**

A mes chers parents

Contents

Notations

Ω: Bounded domain in $\mathbb{R}^N$.
Γ: Topological boundary of Ω.
$x = (x_1, x_1, ..., x_N)$:Generic point of $\mathbb{R}^N$.
$dx = dx_1 dx_1 ... dx_N$: Lebesgue measuring on Ω.
∇u: Gradient of u.
Δu: Laplacien of u.
$f^+, f^- : \max(f, 0), \quad \max(-f, 0)$.
a.e: Almost everywhere.
$p^{'}$: Conjugate of p, i.e $\frac{1}{p} + \frac{1}{p'} = 1$.
$D(\Omega)$: Space of differentiable functions with compact support in Ω.
$D'(\Omega)$: Distribution space.
$C^k(\Omega)$: Space of functions k-times continuously differentiable in Ω.
$C_0(\Omega)$: Space of continuous functions null board in Ω.
$L^p(\Omega)$: Space of functions p-th power integrated on Ω with measure of dx.
$\|f\|_p = \left(\int_\Omega |f(x)|^p \right)^{\frac{1}{p}}$.
$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega), \nabla u \in (L^p(\Omega))^N \right\}$.
$W^{1,p}(\Omega)$: The closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.
$\|u\|_{1,p} = \left(\|u\|_p^p + \|\nabla u\|_p^p \right)^{\frac{1}{p}}$.
$W_0^{1,p}(\Omega)$: The closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.
$W_0^{-1,p'}(\Omega)$: The dual space of $W_0^{1,p}(\Omega)$.
H: Hilbert space.
$H_0^1 = W_0^{1,2}(\Omega)$.
If X is a Banach space, we denote
$L^p(0, T; X) = \left\{ f : (0, T) \to X \text{is measurable; } \int_0^T \|f(t)\|_X^p dt < \infty \right\}$.
$L^\infty(0, T; X) = \left\{ f : (0, T) \to X \text{is measurable; } \sup_{t \in (0,T)} ess \|f(t)\|_X^p \right\}$.
$C^k([0, T]; X)$: Space of functions k-times continuously differentiable for $[0, T] \to X$.
$D([0, T]; X)$: Space of functions continuously differentiable with compact support in $[0, T]$.
$B_X = \{x \in X; \|x\| \leq 1\}$: unit ball.

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Introduction

This thesis is devoted to the study of global existence and asymptotic behavior in time of solutions to nonlinear evolutions equations and systems of hyperbolic and parabolic type. The decreasing of classical energy plays a crucial role in the study of global existence and in the stabilization of various systems. In this thesis, the main objective is to give a global existence and stabilization results. This work consists in three chapters. The first one is concerned preliminaries. The second one, for wave equations with a time-varying delay term. The third one, Decay Property For Solutions In Elastic Solids Without Mechanical Damping.

Chapter 1: Global existence and Asymptotic stability for a coupled viscoelastic wave equation with a time-varying delay term

In this chapter, we consider the following viscoelastic coupled wave equation with a delay term

$$u_{tt} - L_1 u - \int_0^t g_1(t-s) L_1 u(s) ds + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau_2(t)) + f_1(u,v) = 0,$$

$$v_{tt} - L_2 v - \int_0^t g_2(t-s) L_2 v(s) ds + \alpha_1 v_t(x,t) + \alpha_2 v_t(x, t-\tau_2(t)) + f_2(u,v) = 0,$$

in a bounded domain. Under appropriate conditions on μ_1, μ_2, α_1 and α_2, we prove global existence of the solutions by combining the energy method with the Faedo-Galerkin's procedure. Furthermore, we study the asymptotic stability in using an appropriate Lyapunov functional. Finally, we show that the decay rates are the same as those obtained in [23].

Chapter 2: Decay Property For Solutions In Elastic Solids Without Mechanical Damping

In this chapter, we investigate the Cauchy problem for a system of elastic solids with thermal effect. The heat conduction is given by the type III theory of Green and Naghdi. We prove that the dissipation induced by the heat conduction is responsible to the system stabilization, but with slow decay rate.

Chapter 1

Preliminary

In this chapter, we will introduce and state without proofs some important materials needed in the proof of our results,

1.1 Banach Spaces-Definition and properties

We first review some basic facts from calculus in the most important class of linear spaces " Banach spaces".

Definition 1.1.1. . *A Banach space is a complete normed linear space X. Its dual space X' is the linear space of all continuous linear functional $f : X \to \mathbb{R}$.*

Proposition 1.1.1. *(*[33]*) X' equipped with the norm $\|.\|_{X'}$ defined by*

$$\|f\|_{X'} = \sup\{|f(u)| : \|u\| \leq 1\}, \tag{1.1.1}$$

is also a Banach space. We shall denote the value of $f \in X'$ at $u \in X$ by either $f(u)$ or $\langle f, u \rangle_{X',X}$.

Remark 1.1.1. *From X'we construct the bidual or second dual $X'' = (X')'$. Furthermore, with each $u \in X$ we can define $\varphi(u) \in X''$ by $\varphi(u)(f) = f(u), f \in X'$. This satisfies clearly $\|\varphi(x)\| \leq \|u\|$. Moreover, for each $u \in X$ there is an $f \in X'$ with $f(u) = \|u\|$ and $\|f\| = 1$. So it follows that $\|\varphi(x)\| = \|u\|$.*

Definition 1.1.2. . *Since φ is linear we see that*

$$\varphi : X \to X'',$$

is a linear isometry of X onto a closed subspace of X'', we denote this by

$$X \hookrightarrow X''.$$

Definition 1.1.3. . *If φ is onto X'' we say X is reflexive, $X \cong X''$.*

Theorem 1.1.2. *([34]). Let X be Banach space. Then X is reflexive, if and only if,*

$$B_X = \{x \in X : \|x\| \leq 1\},$$

is compact with the weak topology $\sigma(X, X')$. (See the next subsection for the definition of $\sigma(X, X')$).

Definition 1.1.4. . *Let X be a Banach space, and let $(u_n)_{n\in N}$ be a sequence in X. Then u_n converges strongly to u in X if and only if*

$$\lim \|u_n - u\|_X = 0,$$

and this is denoted by $u_n \to u$, or $\lim_{n\to\infty} u_n = u$.

Definition 1.1.5. *The Banach space E is said to be separable if there exists a countable subset D of E which is dense in E, i.e. $\overline{D} = E$.*

Proposition 1.1.3. *If E is reflexive and if F is a closed vector subspace of E, then F is reflexive.*

Corollary 1.1.4. *The following two assertions are equivalent: (i) E is reflexive; (ii) E' is reflexive.*

1.1.1 The weak and weak star topologies

Let X be a Banach space and $f \in X'$. Denote by

$$\begin{aligned} \varphi_f : \quad & X \to \mathbb{R} \\ & x \to \varphi_f(x), \end{aligned} \tag{1.1.2}$$

when f cover X', we obtain a family $(\varphi_f)_{f\in X'}$ of applications to X in $\mathbb{R}$.

Definition 1.1.6. *The weak topology on X, denoted by $\sigma(X, X')$, is the weakest topology on X for which every $(\varphi_f)_{f\in X'}$ is continuous.*

We will define the third topology on X', the weak star topology, denoted by $\sigma(X', X)$. For all $x \in X$. Denote by

$$\begin{aligned} \varphi_f : \quad & X' \to \mathbb{R} \\ & f \to \varphi_x(f) = \langle f, x\rangle_{X',X}, \end{aligned} \tag{1.1.3}$$

when x cover X, we obtain a family $(\varphi_x)_{x\in X'}$ of applications to X' in $\mathbb{R}$.

Definition 1.1.7. . *The weak star topology on X' is the weakest topology on X' for which every $(\varphi_x)_{x\in X'}$ is continuous.*

Remark 1.1.2. *([34]) Since $X \subset X''$, it is clear that, the weak star topology $\sigma(X', X)$ is weakest then the topology $\sigma(X', X'')$, and this later is weakest then the strong topology.*

Definition 1.1.8. *A sequence (u_n) in X is weakly convergent to x if and only if*

$$\lim_{n\to\infty} f(u_n) = f(u),$$

for every $f \in X'$, and this is denoted by $u_n \rightharpoonup u$

Remark 1.1.3. *([34])*

1. *If the weak limit exist, it is unique.*

2. *If $u_n \to u \in X(strongly)$ then $u_n \rightharpoonup u(weakly)$.*

3. *If $dimX < +\infty$, then the weak convergent implies the strong convergent.*

Proposition 1.1.5. *On the compactness in the three topologies in the Banach space X:*

1. *First, the unit ball*
$$B' \equiv \{x \in X : \|x\| \leq 1\}, \tag{1.1.4}$$
in X is compact if and only if $dim(X) < \infty$.

2. *Second, the unit ball B' in X'(The closed subspace of a product of compact spaces) is weakly compact in X' if and only if X is reflexive.*

3. *Third, B' is always weakly star compact in the weak star topology of X'.*

Proposition 1.1.6. *([34]) Let (f_n) be a sequence in X'. We have:*

1. $[f_n \rightharpoonup^* f \text{ in } \sigma(X', X)] \Leftrightarrow [f_n(x) \rightharpoonup^* f(x),\ \forall x \in X]$.

2. *If $f_n \to f(strongly)$ then $f_n \rightharpoonup f, in\ \sigma(X', X'')$,*
If $f_n \rightharpoonup f$ in $\sigma(X', X'')$, then $f_n \rightharpoonup^* f, in\ \sigma(X', X)$.

3. *If $f_n \rightharpoonup^* f$ in $\sigma(X', X)$ then $\|f_n\|$ is bounded and $\|f\| \leq \liminf \|f_n\|$.*

4. *If $f_n \rightharpoonup^* f$ in $\sigma(X', X)$ and $x_n \to x(strongly)$ in X, then $f_n(x_n) \to f(x)$.*

1.1.2 Hilbert spaces

Now, we give some important results on these spaces here.

Definition 1.1.9. *A Hilbert space H is a vectorial space supplied with inner product $\langle u, \upsilon\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.*

Theorem 1.1.7. *(Riesz). If $(H; \langle ., .\rangle)$ is a Hilbert space, $\langle ., .\rangle$ being a scalar product on H, then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $x \in H$ such that $f = \langle x, .\rangle$ and $\|f\|'_H = \|x\|_H$.*

Remark 1.1.4. *From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.*

Theorem 1.1.8. *([34]) Let $(u_n)_{n\in N}$ is a bounded sequence in the Hilbert space H, it posses a subsequence which converges in the weak topology of H.*

Theorem 1.1.9. *([34]) In the Hilbert space, all sequence which converges in the weak topology is bounded.*

Theorem 1.1.10. *([34]) Let $(u_n)_{n\in N}$ be a sequence which converges to u, in the weak topology and $(\upsilon_n)_{n\in N}$ is an other sequence which converge weakly to υ, then*

$$\lim_{n\to\infty} \langle \upsilon_n, u_n\rangle = \langle \upsilon, u\rangle \tag{1.1.5}$$

Theorem 1.1.11. *([34]) Let X be a normed space, then the unit ball*

$$B' \equiv \{x \in X : \|x\| \leq 1\}, \tag{1.1.6}$$

of X' is compact in $\sigma(X', X)$.

1.2 Functional Spaces

1.2.1 The $L^p(\Omega)$ spaces

Definition 1.2.1. *Let $1 \leq p \leq \infty$ and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |f(x)|^p dx < \infty \right\}. \tag{1.2.1}$$

Notation 1.2.1. *If $p = \infty$, we have*

$$\begin{cases} L^{\infty}(\Omega) = \{f : \Omega \to \mathbb{R} \text{ is measurable and there exists a constant } C \text{ such that} \\ \ |f(x)| \leq C \text{ a.e in } \Omega\}. \end{cases}$$

Also, we denote by

$$\|f\|_{\infty} = \Big\{C,\ |f(x)| \leq C \text{ a.e in } \Omega\Big\}. \tag{1.2.2}$$

Notation 1.2.2. *For $p \in \mathbb{R}$ and $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e. $\frac{1}{p} + \frac{1}{q} = 1$.*

Theorem 1.2.3. *([34]) $L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$.*

Remark 1.2.1. *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g\rangle_{L^2(\Omega)} = \int_{\Omega} f(x)g(x)dx, \tag{1.2.3}$$

is a Hilbert space.

Theorem 1.2.4. *([34]) For $1 < p < \infty$, $L^p(\Omega)$ is a reflexive space.*

1.2.2 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Theorem 1.2.5. *([34] Holder's inequality). Let $1 \leq p \leq \infty$. Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^p(\Omega)$ and*

$$\int_{\Omega} |fg| dx \leq \|f\|_p \|g\|_q.$$

Lemma 1.2.6. *([34] Young's inequality). Let $f \in L^p(\mathbb{R})$ and $g \in L^g(\mathbb{R})$ with $1 < p < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$. Then $f * g \in L^r(\mathbb{R})$ and*

$$\|f * g\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}.$$

Lemma 1.2.7. *([34]) Let $1 \leq p \leq r \leq q$, $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$, and $1 \leq \alpha \leq 1$. Then*

$$\|u\|_{L^r} \leq \|u\|_{L^p}^{\alpha} \|u\|_{L^q}^{1-\alpha}.$$

Lemma 1.2.8. *([34]) If $\mu(\Omega) < \infty$, $1 \leq p \leq q \leq \infty$, then $L^q \hookrightarrow L^p$ and*

$$\|u\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p} - \frac{1}{q}} \|u\|_{L^q}.$$

1.2.3 The $W^{m,p}(\Omega)$ spaces

Proposition 1.2.9. *Let Ω be an open domain in $\mathbb{R}^N$. Then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that*

$$\langle T, \varphi\rangle = \int_\Omega f(x)\varphi(x)dx, \text{ for all } \varphi \in D(\Omega),$$

where $1 \leq p \leq \infty$ and it's well-known that f is unique.

Now, we will introduce the Sobolev spaces: The Sobolev space $W^{k,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order k have a finite L^p norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \Big\{ f \in L^p(\Omega); D^\alpha f \in L^p(\Omega). \ \ \forall \alpha; \quad |\alpha| \leq k \Big\}.$$

With this definition, the Sobolev spaces admit a natural norm:

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}, \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^\infty(\Omega)}, \text{ for } p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\| \ . \ \|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)}.$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H^m_0(\Omega)$ (respectively $W^{k,p}_0(\Omega)$).
Now, we introduce a space of functions with values in a space X (a separable Hilbert space). The space $L^2(a, b; X)$ is a Hilbert space for the inner product

$$(f, g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X \, dt$$

We note that $L^\infty(a,b;X) = (L^1(a,b;X))'$. Now, we define the Sobolev spaces with values in a Hilbert space X. For $k \in \ N$, $p \in [1,\infty]$, we set:

$$W^{k,p}(a,b;X) = \left\{ v \in L^p(a,b;X); \frac{\partial v}{\partial x_i} \in L^p(a,b;X). \ \ \forall i \leq k \right\} \ ,$$

The Sobolev space $W^{k,p}(a,b;X)$ is a Banach space with the norm

$$\begin{aligned} \|f\|_{W^{k,p}(a,b;X)} &= \left(\sum_{i=0}^{k} \left\| \frac{\partial f}{\partial x_i} \right\|^p_{L^p(a,b;X)} \right)^{1/p}, \text{ for } p < +\infty \\ \|f\|_{W^{k,\infty}(a,b;X)} &= \sum_{i=0}^{k} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^\infty(a,b;X)}, \qquad \text{for } p = +\infty \end{aligned}$$

The spaces $W^{k,2}(a,b;X)$ form a Hilbert space and it is noted $H^k(0,T;X)$. The $H^k(0,T;X)$ inner product is defined by:

$$(u,v)_{H^k(a,b;X)} = \sum_{i=0}^{k} \int_a^b \left(\frac{\partial u}{\partial x^i}, \frac{\partial v}{\partial x^i} \right)_X dt \ .$$

Theorem 1.2.10. *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

where p^* is given by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ (where $p = n, p^* = \infty$). Moreover there exists a constant $C = C(p,n)$ such that

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corollary 1.2.11. *Let $1 \leq p < n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad \forall q \in [p, p^*]$$

with continuous imbedding.

For the case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad \forall q \in [n, +\infty[$$

Theorem 1.2.12. *Let $p > n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

with continuous imbedding.

Corollary 1.2.13. *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & 1 \leq p < \infty,\ then\ W^{1,p}(\Omega) \subset L^{p^*}(\Omega)\ where\ \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if & p = n,\ then\ W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n,\ then\ W^{1,p}(\Omega) \subset L^\infty(\Omega) \end{array}$$

with continuous imbedding. Moreover, if $p > n$ we have

$$\forall u \in W^{1,p}(\Omega), \quad |u(x) - u(y)| \leq C|x-y|^\alpha \|u\|_{W^{1,p}(\Omega)}\ a.e\ x, y \in \Omega$$

with $\alpha = 1 - \dfrac{n}{p} > 0$ and C is a constant which depend on p, n and Ω. In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Corollary 1.2.14. *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & p < n,\ then\ W^{1,p}(\Omega) \subset L^q(\Omega) \forall q \in [1, p^*[\ where\ \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if & p = n,\ then\ W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n,\ then\ W^{1,p}(\Omega) \subset C(\overline{\Omega}) \end{array}$$

with compact imbedding.

Remark 1.2.2. *We remark in particular that*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

with compact imbedding for $1 \leq p \leq \infty$ and for $p \leq q < p^*$.

Corollary 1.2.15.

$$\begin{array}{ll} if & \dfrac{1}{p} - \dfrac{m}{n} > 0,\ then\ W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)\ where\ \dfrac{1}{q} = \dfrac{1}{p} - \dfrac{m}{n}. \\ if & \dfrac{1}{p} - \dfrac{m}{n} = 0,\ then\ W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [p, +\infty[. \\ if & \dfrac{1}{p} - \dfrac{m}{n} < 0,\ then\ W^{m,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \end{array}$$

with continuous imbedding.

Lemma 1.2.16. *(Sobolev-Poincarés inequality)*

$$If \quad 2 \leq q \leq \frac{2n}{n-2}, n \geq 3 \quad and \quad q \geq 2, n = 1, 2,$$

then

$$\|u\|_q \leq C(q, \Omega)\|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega).$$

Remark 1.2.3. *For all $\varphi \in H^2(\Omega)$, $\Delta\varphi \in L^2(\Omega)$ and for Γ sufficiently smooth, we have*

$$\|\varphi(t)\|_{H^2(\Omega)} \leq C\|\Delta\varphi(t)\|_{L^2(\Omega)}.$$

Proposition 1.2.17. *([34] Green's formula) For all $u \in H^2(\Omega)$, $\upsilon \in H^1(\Omega)$ we have*

$$-\int_\Omega \Delta u \upsilon dx = \int_\Omega \nabla u \nabla \upsilon dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \upsilon d\sigma,$$

where $\dfrac{\partial u}{\partial \eta}$ is a normal derivation of u at Γ.

1.2.4 The $L^p(0,T,X)$ spaces

Let X be a Banach space, denote by $L^p(0,T,X)$ the space of measurable functions

Definition 1.2.2.

$$\begin{aligned} f:\ &]0,T[\to X \\ & t \to f(t). \end{aligned} \tag{1.2.4}$$

such that

$$\left(\int_0^T \|f(t)\|_X^p dt\right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, for\ 1 \leq p < \infty. \tag{1.2.5}$$

If $p = \infty$,

$$\|f\|_{L^p(0,T,X)} = \sup_{t\in]0,T[} ess\|f(t)\|_X. \tag{1.2.6}$$

Theorem 1.2.18. *([34]) The space $L^p(0,T,X)$ is complete.*

We denote by $D'(0,T,X)$ the space of distributions in $]0,T[$ which take its values in X and let us define

$$D'(0,T,X) = \pounds(D]0,T[,X),$$

where $\pounds(\phi,\varphi)$ is the space of the linear continuous applications of ϕ *to* φ. Since $u \in D'(0,T,X)$, we define the distribution derivation as

$$\frac{\partial u}{\partial t}(\varphi) = -u\left(\frac{d\varphi}{dt}\right), \quad \forall\varphi \in D(]0,T[),$$

and since $u \in L^p(0,T,X)$, we have

$$u(\varphi) = \int_0^T u(t)\varphi(t)dt, \quad \forall\varphi \in D(]0,T[),$$

We will introduce some basic results on the $L^p(0,T,X)$ space. These results, will be very useful in the other chapters of this thesis.

Lemma 1.2.19. *([34]) Let $f \in L^p(0,T,X)$ and $\dfrac{\partial f}{\partial t} \in L^p(0,T,X)$, $(1 \le p \le \infty)$, then the function f is continuous from $[0,T]$ to X. i.e. $f \in C^1(0,T,X)$.*

Lemma 1.2.20. *([34]). Let $\varphi =]0,T[\times\Omega$ an open bounded domain in $\mathbb{R}\times\mathbb{R}^n$, and g_μ, g are two functions in $L^q(]0,T[,L^q(\Omega))$, $1 < q < \infty$ such that*

$$\|g_\mu\|_{L^q(]0,T[,L^q(\Omega))} \le C, \forall \mu \in \mathbb{N} \tag{1.2.7}$$

and $g_\mu \to g \quad in \quad \varphi$, then $g_\mu \to g \quad in \quad L^q(\varphi)$.

Theorem 1.2.21. *([34]). $L^p(0,T,X)$ equipped with the norm $\|.\|_{L^q(]0,T[,X)}$, $1 \le p \le \infty$ is a Banach space.*

Proposition 1.2.22. *([34]) Let X be a reflexive Banach space, X' it's dual, and $1 \le p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $L^p(0,T,X)$ is identify algebraically and topologically with $L^q(0,T,X')$.*

Proposition 1.2.23. *([34]) Let X, Y be Banach space, $X \subset Y$ with continuous embedding, then we have*

$$L^p(0,T,X) \subset L^p(0,T,Y),$$

with continuous embedding.

The following compactness criterion will be useful for nonlinear evolution problem, especially in the limit of the nonlinear terms.

Proposition 1.2.24. *([30]) Let B_0, B, B_1 be Banach spaces with $B_0 \subset B \subset B_1$. Assume that the embedding $B_0 \hookrightarrow B$ is compact and $B \hookrightarrow B_1$ are continuous. Let $1 < p, q < \infty$. Assume further that B_0 and B_1 are reflexive. Define*

$$W \equiv \Big\{ u \in L^p(0,T,B_0) : u' \in L^q(0,T,B_1) \Big\}. \tag{1.2.8}$$

Then, the embedding $W \hookrightarrow L^p(0,T,B)$ is compact.

1.2.5 Some Algebraic inequalities

Since our study based on some known algebraic inequalities, we want to recall few of them here.

Lemma 1.2.25. *([34] The Cauchy-Schwartz's inequality) Every inner product satisfies the Cauchy-Schwartz's inequality*

$$\langle x_1, x_2 \rangle \le \|x_1\| \|x_2\|. \tag{1.2.9}$$

The equality sign holds if and only if x_1 and x_1 are dependent.

Lemma 1.2.26. *([34]Young's inequalities). For all $a, b \in \mathbb{R}^+$, we have*

$$ab \leq \alpha a^2 + \frac{1}{4\alpha} b^2 \tag{1.2.10}$$

where α is any positive constant.

Lemma 1.2.27. *([34]) For $a, b \geq 0$, the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{1.2.11}$$

where, $\frac{1}{p} + \frac{1}{q} = 1$.

1.3 Integral Inequalities

We will recall some fundamental integral inequalities introduced by A. Haraux, V. Komornik and A.Guesmia to estimate the decay rate of the energy.

1.3.1 A result of exponential decay

The estimation of the energy decay for some dissipative problems is based on the following lemma:

Lemma 1.3.1. *([35]) Let $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function and assume that there is a constant $A > 0$ such that*

$$\forall t \geq 0, \qquad \int_t^{+\infty} E(\tau)\, d\tau \leq \frac{1}{A} E(t). \tag{1.3.1}$$

Then we have

$$\forall t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}. \tag{1.3.2}$$

Proof of Lemma 1.3.1.
The inequality (1.3.2) is verified for $t \leq \frac{1}{A}$, this follows from the fact that E is a decreasing function. We prove that (1.3.2) is verified for $t \geq \frac{1}{A}$. Introduce the function

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_t^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (1.3.1) we find that

$$\forall t \geq 0, \quad h'(t) + Ah(t) \leq 0.$$

Let

$$T_0 = \sup\{t,\ h(t) > 0\}. \tag{1.3.3}$$

For every $t < T_0$, we have

$$\frac{h'(t)}{h(t)} \leq -A,$$

thus

$$h(0) \leq\ e^{-At} \leq \frac{1}{A}\, E(0)\, e^{-At}, \qquad \text{for} \quad 0 \leq t < T_0. \tag{1.3.4}$$

Since $h(t) = 0$ if $t \geq T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Let $\varepsilon > 0$. As E is positive and decreasing, we deduce that

$$\forall t \geq \varepsilon, \quad E(t) \leq \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} E(\tau)\, d\tau \leq \frac{1}{\varepsilon}\, h(t-\varepsilon) \leq \frac{1}{A\varepsilon}\, E(0)\, e^{\varepsilon t}\, e^{-At}.$$

Choosing $\varepsilon = \frac{1}{A}$, we obtain

$$\forall t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}.$$

The proof of Lemma 1.3.1 is now completed.

1.3.2 A result of polynomial decay

Lemma 1.3.2. *([20]) Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ = [0, +\infty)$) be a non-increasing function and assume that there are two constants $q > 0$ and $A > 0$ such that*

$$\forall t \geq 0, \quad \int_{t}^{+\infty} E^{q+1}(\tau)\, d\tau \leq \frac{1}{A}\, E^q(0) E(t). \tag{1.3.5}$$

Then we have

$$\forall t \geq 0, \quad E(t) \leq E(0) \left(\frac{1+q}{1+A\,q\,t} \right)^{1/q}. \tag{1.3.6}$$

Remark 1.3.1. It is clear that Lemma 1.3.1 is similar to Lemma 1.3.2 in the case of $q = 0$.

Proof of Lemma 1.3.2.
If $E(0) = 0$, then $E \equiv 0$ and there is nothing to prove. Otherwise, replacing the function E by the function $\dfrac{E}{E(0)}$ we may assume that $E(0) = 1$. Introduce the function

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_{t}^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (1.3.5) we find that

$$\forall t \geq 0, \quad -h' \geq (Ah)^{1+q}.$$

where

$$T_0 = \sup\{t,\ h(t) > 0\}.$$

Integrating in $[0, t]$ we obtain that

$$\forall 0 \leq t < T_0, h(t)^{-q} - h(0)^{-\sigma} \geq \sigma\omega^{1+q} t,$$

hence

$$0 \leq t < T_0, \quad h(t) \leq \left(h^{-q}(0) + qA^{1+q}\, t\right)^{-1/q}. \tag{1.3.7}$$

Since $h(t) = 0$ if $t \geq T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Since

$$h(0) \leq \frac{1}{A} E(0)^{1+q} = \frac{1}{A},$$

by (1.3.5), the right-hand side of (1.3.7) is less than or equal to:

$$\left(h^{-q}(0) + qA^{1+q}\, t\right)^{-1/q} \leq \frac{1}{A}(1 + A\, q\, t)^{-1/q}. \tag{1.3.8}$$

From other hand, E being nonnegative and non-increasing, we deduce from the definition of h and the above estimate that:

$$\begin{aligned}\forall s \geq 0,\ E\left(\frac{1}{A} + (q+1)s\right)^{q+1} &\leq \frac{1}{\frac{1}{A} + q + 1} \int_s^{\frac{1}{A}+(q+1)s} E(\tau)^{q+1}\, d\tau \\ &\leq \frac{A}{1 + Aqs} h(s) \leq \frac{A}{1 + Aqs} \frac{1}{A} (1 + Aqs)^{-\frac{1}{q}},\end{aligned}$$

hence

$$\forall S \geq 0, \quad E\left(\frac{1}{A} + (q+1)S\right) \leq \frac{1}{(1 + A\, q\, S)^{1/q}}.$$

Choosing $t = \dfrac{1}{A} + (1+q)s$ then the inequality (1.3.6) follows. Note that letting $q \to 0$ in this theorem we obtain (1.3.6).

1.3.3 New integral inequalities of P. Martinez

The above inequalities are verified only if the energy function is integrable. We will try to resolve this problem by introducing some weighted integral inequalities, so we can estimate the decay rate of the energy when it is slow.

