \text{and } \varphi_2(t) = \big(1-\frac{t}{T}\big)_+^\beta, \tag{4.5}$$

with $\beta>1$, θ is a nonegative parameter which will be chosen suitably later and ϕ is a cut-off non increasing function satisfying

$$\phi(s) = \begin{cases} 1 & \text{if } 0\le s\le 1 \\ 0 & \text{if } s\ge 2 \end{cases}, \quad 0\le\phi\le 1 \text{ everywhere}. \tag{4.6}$$

As tools, we will use the fractional version of integration by parts (See [37]):

$$\int_0^t f(t)D^{\alpha}_{t|T}g(t)dt = \int_0^t \big(D^{\alpha}_{0|t}f(t)\big)g(t)dt \tag{4.7}$$

for all $f, g \in \mathscr{C}([0,T])$ such that $D^{\alpha}_{0|t}f(t)$ and $D^{\alpha}_{t|T}g(t)$ exist and are continuous, as well as the following identity:

$$\big(D^{\alpha}_{0|t} \circ I^{\alpha}_{0|t}\big)(u) = u, \tag{4.8}$$

which holds for all $u \in L^q([0,T])$. The bellow identity (see [37])

$$(-1)^n \partial_t^n D^{\alpha}_{t|T}u(t) = D^{\alpha+n}_{t|T}u(t), n \in \mathbb{N}, \alpha \in]0,1[\tag{4.9}$$

which happens for all $u \in \mathscr{C}^n[0,T]$; $T > 0$, where ∂_t^n is the $n-times$ ordinary derivative operator with respect to t, will be useful to prove Theorem 4.1.2.
In order to estimate some integrals which appear in the weak formulation (4.1), we need to some results about the introduced function φ_2 via the test function φ. Let us state these results in the following lemma:

Lemma 4.2.1. *Given $\beta > 1$, let φ_2 be the function defined by*

$$\varphi_2(t) = \Big(1 - \frac{t}{T}\Big)_+^{\beta}.$$

Then, for all $\alpha \in]0,1[$ we have

$$\begin{aligned} D^{\alpha}_{t|T}\varphi_2(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\beta}(T-t)_+^{\beta-\alpha} \\ &= C_1 T^{-\alpha}\Big(1-\frac{t}{T}\Big)_+^{\beta-\alpha}. \end{aligned}$$

$$\begin{aligned} D^{\alpha+1}_{t|T}\varphi_2(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)}T^{-\beta}(T-t)_+^{\beta-\alpha-1} \\ &= C_2 T^{-\alpha-1}\Big(1-\frac{t}{T}\Big)_+^{\beta-\alpha-1}. \end{aligned}$$

$$\begin{aligned} D^{\alpha+2}_{t|T}\varphi_2(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha-1)}T^{-\beta}(T-t)_+^{\beta-\alpha-2} \\ &= C_3 T^{-\alpha-2}\left(1-\frac{t}{T}\right)_+^{\beta-\alpha-2}. \end{aligned}$$

where

$$C_1 := \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}, \quad C_2 = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)}, \ \textit{and} \ \ C_3 = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha-1)}.$$

Proof. Let $\beta > 1$ and

$$\varphi_2(t) = \Big(1 - \frac{t}{T}\Big)_+^{\beta}.$$

1. For the first identity we have by using the definition (4.4)

$$\begin{aligned} D^{\alpha}_{t|T}\varphi_2(t) &= -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{\varphi_2(s)}{(s-t)^{\alpha}}ds \\ &= -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{(1-\frac{s}{T})^{\beta}}{(s-t)^{\alpha}}ds. \end{aligned}$$

Using Euler's change of variable

$$s \mapsto y = \frac{s-t}{T-t},$$

we show that

$$s = t + (T-t)y \quad \text{and} \quad ds = (T-t)dy.$$

Then, it yields

$$\begin{aligned} D^{\alpha}_{t|T}\varphi_2(t) &= -\frac{T^{-\beta}}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\Big((T-t)^{\beta-\alpha+1}\int_0^1 y^{-\alpha}(1-y)^{\beta}dy\Big) \\ &= \frac{(\beta-\alpha+1)\mathscr{B}(1-\alpha,\beta+1)}{\Gamma(1-\alpha)}T^{-\beta}(T-t)^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha}(1-\frac{t}{T})^{\beta-\alpha}, \end{aligned}$$

where $\mathscr{B}$ is the famous *Beta function* defined by

$$\mathscr{B}(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}\,dt,$$

and satisfies in particular at (See [16] formula 1.24 page 7 for the proof)

$$\mathscr{B}(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},\ u,\, v > 0$$

which we have used here above. In particular we have used the fact that

$$\mathscr{B}(1-\alpha,\beta+1) = \frac{\Gamma(1-\alpha)\Gamma(\beta+1)}{\Gamma(\beta-\alpha+2)}.$$

2. In order to show the second identity of Lemma 4.2.1, we apply directly formula (4.9). In this way, we get for all $t \in [0,T]$

$$\begin{aligned} D^{\alpha+1}_{t|T}\varphi_2(t) &= -\partial_t D^{\alpha}_{t|T}\varphi_2(t) \\ &= -\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha}\partial_t(1-\frac{t}{T})^{\beta-\alpha} \\ &= \frac{(\beta-\alpha)\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha-1}(1-\frac{t}{T})^{\beta-\alpha-1} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)}T^{-\alpha-1}(1-\frac{t}{T})^{\beta-\alpha-1}. \end{aligned} \tag{4.10}$$

3. Finally, For $D^{\alpha+2}_{t|T}\varphi_2(t)$ we have

$$\begin{aligned} D^{\alpha+2}_{t|T}\varphi_2(t) &= \partial_t^2 D^{\alpha}_{t|T}\varphi_2(t) \\ &= \partial_t(\partial_t D^{\alpha}_{t|T}\varphi_2(t)) \\ &= -\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)}T^{-\alpha-1}\partial_t(1-\frac{t}{T})^{\beta-\alpha-1} \\ &= \frac{(\beta-\alpha-1)\Gamma(\beta+1)}{\Gamma(\beta-\alpha)}T^{-\alpha-2}(1-\frac{t}{T})^{\beta-\alpha-2} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha-1)}T^{-\alpha-2}(1-\frac{t}{T})^{\beta-\alpha-2}. \end{aligned} \tag{4.11}$$

This finished the proof of Lemma 4.2.1.

□

4.3 Treatment of the weak formulation (4.1)

4.3.1 Treatment of the left-hand side

Using the formula of integration by parts (4.7) and the identity (4.8) we get

$$\begin{aligned} \int_0^T\int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)\varphi(t,x)dtdx &= \int_0^T\int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)D^{\alpha}_{t|T}\psi(t,x)dtdx \\ &= \int_0^T\int_{\mathbb{R}^n} D^{\alpha}_{0|T}I^{\alpha}_{0|T}(|u|^p)\psi(t,x)dtdx \\ &= \int_0^T\int_{\mathbb{R}^n} |u(t,x)|^p\psi(t,x)dtdx. \end{aligned} \tag{4.12}$$

For the second term of the left-hand side of equality (4.1), we use the results of Lemma 4.2.1 we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)dx &= \int_{\mathbb{R}^n} u_1(x)\varphi_1^r(x)\ D^{\alpha}_{t|T}\varphi_2(t)\big|_{t=0}dx \\ &= C_1T^{-\alpha}\int_{\mathbb{R}^n} u_1(x)\varphi_1^r(x)dx, \end{aligned} \tag{4.13}$$

since

$$D^{\alpha}_{t|T}\varphi_2(t)\big|_{t=0} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha} = C_1T^{-\alpha}.$$

For the third term, we note that

$$\varphi_t(t,x) = \frac{\partial\varphi}{\partial t}(t,x) = -\varphi_1^r(x)D^{\alpha+1}_{t|T}\varphi_2(t),$$

then, after using the results of Lemma 4.2.1 we arrive at

$$\int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)dx = -C_2T^{-\alpha-1}\int_{\mathbb{R}^n} u_0(x)\varphi_1^r(x)dx, \tag{4.14}$$

since

$$D_{t|T}^{\alpha+1}\varphi_2(t)\Big|_{t=0} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)}T^{-\alpha-1} = C_2T^{-\alpha-1}.$$

Finally, due to Lemma 4.2.1, the forth term of the left hand-side of the weak formulation (4.1) is reduced to the following form:

$$\int_{\mathbb{R}^n} g(x)\,|u_0|^{m-1}(x)u_0(x)\varphi(0,x)dx = C_1T^{-\alpha}\int_{\mathbb{R}^n} g(x)\,|u_0|^{m-1}u_0(x)(x)\varphi_1^r(x)dx. \quad (4.15)$$

4.3.2 Treatment of the right-hand side

Now we deal with the terms of the right hand side of the weak formulation (4.1).
So, by using the identity (4.9) we show that

$$\varphi_{tt}(t,x) = \varphi_1^r(x)\partial_t^2 D_{t|T}^{\alpha}\varphi_2(t) = \varphi_1^r(x)D_{t|T}^{\alpha+2}\varphi_2(t).$$

Then we get

$$\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_{tt}(t,x)dtdx = \int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_1^r(x)D_{t|T}^{\alpha+2}\varphi_2(t)dtdx. \quad (4.16)$$

Similarly, since

$$\varphi_t(t,x) = \varphi_1^r(x)\partial_t D_{t|T}^{\alpha}\varphi_2(t) = -\varphi_1^r(x)D_{t|T}^{\alpha+1}\varphi_2(t),$$

we get

$$\int_0^T\int_{\mathbb{R}^n} g(x)|u|^{m-1}u(t,x)\varphi_t(t,x)dtdx = -\int_0^T\int_{\mathbb{R}^n} g(x)\,|u|^{m-1}u(x)\varphi_1^r(x)D_{t|T}^{\alpha+1}\varphi_2(t)dtdx \quad (4.17)$$

Finally for the third term of the right-hand side of the weak formulation (4.1), one can see, firstly, that

$$\Delta\varphi(t,x) = \big(\Delta\varphi_1^r\big)(x)D_{t|T}^{\alpha}\varphi_2(t), \quad \text{for all } t>0 \text{ and } x\in\mathbb{R}^N. \quad (4.18)$$

Next, for all $i = 1,\cdots,N$ we have

$$\frac{\partial\varphi_1^r}{\partial x_i} = r\varphi_1^{r-1}\frac{\partial\varphi_1}{\partial x_i} \quad (4.19)$$

then

$$\frac{\partial^2\varphi_1^r}{\partial x_i^2} = r(r-1)\varphi_1^{r-2}\big(\frac{\partial\varphi_1}{\partial x_i}\big)^2 + r\varphi_1^{r-1}\frac{\partial^2\varphi_1}{\partial x_i^2} \quad (4.20)$$

from which we derive the following identity:

$$\begin{aligned}\Delta\varphi_1^r &= \sum_{i=0}^N \frac{\partial^2\varphi_1^r}{\partial x_i^2}\\ &= r(r-1)\varphi_1^{r-2}\sum_{i=0}^N\big(\frac{\partial\varphi_1}{\partial x_i}\big)^2 + r\varphi_1^{r-1}\sum_{i=0}^N\frac{\partial^2\varphi_1}{\partial x_i^2}\\ &= r(r-1)\varphi_1^{r-2}|\nabla\varphi_1|^2 + r\varphi_1^{r-1}\Delta\varphi_1. \quad (4.21)\end{aligned}$$

