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Existence et non existence de solutions globales pour les équations d'évolution

Soutenue le

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Contents

Notations

Ω: Bounded domain in $\mathbb{R}^N$.

Γ: Topological boundary of Ω.

$x = (x_1, x_1, ..., x_N)$:Generic point of $\mathbb{R}^N$.

$dx = dx_1 dx_1 ... dx_N$: Lebesgue measuring on Ω.

∇u: Gradient of u.

Δu: Laplacien of u.

α: a multi-index $\alpha = (\alpha_1, ...\alpha_N), N \in \mathbb{N}$, with $\alpha_i \geq 0$ for any $i \in 1, ..., N$

$f^+, f^- : \max(f, 0), \quad \max(-f, 0)$.

a.e: Almost everywhere.

q: Conjugate of p, i.e $\frac{1}{p} + \frac{1}{q} = 1$.

$D(\Omega)$: Space of differentiable functions with compact support in Ω.

$D'(\Omega)$: Distribution space.

$C^k(\Omega)$: Space of functions k-times continuously differentiable in Ω.

$C_0(\Omega)$: Space of continuous functions null board in Ω.

$L^p(\Omega)$: Space of functions p-th power integrated on Ω with measure of dx.

$\|f\|_p = \left(\int_\Omega |f(x)|^p\right)^{\frac{1}{p}}$.

$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega), \frac{\partial u}{\partial x_i} \in L^p(\Omega), i = 1, ..., N \right\}$.

$\|u\|_{1,p} = \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{\frac{1}{p}}$.

$W_0^{1,p}(\Omega)$: The closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.

$W_0^{-1,q}(\Omega)$: The dual space of $W_0^{1,p}(\Omega)$.

H: Hilbert space.

$H^1 = W^{1,2}(\Omega)$.

H_0^1 The closure of $D(\Omega)$ in $W^{1,2}(\Omega)$.

If X is a Banach space

$L^p(0, T; X) = \left\{ f :]0, T[\to X \text{ is measurable}; \int_0^T \|f(t)\|_X^p dt < \infty \right\}$.

$L^\infty(0, T; X) = \left\{ f : [0, T] \to X \text{ is measurable}; \sup_{t \in [0,T]} ess \|f(t)\|_X^p \right\}$.

$C^k([0, T]; X)$: Space of functions k-times continuously differentiable for $[0, T] \to X$.

$D([0, T]; X)$: Space of functions continuously differentiable with compact support in $[0, T]$.

$B_X = \{x \in X; \|x\| \leq 1\}$: unit ball.

Chapter 1

General Introduction

In the mid-twentieth century, the theory of partial differential equations was considered the summit of mathematics, due to the difficulty and significance of the problems it solved, and it's existence that came later than most areas of mathematics.

Nowadays, many mathematicians are inclined to look disparagingly at this remarkable area of mathematics as an old-fashioned art of juggling inequalities, or a testing ground for applications of functional analysis.

The principal source of partial differential equations is found in the continuous-medium models of mathematical and theoretical physics.

In this thesis, we address some topics related to the controllability and stability of partial differential equations (PDE).

The controllability problem may be formulated roughly as follows
Consider an evolution system on which we are allowed to act by means of a suitable choice of the control (the right hand side of the system and the boundary conditions...etc). Given a time interval $0 < t < T$, the initial and final states, the goal is to determine whether there exists a control driving to the given initial data and to the given final ones in time T.

This is a classical problem in Control Theory and there is a large literature written on the subject. We refer to the book of J.-L. Lions [47], as an introduction to the case of systems modeled by means of PDE.

Regarding the problem of stabilization, the purpose is to attenuate the vibrations by feedback. it consists of guarantee the decay of the energy of solutions towards 0 in away, more or less fast.
More precisely, we are interested to determine the asymptotic behavior of the energy denoted by $E(t)$ and to give an estimation of the decay rate of the energy.

In order to treat the asymptotic behavior, there are several types of stabilization:
The first type consists to analyzing the energy decay of solutions towards 0 i.e. $E(t) \to 0$ as $t \to \infty$. This is what we name the Strong stabilization.

Concerning the second type, we are interested in the uniform stabilization that is related to the decay of the energy which exponentially tends to 0 i.e.

$$E(t) \leq Ce^{-\gamma t}, \forall t > 0,$$

where C and δ are a positives constant with C depends on the initial data.

In the third type of stabilisation, we study the intermediate situations, in which the energy decay of the solution is not exponential, but polynomial or logarithmic for example:

$$E(t) \leq \frac{C}{(log(1+t))^k}, \forall t > 0,$$

$$E(t) \leq \frac{C'}{t^\alpha}, \forall t > 0,$$

where C, C', α and k are positive constants with C, C' depend on the initial data.

The current thesis presents results of existence and stability of solutions for four evolution problems, and it mainly consists of five chapters. Each chapter is presented as follow:

Chapter 1

Is entirely devoted to the presentation of the definitions, and the results necessary for this work.

First, We mention few basic results: functional spaces including spaces of Sobolev, spaces L^p, and the results of exponential and polynomial decay, and a remind of some methods of existence used in this work.

Chapter 2

In this chapter we study the following wave equation, with damping effects, and a weak internal constant delay in a bounded domain.

$$\begin{cases} u''(x,t) - k_0 \triangle u + \alpha \displaystyle\int_0^t g(t-s) \triangle u(x,s) ds & \\ +\mu_1(t) u'(x,t) + \mu_2(t) u'(x,t-\tau) = 0, & \text{on } \Omega \times]0,+\infty[, \\ u(x,t) = 0, & \text{on } \partial\Omega \times]0,+\infty[, \\ u(x,0) = u_0(x), u_t(x,t) = u_1(x), & \text{on } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \text{on } \Omega \times]0,t[. \end{cases} \tag{1.1}$$

Where Ω is a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}^*$) with a smooth boundary $\partial\Omega$. The initial data u_0, u_1, f_0 belong to a suitable space. Moreover, $\tau > 0$ is the time delay term and μ_1, μ_2 are real functions that will be specified later. Furthermore, k_0 is a positive real number and g is a positive non-increasing function defined on $\mathbb{R}^+$.

With conditions on kernel of the term memory g and the functions μ_1, μ_2, we prove the global existence of solutions by Faedo-Galerkin methods, and we establish the estimation of the decay rate for energy using the multiplier method.

Chapter 3

In the same axis, a multidimensional system of viscoelastic wave equations with dynamic boundary conditions, to the amortization and delay of Kelvin Voigt, were tackled in Chapter 3.

$$
\begin{cases}
u_{tt} - \Delta u - \displaystyle\int_0^t g(t-s)\Delta u(s)ds - \delta\Delta u_t = |u|^{p-1}u, & in \quad \Omega \times (0,+\infty),\\
u = 0, & on \quad \Gamma_0 \times (0,+\infty),\\
 & \\
u_{tt} = -a\big[\frac{\partial u}{\partial \upsilon}(x,t) + \delta\frac{\partial u_t}{\partial \upsilon}(x,t) + \mu_1(t)u_t(x,t) + \mu_2(t)u_t(x,t-\tau)\big], & \\
on \quad \Gamma_1 \times (0,+\infty), & \\
 & \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega,\\
u_t(x,t-\tau) = f_0(x,t-\tau), & on \quad \Gamma_1 \times (0,+\infty).
\end{cases} \tag{1.2}
$$

Where $u = u(x,t)$, $t \geq 0$, $x \in \Omega$ and Δ denote the Laplacian operator with respect to the x variable. Ω is a regular and bounded domain of $\mathbb{R}^N$, $(N \geq 1)$, $\partial\Omega = \Gamma_1 \cup \Gamma_0$, $\Gamma_1 \cap \Gamma_0 = \emptyset$ and $\frac{\partial}{\partial \nu}$ denote the unit outer normal derivative, μ_1 and μ_2 are functions depending on t. Moreover, $\tau > 0$ represents the delay and u_0, u_1, f_0 are given functions belonging to suitable spaces that will be specified later. This type of problems arises (for example) in modeling of longitudinal vibrations in a homogeneous bar on which there are viscous effects. The term Δu_t, indicates that the stress is proportional not only to the strain, but also to the strain rate.

For this problem, we establish a general result of the decay using the Nakao technique.

Chapter 4

The purpose of chapter four is to present a result of existence of solutions for the following viscoelastic plate equation with a constant delay term and logarithmic nonlinearities

$$
\begin{cases}
u_{tt}(x,t) - \Delta^2 u + \phi(x)\left(\alpha\Delta^2 u - \displaystyle\int_0^t g(t-s)\Delta^2 u(x,s)ds\right)\\
+\mu_1(t)u_t(x,t) + \mu_2(t)u_t(x,t-\tau) = u\ln|u|^k \quad \text{in } \mathbb{R}^N\times]0,+\infty[,\\
u(x,t) = 0, \quad \text{on } \partial\mathbb{R}^N\times]0,+\infty[,\\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad \text{in } \mathbb{R}^N,\\
u_t(x,t-\tau) = f_0(x,t-\tau), \quad \text{in } \mathbb{R}^N\times]0,t[.
\end{cases} \tag{1.3}
$$

Where $n \geq 1$, $\phi(x) > 0$ and $(\phi(x))^{-1} = \rho(x)$, such that ρ is a function that will be defined later. The initial datum u_0, u_1, f_0 are given functions belonging to suitable spaces that will be specified later. μ_1, μ_2 are real functions and g is a positive non-increasing function defined on $\mathbb{R}_+$. Moreover $\tau > 0$ represents the time delay term.