Lemma 1.3.3. *([35]) Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^1 function such that*

$$\phi(0) = 0 \quad \textit{and} \quad \phi(t) \to +\infty \quad \textit{when} \quad t \to +\infty. \tag{1.3.9}$$

Assume that there exist $q \geq 0$ and $A > 0$ such that

$$\int_S^{+\infty} E(t)^{q+1} \phi'(t)\, dt \leq \frac{1}{A}\, E(0)^q E(S), \quad 0 \leq S < +\infty. \tag{1.3.10}$$

then we have

$$\begin{aligned} &\textit{if } q > 0, \quad \textit{then } E(t) \leq E(0) \left(\frac{1+q}{1+q\,A\,\phi(t)} \right)^{\frac{1}{q}}, \ \ \forall t \geq 0, \\ &\textit{if } q = 0, \quad \textit{then } E(t) \leq E(0)\, e^{1-A\,\phi(t)}, \ \ \forall t \geq 0. \end{aligned}$$

Proof of Lemma 1.3.3.

This Lemma is a generalization of Lemma 1.3.1. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$, (we notice that ϕ^{-1} has a meaning by the hypotheses assumed on ϕ). f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain f is non-increasing, $f(0) = E(0)$ and if we set $x = \phi(t)$ we obtain for $0 \leq S < T < +\infty$

$$\begin{aligned} \int_{\phi(S)}^{\phi(T)} f(x)^{q+1}\, dx &= \int_{\phi(S)}^{\phi(T)} E\left(\phi^{-1}(x)\right)^{q+1}\, dx = \int_S^T E(t)^{q+1} \phi'(t)\, dt \\ &\leq \frac{1}{A}\, E(0)^q E(S) = \frac{1}{A}\, E(0)^q f(\phi(S)). \end{aligned}$$

Setting $s = \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\forall s \geq 0, \quad \int_s^{+\infty} f(x)^{q+1}\, dx \leq \frac{1}{A}\ E(0)^q f(s).$$

Thanks to Lemma 1.3.1, we deduce the desired results.

1.3.4 Generalized inequalities of A. Guesmia

Lemma 1.3.4. *([20]) Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ differentiable function, $\lambda \in \mathbb{R}_+$ and $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ convex and increasing function such that $\Psi(0) = 0$. Assume that*

$$\begin{aligned} &\int_s^{+\infty} \Psi(E(t))\, dt \leq E(s), \quad \forall s \geq 0. \\ &E'(t) \leq \lambda E(t), \quad \forall t \geq 0. \end{aligned}$$

Then E satisfies the estimate

$$E(t) \leq e^{\tau_0 \lambda T_0} d^{-1}\Big(e^{\lambda(t-h(t))}\Psi\Big(\psi^{-1}\Big(h(t) + \psi(E(0))\Big)\Big)\Big), \quad \forall t \geq 0,$$

where

$$\psi(t) = \int_t^1 \frac{1}{\Psi(s)}\, ds, \quad \forall t > 0,$$

$$d(t) = \begin{cases} \Psi(t) \quad \textit{if} \quad \lambda = 0, \\ \qquad\qquad\qquad\qquad \forall t \geq 0, \\ \displaystyle\int_0^t \frac{\Psi(s)}{s}\, ds \quad \textit{if} \quad \lambda > 0, \end{cases}$$

$$h(t) = \begin{cases} K^{-1}(D(t)), & \textit{if} \quad t > T_0, \\ 0 & \textit{if} \quad t \in [0, T_0], \end{cases}$$

$$K(t) = D(t) + \frac{\psi^{-1}\big(t + \psi(E(0))\big)}{\Psi\big(\psi^{-1}\big(t + \psi(E(0))\big)\big)}\ e^{\lambda t}, \quad \forall t \geq 0,$$

$$D(t) = \int_0^t e^{\lambda s}\, ds, \quad \forall t \geq 0,$$

$$T_0 = D^{-1}\Big(\frac{E(0)}{\Psi(E(0))}\Big), \quad \tau_0 = \begin{cases} 0, & \textit{if } t > T_0, \\ 1, & \textit{if } t \in [0, T_0]. \end{cases}$$

Remark 1.3.2. If $\lambda = 0$ (that is E is non increasing), then we have

$$E(t) \leq \psi^{-1}\Big(h(t) + \psi(E(0))\Big), \quad \forall t \geq 0 \tag{1.3.11}$$

where $\psi(t) = \displaystyle\int_t^1 \frac{1}{\Psi(s)}, ds$ for $t > 0$, $h(t) = 0$ for $0 \leq t \leq \dfrac{E(0)}{\Psi(E(0))}$ and

$$h^{-1}(t) = t + \frac{\psi^{-1}\Big(t + \psi(E(0))\Big)}{\Psi\Big(\psi^{-1}\Big(t + \psi(E(0))\Big)\Big)}, \quad t > 0.$$

This particular result generalizes the one obtained by Martinez ([35]) in the particular case of $\Psi(t) = dt^{p+1}$ with $p \geq 0,\ d > 0$ and improves the one obtained by Eller, Lagnese and Nicaise ([36]).

Proof of Lemma 1.3.4.
Because $E'(t) \leq \lambda E(t)$ imply $E(t) \leq e^{\lambda(t-t_0)}E(t_0)$ for all $t \geq t_0 \geq 0$, then, if $E(t_0) = 0$ for some $t_0 \geq 0$, then $E(t) = 0$ for all $t \geq t_0$, and then there is nothing to prove in this case. So we assume that $E(t) > 0$ for all $t \geq 0$ without loss of generality. Let:

$$L(s) = \int_s^{+\infty} \Psi(E(t))\, dt, \quad \forall s \geq 0.$$

We have, $L(s) \leq E(s)$, for all $s \geq 0$. The function L is positive, decreasing and of class $C^1(\mathbb{R}_+)$ satisfying

$$-L'(s) = \Psi(E(s)) \geq \Psi(L(s)), \quad \forall s \geq 0.$$

The function ψ is decreasing, then

$$\Big(\psi(L(s))\Big)' = \frac{-L'(s)}{\Psi(L(s))} \geq 1, \quad \forall s \geq 0.$$

Integration on $[0,t]$, we obtain

$$\psi(L(t)) \geq t + \psi(E(0)), \quad \forall t \geq 0. \tag{1.3.12}$$

Since Ψ is convex and $\Psi(0) = 0$, we have

$$\Psi(s) \leq \Psi(1)s, \ \forall s \in [0,1] \quad and \quad \Psi(s) \geq \Psi(1)s, \ \forall s \geq 1,$$

then $\lim_{t\to 0} \psi(t) = +\infty$ and $[\psi(E(0)), +\infty[\subset \ Image\,(\psi)$. Then (1.3.12) imply that

$$L(t) \leq \psi^{-1}\Big(t + \psi(E(0))\Big), \quad \forall t \geq 0. \tag{1.3.13}$$

Now, for $s \geq 0$, let

$$f_s(t) = e^{-\lambda t}\int_s^t e^{\lambda \tau}\, d\tau, \quad \forall t \geq s.$$

The function f_s is increasing on $[s, +\infty[$ and strictly positive on $]s, +\infty[$ such that

$$f_s(s) = 0 \quad and \quad f_s'(t) + \lambda f_s(t) = 1, \quad \forall t \geq s \geq 0,$$

and the function d is well defined, positive and increasing such that:

$$d(t) \leq \Psi(t) \quad and \quad \lambda t d'(t) = \lambda\Psi(t), \quad \forall t \geq 0,$$

then

$$\begin{aligned}
\partial_\tau\Big(f_s(\tau)d(E(\tau))\Big) &= f_s'(\tau)d(E(\tau)) + f_s(\tau)E'(\tau)d'(E(\tau))\\
&\leq \Big(1 - \lambda f_s(\tau)\Big)\Psi(E(\tau)) + \lambda f_s(\tau)\Psi(E(\tau))\\
&= \Psi(E(\tau)), \quad \forall \tau \geq s \geq 0.
\end{aligned}$$

Integrating on $[s,t]$, we obtain

$$L(s) \geq \int_s^t \Psi(E(\tau))\, d\tau \geq f_s(t)d(E(t)), \quad \forall t \geq s \geq 0. \tag{1.3.14}$$

Since $\lim_{t\to+\infty} d(s) = +\infty$, $d(0) = 0$ and d is increasing, then (1.3.13) and (1.3.14) imply

$$E(t) \leq d^{-1}\left(\inf_{s\in[0,t[} \frac{\psi^{-1}\Big(s + \psi(E(0))\Big)}{f_s(t)}\right), \quad \forall t > 0. \tag{1.3.15}$$

Now, let $t > T_0$ and

$$J(s) = \frac{\psi^{-1}\Big(s + \psi(E(0))\Big)}{f_s(t)}, \quad \forall s \in [0, t[.$$

The function J is differentiable and we have

$$J'(s) = f_s^{-2}(t)\Big[e^{-\lambda(t-s)}\psi^{-1}\Big(s + \psi(E(0))\Big) - f_s(t)\Psi\Big(\psi^{-1}\Big(s + \psi(E(0))\Big)\Big)\Big].$$

Then

$$J'(s) = 0 \;\Leftrightarrow\; K(s) = D(t) \quad and \quad J'(s) < 0 \;\Leftrightarrow\; K(s) < D(T).$$

Since $K(0) = \dfrac{E(0)}{\Psi(E(0))}$, $D(0) = 0$ and K and D are increasing (because ψ^{-1} is decreasing and $s \mapsto \dfrac{s}{\Psi(s)}$, $s > 0$, is non increasing thanks to the fact that Ψ is convex). Then, for $t > T_0$,

$$\inf_{s\in[0,t[} J(s) = J\Big(K^{-1}(D(t))\Big) = J(h(t)).$$

Since h satisfies $J'(h(t)) = 0$, we conclude from (1.3.15) our desired estimate for $t > T_0$. For $t \in [0, T_0]$, we have just to note that $E'(t) \leq \lambda E(t)$ and the fact that $d \leq \Psi$ implies

$$E(t) \leq e^{\lambda t}E(0) \leq e^{\lambda T_0}E(0) \leq e^{\lambda T_0}\Psi^{-1}\Big(e^{\lambda t}\Psi(E(0))\Big) \leq e^{\lambda T_0}d^{-1}\Big(e^{\lambda t}\Psi(E(0))\Big).$$

Remark 1.3.3. Under the hypotheses of Lemma 1.3.4, we have $\lim_{t\to+\infty} E(t) = 0$. Indeed, we have just to choose $s = \frac{1}{2}t$ in (1.3.15) instead of $h(t)$ and note that $d^{-1}(0) = 0$, $\lim_{t\to+\infty} \psi^{-1}(t) = 0$ and $\lim_{t\to+\infty} f_{\frac{1}{2}t}(t) > 0$.

Lemma 1.3.5. *([Guesmia 20]) Let $E : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a differentiable function, $a : \mathbb{R}^+ \longrightarrow \mathbb{R}^{+*}$ and $\lambda : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ two continuous functions. Assume that there exist $r \geq 0$ such that*

$$\int_s^{+\infty} E^{r+1}(t)dt \leq a(s)E(s), \forall s \geq 0 \tag{1.3.16}$$

$$E'(t) \leq \lambda(t)E(t), \quad \forall t \geq 0 \tag{1.3.17}$$

Then E satisfies for all $t \geq 0$,

$$E(t) \leq \frac{E(0)}{\omega(0)}\omega(h(t)exp(\widetilde{\lambda}(t) - \widetilde{\lambda}(h(t)))exp(-\int_0^{h(t)} \omega(\tau)d\tau), \;\; if\; r = 0$$

and

$$E(t) \leq \omega(h(t))exp(\widetilde{\lambda}(t) - \widetilde{\lambda}(h(t)))\left[\left(\frac{\omega(0)}{E(0)}\right)^r + r\int_0^{h(t)} \omega(\tau)^{r+1}d\tau\right]^{-1/r} \quad if\; r > 0$$

where $\quad \widetilde{\lambda}(t) = \displaystyle\int_0^t \lambda(\tau)d\tau.$

Proof of Lemma 1.3.5.

If $E(s)=0$ or $a(s)=0$ for one $s\geq 0$, the first inequality implies $E(t)=0$ for $t\geq s$,then we can suppose that $E(t)>0$ and $a(t)>0$ for $t\geq 0$.

By putting $\omega=\dfrac{1}{a}$ and $\Psi(s)=\displaystyle\int_s^{+\infty} E^{r+1}(t)dt$, we have

$$\Psi(s)\leq \frac{1}{\omega(s)}E(s),\qquad \forall s\geq 0. \tag{1.3.18}$$

The function Ψ is decreasing, positive and of class C^1 on $\mathbb{R}^+$ and verifies:

$$\Psi'(s)=-E^{r+1}(s)\leq -(\omega(s)\Psi(s))^{r+1},\quad \forall s\geq 0$$

then

$$\Psi(s)\leq \Psi(0)exp\left(\int_0^s \omega(\tau)d\tau\right)\leq \frac{E(0)}{\omega(0)}exp\left(\int_0^s \omega(\tau)d\tau\right)\qquad \text{if } r=0 \tag{1.3.19}$$

$$\Psi(s)\leq \left((\frac{\omega(0)}{E(0)})^r+\int_0^s(\omega(\tau))^{r+1}d\tau\right)^{-1/r}\qquad \text{if } r>0 \tag{1.3.20}$$

Now we put for all $s\geq 0$,

$$f_s(t)=exp(-(r+1)\widetilde{\lambda}(t))\int_s^t exp((r+1)\widetilde{\lambda}(\tau))d\tau,\quad \forall t\geq s \tag{1.3.21}$$

where $f_s(s)=0$ and $f_s'(t)+(r+1)\lambda(t)f_s(t)=1$, $\forall t\geq s\geq 0$. Under the second hypothesis in the lemma, we deduce

$$E^{r+1}(t)\geq \partial_t(f_s(t)E^{r+1}(t));\quad \forall t\geq,\quad s\geq 0 \tag{1.3.22}$$

hence

$$\Psi(s)\geq \int_s^{g(s)} E^{r+1}(t)\geq f_s(g(s))E^{r+1}(g(s)));\quad \forall s\geq 0 \tag{1.3.23}$$

where $g:\mathbb{R}^+\longrightarrow \mathbb{R}^{+*}$ with $I_s(g(s))=0$ and I_s is defined by

$$I_s(t)=(\omega(s))^{r+1}\int_s^t exp((r+1)\widetilde{\lambda}(\tau))d\tau$$

Let $t>g(0)$ and $s=h(t)$ where

$$h(t)=\begin{cases} 0 \quad \text{if} \quad t\in[0,g(0)] \\ \max g^{-1}(t) \quad \text{if} \quad t\in]g(0),+\infty[\end{cases}$$

Hence we have $g(s)=t$ and we deduce from (1.3.23) that for all $t\geq g(0)$, we have

$$\Psi(h(t))\geq f_{h(t)}(t)E^{r+1}(t)=\left(exp(-(r+1)\widetilde{\lambda}(t))\int_{h(t)}^t exp((r+1)\widetilde{\lambda}(\tau))d\tau\right)E^{r+1}(t)$$

We conclude from (1.3.19) and (1.3.20) that for all $t > g(0)$, we obtain

$$E(t) \le \frac{E(0)}{\omega(0)} exp(\widetilde{\lambda}(t)) \left(\int_{h(t)}^{t} exp(\widetilde{\lambda}(\tau)) d\tau \right)^{-1} exp \left(- \int_{0}^{h(t)} \omega(\tau) d\tau \right) \text{ if } r = 0$$

and

$$\begin{aligned} E(t) \le exp(\widetilde{\lambda}(t)) \left(\int_{h(t)}^{t} exp((r+1)\widetilde{\lambda}(\tau)) d\tau \right)^{\frac{-1}{r+1}} \times \\ \left((\tfrac{\omega(0)}{E(0)})^r + r \int_0^{h(t)} (\omega(\tau))^{r+1} d\tau \right)^{\frac{-1}{r(r+1)}} \quad \text{if } r > 0 \end{aligned}$$

The fact that $I_{h(t)}^{t} = I_s(g(s)) = 0$, we get the result of the lemma for $t > g(0)$. If $t \in [0, g(0)]$ the second inequality of the lemma implies that

$$E(t) \le E(0) exp(\widetilde{\lambda}(t))$$

Since $h(t) = 0$ on $[0, g(0)]$ then $E(0)exp(\widetilde{\lambda}(t))$ is identically equal to the left hand side of the results of the lemma.That conclude the proof.

Lemma 1.3.6. *([Guesmia 35]) Let $E : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a differentiable function, $a_1, a_2 \in \mathbb{R}^{+*}$ and $a_3, \lambda, r, p \in \mathbb{R}^+$ such that*

$$a_3 \lambda (r+1) < 1 \quad \textit{and for all} \quad 0 \le s \le T < +\infty,$$

$$\int_s^T E^{r+1}(t) dt \le a_1(s) E(s) + a_2 E^{p+1}(s) + a_3 E^{r+1}(T),$$

$$E'(t) \le \lambda E(t), \qquad \forall t \ge 0$$

Then there exist two positive constants ω and c such that for all $t \ge 0$,

$$E(t) \le c e^{-\omega t}, \ \textit{if } r = 0$$

$$E(t) \le c(1+t)^{-1/r}, \ \textit{if } r > 0 \quad \textit{and } \lambda = 0$$

$$E(t) \le c(1+t)^{\frac{-1}{r(r+1)}}, \ \textit{if } r > 0 \quad \textit{and } \lambda > 0$$

Proof of Lemma 1.3.6

We show that E verifies the inequality (1.3.16). Applying the lemma (1.3.5), we have

$$\begin{aligned} a_3 E^{r+1}(T) &= a_3 \int_s^T E'^{r+1}(t) dt + a_3 E^{r+1}(s) \\ &\le a_3 (r+1) \int_s^T \lambda E^{r+1}(t) dt + a_3 E^{r+1}(s) \end{aligned}$$

Under (1.3.16), we obtain:

$$\int_s^{+\infty} E^{r+1}(t)dt \leq b(s)E(s), \quad \forall s \geq 0 \tag{1.3.24}$$

where

$$b(s) = \frac{a_1 + a_2 E^p(s) + a_3 E^r(s)}{1 - a_3\lambda(r+1)}, \quad \forall s \geq 0$$

We consider the function f_0 defined in (1.3.21)and integrating on $[0, s]$ the inequality

$$E^{r+1}(t) \geq \partial_t(f_0(t)E^{r+1}(t)), \quad \forall t \geq 0$$

we obtain under (1.3.24)

$$b(0)E(0) \geq \int_0^s E^{r+1}(t)dt \geq f_0(s)E^{r+1}(s), \quad \forall s \geq 0$$

then

$$E(s) \leq \left(\frac{b(0)E(0)}{f_0(s)}\right)^{\frac{1}{r+1}}, \quad \forall s \geq 0$$

on the other hand, the conditions of the lemma implies that

$$E(s) \leq E(0)exp(\widetilde{\lambda}(s), \quad \forall s \geq 0$$

Hence

$$E(s) \leq \min\left\{E(0)exp(\widetilde{\lambda}(s), \left(\frac{b(0)E(0)}{f_0(s)}\right)^{\frac{1}{r+1}}\right\} = d(s), \quad \forall s \geq 0$$

d is continuous and positive and

$$b(s) \leq \frac{a_1 + a_2(d(s))^p + a_3(d(s))^r}{1 - a_3\lambda(r+1)}, \quad \forall s \geq 0$$

Hence we can conclude from (1.3.24) the first inequality (1.3.16) of the lemma (1.3.5) with

$$a(s) = \frac{a_1 + a_2(d(s))^p + a_3(d(s))^r}{1 - a_3\lambda(r+1)}, \quad \forall s \geq 0.$$

This completes the proof.

1.4 Existence Methods

1.4.1 Faedo-Galerkin's approximations

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle ., . \rangle$ and the associated norm $\|.\|$.

$$(P) \qquad \begin{cases} u''(t) + A(t)u(t) = f(t), \quad t \in [0, T] \\ (x, 0) = u_0(x), \;\; u'(x, 0) = u_1(x); \end{cases}$$

where u and f are unknown and given function, respectively, mapping the closed interval $[0, T] \subset \mathbb{R}$ into a real separable Hilbert space H. $A(t)$ $(0 \leq t \leq T\ )$ are linear bounded operators in H acting in the energy space $V \subset H$.

Assume that $\langle A(t)u(t), v(t) \rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t; ., .)$ is a bilinear continuous in V. The problem (P) can be formulated as: Found the solution $u(t)$ such that

$$(\widetilde{P}) \qquad \begin{cases} u \in C([0, T]; V), u' \in C([0, T]; H) \\ \langle u''(t), v \rangle + a(t; u(t), v) = \langle f, v \rangle \text{ in } D'(]0, T[) \\ u_0 \in V \;,\;\; u_1 \in H; \end{cases}$$

This problem can be resolved with the approximation process of Fadeo-Galerkin.

Let V_m a sub-space of V with the finite dimension d_m, and let $\{w_{jm}\}$ one basis of V_m such that

1. $V_m \subset V(\dim V_m < \infty), \forall m \in \mathbb{N}$
2. $V_m \to V$ such that, there exist a dense subspace ϑ in V and for all $v \in \vartheta$ we can get sequence $\{u_m\}_{m \in \mathbb{N}} \in V_m$ and $u_m \to u$ in V.
3. $V_m \subset V_{m+1}$ and $\overline{\cup_{m \in \mathbb{N}} V_m} = V$.

we define the solution u_m of the approximate problem

$$(P_m) \qquad \begin{cases} u_m(t) = \displaystyle\sum_{j=1}^{d_m} g_j(t) w_{jm} \\ u_m \in C([0, T]; V_m), u'_m \in C([0, T]; V_m) \quad , u_m \in L^2(0, T; V_m) \\ \langle u''_m(t), w_{jm} \rangle + a(t; u_m(t), w_{jm}) = \langle f, w_{jm} \rangle, \quad 1 \leq j \leq d_m \\ u_m(0) = \displaystyle\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \;,\;\; u'_m(0) = \displaystyle\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \end{cases}$$

where

$$\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \longrightarrow u_0 \text{ in V as } m \longrightarrow \infty$$

$$\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \longrightarrow u_1 \text{ in V as } m \longrightarrow \infty$$

By virtue of the theory of ordinary differential equations, the system (P_m) has unique local solution which is extend to a maximal interval $[0, t_m[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$ to obtain one solution defined for all $t > 0$.

1.4.2 A priori estimation and convergence

Using the following estimation

$$\|u_m\|^2 + \|u'_m\|^2 \leq C \left\{ \|u_m(0)\|^2 + \|u'_m(0)\|^2 + \int_0^T \|f(s)\|^2 ds \right\} \ ; \quad 0 \leq t \leq T$$

and the Gronwall lemma we deduce that the solution u_m of the approximate problem (P_m) converges to the solution u of the initial problem (P). The uniqueness proves that u is the solution.

1.4.3 Gronwall's lemma

Lemma 1.4.1. *Let $T > 0$, $g \in L^1(0,T)$, $g \geq 0$ a.e and c_1, c_2 are positives constants. Let $\varphi \in L^1(0,T)$ $\varphi \geq 0$ a.e such that $g\varphi \in L^1(0,T)$ and*

$$\varphi(t) \leq c_1 + c_2 \int_0^t g(s)\varphi(s) ds \quad a.e \ in \ (0,T).$$

then, we have

$$\varphi(t) \leq c_1 exp \left(c_2 \int_0^t g(s) ds \right) \quad a.e \ in \ (0,T).$$

1.4.4 Semigroups approach

Definition 1.4.1. *([37]). Let X be a Banach space. A one parameter family $T(t)$ for $0 \leq t < \infty$ of bounded linear operators from X into X is a semigroup bounded linear operator on X if*

- *$T(0) = I$, (I is the identity operator on X).*
- *$T(t+s) = T(t).T(s)$ for every $t, s \geq 0$ (the semigroup property).*

A semigroup of bounded linear operators, $T(t)$, is uniformly continuous if

$$\lim_{t\to 0} \|T(t) - I\| = 0.$$

The linear operator A defined by

$$D(A) = \left\{ x \in X; \lim_{t\to 0} \frac{T(t)x - x}{t} \quad exists \right\}$$

and

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt}\Big|_{t=0}, \quad \forall x \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$ and $D(A)$ is the domain of A.

Theorem 1.4.2. *([37])(Lumer-Phillips) Let A be a linear operator with dense domain $D(A)$ in X*

- *If A is dissipative and there is a $\lambda_0 > 0$ such that the range , $R(\lambda_0 I - A) = X$, then A is the infinitesimal generator of a C_0 semigroup of contraction on X.*
- *If A is the infinitesimal generator of a C_0 semigroup of contractions on X then $R(\lambda_0 I - A) = X, \quad \forall \lambda > 0$ and A is dissipative.*

Chapter 2

Global existence and Asymptotic stability for a coupled viscoelastic wave equation with a time-varying delay term

2.1 Introduction

Our main interest lies in the following system of viscoelastic equations :

$$\begin{cases} u_{tt} - L_1 u - \displaystyle\int_0^t g_1(t-s) L_1 u(s) ds + \mu_1 u_t(x,t)) \\ +\mu_2 u_t(x, t-\tau_1(t)) + f_1(u,\upsilon) = 0, \qquad in\ \Omega \times (0,\infty), \\ \upsilon_{tt} - L_2 \upsilon - \displaystyle\int_0^t g_2(t-s) L_2 \upsilon(s) ds + \alpha_1 \upsilon_t(x,t) \\ +\alpha_2 \upsilon_t(x, t-\tau_2(t)) + f_2(u,\upsilon) = 0, \qquad in\ \Omega \times (0,\infty), \\ u(x,t) = 0,\ \upsilon(x,t) = 0, \qquad on \quad \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x),\ \upsilon(x,0) = \upsilon_0(x), \qquad x \in \Omega, \\ u_t(x,0) = u_1(x),\ \upsilon_t(x,0) = \upsilon_1(x), \qquad x \in \Omega, \\ u_t(x, t-\tau_1(t)) = \phi_0(x, t-\tau_1(t)), \qquad x \in \Omega,\ t \geq 0, \\ \upsilon_t(x, t-\tau_2(t)) = \phi_1(x, t-\tau_2(t)),\ \tau_2(t) \neq 0, \qquad x \in \Omega,\ t \geq 0. \end{cases} \tag{2.1.1}$$

Where

$$L_1 u = -div(A_1 \nabla u) = -\sum_{i,j=1}^{N} \left(a_{1i,j}(x) \frac{\partial u}{\partial x_i} \right); \qquad \frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{N} (a_{1i,j}(x) \frac{\partial u}{\partial x_i} \nu_i,$$

and

$$L_2 \upsilon = -div(A_2 \nabla \upsilon) = -\sum_{i,j=1}^{N} \left(a_{2i,j}(x) \frac{\partial \upsilon}{\partial x_i} \right); \qquad \frac{\partial \upsilon}{\partial \nu} = \sum_{i,j=1}^{N} (a_{2i,j}(x) \frac{\partial \upsilon}{\partial x_i} \nu_i.$$

Here Ω is a bounded domain in $\mathbb{R}^n, n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$ and $g_1, g_2 : \mathbb{R}^+ \to \mathbb{R}^+$, $\phi_i(.,.) : \mathbb{R}^2 \to \mathbb{R}$ $i = 1, 2$, are given functions which will be specified later. Moreover $\tau_2(t) > 0$ is a time delay, where $\mu_1, \alpha_1, \alpha_2, \mu_2$ are positive real numbers and the initial data $(u_0, u_1, \phi_0), (\upsilon_0, \upsilon_1, \phi_1)$ belonging to a suitable space. Here u and υ denote the transverse displacements of waves. This problem arises in the theory of viscoelasticity and describes the interaction of two scalar fields (see [15]).
To motivate our work, let us start with the wave equation proposed by the Authors of [3]. They considered the following coupled system of quasilinear viscoelastic equation in canonical form without delay terms in $\Omega \times (0, +\infty)$

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \gamma_1 \Delta u_{tt} + \displaystyle\int_0^t g_1(t-s)\Delta u(s)ds + f_1(x, u) = 0, \\ |\upsilon_t|^\rho \upsilon_{tt} - \Delta \upsilon - \gamma_2 \Delta \upsilon_{tt} + \displaystyle\int_0^t g_2(t-s)\Delta \upsilon(s)ds + f_2(x, u) = 0, \end{cases} \tag{2.1.2}$$

where Ω is a bounded domain in R^n $(n \geq 1)$with a smooth boundary $\partial\Omega, \gamma_1, \gamma_2 \geq 0$ are constants and ρ is a real number such that $0 < \rho < \frac{2n}{(n-2)}$ if $n \geq 3$ or $\rho > 0$ if $n = 1, 2$. The functions u_0, u_1, υ_0 and υ_1 are given initial data. The relaxations functions g_1 and g_2 are continuous functions and $f_1(u, \upsilon), f_2(u, \upsilon)$ represent the nonlinear terms. The authors proved the energy decay result using the perturbed energy method. Many authors considered the initial boundary value problem in $\Omega \times (0, +\infty)$ as follows

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g_1(t-s)\Delta u(s)ds + h_1(u_t) = f_1(x, u), \\ \upsilon_{tt} - \Delta \upsilon + \displaystyle\int_0^t g_2(t-s)\Delta \upsilon(s)ds + h_2(\upsilon_t) = f_2(x, u), \end{cases} \tag{2.1.3}$$

when the viscoelastic terms g_i $(i = 1, 2.)$ are not taken into account in (2.1.3). Rammaha and Sakatusathian [4] obtained several results related to local and global existence of a weak solution. By using the same technique as in [5], they showed that any weak solution blow-up in finite time with negative initial energy. Later Said-Houari [6] extended this blow up result to positive initial energy. Conversely, in the presence of the memory term $(g_i \neq 0$, $i = 1, 2.)$, there are numerous results related to the asymptotic behavior and blow up of solutions of viscoelastic systems. For example, Liang and Gao [7] studied problem (2.1.3) with $h_1(u_t) = -\Delta u_t$, $h_2(\upsilon_t) = -\Delta u_t$. They obtained that, under suitable conditions on the functions $g_i, f_i, i = 1, 2$, and certain initial data in the stable set, the decay rate of the energy functions is exponential. On the contrary, for certain initial data in the unstable set, there are solutions with positive initial energy that blow-up in finite time. For $h_1(u_t) = |u_t|^{m-1}u_t$ and $h_2(\upsilon_t) = |\upsilon_t|^{r-1}\upsilon_t$, Hun and Wang [8] established several results related to local existence, global existence and finite time blow-up (the initial energy $E(0) < 0$).