So, including the identity (4.21) we get

$$\int_0^T \int_{\mathbb{R}^n} u(t,x)\Delta\varphi(t,x)dtdx$$
$$= \int_0^T \int_{\mathbb{R}^n} u(t,x)(r\varphi_1^{r-1}\Delta\varphi_1 + r(r-1)\varphi_1^{r-2}|\nabla\varphi_1|^2)D^{\alpha}_{t|T}\varphi_2(t)dtdx \quad (4.22)$$

Inserting all formulas (4.12), (4.13), (4.14), (4.15), (4.16), (4.17) and (4.22) in the formula (4.1) we obtain

$$\Gamma(\alpha)\int_0^T \int_{\mathbb{R}^n} |u|^p\,\psi(t,x)dtdx + C_1T^{-\alpha}\int_{\mathbb{R}^n} u_1(x)\varphi_1^r(x)dx$$
$$+C_2T^{-\alpha-1}\int_{\mathbb{R}^n} u_0(x)\varphi_1^r(x)dx + \frac{C_1}{m}T^{-\alpha}\int_{\mathbb{R}^n} g(x)|u_0|^{m-1}(x)u_0(x)\varphi_1^r(x)dx$$
$$= \int_0^T \int_{\mathbb{R}^n} u(t,x)\varphi_1^r(x)D^{\alpha+2}_{t|T}\varphi_2(t)dtdx$$
$$-\int_0^T \int_{\mathbb{R}^n} u(t,x)\big(r\varphi_1^{r-1}\Delta\varphi_1 + r(r-1)\varphi_1^{r-2}\left|\nabla\varphi_1\right|^2\big)(x)D^{\alpha}_{t|T}\varphi_2(t)dtdx$$
$$+\frac{1}{m}\int_0^T \int_{\mathbb{R}^n} g(x)\left|u\right|^m(x)\varphi_1^r(x)D^{\alpha+1}_{t|T}\varphi_2(t)dtdx. \quad (4.23)$$

The facts that $r > 1$ and $\varphi_1 \le 1$ allow us to have

$$\left|r\varphi_1^{r-1}\Delta\varphi_1 + r\,(r-1)\,\varphi_1^{r-2}\left|\nabla\varphi_1\right|^2\right| \le \varphi_1^{r-2}\left(|\Delta\varphi_1| + |\nabla\varphi_1|^2\right)$$

and then we get from formula (4.23) the following inequality:

$$\int_0^T \int_{\mathbb{R}^n} |u|^p\psi(t,x)dtdx + CT^{-\alpha}\int_{\mathbb{R}^n} u_1(x)\varphi_1^r(x)dx$$
$$+CT^{-\alpha-1}\int_{\mathbb{R}^n} u_0(x)\varphi_1^r(x)dx + CT^{-\alpha}\int_{\mathbb{R}^n} g(x)|u_0|^m(x)\varphi_1^r(x)dx$$
$$\le C\int_0^T \int_{\mathbb{R}^n} |u(t,x)|\varphi_1^r(x)|D^{\alpha+2}_{t|T}\varphi_2(t)|dtdx$$
$$+C\int_0^T \int_{\mathbb{R}^n} |u(t,x)|\varphi_1^{r-2}\big(\,|\Delta\varphi_1| + |\nabla\varphi_1|^2\,\big)|D^{\alpha}_{t|T}\varphi_2(t)|dtdx$$
$$+C\int_0^T \int_{\mathbb{R}^n} |u|^m\,g(x)\varphi_1^r(x)|D^{\alpha+1}_{t|T}\varphi_2(t)|dtdx, \quad (4.24)$$

for some constant $C > 0$.
At this stage we apply $\varepsilon-$*Young inequality*

$$AB \le \varepsilon A^p + C(\varepsilon)B^q, pq = p+q, p, q > 1 \quad (4.25)$$

to terms of the right-hand side of inequality (4.24) we get

$$\int_0^T \int_{\mathbb{R}^n} |u(t,x)|\varphi_1^r(x)|D^{\alpha+2}_{t|T}\varphi_2(t)|dtdx = \int_0^T \int_{\mathbb{R}^n} u(t,x)\psi^{\frac{1}{p}}\psi^{-\frac{1}{p}}\varphi_1^r(x)|D^{\alpha+2}_{t|T}\varphi_2(t)|dtdx$$
$$\le \varepsilon\int_0^T \int_{\mathbb{R}^n} |u|^p\psi dtdx + C(\varepsilon)\int_0^T \int_{\mathbb{R}^n} \varphi_1^r\varphi_2^{-\frac{1}{p-1}}|D^{\alpha+2}_{t|T}\varphi_2|^{\frac{p}{p-1}}dtdx.$$

Similarly we have

$$\int_0^T\int_{\mathbb{R}^n}|u|\,\varphi_1^{r-2}(|\Delta\varphi_1|+|\nabla\varphi_1|^2)|D_{t|T}^{\alpha}\varphi_2|dtdx$$
$$\leq\varepsilon\int_0^T\int_{\mathbb{R}^n}|u|^p\psi dtdx+C(\varepsilon)\int_0^T\int_{\mathbb{R}^n}\varphi_1^*\varphi_1^{r-2q}\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2|^q\,dtdx$$

with

$$\varphi_1^*=|\Delta\varphi_1|^q+|\nabla\varphi_1|^{2q}.$$

For the third term of the right-hand side, we have after using $\varepsilon-Young$ inequality

$$\int_0^T\int_{\mathbb{R}^n}|g||u|^m\varphi_1^r(x)|D_{t|T}^{\alpha+1}\varphi_2(t)|dtdx=\int_0^T\int_{\mathbb{R}^n}|u|^m\psi^{\frac{m}{p}}\psi^{-\frac{m}{p}}|g|\varphi_1^r(x)|D_{t|T}^{\alpha+1}\varphi_2(t)|dtdx$$
$$\leq\varepsilon\int_0^T\int_{\mathbb{R}^n}|u|^p\psi dtdx+C(\varepsilon)\int_0^T\int_{\mathbb{R}^n}\varphi_1^r|\varphi_2|^{-\frac{m}{p-m}}|g||D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-m}}dtdx.$$

Using the fact that (4.2) implies

$$\int_{\mathbb{R}^n}u_i(x)\varphi_1^r(x)dx>0,\ i=0,1\text{ and }\int_{\mathbb{R}^n}g(x)\,|u_0|^m\,(x)\varphi_1^r(x)dx>0. \tag{4.26}$$

we get from (4.24), (4.26), (4.26) and (4.26), for ε small enough

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}|u|^p\,\psi(t,x)dtdx\ &\leq\ C\left(\int_0^T\int_{\mathbb{R}^n}\varphi_1^r|\varphi_2|^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2|^{\frac{p}{p-1}}dtdx\right.\\&\quad+\int_0^T\int_{\mathbb{R}^n}\varphi_1^*\varphi_1^{r-2q}|\varphi_2|^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2|^qdtdx\\&\quad\left.+\int_0^T\int_{\mathbb{R}^n}g\varphi_1^r|\varphi_2|^{\frac{m}{p-m}}|D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-m}}dtdx\right)\\&\leq\ C\left(I_1+I_2+I_3\right),\end{aligned} \tag{4.27}$$

for some positive constant C where

$$\begin{aligned}I_1&=\int_0^T\int_{\mathbb{R}^n}\varphi_1^r(x)|\varphi_2(t)|^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}}dtdx,\\I_2&=\int_0^T\int_{\mathbb{R}^n}\varphi_1^*(x)\varphi_1^{r-2q}(x)|\varphi_2(t)|^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2(t)|^qdtdx,\\I_3&=\int_{\mathbb{R}^n}g(x)\varphi_1^r(x)|\varphi_2(t)|^{\frac{m}{p-m}}|D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-m}}dtdx.\end{aligned}$$

In order to estimate the integrals I_1, I_2 and I_3 we consider the scaled coordinates

$$x=T^{\frac{\theta}{2}}y\quad\text{and}\quad t=T\tau, \tag{4.28}$$

where θ is the parameter which appears in the definition of φ_1, and noting that I_1, I_2 and I_3 are null outside Ω_T such that

$$\Omega_T:=\{x\in\mathbb{R}^N,\quad|x|^2\leq2T^\theta\}=\text{supp}\varphi_1.$$

By using Fubini's theorem (See e.g. Theorem 1.1.7 in [42]) we get

$$\begin{aligned} I_1 &= \Big(\int_{\Omega_T} \varphi_1^r(x)dx\Big)\Big(\int_0^T \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}}dt\Big) \\ &= J_{11}J_{12}. \end{aligned} \tag{4.29}$$

For J_{11} we note that

$$dx = T^{N\frac{\theta}{2}}dy,$$

then we have

$$J_{11} = \int_{\Omega_T} \varphi_1^r(x)dx = T^{\frac{N\theta}{2}} \int_0^2 \phi^r(y^2)dy = CT^{\frac{N\theta}{2}}. \tag{4.30}$$

For J_{12}, we obtain after using Lemma 4.2.1

$$\begin{aligned} J_{12} &= \int_0^T |\varphi_2(t)|^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}}dt \\ &= T^{1-(\alpha+2)\frac{p}{p-1}} \int_0^1 (1-\tau)^{\beta-(\alpha+2)\frac{p}{p-1}}d\tau \\ &= CT^{1-(\alpha+2)\frac{p}{p-1}}. \end{aligned} \tag{4.31}$$

Combining (4.30) and (4.31) into (4.29) we get

$$I_1 = CT^{-(\alpha+2)\frac{p}{p-1}+\frac{N\theta}{2}+1}. \tag{4.32}$$

In the same way we have

$$\begin{aligned} I_2 &= \Big(\int_{\Omega_T} \varphi_1^*(x)\varphi_1^{r-2q}(x)dx\Big)\Big(\int_0^T \varphi_2^{-\frac{1}{p-1}}(t)|D_{t|T}^{\alpha}\varphi_2(t)|^q dt\Big) \\ &= J_{21}J_{22}. \end{aligned} \tag{4.33}$$

So, after replacing q by its value $q = \frac{p}{p-1}$ we get

$$J_{21} = \int_{\Omega_T} \big(|\Delta\varphi_1|^{\frac{p}{p-1}} + |\nabla\varphi_1|^{2\frac{p}{p-1}}\big)\varphi_1^{r-2\frac{p}{p-1}}dx = CT^{-\theta\frac{p}{p-1}+\frac{N\theta}{2}}, \tag{4.34}$$

and

$$J_{22} = \int_0^T \varphi_2(t)^{-\frac{1}{p-1}} |D_{t|T}^{\alpha}\varphi_2(t)|^{\frac{p}{p-1}}dt = CT^{-\alpha\frac{p}{p-1}+1}. \tag{4.35}$$

Pluging (4.34) and (4.35) into (4.33) we find

$$I_2 = \int_0^T \int_{\Omega_T} \varphi_1^*\varphi_1^{r-2q}\varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha}\varphi_2|^q dtdx = CT^{-(\alpha+\theta)\frac{p}{p-1}+\frac{N\theta}{2}+1}. \tag{4.36}$$