Under certain conditions, we prove the global existence. Also, we establish a general rate of decrease in solutions.

Chapter 5

The last research problem is presented in chapter five. We investigate the decay properties of solutions for the initial boundary problem value of a nonlinear wave equation of the form

$$\begin{cases} (|u_t|^{\gamma-2}u_{tt}) - Lu - \displaystyle\int_0^t g(t-s)Lu(s)ds + \mu_1 u_t(x,t) & \\ + \int_{\tau_1}^{\tau_2} \mu_2(s)u_t(x,t-s)ds = 0, & \text{in } \Omega\times]0,+ infty[, \\ u(x,t) = 0, & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \\ u_t(x,-t) = f_0(x,t), & \text{in } \Omega\times]0,\tau_2[. \end{cases} \tag{1.4}$$

Where Ω is a bounded domain in $\mathbb{R}^N$, $N \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$. The constant τ_1 is nonnegative such that $\tau_1 < \tau_2$ and $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function. The initial datum (u_0, u_1, f_0) belong to a suitable functional space, where $Lu = -div(A\nabla u) = -\displaystyle\sum_{i,j=1}^{N}\left(a_{i,j}(x)\frac{\partial u}{\partial x_i}\right)$ and $A = (a_{i,j}(x))$ is a matrix that will be specified later. We prove the result of the energy decay by constructing a suitable Lyapunov function.

Chapter 2

Preliminary

In this chapter, we will introduce and state without proofs some important materials needed in the proof of our results.

2.1 Banach Spaces - Definition and properties

We first review some basic facts from calculus in the most important class of linear spaces " Banach spaces".

Definition 2.1.1 *A Banach space is a complete normed linear space X. Its dual space X' is the linear space of all continuous linear functional $f : X \to \mathbb{R}$.*

Proposition 2.1.1 *([71]) X' equipped with the norm $\|.\|_{X'}$ defined by*

$$\|f\|_{X'} = \sup\Big\{|f(u)|, \quad \|u\| \leq 1\Big\}, \tag{2.1}$$

is also a Banach space. We shall denote the value of $f \in X'$ at $u \in X$ by either $f(u)$ or $\langle f, u\rangle_{X',X}$.

From X' we construct the bidual or second dual $X'' = (X')'$. Furthermore, with each $u \in X$ we can define $\varphi(u) \in X''$ by $\varphi(u)(f) = f(u), f \in X'$. This satisfies clearly $\|\varphi(u)\| \leq \|u\|$. Moreover, for each $u \in X$ there is an $f \in X'$ with $f(u) = \|u\|$ and $\|f\| = 1$. So it follows that $\|\varphi(u)\| = \|u\|$.

Definition 2.1.2 *Since φ is linear we see that*

$$\varphi : X \to X'',$$

is a linear isometry of X onto a closed subspace of X'', we denote this by

$$X \hookrightarrow X''.$$

Definition 2.1.3 *If φ is onto X'' we say X is reflexive, $X \cong X''$.*

Theorem 2.1.1 *(Kakutani)([12]) Let X be Banach space. Then X is reflexive, if and only if,*

$$B_X = \Big\{x \in X : \|x\| \leq 1\Big\},$$

is compact with the weak topology $\sigma(X, X')$. (See the next subsection for the definition of $\sigma(X, X')$).

Definition 2.1.4 *Let X be a Banach space, and let $(u_n)_{n\in N}$ be a sequence in X. Then u_n converges strongly to u in X if and only if*

$$\lim_{n\to+\infty} \|u_n - u\|_X = 0,$$

and this is denoted by $u_n \to u$ or $\lim\limits_{n\to+\infty} u_n = u$.

Definition 2.1.5 *The Banach space X is said to be separable if there exists a countable subset D of X which is dense in X, i.e. $\overline{D} = X$.*

Proposition 2.1.2 *([12]) If X is reflexive and if F is a closed vector subspace of X, then F is reflexive.*

Corollary 2.1.1 *([12]) The following two assertions are equivalent:*
(i) X is reflexive. (ii) X' is reflexive.

2.1.1 The weak and weak star topologies

Let X be a Banach space and $f \in X'$. Denote by

$$\begin{aligned} \varphi_f : X &\to \mathbb{R} \\ x &\to \varphi_f(x), \end{aligned} \tag{2.2}$$

when f cover X', we obtain a family $(\varphi_f)_{f\in X'}$ of applications to X in $\mathbb{R}$.

Definition 2.1.6 *The weak topology on X, denoted by $\sigma(X, X')$, is the weakest topology on X, for which every $(\varphi_f)_{f \in X'}$ is continuous.*

We will define the third topology on X', the weak star topology, denoted by $\sigma(X', X)$. For all $x \in X$, denote by

$$\begin{aligned} \varphi_x : X' &\to \mathbb{R} \\ f &\to \varphi_x(f) = \langle f, x \rangle_{X', X}, \end{aligned} \tag{2.3}$$

when x cover X, we obtain a family $(\varphi_x)_{x \in X}$ of applications to X' in $\mathbb{R}$.

Definition 2.1.7 *The weak star topology on X' is the weakest topology on X' for which every $(\varphi_x)_{x \in X}$ is continuous.*

Since $X \subset X''$, it is clear that, the weak star topology $\sigma(X', X)$ is weakest then the topology $\sigma(X', X'')$, and this later is weakest then the strong topology.

Definition 2.1.8 *A sequence (u_n) in X is weakly convergent to u if and only if*

$$\lim_{n \to \infty} f(u_n) = f(u),$$

for every $f \in X'$, and this is denoted by $u_n \rightharpoonup u$.

Remark 2.1.1 *([12])*

1. *If $u_n \to u \in X$ (strongly) then $u_n \rightharpoonup u$ (weakly).*
2. *If $dimX < +\infty$, then the weak convergent equivalent the strong convergent.*

Proposition 2.1.3 *([71]) On the compactness in the three topologies in the Banach space X:*

1. *First, the unit ball*
$$B \equiv \Big\{ x \in X : \|x\| \leq 1 \Big\}, \tag{2.4}$$
in X is compact if and only if $dim(X) < \infty$.
2. *Second, the unit ball B' in X'(The closed subspace of a product of compact spaces) is weakly compact in X' if and only if X is reflexive.*
3. *Third, B' is always weakly star compact in the weak star topology of X'.*

Proposition 2.1.4 *([12]) Let (f_n) be a sequence in X'. We have:*

1. $[f_n \rightharpoonup^* f \text{ in } \sigma(X', X)] \Leftrightarrow [f_n(x) \rightharpoonup^* f(x),\ \forall x \in X]$.

2. *If* $f_n \to f$(*strongly*) *then* $f_n \rightharpoonup f$, *in* $\sigma(X', X'')$,
   If $f_n \rightharpoonup f$ *in* $\sigma(X', X'')$, *then* $f_n \rightharpoonup^* f$, *in* $\sigma(X', X)$.

3. *If* $f_n \rightharpoonup^* f$ *in* $\sigma(X', X)$ *then* $\|f_n\|$ *is bounded and* $\|f\| \leq \liminf \|f_n\|$.

4. *If* $f_n \rightharpoonup^* f$ *in* $\sigma(X', X)$ *and* $x_n \to x$(*strongly*) *in* X, *then* $f_n(x_n) \to f(x)$.

2.1.2 Hilbert spaces

Definition 2.1.9 *A Hilbert space H, is a vectorial space supplied with inner product $\langle u, \upsilon \rangle$, such that $\|u\| = \sqrt{\langle u, u \rangle}$ is the norm which let H complete.*

Theorem 2.1.2 *([12])(Riesz)*
If $(H; \langle ., . \rangle)$ is a Hilbert space, $\langle ., . \rangle$ being a scalar product on H, then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $u \in H$ such that $f = \langle u, . \rangle$ and $\|f\|_{H'} = \|u\|_H$.