This latter has been improved by Messaoudi and Said-Houari [14] by considering a larger class of initial data for which the initial energy can take positive values. On the other hand, Muhammad I. Mustafa [26] considered the following problem in $\Omega \times (0, +\infty)$

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + f_1(\upsilon, u) = 0, \\ \upsilon_{tt} - \Delta \upsilon + \int_0^t g_2(t-s)\Delta \upsilon(s)ds + f_2(\upsilon, u) = 0, \end{cases} \tag{2.1.4}$$

and proved the well-posedness and energy decay result for wider class of relaxation functions. The author in [24] have studied the following problem in $\Omega \times (0, +\infty)$

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + \left(|u|^k + |\upsilon|^q\right)|u_t|^{m-1}u_t = f_1(\upsilon, u), \\ \upsilon_{tt} - \Delta \upsilon + \int_0^t h(t-s)\Delta \upsilon(s)ds + \left(|u|^\theta + |\upsilon|^\rho\right)|\upsilon_t|^{r-1}\upsilon_t = f_2(\upsilon, u), \end{cases} \tag{2.1.5}$$

with degenerate damping and source terms in a bounded domain. Under some assumptions on the relaxation functions, degenerate damping and source terms, he obtained the decay rate of the energy function for certain initial data.
It is widely known that delay effects, which arise in many practical problems, source of some instabilities, in this way Datko and Nicaise ([11, 20, 21]) showed that a small delay in a boundary control turns a well-behave hyperbolic system into a wild one which in turn, becomes a source of instability, where they proved that the energy is exponentially stable under the condition

$$\mu_2 < \mu_1.$$

Motivated by the previous works, in the present paper, we analyze the influence of the viscoelastic, damping and delay terms on the solutions to (2.1.1). Under suitable assumptions on the functions $g_i(.)$, $f_i(.,.)$ $(i = 1.2)$, the initial data and the parameters in the equations, to the best of our knowledge, there is no result concerning coupled system with the presence of delay term and elliptic operator. We establish several results concerning local and global existence, asymptotic stability and the boundedness of the solutions to (2.1.1).
Our work is organized as follows. In section 2, we present the preliminaries and some lemmas. In section 3, the existence result is obtained. Finally in section 4, decay property is derived.

2.2 Preliminary Results

In this section, we present some material and asumptions for the proof of our results. We will use embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 \leq q \leq \frac{2n}{n-2}$, if $n \geq 3$ and $q \geq 2$, if $n = 1, 2$; and

$L^r(\Omega) \hookrightarrow L^q(\Omega)$, for $q < r$. We will use, in this case, the same embedding constant denoted by c_s such that

$$\|\nu\|_q \leq c_s\|\nabla\nu\|_2, \quad \|\nu\|_q \leq c_s\|\nu\|_r \quad \text{for} \quad \nu \in H_0^1(\Omega).$$

For studying the problem (2.1.1) we will need the following assumptions. For the relaxation function g_i for $i = 1, 2$. We assume

(A_0) : $g_1(t)$, $g_2(t)$: $[0, \infty) \to [0, \infty)$ are of class C^2 and satisfying, for $s \geq 0$

$$g_1(0) = g_{10} > 0,\ 1 - \int_0^\infty g_1(s)ds = l_1 > 0, \quad g_2(0) = g_{20} > 0,\ 1 - \int_0^\infty g_2(s)ds = l_2 > 0,$$

and there exist a nonincreasing functions $\zeta_1(t)$ and $\zeta_2(t)$ such that

$$g_1'(t) \leq -\zeta_1(t)g_1(t), \quad g_2'(t) \leq -\zeta_2(t)g_2(t),\ \forall t \geq 0. \tag{2.2.1}$$

(A_1) : The matrix $A_1 = (a_{1i,j}(x))$, $A_2 = (a_{2i,j}(x))$, where $a_{1i,j}, a_{2i,j} \in C^1(\overline{\Omega})$, are symmetric and there exists a constants $a_{01}, a_{02} > 0$ such that for all $x \in \overline{\Omega}$ and $\eta = (\eta_1, ...\eta_N) \in \mathbb{R}^N$ we have

$$\sum_{i,j=1}^{N} a_{1i,j}(x)\eta_j\eta_i \geq a_{01}|\eta|^2, \quad \sum_{i,j=1}^{N} a_{2i,j}(x)\eta_j\eta_i \geq a_{02}|\eta|^2. \tag{2.2.2}$$

(A_2) : We take $f_1(u, \upsilon)$, $f_2(u, \upsilon)$ as in [23]

$$f_1(u, \upsilon) = a|u + \upsilon|^{p-1}(u + \upsilon) + b|u|^{\frac{p-3}{2}}|\upsilon|^{\frac{p+1}{2}}u, \tag{2.2.3}$$

$$f_2(u, \upsilon) = a|u + \upsilon|^{p-1}(u + \upsilon) + b|\upsilon|^{\frac{p-3}{2}}|u|^{\frac{p+1}{2}}\upsilon. \tag{2.2.4}$$

With $a, b > 0$. Further, one can easily verify that

$$uf_1(u, \upsilon) + \upsilon f_2(u, \upsilon) = (p + 1)F(u, \upsilon), \forall(u, \upsilon) \in \mathbb{R}^2.$$

Where

$$F(u, \upsilon) = \frac{1}{(p+1)}\left(a|u + \upsilon|^{p+1} + 2b|u\upsilon|^{\frac{p+1}{2}}\right), \quad f_1(u, \upsilon) = \frac{\partial F}{\partial u}, \quad f_2(u, \upsilon) = \frac{\partial F}{\partial \upsilon}.$$

And there exists C, such that

$$\left|\frac{\partial f_i}{\partial u}(u, \upsilon)\right| + \left|\frac{\partial f_i}{\partial \upsilon}(u, \upsilon)\right| \leq C\left(|u|^{p-1} + |\upsilon|^{p-1}\right), \quad i = 1, 2 \quad where \quad 1 \leq p < 6.$$

(A_3) :

$$if\ n = 1, 2; \quad p \geq 3 \quad if\ n = 3; \quad p = 3. \tag{2.2.5}$$

(A_4) : τ_i is a function such that

$$\tau_i \in W^{2.\infty}([0, T]), \quad \forall\ T > 0, \quad i = 1, 2, \tag{2.2.6}$$

$$0 < \tau_0 \leq \tau_2(t) \leq \tau_1, \ \forall t > 0, \tag{2.2.7}$$

$$\tau_1'(t) \leq d < 1, \ \tau_2'(t) \leq d' < 1, \tag{2.2.8}$$

where τ_0 and τ_1 are two positive constants.
(A_5) :

$$\mu_2 < \sqrt{1-d}\mu_1, \tag{2.2.9}$$

$$\alpha_2 < \sqrt{1-d'}\alpha_1. \tag{2.2.10}$$

As in [28] we choose ξ_1 and ξ_2 such that

$$\frac{\mu_2}{\sqrt{1-d}} < \xi_1 < 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}, \tag{2.2.11}$$

$$\frac{\alpha_2}{\sqrt{1-d'}} < \xi_2 < 2\alpha_1 - \frac{\alpha_2}{\sqrt{1-d'}}. \tag{2.2.12}$$

Lemma 2.2.1. *([23]) Suppose that (2.2.5) holds. Then there exists $\rho > 0$ such that for any $(u, \upsilon) \in (H_0^1(\Omega))^2$, we have*

$$\|u+\upsilon\|_{p+1}^{p+1} + 2\|u\upsilon\|_{\frac{p+1}{2}}^{\frac{p+1}{2}} \leq \rho\left(l_1\|\nabla u\|_2^2 + l_2\|\nabla \upsilon\|_2^2\right)^{\frac{p+1}{2}}.$$

Lemma 2.2.2. *([23]) There exist two positive constants c_0 and c_1 such that*

$$c_0\Big(|u|^{p+1} + |\upsilon|^{p+1}\Big) \leq F(u,\upsilon) \leq c_1\Big(|u|^{p+1} + |\upsilon|^{p+1}\Big), \quad \forall (u,\upsilon) \in \mathbb{R}^2.$$

Remark 2.2.1. *for seeking of simplicity, we take $a = b = 1$ in* $(2.2.3) - (2.2.4)$*.*

2.3 Global existence

In order to prove the existence of solutions of problem (2.1.1), we introduce the new variables z_1, z_2 as in [28]

$$z_1(x, k_1, t) = u_t(x, t - \tau_1(t)k_1), \qquad x \in \Omega, \ k_1 \in (0,1),$$

$$z_2(x, k_2, t) = \upsilon_t(x, t - \tau_2(t)k_2), \qquad x \in \Omega, \ k_2 \in (0,1),$$

which implies that

$$\tau_1(t)z_{1t}(x, k_1, t) + (1 - \tau_1'(t))z_{k_1}(x, k_1, t) = 0, \quad \text{in} \quad \Omega \times (0,1) \times (0,\infty),$$

$$\tau_2(t)z_{2t}(x, k_2, t) + (1 - \tau_2'(t))z_{k_2}(x, k_2, t) = 0, \quad \text{in} \quad \Omega \times (0,1) \times (0,\infty),$$

therefore, problem (2.1.1) is equivalent to

$$
\begin{cases}
u_{tt} - L_1 u + \displaystyle\int_0^t g_1(t-s) L_1 u(s) ds & \\
+\mu_1 u_t(x,t) + \mu_2 z_1(x,1,t) + f_1(u,\upsilon) = 0, & in\ \Omega \times (0,\infty), \\
\upsilon_{tt} - L_2 \upsilon + \displaystyle\int_0^t g_2(t-s) L_2 \upsilon(s) ds & \\
+\alpha_1 \upsilon_t(x,t) + \alpha_2 z_2(x,1,t) + f_2(u,\upsilon) = 0, & in\ \Omega \times (0,\infty), \\
& \\
\tau_1(t) z_{1t}(x,k_1,t) + (1-\tau_1'(t)) z_{k_1}(x,k_1,t) = 0, & in \quad \Omega \times (0,1) \times (0,\infty), \\
\tau_2(t) z_{2t}(x,k_2,t) + (1-\tau_2'(t)) z_{k_2}(x,k_2,t) = 0, & in \quad \Omega \times (0,1) \times (0,\infty), \\
z_1(x,0,t) = u_t(x,t), x \in \Omega, & t > 0, \\
z_2(x,0,t) = \upsilon_t(x,t), x \in \Omega, & t > 0, \\
z_1(x,k_1,0) = \phi_0(x, -\tau_2(0) k_1), & x \in \Omega, \\
z_2(x,k_2,0) = \phi_1(x, -\tau_2(0) k_2), & x \in \Omega, \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\
\upsilon(x,0) = \upsilon_0(x), \upsilon_t(x,0) = \upsilon_1(x), & x \in \Omega, \\
u(x,t) = 0, \upsilon(x,t) = 0,\ x \in \partial\Omega, & t \geq 0.
\end{cases}
\tag{2.3.1}
$$

In the following, we will give sufficient conditions for the well-posedness of problem (2.3.1) by using the Fadeo-Galerkin's method.

Theorem 2.3.1. *Let* $(u_0, \upsilon_0) \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right)^2, (u_1, \upsilon_1) \in (H_0^1(\Omega))^2$ *and* $(\phi_0, \phi_1) \in \left(L^2(\Omega \times (0,1)\right)^2$ *satisfying the compatibility conditions*

$$
\phi_0(.,0) = u_1, \quad \phi_1(.,0) = \upsilon_1.
$$

Assume that the hypotheses $(A_0) - (A_5)$ *hold. Then there exists a unique weak solution* $((u,z_1),(\upsilon,z_2))$ *of (2.3.1) such that for* $T > 0$ *we have*

$$
u, \upsilon \in C\left([-\tau_1(0), T]; H^2(\Omega) \cap H_0^1(\Omega)\right) \cap C^1\left([-\tau_2(0), T]; L^2(\Omega)\right),
$$

$$
u_t, \upsilon_t \in L^2\left([-\tau_2(0), T]; H_0^1(\Omega)\right) \cap L^2\left([-\tau_2(0), T] \times \Omega\right).
$$

Proof. We use Faedo-Galerkin's method to construct approximate solution. Let $T > 0$ be fixed and denote by V_n the space generated by the set $\{w_n, n \in N\}$ is a basis of $H^2(\Omega) \cap H_0^1(\Omega)$. We define also for $1 \leq j \leq n$, the sequence $\varphi_j(x,n)$ as follows $\varphi_j(x,0) = w_j(x)$. Then we may extend $\varphi_j(x,0)$ by $\varphi_j(x,n)$ over $L^2(\Omega \times [0,1])$ and denote Z_n to be the space generated by $\{\varphi_1, ... \ \varphi_n\}$, $(n = 1, 2, 3...)$. We construct approximate solutions $(u^n(t), z_1^n(t), \upsilon^n(t), z_2^n(t))$ $(n = 1, 2, 3...)$ in the form

$$u^n(t) = \sum_{j=1}^{n} u_{n,j}(t) w_j(x), \quad z_1^n(t) = \sum_{j=1}^{n} z_{1n,j}(t)\varphi_j(x, k_1),$$

$$\upsilon^n(t) = \sum_{j=1}^{n} \upsilon_{n,j}(t) w_j(x), \quad z_2^n(t) = \sum_{j=1}^{n} z_{2n,j}(t)\varphi_j(x, k_2),$$

where $((u^n(t), z_1^n(t)), (\upsilon^n(t), z_2^n(t)))$ are the solutions of the following approximate problem corresponding to (2.3.1) then $((u^n(t), z_1^n(t)), (\upsilon^n(t), z_2^n(t)))$ verify the following system of ODEs:

$$\begin{aligned} &\langle u_{tt}^n(t), w_j\rangle_\Omega + a_1(u^n(t), w_j) + \left\langle \int_0^t g_1(t-s) A_1 \nabla u^n(s) ds, \nabla w_j \right\rangle_\Omega \\ &+ \langle \mu_1 u_t^n(x,t), w_j\rangle_\Omega + \langle \mu_2 z_1^n(x,1,t), w_j\rangle_\Omega + \langle f_1(u^n(t), \upsilon^n(t)), w_j\rangle_\Omega = 0, \end{aligned} \tag{2.3.2}$$

and

$$\begin{aligned} &\langle \upsilon_{tt}^n(t), w_j\rangle_\Omega + a_2(\upsilon^n(t), w_j) + \left\langle \int_0^t g_2(t-s) A_2 \nabla \upsilon^n(s) ds, \nabla w_j \right\rangle_\Omega \\ &+ \langle \alpha_1 \upsilon_t^n(x,t), w_j\rangle_\Omega + \langle \alpha_2 z_2^n(x,1,t), w_j\rangle_\Omega + \langle f_2(u^n(t), \upsilon^n(t)), w_j\rangle_\Omega = 0, \end{aligned} \tag{2.3.3}$$

for $j = 1.....n$. More specifically

$$\begin{aligned} u^n(0) &= \sum_{j=1}^{n} u_{n,j}(0) w_j, \upsilon^n(0) = \sum_{j=1}^{n} \upsilon_{n,j}(0) w_j, u_t^n(0) \\ &= \sum_{j=1}^{n} u'_{n,j}(0) w_j, \upsilon_t^n(0) = \sum_{j=1}^{n} \upsilon'_{n,j}(0) w_j, \end{aligned} \tag{2.3.4}$$

where

$$u^{n,j}(0) = \langle u^0, w_j\rangle, \upsilon^{n,j}(0) = \langle \upsilon^0, w_j\rangle, \upsilon_t^{n,j}(0) = \langle \upsilon^1, w_j\rangle, \upsilon_t^{n,j}(0) = \langle \upsilon^1, w_j\rangle,$$

$j = 1, ..., n$. Obviously, $u^n(0) \to u^0$, $\upsilon^n(0) \to \upsilon^0$ strongly in $H_0^1(\Omega)$, $u_t^n(0) \to u^1$, $\upsilon_t^n(0) \to \upsilon^1$ strongly in $L^2(\Omega)$ as $n \to \infty$.

$$\left\langle \tau_2(t) z_{1t}^n(x, k_1, t) + (1 - \tau_2'(t)) z_{1k_1}^n(x, k_1, t), \varphi_j \right\rangle_\Omega = 0, \tag{2.3.5}$$

$$\left\langle \tau_2(t) z_{2t}^n(x,k_2,t) + (1-\tau_2'(t)) z_{2k_2}^n(x,k_2,t), \varphi_j \right\rangle_\Omega = 0, \tag{2.3.6}$$

$$z_1^n(0) = z_1^n \to \phi_0,\ z_2^n(0) = z_2^n \to \phi_1 \ in\ L^2\left(\Omega \times (0,1)\right). \tag{2.3.7}$$

For $j = 1.....n$. Where

$$a_1(\psi(t), \phi(t)) = \sum_{i,j=1}^{N} \int_\Omega a_{1i,j}(x) \frac{\partial \psi(t)}{\partial x_j} \frac{\partial \phi(t)}{\partial x_i} dx = \int_\Omega A_1 \nabla \psi(t) \phi(t) dx,$$

$$a_2(\psi(t), \phi(t)) = \sum_{i,j=1}^{N} \int_\Omega a_{2i,j}(x) \frac{\partial \psi(t)}{\partial x_j} \frac{\partial \phi(t)}{\partial x_i} dx = \int_\Omega A_2 \nabla \psi(t) \phi(t) dx.$$

By using hypothesis (A_1), we verify that the bilinear forms $a_1(.,.), a_2(.,.) : H_0^1(\Omega) \times H_0^1(\Omega) \to R$ are symmetric and continuous. On the other hand, from (2.2.2) for $\zeta = \nabla \psi$, we get

$$a_1(\psi(t), \psi(t)) \geq a_{01} \int_\Omega \sum_{i,j=1}^{N} \left| \frac{\partial \psi}{\partial x_i} \right|^2 dx = a_{01} \|\nabla \psi(t)\|_2^2, \tag{2.3.8}$$

$$a_2(\psi(t), \psi(t)) \geq a_{02} \int_\Omega \sum_{i,j=1}^{N} \left| \frac{\partial \psi}{\partial x_i} \right|^2 dx = a_{02} \|\nabla \psi(t)\|_2^2. \tag{2.3.9}$$

Which implies that $a_1(.,.)$, $a_2(.,.)$ are coercive. We will utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for $(u^n(t), z_1^n(t), \upsilon^n(t), z_2^n(t))$ such that $n \in N$.

Estimate 1. Multiplying equation (2.3.2) by $u_{n,j}'(t)$ and the equation (2.3.3) by $\upsilon_{n,j}'(t)$ then summing with respect to j, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|u_t^n(t)\|_2^2 + \|\upsilon_t^n(t)\|_2^2 + a_1(u^n(t), u_t^n(t)) + a_2(\upsilon^n(t), \upsilon_t^n(t))\right] \\
&+ \frac{1}{2}\frac{d}{dt}\int_\Omega F(u^n(t), \upsilon^n(t)) dx + \frac{\mu_1}{2}\|u_t^n(t)\|_2^2 + \frac{\alpha_1}{2}\|\upsilon_t^n(t)\|_2^2 \\
&+ \mu_2 \int_\Omega z_1(x,1,t) u_t^n(x,t) dx + \alpha_2 \int_\Omega z_2(x,1,t) \upsilon_t^n(x,t) dx \\
&- \int_0^t g_1(t-s) \int_\Omega A_1 \nabla u^n(s) \nabla u_t^n(t) dx ds \\
&- \int_0^t g_2(t-s) \int_\Omega A_2 \nabla \upsilon^n(s) \nabla \upsilon_t^n(t) dx ds = 0.
\end{aligned} \tag{2.3.10}$$

Note that

$$a_1(u^n(t), u_t^n(t)) = \frac{1}{2}\frac{d}{dt} a_1(u^n(t), u^n(t)), \tag{2.3.11}$$

$$a_2(\upsilon^n(t), \upsilon_t^n(t)) = \frac{1}{2}\frac{d}{dt} a_2(\upsilon^n(t), \upsilon^n(t)). \tag{2.3.12}$$

Following the same technique as in [31], we can obtain

$$
\begin{aligned}
&\int_0^t g_1(t-s)\int_\Omega A_1\nabla u^n(s)\nabla u_t^n(t)dxds\\
&=\sum_{i,j=1}^N\int_0^t\int_\Omega g_1(t-s)a_{1i,j}(x)\frac{\partial u^n(s)}{\partial x_j}\frac{\partial u_t^n(t)}{\partial x_i}dxds\\
&=\sum_{i,j=1}^N\int_0^t\int_\Omega g_1(t-s)a_{1i,j}(x)\frac{\partial u^n(t)}{\partial x_i}\frac{\partial u_t^n(t)}{\partial x_i}dxds\\
&-\sum_{i,j=1}^N\int_0^t\int_\Omega g_1(t-s)a_{1i,j}(x)\left(\frac{\partial u^n(t)}{\partial x_i}-\frac{\partial u_t^n(s)}{\partial x_j}\right)\frac{\partial u_t^n(t)}{\partial x_i}dxds\\
&=\frac{1}{2}\int_0^t g_1(t-s)\left(\frac{d}{dt}a_1(u^n(t),u^n(t))ds\right)\\
&-\frac{1}{2}\int_0^t g_1(t-s)\left(\frac{d}{dt}a_1(u^n(t)-u^n(s),u^n(t)-u^n(s))ds\right)\\
&=\frac{1}{2}\frac{d}{dt}\left(\int_0^t g_1(t-s)a_1(u^n(t),u^n(t)\right)ds\\
&-\frac{1}{2}\frac{d}{dt}\left(\int_0^t g_1(t-s)a_1(u^n(t)-u^n(s),u^n(t)-u^n(s))\right)ds\\
&-\frac{1}{2}g_1(t)a_1(u^n(t),u^n(t))+\frac{1}{2}\int_0^t g'(t-s)a_1(u^n(t)-u^n(s),u^n(t)-u^n(s))ds\\
&=-\frac{1}{2}\frac{d}{dt}(g_1\circ u^n)(t)+\frac{1}{2}(g_1'\circ u^n)(t)\\
&+\frac{1}{2}\frac{d}{dt}\left[a_1(u^n(t),u^n(t))\int_0^t g_1(s)ds\right]-\frac{1}{2}g_1(t)a_1(u^n(t),u^n(t)),
\end{aligned}
\tag{2.3.13}
$$

where

$$
(g_1\circ u^n)(t)=\int_0^t g_1(t-s)a_1(u^n(t)-u^n(s),u^n(t)-u^n(s))ds. \tag{2.3.14}
$$

In the same way

$$
\begin{aligned}
&\int_0^t g_2(t-s)\int_\Omega A_2\nabla\upsilon^n(s)\nabla\upsilon_t^n(t)dxds=-\frac{1}{2}\frac{d}{dt}(g_2\circ\upsilon^n)(t)+\frac{1}{2}(g_2'\circ\upsilon^n)(t)\\
&+\frac{1}{2}\frac{d}{dt}\left[a_2(\upsilon^n(t),\upsilon^n(t))\int_0^t g_2(s)ds\right]-\frac{1}{2}g_2(t)a_2(\upsilon^n(t),\upsilon^n(t)).
\end{aligned}
\tag{2.3.15}
$$

Inserting (2.3.11)- (2.3.15) in (2.3.10) and integrating over $(0,t)$, we get

$$
\begin{aligned}
&\frac{1}{2}\|u_t^n(t)\|_2^2+\frac{1}{2}\|\upsilon_t^n(t)\|_2^2+\int_\Omega F(u^n(t),\upsilon^n(t))dx\\
&+\frac{1}{2}\left(1-\int_0^t g_1(s)ds\right)a_1(u^n(t),u^n(t))\\
&+\frac{1}{2}\left(1-\int_0^t g_2(s)ds\right)a_2(\upsilon^n(t),\upsilon^n(t))+\frac{1}{2}(g_1ou^n)(t)+\frac{1}{2}(g_2o\upsilon^n)(t)\\
&+\mu_1\int_0^t\|u_s^n(s)\|_2^2ds+\alpha_1\int_0^t\|\upsilon_s^n(s)\|_2^2ds+\mu_2\int_0^t\int_\Omega z_1^n(x,1,s)u_s^n(s)dxds\\
&+\alpha_2\int_0^t\int_\Omega z_2^n(x,1,s)\upsilon_s^n(s)dxds+\frac{1}{2}\int_0^t g_1(s)a_1(u_t^n(t),u_t^n(t))ds\\
&+\frac{1}{2}\int_0^t g_2(s)a_2(\upsilon_t^n(t),\upsilon_t^n(t))ds-\frac{1}{2}\int_0^t(g_1'ou^n)(s)ds-\frac{1}{2}\int_0^t(g_2'o\upsilon^n)(s)ds=0.
\end{aligned}
\tag{2.3.16}
$$

Now, we multiply (2.3.5) by $\xi_1(t)e^{-k_1\tau_2(t)}z_{1n,j}(t)$, summing with respect to j and integrating over $\Omega\times(0,1)$ to obtain

$$
\begin{aligned}
&\xi_1(t)e^{-k_1\tau_2(t)}\int_\Omega\int_0^1 z_{1t}^n z_1^n(x,k_1,t)dk_1dx\\
&=-\tfrac{\xi_1(t)e^{-k_1\tau_2(t)}}{2\tau_0}\int_\Omega\int_0^1(1-\tau_2'(t)k_1)\frac{\partial}{\partial k_1}\left(z_1^n(x,k_1,t)\right)^2dk_1dx.
\end{aligned}
\tag{2.3.17}
$$

Consequently,

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\xi_1(t)e^{-k_1\tau_2(t)}}{2}\int_\Omega\int_0^1(z_1^n(x,k_1,t))^2dk_1dx\right)\\
&=-\tfrac{\xi_1(t)}{2}\int_0^1\int_\Omega\frac{\partial}{\partial k_1}((1-\tau_2'(t)k_1)e^{-k_1\tau_2(t)}(z_1^n(x,k_1,t))^2dk_1dx\\
&+\tfrac{\xi_1'(t)e^{-k_1\tau_2(t)}}{2}\int_0^1\int_\Omega(z_1^n(x,k_1,t))^2dk_1dx\\
&=\tfrac{\xi_1(t)}{2}\int_\Omega\left[(z_1^n(x,0,t))^2-(z_1^n(x,1,t))^2\right]e^{-\tau_2(t)}dx\\
&+\tfrac{\xi_1(t)\tau_2'(t)e^{-\tau_2(t)}}{2}\int_\Omega(z_1^n(x,1,t))^2dx+\frac{\xi_1'(t)e^{-k_1\tau_2(t)}}{2}\int_0^1\int_\Omega(z_1^n(x,k_1,t))^2dk_1dx.
\end{aligned}
\tag{2.3.18}
$$

In the same way for (2.3.6), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\xi_2(t)e^{-k_2\tau_2(t)}}{2}\int_\Omega\int_0^1(z_2^n(x,k_2,t))^2dk_2dx\right)\\
&=\tfrac{\xi_2(t)}{2}\int_\Omega\left[(z_2^n(x,0,t))^2-(z_2^n(x,1,t))^2\right]e^{-\tau_2(t)}dx\\
&+\tfrac{\xi_2(t)\tau_2'(t)e^{-\tau_2(t)}}{2}\int_\Omega(z_2^n(x,1,t))^2dx+\frac{\xi_2'(t)e^{-k_2\tau_2(t)}}{2}\int_0^1\int_\Omega(z_2^n(x,k_2,t))^2dk_2dx.
\end{aligned}
\tag{2.3.19}
$$