For the third integral we have

$$I_3 = \int_0^T \int_{\Omega_T} \varphi_1^r(t)|\varphi_2(t)|^{-\frac{m}{p-m}}|g(x)||D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-m}}dtdx \tag{4.37}$$

$$= \Big(\int_{\Omega_T} |g(x)|\varphi_1^r(x)dx\Big)\Big(\int_0^T |\varphi_2(t)|^{-\frac{m}{p-m}}|D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-m}}dt\Big) = J_{31}J_{32}.$$

By using the mean value theorem for integrals, we get for J_{31}

$$\begin{aligned} J_{31} &= |g(\xi)| \int_{\Omega_T} \varphi_1^r(x)dx \quad \text{ for some } \xi \in \Omega_T \\ &= T^{\frac{N\theta}{2}} |g(\xi)| \int_0^2 \phi^r(y^2)dy = CT^{\frac{N\theta}{2}}. \end{aligned} \tag{4.38}$$

where we have used the fact that g is bounded. For J_{32}, we have

$$J_{32} = \int_0^T |\varphi_2(t)|^{-\frac{m}{p-m}} |D_{t|T}^{\alpha+1} \varphi_2(t)|^{\frac{p}{p-m}} dt = CT^{-(\alpha+1)\frac{p}{p-m}+1}. \tag{4.39}$$

Hence inserting (4.38) and (4.39) in 4.37 we obtain

$$I_3 = \int_0^T \int_{\Omega_T} \varphi_1^r |\varphi_2|^{-\frac{m}{p-m}} |g| |D_{t|T}^{\alpha+1} \varphi_2(t)|^{\frac{p}{p-m}} dtdx = CT^{-(\alpha+1)\frac{p}{p-m}+\frac{N\theta}{2}+1}. \tag{4.40}$$

Finally, by including (4.32),(4.36) and (4.40) into (4.27) we get

$$\begin{aligned} &\int_0^T \int_{\Omega_T} |u|^p \psi(t,x) dtdx \\ &\quad \leq C\big(T^{-(\alpha+2)\frac{p}{p-1}+\frac{N\theta}{2}+1} + T^{-(\alpha+\theta)\frac{p}{p-1}+\frac{N\theta}{2}+1} + T^{-(\alpha+1)\frac{p}{p-m}+\frac{N\theta}{2}+1}\big). \end{aligned} \tag{4.41}$$

Now, since θ is arbitrary and it should be, only, non-negative, we choose it as follows

$$\theta = \frac{(p-1)(\alpha+1)}{p-m} - \alpha > 0 \quad \text{since } p > m. \tag{4.42}$$

This choice of θ allows us, in particular, to have

$$-(\alpha+\theta)\frac{p}{p-1} + \frac{N\theta}{2} + 1 = -(\alpha+1)\frac{p}{p-m} + \frac{N\theta}{2} + 1. \tag{4.43}$$

Then by using (4.43) we get from (4.41) the estimate

$$\int_0^T \int_{\Omega_T} |u(t,x)|^p \psi(t,x) dtdx \leq CT^{\sigma}, \tag{4.44}$$

where

$$\sigma = \max\left\{ -(\alpha+2)\frac{p}{p-1} + \frac{N\theta}{2} + 1,\ -(\alpha+1)\frac{p}{p-m} + \frac{N\theta}{2} + 1 \right\}.$$

4.3.3 Deriving of the results

At this stage we distinguish between two principle cases and each case itself is divided into two sub-cases as it will be explained hereafter.

1. Case $\sigma \leq 0$

This case itself is divided into two subcases as follows:
1. *i.* Case $\sigma < 0$.

In this case, we note that $\sigma < 0$ is equivalent to

$$-(\alpha+2)\frac{p}{p-1}+\frac{N\theta}{2}+1<0$$

and

$$-(\alpha+1)\frac{p}{p-m}+1+\frac{N\theta}{2}<0$$

too. So the condition

$$-(\alpha+2)\frac{p}{p-1}+\frac{N\theta}{2}+1<0$$

is equivalent to

$$N < -\frac{2p(\alpha+1)+2}{\left(\frac{(\alpha+1)(p-1)}{m-p}+\alpha\right)(p-1)} \tag{4.45}$$

and the condition

$$-(\alpha+1)\frac{p}{p-m}+1+\frac{N\theta}{2}<0$$

is equivalent to

$$N < \frac{2\left(\alpha p+m\right)}{\alpha\left(m-1\right)+\left(p-1\right)} \tag{4.46}$$

where we have replaced θ by its value. Both conditions (4.45) and (4.45) on N give

$$N < \min\left\{\frac{2(\alpha p+m)}{\alpha(m-1)+(p-1)}, -\frac{2p(\alpha+1)+2}{\left(\frac{(\alpha+1)(p-1)}{m-p}+\alpha\right)(p-1)}\right\} \tag{4.47}$$

If we come back and replace α by its value $1-\gamma$ in (4.47) we get

$$N < \min\left\{\frac{2((1-\gamma)p+m)}{(1-\gamma)(m-1)+(p-1)}, -\frac{2p(2-\gamma)+2}{(\frac{(2-\gamma)(p-1)}{m-p}+(1-\gamma))(p-1)}\right\}. \tag{4.48}$$

Then if the condition (4.47) (or equivalently 4.48)) is satisfied, we pass to the limit as $T \to +\infty$ in (4.44) we get

$$\lim_{T\to+\infty}\int_0^T\int_{\Omega_T}|u(t,x)|^p\psi(t,x)dtdx=0.$$

Using the dominated convergence theorem of Lebesgue (E.g. Theorem 1.1.4 in [26]), the continuity of u with respect to t and x and the fact that

$$\lim_{T\to+\infty}\psi(t,x)=1 \tag{4.49}$$

we get

$$\int_0^{+\infty}\int_{\mathbb{R}^N}|u(t,x)|^p dtdx = 0$$

which implies that $u\equiv 0$ and this is a contradiction.
1. ii. Subcase of $\sigma = 0$.
First, noting that $\sigma = 0$ is equivalent to

$$N = \min\left\{\frac{2((1-\gamma)p+m)}{(1-\gamma)(m-1)+(p-1)}, -\frac{2p(2-\gamma)+2}{(\frac{(2-\gamma)(p-1)}{m-p}+(1-\gamma))(p-1)}\right\}. \tag{4.50}$$

Then, taking the limit as $T\to\infty$ in (4.41) with the consideration $\sigma = 0$ we get

$$\int_0^{+\infty}\int_{\mathbb{R}^N}|u(t,x)|^p dtdx \le C.$$

This means that

$$u\in L^p((0,+\infty);L^p(\mathbb{R}^N)),$$

From which we may deduce that

$$\lim_{R\to\infty}\int_0^{+\infty}\int_{\Delta_R}|u(t,x)|^p\psi(t,x)dtdx = 0 \tag{4.51}$$

where

$$\Delta_R := \left\{x\in\mathbb{R}^N : R^\theta < |x|^2 \le 2R^\theta\right\}$$

and θ is defined by (4.42).
Now, fixing arbitrarily R in $]0,T[$ for some $T>0$ such that we don't have, simultaneously, $R\to+\infty$ when $T\to+\infty$, and taking in this time

$$\varphi_1(x) = \phi\Big(\frac{|x|^2}{T^{\frac{\theta}{2}}R^{-\frac{\theta}{2}}}\Big)$$

where ϕ is the function defined by (4.6).
Using Hölder's inequality

$$\int_X uvd\mu \le \Big(\int_X u^p d\mu\Big)^{\frac{1}{p}}\Big(\int_X v^q d\mu\Big)^{\frac{1}{q}}, \tag{4.52}$$

which happens for all $u\in L^p(X)$ and $v\in L^q(X)$ such that $p,q>1, pq = p+q$, instead of the $\varepsilon-$*Young's* one to estimate integral I_2 in (4.27) on the set

$$\Omega_{TR^{-1}} := \left\{x\in\mathbb{R}^N : |x|^2\le 2T^\theta R^{-\theta}\right\} = \mathrm{supp}\varphi_1$$

and noting that $\Delta_{TR^{-1}}\subset\Omega_{TR^{-1}}$ and the support of $\Delta\varphi_1$ is contained in $\Delta_{TR^{-1}}$ where

$$\Delta_{TR^{-1}} := \left\{x\in\mathbb{R}^N : (TR^{-1})^\theta < |x|^2 \le 2(TR^{-1})^\theta\right\}$$

and θ is always given by (4.42), we get

$$\int_0^T \int_{\Omega_{TR^{-1}}} |u|\varphi_1^{r-2}\big(|\Delta\varphi_1|^2 + |\nabla\varphi_1|^2\big)|D^\alpha_{t|T}\varphi_2| dtdx$$
$$\leq \Big(\int_0^T \int_{\Delta_{TR^{-1}}} |u|^p \psi dtdx\Big)^{\frac{1}{p}} \Big(\int_0^T \int_{\Delta_{TR^{-1}}} \psi^{\frac{q}{p}} \varphi_1^{r-2q} \left(|\Delta\varphi_1|^q + |\nabla\varphi_1|^{2q}\right) |D^\alpha_{t|T}\varphi_2|^q dtdx\Big)^{\frac{1}{q}}. \tag{4.53}$$

Recalling the integrals I_1, I_3 which appear in formula (4.27) and $\tilde{I}_2$ such that

$$\tilde{I}_2 := \Big(\int_0^T \int_{\Delta_{TR^{-1}}} \psi^{\frac{q}{p}} \varphi_1^{r-2q} \left(|\Delta\varphi_1|^q + |\nabla\varphi_1|^{2q}\right) \left|D^\alpha_{t|T}\varphi_2\right|^q dtdx\Big)^{\frac{1}{q}}$$

to estimate them thanks to the scaled variables

$$x = T^{\frac{\theta}{2}} R^{-\frac{\theta}{2}} y, \quad t = T\tau$$

on the set $\Omega_{TR^{-1}}$ we get firstly

$$I_1 + I_3 \leq C\left(T^{1-(\alpha+2)\frac{p}{p-1}+N\frac{\theta}{2}} + T^{1-(\alpha+1)\frac{p}{p-m}+N\frac{\theta}{2}}\right) R^{-\frac{N\theta}{2}} \tag{4.54}$$

and using the hypothesis $\sigma = 0$ we get from (4.54)

$$I_1 + I_3 \leq CR^{-N\theta/2} \tag{4.55}$$

Computing the integral $\tilde{I}_2$ using the same scaled variables and the same form of function φ_1 then, we get from (4.27) after using the estimate (4.55)

$$\int_0^T \int_{\Omega_{TR^{-1}}} |u|^p \psi dtdx \leq CR^{-N\theta/2} + CR^{\theta-\frac{N\theta}{2q}} \Big(\int_0^T \int_{\Delta_{TR^{-1}}} |u|^p \,\psi dtdx\Big)^{\frac{1}{p}}. \tag{4.56}$$

Next taking the limit as $T \to +\infty$ in (4.56). Using (4.51) and (4.49) we arrive at

$$\int_0^\infty \int_{\mathbb{R}^N} |u(t,x)|^p dtdx \leq CR^{-N\theta/2}.$$

This means that necessarily $R \to +\infty$ and this contradicts our hypothesis.
Finally, it remains to note that both conditions (4.48) and (4.50) are equivalent to

$$N \leq \min\left\{\frac{2((1-\gamma)p+m)}{(1-\gamma)(m-1)+(p-1)}, -\frac{2p(2-\gamma)+2}{\left(\frac{(2-\gamma)(p-1)}{m-p} + (1-\gamma)\right)(p-1)}\right\}.$$

The second main case is the following
Case 2: $p \leq \frac{1}{\gamma}$.
Even this case is divided into two subcases as follows:
2. i. Subcase $p < \frac{1}{\gamma}$.