Remark 2.1.2 *From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.*

Theorem 2.1.3 *([12]). Let $(u_n)_{n \in N}$ is a bounded sequence in the Hilbert space H, it posses a subsequence which converges in the weak topology of H.*

Theorem 2.1.4 *([12]). In the Hilbert space, all sequence which converges in the weak topology is bounded.*

Theorem 2.1.5 *([12]). Let $(u_n)_{n \in N}$ be a sequence which converges to u, in the weak topology and $(\upsilon_n)_{n \in N}$ is an other sequence which converge weakly to υ, then*

$$\lim_{n \to \infty} \langle \upsilon_n, u_n \rangle = \langle \upsilon, u \rangle. \tag{2.5}$$

Theorem 2.1.6 *([12])(Banach-Alaoglu-Bourbaki). Let X be a normed space, then the following unit ball of X' is compact in $\sigma(X', X)$*

$$B' \equiv \Big\{ x \in X'; \|x\| \leq 1 \Big\}. \tag{2.6}$$

2.2 Functional Spaces

2.2.1 The $L^p(\Omega)$ spaces

Definition 2.2.1 *Let $1 \leq p < \infty$ and let Ω be an open domain in $\mathbb{R}^N$, $N \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |f(x)|^p dx < \infty \right\}. \tag{2.7}$$

If $p = \infty$, we have

$$L^\infty(\Omega) = \Big\{ f : \Omega \to \mathbb{R} \text{ is measurable and there exists a constant } C \text{ such that } \\ |f(x)| \leq C \text{ a.e in } \Omega \Big\}.$$

Notation 2.2.1 *We denote*

$$\|f\|_p = \left[\int_\Omega |f(x)|^p dx \right]^{\frac{1}{p}}, \tag{2.8}$$

$$\|f\|_\infty = \inf \Big\{ C,\ |f(x)| \leq C \text{ a.e in } \Omega \Big\}. \tag{2.9}$$

Notation 2.2.2 *For $1 \leq p \leq \infty$, we denote by q the conjugate of pi.e. $\frac{1}{p} + \frac{1}{q} = 1$.*

Theorem 2.2.1 *([12]) L^p is a vectorial space, and $\|.\|_p$ is a norm for all $1 \leq p \leq \infty$.*

Theorem 2.2.2 *([12])(Fischer-Riesz) L^p is a Banach space for all $1 \leq p \leq \infty$.*

Remark 2.2.1 *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x) g(x) dx, \tag{2.10}$$

is a Hilbert space.

Theorem 2.2.3 *([12]) For $1 < p < \infty$, $L^p(\Omega)$ is a reflexive space.*

2.2.2 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Theorem 2.2.4 *([12]) (Holder's inequality). Let $1 \leq p \leq \infty$. Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and*

$$\int_\Omega |fg|dx \leq \|f\|_p \|g\|_q.$$

Lemma 2.2.1 *([12]). Let $f \in L^p(\Omega) \cap L^q(\Omega)$, where $1 \leq p \leq r \leq q$, $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$, and $0 \leq \alpha \leq 1$. Then*

$$\|f\|_{L^r} \leq \|f\|_{L^p}^{\alpha} \|f\|_{L^q}^{1-\alpha}.$$

Proposition 2.2.1 *([88]) If $\mu(\Omega) < \infty$, $1 < p < q < \infty$, then $L^q \hookrightarrow L^p$ and, if $f \in L^q(\Omega)$*

$$\|f\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q},$$

if $f \in L^\infty(\Omega)$ then $f \in L^p(\Omega)$ and

$$\|f\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p}} \|u\|_{L^\infty}.$$

Lemma 2.2.2 *([12])(Young's inequality). Let $f \in L^p(\mathbb{R}^{\mathbb{N}})$ and $g \in L^q(\mathbb{R}^{\mathbb{N}})$ with $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$. Then $f * g \in L^r(\mathbb{R}^{\mathbb{N}})$ and*

$$\|f * g\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

2.2.3 The $W^{m,p}(\Omega)$ spaces

The theory of Sobolev spaces has been developed by generalizing the notion of classical derivatives and introducing the idea of weak or generalized derivatives.

• **Definition and basic properties**. Let Ω be an open subset of $\mathbb{R}^N$ and $N \in \mathbb{N}$. Let α a multi-index where $\alpha = (\alpha_1, ..., \alpha_N)$ with $\alpha_i \geq 0$ for any $i \in 1, ..., N, |\alpha| = \sum_{i=1}^{d} \alpha_i$ and $D^\alpha = D_1^{\alpha_1} ... D_N^{\alpha_N}$ with $D_i = \frac{\partial}{\partial x_i}$.

Proposition 2.2.2 *([53]) Let Ω be an open domain in $\mathbb{R}^N$. Then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that*

$$\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx, \text{ for all } \varphi \in D(\Omega),$$

where $1 \leq p \leq \infty$ and it's well-known that f is unique.

Now, we will introduce the Sobolev spaces. Let $1 \leq p \leq \infty$, and $k \in \mathbb{N}$, the Sobolev space $W^{k,p}(\Omega)$ is the space of functions $f \in L^p(\Omega)$. which have generalized up to order k such that $D^\alpha f \in L^p(\Omega)$ for all $|\alpha| \leq k$. For $k = 0$, we set $W^{0,p}(\Omega) = L^p(\Omega)$.

$$W^{k,p}(\Omega) = \Big\{ f \in L^p(\Omega),\ D^\alpha f \in L^p(\Omega),\ \ \forall \alpha \in \mathbb{N}^N,\ |\alpha| \leq k \Big\}.$$

The space $W^{k,p}(\Omega)$ becomes a Banach space with the norm

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \Big(\displaystyle\sum_{|\alpha| \leq k} \|D^\alpha f\|^p_{L^p(\Omega)} \Big)^{1/p}, \text{ for } 1 \leq p < +\infty, \\ \\ \displaystyle\max_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}, \text{ for } p = +\infty. \end{cases}$$

$W^{k,p}(\Omega)$ is a reflexive space for $1 < p < \infty$, and a separable space for $1 \leq p < \infty$.

Definition 2.2.2 *Let $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, then the Sobolev space $W_0^{k,p}(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ in the norm of the space $W^{k,p}(\Omega)$.*
It follows from the definition above that the space $W_0^{k,p}(\Omega)$ is a Banach space with the norm $\|.\|_{W^{k,p}(\Omega)}$. We write $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.

For $1 \leq p \leq \infty$, the dual space of $W^{k,p}(\Omega)$ is denoted by $W^{-k,q}(\Omega)$ where q is the conjugate exponent of p. We usually use the notation $W^{-1,2}(\Omega) = H^{-1}(\Omega)$ Moreover, for $k, l \in \mathbb{N}, k \leq l$ we have the inclusions

$$C_0^\infty(\Omega) \subset H_0^1(\Omega) \subset H_0^k(\Omega) \subset L^2(\Omega) \subset H^{-k}(\Omega) \subset H^{-l}(\Omega) \subset (C_0^\infty(\Omega))'.$$

Each of these spaces being dense in the following one.
• **Embedding results**. We turn now on embedding and compact embedding results concerning the Sobolev spaces.

Theorem 2.2.5 *([53]) Let Ω be an open bounded set of $\mathbb{R}^N$ with a Lipschitz boundary. For nonnegative integers k, l such that $0 \leq l \leq k$, we have the continuous embedding $W^{k,p}(\Omega) \hookrightarrow W^{l,p}(\Omega)$ for all $1 \leq p \leq \infty$. Moreover, for $k \geq 0$, we have $W^{k,r}(\Omega) \hookrightarrow W^{k,p}(\Omega)$ for all $1 \leq p \leq r \leq \infty$.*

Definition 2.2.3 *If $1 \leq p < N$, the Sobolev conjugate of p is defined as*

$$p^* = \frac{Np}{N-p},$$

Equivalently $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$. Also $p^* > p$.

Theorem 2.2.6 *([12]) (Gagliardo-Nirenberg-Sobolev Inequality). Let $1 \leq p < N$. Then there exists a constant $C > 0$ (depending on p and N) such that*

$$\|f\|_{L^{p^*}(\mathbb{R}^N)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^N)}, \quad \forall f \in W^{1,p}(\mathbb{R}^N).$$

In particular, we have the continuous imbedding

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N).$$

Corollary 2.2.1 *([12]) For any $1 \leq p < N$, $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is continuously imbedded, for all $r \in [p, p^*]$.*

Theorem 2.2.7 *([12])Equality case, $p = N$, $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for all $r \in [N, \infty[$.*

Theorem 2.2.8 *([12]) Let $p > N$, $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$.*

We now extend the results to proper subsets of $\mathbb{R}^N$.

Corollary 2.2.2 *([12]) Let Ω a bounded domain in $\mathbb{R}^N$ with C^1 boundary and $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have with continuous imbedding*

$$\begin{array}{ll} If & 1 \leq p < \infty, \text{ then } W^{1,p}(\Omega) \subset L^{p^*}(\Omega), \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}. \\ If & p = N, \text{ then } W^{1,p}(\Omega) \subset L^r(\Omega), \quad \forall r \in [p, +\infty[. \\ If & p > N, \text{ then } W^{1,p}(\Omega) \subset L^\infty(\Omega). \end{array}$$

Moreover, If $p > N$ we have

$$\forall f \in W^{1,p}(\Omega), \quad |f(x) - f(y)| \leq C|x-y|^\delta \|f\|_{W^{1,p}(\Omega)} \text{ a.e with } x, y \in \Omega,$$

with $\delta = 1 - \frac{N}{p} > 0$ and C is a constant which depend on p, N and Ω. In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Theorem 2.2.9 *([53])(Rellich-Kondrachov). Let Ω a bounded domain in $\mathbb{R}^N$ with C^1 boundary and $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have with compact imbedding*

$$\begin{array}{ll} If & p < N, \text{ then } W^{1,p}(\Omega) \subset L^r(\Omega), \forall r \in [1, p^*[\quad where \ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}. \\ If & p = N, \text{ then } W^{1,p}(\Omega) \subset L^r(\Omega), \ \forall r \in [p, +\infty[. \\ If & p > N, \text{ then } W^{1,p}(\Omega) \subset C(\overline{\Omega}). \end{array}$$

Remark 2.2.2 *We remark in particular that*

$$W^{1,p}(\Omega) \subset L^r(\Omega),$$

with compact imbedding for $1 \leq p \leq \infty$ and for $p \leq r < p^*$.