Due to Young's inequality, we have

$$\mu_2 \int_\Omega z_1^n(x,1,t)u_t^n(x,t)dx \leq \frac{\mu_2}{2\sqrt{1-d}}\|u_t^n(t)\|_2^2 + \frac{\mu_2\sqrt{1-d}}{2}\|z_1^n(x,1,t)\|_2^2, \tag{2.3.20}$$

$$\alpha_2 \int_\Omega z_2^n(x,1,t)\upsilon_t^n(x,t)dx \leq \frac{\alpha_2}{2\sqrt{1-d'}}\|\upsilon_t(t)\|_2^2 + \frac{\alpha_2\sqrt{1-d'}}{2}\|z_2^n(x,1,t)\|_2^2. \tag{2.3.21}$$

Under the fact that

$$\int_0^t (g_1' \circ u_s^n)(s)ds \leq 0, \qquad \int_0^t (g_2' \circ \upsilon_s^n)(s)ds \leq 0,$$

we conclude that

$$(g_1 \circ u_t^n)(t) - \int_0^t (g_1' \circ u_s^n)(s)ds + \int_0^t g_1(s)a_1(u_t^n(t), u_t^n(t))ds \geq 0, \tag{2.3.22}$$

$$(g_2 \circ \upsilon_t^n)(t) - \int_0^t (g_2' \circ \upsilon_s^n)(s)ds + \int_0^t g_2(s)a_2(\upsilon_t^n(t), \upsilon_t^n(t))ds \geq 0. \tag{2.3.23}$$

Summing (2.3.16), (2.3.18), (2.3.19), (2.3.20), (2.3.21), (2.3.22) and (2.3.23),we get

$$\begin{aligned}
&E^n(t) + \sigma_1 \int_0^t \|u_s^n(s)\|_2^2 ds + \sigma_2 \int_0^t \|\upsilon_s^n(s)\|_2^2 ds \\
&+\sigma_3 \int_0^t \|z_1^n(x,1,s)\|_2^2 ds + \sigma_4 \int_0^t \|z_2^n(x,1,s)\|_2^2 ds \\
&-\frac{1}{2}\int_0^t (g_1' o u)(s)ds + \frac{1}{2}\int_0^t g_1(s)a_1(u(s),u(s))ds \\
&-\frac{1}{2}\int_0^t (g_2' o \upsilon)(s)ds + \frac{1}{2}\int_0^t g_2(s)a_2(\upsilon(s),\upsilon(s))ds \\
&\leq \frac{1}{2}\left[\|u^{1n}\|_2^2 + \|u^{1n}\|_2^2 + \int_\Omega F(u^n(0),\upsilon^n(0))dx\right] = E^n(0).
\end{aligned} \tag{2.3.24}$$

Such that

$$\sigma_1 = \left(\mu_1 - \frac{\xi_1}{2} - \frac{\mu_2}{2\sqrt{1-d}}\right), \quad \sigma_3 = \left(\frac{\xi_1(1-\tau_2'(t))}{2} - \frac{\mu_2\sqrt{1-d}}{2}\right),$$

$$\sigma_2 = \left(\alpha_1 - \frac{\xi_2}{2} - \frac{\alpha_2}{2\sqrt{1-d'}}\right), \quad \sigma_4 = \left(\frac{\xi_2(1-\tau_2'(t))}{2} - \frac{\alpha_2\sqrt{1-d'}}{2}\right).$$

Where $E(t)$ is the energy of the solution defined by the following formula

$$\begin{aligned}
E^n(t) &= \frac{1}{2}\|u_t^n(t)\|_2^2 + \frac{1}{2}\|\upsilon_t^n(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right)a_1(u^n(t),u^n(t)) \\
&+\frac{1}{2}\left(1 - \int_0^t g_2(s)ds\right)a_2(\upsilon^n(t),\upsilon^n(t)) + \frac{1}{2}(g_1 o u^n)(t) \\
&+\frac{e^{-k_1\tau_2(t)}\xi_1(t)}{2}\int_\Omega\int_0^1 (z_1^n(x,k_1,t))^2 dk_1 dx + \frac{1}{2}(g_2 o \upsilon^n)(t) \\
&+\frac{e^{-k_2\tau_2(t)}\xi_2(t)}{2}\int_\Omega\int_0^1 (z_2^n(x,k_2,t))^2 dk_2 dx + \int_\Omega F(u^n(t),\upsilon^n(t))dx.
\end{aligned} \tag{2.3.25}$$

We shall prove that the problem (2.3.2)−(2.3.6) admits a local solution in $[0, t_m)$, $0 < t_m < T$, for an arbitrary $T > 0$. The extension of the solution to the whole interval $[0, T]$ is a consequence of the estimates below.

Estimate 2. As in [26] replacing w_j by $-\Delta w_j$ in (2.3.2)-(2.3.6), multiplying the equation (2.3.2) by $u'_{n,j}(t)$ and equation (2.3.3) by $\upsilon'_{n,j}(t)$, summing over j from 1 to n then using (2.3.13)and (2.3.15), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|\nabla u_t^n(t)\|_2^2 + a_{01}\left(1-\int_0^t g_1(s)ds\right)\|\Delta u^n(t)\|_2^2 + (g_1 o \Delta u^n)(t)\right]\\&+\frac{1}{2}g_1(t)\|\Delta u^n(t)\|_2^2 - \frac{1}{2}(g_1' o \Delta u^n)(t) + \frac{\mu_1}{2}\|\nabla u_t^n(t)\|_2^2\\&+\mu_2\int_\Omega z_1^n(x,1,t)\Delta u_t^n(t)dx + \int_\Omega f_1(u^n(t),\upsilon^n(t))\Delta u^n(t)dx = 0,\end{aligned} \tag{2.3.26}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|\nabla \upsilon_t^n(t)\|_2^2 + a_{02}\left(1-\int_0^t g_2(s)ds\right)\|\Delta \upsilon^n(t)\|_2^2 + (g_2 o \Delta \upsilon^n)(t)\right]\\&+\frac{1}{2}g_2(t)\|\Delta \upsilon^n(t)\|_2^2 - \frac{1}{2}(g_2' o \Delta \upsilon^n)(t) + \frac{\alpha_1}{2}\|\nabla \upsilon_t^n(t)\|_2^2\\&+\alpha_2\int_\Omega z_2^n(x,1,t)\Delta \upsilon_t^n(t)dx + \int_\Omega f_2(u^n(t),\upsilon^n(t))\Delta \upsilon^n(t)dx = 0.\end{aligned} \tag{2.3.27}$$

Replacing φ_j by $-\Delta\varphi_j$ in (2.3.5)-(2.3.6), multiplying (2.3.5) by $z_{1n,j}(t)$, summing over j, it follows that

$$\left(\frac{\tau_2(t)}{2(1-\tau_2'(t)k_1)}\right)\frac{d}{dt}\|\nabla z_1^n(t)\|_2^2 + \frac{1}{2}\frac{d}{dk_1}\|\nabla z_1^n(t)\|_2^2 = 0. \tag{2.3.28}$$

Then

$$\frac{1}{2}\frac{d}{dt}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\|\nabla z_1^n(t)\|_2^2\right) - \frac{1}{2}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)'\|\nabla z_1^n(t)\|_2^2 + \frac{1}{2}\frac{d}{dk_1}\|\nabla z_1^n(t)\|_2^2 = 0. \tag{2.3.29}$$

In the same way for (2.3.6)

$$\frac{1}{2}\frac{d}{dt}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\|\nabla z_2^n(t)\|_2^2\right) - \frac{1}{2}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)'\|\nabla z_2^n(t)\|_2^2 + \frac{1}{2}\frac{d}{dk_2}\|\nabla z_2^n(t)\|_2^2 = 0. \tag{2.3.30}$$

Using Young's inequality, summing (2.3.26), (2.3.29) and (2.3.30) then integrating over $(0, t)$, we get

$$\begin{aligned}
&\frac{1}{2}\left[\|\nabla u_t^n(t)\|_2^2+\|\nabla \upsilon_t^n(t)\|_2^2+a_{01}\left(1-\int_0^t g_1(s)ds\right)\|\Delta u^n(t)\|_2^2\right]\\
&+\int_0^1 \frac{\tau_2(t)}{1-\tau_2'(t)k_1}\|\nabla z_{1t}^n(x,k_1,t)\|_{L^2(\Omega)}^2 dk_1-\frac{1}{2}\int_0^t (g_1' o\Delta \upsilon^n)(s)ds\\
&+\int_0^1 \frac{\tau_2(t)}{1-\tau_2'(t)k_2}\|\nabla z_{2t}^n(x,k_2,t)\|_{L^2(\Omega)}^2 dk_2+\frac{1}{2}(g_2 o\Delta \upsilon^n)(t)\\
&+\frac{a_{02}}{2}\left(1-\int_0^t g_2(s)ds\right)\|\Delta \upsilon^n(t)\|_2^2+\frac{1}{2}(g_1 o\Delta u^n)(t)\\
&+\frac{1}{2}\int_0^t g_1(s)\|\Delta u^n(s)\|_2^2 ds+\frac{1}{2}\int_0^t g_2(s)\|\Delta \upsilon^n(s)\|_2^2 ds\\
&-\frac{1}{2}\int_0^t (g_2' o\Delta \upsilon^n)(s)ds+\frac{\mu_1}{2}\int_0^t \|\nabla u_s^n(s)\|_2^2 ds+\frac{\alpha_1}{2}\int_0^t \|\nabla \upsilon_s^n(s)\|_2^2 ds\\
&+\mu_2\int_0^t\int_\Omega |\nabla z_1^n(x,1,s)|^2 dsdx+\mu_2\int_0^t \|\nabla u_s^n(s)\|_2^2 ds\\
&+\alpha_2\int_0^t\int_\Omega |\nabla z_2^n(x,1,s)|^2 dsdx+\alpha_2\int_0^t \|\nabla \upsilon_s^n(s)\|_2^2 ds\\
&\le \int_\Omega \begin{pmatrix} f_1(u^n,\upsilon^n)\Delta u^n - f_1(u^0,\upsilon^0)\Delta u^0 \\ \\ f_2(u^n,\upsilon^n)\Delta \upsilon^n - f_2(u^0,\upsilon^0)\Delta \upsilon^0 \end{pmatrix} dx\\
&-\int_0^t\int_\Omega \begin{pmatrix} \frac{\partial}{\partial u} f_1(u^n,\upsilon^n)u_s^n\Delta u^n+\frac{\partial}{\partial \upsilon} f_1(u^n,\upsilon^n)\upsilon_s^n\Delta u^n \\ \\ +\frac{\partial}{\partial \upsilon} f_2(u^n,\upsilon^n)u_s^n\Delta \upsilon^n+\frac{\partial}{\partial \upsilon} f_2(u^n,\upsilon^n)\upsilon_s^n\Delta \upsilon^n \end{pmatrix} dxds\\
&+\int_0^t\int_0^1\left(\frac{\tau_2(s)}{1-\tau_2'(s)k_1}\right)\|\nabla z_{1s}^n(x,k_1,s)\|_{L^2(\Omega)}^2 dk_1+\|\Delta u^{1n}\|_2^2\\
&+\int_0^t\int_0^1\left(\frac{\tau_2(s)}{1-\tau_2'(s)k_2}\right)\|\nabla z_{2s}^n(x,k_2,s)\|_{L^2(\Omega)}^2 dk_2+\|\Delta \upsilon^{1n}\|_2^2\\
&+\int_0^t \|\nabla u_s^n(s)\|_2^2 ds+\int_0^t \|\nabla \upsilon_s^n(s)\|_2^2 ds+\frac{1}{2}\|\nabla u^{1n}\|_2^2+\frac{1}{2}\|\nabla \upsilon^{1n}\|_2^2.
\end{aligned} \tag{2.3.31}$$

Where $c_0=\frac{1}{2}\|\nabla u^{1n}\|_2^2+\frac{1}{2}\|\nabla \upsilon^{1n}\|_2^2+\|\Delta u^{1n}\|_2^2+\|\Delta \upsilon^{1n}\|_2^2$ is a positive constant. We just need to estimate the right hand terms of (2.3.31). Applying Holder's inequality and Sobolev's embedding theorem inequality, we infer

$$\begin{aligned}
&\left|\int_\Omega f_1(u^n(t),\upsilon^n(t))\Delta u^n(t)dx\right| \le \int_\Omega \left(|u^n|^p+|\upsilon^n|^p+|u^n|^{\frac{p-1}{2}}|\upsilon^n|^{\frac{p+1}{2}}\right)|\Delta u^n|dx\\
&\le C\left(\|u^n\|_{2p}^p+\|\upsilon^n\|_{2p}^p+\|u^n\|_{\frac{p-1}{2}}^{3(p-1)}\|u^n\|_{\frac{p+1}{2}}^{\frac{3(p+1)}{2}}\right)\|\Delta u^n\|_2\\
&\le C\left(\|\nabla u^n\|_2^p+\|\nabla \upsilon^n\|_2^p+\|\nabla u^n\|_2^{\frac{p-1}{2}}\|\nabla \upsilon^n\|_2^{\frac{p+1}{2}}\right)\|\Delta u^n\|_2\\
&\le C\left(\|\Delta u^n\|_2^2+\|\nabla u^n\|_2^{2p}+\|\nabla \upsilon^n\|_2^{2p}+\|\nabla u^n\|_2^{p-1}\|\nabla u^n\|_2^{p+1}\right)\\
&\le C\|\Delta u^n\|_2^2+c.
\end{aligned} \tag{2.3.32}$$

Likewise, we obtain

$$\left|\int_\Omega f_2(u^n(t),\upsilon^n(t))\Delta\upsilon^n(t)dx\right| \leq C\|\Delta\upsilon^n\|_2^2+c. \tag{2.3.33}$$

Now we estimate $I=\displaystyle\int_\Omega \frac{\partial}{\partial u}f_1(u^n(t),\upsilon^n(t))u_t^n(t)\Delta u^n(t)dx$, then, by (A_2) and Young's inequality, we get

$$\begin{aligned}|I| &\leq c\int_\Omega \left(|u^n|^{p-1}+|\upsilon^n|^{p-1}\right)|u_t^n||\Delta u^n|dx\\ &\leq c\|u_t^n\|_2\|u^n\|_{2p}^{p-1}\|\Delta u^n\|_{2p}+\|\upsilon^n\|_{2p}^{p-1}\|u_t^n\|_{2p}\|\Delta u^n\|_2.\end{aligned} \tag{2.3.34}$$

Then

$$\begin{aligned}&|I|\leq c\left(\|\nabla u^n\|_2^{p-1}+\|\nabla\upsilon^n\|_2^{p-1}\right)\|\nabla u_t^n\|_2\|\Delta u^n\|_2\\ &\leq c\|\nabla u_t^n\|_2\|\Delta u^n\|_2\leq c\|\Delta u^n\|_2^2+c\|\nabla u_t^n\|_2^2.\end{aligned} \tag{2.3.35}$$

Then, we infer from (2.3.32) – (2.3.35) and using Gronwall's lemma, we deduce that

$$\begin{aligned}&\|\nabla u_t^n(t)\|_2^2+\|\nabla\upsilon_t^n(t)\|_2^2+a_{01}\left(1-\int_0^t g_1(s)ds\right)\|\Delta u^n(t)\|_2^2\\ &+a_{02}\left(1-\int_0^t g_2(s)ds\right)\|\Delta\upsilon^n(t)\|_2^2\\ &+\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|\nabla z_{1t}^n(x,k_1,t)\|_{L^2(\Omega)}^2dk_1\\ &+\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|\nabla z_{2t}^n(x,k_2,t)\|_{L^2(\Omega)}^2dk_2\\ &\leq e^{cT}\left(\|\nabla u_t^n(0)\|_2^2+\|\nabla\upsilon_t^n(0)\|_2^2+\|\Delta u^n(0)\|_2^2+\|\Delta\upsilon^n(0)\|_2^2\right)\\ &+e^{cT}\left(\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|\nabla z_{1t}^n(x,k_1,0)\|_{L^2(\Omega)}^2dk_1\right)\\ &+e^{cT}\left(\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|\nabla z_{2t}^n(x,k_2,0)\|_{L^2(\Omega)}^2dk_2\right),\end{aligned} \tag{2.3.36}$$

we have also from (2.3.24)

$$\begin{aligned}&\|u_t^n(t)\|_2^2+\|\upsilon_t^n(t)\|_2^2+\|\nabla u^n(t)\|_2^2+\|\nabla\upsilon^n(t)\|_2^2+(g_1ou^n)(t)\\ &+\int_0^1\int_\Omega z_1^n(x,1,s)dxds+\int_0^1\int_\Omega z_2^n(x,1,s)dxds+(g_2o\upsilon^n)(t)\\ &+\int_0^1\int_\Omega z_1^n(x,k_1,s)dxds+\int_0^1\int_\Omega z_2^n(x,k_2,s)dxds+\int_\Omega F(u,\upsilon)dx\leq C_1,\end{aligned} \tag{2.3.37}$$

where C_1 is a positive constant depending on the parameter $E(0)$.

Estimate 3. First, we estimate $(u_n''(0))$ and $(\upsilon_n''(0))$ in (2.3.2)-(2.3.3) and taking $t=0$, we obtain

$$\begin{aligned}&\|u_{tt}^n(0)\|_2^2+\|\upsilon_{tt}^n(0)\|_2^2\leq\|a_{01}\ u^{0n}\|_2^2+\mu_1\|u^{1n}\|_2^2+\mu_2\|z_1^{0n}\|_2^2\\ &+a_{02}\|\upsilon^{0n}\|_2^2+\alpha_1\|\upsilon^{1n}\|_2^2+\alpha_2\|z_2^{0n}\|_2^2\leq a_{01}\|u^0\|_2^2+\mu_1\|u^1\|_2^2+\mu_2\|z_1^0\|_2^2\\ &+a_{02}\|\upsilon^0\|_2^2+\alpha_1\|\upsilon^1\|_2^2+\alpha_2\|z_2^0\|_2^2\leq C.\end{aligned} \tag{2.3.38}$$

Where C is a positive constant. Now, differentiating (2.3.2) and (2.3.3) with respect to t, we have

$$\begin{aligned}&\left\langle \sum_{i,j=1}^{N} a_{1ij}(x)\left(\int_0^t g_1(t-s)\nabla u^n(t)ds\right)', \nabla w_j\right\rangle_\Omega\\ &+\left\langle \mu_1\left(u_t^n(x,t)\right)', w_j\right\rangle_\Omega + \langle u_{ttt}^n(t), w_j\rangle_\Omega + a_1(u_t^n(t), w_j)\\ &+\left\langle \mu_2\left(z_1^n(x,1,t)\right)', w_j\right\rangle_\Omega + \langle (Df_1(u^n(t), v^n(t)), w_j\rangle_\Omega = 0,\end{aligned} \tag{2.3.39}$$

and

$$\begin{aligned}&\left\langle \sum_{i,j=1}^{N} a_{2ij}(x)\left(\int_0^t g_2(t-s)\nabla v^n(t)ds\right)', \nabla w_j\right\rangle_\Omega\\ &+\langle v_{ttt}^n(t), w_j\rangle_\Omega + a_2(v_t^n(t), w_j) + \left\langle \alpha_1\left(v_t^n(x,t)\right)', w_j\right\rangle_\Omega\\ &+\left\langle \alpha_2\left(z_2^n(x,1,t)\right)', w_j\right\rangle_\Omega + \langle (Df_2(u^n(t), v^n(t)), w_j\rangle_\Omega = 0.\end{aligned} \tag{2.3.40}$$

Multiplying (2.3.39) by $u''_{n,j}(t)$ and (2.3.40) by $v''_{n,j}(t)$, summing over j from 1 to n, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_{tt}^n(t)\|_2^2 + a_1(u_t^n(t), u_t^n(t)) + \mu_1\|u_{tt}^n(t)\|_2^2 + \mu_2\langle u_{tt}^n(t), z_{1t}^n(x,1,t)\rangle_\Omega\\ &-g_1(0)\frac{d}{dt}A_1\langle \nabla u^n(t), \nabla u_t^n(t)\rangle_\Omega + g_1(0)a_1(u_t^n(t), u_t^n(t))\\ &-\frac{d}{dt}\int_0^t A_1 g_1'(t-s)\langle \nabla u_s^n(s), \nabla u_t^n(t)\rangle_\Omega ds + \langle Df_1(u^n(t), v^n(t)), u_{tt}^n(t)\rangle_\Omega\\ &-\int_0^t A_1 g_1''(t-s)\langle \nabla u_s^n(s), \nabla u_t^n(t)\rangle_\Omega ds + g_1'(0)\langle A_1\nabla u_t^n(t), \nabla u_t^n(t)\rangle_\Omega.\end{aligned} \tag{2.3.41}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v_{tt}^n(t)\|_2^2 + a_2(v_t^n(t), v_t^n(t)) + \alpha_1\|v_{tt}^n(t)\|_2^2 + \alpha_2\langle v_{tt}^n(t), z_{2t}^n(x,1,t)\rangle_\Omega\\ &-g_2(0)\frac{d}{dt}\langle A_2\nabla v^n(t), \nabla v_t^n(t)\rangle_\Omega + g_2(0)a_2(v_t^n(t), v_t^n(t))\\ &-\frac{d}{dt}\int_0^t A_2 g_2'(t-s)\langle \nabla v_s^n(s), \nabla v_t^n(t)\rangle_\Omega ds + \langle Df_2(u^n(t), v^n(t)), v_{tt}^n(t)\rangle_\Omega ds\\ &-\int_0^t A_2 g_2''(t-s)\langle \nabla v_s^n(s), \nabla v_t^n(t)\rangle_\Omega ds + g_2'(0)\langle A_2\nabla v_t^n(t), \nabla v_t^n(t)\rangle_\Omega.\end{aligned} \tag{2.3.42}$$

Differentiating (2.3.5) with respect to t, we get

$$\left(\frac{\tau_2(t)}{(1-\tau_2'(t)k_1}\right) z_{1t}^n(t) + \frac{\tau_2(t)}{(1-\tau_2'(t)k_1)} z_{1tt}^n(t) + \frac{\partial}{\partial k_1}\|z_{1t}^n(t)\|_2^2 = 0. \tag{2.3.43}$$

Multiplying (2.3.43) by $z'_{1n,j}(t)$, summing over j from 1 to n, it follows that

$$\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)'\|z_{1t}^n(t)\|_2^2 + \frac{1}{2}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\frac{d}{dt}\|z_{1t}^n(t)\|_2^2 + \frac{1}{2}\frac{d}{dk_1}\|z_{1t}^n(t)\|_2^2 = 0. \tag{2.3.44}$$

Then, we have

$$\frac{1}{2}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)'\|z_{1t}^n(t)\|_2^2+\frac{1}{2}\frac{d}{dt}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\|z_{1t}^n(t)\|_2^2\right)+\frac{1}{2}\frac{d}{dk_1}\|z_{1t}^n(t)\|_2^2=0. \quad (2.3.45)$$

In the same way for (2.3.6), we get

$$\frac{1}{2}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)'\|z_{2t}^n(t)\|_2^2+\frac{1}{2}\frac{d}{dt}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\|z_{2t}^n(t)\|_2^2\right)+\frac{1}{2}\frac{d}{dk_2}\|z_{2t}^n(t)\|_2^2=0. \quad (2.3.46)$$

Taking the sum of (2.3.41), (2.3.42), (2.3.44) and (2.3.46), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|u_{tt}^n(t)\|_2^2+\|\upsilon_{tt}^n(t)\|_2^2+a_1(u_t^n(t),u_t^n(t))+a_2(\upsilon_t^n(t),\upsilon_t^n(t))\right]\\
&+\frac{1}{2}\frac{d}{dt}\left(\int_0^1\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|z_{1t}^n(t)\|_2^2dk_1+\mu_1\|u_{tt}^n(t)\|_2^2+\alpha_1\|\upsilon_{tt}^n(t)\|_2^2\\
&+\frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|z_{2t}^n(t)\|_2^2dk_2+g_1(0)a_1(u_t^n(t),u_t^n(t))\\
&+g_2(0)a_2(\upsilon_t^n(t),\upsilon_t^n(t))+\frac{1}{2}\|z_{1t}^n(x,1,t)\|_2^2+\frac{1}{2}\|z_{2t}^n(x,1,t)\|_2^2\\
&=-\frac{1}{2}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)'\|z_{1t}^n(t)\|_2^2dk_1+\frac{1}{2}\|u_{tt}^n(t)\|_2^2\\
&-\frac{1}{2}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)'\|z_{2t}^n(t)\|_2^2dk_2+\frac{1}{2}\|\upsilon_{tt}^n(t)\|_2^2\\
&+\langle Df_1(u^n(t),\upsilon^n(t)),u_{tt}^n(t)\rangle_\Omega+\langle Df_2(u^n(t),\upsilon^n(t)),\upsilon_{tt}^n(t)\rangle_\Omega\\
&+g_1(0)\frac{d}{dt}\langle A_1\nabla u^n(t),\nabla u_t^n(t)\rangle_\Omega+g_2(0)\frac{d}{dt}\langle A_2\nabla\upsilon^n(t),\nabla\upsilon_t^n(t)\rangle_\Omega\\
&-g_1'(0)\langle A_1\nabla u_t^n(t),\nabla u_t^n(t)\rangle_\Omega-\int_0^t A_1g_1''(t-s)\langle\nabla u^n(s),\nabla u_t^n(t)\rangle_\Omega\,ds\\
&-\frac{d}{dt}\int_0^t A_2g_2'(t-s)\langle\nabla\upsilon_s^n(s),\nabla\upsilon_t^n(t)\rangle_\Omega\,ds-g_2'(0)\langle A_2\nabla\upsilon_t^n(t),\nabla\upsilon_t^n(t)\rangle_\Omega\\
&-\int_0^t A_2g_2''(t-s)\langle\nabla\upsilon^n(s),\nabla\upsilon_t^n(t)\rangle_\Omega\,ds-\mu_2\langle u_{tt}^n(t),z_{1t}^n(x,1,t)\rangle_\Omega\\
&-\frac{d}{dt}\int_0^t A_1g_1'(t-s)\langle\nabla u_s^n(s),\nabla u_t^n(t)\rangle_\Omega\,ds-\alpha_2\langle\upsilon_{tt}^n(t),z_{2t}^n(x,1,t)\rangle_\Omega.
\end{aligned} \quad (2.3.47)$$

Using Holder, Young's inequalities and the same technique as in [29], we conclude the following estimates

$$
\begin{aligned}
&g_1'(0)A_1\left\langle \nabla u^n(t), \nabla u_t^n(t)\right\rangle_\Omega = \sum_{i,j=1}^{N} g_1'(0)a_{1ij}(x)\left\langle \frac{\partial u^n(t)}{\partial x_j}, \frac{\partial u_t^n(t)}{\partial x_i}\right\rangle_\Omega \\
&\leq \sum_{i,j=1}^{N}\frac{(g_1'(0))^2}{2\mu}\int_\Omega\left|\frac{\partial u^n(t)}{\partial x_j}\right|^2 dx + 2\mu\sum_{i,j=1}^{N}\int_\Omega\left|a_{1ij}(x)\frac{\partial u_t^n(t)}{\partial x_i}\right|^2 dx \\
&\leq \tfrac{(g_1'(0))^2}{2\mu}\|\nabla u^n(t)\|_2^2 + 2\mu\max_{1\leq i\leq N}\left(\sum_{j=1}^{N}\|a_{1ij}\|_\infty^2\right)\|\nabla u_t^n(t)\|_2^2 \\
&\leq \tfrac{(g_1'(0))^2}{2\mu}\|\nabla u^n(t)\|_2^2 + 2a_{11}\mu\|\nabla u_t^n(t)\|_2^2,
\end{aligned}
\tag{2.3.48}
$$

where

$$
a_{11} = \max_{1\leq i\leq N}\left(\sum_{j=1}^{N}\|a_{1ij}\|_\infty^2\right), \quad a_{22} = \max_{1\leq i\leq N}\left(\sum_{j=1}^{N}\|a_{2ij}\|_\infty^2\right).
$$