In this case we recall the principle estimate (4.27) and we take

$$\varphi_1(x) = \phi\Big(\frac{|x|^2}{R^\theta}\Big)$$

where ϕ is the function defined by (4.6) and R is a fixed positive number. Since the integrals I_1, I_2 and I_3 are null outside suppφ_1, then it is enough to estimate them on the set

$$\Sigma_R := \big\{x \in \mathbb{R}^N : |x| \leq 2R^{\theta/2}\big\} = \text{supp}\varphi_1.$$

We may do so by using the scaled variables

$$x = R^{\frac{\theta}{2}} y, \quad t = T\tau$$

as follows:

- For the first integral we estimate

$$\begin{aligned} I_1 &= \Big(\int_{\Sigma_R} \varphi_1^r(x)dx\Big)\Big(\int_0^T \varphi_2^{-\frac{1}{p-1}}(t)|D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}}dt\Big) \\ &= \Big(R^{N\theta/2}\int_0^2 \phi^r(y^2)dy\Big)\Big(T^{1-(\alpha+2)\frac{p}{p-1}}\int_0^1 (1-\tau)^{-\frac{\beta}{p-1}+(\beta-\alpha-2)\frac{p}{p-1}}d\tau\Big) \\ &= CR^{\frac{N\theta}{2}}T^{1-(\alpha+2)\frac{p}{p-1}}. \end{aligned} \tag{4.57}$$

- For I_2, we find

$$\begin{aligned} I_2 &= \Big(\int_{\Sigma_T} \varphi^*(x)_1\varphi(x)_1^{r-2q}dx\Big)\Big(\int_0^T \varphi_2(t)^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2(t)|^q dt\Big) \\ &= CR^{\frac{N\theta}{2}-\theta\frac{p}{p-1}}T^{1-\alpha\frac{p}{p-1}}. \end{aligned} \tag{4.58}$$

- Finally, after using the mean value theorem for integrals and Fubini's theorem we obtain

$$\begin{aligned} I_3 &= \Big(\int_{\Sigma_T} |g(x)|\varphi_1^r(x)dx\Big)\Big(\int_0^T |\varphi_2(t)|^{-\frac{m}{p-m}}|D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-m}}dt\Big) \\ &= CR^{\frac{N\theta}{2}}T^{1-(\alpha+1)\frac{p}{p-m}}. \end{aligned} \tag{4.59}$$

Including the estimates (4.57), (4.58) and (4.59) into (4.27) we get

$$\begin{aligned} &\int_0^T\int_{\Sigma_R} |u(t,x)|^p\psi(t,x)dtdx \\ &\qquad = CR^{\frac{N\theta}{2}}\big(T^{1-(\alpha+2)\frac{p}{p-1}} + T^{1-(\alpha+1)\frac{p}{p-m}}\big) + CR^{(\frac{N}{2}-\frac{p}{p-1})\theta}T^{1-\alpha\frac{p}{p-1}}. \end{aligned} \tag{4.60}$$

Noting that if $p < \frac{1}{\gamma}$ then all the exponents of T in (4.60) are negative, therefore by passing to the limit in (4.60) as $T \to +\infty$ with using the fact that

$$\lim_{T\to+\infty}\psi(t,x) = \varphi_1(x) \tag{4.61}$$

we get

$$\int_0^{+\infty}\int_{\Sigma_R}|u(t,x)|^p\varphi_1(x)dtdx = 0. \tag{4.62}$$

Next, taking the limit in (4.62) as $R \to +\infty$ with using the fact that

$$\lim_{R\to+\infty}\varphi_1(x) = 1 \tag{4.63}$$

we obtain

$$\int_0^{+\infty}\int_{\mathbb{R}^N}|u(t,x)|^p dtdx = 0.$$

This implies that $u \equiv 0$ which is a contradiction.
2. ii. Subcase $p = \frac{1}{\gamma}$.
In this case we assume furthermore that if $N \geq 3$ then

$$\frac{N}{2} - \frac{p}{p-1} < 0. \tag{4.64}$$

which is equivalent to

$$p \leq \frac{N}{N-2}. \tag{4.65}$$

Under this assumption, we have

$$1-(\alpha+2)\frac{p}{p-1} = -\frac{2}{\alpha} < 0, \tag{4.66}$$

$$1-(\alpha+1)\frac{p}{p-m} = \frac{m\alpha-m-\alpha}{m\alpha-m+1} < 0, \tag{4.67}$$

$$1-\alpha\frac{p}{p-1} = 0. \tag{4.68}$$

First, we take the limit as $T \to \infty$ in (4.60). Using (4.61) with taking into account the considerations (4.66), (4.67) and (4.68) we obtain

$$\int_0^{\infty}\int_{\Sigma_R}|u(t,x)|^p\varphi_1(x)dtdx = CR^{(\frac{N}{2}-\frac{p}{p-1})\theta}. \tag{4.69}$$

Next, passing to the limit as $R \to \infty$ in (4.69). After using (4.63), the condition (4.64) and the fact that $\theta > 0$, we get

$$\int_0^{\infty}\int_{\mathbb{R}^N}|u(t,x)|^p dtdx = 0.$$

This implies that $u \equiv 0$ and this is a contradiction. By this step ,the proof of Theorem 4.1.2 is achieved. □

Remark 4.3.4. *The upper bound of p in (4.65) is known as Ggaliardo-Nirenberg exponent and denoted usually by p_{GN}. This upper bound appears in particular when we use the linear estimates of Matsumura type and Gagliardo-Nirenberg inequality to proof global in time existence and local existence results to Problem (3.1)-(3.2) (See for more details E.g [7] and [25]).*

Chapter 5

Blow-up for Fractional damped wave equations with nonlinear memory

The focus of this chapter is to study the blow-up phenomenon to the following Cauchy problem of a wave equations with fractional damping and non linear memory

$$u_{tt} - \Delta u + D^{\sigma}_{0|t} u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau,\cdot)|^p d\tau, \tag{5.1}$$

$$u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N. \tag{5.2}$$

in the multi-dimensional real space $\mathbb{R}^n$. Where $p > 1, \;\; 0 < \gamma < 1$ and Δ is the classical Laplace operator, $\sigma \in]0,1[$ and $D^{\sigma}_{0|t}$ is the right hand side fractional operator of Riemann-Liouville defined by (2.18).

5.1 Definition and Statement of problem

For further considerations, in particular, to not be outside the subject of this thesis and don't encumber it, we restrict our selves only on the blow-up results to Problem (5.1)-(5.2).
The method which we will use to proof our results is the same method used to study Problem (3.1)-(3.2), that is the *test function methode*.

Before starting the study of blow-up, let us rewrite Problem (5.1)-(5.2) in the following form

$$u_{tt} - \Delta u + D^{\sigma}_{0|t} u_t = \Gamma(\alpha) I^{\alpha}_{0|t}(|u|^p). \tag{5.3}$$

subjected to the initial data

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \tag{5.4}$$

where $\alpha = 1-\gamma$, $\sigma \in]0,1[$, $p > 1$ and $I^{\alpha}_{0|t}$ is the fractional integral of order α ($\alpha \in]0,1[$) defined by (3.5) and Δ is the usual Laplace operator defined by (3.3).

5.2 Main result

Due to (5.3), the weak solution of problem (5.3)-(5.4) is defined as follows:

Definition 5.2.1 (weak solution). *Let $T > 0$ and $\gamma \in]0, 1[$. A weak solution for the Cauchy problem (5.3)-(5.4) on $\mathbb{R}_+ \times \mathbb{R}^n$ with the initial data $u_0, u_1 \in L^1_{loc}\left(\mathbb{R}^N\right)$ is a locally integrable function $u \in L^p\big((0,T), L^p_{loc}(\mathbb{R}^N)\big)$ satisfies at*

$$\begin{aligned}\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)\varphi(t,x)dtdx + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)dx - \int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)dx \\ + \int_{\mathbb{R}^n} u_0(x) D^{\sigma}_{t|T}\varphi(t,x)_{|t=0}\,dx = \int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_{tt}(t,x)dtdx \\ + \int_0^T\int_{\mathbb{R}^n} u(t,x)D^{\sigma+1}_{t|T}\varphi(t,x)dtdx - \int_0^T\int_{\mathbb{R}^n} u(t,x)\Delta\varphi(t,x)dtdx \qquad (5.5)\end{aligned}$$

for all non-negative test function $\varphi \in \mathscr{C}^2([0,T]\times\mathbb{R}^N)$ such that $\varphi(T,\cdot) = \varphi_t(T,\cdot) = D^{\sigma}_{t|T}\varphi(T,\cdot) = 0$ and $\alpha = 1-\gamma$.

Our main result concerning the blow-up for Problem (5.3)-(5.4) is the following:
For all $\gamma,\ \sigma \in]0, 1[$ and $N \in \mathbb{N}$, we put

$$p_\gamma(\sigma) = 1 + \frac{2(2-\gamma)+2\sigma}{(N-2+2\gamma+(N-2)\sigma)_+} \qquad (5.6)$$

and

$$p^* = \max\{p_\gamma(\sigma), \gamma^{-1}\}. \qquad (5.7)$$

Then we have the following theorem

Theorem 5.2.2. *Let $0 < \gamma < 1$, $p \in (1,\infty)$ for $N = 1, 2$ and $1 < p < \frac{N}{N-2}$ for $N \geq 3$. Assume that $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ and satisfy at*

$$\int_{\mathbb{R}^n} u_i(x)dx > 0, \quad i = 0, 1. \qquad (5.8)$$

Then, if $p \leq p^*$ then the solutions of the Cauchy problem (5.3)-(5.4) does not exist globally in time.

5.3 Proof of the main result

The Theorem (5.2.2) will be demonstrated by absurd, So we suppose that u is a global non trivial weak solution to problem (5.3)-(5.4) and we show that this leads to a contradiction. In order to prove Theorem 3.2.1 we need also to some results we will give them in the following section.