• Generalized Sobolev Imbedding

We shall now generalize the results of previous section to all derivative orders of $k \geq 2$.

Theorem 2.2.10 *([53]). Let $k \geq 1$ be an integer and $1 \leq p < \infty$ and. Then*

1. *If $p < N/k$, then $W^{k,p}(\mathbb{R}^N) \subset L^r(\mathbb{R}^N)$ for all $r \in [p, Np/(N - pk)]$.*

2. *If $p = N/k$ then $W^{k,n/k}(\mathbb{R}^N) \subset L^r(R^N)$ for all $r \in [N/k, \infty)$.*

3. *If $p > N/k$, then $W^{k,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$,*

with continuous imbedding.

Lemma 2.2.3 *([12])(Sobolev-Poincaré's inequality). Let Ω be a bounded open subset of $\mathbb{R}^N$, then there is a constant $C(p, \Omega)$(depending on p and Ω) such that*

$$If \quad 2 \leq p \leq \frac{2N}{N-2}, \quad N \geq 3 \quad and \quad q \geq 2, \quad N = 1, 2,$$

then

$$\|u\|_{L^p(\Omega)} \leq C(p, \Omega)\|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

Remark 2.2.3 *For all $\varphi \in H^2(\Omega)$, $\Delta\varphi \in L^2(\Omega)$ and for Γ sufficiently smooth, we have*

$$\|\varphi(t)\|_{H^2(\Omega)} \leq C\|\Delta\varphi(t)\|_{L^2(\Omega)}.$$

Proposition 2.2.3 *([12] Green's formula). For all $u \in H^2(\Omega)$, $\upsilon \in H^1(\Omega)$ we have*

$$-\int_{\Omega} \Delta u \upsilon dx = \int_{\Omega} \nabla u \nabla \upsilon dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \upsilon d\sigma,$$

where $\frac{\partial u}{\partial \eta}$ is a normal derivation of u at Γ.

2.2.4 The $L^p(0,T,X)$ spaces

Definition 2.2.4 *Let X be a Banach space, denote by $L^p(0,T,X)$ the space of measurable functions*

$$\begin{aligned} f :]0,T[&\to X \\ t &\mapsto f(t), \end{aligned} \tag{2.11}$$

such that

$$\left(\int_0^T \|f(t)\|_X^p dt \right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, \text{ for } 1 \leq p < \infty. \tag{2.12}$$

If $p = \infty$

$$\|f\|_{L^\infty(0,T,X)} = \sup_{t \in]0,T[} ess \|f(t)\|_X. \tag{2.13}$$

Theorem 2.2.11 *([47]). The space $L^p(0,T,X)$ is complete.*

We denote by $\mathcal{D}'(0,T,X)$ the space of distributions in $]0,T[$ which take its values in X and let us define

$$D'(0,T,X) = \mathcal{L}(\mathcal{D}]0,T[,X),$$

where $\mathcal{L}(E,F)$ is the space of the linear continuous applications of E to F. Since $u \in \mathcal{D}'(0,T,X)$, we define the distribution derivation as

$$\frac{\partial u}{\partial t}(\varphi) = -u\left(\frac{d\varphi}{dt}\right), \quad \forall \varphi \in \mathcal{D}(]0,T[),$$

and since $u \in L^p(0,T,X)$, we have

$$u(\varphi) = \int_0^T u(t)\varphi(t)dt, \quad \forall \varphi \in \mathcal{D}(]0,T[).$$

We will introduce some basic results on the $L^p(0,T,X)$ space. These results, will be very useful in the other chapters of this thesis.

Lemma 2.2.4 *([47]). Let $f \in L^p(0,T,X)$ and $\frac{\partial f}{\partial t} \in L^p(0,T,X)$, $(1 \leq p \leq \infty)$, then the function f is continuous from $[0,T] \to X$. i.e. $f \in C^1(0,T,X)$.*

Lemma 2.2.5 *(47]). Let $\mathcal{V} =]0,T[\times\Omega$ an open bounded domain in $\mathbb{R}\times\mathbb{R}^n$, and g_μ, g are two functions in $L^q(]0,T[,L^q(\Omega))$, $1 < q < \infty$ such that*

$$\|g_\mu\|_{L^q(]0,T[,L^q(\Omega))} \leq C, \forall \mu \in \mathbb{N}, \tag{2.14}$$

and $g_\mu \to g \quad in \quad \mathcal{V}$, then $g_\mu \rightharpoonup g \quad in \quad L^q(\mathcal{V})$.

Theorem 2.2.12 *([47]). $L^p(0,T,X)$ equipped with the norm $\|.\|_{L^q(]0,T[,X)}$, $1 \leq p \leq \infty$ is a Banach space.*

Proposition 2.2.4 *([47]). Let X be a reflexive Banach space, X' it's dual, and $1 \leq p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $L^p(0,T,X)$ is identify algebraically and topologically with $L^q(0,T,X')$.*

Proposition 2.2.5 *([47]) Let X, Y be Banach space, $X \subset Y$ with continuous embedding, then we have with continuous embedding*

$$L^p(0,T,X) \subset L^p(0,T,Y),$$

The following compactness criterion will be useful for nonlinear evolution problem, especially in the limit of the nonlinear terms.

Proposition 2.2.6 *([47])*
Let B_0, B, B_1 be Banach spaces with $B_0 \subset B \subset B_1$. Assume that the embedding $B_0 \hookrightarrow B$ is compact and $B \hookrightarrow B_1$ is continuous. Let $1 < p, q < \infty$. Assume further that B_0 and B_1 are reflexive. Define

$$W \equiv \Big\{u \in L^p(0,T,B_0) : u' \in L^q(0,T,B_1)\Big\}. \tag{2.15}$$

Then, the embedding $W \hookrightarrow L^p(0,T,B)$ is compact.

2.2.5 Some Algebraic inequalities

Since our study based on some known algebraic inequalities, we want to recall few of them here.

Lemma 2.2.6 *(The Cauchy-Schwartz's inequality) Every inner product satisfies the Cauchy-Schwartz's inequality*

$$\langle x_1, x_2 \rangle \leq \|x_1\| \|x_2\|. \tag{2.16}$$

The equality sign holds if and only if x_1 and x_1 are dependent.

Lemma 2.2.7 *([12]Young's inequalities). For all $a, b \in \mathbb{R}_+$, we have*

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \tag{2.17}$$

where $C_\varepsilon = \varepsilon^{-\frac{1}{p-1}}$.

Lemma 2.2.8 *([12]) For $a, b \geq 0$, the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{2.18}$$

where, $\frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty$.

2.3 Integral Inequalities

We will recall some fundamental integral inequalities introduced by A. Haraux, V. Komornik , P.Martinez and A.Guesmia to estimate the decay rate of the energy.

2.3.1 A result of exponential decay

The estimation of the energy decay for some dissipative problems is based on the following lemma

Lemma 2.3.1 *([44]) Let $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function and assume that there is a constant $A > 0$ such that*

$$\forall t \geq 0, \qquad \int_t^{+\infty} E(\tau)\, d\tau \leq \frac{1}{A}\, E(t). \tag{2.19}$$

Then we have

$$\forall t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}. \tag{2.20}$$

Proof 2.3.1 *The inequality (2.20) is verified for $t \leq \frac{1}{A}$, this follows from the fact that E is a decreasing function. We prove that (2.20) is verified for $t \geq \frac{1}{A}$. Introduce the function*

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_t^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (2.19) we find that

$$\forall t \geq 0, \quad h'(t) + Ah(t) \leq 0.$$

Let

$$T_0 = \sup\{t,\ h(t) > 0\}. \tag{2.21}$$

For every $t < T_0$, we have

$$\frac{h'(t)}{h(t)} \leq -A,$$

thus

$$h(0) \leq\ e^{-At} \leq \frac{1}{A}\, E(0)\, e^{-At}, \qquad for \quad 0 \leq t < T_0. \tag{2.22}$$

Since $h(t) = 0$ if $t \geq T_0$, this inequality holds in fact for every $t \in R_+$. Let $\varepsilon > 0$. As E is positive and decreasing, we deduce that

$$\forall t \geq \varepsilon, \quad E(t) \leq \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} E(\tau)\, d\tau \leq \frac{1}{\varepsilon}\, h(t-\varepsilon) \leq \frac{1}{A\varepsilon}\, E(0)\, e^{\varepsilon t}\, e^{-At}.$$

Choosing $\varepsilon = \frac{1}{A}$, we obtain

$$\forall t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}.$$

The proof of Lemma 2.3.1 is now completed.