And

$$
g_2'(0)A_2\left\langle \nabla \upsilon^n(t), \nabla \upsilon_t^n(t)\right\rangle_\Omega \leq \frac{(g_2'(0))^2}{2\mu}\|\nabla \upsilon^n(t)\|_2^2 + 2a_{22}\mu\|\nabla \upsilon_t^n(t)\|_2^2, \tag{2.3.49}
$$

$$
\begin{aligned}
&\int_0^t A_1 g_1''(t-s)\left\langle \nabla u^n(s), \nabla u_t^n(t)\right\rangle_\Omega ds \\
&= \int_0^t \sum_{j=1}^{N} a_{1ij}(x)g_1''(t-s)\left\langle \nabla u^n(s), \nabla u_t^n(t)\right\rangle_\Omega ds \\
&\leq \sum_{j=1}^{N}\frac{a_{11}}{\epsilon}\int_\Omega\left|\frac{\partial u_t^n(t)}{\partial x_i}\right|^2 dx + 2\epsilon\|g_1''\|_{L^1}\sum_{j=1}^{N}\int_0^t\int_\Omega\left|\frac{\partial u^n(s)}{\partial x_j}\right|^2 dxds \\
&\leq \frac{a_{11}}{\epsilon}\|\nabla u_t^n(t)\|_2^2 + 2\epsilon\|g_1''\|_{L^1}\int_0^t\|\nabla u^n(s)\|_2^2 ds,
\end{aligned}
\tag{2.3.50}
$$

$$
\begin{aligned}
\left|\int_0^t A_2 g_2''(t-s)\left\langle \nabla \upsilon^n(s), \nabla \upsilon_t^n(t)\right\rangle_\Omega ds\right| &\leq \frac{a_{22}}{\epsilon}\|\nabla \upsilon_t^n(t)\|_2^2 \\
&\quad + 2\epsilon\|g_2''\|_{L^1}\int_0^t\|\nabla \upsilon^n(s)\|_2^2 ds.
\end{aligned}
\tag{2.3.51}
$$

Using (A_2) and the Sobolev's embedding, gives us

$$
\begin{aligned}
&|\langle Df_1 u^n(t), \upsilon^n(t)), u_{tt}^n(t)\rangle_\Omega| \leq C\left[(\|u^n\|^{p-1} + \|\upsilon^n\|^{p-1})\|u_t^n\|_2\right]\|u_{tt}^n\|_2 \\
&+C\left[(\|u^n\|^{p-1} + \|\upsilon^n\|^{p-1})\|\upsilon_t^n\|_2\right]\|u_{tt}^n\|_2 \\
&\leq C\left[\|u^n\|_2^{2(p-1)} + \|\upsilon^n\|_2^{2(p-1)} + \|u_t^n\|_2^2 + \|\upsilon_t^n\|_2^2\right]\|u_{tt}^n\|_2 \leq \|u_{tt}^n\|_2^2 + c,
\end{aligned}
\tag{2.3.52}
$$

where c is a positive constant. In the same way, we obtain

$$
|\langle Df_2(u^n(t), \upsilon^n(t))\upsilon_{tt}^n\rangle_\Omega| \leq \|\upsilon_{tt}^n\|_2^2 + c. \tag{2.3.53}
$$

Replacing (2.3.48)-(2.3.53) in (2.3.47), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\|u_{tt}^n(t)\|_2^2 + \|\upsilon_{tt}^n(t)\|_2^2 + a_1(u_t^n(t), u_t^n(t)) + a_2(\upsilon_t^n(t), \upsilon_t^n(t))\Big] \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|z_{1t}^n(t)\|_2^2 dk_1 + \frac{1}{2}\|z_{1t}^n(x,1,t)\|_2^2 \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|z_{2t}^n(t)\|_2^2 dk_2 + \frac{1}{2}\|z_{2t}^n(x,1,t)\|_2^2 \\
&\quad + g_1(0)a_{11}(u_t^n(t), u_t^n(t)) + g_2(0)a_{22}(\upsilon_t^n(t), \upsilon_t^n(t)) \\
&\leq c\|u_{tt}^n(t)\|_2^2 + c\|\upsilon_{tt}^n(t)\|_2^2 + c\left(\int_0^1 \left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|z_{1t}^n(t)\|_2^2 dk_1\right) \\
&\quad + c\left(\int_0^1 \left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|z_{2t}^n(t)\|_2^2 dk_2\right) + 2\mu a_{22}\|\nabla \upsilon_t^n(t)\|_2^2 \\
&\quad + \frac{(g_1'(0))^2}{2\mu}\|\nabla u^n(t)\|_2^2 + 2\mu a_{11}\|\nabla u_t^n(t)\|_2^2 + \frac{(g_2'(0))^2}{2\mu}\|\nabla \upsilon^n(t)\|_2^2 \\
&\quad + \epsilon\|g_1''\|_{L^1}\int_0^t A_1 g_1''(t-s)\|\nabla u^n(s)\|_2^2 ds + \frac{a_{11}}{\epsilon}\|\nabla u_t^n(t)\|_2^2 \\
&\quad + \epsilon\|g_2''\|_{L^1}\int_0^t A_2 g_2''(t-s)\|\nabla \upsilon^n(s)\|_2^2 ds + \frac{a_{22}}{\epsilon}\|\nabla \upsilon_t^n(t)\|_2^2 \\
&\quad - \frac{d}{dt}\int_0^t A_1 g_1'(t-s)\left\langle \nabla u_s^n(s), \nabla u_t^n(t)\right\rangle_\Omega ds + g_1(0)\frac{d}{dt}A_1\left\langle \nabla u^n(t), \nabla u_t^n(t)\right\rangle_\Omega \\
&\quad - \frac{d}{dt}\int_0^t A_2 g_2'(t-s)\left\langle \nabla \upsilon_s^n(s)\nabla \upsilon_t^n(t)\right\rangle_\Omega ds + g_2(0)\frac{d}{dt}A_2\left\langle \nabla \upsilon^n(t), \nabla \upsilon_t^n(t)\right\rangle_\Omega .
\end{aligned}
\tag{2.3.54}
$$

Integrating the last inequality over $(0,t)$ and using Gronwall's lemma, we get

$$
\begin{aligned}
&\|u_{tt}^n(t)\|_2^2+\|\upsilon_{tt}^n(t)\|_2^2+a_1(u_t^n(t),u_t^n(t))+a_2(\upsilon_t^n(t),\upsilon_t^n(t))\\
&+\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|z_{1t}^n(t)\|_2^2dk_1+\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|z_{2t}^n(t)\|_2^2dk_2\\
&\leq\|u_{tt}^n(0)\|_2^2+\|\upsilon_{tt}^n(0)\|_2^2+a_1(u_t^n(0),u_t^n(0))+a_2(\upsilon_t^n(0),\upsilon_t^n(0))\\
&+\int_0^1\left(\frac{\tau_2(0)}{1-\tau_2'(0)k_1}\right)\|z_{1t}^n(x,k_1,0)\|_2^2dk_1+g_1(0)a_1(u^n(t),u_t^n(t))\\
&+\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|z_{2t}^n(x,k_2,0)\|_2^2dk_2+g_2(0)a_2(\upsilon^n(t),\upsilon_t^n(t))\\
&-\int_0^t g_1^{'}(t-s)A_1\left\langle\nabla u_s^n(s),\nabla u_t^n(s)\right\rangle_\Omega ds-g_1(0)a_1(u_t^n(0),u_t^n(0))\\
&-\int_0^t g_2^{'}(t-s)A_2\left\langle\nabla \upsilon_s^n(s),\nabla \upsilon_t^n(t)\right\rangle_\Omega ds-g_2(0)a_2(\upsilon_t^n(0),\upsilon_t^n(0))\\
&+\left(\frac{1}{4\epsilon}+\frac{g_1'(0)^2}{4\epsilon}\right)\int_0^t\|\nabla u_t^n(s)\|_2^2ds+\left(\frac{1}{4\epsilon}+\frac{g_2'(0)^2}{4\epsilon}\right)\int_0^t\|\nabla \upsilon_t^n(s)\|_2^2ds\\
&+(\epsilon+\epsilon\|g_1\|_{L^1}^2)\int_0^t\|\nabla u^n(s)\|_2^2ds+(\epsilon+\epsilon\|g_2\|_{L^1}^2)\int_0^t\|\nabla \upsilon^n(s)\|_2^2ds\\
&+\int_0^t\left(\|u_{ss}^n(s)\|_2^2+\|\upsilon_{ss}^n(s)\|_2^2+a_1(u_s^n(s),u_s^n(s))+a_2(\upsilon_s^n(s),\upsilon_s^n(s))\right)ds\\
&+\int_0^t\left(\int_0^1\left(\frac{\tau_2(s)}{1-\tau_2'(s)k_1}\right)\|z_{1t}^n(x,k_1,s)\|_2^2dk_1\right)ds\\
&+\int_0^t\left(\int_0^1\left(\frac{\tau_2(s)}{1-\tau_2'(s)k_2}\right)\|z_{2t}^n(x,k_2,s)\|_2^2dk_2\right)ds.
\end{aligned}
\tag{2.3.55}
$$

We have to estimate the right hand side of (2.3.55)

$$
\begin{aligned}
&g_1(0)A_1\left\langle\nabla u^n(t),\nabla u_t^n(t)\right\rangle_\Omega=\sum_{i,j=1}^N g_1(0)a_{1ij}(x)\left\langle\frac{\partial u^n(t)}{\partial x_j},\frac{\partial u_t^n(t)}{\partial x_i}\right\rangle_\Omega\\
&\leq\sum_{i,j=1}^N\frac{(g_1(0))^2}{2\mu}\int_\Omega\left|\frac{\partial u^n(t)}{\partial x_j}\right|^2dx+2\mu\sum_{i,j=1}^N\int_\Omega\left|a_{1ij}(x)\frac{\partial u_t^n(t)}{\partial x_i}\right|^2dx\\
&\leq\tfrac{(g_1(0))^2}{2\mu}\|\nabla u^n(t)\|_2^2+2\mu\max_{1\leq i\leq N}\left(\sum_{j=1}^N\|a_{1ij}\|_\infty^2\right)\|\nabla u_t^n(t)\|_2^2\\
&\leq\tfrac{(g_1(0))^2}{2\mu}\|\nabla u^n(t)\|_2^2+2a_{11}\mu\|\nabla u_t^n(t)\|_2^2.
\end{aligned}
\tag{2.3.56}
$$

In the same way

$$
g_2(0)\left\langle\nabla \upsilon^n(t),\nabla \upsilon_t^n(t)\right\rangle_\Omega\leq\frac{(g_2(0))^2}{2\mu}\|\nabla \upsilon^n(t)\|_2^2+2a_{22}\mu\|\nabla \upsilon_t^n(t)\|_2^2,
\tag{2.3.57}
$$

as previously, we can obtain

$$\left|\int_0^t g_1'(t-s)A_1 \langle \nabla u^n(s), \nabla u_t^n(t)\rangle_\Omega \, ds\right| \le \epsilon \|\nabla u_t^n(t)\|_2^2 + \frac{a_{11}\|g_1\|_{L^1}\|g_1\|_{L^\infty}}{4\epsilon}\int_0^t \|\nabla u^n(s)\|_2^2 ds, \tag{2.3.58}$$

$$\left|\int_0^t g_2'(t-s)A_2 \langle \nabla \upsilon^n(s), \nabla \upsilon_t^n(t)\rangle_\Omega \, ds\right| \le \epsilon \|\nabla \upsilon_t^n(t)\|_2^2 + \frac{a_{22}\|g_2\|_{L^1}\|g_2\|_{L^\infty}}{4\epsilon}\int_0^t \|\nabla \upsilon^n(s)\|_2^2 ds. \tag{2.3.59}$$

Replacing (2.3.56)-(2.3.59) in (2.3.55), after choosing ϵ small enough and using Gronwall's lemma, we obtain

$$\begin{aligned} &\|u_{tt}^n(t)\|_2^2 + \|\upsilon_{tt}^n(t)\|_2^2 + \int_0^1 \left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|z_{1t}^n(x,k_1,t)\|_2^2 dk_1 \\ &+ \int_0^1 \left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|z_{2t}^n(x,k_2,t)\|_2^2 dk_2 + a_{01}\|\nabla u_t^n(t)\|_2^2 + a_{02}\|\nabla \upsilon_t^n(t)\|_2^2 \le M. \end{aligned} \tag{2.3.60}$$

Where M is some positive constant. Therefore, from (2.3.24), (2.3.36) and (2.3.60), we conclude that

$$u^n \text{ is bounded in } L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \tag{2.3.61}$$

$$\upsilon^n \text{ is bounded in} L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \tag{2.3.62}$$

$$u_t^n \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)), \tag{2.3.63}$$

$$\upsilon_t^n \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)), \tag{2.3.64}$$

$$u_{tt}^n \text{ is bounded in } L^\infty(0,T;L^2(\Omega)), \tag{2.3.65}$$

$$\upsilon_{tt}^n \text{ is bounded in } L^\infty(0,T;L^2(\Omega)), \tag{2.3.66}$$

$$z_1^n(x,1,t) \text{ is bounded in} L^2(\Omega\times(0,T)), \tag{2.3.67}$$

$$z_2^n(x,1,t) \text{ is bounded in } L^2(\Omega\times(0,T)), \tag{2.3.68}$$

$$\tau_1(t)z_{1t}^n(x,k_1,t) \text{ is bounded in } L^\infty(0,T;L^1(\Omega\times(0,1)), \tag{2.3.69}$$

$$\tau_2(t)z_{2t}^n(x,k_2,t) \text{ is bounded in } L^\infty(0,T;L^1(\Omega\times(0,1)), \tag{2.3.70}$$

$$z_{1t}^n(x,1,t) \text{ is bounded in } L^\infty(0,T;L^2(\Omega\times(0,T))), \tag{2.3.71}$$

$$z_{2t}^n(x,,t) \text{ is bounded in } L^\infty(0,T;L^2(\Omega\times(0,T))). \tag{2.3.72}$$

Applying Dunford-Pettis theorem, we deduce from (2.3.61)-(2.3.73) that there exists a subsequence $(u^n, z_1^n), (\upsilon^n, z_2^n)$ such that

$$u^n \to u \text{ weakly star in } L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega)), \tag{2.3.73}$$

$$\upsilon^n \to \upsilon \text{ weakly star in } L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega)), \tag{2.3.74}$$

$$u_t^n \to u_t \text{ weakly star in } L^\infty(0,T; H_0^1(\Omega)), \tag{2.3.75}$$

$$\upsilon_t^n \to \upsilon_t \text{ weakly star in } L^\infty(0,T; H_0^1(\Omega)), \tag{2.3.76}$$

$$u_t^n \to \chi_1 \text{ weakly star in } L^2(\Omega \times (0,T)), \tag{2.3.77}$$

$$\upsilon_t^n \to \chi_2 \text{ weakly star in } L^2(\Omega \times (0,T)), \tag{2.3.78}$$

$$u_{tt}^n \to u_{tt} \text{ weakly star in } L^\infty(0,T; L^2(\Omega)), \tag{2.3.79}$$

$$\upsilon_{tt}^n \to \upsilon_{tt} \text{ weakly star in } L^\infty(0,T; L^2(\Omega)), \tag{2.3.80}$$

$$z_1^n(x,k_1,t) \to z_1(x,k_1,t) \text{ weakly star in } L^\infty(0,T; H_0^1(\Omega; L^2(0,1)), \tag{2.3.81}$$

$$z_2^n(x,k_2,t) \to z_2(x,k_2,t) \text{ weakly star in } L^\infty(0,T; H_0^1(\Omega; L^2(0,1)), \tag{2.3.82}$$

$$\tau_1(t) z_{1t}^n(x,k_1,t) \to z_{1t}^n(x,k_1,t) \text{ weakly star in } \ L^\infty(0,T; L^2(\Omega \times (0,T))), \tag{2.3.83}$$

$$\tau_2(t) z_{2t}^n(x,k_2,t) \to z_{2t}^n(x,k_2,t) \text{ weakly star in } \ L^\infty(0,T; L^2(\Omega \times (0,T))), \tag{2.3.84}$$

$$z_{1t}^n(x,1,t) \to \psi_1 \text{ weakly star in } L^2(\Omega \times (0,1)), \tag{2.3.85}$$

$$z_{2t}^n(x,1,t) \to \psi_2 \text{ weakly star in } L^2(\Omega \times (0,1)). \tag{2.3.86}$$

Further, by Aubin's lemma [30], it follows from (2.3.73)-(2.3.78) that there exists a subsequence (u^n, υ^n) still represented by the same notation, such that

$$u^n \to u \text{ strongly in } \ L^2(0,T; L^2(\Omega)), \tag{2.3.87}$$

$$\upsilon^n \to \upsilon \text{ strongly in } L^2(0,T; L^2(\Omega)). \tag{2.3.88}$$

Then

$$u^n \to u \text{ and } \ \upsilon^n \to \upsilon \text{ a.e in } (0,T) \times \Omega, \tag{2.3.89}$$

$$u_t^n \to u_t \text{ and } \ \upsilon_t^n \to \upsilon_t \text{ a.e in } (0,T) \times \Omega. \tag{2.3.90}$$

Analysis of the nonlinear term.

$$\begin{aligned}
&\|f_1(u^n,\upsilon^n)\|_{L^2(\Omega\times(0,T))} \leq \int_0^T\int_\Omega (|u^n(s)|^p + |\upsilon^n(s)|^p)dsdx\\
&+\int_0^T\int_\Omega (|u^n(s)|^{\frac{p-1}{2}}|\upsilon^n(s)|^{\frac{p+1}{2}})dsdx\\
&\leq c_s^p\int_0^T \|\nabla u^n(s)\|^p ds + c_s^p\int_0^T \|\nabla \upsilon^n(s)\|^p ds,\\
&+c_s^{\frac{p-1}{2}}\int_0^T \|\nabla u^n(s)\|^{\frac{p-1}{2}}ds + c_s^{\frac{p+1}{2}}\int_0^T \|\nabla \upsilon^n(s)\|^{\frac{p+1}{2}}ds\\
&\leq 2c_s^pTC_1^P + c_s^{\frac{p-1}{2}}TC_1^{\frac{p-1}{2}}TC_1^{\frac{p-1}{2}} + c_s^{\frac{p+1}{2}}TC_1^{\frac{p+1}{2}}TC_1^{\frac{p+1}{2}} = C.
\end{aligned} \tag{2.3.91}$$

Where C is a positive constant. In the same way for $f_2(u^n,\upsilon^n)$

$$\|f_2(u^n,\upsilon^n)\|_{L^2(\Omega\times(0,T))} \leq C. \tag{2.3.92}$$

From the (2.3.91) and (2.3.92), we deduce that

$$\begin{aligned}
f_1(u^n,\upsilon^n) \to f_1(u,\upsilon) \textit{ weakly in } L^2(0,T;L^2(\Omega)),\\
f_1(u^n,\upsilon^n) \to f_1(u,\upsilon) \textit{ weakly in } L^2(0,T;L^2(\Omega)).
\end{aligned} \tag{2.3.93}$$

For suitable functions $(u,\upsilon) \in (L^\infty(0,T;H_0^1(\Omega)))^2$, $(z_1,z_2) \in (L^\infty(0,T;L^2(\Omega\times(0.1))))^2$, $\psi_1,\psi_2 \in L^2(\Omega\times(0,T))$, $(\chi_1,\chi_2) \in L^2(\Omega\times(0,T))^2$, $\xi \in L^\infty((0,T);L^2(\Omega))$. We have to show that $((u,z_1),(\upsilon,z_2))$ is a solution of (2.3.2) – (2.3.6). Using the embedding

$$L^\infty(0,T;H_0^1(\Omega)) \hookrightarrow L^2(0,T;H_0^1(\Omega)); \quad H^1((0,T)\times\Omega) \hookrightarrow L^2((0,T)\times\Omega).$$

From (2.3.63)-(2.3.64) we have that u_t^n and υ_t^n are bounded in

$$L^\infty((0,T);H_0^1(\Omega)) \hookrightarrow L^2((0,T);H_0^1(\Omega)),$$

then u_{tt}^n and υ_{tt}^n are bounded in

$$L^\infty((0,T);L^2(\Omega)) \hookrightarrow L^2((0,T);L^2(\Omega)).$$

Consequently, u_t^n, υ_t^n are bounded in $H^1((\Omega)\times(0,T))$. Using Aubin-Lions theorem [30], we can extract a subsequence (u^ξ) of (u^n) and (υ^ξ) of (υ^n) such that

$$u_t^\xi \to u_t \textit{ strongly in } L^2(\Omega\times(0,T)), \tag{2.3.94}$$

$$\upsilon_t^\xi \to \upsilon_t \textit{ strongly in } L^2(\Omega\times(0,T)), \tag{2.3.95}$$

therefore

$$u_t^\xi \to u_t \textit{ strongly and a.e. in } (\Omega\times(0,T)), \tag{2.3.96}$$

$$\upsilon_t^{\xi} \to \upsilon_t \textit{ strongly and a.e. in } (\Omega \times (0,T)). \tag{2.3.97}$$

Similarly

$$z_1^{\xi} \to z_1 \textit{ strongly in } L^2(0,T;L^2(\Omega \times (0,1)), \tag{2.3.98}$$

$$z_2^{\xi} \to z_2 \textit{ strongly in } L^2(0,T;L^2(\Omega \times (0,1)). \tag{2.3.99}$$

It follows at once from the convergence (2.3.73), (2.3.74),(2.3.79), (2.3.80), (2.3.83), (2.3.84), (2.3.94), and (2.3.95) for each fixed $\vartheta \in L^2(0,T,L^2(\Omega))$, $\sigma \in L^2(0,T,L^2(\Omega) \times (0,1))$ as $\xi \to \infty$ permits us to deduce that

$$\begin{aligned}
&\int_0^T \left[\int_\Omega u_{tt}^{\xi}(t)\vartheta dx + \int_\Omega A_1 \nabla u^{\xi}(t) \nabla \vartheta dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \int_0^t g_1(t-s) A_1 \nabla u^{\xi}(t) \nabla \vartheta ds dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \mu_1 u_t^{\xi}(t)\vartheta dx + \int_\Omega \mu_2 z_1^{\xi}(x,1,t)\vartheta dx = \int_\Omega f_1(u^{\xi}(t)\upsilon^{\xi}(t))\vartheta dx\right] dt \\
&\to \int_0^T \left[\int_\Omega u_{tt}(t)\vartheta dx + \int_\Omega A_1 \nabla u(t) \nabla \vartheta dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \int_0^t g_1(t-s) A_1 \nabla u(t) \nabla \vartheta ds dx dt\right] dt \\
&+ \int_0^T \left[\int_\Omega \mu_1 u_t(t)\vartheta dx + \int_\Omega \mu_2 z_1(x,1,t)\vartheta dx = \int_\Omega f_1(u(t),\upsilon(t))\vartheta dx\right] dt,
\end{aligned} \tag{2.3.100}$$

and

$$\begin{aligned}
&\int_0^T \left[\int_\Omega \upsilon_{tt}^{\xi}(t)\vartheta dx dt + \int_\Omega A_2 \nabla \upsilon^{\xi}(t) \nabla \vartheta dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \int_0^t A_2 g_2(t-s) \nabla \upsilon^{\xi}(t) \nabla \vartheta ds dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \mu_1 \upsilon_t^{\xi}(t)\vartheta dx + \int_\Omega \mu_2 z_2^{\xi}(x,1,t)\vartheta dx = \int_\Omega f_2(u^{\xi}(t)\upsilon^{\xi}(t))\vartheta dx\right] dt \\
&\to \int_0^T \left[\int_\Omega \upsilon_{tt}(t)\vartheta dx + \int_\Omega A_2 \nabla \upsilon(t) \nabla \vartheta dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \int_0^t g_2(t-s) A_2 \nabla \upsilon(t) ds \nabla \vartheta dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \alpha_1 \upsilon_t(t)\vartheta dx + \int_\Omega \alpha_2 z_2(x,1,t)\vartheta dx = \int_\Omega f_2(u(t),\upsilon(t))\vartheta dx\right] dt.
\end{aligned} \tag{2.3.101}$$

exploiting the convergence (2.3.98) and (2.3.99) we deduce

$$\begin{aligned}&\int_0^T\int_0^1\int_\Omega\left(\tau_2(t)\frac{\partial}{\partial t}z_1^\xi+(1-\tau_2^{'}(t)k_1)\frac{\partial}{\partial k_1}z_1^\xi\right)\sigma dxdk_1dt\\&\rightarrow\int_0^T\int_0^1\int_\Omega\left(\tau_2(t)\frac{\partial}{\partial t}z_1+(1-\tau_2^{'}(t)k_1)\frac{\partial}{\partial k_1}z_1\right)\sigma dxdk_1dt,\end{aligned}\tag{2.3.102}$$

$$\begin{aligned}&\int_0^T\int_0^1\int_\Omega\left(\tau_2(t)\frac{\partial}{\partial t}z_2^\xi+(1-\tau_2^{'}(t)k_2)\frac{\partial}{\partial k_2}z_2^\xi\right)\sigma dxdk_2dt\\&\rightarrow\int_0^T\int_0^1\int_\Omega\left(\tau_2(t)\frac{\partial}{\partial t}z_2+(1-\tau_2^{'}(t)k_2)\frac{\partial}{\partial k_2}z_2\right)\sigma dxdk_2dt.\end{aligned}\tag{2.3.103}$$

Uniqueness. Let (u_1, υ_1) and (u_2, υ_2) be two solutions of problem (23). Then $(w, q) = (u_1, \upsilon_1) - (u_2, \upsilon_2)$ and we put also $\tilde{w} = u_1'(x, t - k_1\tau_2(t)) - u_2'(x, t - k_1\tau_2(t))$, $\tilde{q} = \upsilon_1'(x, t - k_2\tau_2(t)) - \upsilon_2'(x, t - k_2\tau_2(t))$. Multiplying the first equation in (2.3.1) by w', integrating over Ω and using integration by parts, we get

$$\begin{aligned}&\frac{d}{dt}\left(\|w'(t)\|_2^2+\left(1-\int_0^t g_1(s)ds\right)a_1(w(t),w(t))+(g_1ow)(t)\right)\\&+\mu_1\|w'(t)\|_2^2+\frac{1}{2}\|\tilde{w}(x,1,t)\|_2^2+g_1(t)a_1(w(t),w(t))-(g_1'\circ w)(t)\\&=-\mu_2\int_\Omega\tilde{w}(x,1,t)w'(t)dx+\frac{1}{2}\|w'(t)\|_2^2+\int_\Omega\left[f_1(u_1,\upsilon_1)-f_1(u_2,\upsilon_2)\right]w'(t)dx.\end{aligned}\tag{2.3.104}$$

in the same way for second equation in (2.3.1). Multiplying the second equation in (2.3.1) by q', integrating over Ω and using integration by parts, we get

$$\begin{aligned}&\frac{d}{dt}\left(\|q'(t)\|_2^2+\left(1-\int_0^t g_2(s)ds\right)a_2(q(t),q(t))+(g_2oq)(t)\right)\\&+\alpha_1\|q'(t)\|_2^2+\tfrac{1}{2}\|\tilde{q}(x,1,t)\|_2^2+g_2(t)a_2(q(t),q(t))-(g_2'\circ q)(t)\\&=-\alpha_2\int_\Omega\tilde{q}(x,1,t)q'dx+\frac{1}{2}\|q'(t)\|_2^2+\int_\Omega\left[f_2(u_1,\upsilon_1)-f_2(u_2,\upsilon_2)\right]q'(t)dx.\end{aligned}\tag{2.3.105}$$

Multiplying the third equation in (2.3.1) by $\tilde{w}$, integrating over $\Omega \times (0, 1)$, we get

$$\frac{1}{2}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\frac{d}{dt}\|\tilde{w}(t)\|_2^2+\frac{1}{2}\frac{d}{dk_1}\|\tilde{w}(t)\|_2^2=0.\tag{2.3.106}$$

Then

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\|\tilde{w}(t)\|_2^2\right)dk_1-\frac{1}{2}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)'\|\tilde{w}(t)\|_2^2\,dk_1\\&+\frac{1}{2}\left(\|\tilde{w}(x,1,t)\|_2^2-\|w'(t)\|_2^2\right)=0.\end{aligned}\tag{2.3.107}$$

In the same way for the forth equation in (2.3.1), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\|\tilde{q}(t)\|_2^2\right)dk_2-\frac{1}{2}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)'\|\tilde{q}(t)\|_2^2\,dk_2\\&+\frac{1}{2}\left(\|\tilde{q}(x,1,t)\|_2^2-\|q'(t)\|_2^2\right)=0.\end{aligned}\tag{2.3.108}$$

Combining (2.3.104)-(2.3.105) and (2.3.106)-(2.3.108), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\begin{array}{c}\|w'(t)\|_2^2+\|q'(t)\|_2^2+\left(1-\int_0^t g_1(s)ds\right)a_2(w(t),w(t))\\+\left(1-\int_0^t g_2(s)ds\right)a_1(q(t),q(t))+(g_1ow)(t)+(g_2oq)(t)\end{array}\right)dx\\&+\frac{1}{2}\frac{d}{dt}\left(\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\|\tilde{w}(t)\|_2^2\right)dk_1+\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\|\tilde{q}(t)\|_2^2\right)dk_2\right)\\&+\mu_1\|\omega'(t)\|_2^2+\alpha_1\|q'(t)\|_2^2+\tfrac{1}{2}\|\tilde{q}(x,1,t)\|_2^2+\tfrac{1}{2}\|\tilde{w}(x,1,t)\|_2^2\\&=-\mu_2\int_\Omega\tilde{w}(x,1,t)w'(t)dx-\alpha_2\int_\Omega\tilde{q}(x,1,t)q'(t)dx+\frac{1}{2}\|w'(t)\|_2^2+\frac{1}{2}\|q'(t)\|_2^2\\&+\int_\Omega\left[f_1(u_1,\upsilon_1)-f_1(u_2,\upsilon_2)\right]w'(t)dx+\int_\Omega\left[f_2(u_1,\upsilon_1)-f_2(u_2,\upsilon_2)\right]q'(t)dx.\end{aligned}\tag{2.3.109}$$

$$\left|\int_\Omega\left[f_1(u_1,\upsilon_1)-f_1(u_2,\upsilon_2)\right]w'dx\right|\le C\left(\|w'(t)\|_2^2+\|\nabla w(t)\|_2^2+\|\nabla q(t)\|_2^2\right),\tag{2.3.110}$$

$$\left|\int_\Omega\left[f_2(u_1,\upsilon_1)-f_2(u_2,\upsilon_2)\right]w'dx\right|\le C\left(\|q'(t)\|_2^2+\|\nabla w(t)\|_2^2+\|\nabla q(t)\|_2^2\right).\tag{2.3.111}$$

We set

$$\begin{aligned}Y(t)&=\|w'(t)\|_2^2+\|q'(t)\|_2^2+\|\nabla w(t)\|_2^2+\|\nabla q(t)\|_2^2\\&\quad+\int_0^1\left(\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|\tilde{w}(t)\|_2^2\right)dk_1+\int_0^1\left(\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|\tilde{q}(t)\|_2^2\right)dk_2,\end{aligned}$$

then the equality (2.3.109) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}Y(t)+\mu_1\|\omega'(t)\|_2^2+\alpha_1\|q'(t)\|_2^2+\frac{1}{2}\|\tilde{q}(x,1,t)\|_2^2+\frac{1}{2}\|\tilde{w}(x,1,t)\|_2^2\\&\le\|\tilde{w}(t)\|_2\|\nabla w'(t)\|_2+\|\tilde{q}(t)\|_2\|\nabla q'(t)\|_2+\frac{1}{2}\|w'(t)\|_2^2\\&+\|q'(t)\|_2^2+\|\nabla w(t)\|_2^2+\|\nabla q(t)\|_2^2,\end{aligned}\tag{2.3.112}$$

hence

$$\frac{1}{2}\frac{d}{dt}Y(t)\le Y(t).\tag{2.3.113}$$

Integrating the last equality and using the Gronwall's lemma we get

$$\begin{aligned}&\|w'(t)\|_2^2+\|q'(t)\|_2^2+a_{01}l_1\|\nabla w(t)\|_2^2+a_{02}l_2\|\nabla q(t)\|_2^2\\&+\int_0^1\left(\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|\tilde{w}(t)\|_2^2\right)dk_1+\int_0^1\left(\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|\tilde{q}(t)\|_2^2\right)dk_2=0.\end{aligned}$$

This completes our proof of existence and uniqueness of the weak solution. □

Remark 2.3.1. *By virtue of the theory of ordinary differential equations, the system* (2.3.2)–(2.3.6) *has local solution which is extended to a maximal interval* $[0, T_k[$ *with* ($0 < T_k \leq +\infty$).