5.3.1 Preliminary results

Choosing a test function φ, defined, for some $T > 0$, as follows:

$$\varphi(t,x) = D^{\alpha}_{t|T}\psi(t,x) = \varphi_1^r(x) D^{\alpha}_{t|T}\varphi_2(t), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N \tag{5.9}$$

where $r > 1$ and $D^{\alpha}_{t|T}$ is the right fractional derivative operator of order α in the sense of Riemann-Liouville defined by (4.4), the functions φ_1 and φ_2 are given by

$$\varphi_1(x) = \phi\Big(\frac{x^2}{T^\theta}\Big), \quad \textit{and} \;\; \varphi_2(t) = \Big(1 - \frac{t}{T}\Big)_+^{\beta} \tag{5.10}$$

with $\beta > 1$, θ is a positive constant which will be chosen suitably later and ϕ is a cut-off non increasing function satisfying at

$$\phi(s) = \begin{cases} 1 & \text{if} \quad 0 \le s \le 1 \\ 0 & \text{if} \qquad\quad s \ge 2 \end{cases}, \quad 0 \le \phi \le 1 \text{ everywhere.} \tag{5.11}$$

We also denote by Ω_T for the support of φ_1, that is

$$\Omega_T := \text{supp}\varphi_1 = \{x \in \mathbb{R}^N, |x|^2 \le 2T^\theta\}, \tag{5.12}$$

and by Δ_T for the set containing the support of $\Delta\varphi_1$ which is defined by

$$\Delta_T = \big\{x \in \mathbb{R}^N, T^\theta \le |x|^2 \le 2T^\theta\big\}. \tag{5.13}$$

In addition to the proprieties of the fractional derivatives (4.7), 4.8) and (4.9), we need to the following identity

$$D^{\sigma}_{t|T}\big(D^{\alpha}_{t|T}\big) = D^{\sigma+\alpha}_{t|T} \tag{5.14}$$

and the following lemma in addition to Lemma 4.2.1.

Lemma 5.3.2. *Given $\beta > 1$, let φ_2 be the function defined by*

$$\varphi_2(t) = (1 - \frac{t}{T})_+^{\beta}.$$

Then, for all $\alpha, \sigma \in]0,1[$ we have

$$\begin{aligned} D^{\sigma+\alpha}_{t|T}\varphi_2(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\sigma-\alpha)} T^{-\beta}(T-t)_+^{\beta-\alpha-\sigma} \\ &= CT^{-\sigma-\alpha}(1 - \frac{t}{T})_+^{\beta-\alpha-\sigma}. \end{aligned}$$

where

$$C = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\sigma-\alpha)}.$$

Proof of Lemma 5.3.2. The proof of Lemma (5.3.2) is similar to the proof of Lemma 4.2.1. So, by using the results of Lemma 4.2.1, we have for all $\alpha \in (0,1)$

$$\begin{aligned} D^{\alpha}_{t|T}\varphi_2(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha}(1-\frac{t}{T})^{\beta-\alpha} && (5.15)\\ &= C_{\alpha,\beta}(T)(1-\frac{t}{T})^{\beta-\alpha}, && (5.16)\end{aligned}$$

where

$$C_{\alpha,\beta}(T) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha}.$$

Applying the identity (5.14) with noting that $C_{\alpha,\beta}(T)$ is independent of t, we get after using Lemma 4.2.1 again with $\beta-\alpha$ instead of β the desired result. This ends the proof. □

5.3.3 Study of the weak formulation (5.5)

Study of the left-hand side

Introducing the test function defined by (5.9) into the weak formulation (5.5). After using the formula of integration by parts (4.7) and the identity (4.8), for the first term of the left hand side becomes

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)\varphi(t,x)dtdx &= \int_0^T\int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)D^{\alpha}_{t|T}\psi(t,x)dtdx\\ &= \int_0^T\int_{\mathbb{R}^n} D^{\alpha}_{0|T}I^{\alpha}_{0|T}(|u|^p)\psi(t,x)dtdx\\ &= \int_0^T\int_{\mathbb{R}^n} |u|^p\psi(t,x)dtdx. && (5.17)\end{aligned}$$

For the second term of the left-hand side of equality (5.5), we get after using Lemma 4.2.1

$$\begin{aligned}\int_{\mathbb{R}^n} u_1(x)\varphi(0,x)dx &= \int_{\mathbb{R}^n} u_1(x)\varphi_1^r(x)D^{\alpha}_{t|T}\varphi_2(t)_{|t=0}\,dx\\ &= C_1T^{-\alpha}\int_{\mathbb{R}^n} u_1(x)\varphi_1^r(x)dx, && (5.18)\end{aligned}$$

since

$$D^{\alpha}_{t|T}\varphi_2(t)_{|t=0} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha} = C_1T^{-\alpha}.$$

For the third term, we note that

$$\varphi_t(t,x) = \frac{\partial\varphi}{\partial t}(t,x) = -\varphi_1^r(x)D^{\alpha+1}_{t|T}\varphi_2(t).$$

Then, by using Lemma 4.2.1, we get the following estimate

$$\int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)dx = C_2T^{-\alpha-1}\int_{\mathbb{R}^n} u_0(x)\varphi_1^r(x)dx, \qquad (5.19)$$

since

$$D^{\alpha+1}_{t|T}\varphi_2(t)_{|t=0} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)}T^{-\alpha-1} = C_2T^{-\alpha-1}.$$

Using Lemma 5.3.2, then, the following estimate will be derived for the forth term of the left hand-side of the weak formulation(5.5)

$$\int_{\mathbb{R}^n} u_0(x)D^{\sigma}_{t|T}\varphi(t,x)_{|t=0}\,dx = CT^{-\sigma-\alpha}\int_{\mathbb{R}^n} u_0(x)\varphi_1^r(x)dx, \tag{5.20}$$

since

$$D^{\sigma}_{t|T}\varphi(t,x)\Big|_{t=0} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\sigma-\alpha+1)}T^{-\sigma-\alpha}\varphi_1^r(x) = CT^{-\sigma-\alpha}\varphi_1^r(x). \tag{5.21}$$

Treatment of the right-hand side

Using the identity (4.9) we have

$$\varphi_{tt}(t,x) = \frac{\partial^2\varphi}{\partial t^2}(t,x) = \varphi_1^r(x)\partial_t^2 D^{\alpha}_{t|T}\varphi_2(t) = \varphi_1^r(x)D^{\alpha+2}_{t|T}\varphi_2(t).$$

Therefore

$$\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_{tt}(t,x)dtdx = \int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_1^r(x)D^{\alpha+2}_{t|T}\varphi_2(t)dtdx. \tag{5.22}$$

Using formula (5.14) and (4.9) we show firstly that

$$D^{\sigma+1}_{t|T}\varphi(t,x) = -\varphi_1^r(x)D^{\sigma+\alpha+1}_{t|T}\varphi_2(t),$$

therefore, we may derive the following estimate

$$\int_0^T\int_{\mathbb{R}^n} u(t,x)D^{\sigma+1}_{t|T}\varphi(t,x)dtdx = -\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_1^r(x)D^{\sigma+\alpha+1}_{t|T}\varphi_2(t)dtdx. \tag{5.23}$$

Finally, for the third term of the right-hand side of the formulation (5.5), we get after using formula (4.21)

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n} u(t,x)\Delta\varphi(t,x)dtdx\\ &\quad= \int_0^T\int_{\mathbb{R}^n} u(t,x)(r\varphi_1^{r-1}\Delta\varphi + r(r-1)\varphi_1^{r-2}|\nabla\varphi_1|^2)D^{\alpha}_{t|T}\varphi_2(t)dtdx. \end{aligned}\tag{5.24}$$

Inserting all the formulas (5.17), (5.18), (5.19), (5.20), (5.22), (5.23) and (5.24) in the weak formulation (5.5) we get

$$\begin{aligned}\Gamma(\alpha)&\int_0^T\int_{\mathbb{R}^n}|u|^p\,\psi(t,x)dtdx + C_1T^{-\alpha}\int_{\mathbb{R}^n}u_1(x)\varphi_1^r(x)dx + C_2T^{-\alpha-1}\int_{\mathbb{R}^n}u_0(x)\varphi_1^r(x)dx\\ &+CT^{-\sigma-\alpha}\int_{\mathbb{R}^n}u_0(x)\varphi_1^r(x)dx = \int_0^T\int_{\mathbb{R}^n}u(t,x)\varphi_1^r(x)D^{\alpha+2}_{t|T}\varphi_2(t)dtdx\\ &-\int_0^T\int_{\mathbb{R}^n}u(t,x)\left(r\varphi_1^{r-1}\Delta\varphi_1 + r(r-1)\varphi_1^{r-2}\left|\nabla\varphi_1\right|^2\right)(x)D^{\alpha}_{t|T}\varphi_2(t)dtdx\\ &+\int_0^T\int_{\mathbb{R}^n}u(t,x)\varphi_1^r(x)D^{\sigma+\alpha+1}_{t|T}\varphi_2(t)dtdx.\end{aligned}\tag{5.25}$$

The facts that $\varphi_1 \leq 1$ and

$$\left|r\varphi_1^{r-1}\Delta\varphi + r(r-1)\varphi^{r-2}\left|\nabla\varphi_1\right|^2\right| \leq \varphi^{r-2}(|\Delta\varphi_1| + |\nabla\varphi_1|^2)$$

allow us to have from the formula (5.25) the following inequality:

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} |u|^p\,\psi(t,x)dtdx + CT^{-\alpha}\int_{\mathbb{R}^n} u_1(x)\varphi_1^r(x)dx + CT^{-\alpha-1}\int_{\mathbb{R}^n} u_0(x)\varphi_1^r(x)dx \\
&\quad + CT^{-\sigma-\alpha}\int_{\mathbb{R}^n} u_0(x)\varphi_1^r(x)dx \leq C\int_0^T\int_{\mathbb{R}^n} |u(t,x)|\varphi_1^r(x)|D_{t|T}^{\alpha+2}\varphi_2(t)|dtdx \\
&\quad + C\int_0^T\int_{\mathbb{R}^n} |u(t,x)|\varphi_1^{r-2}\big(|\Delta\varphi_1| + |\nabla\varphi_1|^2\big)|D_{t|T}^{\alpha}\varphi_2|(t)dtdx \\
&\quad + \int_0^T\int_{\mathbb{R}^n} |u(t,x)|\varphi_1^r(x)|D_{t|T}^{\sigma+\alpha+1}\varphi_2(t)|dtdx, \qquad (5.26)
\end{aligned}$$

for some constant $C > 0$. Next, applying $\varepsilon-$*Young inequality* (4.25) to the terms of the right-hand side of inequality (5.26) we obtain

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} |u(t,x)|\,\varphi_1^r(x)|D_{t|T}^{\alpha+2}\varphi_2(t)|dtdx \qquad (5.27)\\
&\quad = \int_0^T\int_{\mathbb{R}^n} |u(t,x)|\psi^{\frac{1}{p}}\psi^{-\frac{1}{p}}\varphi_1^r(x)|D_{t|T}^{\alpha+2}\varphi_2(t)|dtdx \\
&\quad \leq \varepsilon\int_0^T\int_{\mathbb{R}^n} |u|^p\,\psi dtdx + C(\varepsilon)\int_0^T\int_{\mathbb{R}^n} \varphi_1^r\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2|^{\frac{p}{p-1}}dtdx. \qquad (5.28)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} |u|\varphi_1^{r-2}\big(|\Delta\varphi_1| + |\nabla\varphi_1|^2\big)|D_{t|T}^{\alpha}\varphi_2|dtdx \\
&\quad \leq \varepsilon\int_0^T\int_{\mathbb{R}^n} |u|^p\,\psi dtdx + C(\varepsilon)\int_0^T\int_{\mathbb{R}^n} \lambda(\varphi_1)\varphi_1^{r-2\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2|^{\frac{p}{p-1}}dtdx. \qquad (5.29)
\end{aligned}$$

where

$$\lambda(\varphi_1) = |\Delta\varphi_1|^q + |\nabla\varphi_1|^{2q}$$

For the third term of the right-hand side we have

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} |u(t,x)|\varphi_1^r(x)|D_{t|T}^{\sigma+\alpha+1}\varphi_2(t)|dtdx \qquad (5.30)\\
&\quad \leq \int_0^T\int_{\mathbb{R}^n} |u(t,x)|^p\psi(t,x)dtdx + \int_0^T\int_{\mathbb{R}^n} \varphi_1^r(x)\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\sigma+\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}}dtdx
\end{aligned}$$