2.3.2 A result of polynomial decay

Lemma 2.3.2 *([44]) Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and assume that there are two constants $q > 0$ and $A > 0$ such that*

$$\forall t \geq 0, \quad \int_t^{+\infty} E^{q+1}(\tau)\, d\tau \leq \frac{1}{A}\, E^q(0)E(t). \tag{2.23}$$

Then we have

$$\forall t \geq 0, \quad E(t) \leq E(0) \left(\frac{1+q}{1+A\,q\,t} \right)^{1/q}. \tag{2.24}$$

Remark 2.3.1 It is clear that Lemma 2.3.1 is similar to Lemma 2.3.2 in the case of $q = 0$.

Proof 2.3.2 *If $E(0) = 0$, then $E \equiv 0$ and there is nothing to prove. Otherwise, replacing the function E by the function $\frac{E}{E(0)}$ we may assume that $E(0) = 1$. Introduce the function*

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_t^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (2.23) we find that

$$\forall t \geq 0, \quad -h' \geq (Ah)^{1+q},$$

where

$$T_0 = \sup\{t,\ h(t) > 0\}.$$

Integrating in $[0,t]$ we obtain that

$$\forall 0 \leq t < T_0, h(t)^{-q} - h(0)^{-\sigma} \geq \sigma\omega^{1+q}t,$$

hence

$$0 \leq t < T_0, \quad h(t) \leq \left(h^{-q}(0) + qA^{1+q}\, t \right)^{-1/q}. \tag{2.25}$$

Since $h(t) = 0$ if $t \geq T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Since

$$h(0) \leq \frac{1}{A} E(0)^{1+q} = \frac{1}{A},$$

by (2.23), the right-hand side of (2.25) is less than or equal to:

$$\left(h^{-q}(0) + qA^{1+q}\, t\right)^{-1/q} \leq \frac{1}{A}(1 + A\, q\, t)^{-1/q}. \tag{2.26}$$

From other hand, E being nonnegative and non-increasing, we deduce from the definition of h and the above estimate that:

$$\begin{array}{rcl} \forall s \geq 0,\; E\left(\frac{1}{A} + (q+1)s\right)^{q+1} & \leq & \frac{1}{\frac{1}{A}+q+1} \int_s^{\frac{1}{A}+(q+1)s} E(\tau)^{q+1}\, d\tau \\ & \leq & \frac{A}{1+Aqs} h(s) \leq \frac{A}{1+Aqs} \frac{1}{A} (1 + Aqs)^{-\frac{1}{q}}, \end{array}$$

hence

$$\forall S \geq 0, \quad E\left(\frac{1}{A} + (q+1)S\right) \leq \frac{1}{(1 + A\, q\, S)^{1/q}}.$$

Choosing $t = \frac{1}{A} + (1+q)s$ then the inequality (2.24) follows.Note that letting $q \to 0$ in this theorem we obtain (2.24).

2.3.3 New integral inequalities of P. Martinez

The above inequalities are verified only if the energy function is integrable. We will try to resolve this problem by introducing some weighted integral inequalities, so we can estimate the decay rate of the energy when it is slow.

Lemma 2.3.3 *([52]) Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^1 function such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{when} \quad t \to +\infty. \tag{2.27}$$

Assume that there exist $q \geq 0$ and $A > 0$ such that

$$\int_S^{+\infty} E(t)^{q+1} \phi'(t)\, dt \leq \frac{1}{A} E(0)^q E(S), \quad 0 \leq S < +\infty. \tag{2.28}$$

then we have

$$\text{If } q > 0, \quad \text{then } E(t) \leq E(0) \left(\frac{1+q}{1 + q\, A\, \phi(t)}\right)^{\frac{1}{q}}, \;\; \forall t \geq 0,$$

$$\text{If } q = 0, \quad \text{then } E(t) \leq E(0)\, e^{1 - A\, \phi(t)}, \;\; \forall t \geq 0.$$

Proof of Lemma 2.3.3.

This Lemma is a generalization of Lemma 2.3.1. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$, (we notice that ϕ^{-1} has a meaning by the hypotheses assumed on ϕ). f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain f is non-increasing, $f(0) = E(0)$ and if we set $x = \phi(t)$ we obtain for $0 \leq S < T < +\infty$

$$\begin{aligned}
&\int_{\phi(S)}^{\phi(T)} f(x)^{q+1} dx = \int_{\phi(S)}^{\phi(T)} E\left(\phi^{-1}(x)\right)^{q+1} dx = \int_S^T E(t)^{q+1}\phi'(t)dt \\
&\leq \frac{1}{A}\, E(0)^q E(S) = \frac{1}{A}\, E(0)^q f(\phi(S)).
\end{aligned}$$

Setting $s = \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\forall s \geq 0, \quad \int_s^{+\infty} f(x)^{q+1}\, dx \leq \frac{1}{A}\ E(0)^q f(s).$$

Thanks to Lemma 2.3.1, we deduce the desired results.

2.3.4 Generalized inequalities of A. Guesmia

Lemma 2.3.4 *([33]) Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ differentiable function, $\lambda \in \mathbb{R}_+$ and $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ convex and increasing function such that $\Psi(0) = 0$. Assume that*

$$\int_s^{+\infty} \Psi(E(t))\, dt \leq E(s), \quad \forall s \geq 0,$$

$$E'(t) \leq \lambda E(t), \quad \forall t \geq 0.$$

Then E satisfies the estimate

$$E(t) \leq e^{\tau_0 \lambda T_0} d^{-1}\Big(e^{\lambda(t-h(t))}\Psi\Big(\psi^{-1}\Big(h(t) + \psi(E(0))\Big)\Big)\Big), \quad \forall t \geq 0,$$

where

$$\psi(t) = \int_t^1 \frac{1}{\Psi(s)}\, ds, \quad \forall t > 0,$$

$$d(t) = \begin{cases} \Psi(t) & if \quad \lambda = 0, \\ & \qquad\qquad\qquad \forall t \geq 0, \\ \int_0^t \frac{\Psi(s)}{s}\, ds & if \quad \lambda > 0, \end{cases}$$

$$h(t) = \begin{cases} K^{-1}(D(t)), & if \quad t > T_0, \\ \\ 0 & if \quad t \in [0, T_0], \end{cases}$$

$$K(t) = D(t) + \frac{\psi^{-1}\big(t + \psi(E(0))\big)}{\Psi\big(\psi^{-1}\big(t + \psi(E(0))\big)\big)}\; e^{\lambda t}, \quad \forall t \geq 0,$$

$$D(t) = \int_0^t e^{\lambda s}\, ds, \quad \forall t \geq 0,$$

$$T_0 = D^{-1}\Big(\frac{E(0)}{\Psi(E(0))}\Big), \quad \tau_0 = \begin{cases} 0, & if \quad\; t > T_0, \\ \\ 1, & if \quad t \in [0, T_0]. \end{cases}$$

Remark 2.3.2 If $\lambda = 0$ (that is E is non increasing), then we have

$$E(t) \leq \psi^{-1}\Big(h(t) + \psi(E(0))\Big), \quad \forall t \geq 0 \tag{2.29}$$

where $\psi(t) = \int_t^1 \frac{1}{\Psi(s)} ds$ for $t > 0$, $h(t) = 0$ for $0 \leq t \leq \frac{E(0)}{\Psi(E(0))}$ and

$$h^{-1}(t) = t + \frac{\psi^{-1}\Big(t + \psi(E(0))\Big)}{\Psi\Big(\psi^{-1}\Big(t + \psi(E(0))\Big)\Big)}, \quad t > 0.$$

This particular result generalizes the one obtained by Martinez ([52]) in the particular case of $\Psi(t) = dt^{p+1}$ with $p \geq 0,\ d > 0$ and improves the one obtained by Eller, Lagnese and Nicaise .