Now we will prove that the solution obtained above is global and bounded in time, for this purpose, we define

$$\begin{aligned} I(t) &= \xi_1(t)e^{-k_1\tau_2(t)}\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx + (g_1\circ u)(t) + (g_2\circ \upsilon)(t) \\ &+ \xi_2(t)e^{-k\tau_2(t)}\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx + (p+1)\int_\Omega F(u,\upsilon)dx \\ &+ \left(1-\int_0^t g_1(s)ds\right)a_1(u(t),u(t)) + \left(1-\int_0^t g_2(s)ds\right)a_2(\upsilon(t),\upsilon(t)), \end{aligned} \tag{2.3.114}$$

and

$$\begin{aligned} J(t) &= \frac{\xi_1(t)e^{-k_1\tau_2(t)}}{2}\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx + \frac{1}{2}(g_1\circ u)(t) + \frac{1}{2}(g_2\circ \upsilon)(t) \\ &+ \frac{\xi_2(t)e^{-k_2\tau_2(t)}}{2}\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx + \int_\Omega F(u,\upsilon)dx \\ &+ \frac{1}{2}\left(1-\int_0^t g_1(s)ds\right)a_1(u(t),u(t)) + \frac{1}{2}\left(1-\int_0^t g_2(s)ds\right)a_2(\upsilon(t),\upsilon(t)). \end{aligned} \tag{2.3.115}$$

Remark 2.3.2. *From the definition of $E(t)$ by (2.3.115), we observe that*

$$E(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|\upsilon_t(t)\|_2^2\right) + J(t). \tag{2.3.116}$$

Definition 2.3.1. *Let $(u_0,\upsilon_0) \in (H_0^1(\Omega))^2$, $(u_1,\upsilon_1) \in (L^2(\Omega))^2$ and $(\phi_0,\phi_1) \in (L^2(\Omega\times(0,1)))^2$. We denote by $((u,z_1),(\upsilon,z_2))$ the solution to the problem (3.1). We define*

$$T^* = \sup\Big\{T > 0, ((u,z_1),(\upsilon,z_2)) \quad \textit{exists} \quad \textit{on } [0,T]\Big\}.$$

If $T^* = \infty$, we say that the solution of (2.3.1) is global.

Lemma 2.3.2. *Let $((u,z_1),(\upsilon,z_2))$, be the solution of problem (2.3.1). Assume further that $I(0) > 0$ and*

$$\alpha = \rho\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}} < 1. \tag{2.3.117}$$

Then $I(t) > 0\ \forall\ t$.

Proof. Since $I(0) > 0$, then there exists (by continuity of $u(t)$), there exists a time $t_1 > 0$ such that

$$I(t) \geq 0, \quad \forall\, t \in (0,t_1). \tag{2.3.118}$$

Let

$$\mathfrak{F} = \Big\{ I(t_0 = 0 \ and \ I(t) > 0, \ 0 \leq t < t_0 \Big\}.$$

From (2.3.16) and (2.3.17), we haver $\forall \ t \in [0, t_0]$

$$\begin{aligned} J(t) &\geq \frac{p-1}{2(p+1)} \left[l_1 a_1(u(t), u(t)) + l_2 a_2(\upsilon(t), u(t)) \right] \\ &+ \frac{p-1}{2(p+1)} \left[\xi_1(t) e^{-k_1 \tau_2(t)} \int_0^1 \int_\Omega z_1^2(x, k_1, t) dk_1 dx \right] \\ &+ \frac{p-1}{2(p+1)} \left[\xi_2(t) e^{-k_2 \tau_2(t)} \int_0^1 \int_\Omega z_2^2(x, k_2, t) dk_2 dx \right] \\ &+ \frac{p-1}{2(p+1)} \left[(g_1 \circ u)(t) + (g_2 \circ \upsilon)(t) \right] + \frac{1}{p+1} I(t) \\ &\geq \frac{p-1}{2(p+1)} \left[l_1 a_1(u(t), u(t)) + l_2 a_2(\upsilon(t), \upsilon(t)) \right]. \end{aligned} \tag{2.3.119}$$

Thus by (2.3.116) and (2.3.119) and the fact that $(g_1 \circ u)(t) + (g_2 \circ \upsilon)(t) > 0$, we deduce

$$\begin{aligned} l_1 a_1(u(t), u(t)) + l_2 a_2(\upsilon(t), \upsilon(t)) \ &\leq \tfrac{2(p+1)}{p-1} J(t) \leq \tfrac{2(p+1)}{(p-1)} E(t) \\ &\leq \tfrac{2(p+1)}{(p-1)} E(0), \ \ \forall \ t \in [0, t_0]. \end{aligned} \tag{2.3.120}$$

Employing lemma 2.2.1, we obtain

$$\begin{aligned} &(p+1) \int_\Omega F(u(t_0), \upsilon(t_0)) dx \leq \rho \left(l_1 \|\nabla u(t_0)\|_2^2 + l_2 \|\nabla \upsilon(t_0)\|_2^2 \right)^{\frac{p+1}{2}} \\ &\leq \rho \left(\tfrac{2(p+1)}{p-1} \right)^{\frac{p-1}{2}} (l_1 \|\nabla u(t_0)\|_2^2 + l_2 \|\nabla \upsilon(t_0)\|_2^2) \\ &= \alpha (l_1 \|\nabla u(t_0)\|_2^2 + l_2 \|\nabla \upsilon(t_0)\|_2^2) \\ &< (l_1 \|\nabla u(t_0)\|_2^2 + l_2 \|\nabla \upsilon(t_0)\|_2^2) \\ &\leq \tfrac{l_1}{a_{01}} a_1(u(t_0), u(t_0)) + \tfrac{l_2}{a_{02}} a_2(\upsilon(t_0), \upsilon(t_0)). \end{aligned} \tag{2.3.121}$$

By exploiting lemma 2.2.2. Hence, we conclude from (2.3.121) that $I(t) > 0$ on $[0, t_0]$ which contradicts thus $I(t) > 0$ on $[0, T]$, which completes the proof. □

Theorem 2.3.3. *Let* $(u_0, \upsilon_0) \in (H_0^1(\Omega))^2$, $(u_1, \upsilon_1) \in (L^2(\Omega))^2$, $(\phi_0, \phi_1) \in (L^2(\Omega \times (0,1)))^2$. *Suppose that (140) and* $I(0) > 0$ *hold. Then the solution of (2.3.1) is global and bounded.*

Proof. To prove Theorem 2.3.3, using the definition of T^*, we have to verify that

$$a_1(u(t), u(t)) + a_2(\upsilon(t), \upsilon(t))$$

is uniformly bounded in time. To do this, we use (2.3.116) to get

$$\begin{aligned} E(0) \ &\geq E(t) = J(t) + \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} \|\upsilon_t(t)\|^2 \\ &\geq \left(\tfrac{2(p+1)}{p-1} \right) \left[a_1(u(t), u(t)) + a_2(\upsilon(t), \upsilon(t)) \right] + \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} \|\upsilon_t(t)\|_2^2. \end{aligned} \tag{2.3.122}$$

Therefore

$$a_1(u(t), u(t)) + a_2(\upsilon(t), \upsilon(t)) \leq CE(0),$$

where C is positive constant, which depends only on p. Thus, we obtain the global existence result. This completes the proof. $\square$

2.4 Asymptotic stability

In this section we prove the asymptotic stability result by constructing a suitable Lyapunov functional. Now we define the following functional

$$L(t) = ME(t) + \epsilon\psi(t) + \varphi(t) + \epsilon I(t), \tag{2.4.1}$$

$$\psi(t) = \int_\Omega uu_t dx + \int_\Omega \upsilon\upsilon_t dx, \tag{2.4.2}$$

$$\varphi(t) = -\int_\Omega u_t \int_0^t g_1(t-s)(u(t) - u(s))dsdx - \int_\Omega \upsilon_t \int_0^t g_2(t-s)(\upsilon(t) - \upsilon(s))dsdx, \tag{2.4.3}$$

$$I(t) = \int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)} u_t^2(x,s)dxds + \int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)} \upsilon_t^2(x,s)dxds. \tag{2.4.4}$$

Remark 2.4.1. *We can easily see that*

$$\int_{t-\tau_2(t)}^t \int_\Omega e^{\lambda(s-t)} u_t^2(x,s)dxds = \int_0^1 \int_\Omega e^{-k_1\tau_2(t)} z_1^2(x, k_1, t)dxdk_1,$$

after using a change of variables, $t - \tau_2(t) = s$. We use the same way for the second term in (2.4.4).

In order to show our stability result, we need the following Lemmas.

Lemma 2.4.1. *([24]) Let $((u, z_1), (\upsilon, z_2))$ be the solution of problem (2.3.1) and assume that (140) holds. Then, for $\gamma \geq 0$, we have*

$$\begin{aligned}\int_\Omega \left(\int_0^t g_1(t-s)(u(t) - u(s))ds\right)^{\gamma+1} dx \leq & \left(\frac{4(p+1)E(0)}{l_1(p-1)}\right)^{\frac{\gamma}{2}} \\ & \times (1-l_1)^{\gamma+1} c_s^{\gamma+1} (g_1 \ o \ u)(t),\end{aligned} \tag{2.4.5}$$

and

$$\begin{aligned}\int_\Omega \left(\int_0^t g_2(t-s)(\upsilon(t) - \upsilon(s))ds\right)^{\gamma+1} dx \leq & \left(\frac{4(p+1)E(0)}{l_2(p-1)}\right)^{\frac{\gamma}{2}} \\ & \times (1-l_2)^{\gamma+1} c_s^{\gamma+1} (g_2 \ o \ \upsilon)(t).\end{aligned} \tag{2.4.6}$$

Lemma 2.4.2. *([24]) Suppose that* $(A_0)-(A_5)$ *hold. Let* $(u_0,\upsilon_0)\in(H_0^1(\Omega))^2$, $(u_1,\upsilon_1)\in(L^2(\Omega))^2$ *be given and satisfying (3.118). Then there exist two positive constants* η_1 *and* η_2 *such that for any* $\delta>0$ *and for all* $t\geq 0$,

$$\begin{aligned}\int_\Omega f_1(u,\upsilon)\int_0^t g_1(t-s)(u(t)-u(s))dsdx &\leq \eta_1\delta(l_1\|\nabla u\|_2^2+l_2\|\nabla\upsilon\|_2^2)\\ &\quad+\frac{(1-l_1)c_s^2}{4\delta}(g_2\ o\ u)(t),\end{aligned} \tag{2.4.7}$$

and

$$\begin{aligned}\int_\Omega f_2(u,\upsilon)\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx &\leq \eta_2\delta(l_1\|\nabla u\|_2^2+l_2\|\nabla\upsilon\|_2^2)\\ &\quad+\frac{(1-l_2)c_s^2}{4\delta}(g_2\ o\ \upsilon)(t).\end{aligned} \tag{2.4.8}$$

Lemma 2.4.3. *There exists two positive constants* λ_1,λ_2 *depending on* ϵ *and* M *such that for all* $t>0$

$$\lambda_1E(t)\leq L(t)\leq\lambda_2E(t), \tag{2.4.9}$$

for M *sufficiently large.*

Proof. Thank's to the Holder and Young's inequalities, we have

$$|\psi(t)|\leq\omega\|u\|_2^2+\frac{1}{4\omega}\|u_t\|_2^2+\omega\|\upsilon\|_2^2+\frac{1}{4\omega}\|\upsilon_t\|_2^2, \tag{2.4.10}$$

and

$$\begin{aligned}\varphi(t)&=|-\int_\Omega u_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx\\ &-\int_\Omega \upsilon_t\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx|\\ &\leq\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\int_\Omega\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)^2dx\\ &+\frac{1}{2}\|\upsilon_t\|_2^2+\frac{1}{2}\int_\Omega\left(\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))ds\right)^2dx\\ &\leq\frac{1}{2}\left(\|u_t\|_2^2+(1-l_1)c_s^2\int_0^t g_1(t-s)a_1(u(t)-u(s),u(t)-u(s))ds\right)\\ &+\frac{1}{2}\left(\|\upsilon_t\|_2^2+(1-l_2)c_s^2\int_0^t g_2(t-s)a_2(\upsilon(t)-\upsilon(s),\upsilon(t)-\upsilon(s))ds\right)\\ &\leq\frac{1}{2}\left(\|u_t\|_2^2+(1-l_1)c_s^2\left(\frac{2\beta E(0)}{l_1}\right)(g_1ou)(t)\right)\\ &+\frac{1}{2}\left(\|\upsilon_t\|_2^2+(1-l_2)c_s^2\left(\frac{2\beta E(0)}{l_2}\right)(g_2o\upsilon)(t)\right).\end{aligned} \tag{2.4.11}$$

It follows from (2.4.4) that $\forall c > 0$

$$\begin{aligned}
|I(t)| &\leq \left|\xi_1(t)\int_\Omega\int_0^1 e^{-k_1\tau_2(t)}z_1^2(x,k_1,t)dk_1dx\right| \\
&+ \left|\xi_2(t)\int_\Omega\int_0^1 e^{-k_2\tau_2(t)}z_2^2(x,k_2,t)dk_2dx\right| \\
&\leq c\xi_1(t)e^{-k_1\tau_2(t)}\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx \\
&+ c\xi_2(t)e^{-k_2\tau_2(t)}\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx.
\end{aligned} \tag{2.4.12}$$

Hence, combining (2.4.10)-(2.4.12). This yields

$$\begin{aligned}
&|L(t) - ME(t)| = \epsilon\psi(t) + \varphi(t) + \epsilon I(t) \leq \epsilon\omega c_s^2 a_1(u(t),u(t)) \\
&+ \epsilon\omega c_s^2 a_2(\upsilon(t),\upsilon(t)) + \left(\frac{\epsilon}{4\omega}+\frac{1}{2}\right)\|u_t\|_2^2 + \left(\frac{\epsilon}{4\omega}+\frac{1}{2}\right)\|\upsilon_t\|_2^2 \\
&+ c\xi_1(t)e^{-k_1\tau_2(t)}\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx + \frac{(1-l_1)c_s^2}{2}\left(\frac{2\beta E(0)}{l_1}\right)(g_1ou)(t) \\
&+ c\xi_2(t)e^{-k_2\tau_2(t)}\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx + \frac{(1-l_2)c_s^2}{2}\left(\frac{2\beta E(0)}{l_2}\right)(g_2o\upsilon)(t).
\end{aligned} \tag{2.4.13}$$

Where $c_1 = \epsilon\omega c_s^2, c_2 = \epsilon\omega c_s^2,\ \ c_3 = \left(\frac{\epsilon}{4\omega}+\frac{1}{2}\right),\ c_4 = \left(\frac{\epsilon}{4\omega}+\frac{1}{2}\right),\ c_5 = \frac{(1-l_1)c_s^2}{2}\left(\frac{2\beta E(0)}{l_1}\right),$ $c_6 = \frac{(1-l_2)c_s^2}{2}\left(\frac{2\beta E(0)}{l_2}\right)$, $c_7 = c_8 = c$. Finally we obtain

$$|L(t) - ME(t)| \leq c_9 E(t), \tag{2.4.14}$$

where $c_9 = max(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$. Thus, from the definition of $E(t)$ and selecting M sufficiently large to get

$$\lambda_2 E(t) \leq L(t) \leq \lambda_1 E(t). \tag{2.4.15}$$

Where $\lambda_1 = (M - c_9)$, $\lambda_2 = (M + c_9)$. This completes the proof. □

Lemma 2.4.4. *The functional defined in (2.4.4) satisfies*

$$\begin{aligned}
&\frac{dI(t)}{dt} \leq \frac{\xi_1(t)}{2\tau_0}\|u_t\|_2^2 - \xi_1(t)\frac{c}{2\tau_1}\int_\Omega z_1^2(x,1,t)dx + \frac{\xi_2(t)}{2\tau_0}\|\upsilon_t\|_2^2 \\
&- \xi_2(t)\frac{c}{2\tau_1}\int_\Omega z_2^2(x,1,t)dx - \xi_2(t)k_2e^{-\tau_2(t)k_2}\tau_0\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx \\
&- \xi_1(t)k_1e^{-\tau_2(t)k_1}\tau_0\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx.
\end{aligned} \tag{2.4.16}$$

Where τ_0, τ_1 are a positive constants.

Proof. Taking derivative of (2.4.4) and using the same technique as (2.3.18) produces

$$
\begin{aligned}
\frac{dI(t)}{dt} &= \frac{d}{dt}\left[\xi_1(t)e^{-k_1\tau_2(t)}\int_\Omega\int_0^1 z_1^2(x,k_1,t))dk_1dx\right] \\
&+ \frac{d}{dt}\left[\xi_2(t)e^{-k_2\tau_2(t)}\int_\Omega\int_0^1 z_2^2(x,k_2,t))dk_2dx\right] \\
&= \xi_1'(t)e^{-\tau_2(t)k_1}\int_\Omega\int_0^1 z_1^2(x,k_1,t))dkdx - \xi_1(t)k_1e^{-\tau_2(t)k_1}\tau_2'(t)\int_\Omega\int_0^1 z_2^2(x,k_1,t))dk_1dx \\
&+ \xi_2'(t)e^{-\tau_2(t)k_2}\int_\Omega\int_0^1 z_2^2(x,k_2,t))dk_2dx - \xi_2(t)k_2e^{-\tau_2(t)k_2}\tau_2'(t)\int_\Omega\int_0^1 z_2^2(x,k_2,t))dk_2dx \\
&+ \frac{1}{\tau_2(t)}e^{-\tau_2(t)k_1}\tau_2(t)\xi_1(t)\int_\Omega\int_0^1 \frac{d}{dt}z_1^2(x,k_1,t))dk_1dx \\
&+ \frac{1}{\tau_2(t)}e^{-\tau_2(t)k_2}\tau_2(t)\xi_2(t)\int_\Omega\int_0^1 \frac{d}{dt}z_2^2(x,k_2,t))dk_2dx \\
&= \xi_1'(t)e^{-\tau_2(t)k_1}\int_\Omega\int_0^1 z_1^2(x,k_1,t))dk_1dx - \xi_1(t)k_1e^{-\tau_2(t)k_1}\tau_2'(t)\int_\Omega\int_0^1 z_1^2(x,k_1,t))dk_1dx \\
&+ \xi_2'(t)e^{-\tau_2(t)k_2}\int_\Omega\int_0^1 z_2^2(x,k_2,t))dk_2dx - \xi_2(t)k_2e^{-\tau_2(t)k_2}\tau_2'(t)\int_\Omega\int_0^1 z_2^2(x,k_2,t))dk_2dx \\
&+ \frac{1}{\tau_2(t)}e^{-\tau_2(t)k_1}\xi_1(t)\int_\Omega\int_0^1 \frac{\partial}{\partial k_1}(1-\tau_2'(t)k_1)z_1^2(x,k_1,t))dk_1dx \\
&+ \frac{1}{\tau_2(t)}e^{-\tau_2(t)k_2}\xi_2(t)\int_\Omega\int_0^1 \frac{\partial}{\partial k_2}(1-\tau_2'(t)k_2)z_2^2(x,k_2,t))dk_2dx \\
&\leq -\xi_1(t)k_1e^{-\tau_2(t)k_1}\tau_2'(t)\int_\Omega\int_0^1 z_1^2(x,k_1,t))dk_1dx - \xi_2(t)k_1e^{-\tau_2(t)k_1}\tau_2'(t)\int_\Omega\int_0^1 z_2^2(x,k_1,t))dk_1dx \\
&+ \frac{1}{\tau_2(t)}\xi_1(t)\int_\Omega[z_1^2(x,0,t)) - z_1^2(x,1,t)]dx + \xi_1(t)\frac{\beta}{\tau_2(t)}\int_\Omega z_1^2(x,1,t)dx \\
&+ \frac{1}{\tau_2(t)}\xi_2(t)\int_\Omega[z_2^2(x,0,t)) - z_2^2(x,1,t)]dx + \xi_1(t)\frac{\beta}{\tau_2(t)}\int_\Omega z_1^2(x,1,t)dx \\
&\leq \frac{\xi_1(t)}{2\tau_0}\|u_t\|_2^2 - \xi_1(t)\frac{c}{2\tau_1}\int_\Omega z_1^2(x,1,t)dx \\
&- \xi_1(t)k_1e^{-\tau_2(t)k_1}\tau_0\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx + \frac{\xi_2(t)}{2\tau_0}\|v_t\|_2^2 \\
&- \xi_2(t)\frac{c}{2\tau_1}\int_\Omega z_2^2(x,1,t)dx - \xi_2(t)k_2e^{-\tau_2(t)k_2}\tau_0\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx.
\end{aligned}
\tag{2.4.17}
$$

This ends the proof. □

Lemma 2.4.5. *The functional defined in (2.4.2) satisfies*

$$\begin{aligned}
&\frac{d\psi(t)}{dt} \leq \left(1+\frac{\mu_1}{4\beta}\right)\|u_t\|_2^2 + \left(1+\frac{\alpha_1}{4\beta}\right)\|\upsilon_t\|_2^2 - (p+1)\int_\Omega F(u(t),\upsilon(t))dx \\
&+\left[\mu_1\left(1+\tfrac{2\beta c_s^2}{a_{01}}\right)-l_1\right]a_1(u(t),u(t)) + \left[\alpha_2\left(1+\tfrac{2\beta c_s^2}{a_{01}}\right)-l_2\right]a_2(\upsilon(t),\upsilon(t)) \\
&+\tfrac{\mu_1}{4\beta}\|z_1(x,1,t)\|_2^2 + \tfrac{\alpha_2}{4\beta}\|z_2(x,1,t)\|_2^2 + \tfrac{N}{4a_{01}\mu}(1-l_1)(g_1\circ u)(t) \\
&+\tfrac{N}{4a_{02}\mu}(1-l_2)(g_2\circ\upsilon)(t)
\end{aligned} \tag{2.4.18}$$

Proof. Taking derivative of (2.4.2) and using (2.3.1), gives us

$$\begin{aligned}
&\frac{d\psi(t)}{dt} = \int_\Omega u_{tt}udx + \int_\Omega u_{tt}udx + \|u_t\|_2^2 + \|\upsilon_t\|_2^2 \\
&= \|u_t\|_2^2 + \|\upsilon_t\|_2^2 - a_1(u(t),u(t)) - a_2(\upsilon(t),\upsilon(t)) \\
&+\int_\Omega\int_0^t g_1(t-s)A_1\nabla u(s)\nabla u(t)dsdx \\
&+\int_\Omega\int_0^t g_2A_2(t-s)\nabla\upsilon(s)\nabla\upsilon(t)dsdx - \mu_2\int_\Omega z_1(x,1,t)udx - \mu_1\int_\Omega u_tudx \\
&-\alpha_2\int_\Omega z_2(x,1,t)\upsilon dx - \alpha_1\int_\Omega \upsilon_t\upsilon dx - (p+1)\int_\Omega F(u(t),\upsilon(t))dx,
\end{aligned} \tag{2.4.19}$$

following [31], yields

$$\begin{aligned}
&\int_\Omega A_1\int_0^t g_1(t-s)(\nabla u(t)\nabla u(s)dsdx \\
&= \sum_{i,j=1}^{N}\int_0^t g_1(t-s)\int_\Omega a_{1ij}(x)\frac{\partial u(t)}{\partial x_j}\left(\frac{\partial u(s)}{\partial x_i}-\frac{\partial u(t)}{\partial x_i}+\frac{\partial u(t)}{\partial x_i}\right)dxds \\
&= \sum_{i,j=1}^{N}\int_\Omega\int_0^t a_{1ij}(x)\frac{\partial u(t)}{\partial x_j}\frac{\partial u(t)}{\partial x_i}dsdx \\
&+\sum_{i,j=1}^{N}\int_\Omega\int_0^t\left(g_1(t-s)a_{1ij}(x)\frac{\partial u(t)}{\partial x_j}\left(\frac{\partial u(s)}{\partial x_i}-\frac{\partial u(t)}{\partial x_i}\right)\right)dsdx \\
&\leq (1-l_1)a_1(u(t),u(t)) + \mu\sum_{i,j=1}^{N}\int_\Omega\left(a_{1ij}(x)\frac{\partial u(s)}{\partial x_j}ds\right)^2dx \\
&+\frac{1}{\mu}\sum_{i,j=1}^{N}\int_\Omega\left(\int_0^t g_1(t-s)\left(\frac{\partial u(s)}{\partial x_i}-\frac{\partial u(t)}{\partial x_i}\right)ds\right)^2dx \\
&\leq \left[(1-l_1)+\tfrac{\mu a_{11}}{a_{01}}\right]a_1(u(t),u(t)) + \tfrac{N}{4a_{01}\mu}(1-l)(g_1ou)(t),
\end{aligned} \tag{2.4.20}$$

in the same way

$$\begin{aligned}
\int_\Omega A_2\int_0^t g_2(t-s)(\nabla\upsilon(t)\nabla\upsilon(s)dsdx &\leq \left[(1-l_2)+\frac{\mu a_{22}}{a_{02}}\right]a_{22}(u(t),u(t)) \\
&+\frac{N}{4a_{02}\mu}(1-l_2)(g_2o\upsilon)(t),
\end{aligned} \tag{2.4.21}$$

for the seventh, eight, ninth and tenth term in (2.4.19), Holder, Young's inequalities to get

$$\left|\int_\Omega u_t u dx\right| \leq \frac{\beta c_s^2}{a_{01}} a_1(u(t),u(t)) + \frac{1}{4\beta}\|u_t\|_2^2, \tag{2.4.22}$$

$$\left|\int_\Omega \upsilon_t \upsilon dx\right| \leq \frac{\beta c_s^2}{a_{02}} a_2(\upsilon(t),\upsilon(t)) + \frac{1}{4\beta}\|\upsilon_t\|_2^2, \tag{2.4.23}$$

$$\left|\int_\Omega z_1(x,1,t) u dx\right| \leq \frac{\beta c_s^2}{a_{01}} a_1(u(t),u(t)) + \frac{1}{4\beta}\int_\Omega z_1^2(x,1,t)dx, \tag{2.4.24}$$

$$\left|\int_\Omega z_2(x,1,t) \upsilon dx\right| \leq \frac{\beta c_s^2}{a_{02}} a_1(\upsilon(t),\upsilon(t)) + \frac{1}{4\beta}\int_\Omega z_2^2(x,1,t)dx. \tag{2.4.25}$$