Using the fact that (5.8) implies that

$$\int_{\mathbb{R}^n} u_i(x)\varphi^r(x)dx > 0,\;\; i = 0, 1,$$

we get from (5.26), (5.27), (5.29) and (5.30), for ε small enough

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}|u|^p\psi(t,x)dtdx &\le C\Big(\int_0^T\int_{\mathbb{R}^n}\varphi_1^r\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2|^{\frac{p}{p-1}}dtdx\\ &+\int_0^T\int_{\mathbb{R}^n}\lambda(\varphi_1)\varphi_1^{r-2\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2|^{\frac{p}{p-1}}dtdx\\ &+\int_0^T\int_{\mathbb{R}^n}\varphi_1^r(x)\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\sigma+\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}}dtdx\Big)\\ &\le C\left(I_1+I_2+I_3\right) \qquad (5.31)\end{aligned}$$

for some positive constant C where

$$I_1=\int_0^T\int_{\mathbb{R}^n}\varphi_1^r\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2|^{\frac{p}{p-1}}dtdx, \qquad (5.32)$$

$$I_2=\int_0^T\int_{\mathbb{R}^n}\lambda(\varphi_1)\varphi_1^{r-2\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2|^{\frac{p}{p-1}}dtdx, \qquad (5.33)$$

$$I_3=+\int_0^T\int_{\mathbb{R}^n}\varphi_1^r(x)\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\sigma+\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}}dtdx. \qquad (5.34)$$

Now, to estimate the integrals I_1, I_2 and I_3 we consider the scaled variables

$$x=T^{\frac{\theta}{2}}y \quad \text{and} \quad t=T\tau \qquad (5.35)$$

and noting that they are null outside Ω_T (defined by (5.12)). Then by using Fubini's theorem, we get, for I_1

$$\begin{aligned}\int_0^T\int_{\Omega_T}\varphi_1^r\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2|^{\frac{p}{p-1}}dtdx &=\Big(\int_{\Omega_T}\varphi_1^r dx\Big)\Big(\int_0^T\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2|^{\frac{p}{p-1}}dt\Big)\\ &=J_{11}J_{12} \qquad (5.36)\end{aligned}$$

we have

$$J_{11}=\int_{\Omega_T}\varphi_1^r(x)dx=T^{\frac{N\theta}{2}}\int_0^2\phi^r(y^2)dy=CT^{\frac{N\theta}{2}}, \qquad (5.37)$$

and after using Lemma 4.2.1, we get

$$J_{12}=\int_0^T\varphi_2^{-\frac{1}{p-1}}|D_{tT}^{\alpha+2}\varphi_2|^{\frac{p}{p-1}}dt=CT^{1-(\alpha+2)\frac{p}{p-1}}. \qquad (5.38)$$

Combining (5.37) and (5.38) into (5.36) we find

$$\int_0^T\int_{\Omega_T}\varphi_1^r\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2|^{\frac{p}{p-1}}dtdx=CT^{-(\alpha+2)\frac{p}{p-1}+\frac{N\theta}{2}+1}. \qquad (5.39)$$

In the same way we have, for I_2

$$\begin{aligned}\int_0^T\int_{\Omega_T}\lambda(\varphi_1)\varphi_1^{r-2q}\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2|^q dtdx &=\Big(\int_{\Omega_T}\lambda(\varphi_1)\varphi_1^{r-2q}dx\Big)\Big(\int_0^T\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2|^q dt\Big)\\ &=J_{21}J_{22}. \qquad (5.40)\end{aligned}$$

So, if we replace q by its value $\frac{p}{p-1}$ we obtain

$$J_{21} = \int_{\Omega_T} \left(|\Delta\varphi_1|^{\frac{p}{p-1}} + |\nabla\varphi_1|^{2\frac{p}{p-1}} \right) \varphi_1^{r-2\frac{p}{p-1}} dx = CT^{-\theta\frac{p}{p-1}+\frac{N\theta}{2}} \tag{5.41}$$

and by Lemma 4.2.1 we find the estimate

$$J_{22} = \int_0^T \varphi_2^{-\frac{1}{p-1}} |D^\alpha_{t|T}\varphi_2|^{\frac{p}{p-1}} dt = CT^{-\alpha\frac{p}{p-1}+1}. \tag{5.42}$$

By including (4.34) and (4.35) into (5.40) we find

$$\int_0^T \int_{\Omega_T} \lambda(\varphi_1)\varphi_1^{r-2q}\varphi_2^{-\frac{1}{p-1}} |D^\alpha_{t|T}\varphi_2|^q dtdx = CT^{-(\alpha+\theta)\frac{p}{p-1}+\frac{N\theta}{2}+1}. \tag{5.43}$$

Now we deal with I_3. We have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \varphi_1^r(x)\varphi_2^{-\frac{1}{p-1}} |D^{\sigma+\alpha+1}_{t|T}\varphi_2(t)|^{\frac{p}{p-1}} dtdx &= \Big(\int_{\Omega_T} \varphi_1^r dx\Big)\Big(\int_0^T |\varphi_2|^{-\frac{1}{p-1}} |D^{\sigma+\alpha+1}_{t|T}\varphi_2(t)|^{\frac{p}{p-1}} dt\Big) \\ &= J_{31}J_{32}. \end{aligned} \tag{5.44}$$

The integral J_{31} is the same as J_{11}, so we have directly

$$J_{31} = J_{11} = CT^{\frac{N\theta}{2}} \tag{5.45}$$

It remains to estimate J_{32}. Recalling Lemma 5.3.2 we may estimate

$$J_{32} = \int_0^T |\varphi_2|^{-\frac{1}{p-1}} |D^{\sigma+\alpha+1}_{t|T}\varphi_2(t)|^{\frac{p}{p-1}} dt = CT^{1-(\sigma+\alpha+1)\frac{p}{p-1}}. \tag{5.46}$$

Hence, after replacing (4.38) and (4.39) in 4.37 we get

$$\int_0^T \int_{\mathbb{R}^n} \varphi_1^r(x)\varphi_2^{-\frac{1}{p-1}} |D^{\sigma+\alpha+1}_{t|T}\varphi_2(t)|^{\frac{p}{p-1}} dtdx = CT^{-(\sigma+\alpha+1)\frac{p}{p-1}+1+\frac{N\theta}{2}}. \tag{5.47}$$

Finally, we introduce (5.39),(5.43) and (5.47) into (5.31) we obtain

$$\begin{aligned} &\int_0^T \int_{\Omega_T} |u|^p\psi(t,x) dtdx \\ &\quad \le C\big(T^{-(\alpha+2)\frac{p}{p-1}+\frac{N\theta}{2}+1} + T^{-(\alpha+\theta)\frac{p}{p-1}+\frac{N\theta}{2}+1} + T^{-(\sigma+\alpha+1)\frac{p}{p-1}+1+\frac{N\theta}{2}}\big). \end{aligned} \tag{5.48}$$

Now, since θ is arbitrary and it must only be positive, we choose it as follows:

$$\theta = \sigma + 1 > 0 \quad \text{since } \sigma \in]0,1[.$$

This choice of θ allows us to have

$$-(\alpha+\theta)\frac{p}{p-1} + \frac{N\theta}{2} + 1 = -(\sigma+\alpha+1)\frac{p}{p-1} + 1 + \frac{N\theta}{2}. \tag{5.49}$$

Taking into account the consideration (5.49) we get from (5.48)

$$\int_0^T \int_{\Omega_T} |u|^p \, \psi(t,x) dt dx \leq C T^{\delta}. \tag{5.50}$$

where

$$\delta = \max\left(-(\alpha+2)\frac{p}{p-1} + \frac{N\theta}{2} + 1, -(\sigma+\alpha+1)\frac{p}{p-1} + \frac{N\theta}{2} + 1\right).$$

Using the facts that σ, $\alpha < 1$ and $p > 1$ we find

$$\delta = -(\sigma+\alpha+1)\frac{p}{p-1} + \frac{N\theta}{2} + 1. \tag{5.51}$$

Recalling that $\theta = \sigma + 1$ we have

$$\delta = -(\sigma+\alpha+1)\frac{p}{p-1} + (\sigma+1)\frac{N}{2} + 1. \tag{5.52}$$

At this stage, in order to prove the first result in Theorem 3.2.1, we distinguish two cases according to δ.

Case of $p \leq p_\gamma(\sigma)$ This case itself is divided into two subcases as follows :

1. i. Subcase of $p < p_\gamma(\sigma)$. In this case, one can remark that the condition $p < p_\gamma(\sigma)$ is equivalent to $\delta < 0$, then we pass to the limit as $T \to \infty$ in (5.50), we get

$$\lim_{T\to+\infty} \int_0^T \int_{\Omega_T} |u|^p \, \psi(t,x) dt dx = 0. \tag{5.53}$$

Using the dominated convergence theorem of Lebesgue (*Theorem 1.1.4* in [**?**]), the continuity of u with respect to t and x and the fact that

$$\lim_{T\to+\infty} \psi(t,x) = 1, \tag{5.54}$$

we get

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p dt dx = 0,$$

which implies that

$$u = 0$$

and this is a contradiction because we have supposed that the solution u is not trivial.