Proof. Because $E'(t) \leq \lambda E(t)$ imply $E(t) \leq e^{\lambda(t-t_0)} E(t_0)$ for all $t \geq t_0 \geq 0$, then, if $E(t_0) = 0$ for some $t_0 \geq 0$, then $E(t) = 0$ for all $t \geq t_0$, and then there is nothing to prove in this case. So we assume that $E(t) > 0$ for all $t \geq 0$ without loss of generality. Let:

$$L(s) = \int_s^{+\infty} \Psi(E(t))\, dt, \quad \forall s \geq 0.$$

We have, $L(s) \leq E(s)$, for all $s \geq 0$. The function L is positive, decreasing and of class $C^1(R_+)$ satisfying

$$-L'(s) = \Psi(E(s)) \geq \Psi(L(s)), \quad \forall s \geq 0.$$

The function ψ is decreasing, then

$$\Big(\psi(L(s))\Big)' = \frac{-L'(s)}{\Psi(L(s))} \geq 1, \quad \forall s \geq 0.$$

Integration on $[0,t]$, we obtain

$$\psi(L(t)) \geq t + \psi(E(0)), \quad \forall t \geq 0. \tag{2.30}$$

Since Ψ is convex and $\Psi(0) = 0$, we have

$$\Psi(s) \leq \Psi(1)s, \ \forall s \in [0,1] \quad and \quad \Psi(s) \geq \Psi(1)s, \ \forall s \geq 1,$$

then $\lim_{t\to 0} \psi(t) = +\infty$ and $[\psi(E(0)), +\infty[\subset Image\,(\psi)$. Then (2.30) imply that

$$L(t) \leq \psi^{-1}\Big(t + \psi(E(0))\Big), \quad \forall t \geq 0. \tag{2.31}$$

Now, for $s \geq 0$, let

$$f_s(t) = e^{-\lambda t}\int_s^t e^{\lambda\tau}\,d\tau, \quad \forall t \geq s.$$

The function f_s is increasing on $[s, +\infty[$ and strictly positive on $]s, +\infty[$ such that

$$f_s(s) = 0 \quad and \quad f_s'(t) + \lambda f_s(t) = 1, \quad \forall t \geq s \geq 0,$$

and the function d is well defined, positive and increasing such that:

$$d(t) \leq \Psi(t) \quad and \quad \lambda t d'(t) = \lambda\Psi(t), \quad \forall t \geq 0,$$

then

$$\begin{aligned} \partial_\tau\Big(f_s(\tau)d(E(\tau))\Big) &= f_s'(\tau)d(E(\tau)) + f_s(\tau)E'(\tau)d'(E(\tau)) \\ &\leq \Big(1 - \lambda f_s(\tau)\Big)\Psi(E(\tau)) + \lambda f_s(\tau)\Psi(E(\tau)) \\ &= \Psi(E(\tau)), \quad \forall \tau \geq s \geq 0. \end{aligned}$$

Integrating on $[s,t]$, we obtain

$$L(s) \geq \int_s^t \Psi(E(\tau))\,d\tau \geq f_s(t)d(E(t)), \quad \forall t \geq s \geq 0. \tag{2.32}$$

Since $\lim_{t\to+\infty} d(s) = +\infty$, $d(0) = 0$ and d is increasing, then (2.31) and (2.32) imply

$$E(t) \le d^{-1}\left(\inf_{s\in[0,t[} \frac{\psi^{-1}\Big(s+\psi(E(0))\Big)}{f_s(t)}\right), \quad \forall t > 0. \tag{2.33}$$

Now, let $t > T_0$ and

$$J(s) = \frac{\psi^{-1}\Big(s+\psi(E(0))\Big)}{f_s(t)}, \quad \forall s \in [0,t[.$$

The function J is differentiable and we have

$$J'(s) = f_s^{-2}(t)\Big[e^{-\lambda(t-s)}\psi^{-1}\Big(s+\psi(E(0))\Big) - f_s(t)\Psi\Big(\psi^{-1}\Big(s+\psi(E(0))\Big)\Big)\Big].$$

Then

$$J'(s) = 0 \;\Leftrightarrow\; K(s) = D(t) \quad \textit{and} \quad J'(s) < 0 \;\Leftrightarrow\; K(s) < D(T).$$

Since $K(0) = \frac{E(0)}{\Psi(E(0))}$, $D(0) = 0$ and K and D are increasing (because ψ^{-1} is decreasing and $s \mapsto \frac{s}{\Psi(s)}$, $s > 0$, is non increasing thanks to the fact that Ψ is convex). Then, for $t > T_0$,

$$\inf_{s\in[0,t[} J(s) = J\Big(K^{-1}(D(t))\Big) = J(h(t)).$$

Since h satisfies $J'(h(t)) = 0$, we conclude from (2.33) our desired estimate for $t > T_0$.
For $t \in [0, T_0]$, we have just to note that $E'(t) \le \lambda E(t)$ and the fact that $d \le \Psi$ implies

$$E(t) \le e^{\lambda t}E(0) \le e^{\lambda T_0}E(0) \le e^{\lambda T_0}\Psi^{-1}\Big(e^{\lambda t}\Psi(E(0))\Big) \le e^{\lambda T_0}d^{-1}\Big(e^{\lambda t}\Psi(E(0))\Big).$$

Remark 2.3.3 Under the hypotheses of Lemma 2.3.4, we have $\lim\limits_{t\to+\infty} E(t) = 0$. Indeed, we have just to choose $s = \frac{1}{2}t$ in (2.33) instead of $h(t)$ and note that $d^{-1}(0) = 0$, $\lim\limits_{t\to+\infty} \psi^{-1}(t) = 0$ and $\lim\limits_{t\to+\infty} f_{\frac{1}{2}t}(t) > 0$.

Lemma 2.3.5 *([Guesmia [33]) Let $E:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ be a differentiable function, $a:\mathbb{R}_+\longrightarrow\mathbb{R}_+^*$ and $\lambda:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ two continuous functions. Assume that there exist $r\geq 0$ such that*

$$\int_s^{+\infty} E^{r+1}(t)dt \leq a(s)E(s), \forall s\geq 0, \tag{2.34}$$

$$E'(t)\leq \lambda(t)E(t), \qquad \forall t\geq 0. \tag{2.35}$$

Then E satisfies for all $t\geq 0$,

$$E(t)\leq \frac{E(0)}{\omega(0)}\omega(h(t)exp(\widetilde{\lambda}(t)-\widetilde{\lambda}(h(t)))exp(-\int_0^{h(t)}\omega(\tau)d\tau), \;\; if\; r=0$$

$$E(t)\leq \omega(h(t))exp(\widetilde{\lambda}(t)-\widetilde{\lambda}(h(t)))\left[\left(\frac{\omega(0)}{E(0)}\right)^r + r\int_0^{h(t)}\omega(\tau)^{r+1}d\tau\right]^{-1/r} \quad and\; if\; r>0$$

where $\quad \widetilde{\lambda}(t)=\int_0^t \lambda(\tau)d\tau.$

Proof of Lemma 2.3.5.

If $E(s)=0$ or $a(s)=0$ for one $s\geq 0$, the first inequality implies $E(t)=0$ for $t\geq s$,then we can suppose that $E(t)>0$ and $a(t)>0$ for $t\geq 0$.
By putting $\omega=\frac{1}{a}$ and $\Psi(s)=\int_s^{+\infty}E^{r+1}(t)dt$, we have

$$\Psi(s)\leq \frac{1}{\omega(s)}E(s), \qquad \forall s\geq 0. \tag{2.36}$$

The function Ψ is decreasing, positive and of class C^1 on R^+ and verifies:

$$\Psi'(s)=-E^{r+1}(s)\leq -(\omega(s)\Psi(s))^{r+1}, \quad \forall s\geq 0$$

then

$$\Psi(s)\leq \Psi(0)exp\left(\int_0^s\omega(\tau)d\tau\right)\leq \frac{E(0)}{\omega(0)}exp\left(\int_0^s\omega(\tau)d\tau\right) \qquad \text{if } r=0 \tag{2.37}$$

$$\Psi(s)\leq \left((\frac{\omega(0)}{E(0)})^r+\int_0^s(\omega(\tau))^{r+1}d\tau\right)^{-1/r} \qquad \text{if } r>0 \tag{2.38}$$

Now we put for all $s \geq 0$,

$$f_s(t) = exp(-(r+1)\widetilde{\lambda}(t)) \int_s^t exp((r+1)\widetilde{\lambda}(\tau))d\tau, \quad \forall t \geq s \tag{2.39}$$

where $f_s(s) = 0$ and $f_s'(t) + (r+1)\lambda(t)f_s(t) = 1$, $\forall t \geq s \geq 0$. Under the second hypothesis in the lemma, we deduce

$$E^{r+1}(t) \geq \partial_t(f_s(t)E^{r+1}(t)); \quad \forall t \geq, \quad s \geq 0 \tag{2.40}$$

hence

$$\Psi(s) \geq \int_s^{g(s)} E^{r+1}(t) \geq f_s(g(s))E^{r+1}(g(s))); \quad \forall s \geq 0 \tag{2.41}$$

where $g : R^+ \longrightarrow R^{+*}$ with $I_s(g(s)) = 0$ and I_s is defined by

$$I_s(t) = (\omega(s))^{r+1} \int_s^t exp((r+1)\widetilde{\lambda}(\tau))d\tau$$

Let $t > g(0)$ and $s = h(t)$ where

$$h(t) = \begin{cases} 0 \quad \text{if} \quad t \in [0, g(0)] \\ \max g^{-1}(t) \quad \text{if} \quad t \in]g(0), +\infty[\end{cases}$$

Hence we have $g(s) = t$ and we deduce from (2.41) that for all $t \geq g(0)$, we have

$$\Psi(h(t)) \geq f_{h(t)}(t)E^{r+1}(t) = \left(exp(-(r+1)\widetilde{\lambda}(t)) \int_{h(t)}^t exp((r+1)\widetilde{\lambda}(\tau))d\tau \right) E^{r+1}(t)$$

We conclude from (2.37) and (2.38) that for all $t > g(0)$, we obtain

$$E(t) \leq \frac{E(0)}{\omega(0)} exp(\widetilde{\lambda}(t)) \left(\int_{h(t)}^t exp(\widetilde{\lambda}(\tau))d\tau \right)^{-1} exp\left(-\int_0^{h(t)} \omega(\tau)d\tau \right) \text{ if } r = 0$$

and

$$E(t) \leq exp(\widetilde{\lambda}(t)) \left(\int_{h(t)}^t exp((r+1)\widetilde{\lambda}(\tau))d\tau \right)^{\frac{-1}{r+1}} \times$$
$$\left((\tfrac{\omega(0)}{E(0)})^r + r\int_0^{h(t)} (\omega(\tau))^{r+1}d\tau \right)^{\frac{-1}{r(r+1)}} \quad \text{if } r > 0.$$

The fact that $I^t_{h(t)} = I_s(g(s)) = 0$, we get the result of the lemma for $t > g(0)$. If $t \in [0, g(0)]$ the second inequality of the lemma implies that

$$E(t) \leq E(0)exp(\widetilde{\lambda}(t)).$$

Since $h(t) = 0$ on $[0, g(0)]$ then $E(0)exp(\widetilde{\lambda}(t))$ is identically equal to the left hand side of the results of the lemma.That conclude the proof.