Inserting (2.4.20)-(2.4.25), we get finally

$$\begin{aligned}
&\frac{d\psi(t)}{dt} \leq \left(1+\frac{\mu_1}{4\beta}\right)\|u_t\|_2^2 + \left(1+\frac{\alpha_1}{4\beta}\right)\|\upsilon_t\|_2^2 - (p+1)\int_\Omega F(u(t),\upsilon(t))dx \\
&+\left[\mu_1\left(1+\tfrac{2\beta c_s^2}{a_{01}}\right) - l_1\right] a_1(u(t),u(t)) + \left[\alpha_2\left(1+\tfrac{2\beta c_s^2}{a_{01}}\right) - l_2\right] a_2(\upsilon(t),\upsilon(t)) \\
&+\frac{\mu_1}{4\beta}\|z_1(x,1,t)\|_2^2 + \frac{\alpha_2}{4\beta}\|z_2(x,1,t)\|_2^2 + \frac{N}{4a_{01}\mu}(1-l_1)(g_1\circ u)(t) \\
&+\frac{N}{4a_{02}\mu}(1-l_2)(g_2\circ \upsilon)(t).
\end{aligned} \tag{2.4.26}$$

□

Lemma 2.4.6. *The functional defined in (2.4.3) satisfies*

$$\begin{aligned}
&\frac{d\varphi(t)}{dt} \leq (\beta+\mu_1-g_{10})\|u_t\|_2^2 + (\beta+\alpha_1-g_{20})\|\upsilon_t\|_2^2 \\
&+\left\{\tfrac{\beta}{a_{01}} + \tfrac{a_{11}\beta}{a_{01}}(1-l_1^2) + \left(\tfrac{2\lambda\delta}{a_{01}} - \tfrac{2a_{11}\beta}{a_{01}}\right)\right\} a_1(u(t),u(t)) \\
&+\left\{\tfrac{\beta}{a_{02}} + \tfrac{a_{22}\beta}{a_{02}}(1-l_2^2) + \left(\tfrac{2\lambda\delta}{a_{02}} - \tfrac{2a_{22}\beta}{a_{02}}\right)\right\} a_2(\upsilon(t),\upsilon(t)) \\
&+(1-l_1)\left\{\left\{\tfrac{1}{a_{01}\beta}\left(\tfrac14 + 2\beta a_{11} + \tfrac{N}{4}\right) + \tfrac{c_s^2}{4\delta}\right\} + (1-l_1)c_s^2(\mu_1+\mu_2)\right\} g_1\circ u)(t) \\
&+(1-l_2)\left\{\left\{\tfrac{1}{a_{02}\beta}\left(\tfrac14 + 2\beta a_{22} + \tfrac{N}{4}\right) + \tfrac{c_s^2}{4\delta}\right\} + (1-l_2)c_s^2(\alpha_1+\alpha_2)\right\} g_2\circ \upsilon)(t) \\
&+\mu_2\|z_1(x,1,t)\|_2^2 + \alpha_2\|z_2(x,1,t)\|_2^2 + \tfrac{g_1(0)c_s^2}{4\beta}(-g_1'\circ u)(t) + \tfrac{g_2(0)c_s^2}{4\beta}(-g_2'\circ \upsilon)(t).
\end{aligned} \tag{2.4.27}$$

Proof. Taking derivative of (2.4.3) and using (2.3.1), we obtain

$$
\begin{aligned}
\frac{d\varphi(t)}{dt} &= -\int_\Omega u_{tt}\int_0^t g_1(t-s)(u(t)-u(s))dsdx - \left(\int_0^t g_1(s)ds\right)\int_\Omega u_t^2dx \\
&- \int_\Omega u_t\int_0^t g_1'(t-s)(u(t)-u(s))dsdx - \int_\Omega \upsilon_{tt}\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx \\
&- \int_\Omega \upsilon_t\int_0^t g_2'(t-s)(\upsilon(t)-\upsilon(s))dsdx - \left(\int_0^t g_2(s)ds\right)\int_\Omega \upsilon_t^2dx \\
&= \sum_{i,j=1}^N\int_\Omega a_{1ij}(x)\frac{\partial u(t)}{\partial x_j}\left(\int_0^t g_1(t-s)\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u(s)}{\partial x_i}\right)ds\right)dx \\
&+ \sum_{i,j=1}^N\int_\Omega a_{2ij}(x)\frac{\partial \upsilon(t)}{\partial x_j}\left(\int_0^t g_2(t-s)\left(\frac{\partial \upsilon(t)}{\partial x_i}-\frac{\partial \upsilon(s)}{\partial x_i}\right)ds\right)dx \\
&- \sum_{i,j=1}^N\int_\Omega \left(\int_0^t g_1(t-s)\frac{\partial u(s)}{\partial x_i}ds\right)\left(\int_0^t g_1(t-s)\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u(s)}{\partial x_i}\right)ds\right)dx \\
&- \sum_{i,j=1}^N\int_\Omega \left(\int_0^t g_2(t-s)\frac{\partial \upsilon(s)}{\partial x_i}ds\right)\left(\int_0^t g_2(t-s)\left(\frac{\partial \upsilon(t)}{\partial x_i}-\frac{\partial \upsilon(s)}{\partial x_i}\right)ds\right)dx \\
&- \int_\Omega f_1(u(t),\upsilon(t))\left(\int_0^t g_1(t-s)u(t)-u(s)ds\right)dx \\
&- \int_\Omega f_2(u(t),\upsilon(t))\left(\int_0^t g_2(t-s)\upsilon(t)-\upsilon(s)ds\right)dx \\
&+ \int_\Omega \mu_1u(t)\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&+ \int_\Omega \alpha_1\upsilon(t)\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx \\
&+ \int_\Omega \mu_2z_1(x,1,t)\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&+ \int_\Omega \alpha_2z_2(x,1,t)\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx \\
&- \int_\Omega u_t\int_0^t g_1'(t-s)(u(t)-u(s))dsdx - \left(\int_0^t g_1(s)ds\right)\int_\Omega u_t^2dx \\
&- \int_\Omega \upsilon_t\int_0^t g_2'(t-s)(\upsilon(t)-\upsilon(s))dsdx - \left(\int_0^t g_2(s)ds\right)\int_\Omega \upsilon_t^2dx.
\end{aligned}
\tag{2.4.28}
$$

Using Young's inequality and the embedding $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$, we infer

$$
\begin{aligned}
&\sum_{i,j=1}^N\int_\Omega a_{1ij}(x)\frac{\partial u(t)}{\partial x_j}\left(\int_0^t g_1(t-s)\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u(s)}{\partial x_i}\right)ds\right)dx \\
&+ \sum_{i,j=1}^N\int_\Omega a_{2ij}(x)\frac{\partial \upsilon(t)}{\partial x_j}\left(\int_0^t g_2(t-s)\left(\frac{\partial \upsilon(t)}{\partial x_i}-\frac{\partial \upsilon(s)}{\partial x_i}\right)ds\right)dx \\
&\le \tfrac{\beta}{a_{01}}a_1(u(t),u(t)) + \tfrac{\beta}{a_{02}}a_2(\upsilon(t),\upsilon(t)) + \tfrac{(1-l_1)}{4a_{01}\beta}(g_1ou)(t) + \tfrac{(1-l_2)}{4a_{02}\beta}(g_2o\upsilon)(t),
\end{aligned}
\tag{2.4.29}
$$

and

$$\begin{array}{l}\left|\sum_{i,j=1}^{N}\int_{\Omega}\left(\int_0^t g_1(t-s)\frac{\partial u(s)}{\partial x_i}ds\right)\left(\int_0^t g_1(t-s)\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u(s)}{\partial x_i}\right)ds\right)dx\right| \\ +\left|\sum_{i,j=1}^{N}\int_{\Omega}\left(\int_0^t g_2(t-s)\frac{\partial \upsilon(s)}{\partial x_i}ds\right)\left(\int_0^t g_2(t-s)\left(\frac{\partial \upsilon(t)}{\partial x_i}-\frac{\partial \upsilon(s)}{\partial x_i}\right)ds\right)dx\right| \\ \leq \beta\sum_{i,j=1}^{N}\int_{\Omega}\left(\int_0^t g_1(t-s)\frac{\partial u(s)}{\partial x_i}ds\right)^2dx \\ +\frac{1}{\beta}\sum_{i,j=1}^{N}\int_{\Omega}\left(\int_0^t g_1(t-s)\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u(s)}{\partial x_i}\right)ds\right)^2dx \\ +\beta\sum_{i,j=1}^{N}\int_{\Omega}\left(\int_0^t g_2(t-s)\frac{\partial \upsilon(s)}{\partial x_i}ds\right)^2dx \\ +\frac{1}{\beta}\sum_{i,j=1}^{N}\int_{\Omega}\left(\int_0^t g_2(t-s)\left(\frac{\partial \upsilon(t)}{\partial x_i}-\frac{\partial \upsilon(s)}{\partial x_i}ds\right)\right)^2dx \\ \leq \frac{a_{11}\beta}{a_{01}}(1-l_1)^2a_1(u(t),u(t))+\frac{a_{22}\beta}{a_{01}}(1-l_2)^2a_2(\upsilon(t),\upsilon(t)) \\ +\frac{(1-l_1)}{a_{01}}\left[2\beta a_{11}+\frac{N}{4\beta}\right](g_1\circ u)(t)+\frac{(1-l_2)}{a_{02}}\left[2\beta a_{22}+\frac{N}{4\beta}\right](g_2\circ \upsilon)(t).\end{array} \tag{2.4.30}$$

From the lemma 2.4.2, we deduce

$$\begin{array}{l}\int_{\Omega}f_1(u,\upsilon)\int_0^t g_1(t-s)(u(t)-u(s))dsdx \leq \lambda\delta\left(l_1\|\nabla u\|_2^2+l_2\|\nabla \upsilon\|_2^2\right) \\ +\frac{(1-l_1)c_s^2}{4\delta}(g_2ou)(t) \leq \frac{\lambda\delta l_1}{a_{01}}a_1(u(t),u(t))+\frac{\lambda\delta l_2}{a_{02}}a_2(\upsilon(t),\upsilon(t))+\frac{(1-l_1)c_s^2}{4\delta}(g_1ou)(t),\end{array} \tag{2.4.31}$$

also

$$\begin{array}{l}\int_{\Omega}f_2(u,\upsilon)\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx \leq \lambda\delta\left(l_1\|\nabla u\|_2^2+l_2\|\nabla \upsilon\|_2^2\right) \\ +\frac{(1-l_2)c_s^2}{4\delta}(g_2o\upsilon)(t) \leq \frac{\lambda\delta l_2}{a_{02}}a_2(\upsilon(t),\upsilon(t))+\frac{\lambda\delta l_1}{a_{01}}a_1(u(t),u(t))+\frac{(1-l_2)c_s^2}{4\delta}(g_2o\upsilon)(t).\end{array} \tag{2.4.32}$$

Since $g_1,\ g_2$ are positive, continuous and $g_1(0)>0$, $g_2(0)>0$ for any t_0, we have

$$\int_0^t g_1(s)ds \geq \int_0^{t_0} g_1(s)ds = g_{10}, \quad \forall t \geq t_0, \tag{2.4.33}$$

$$\int_0^t g_2(s)ds \geq \int_0^{t_0} g_2(s)ds = g_{20}, \quad \forall t \geq t_0, \tag{2.4.34}$$

then we use (2.4.33) and (2.4.34) to get

$$\begin{array}{l}\int_{\Omega}u_t\int_0^t g_1'(t-s)(u(t)-u(s))dsdx-\left(\int_0^t g_1(s)ds\right)\int_{\Omega}u_t^2dx \\ \leq \beta\|u_t\|_2^2+\frac{g_1(0)c_s^2}{4\beta}(-g_1'ou)(t)-g_{10}\|u_t\|_2^2,\end{array} \tag{2.4.35}$$

and

$$\begin{aligned}&\int_\Omega \upsilon_t \int_0^t g_2'(t-s)(\upsilon(t)-\upsilon(s))dsdx - \left(\int_0^t g_2(s)ds\right)\int_\Omega \upsilon_t^2 dx \\ &\le \beta\|\upsilon_t\|_2^2 + \tfrac{g_2(0)c_s^2}{4\beta}(-g_2' o \upsilon)(t) - g_{20}\|\upsilon_t\|_2^2.\end{aligned} \tag{2.4.36}$$

From the lemma 2.4.1 by taking $\gamma = 0$, we have, for $\delta > 0$,

$$\left|-\int_\Omega \mu_1 u_t \int_0^t g_1(t-s)(u(t)-u(s))dsdx\right| \le \mu_1\|u_t\|_2^2 + \mu_1(1-l_1)^2 c_s^2 (g_1 o u)(t), \tag{2.4.37}$$

$$\left|-\int_\Omega \alpha_2 \upsilon_t \int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx\right| \le \alpha_2\|\upsilon_t\|_2^2 + \alpha_2(1-l_2)^2 c_s^2 (g_2 o \upsilon)(t), \tag{2.4.38}$$

$$\begin{aligned}\left|-\int_\Omega \mu_2 z_1(x,1,t) \int_0^t g_1(t-s)(u(t)-u(s))dsdx\right| &\le \mu_2 \int_\Omega z_1^2(x,1,t)dx \\ &\quad + \mu_2(1-l_1)^2 c_s^2 (g_1 o u)(t),\end{aligned} \tag{2.4.39}$$

and

$$\left|-\int_\Omega \alpha_2 z_2(x,1,t) \int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx\right| \le \alpha_2 \int_\Omega z_2^2(x,1,t)dx + \alpha_2(1-l_1)^2 c_s^2 (g_2 o \upsilon)(t). \tag{2.4.40}$$

A substitution of (2.4.36)-(2.4.40) into (2.4.28) yields

$$\begin{aligned}\frac{d\varphi(t)}{dt} &\le (\beta + \mu_1 - g_{10})\|u_t\|_2^2 + (\beta + \alpha_1 - g_{20})\|\upsilon_t\|_2^2 \\ &+ \left\{\tfrac{\beta}{a_{01}} + \tfrac{a_{11}\beta}{a_{01}}(1-l_1^2) + \left(\tfrac{2\lambda\delta}{a_{01}} - \tfrac{2a_{11}\beta}{a_{01}}\right)\right\} a_1(u(t),u(t)) \\ &+ \left\{\tfrac{\beta}{a_{02}} + \tfrac{a_{22}\beta}{a_{02}}(1-l_2^2) + \left(\tfrac{2\lambda\delta}{a_{02}} - \tfrac{2a_{22}\beta}{a_{02}}\right)\right\} a_2(\upsilon(t),\upsilon(t)) \\ &+(1-l_1)\left\{\left\{\tfrac{1}{a_{01}\beta}\left(\tfrac{1}{4} + 2\beta a_{11} + \tfrac{N}{4}\right) + \tfrac{c_s^2}{4\delta}\right\} + (1-l_1)c_s^2(\mu_1+\mu_2)\right\} g_1 \circ u)(t) \\ &+(1-l_2)\left\{\left\{\tfrac{1}{a_{02}\beta}\left(\tfrac{1}{4} + 2\beta a_{22} + \tfrac{N}{4}\right) + \tfrac{c_s^2}{4\delta}\right\} + (1-l_2)c_s^2(\alpha_1+\alpha_2)\right\} g_2 \circ \upsilon)(t) \\ &+\mu_2\|z_1(x,1,t)\|_2^2 + \alpha_2\|z_2(x,1,t)\|_2^2 + \tfrac{g_1(0)c_s^2}{4\beta}(-g_1' \circ u)(t) + \tfrac{g_2(0)c_s^2}{4\beta}(-g_2' \circ \upsilon)(t).\end{aligned} \tag{2.4.41}$$

□

Theorem 2.4.7. *Let $(u_0, , \upsilon_0) \in (H_0^1(\Omega) \cap H^2(\Omega))^2, (u_1, \upsilon_1) \in (H_0^1(\Omega))^2$ be given. Assume that $(A_0) - (A_5)$ hold. Then, for each $t_0 > 0$, there exist strictly positive constants K and κ such that the solution of (2.4.1) satisfies*

$$E(t) \le K e^{-\alpha \int_{t_0}^t \zeta(s)ds}, \quad for \quad t \ge t_0. \tag{2.4.42}$$

Proof. Taking a derivative of (2.4.1) and using the lemmas 2.4.4, 2.4.5 and 2.4.6, we infer

$$
\begin{aligned}
\frac{dL(t)}{dt} \leq &-\left(a_1 M - \epsilon\left(1+\frac{\mu_1}{4\beta}-\frac{\xi_1(t)}{2\tau_0}\right)+g_{10}-\mu_1-\beta\right)\|u_t\|_2^2 \\
&-\left(a_2 M - \epsilon\left(1+\frac{\alpha_1}{4\beta}-\frac{\xi_2(t)}{2\tau_0'}\right)+g_{20}-\alpha_1-\beta\right)\|\upsilon_t\|_2^2 \\
&-\left\{\frac{a_{11}\beta}{a_{01}}(1+l_1^2)-\frac{\beta}{a_{01}}-\frac{2\lambda\delta}{a_{01}}-\epsilon\left\{\mu_2\left(1+\frac{2\beta c_s^2}{a_{01}}\right)+l_1\right\}\right\}a_1(u(t),u(t)) \\
&-\left\{\frac{a_{22}\beta}{a_{01}}(1+l_2^2)-\frac{\beta}{a_{02}}-\frac{2\lambda\delta}{a_{02}}-\epsilon\left\{\alpha_2\left(1+\frac{2\beta c_s^2}{a_{02}}\right)+l_2\right\}\right\}a_2(\upsilon(t),\upsilon(t)) \\
&+(1-l_1)\left\{\frac{1}{a_{01}\beta}\left(\frac{1}{4}+2\beta a_{11}+\frac{N}{4}+\frac{\epsilon}{4\mu}\right)+\frac{c_s^2}{4\delta}\right\}(g_1\circ u)(t) \\
&+(1-l_1)^2(\mu_1+\mu_2)(g_1\circ u)(t)-(Ma_3+\mu_1)\|z_1(x,1,t)\|_2^2 \\
&+(1-l_2)\left\{\frac{1}{a_{02}\beta}\left(\frac{1}{4}+2\beta a_{22}+\frac{N}{4}+\frac{\epsilon}{4\mu}\right)+\frac{c_s^2}{4\delta}\right\}(g_2\circ \upsilon)(t) \\
&+(1-l_2)^2(\alpha_1+\alpha_2)(g_2\circ\upsilon)(t)+\left(\frac{M}{2}-\frac{g_2(0)c_s^2}{4\beta}\right)(g_2' o \upsilon)(t) \\
&+\left(\epsilon\left(\frac{\mu_2}{4\beta}-\xi_1(t)\frac{c}{2\tau_1}\right)\right)\|z_1(x,1,t)\|_2^2-\epsilon(p+1)\int_\Omega F(u(t),\upsilon(t))dx \\
&-\left((Ma_4+\alpha_1)-\epsilon\left(\frac{\alpha_2}{4\beta}-\xi_2(t)\frac{c}{2\tau_1}\right)\right)\|z_2(x,1,t)\|_2^2 \\
&-\epsilon\xi_1(t)k_1\tau_0 e^{-\tau_2(t)k_1}\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx+\left(\frac{M}{2}-\frac{g_1(0)c_s^2}{4\beta}\right)(g_1' o u)(t) \\
&-\epsilon\xi_2(t)k_1\tau_0 e^{-\tau_2(t)k_2}\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx.
\end{aligned}
\tag{2.4.43}
$$

At this point, we choose M so large such that

$$\eta_1=\left(a_1 M - \epsilon\left(1+\frac{\mu_1}{4\beta}-\frac{\xi_1(t)}{2\tau_0}\right)+g_{10}-\mu_1-\beta\right)>0,$$

$$\eta_2=\left(a_2 M - \epsilon\left(1+\frac{\alpha_1}{4\beta}-\frac{\xi_2(t)}{2\tau_0'}\right)+g_{20}-\alpha_1-\beta\right)>0,$$

$$\eta_3=\left(\frac{M}{2}-\frac{g_1(0)c_s^2}{4\beta}\right)>0,$$

$$\eta_4=\left(\frac{M}{2}-\frac{g_2(0)c_s^2}{4\beta}\right)>0.$$

Then we choose ϵ sufficiently small such that

$$\eta_5=\left\{\frac{a_{11}\beta}{a_{01}}(1+l_1^2)-\frac{\beta}{a_{01}}-\frac{2\lambda\delta}{a_{01}}-\epsilon\left\{\mu_2\left(1+\frac{2\beta c_s^2}{a_{01}}\right)+l_1\right\}\right\}>0,$$

$$\eta_6 = \left\{ \frac{a_{22}\beta}{a_{01}}(1+l_2^2) - \frac{\beta}{a_{02}} - \frac{2\lambda\delta}{a_{02}} - \epsilon \left\{ \alpha_2 \left(1 + \frac{2\beta c_s^2}{a_{02}} \right) + l_2 \right\} \right\} > 0,$$

and (2.4.43) remains valid. Hence for all $t \geq t_0$, we arrive at

$$\begin{aligned} &\frac{dL(t)}{dt} \leq -\eta_1\|u_t\|_2^2 - \eta_2\|\upsilon_t\|_2^2 - \eta_3(g_1'ou)(t) + \eta_4(g_2'o\upsilon)(t) \\ &-\epsilon(p+1)\int_\Omega F(u(t),\upsilon(t))dx - \eta_5 a_1(u(t),u(t)) - \eta_6 a_2(\upsilon(t),\upsilon(t)) \\ &+\eta_7(g_1\circ u)(t) + \eta_8(g_2\circ\upsilon)(t) - \eta_9\|z_1(x,1,t)\|_2^2 - \eta_{10}\|z_2(x,1,t)\|_2^2 \\ &-\eta_{11}\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx - \eta_{12}\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx, \end{aligned} \tag{2.4.44}$$

which yields

$$\frac{dL(t)}{dt} \leq -\eta_{14}E(t) + \eta_{15}\left(g_1\circ u)(t) + g_2\circ\upsilon)(t)\right), \quad \forall t \geq t_0, \tag{2.4.45}$$

where $\eta_i, i = 5,6,7..$ are some positive constants. Multiplying the above inequality by $\zeta = \min\{\zeta_1,\zeta_2\}$ and exploiting (A_0), we get, for all $t \geq t_0$

$$\zeta(t)L'(t) \leq -\eta_{14}\zeta(t)E(t) + \zeta(t)\eta_{15}((g_1\circ u)(t) + g_2\circ\upsilon)(t)).$$

Since $g_1'(t) \leq -\zeta(t)g_1(t)$ and $g_2'(t) \leq -\zeta(t)g_2(t)$ and using the fact that

$$-(g_1'\circ u(t) + (g_2'\circ\upsilon(t)) \leq -2E'(t),$$

by (2.3.24), we get

$$\begin{aligned} \zeta(t)L'(t) &\leq -\eta_{14}\zeta(t)E(t) - \eta_{15}\left(g_1'\circ u)(t) + g_2'\circ\upsilon)(t)\right) \\ &\leq -\eta_{14}\zeta(t)E(t) - 2\eta_{15}E'(t), \forall\, t \geq t_0. \end{aligned} \tag{2.4.46}$$

Define $\chi(t) = \zeta(t)L(t) + 2\eta_{15}E(t)$, which is equivalent to $E(t)$ and $\zeta'(t) \leq 0$, $\forall t \geq 0$, we obtain

$$\begin{aligned} \chi'(t) &\leq \zeta'(t)L(t) - \eta_{14}\zeta(t)E(t) \\ &\leq -\alpha\zeta(t)E(t), \quad \forall\, t \geq t_0. \end{aligned} \tag{2.4.47}$$

Integrating the last inequality over (t_0, t), we conclude that

$$\chi(t) \leq \chi(0)e^{-\alpha\int_{t_0}^t \zeta(s)ds}. \tag{2.4.48}$$

Then, the equivalent relation between $\chi(t)$ and $E(t)$ yields

$$E(t) \leq Ke^{-\alpha\int_{t_0}^t \zeta(s)ds}. \tag{2.4.49}$$

This completes the proof. □

Remark 2.4.2. *We illustrate the energy decay rate given by Theorem 4.1 through the following examples which are introduced in [23, 24].*

1. *If* $g_1(t) = a_1 e^{-b_1(1+t)^{\nu_1}}$, $g_2(t) = \frac{a_2}{(1+t)^{\nu_2}}$, *for* $a_i > 0$ *and* $\nu_i > 0$, *then* $\zeta_1(t) = b_1\nu_1(1+t)^{\nu_1-1}$ *and* $\zeta_2(t) = \frac{\nu_2}{1+t}$ *satisfy the condition* (2.1). *Thus (2.4.49) gives the estimate*

$$E(t) \leq K(1+t)^{-\alpha}$$

2. *If* $g_1(t) = a_1 e^{-(1+t)^{\nu_1}}$, $g_2(t) = a_2 e^{-(1+t)^{\nu_2}}$ *for* $a_i, \nu_i > 0 (i = 1, 2)$, *then* $\zeta_i(t) = \nu_i(1+t)^{\min(0,\nu_i-1)}$ *satisfies the condition* (2.1). *Thus (2.4.49) gives the estimate*

$$E(t) \leq K e^{-\alpha(1+t)^{\min(1,\nu_1,\nu_2)}}$$

3. *If* $g_1(t) = a_1 e^{-(\ln(1+t))^{\nu_1}}$, $g_2(t) = a_2 e^{-(\ln(1+t))^{\nu_2}}$ *for* $a_i > 0$ *and* $\nu_i > 1 (i = 1, 2)$, *then* $\zeta_i(t) = \frac{\nu_i(\ln(1+t))^{\nu_i-1}}{1+t} (i = 1, 2)$ *satisfies the condition* (2.1) . *Thus (2.4.49) gives the estimate*

$$E(t) \leq K e^{-\alpha(\ln(1+t))^{\min(\nu_1,\nu_2)}}$$

Chapter 3

Decay Property For Solutions In Elastic Solids Without Mechanical Damping

3.1 Introduction

In this paper, we consider the one-dimensional linear system of a homogeneous and isotropic elastic solid with the heat conduction given by:

$$\begin{cases} \rho_1 u_{tt} - \mu u_{xx} - b\varphi_x = 0 \\ \\ \rho\kappa\varphi_{tt} - \alpha\varphi_{xx} + bu_x + a\varphi + \beta\theta_x = 0 \\ \\ \theta_{tt} - \delta\theta_{xx} + \gamma\varphi_{ttx} - k\theta_{txx} = 0 \end{cases} \tag{3.1.1}$$

with the initial data

$$(u, u_t, \varphi, \varphi_t, \theta, \theta_t)(x, 0) = (u_0, u_1, \varphi_0, \varphi_1, \theta_0, \theta_1), \tag{3.1.2}$$

where u is the longitudinal displacement, φ is the volume fraction, $\rho_1 > 0$, $\rho > 0$ are the mass density, $\kappa > 0$ is the equilibrated inertia and μ, α, a are the constitutive constants which are positive and satisfy

$$\mu a > b^2. \tag{3.1.3}$$

To motivate our work, let us start with the linear theory of elastic materials that has been established by Cowin and Nunziato [40]. The one-dimensional porous-elastic has the form

$$\begin{cases} \rho_1 u_{tt} - \mu u_{xx} - b\varphi_x = 0 \\ \\ \rho\kappa\varphi_{tt} - \alpha\varphi_{xx} + bu_x + a\varphi + \tau\varphi_t = 0 \end{cases} \tag{3.1.4}$$

with initial conditions and mixed boundary conditions and showed that the damping in the porous equation ($\tau\varphi_t$) is not strong enough to obtain an exponential decay. Only the slow decay has been proved. Many authors considered the linear elastic materials problem as in (3.1.4) where the decay of solutions to the problems in elasticity have been investigated in previous works, see ([38],[39],[40],[41],[42]).

Very Recently, L. Djouamai and B. Said-Houari [44] considered the same problem as above

$$\begin{cases} \rho_1 u_{tt} - \mu u_{xx} - b\varphi_x = 0 \\ \\ \rho\kappa\varphi_{tt} - \alpha\varphi_{xx} + bu_x + a\varphi + \tau\varphi_t = 0 \end{cases}$$

and proved that a linear porous dissipation leads to decay rates of regularity-loss of the solution. They showed the decay estimates with the very restriction on the initial data and also proved that any additional mechanical damping is enough to stabilize the system.