1. ii. Subcase of $p = p_\gamma(\sigma)$

First, we remark that the condition $p = p_\gamma(\sigma)$ is equivalent to $\delta = 0$,

Next, taking the limit as $T \to \infty$ in (5.50) with the consideration $\delta = 0$ we get

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p \, dt dx < +\infty$$

from which we can deduce that

$$\lim_{T\to\infty}\int_0^{+\infty}\int_{\Delta_T}|u|^p\,\psi dtdx=0 \tag{5.55}$$

where Δ_T is defined by (5.13).
Fixing arbitrarily R in $]0,T[$ for some $T>0$ such that when $T\to\infty$ we don't have $R\to\infty$ at the same time, and taking in this time

$$\varphi_1(x)=\phi\Big(\frac{|x|^2}{T^{\frac{\theta}{2}}R^{-\frac{\theta}{2}}}\Big) \tag{5.56}$$

where θ is an arbitrary positive constant and ϕ is the cut-off function defined by (4.6).
Recalling the estimate (5.31) and applying Hölder's inequality (4.52) instead of the $\varepsilon-$*Young's* one to estimate the integral I_2 on the set

$$\Omega_{TR^{-1}}:=\big\{x\in\mathbb{R}^N:|x|^2\leq 2T^\theta R^{-\theta}\big\}=\text{supp}\varphi_1$$

with noting that

$$\text{supp}\Delta\varphi_1\subset\Delta_{TR^{-1}}\subset\Omega_{TR^{-1}}$$

where

$$\Delta_{TR^{-1}}:=\big\{x\in\mathbb{R}^N:T^\theta R^{-\theta}\leq|x|^2\leq 2T^\theta R^{-\theta}\big\} \tag{5.57}$$

we get

$$\begin{aligned}\int_0^T\int_{\Omega_{TR^{-1}}}|u|\varphi_1^{r-2}\big(|\Delta\varphi_1|^2+|\nabla\varphi_1|^2\big)|D^\alpha_{t|T}\varphi_2|dtdx&\leq\Big(\int_0^T\int_{\Delta_{TR^{-1}}}|u|^p\,\psi dtdx\Big)^{\frac{1}{p}}\\&\times\Big(\int_0^T\int_{\Delta_{TR^{-1}}}\psi^{\frac{q}{p}}\varphi_1^{r-2q}(|\Delta\varphi_1|^q+|\nabla\varphi_1|^{2q})|D^\alpha_{t|T}\varphi_2|^q dtdx\Big)^{\frac{1}{q}}.\end{aligned} \tag{5.58}$$

Recalling the integrals I_1 and I_3 which appear in the estimate (5.31) and $\tilde{I}_2$ such that

$$\tilde{I}_2=\Big(\int_0^T\int_{\Delta_{TR^{-1}}}\psi^{\frac{q}{p}}\varphi_1^{r-2q}\left(|\Delta\varphi_1|^q+|\nabla\varphi_1|^{2q}\right)\left|D^\alpha_{t|T}\varphi_2\right|^q dtdx\Big)^{\frac{1}{q}}$$

to estimate them using the change of variables

$$x=T^{\frac{\theta}{2}}R^{-\frac{\theta}{2}}y\ \text{ and } t=T\tau$$

on the set $\Omega_{TR^{-1}}$. Thanks to Fubini's theorem, as it is explained in the first case, we get firstly

$$I_1+I_3\leq C\big(T^{-(\alpha+2)\frac{p}{p-1}+N\frac{\theta}{2}+1}+T^{-(\sigma+\alpha+1)\frac{p}{p-1}+\frac{N\theta}{2}+1}\big)R^{-N\theta}. \tag{5.59}$$

Taking into account the hypothesis $\delta=0$ we get from (5.59)

$$I_1+I_3\leq CR^{-N\theta/2} \tag{5.60}$$

Calculating the integral $\tilde{I}_2$ using the same change of variables and the same form of function φ_1 we get from (5.31) after using (5.60)

$$\int_0^T \int_{\Omega_{TR^{-1}}} |u|^p \, \psi dtdx \leq CR^{-N\theta/2} + CR^{\theta-\frac{N\theta}{2q}} \Big(\int_0^T \int_{\Delta_{TR^{-1}}} |u|^p \, \psi dtdx \Big)^{\frac{1}{p}} \tag{5.61}$$

Passing to the limit as $T \to +\infty$ in (5.61). After using (4.51) and the fact that

$$\lim_{T\to+\infty} \psi(t,x) = 1$$

we get

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p \, dtdx \leq CR^{-N\theta/2}$$

which means that necessarily $R \to +\infty$ and this is a contradiction.
Now we deal with the second main result in Theorem 3.2.1.
Case of $p \leq \frac{1}{\gamma}$.
Even this case is divided into two subcases as follows
2. i. Subcase of $p < \frac{1}{\gamma}$.
Recalling again the estimate (5.31) and we take in this case

$$\varphi_1(x) = \phi\big(\frac{|x|^2}{R^\theta}\big)$$

where ϕ is the function defined by (4.6) and R is a fixed positive number
To estimate the generalized integrals I_1, I_2 and I_3 (page (52)) with respect to x on the set

$$\Sigma_R := \left\{ x \in \mathbb{R}^N : |x| \leq 2R^{\theta/2} \right\} = supp\varphi_1$$

we consider the scaled variables

$$x = R^{\frac{\theta}{2}} y \ \text{ and } t = T\tau.$$

So, for the first integral, by using Fubini's theorem, we have

$$\begin{aligned}
\int_0^T \int_{\Sigma_R} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2} \varphi_2|^{\frac{p}{p-1}} dtdx &= \Big(\int_{\Sigma_R} \varphi_1^r dx \Big) \Big(\int_0^T \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2} \varphi_2|^{\frac{p}{p-1}} dt \Big) \\
&= \Big(R^{N\theta/2} \int_0^1 \phi^r(y^2) dy \Big) \\
&\quad \times \Big(T^{1-(\alpha+2)\frac{p}{p-1}} \int_0^T (1-\tau)^{-\frac{\beta}{p-1}+(\beta-\alpha-2)\frac{p}{p-1}} d\tau \Big) \\
&= CR^{\frac{N\theta}{2}} T^{1-(\alpha+2)\frac{p}{p-1}}.
\end{aligned} \tag{5.62}$$

In the same way, we get, for the second one the estimate

$$\begin{aligned}
&\int_0^T \int_{\Sigma_R} \lambda(\varphi_1) \varphi_1^{r-2q} \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha} \varphi_2|^q dtdx \\
&\qquad = \Big(\int_{\Sigma_T} \lambda(\varphi_1) \varphi_1^{r-2q} dx \Big) \Big(\int_0^T \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha} \varphi_2|^q dt \Big) \\
&\qquad = CR^{\frac{N\theta}{2} - \theta\frac{p}{p-1}} T^{1-\alpha\frac{p}{p-1}}.
\end{aligned} \tag{5.63}$$

Finally, for the third integral, we get

$$\int_0^T \int_{\mathbb{R}^n} \varphi_1^r(x)\varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\sigma+\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}} dtdx = CR^{\frac{N\theta}{2}} T^{-(\sigma+\alpha+1)\frac{p}{p-1}+1}. \tag{5.64}$$

Taking account the estimates (5.62), (5.63) and (5.64) we get from (5.31) the estimate

$$\begin{aligned}\int_0^T \int_{\Sigma_R} &|u|^p \psi(t,x)dtdx \\ &= CR^{\frac{N\theta}{2}}\left(T^{1-(\alpha+2)\frac{p}{p-1}} + T^{-(\sigma+\alpha+1)\frac{p}{p-1}+1}\right) + CR^{\left(\frac{N}{2}-\frac{p}{p-1}\right)\theta} T^{1-\alpha\frac{p}{p-1}}. \end{aligned} \tag{5.65}$$

First, noting that $p < \frac{1}{\gamma}$ is equivalent to

$$-\alpha\frac{p}{p-1} < 0. \tag{5.66}$$

This also implies that

$$(\alpha+2)\frac{p}{p-1} > \alpha\frac{p}{p-1} \quad \text{and} \quad (\sigma+\alpha+1)\frac{p}{p-1} > \alpha\frac{p}{p-1}. \tag{5.67}$$

Taking the limit as $T \to +\infty$ in (5.65) with the considerations (5.66) and (5.67) and the fact that

$$\lim_{T\to+\infty} \psi(t,x) = \varphi_1^r(x) \tag{5.68}$$

we get

$$\int_0^{+\infty} \int_{\Sigma_R} |u|^p \varphi_1^r(x)dtdx = 0. \tag{5.69}$$

Finally, by taking the limit in (5.69) as $R \to +\infty$, with using the fact that $\lim_{R\to+\infty} \varphi_1^r(x) = 1$ we arrive at

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p dtdx = 0.$$

This implies that $u = 0$ which is contradiction.

2. ii. Subcase of $p = \frac{1}{\gamma}$

Recalling the estimate (5.65). We assume, in this case, that if $N \geq 3$ then p should satisfy at

$$p < \frac{N}{N-2} \tag{5.70}$$

First, we observe that (5.70) implies that

$$\frac{N}{2} - \frac{p}{p-1} < 0 \tag{5.71}$$

Recalling that $\alpha = 1-\gamma$, we show that the assumption $p = \frac{1}{\gamma}$ leads to

$$1 - \alpha\frac{p}{p-1} = 0 \tag{5.72}$$

as well as

$$1-(\sigma+\alpha+1)\frac{p}{p-1}=-(\sigma+1)\frac{p}{p-1}<0, \tag{5.73}$$

$$1-(\alpha+2)\frac{p}{p-1}=-\frac{2}{\alpha}<0. \tag{5.74}$$

Hence, by taking the limit as $T\to\infty$ in (5.65) with the considerations (5.73), (5.72) and (5.68) we obtain the estimate

$$\int_0^\infty\int_{\Sigma_R}|u|^p\varphi_1^r(x)dtdx=CR^{(\frac{N}{2}-\frac{p}{p-1})\theta}. \tag{5.75}$$

Finally, one can remark that if $N=1,2$ then $\frac{N}{2}-\frac{p}{p-1}<0$ for all $p>1$ (without any upper bound), then by taking the limit as $R\to\infty$ in (5.75) after using the facts that $\theta>0$ and $\lim\limits_{R\to+\infty}\varphi_1^r(x)=1$ we get

$$\int_0^\infty\int_{\mathbb{R}^N}|u|^p\,dtdx=0 \tag{5.76}$$

which implies that $u=0$ and this is a contradiction.

If $N\geq 3$ then $\frac{N}{2}-\frac{p}{p-1}$ is negative if, and only if the condition (5.70) or equivalently (5.71) is satisfied. So, if the condition (5.71) is fulfilled, then by passing to the limit as $R\to\infty$ in (5.75) we get (5.76). This achieves the proof of Theorem 3.2.1.

Remark 5.3.4. *We remark that the condition (5.70) is needed only in the case of* $p=\frac{1}{\gamma}$ *and* $N\geq 3$ *and not otherwise. The upper bound* $\frac{N}{N-2}$ *of* p *is known as the exponent of Gagliardo-Nirenberg and denoted in general* p_{GN} *which is used to prove the existence of (local and global) solutions, in particular to apply the Gagliardo-Nirenberg inequality.*
We, also, remark that the condition (5.70) is equivalent to

$$\frac{N-2}{N}<\gamma<1,$$

and has really a meaning only if $N\geq 3$. *For* $N=1,2$ *this condition becomes trivial.*

Chapter 6

Conclusion and concluding remarks

First, one can show that if $\sigma \to 0$ in (5.6) then $p_\gamma(\sigma) \to p_\gamma$. This means that we find the same critical exponent obtained by the author in ([2]) and this is reasonable because if $\sigma = 0$ then $D^\sigma_{0|t} u_t = u_t$. In other word, our results are more general than the results of ([2]). In the other hand, thanks to the presence of the damping term u_t in the model studied in ([2]), one can show using Fourier transform, for example, or, even, by scaling argument (See Introduction), that the model studied in ([2]) has the same behaviour with the model studied by the authors in ([39]) as $t \to +\infty$. For this reason, the two problems have the same critical exponent p^*.
As a proposal for open subjects, we see that

$$\lim_{\gamma \to 1} \int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p ds = C|u(s,\cdot)|^p \tag{6.1}$$

where in fact $C = \frac{1}{\Gamma(1-\gamma)}$. Then, one can show that the critical exponent of the model

$$u_{tt} - \Delta u + D^\sigma_{0|t} u_t = |u|^p \tag{6.2}$$

is

$$p(\sigma) = 1 + \frac{2 + 2\sigma}{(N + (N-2)\sigma)_+}, \tag{6.3}$$

since

$$\lim_{\gamma \to 1} p_\gamma(\sigma) = 1 + \frac{2 + 2\sigma}{(N + (N-2)\sigma)_+}.$$

In the particular case $\sigma = 0$ in (6.3), we find the critical exponent founded by Fujita, since

$$\lim_{\sigma \to 0} p(\sigma) = p_{Fuj} := 1 + \frac{2}{N} \tag{6.4}$$

The result (6.4) is very reasonable because if $\sigma \to 0$, then the model (6.2) tends the semi-linear damped wave equation

$$u_{tt} - \Delta u + u_t = |u(s,\cdot)|^p \tag{6.5}$$

and the parabolic model which corresponds to (6.5) (See Introduction) is

$$u_t - \Delta u = |u|^p \tag{6.6}$$

For this reason we find the same critical exponent of Fujita founded by Fujita in his pioneering works (See [14]).