Lemma 2.3.6 *([Guesmia 33]) Let $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a differentiable function, $a_1, a_2 \in \mathbb{R}^*_+$ and $a_3, \lambda, r, p \in \mathbb{R}_+$ such that*

$$a_3\lambda(r+1) < 1 \quad \textit{and for all} \quad 0 \leq s \leq T < +\infty,$$

$$\int_s^T E^{r+1}(t)dt \leq a_1(s)E(s) + a_2E^{p+1}(s) + a_3E^{r+1}(T),$$

$$E'(t) \leq \lambda E(t), \qquad \forall t \geq 0.$$

Then there exist two positive constants ω and c such that for all $t \geq 0$,

$$E(t) \leq ce^{-\omega t}, \ \textit{if } r = 0$$

$$E(t) \leq c(1+t)^{-1/r}, \ \textit{if } r > 0 \quad \textit{and } \lambda = 0$$

$$E(t) \leq c(1+t)^{\frac{-1}{r(r+1)}}, \ \textit{if } r > 0 \quad \textit{and } \lambda > 0$$

Proof of Lemma 2.3.6.

We show that E verifies the inequality (2.34). Applying the lemma (2.3.5), we have

$$a_3E^{r+1}(T) = a_3 \int_s^T E'^{r+1}(t)dt + a_3E^{r+1}(s) \leq a_3(r+1)\int_s^T \lambda E^{r+1}(t)dt + a_3E^{r+1}(s).$$

Under (2.34), we obtain

$$\int_s^{+\infty} E^{r+1}(t)dt \leq b(s)E(s), \qquad \forall s \geq 0, \tag{2.42}$$

where

$$b(s) = \frac{a_1 + a_2 E^p(s) + a_3 E^r(s)}{1 - a_3\lambda(r+1)}, \qquad \forall s \geq 0.$$

We consider the function f_0 defined in (2.39)and integrating on $[0, s]$ the inequality

$$E^{r+1}(t) \geq \partial_t(f_0(t)E^{r+1}(t)), \quad \forall t \geq 0,$$

we obtain under (2.42)

$$b(0)E(0) \geq \int_0^s E^{r+1}(t)dt \geq f_0(s)E^{r+1}(s), \qquad \forall s \geq 0,$$

then

$$E(s) \leq \left(\frac{b(0)E(0)}{f_0(s)}\right)^{\frac{1}{r+1}}, \qquad \forall s \geq 0.$$

On the other hand, the conditions of the lemma implies that

$$E(s) \leq E(0)exp(\widetilde{\lambda}(s), \qquad \forall s \geq 0.$$

Hence

$$E(s) \leq \min\left\{E(0)exp(\widetilde{\lambda}(s), \left(\frac{b(0)E(0)}{f_0(s)}\right)^{\frac{1}{r+1}}\right\} = d(s), \qquad \forall s \geq 0.$$

d is continuous and positive and

$$b(s) \leq \frac{a_1 + a_2(d(s))^p + a_3(d(s))^r}{1 - a_3\lambda(r+1)}, \qquad \forall s \geq 0.$$

Hence we can conclude from (2.42) the first inequality (2.34) of the lemma (2.3.5) with

$$a(s) = \frac{a_1 + a_2(d(s))^p + a_3(d(s))^r}{1 - a_3\lambda(r+1)}, \qquad \forall s \geq 0.$$

This completes the proof.

2.4 Existence Methods

2.4.1 Faedo-Galerkin's approximations

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle ., . \rangle$ and the associated norm $\|.\|$.

$$(P) \qquad \begin{cases} u''(t) + A(t)u(t) = f(t), & t \in [0, T] \\ (x, 0) = u_0(x), \; u'(x, 0) = u_1(x); \end{cases}$$

where u and f are unknown and given function, respectively, mapping the closed interval $[0, T] \subset \mathbb{R}$ into a real separable Hilbert space H. $A(t)$ $(0 \leq t \leq T\ )$ are linear bounded operators in H acting in the energy space $V \subset H$.

Assume that $\langle A(t)u(t), v(t) \rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t; ., .)$ is a bilinear continuous in V. The problem (P) can be formulated as: Found the solution $u(t)$ such that

$$(\widetilde{P}) \qquad \begin{cases} u \in C([0, T]; V), u' \in C([0, T]; H) \\ \langle u''(t), v \rangle + a(t; u(t), v) = \langle f, v \rangle \text{ in } D'(]0, T[) \\ u_0 \in V \ , \ u_1 \in H. \end{cases}$$

This problem can be resolved with the approximation process of Fadeo-Galerkin.

Let V_m a sub-space of V with the finite dimension d_m, and let $\{w_{jm}\}$ one basis of V_m such that

1. $V_m \subset V (\dim V_m < \infty), \forall m \in N$
2. $V_m \to V$ such that, there exist a dense subspace ϑ in V and for all $\upsilon \in \vartheta$ we can get sequence $\{u_m\}_{m \in N} \in V_m$ and $u_m \to u$ in V.
3. $V_m \subset V_{m+1}$ and $\overline{\cup_{m \in N} V_m} = V$.

We define the solution u_m of the approximate problem

$$(P_m) \qquad \begin{cases} u_m(t) = \sum_{j=1}^{d_m} g_j(t) w_{jm} \\ u_m \in C([0, T]; V_m), u'_m \in C([0, T]; V_m) \quad , u_m \in L^2(0, T; V_m) \\ \langle u''_m(t), w_{jm} \rangle + a(t; u_m(t), w_{jm}) = \langle f, w_{jm} \rangle, \quad 1 \leq j \leq d_m \\ u_m(0) = \sum_{j=1}^{d_m} \xi_j(t) w_{jm} \ , \ u'_m(0) = \sum_{j=1}^{d_m} \eta_j(t) w_{jm} \end{cases}$$

where

$$\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \longrightarrow u_0 \text{ in V as } m \longrightarrow \infty$$

$$\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \longrightarrow u_1 \text{ in V as } m \longrightarrow \infty.$$

By virtue of the theory of ordinary differential equations, the system (P_m) has unique local solution which is extend to a maximal interval $[0, t_m[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$ to obtain one solution defined for all $t > 0$.

2.4.2 A priori estimation and convergence

Using the following estimation

$$\|u_m\|^2 + \|u'_m\|^2 \leq C \left\{ \|u_m(0)\|^2 + \|u'_m(0)\|^2 + \int_0^T \|f(s)\|^2 ds \right\} \ ; \quad 0 \leq t \leq T,$$

and the Gronwall lemma we deduce that the solution u_m of the approximate problem (P_m) converges to the solution u of the initial problem (P). The uniqueness proves that u is the solution.

Lemma 2.4.1 *(Gronwall's lemma). Let $T > 0$, $g \in L^1(0,T)$, $g \geq 0$ a.e and c_1, c_2 are positives constants. Let $\varphi \in L^1(0,T)$ $\varphi \geq 0$ a.e such that $g\varphi \in L^1(0,T)$ and*

$$\varphi(t) \leq c_1 + c_2 \int_0^t g(s)\varphi(s) ds \quad a.e \ in \ (0,T),$$

then, we have

$$\varphi(t) \leq c_1 exp \left(c_2 \int_0^t g(s) ds \right) \quad a.e \ in \ (0,T).$$

2.4.3 Semigroups approach

Definition 2.4.1 *([88]). Let X be a Banach space. A one parameter family $T(t)$ for $0 \leq t < \infty$ of bounded linear operators from X into X is a semigroup bounded linear operator on X if*

- *$T(0) = I$, (I is the identity operator on X).*

- *$T(t+s) = T(t).T(s)$ for every $t, s \geq 0$ (the semigroup property).*

A semigroup of bounded linear operators, $T(t)$, is uniformly continuous if

$$\lim_{t \to 0} \|T(t) - I\| = 0.$$

The linear operator A defined by

$$D(A) = \left\{ x \in X; \lim_{t \to 0} \frac{T(t)x - x}{t} \quad exists \right\}$$

and

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt}\Big|_{t=0}, \quad \forall x \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$ and $D(A)$ is the domain of A.