Motivated by the previous works, in the present paper, we consider the decay rates for the solution of (3.1.1), from which the estimates in [44] are only particular cases. In order to prove our result, we apply the energy method in the Fourier space to obtain the pointwise estimate for the Fourier image. Our aim is attained by combining the pointwise estimate, Plancherel theorem and some integral estimates.

We introduce the notation used in this chapter. Throughout this work, $\|.\|_{L^q}$ and $\|.\|_{H^1}$ stand for the $L^q(\mathbb{R})$-norm $(2 \leq q \leq \infty)$ and the $H^1(\mathbb{R})$-norm. We denote by $\hat{f}$ the Fourier transform of f :

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}dx, \quad f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{i\xi x}d\xi,$$

and $c_n,\ c(\epsilon_n),\ \delta_n$ for $n = 1, 2...$ are all positive constants.

3.2 The energy method in the Fourier space

In order to exhibit the dissipative nature of system (3.1.1) and following Reference [51], we use the following transformation

$$\widetilde{\theta}(x,t) = \int_0^t \theta(x,s)ds + \chi(x), \tag{3.2.1}$$

with a function $\chi = \chi(x)$ satisfying

$$\delta\chi'' = \theta_1 - k\theta_0'' + \gamma\varphi_1'.$$

Then, we get from (3.1.1) (by writing, for simplicity θ instead of $\widetilde{\theta}$)

$$\begin{cases} \rho_1 u_{tt} - \mu u_{xx} - b\varphi_x = 0 \\ \\ \rho\kappa\varphi_{tt} - \alpha\varphi_{xx} + bu_x + a\varphi + \beta\theta_{tx} = 0 \\ \\ \theta_{tt} - \delta\theta_{xx} + \gamma\varphi_{tx} - k\theta_{txx} = 0 \end{cases} \tag{3.2.2}$$

with the initial data

$$(u, u_t, \varphi, \varphi_t, \theta, \theta_t)(x, 0) = (u, u_t, \varphi, \varphi_t, \widetilde{\theta}, \widetilde{\theta}_t)(x, 0). \tag{3.2.3}$$

Our goal in this section is to apply the energy method in the Fourier space and prove the decay rate of the Fourier image of the solution of (3.2.2). Let us first transform our problem to a first order system, by introducing the new variables:

$$\upsilon = u_x, \quad h = u_t, \quad z = \varphi_x, \quad y = \varphi_t, \quad \eta = \theta_t, \quad \omega = \theta_x, \tag{3.2.4}$$

then the above system takes the form

$$\begin{cases} \rho_1 h_t - \mu\upsilon_x - bz = 0 \\ \rho_2 y_t - \alpha z_x + b\upsilon + a\varphi + \beta\eta_x = 0 \\ \rho_3 \eta_t - \delta\omega_x + \gamma y_x - k\eta_{xx} = 0 \\ \omega_t - \eta_x = 0 \\ z_t - y_x = 0 \\ \upsilon_t - h_x = 0 \\ \varphi_t - y = 0 \end{cases} \tag{3.2.5}$$

with the initial data

$$(h, y, \eta, \omega, z, \upsilon, \varphi)(x, 0) = (h_0, y_0, \eta_0, \omega_0, z_0, \upsilon_0, \varphi_0). \tag{3.2.6}$$

The system (3.2.5) is a hyperbolic-parabolic system and can be written in the matrix form

$$\begin{cases} U_t + AU_x + LU = BU_{xx} \\ U(x, 0) = U_0 \end{cases} \tag{3.2.7}$$

with $U = (h, y, \eta, \omega, z, \upsilon, \varphi)^T$, $\quad U_0 = (h_0, y_0, \eta_0, \omega_0, z_0, \upsilon_0, \varphi_0)^T$and A, L and B are matrices defined as follows

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & \beta & 0 & -\alpha & 0 & 0 \\ 0 & \gamma & 0 & -\delta & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Indeed, taking the Fourier transform of (3.2.7), we get

$$\begin{cases} \hat{U}_t + iA\xi\hat{U} + L\hat{U} = -B\xi^2\hat{U} \\ \hat{U}(x,0) = \hat{U}_0 \end{cases} \tag{3.2.8}$$

then (3.2.8) takes the form:

$$\begin{cases} \hat{U}_t(\xi,t) = \Lambda(\xi)\hat{U}(\xi,t) \\ \hat{U}(x,0) = \hat{U}_0 \end{cases} \tag{3.2.9}$$

where $\Lambda(\xi) = -L - i\xi A - \xi^2 B$. Consequently, solving the above first order ordinary differential equation, we get

$$\hat{U}(\xi,t) = e^{\Lambda(\xi)t}\hat{U}_0(\xi), \tag{3.2.10}$$

computing the term $e^{\Lambda(\xi)t}$ is a challenging problem and in many situations this cannot be done. Consequently, in order to show the asymptotic behavior of the solution, it suffices to find a function $\zeta(\xi)$ such that

$$\left| e^{\Lambda(\xi)t} \right| \le Ce^{-\zeta(\xi)t}. \tag{3.2.11}$$

Taking the Fourier transform of the system (3.2.5) with respect to x yields

$$\begin{cases} \rho_1\hat{h}_t - i\xi\mu\hat{v} - b\hat{z} = 0 \\ \rho_2\hat{y}_t - i\alpha\xi\hat{z} + b\hat{v} + a\hat{\varphi} + i\xi\beta\hat{\eta} = 0 \\ \rho_3\hat{\eta}_t - i\xi\delta\hat{\omega} + i\xi\gamma\hat{y} + k\xi^2\hat{\eta} = 0 \\ \hat{\omega}_t - i\xi\hat{\eta} = 0 \\ \hat{z}_t - i\xi\hat{y} = 0 \\ \hat{v}_t - i\xi\hat{h} = 0 \\ \hat{\varphi}_t - \hat{y} = 0 \end{cases} \tag{3.2.12}$$

with the initial data

$$(\hat{h}, \hat{y}, \hat{\eta}, \hat{\omega}, \hat{z}, \hat{v}, \hat{\varphi})(\xi, 0) = (\hat{h_0}, \hat{y_0}, \hat{\eta_0}, \hat{\omega_0}, \hat{z_0}, \hat{v_0}, \hat{\varphi_0})(\xi). \tag{3.2.13}$$

The energy functional associated to the system (3.2.5) is defined as

$$\hat{E}(\xi, t) = \frac{1}{2}\left\{|\hat{h}|^2 + |\hat{y}|^2 + |\hat{\eta}|^2 + |\hat{\omega}|^2 + |\hat{z}|^2 + |\hat{v}|^2 + |\hat{\varphi}|^2 + 2bRe(\hat{v}\bar{\hat{\varphi}})\right\} \tag{3.2.14}$$

Lemma 3.2.1. *Let $\hat{U}(\xi, t) = (\hat{h}, \hat{y}, \hat{\eta}, \hat{\omega}, \hat{z}, \hat{v}, \hat{\varphi})(\xi, t)$ be the solution of (3.2.12), then for any $t \geq 0$, the identity*

$$\frac{d}{dt}\hat{E}(\xi, t) = -\beta k\xi^2|\hat{\eta}|^2 \tag{3.2.15}$$

is fulfilled. Also, assume that (3.1.3) holds, then there exist two positive constants c_0 and c_1 that satisfy the following inequality

$$c_0\left|\hat{U}(\xi, t)\right|^2 \leq \hat{E}(\xi, t) \leq c_1\left|\hat{U}(\xi, t)\right|^2. \tag{3.2.16}$$

Proof. Multiplying the first equation in (3.2.12) by $\bar{\hat{h}}$, the second equation by $\bar{\hat{y}}$, the third equation by $\bar{\hat{\eta}}$, the fourth equation by $\bar{\hat{\omega}}$, the fifth equation by $\bar{\hat{z}}$, the sixth by $\bar{\hat{v}}$, and the seventh equation by $\bar{\hat{\varphi}}$, we get

$$\frac{d}{dt}\frac{1}{2}\left\{|\hat{h}|^2 + |\hat{y}|^2 + |\hat{\eta}|^2 + |\hat{\omega}|^2 + |\hat{z}|^2 + |\hat{v}|^2 + |\hat{\varphi}|^2\right\} + bRe(\hat{v}\bar{\hat{y}} - \hat{z}\bar{\hat{h}}) + k\beta\xi^2|\hat{\eta}|^2 = 0 \tag{3.2.17}$$

recalling that as in [44], $\hat{v} = i\xi\hat{u}, \quad \hat{h} = \hat{u}_t, \quad \hat{z} = i\xi\hat{\varphi}, \quad \hat{y} = \hat{\varphi}_t$, we get

$$\begin{aligned} Re(\hat{v}\bar{\hat{y}} - \hat{z}\bar{\hat{h}}) &= Re(i\xi\hat{u}_t\bar{\hat{\varphi}}_t - i\xi\hat{\varphi}\bar{\hat{u}}_t) \\ &= \frac{d}{dt}\left\{Re(i\xi\hat{u}\bar{\hat{\varphi}})\right\} = \frac{d}{dt}Re(\hat{v}\bar{\hat{\varphi}}). \end{aligned} \tag{3.2.18}$$

Inserting (3.2.18) into (3.2.17), so the assertion (3.2.15) holds.
In order to prove (3.2.16), we use Young's inequality for the last term in (2.18) for any $\epsilon > 0$

$$\left|2bRe(\hat{v}\bar{\hat{\varphi}})\right| \leq 2b\epsilon|\hat{v}|^2 + 2b\frac{1}{4\epsilon}|\bar{\hat{\varphi}}|^2, \tag{3.2.19}$$

inserting (3.2.19) into (3.2.14), we obtain (3.2.16) with

$$c_0 = \frac{1}{2}\min\left\{1, (1 - 2b\epsilon), (1 - \frac{2b}{4\epsilon})\right\} > 0,$$

$$c_1 = \frac{1}{2}\max\left\{1, (1 + 2b\epsilon), (1 + \frac{2b}{4\epsilon})\right\} > 0.$$

Hence the proof is completed. □

Proposition 3.2.2. *Let $\hat{U}(\xi,t) = (\hat{h},\hat{y},\hat{\eta},\hat{\omega},\hat{z},\hat{v},\hat{\varphi})(\xi,t)$ be the solution of the system (3.2.12). Assume that (3.1.3) holds. Then for any $t \geq 0$ and $\xi \in \mathbb{R}$, we have the following pointwise estimates*

$$\left|\hat{U}(\xi,t)\right|^2 \leq Ce^{-c\zeta(\xi)t}\left|\hat{U}(\xi,0)\right|^2, \tag{3.2.20}$$

where

$$\zeta(\xi) = \frac{\xi^2}{(1+\xi^4)}. \tag{3.2.21}$$

Here C and c are two positive constants.

Proof. Multiplying first equation in (3.2.12) by $-\bar{\hat{z}}$ and fifth equation by $\rho_1\bar{\hat{h}}$, we get

$$-\rho_1\bar{\hat{h}}_t\bar{\hat{z}} + i\xi\hat{v}\bar{\hat{z}} + b|\hat{z}|^2 = 0 \tag{3.2.22}$$

and

$$-\rho_1\hat{z}_t\bar{\hat{h}} + \rho_1 i\xi\hat{y}\bar{\hat{h}} = 0 \tag{3.2.23}$$

Adding the above two equations and taking the real parts, we obtain

$$-\frac{d}{dt}Re(\rho_1\hat{h}\bar{\hat{z}}) - Re(\rho_1\mu i\xi\hat{v}\bar{\hat{z}}) + b|\hat{z}|^2 + Re(\rho_1 i\xi\hat{y}\bar{\hat{h}}) = 0, \tag{3.2.24}$$

by using Young's inequality, we get

$$Re(\rho_1\mu i\xi\hat{v}\bar{\hat{z}}) \leq c(\epsilon_1)\xi^2|\hat{z}|^2 + \epsilon_1\frac{\xi^2}{1+\xi^2}|\hat{v}|^2 \tag{3.2.25}$$

$$Re(\rho_1 i\xi\hat{y}\bar{\hat{h}}) \leq c(\epsilon_1)\frac{\xi^2}{1+\xi^2}|\hat{y}|^2 + \epsilon_1\xi^2(1+\xi^2)|\hat{h}|^2, \tag{3.2.26}$$

inserting (3.2.25)-(3.2.26) in (3.2.24), we obtain

$$\begin{aligned}\frac{d}{dt}F_1(\xi,t) + b|\hat{z}|^2 \leq\ & c(\epsilon_1)\xi^2|\hat{z}|^2 + \epsilon_1\frac{\xi^2}{1+\xi^2}|\hat{v}|^2 \\ & + c(\epsilon_1)\frac{\xi^2}{1+\xi^2}|\hat{y}|^2 + \epsilon_1\xi^2(1+\xi^2)|\hat{h}|^2,\end{aligned} \tag{3.2.27}$$

noting that $F_1(\xi,t) = -Re(\epsilon_1\hat{h}\bar{\hat{z}})$. Now, by the same procedure multiplying the second equation by $\hat{h}\bar{\hat{v}}$ and sixth equation by $\rho_2\hat{h}\bar{\hat{y}}$, we get

$$\rho_2\hat{y}_t\bar{\hat{v}} - i\alpha\hat{z}\bar{\hat{v}} + \rho_2|\hat{v}|^2 + a\hat{\varphi}\bar{\hat{v}} + i\xi\hat{\eta}\bar{\hat{v}} = 0 \tag{3.2.28}$$

and

$$\rho_2\hat{v}_t\bar{\hat{y}} - i\xi\hat{h}\bar{\hat{y}} = 0, \tag{3.2.29}$$

summing up (3.2.28)-(3.2.29) and taking the real parts, we have

$$\frac{d}{dt}F_2(\xi,t) - Re(i\alpha\hat{z}\bar{\hat{v}}) + \rho_2|\hat{v}|^2 + Re(\hat{\varphi}\bar{\hat{v}}) + Re(i\xi\beta\hat{\eta}\bar{\hat{v}}) - Re(i\xi\rho_2\hat{h}\bar{\hat{y}}) = 0, \tag{3.2.30}$$

such that $F_2(\xi,t) = Re(\hat{y}\bar{\hat{v}})$. By exploiting the Young's inequality:

$$Re(i\alpha\hat{z}\bar{\hat{v}}) \leq c(\epsilon_2)\xi^2|\hat{z}|^2 + \epsilon_2\frac{\xi^2}{\xi^2+1}|\hat{v}|^2, \tag{3.2.31}$$

$$Re(\hat{\varphi}\bar{\hat{v}}) \leq c(\epsilon_2)\frac{\xi^2}{\xi^2+1}|\hat{\varphi}|^2 + \xi^2\epsilon_2|\hat{v}|^2, \tag{3.2.32}$$

and inserting (3.2.31)-(3.2.32) into (3.2.30) this produces

$$\frac{d}{dt}F_3(\xi,t) + \xi^2C(\epsilon_1,\epsilon_2,\alpha)|\hat{z}|^2 - \xi^2|\hat{y}|^2 \leq \epsilon_2\frac{\xi^2}{\xi^2+1}|\hat{\varphi}|^2 + \epsilon_3\frac{\xi^2}{\xi^2+1}|\hat{\eta}|^2, \tag{3.2.33}$$

such that $F_3(\xi,t) = Re(\rho_2\hat{y}\bar{\hat{v}})$. Multiplying the second equation by $\bar{\hat{\varphi}}$ and seventh by $\rho_2\bar{\hat{y}}$

$$\rho_2\hat{y}_t\bar{\hat{\varphi}} - i\xi\alpha\hat{z}\bar{\hat{\varphi}} + b\hat{v}\bar{\hat{\varphi}} + a|\hat{\varphi}|^2 + i\xi\beta\hat{\eta}\bar{\hat{\varphi}} = 0, \tag{3.2.34}$$

$$\rho_2\hat{\varphi}_t\bar{\hat{y}} - \rho_2|\hat{y}|^2 = 0. \tag{3.2.35}$$

Adding the above two equations and taking the real parts, we obtain

$$\frac{d}{dt}F_4(\xi,t) + a|\hat{\varphi}|^2 - \rho_2|\hat{y}|^2 - Re(\alpha i\xi\hat{z}\bar{\hat{\varphi}}) + Re(b\hat{v}\bar{\hat{\varphi}}) + Re(i\xi\beta\hat{\eta}\bar{\hat{\varphi}}) = 0. \tag{3.2.36}$$

Using Young's inequality for the forth, fifth and sixth terms, we get

$$\begin{aligned}&\frac{d}{dt}F_4(\xi,t) + (a-\epsilon_7)|\hat{\varphi}|^2 - \rho_2|\hat{y}|^2 \leq \xi^2|\hat{\varphi}|^2 + \frac{\xi^2}{\xi^2+1}|\hat{z}|^2 + c(\epsilon_7)|\hat{v}|^2\\ &+ \xi^2|\hat{\eta}|^2 + \frac{\xi^2}{\xi^2+1}|\hat{\varphi}|^2\end{aligned} \tag{3.2.37}$$

such that $F_4(\xi,t) = Re(\rho_2\hat{y}\bar{\hat{\varphi}})$. Multiplying the first equation in (3.2.12) by $(-i\xi\rho_1\bar{\hat{\varphi}})$ and the sixth equation by $(i\xi\rho_1\bar{\hat{h}})$, we obtain

$$-i\xi\rho_1\hat{h}_t\bar{\hat{v}} + \xi^2\rho\mu|\hat{v}|^2 + i\xi\rho_1 b\hat{z}\bar{\hat{v}} = 0, \tag{3.2.38}$$

$$i\xi\rho_1\hat{v}_t\bar{\hat{h}} + \xi^2\rho_1|\hat{h}|^2 = 0, \tag{3.2.39}$$

adding (3.2.38)-(3.2.39), we get

$$\frac{d}{dt}F_5(\xi,t) + \xi^2\rho_1\mu|\hat{v}|^2 + \xi^2\rho_1|\hat{h}|^2 + Re(i\xi\rho_1 b\hat{z}\bar{\hat{v}}) = 0. \tag{3.2.40}$$

Young's inequality for the forth term produces

$$\frac{d}{dt}F_5(\xi,t)+\xi^2(\rho_1\mu-c(\epsilon_6))|\hat{v}|^2\leq\xi^2\rho_1|\hat{h}|^2+\epsilon_6\frac{\xi^2}{\xi^2+1}|\hat{z}|^2, \tag{3.2.41}$$

where $F_5(\xi,t)=Re(\rho_1 i\xi\hat{h}\bar{\hat{v}})$. Now, multiplying the forth equation in (2.12) by $(-\rho_2 i\xi\bar{\hat{z}})$ and the fifth equation by $(-\rho_2 i\xi\bar{\hat{y}})$, we have

$$-\rho_2\xi\hat{y}_t\bar{\hat{z}}+\xi\rho_2\alpha|\hat{z}|^2-ib\xi\rho_2\hat{v}\bar{\hat{z}}-ai\xi\rho_2\hat{\varphi}\bar{\hat{z}}-i\xi^2\rho\beta\hat{\eta}\bar{\hat{z}}=0, \tag{3.2.42}$$

$$\xi\rho_2 i\hat{z}_t\bar{\hat{y}}-\rho_2\xi^2|\hat{y}|^2=0, \tag{3.2.43}$$

summing up (3.2.42)-(3.2.43) and using Young's inequality, we get

$$\begin{aligned}&\frac{d}{dt}F_6(\xi,t)+\xi^2(\alpha\rho_2-1-2\delta_4)|\hat{z}|^2-\xi^2\rho_2|\hat{y}|^2+\xi^2|\hat{\varphi}|^2\\&\leq\frac{\xi^2}{\xi^2+1}|\hat{v}|^2+\frac{\xi^2}{\xi^2+1}|\hat{\eta}|^2,\end{aligned} \tag{3.2.44}$$

such that $F_6(\xi,t)=Re(\rho_2 i\xi\hat{y}\bar{\hat{z}})$. By the same procedure, multiplying the forth equation in (3.2.12) by $(-\rho_3 i\xi\bar{\hat{\eta}})$ and the third equation by $(-\rho_3 i\xi\bar{\hat{\omega}})$, we obtain

$$-\rho_3 i\xi\hat{\omega}_t\bar{\hat{\eta}}-\rho_3\xi^2|\hat{\eta}|^2=0, \tag{3.2.45}$$

$$\rho_3 i\xi\hat{\eta}_t\bar{\hat{\omega}}+\xi^2\delta|\hat{\omega}|^2-\xi^2\gamma\hat{y}\bar{\hat{\omega}}+ki\xi^3\hat{\eta}\bar{\hat{\omega}}=0, \tag{3.2.46}$$

summing up (3.2.45)-(3.2.46) and using Young's inequality and taking the real parts, we get

$$\frac{d}{dt}F_7(\xi,t)+\xi^2(\delta-c(\epsilon_7)\gamma-\epsilon_8)|\hat{\omega}|^2+\xi^2\epsilon_7|\hat{y}|^2\leq c(\epsilon_8)\frac{\xi^2}{\xi^2+1}|\hat{\eta}|^2, \tag{3.2.47}$$

where $F_7(\xi,t)=Re(\rho_3 i\xi\hat{\omega}\bar{\hat{\eta}})$. Now we define the Lyapunov functional

$$\hat{\mathcal{L}}(t)=(1+\xi^2+\xi^4)\hat{E}(\xi,t)+\frac{1}{\xi^4+1}\left\{F_1+F_2\right\}+\frac{\xi^2}{\xi^4+1}\left\{F_3+F_4+F_5+F_6+F_7\right\}. \tag{3.2.48}$$

Taking the derivative of (3.2.48) with respect to t and by using the formulas (3.2.27), (3.2.30) (3.2.33), (3.2.37), (3.2.41), (3.2.44), (3.2.47) and invoking the fact that $\frac{\xi^4}{(\xi^4+1)(\xi^2+1)}\leq\frac{\xi^4}{(\xi^4+1)}$, we arrive at

$$\begin{aligned}&\frac{d}{dt}\hat{\mathcal{L}}(t)+\frac{\xi^2}{\xi^4+1}\left\{c(\epsilon_1)-\alpha-c(\epsilon_2)-c(\epsilon_3)-c(\epsilon_4)-\epsilon_6-c(\epsilon_9)\right\}|\hat{z}|^2\\&+\frac{\xi^4}{\xi^4+1}\left\{\epsilon_1-\delta_3+c(\epsilon_6)+(\rho_2-\epsilon_{10})+\epsilon_9+c(\epsilon_{11})\right\}|\hat{v}|^2\\&+\frac{\xi^2}{\xi^4+1}\left\{\delta_4-c(\epsilon_1)+\epsilon_7+\epsilon_{12}\right\}|\hat{y}|^2+\frac{\xi^4}{\xi^4+1}\left\{c(\epsilon_{12})-\alpha_4-\epsilon_1\right\}|\hat{h}|^2\\&+\frac{\xi^2}{\xi^4+1}\left\{c(\epsilon_{10})-\epsilon_2\right\}|\hat{\varphi}|^2+\frac{\xi^4}{\xi^4+1}\left\{c(\epsilon_7)\gamma-\alpha+\epsilon_8\right\}|\hat{\omega}|^2\\&\leq\xi^2(1+\xi^2+\xi^4)\left\{c(\epsilon_8)-\epsilon_{11}-\epsilon_4-\beta k\right\}|\hat{\eta}|^2.\end{aligned} \tag{3.2.49}$$

On the other hand, it is a straightforward, we may find λ_1 and λ_2 such that

$$\lambda_1\xi^2(1+\xi^2+\xi^4)\hat{E}(\xi,t) \leq \hat{\mathcal{L}}(\xi,t) \leq \lambda_2\xi^2(1+\xi^2+\xi^4)\hat{E}(\xi,t), \quad t \geq 0. \tag{3.2.50}$$

Consequently, having fixed the constants as above, we deduce that there exists a positive constants $\gamma_0 > 0$, such that

$$\frac{d}{dt}\hat{\mathcal{L}}(\xi,t) + \gamma_0\phi(\xi,t) \leq 0. \quad \forall\, t \geq 0, \tag{3.2.51}$$

where

$$\phi(\xi,t) \geq \frac{\xi^2}{\xi^4+1}\left\{|\hat{z}|^2 + |\hat{v}|^2 + |\hat{y}|^2 + |\hat{h}|^2 + |\hat{\varphi}|^2 + |\hat{\omega}|^2\right\}, \tag{3.2.52}$$

exploiting (3.2.51), (3.2.52) and (3.2.50), we get

$$\frac{d}{dt}\hat{\mathcal{L}}(\xi,t) + \frac{\gamma_0}{\lambda_2}\zeta(\xi)\hat{\mathcal{L}}(\xi,t) \leq 0, \quad \forall\, t \geq 0, \tag{3.2.53}$$

where $\zeta(\xi)$ is defined in (3.2.21). Integrating (3.2.53) with respect to t, we find

$$\hat{\mathcal{L}}(\xi,t) \leq \hat{\mathcal{L}}(\xi,0)e^{\frac{\gamma_0}{\lambda_2}\zeta(\xi)t} \quad \forall\, t \geq 0, \tag{3.2.54}$$

exploiting (3.2.50) once again, then the first estimate in (3.2.20) holds. □

Theorem 3.2.3. *Let s be a nonnegative integer and $\hat{U}_0 = (\hat{h}_0, \hat{y}_0, \hat{\eta}_0, \hat{\omega}_0, \hat{z}_0, \hat{v}_0, \hat{\varphi}_0)^T \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Assume that (1.3) holds. Then the solution $\hat{U} = (\hat{h}, \hat{y}, \hat{\eta}, \hat{\omega}, \hat{z}, \hat{v}, \hat{\varphi})^T$ of the system (3.2.12) satisfies the following decay estimates:*

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{\frac{-1}{2}-k}\|\partial_x^k U_0\|_{L^1} + C(1+t)^{-l}\|\partial^{k+l}U_0\|_{L^2} \tag{3.2.55}$$

for $k+l \leq s$. Here C is a positive constant.

Proof. First, we can see that the functions $\zeta(\xi)$ satisfy

$$\zeta(\xi) \geq \begin{cases} c_1\xi^2 & if\ |\xi| \leq 1, \\ \\ c_2\xi^{-2} & if\ |\xi| \geq 1. \end{cases}$$

Making use of the Plancherel theorem, observing that $|\hat{U}(\xi,t)|^2$ and $\hat{E}(\xi,t)$ are equivalent, we get

$$\begin{aligned} &\|\partial_x^k U(t)\|_{L^2} \leq \int_{\mathbb{R}} |\xi|^{2k}|\hat{U}(\xi,t)|^2 d\xi \leq C\int_{\mathbb{R}} |\xi|^{2k}e^{-C_4\zeta(\xi)t}|\hat{U}(\xi,0)|^2 d\xi \\ &= C\int_{|\xi|\leq 1} |\xi|^{2k}e^{-C_4\zeta(\xi)t}|\hat{U}(\xi,0)|^2 d\xi + C\int_{|\xi|\geq 1} |\xi|^{2k}e^{-C_4\zeta(\xi)t}|\hat{U}(\xi,0)|^2 d\xi = I_1 + I_2. \end{aligned} \tag{3.2.56}$$

For the first integral, we have

$$I_1 \leq \sup_{|\xi|\leq 1}\left\{\left|\hat{U}(\xi,0)\right|^2\right\}\int_{|\xi|\leq 1} C|\xi|^{2k}e^{-c\xi^2 t}d\xi \leq C\|\hat{U}_0(t)\|^2_{L^\infty}\int_{|\xi|\leq 1} C|\xi|^{2k}e^{-c\xi^2 t}d\xi. \quad (3.2.57)$$

Using the inequality

$$\int_0^1 C|\xi|^{\sigma}e^{-c\xi^2 t}d\xi \leq C(1+t)^{\frac{-(\sigma+1)}{2}}, \quad (3.2.58)$$

we deduce that

$$I_1 \leq C(1+t)^{\frac{-1}{2}-k}\|\hat{U}_0(t)\|^2_{L^1}, \quad (3.2.59)$$

exploiting the inequality

$$\sup_{|\xi|\geq 1}\left\{|\xi|^{-2l}e^{-c\xi^2 t}\right\} \leq C(1+t)^{-l}, \quad (3.2.60)$$

we obtain

$$\begin{aligned} I_2 &\leq C\sup_{|\xi|\geq 1}\left\{|\xi|^{-2l}e^{-c\xi^2 t}\right\}\int_{|\xi|\geq 1} C|\xi|^{2(k+1)}\left|\hat{U}(\xi,0)\right|^2 d\xi \\ &\leq C(1+t)^{-l}\|\partial_x^{k+1}U_0\|^2_{L^2}. \end{aligned} \quad (3.2.61)$$

Combining (3.2.57) and (3.2.61), we get the desired result. Hence the proof is completed. □

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