As a second proposal for open subject, taking into account the consideration (6.1), one can find sufficient conditions of non-existence of global in time solution for the following weighted damped wave models

$$u_{tt} - \Delta u + g(\cdot)|u|^{m-1}u_t = |u|^p, \; p > 1, \; m > 1. \tag{6.7}$$

by taking $\gamma = 1$ in Theorem 4.1.2. One can prove a such result by the same method with a suitable test function. In [21], the authors studied a model more general then (6.7), but only, in one dimensional real space, that is $N = 1$.

Annexe

Proof of Lemma 2.3.5

Let $s > 1$. By using the definition of the norm in H^s, we have

$$\|fu\|_{H^s(\mathbb{R}^N)} = \sum_{|\beta|\leq s} \left\|D^\beta(fu)\right\|_{L^2(\mathbb{R}^N)}. \tag{6.8}$$

By Leibniz formula (2.31) we have for all $\alpha \in \mathbb{N}^N$

$$\partial^\alpha(fu) = \sum_{\beta\leq\alpha} C_\alpha^\beta \partial^\beta f \partial^{\alpha-\beta} u, \quad C_\alpha^\beta = \frac{\alpha!}{\beta!(\alpha-\beta)!}. \tag{6.9}$$

Then, we have by triangular inequality, for all $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leq s-1$

$$|\partial^\alpha(fu)|^2 \leq |\partial^\alpha(f)u|^2 + |\partial^{\alpha-1}(f)\partial u|^2 + \cdots + |f\partial^\alpha u|^2. \tag{6.10}$$

Therefore

$$\begin{aligned}
\int_{\mathbb{R}^N} |\partial^\alpha(fu)|^2 dx \quad & \leq \int_{\mathbb{R}^N} |\partial^\alpha(f)u|^2 dx + \int_{\mathbb{R}^N} |\partial^{\alpha-1}(f)\partial u|^2 dx + \cdots + \int_{\mathbb{R}^N} |f\partial^\alpha u|^2 dx \\
& \leq \sup_{x\in\mathbb{R}^N} |\partial^\alpha f(x)| \int_{\mathbb{R}^N} |u|^2 dx + \cdots + \sup_{x\in\mathbb{R}^N} |f(x)| \int_{\mathbb{R}^N} |\partial^\alpha u|^2 dx \\
& \leq \max\{\sup_{x\in\mathbb{R}^N} |\partial^\alpha f(x)|, \cdots, \sup_{x\in\mathbb{R}^N} |f(x)|\}\Big(\int_{\mathbb{R}^N} |u|^2 dx + \cdots + \int_{\mathbb{R}^N} |\partial^\alpha u|^2 dx\Big) \\
& \leq \max_{|\alpha|\leq s-1} (\sup_{x\in\mathbb{R}^N} |\partial^\alpha f(x)|)\|u\|_{H^{s-1}}^2. \qquad (6.11)
\end{aligned}$$

Since $u \in H^{s-1}$ and f is bounded with all its derivatives (at least up to the $([s]+1)^{th}$ one) we get from (6.11)

$$\partial^\alpha(fu) \in L^2(\mathbb{R}^N) \tag{6.12}$$

for all $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leq s-1$.
Consequently, we get from (6.12)

$$fu \in H^{s-1}.$$

Next by using (6.11) we get firstly

$$\|\partial^\alpha(fu)\|_{L^2} \leq \sqrt{\max_{|\alpha|\leq s-1} (\sup_{x\in\mathbb{R}^N} |\partial^\alpha f(x)|)}\|u\|_{H^{s-1}} \tag{6.13}$$

and then

$$\begin{aligned}
\|fu\|_{H^{s-1}} \quad & = \quad \sum_{|\alpha|\leq s-1} \|\partial^\alpha(fu)\|_{L^2} \\
& \leq \quad C\sqrt{\max_{|\alpha|\leq s-1} (\sup_{x\in\mathbb{R}^N} |\partial^\alpha f(x)|)}\|u\|_{H^{s-1}} \qquad (6.14)
\end{aligned}$$

for some $C > 0$ since the sum is finite in (6.14).

Fractional Laplacian Operator

In this section, we give some preliminary properties on the fractional Laplacian that we have used in the definition of Sobolev spaces and in the proof of Lemma of Ju.

Definition 6.0.1. *The fractional Laplacian is defined for $v \in H^s(\mathbb{R}^N)$ by*

$$\mathscr{F}\left((-\Delta)^s v\right) = |\xi|^{2s} \mathscr{F}(v),$$

where $\mathscr{F}$ denotes the Fourier transform.

The fractional Laplacian operator $(-\Delta)^s$ is a pseudo-differential operator of symbol $p(x,\xi) = |\xi|^{2s}$, as it can be seen by taking inverse Fourier transform in the above formula.
The following definition is equivalent to the above one (see [30] for a proof).

Definition 6.0.2. *For all $x \in \mathbb{R}^N$,*

$$(-\Delta)^s v(x) = C_{n,s} P.V. \int_{\mathbb{R}^N} \frac{v(x) - v(t)}{|x-t|^{N+2s}} dt,$$

where $C_{n,s}$ is a normalizing coefficient, P.V. stands for the Cauchy principal value and v is taken e.g. in $\mathscr{S}(\mathbb{R}^N)$ in order to define the (singular) integral in the usual sense.

The following pointwise estimate given in N. Ju [29] for $N = 2$, makes use of the Riesz potential representation of $(-\Delta)^{\delta/2}$ is motivated by the proof of the Proposition 3.2 of A. Córdoba and D. Córdoba [1], which we have used in the proof of the blow-up theorem.

Lemma 6.0.3 (Lemma of Ju-Cordoba). *Let $0 \leq \delta \leq 2$ and $\ell \geq 1$. Then, for any $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^N)$such that $\varphi \geq 0$ and all $x \in \mathbb{R}^N$ the following inequality holds*

$$(-\Delta)^{\delta/2} \psi^\ell(x) \leq l \psi^{\ell-1}(x) (-\Delta)^{\delta/2} \psi(x).$$

Proof. (See Proposition 3.3 in [29]).
The cases $\delta = 0$ and 1 are obvious. If $\delta \in (0,1)$, we have

$$(-\Delta)^{\delta/2} \psi(x) = -C_N(\delta) \int_{\mathbb{R}^N} \frac{\psi(x+z) - \psi(x)}{|z|^{N+\delta}} dz, \text{ for all } x \in \mathbb{R}^N, \tag{6.15}$$

where

$$C_N(\delta) = \delta \frac{\Gamma(\frac{N+\delta}{2})}{2\pi^{\frac{N}{2}+\delta} \Gamma(1-\frac{\delta}{2})}.$$

Then

$$\psi^{\ell-1}(x) (-\Delta)^{\delta/2} \psi(x) = -C_N(\delta) \int_{\mathbb{R}^N} \frac{\psi^{\ell-1}(x)\psi(x+z) - \psi^\ell(x)}{|z|^{N+\delta}} dz. \tag{6.16}$$

The case $\ell = 1$ is clear (See (4.21)). If $\ell > 1$, then by *Young's inequality* (4.25) we have

$$\psi^{\ell-1}(x)\psi(x+z) \leq \frac{\ell-1}{\ell}\psi^{\ell}(x) + \frac{1}{\ell}\psi^{\ell}(x+z).$$

Therefore

$$\psi^{\ell-1}(x)(-\Delta)^{\delta/2}\psi(x) \geq \frac{-C_N(\delta)}{\ell}\int_{\mathbb{R}^N}\frac{\psi^{\ell}(x+z)-\psi^{\ell}(x)}{|z|^{N+\delta}}dz = \frac{1}{\ell}(-\Delta)^{\delta/2}\psi^{\ell}(x). \quad (6.17)$$

If $\delta \in (1,2)$, we have

$$(-\Delta)^{\delta/2}\psi^{\ell}(x) = -C_N(\delta)\int_{\mathbb{R}^N}\frac{\psi(x+z)-\psi(x)-\nabla\psi(x).z}{|z|^{N+\delta}}dz, \forall x \in \mathbb{R}^N. \quad (6.18)$$

Then

$$\psi^{\ell-1}(x)(-\Delta)^{\delta/2}\psi^{\ell}(x) \geq -C_N(\delta)\int_{\mathbb{R}^N}\frac{\frac{1}{\ell}(\psi^{\ell}(x+z)-\psi^{\ell}(x))-(\nabla\psi(x).z)\psi^{\ell-1}(x)}{|z|^{N+\delta}}dz$$

$$= \frac{-C_N(\delta)}{\ell}\int_{\mathbb{R}^N}\frac{(\psi^{\ell}(x+z)-\psi^{\ell}(x))-\nabla\psi^{\ell}(x).z}{|z|^{N+\delta}}dz = \frac{1}{\ell}(-\Delta)^{\delta/2}\psi^{\ell}(x). \quad (6.19)$$

□

In an open bounded domain Ω, we denote by $\Delta_D^{\beta/2}$ for the fractional Laplacian in Ω with Dirichlet condition. We have

Lemma 6.0.4. *Let λ_k, $(k = 1, ..., +\infty)$ be the eigenvalues for the Laplacian operator in $L^2(\Omega)$ and let φ_k be the eigenfunction corresponding to λ_k. Then*

$$\Delta_D^{\beta/2}\varphi_k = \lambda_k^{\beta/2}\varphi_k$$

and

$$D(\Delta_D^{\beta/2}) = \left\{ u \in L^2(\Omega) \text{ such that } \|\Delta_D^{\beta/2}u\|_{L^2(\Omega)} := \sum_{k=1}^{+\infty}|\lambda_k^{\beta/2}\langle u,\varphi_k\rangle|^2 < +\infty \right\}.$$

So, for $u \in D(\Delta_D^{\beta/2})$ we have

$$\Delta_D^{\beta/2}u = \sum_{k=1}^{+\infty}\lambda_k^{\beta/2}\langle u,\varphi_k\rangle\varphi_k,$$

and then we get the following form of integration by parts

$$\int_{\Omega}u(x)\Delta_D^{\beta/2}v(x)dx = \int_{\Omega}v(x)\Delta_D^{\beta/2}u(x)dx$$

for all $u, v \in D(\Delta_D^{\beta/2})$.

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