Theorem 2.4.1 *([88])(Lumer-Phillips). Let A be a linear operator with dense domain $D(A)$ in X*

- *If A is dissipative and there is a $\lambda_0 > 0$ such that the range , $R(\lambda_0 I - A) = X$, then A is the infinitesimal generator of a C_0 semigroup of contraction on X.*

- *If A is the infinitesimal generator of a C_0 semigroup of contractions on X then $R(\lambda_0 I - A) = X, \quad \forall \lambda > 0$ and A is dissipative.*

Chapter 3

Global existence and asymptotic behavior of solutions for a viscoelastic wave equation with a constant delay term

3.1 Introduction

In this chapter we consider the following Cauchy problem of the form

$$\begin{cases} u''(x,t) - k_0\Delta u + \alpha \displaystyle\int_0^t g(t-s)\Delta u(x,s)ds & \\ +\mu_1(t)u'(x,t) + \mu_2(t)u'(x,t-\tau) = 0, & \text{on } \Omega\times]0,+\infty[\\ u(x,t) = 0, & \text{on } \partial\Omega\times]0,+\infty[\\ u(x,0) = u_0(x), u_t(x,t) = u_1(x), & \text{on } \Omega \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \text{on } \Omega\times]0,t[\end{cases} \tag{3.1}$$

Where Ω is a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}^*$) with a smooth boundary $\partial\Omega$. The initial data u_0, u_1, f_0 belong to a suitable space. Moreover, $\tau > 0$ is the time delay term and $\mu_1,\ \mu_2$ are real functions that will be specified later. Furthermore k_0 is a positive real number and g is a positive non-increasing function defined on $\mathbb{R}_+$.

In recent years, the PDEs with time delay effects have become an active area of research.

Many authors have focused on this problem (see ([8],[2],[3],[17],[61],[79]).
The presence of delay may lead to a source of instability. In [22] for example, R. Datko, J. Lagnese and M. P. Polis proved that a small delay may destabilize a system.
S. Nicaise, C. Pignotti studied in [56] the wave equation with a linear internal damping term with constant delay and determined suitable relations between μ_0 and $\mu_1 > 0$ in which the stability or alternatively instability takes place.
After that, they studied in [60] the stabilization problem by interior damping of the wave equation with boundary time-varying delay in a bounded and smooth domain. By introducing suitable Lyapunov functionals, exponential stability estimates are obtained if the time delay effect is appropriately compensated by the internal damping.
It is worth mentioning that Z. Y. Zhang et al. [80] recently have investigated global existence and uniform decay for wave equation with dissipative term and boundary damping. Under some assumptions on nonlinear feedback function. They have obtain the results by means of Galerkin method and the multiplier technique. More precisely, they introduced a new variables and transformed the boundary value problem into an equivalent one with zero initial data by argument of compactly and monotonicity. More details are present in [82].
Later on, Zhang et al. studied in [84] the well posedness and uniform stability of strong and weak solutions of the nonlinear generalized dissipative Klein-Gordon equation with nonlinear damped boundary conditions. Also, the authors proved the well posedness by means of nonlinear semigroup method and obtain the uniform stabilization by using the perturbed energy functional method. In another works, Zai-Yun Zhang and al ([80],[82]) considered a more general problem than (1.1). Their proof of the existence is based on the Galerkin approximation. For strong solutions, their approximation requires a change of variables to transform the main problem into an equivalent problem with initial value equals zero. Especially, they overcome some difficulties, that is, the presence of nonlinear terms and nonlinear boundary damping bringing up serious difficulties when passing to the limit, by combining arguments of compactly and monotonicity.
F. Tahamtani and A. Peyravi [76] investigated the nonlinear viscoelastic wave equation with dissipative boundary conditions:

$$u'' - k_0 \triangle u + \alpha \int g(t-s) div[a(s) \nabla u(s)] ds + (k_1 + b(x)|u'|^{m-2}) u' = |u^{p-2}| u$$

They showed that the solutions blow up in finite time under some restrictions on initial data and for arbitrary initial energy in some case. In another case, they proved a nonexistence result when the initial energy is less than potential well depth.
Wenjun Liu [48] studied the weak viscoelastic equation with an internal time-varying delay term

$$u''(x,t) - k_0 \triangle u + \alpha(t) \int g(t-s) \triangle u(x,s) ds + a_0 u'(x,t) + a_1 u'(x, t - \tau(t)) = 0$$

in a bounded domain. By introducing suitable energy and Lyapunov functionals, he establishes a general decay rate estimate for the energy under suitable assumptions.
A. Benaissa, A. Benguessoum and S. A. Messaoudi [5] considered the wave equation with a weak internal constant delay term:

$$u''(x,t) - \triangle u + \mu_1(t) u'(x,t) + \mu_2(t) u'(x, t-\tau) = 0 \text{ on } \times [0, +\infty[$$

In a bounded domain. Under appropriate conditions on μ_1 and μ_2, they proved global existence of solutions by the Faedo-Galerkin method and establish a decay rate estimate for the energy by using the multiplier method.
However, according to our best knowledge, in the present work, we have to treat Eq.(1.1) with a delay term and it is not considered in the literature. The proof of the existence is based on the Galerkin approximation. The content of this chapter is organized as follows. In Section 2, we provide assumptions that will be used later. We state and prove the existence result. In Section 3, we establish the energy decay result that is given in Theorem 5.2.3.

3.2 Preliminary Results

In the following, we will give sufficient conditions and assumptions that guarantee that the problem 3.1 has a global solution.
$(H1)$ g is a positive bounded function satisfying

$$k_0 - \alpha \int_0^\infty g(s) ds = l > 0, \quad \alpha > 0,$$

and there exists a positive non-increasing function η such that for $t > 0$ we have

$$g'(t) \leq -\eta(t)g(t), \quad \eta(t) > 0$$

$(H2)$ $\mu_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing function of class $C^1(\mathbb{R}_+)$ satisfying

$$\left|\frac{\mu_1'(t)}{\mu_1(t)}\right| \leq M, \tag{3.2}$$

$(H3)$ $\mu_2 : \mathbb{R}_+ \to \mathbb{R}$ is a function of class $C^1(\mathbb{R}_+)$, which is not necessarily positive or monotone, such that

$$|\mu_2(t)| \leq \beta\mu_1(t), \tag{3.3}$$

$$|\mu_2'(t)| \leq \bar{M}\mu_1(t), \tag{3.4}$$

for some $0 < \beta < 1$ and $\bar{M} > 0$. For the relaxation function g we assume

We also need the following technical Lemmas in the course of our investigation.

Lemma 3.2.1 (52) *For any $g \in C^1$ and $\varphi \in H_0^1(0,T)$ we have*

$$\int_0^t \int_\Omega g(t-s)\varphi\varphi_t dxds = -\frac{d}{dt}\left(\frac{1}{2}(g \circ \varphi)(t) - \frac{1}{2}\int_0^t g(s)ds\|\varphi\|_2^2\right) - \frac{1}{2}g(t)\|\varphi\|_2^2 + \frac{1}{2}(g' \circ \varphi)(t),$$

where

$$(g \circ \varphi)(t) = \int_0^t \int_\Omega g(t-s)|\varphi(s) - \varphi(t)|^2 dx\ ds.$$

In order to prove the existence of solutions of the problem (3.1) we introduce like in [80] the unknown auxiliary

$$z(x,\rho,t) = u'(x,t-\tau\rho), \quad x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Then we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0.$$

Therefore the problem (3.1) takes the form

$$
\begin{cases}
u''(x,t) - k_0\Delta u(x,t) + \alpha \displaystyle\int_0^t g(t-s)\Delta u(x,s)ds & \\
+\mu_1(t)u'(x,t) + \mu_2(t)z(x,1,t) = 0, & \text{on } \Omega\times]0,+\infty[\\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & x\in\Omega, \rho\in(0,1), t>0 \\
u(x,t) = 0, & \text{on } \partial\Omega\times]0,+\infty[\\
u(x,0) = u_0, u'(x,t) = u_1, & \text{on } \Omega \\
z(x,\rho,0) = f_0(x,-\tau\rho) & \text{on } \Omega\times]0,t[
\end{cases}
\tag{3.5}
$$

Now, we are in the position to state our main result, namely the theorem of global existence.

Theorem 3.2.1 *Let $(u_0, u_1, f_0) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1))$ be given. Assume that assumptions (H1) -(H3) are fulfilled. Then the problem (3.5) admits a unique global weak solution (u,z) satisfying*

$$
u \in C([0,T]; H_0^1(\Omega)),\ \ u' \in C([0,T]; H_0^1(\Omega)),\ \ z \in C([0,T]; L^2(\Omega \times (0,1))
$$

To prove this theorem, we need the following lemma. First we define the energy associated to the solution of the problem (3.5) by

$$
\begin{aligned}
E(t) = &\frac{1}{2}\|u'\|_2^2 + \left(\frac{k_0}{2} - \frac{\alpha}{2}\int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\
&+ \frac{\alpha}{2}(g\circ\nabla u)(t) + \frac{1}{2}\xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx,
\end{aligned}
\tag{3.6}
$$

where ξ is non-increasing function such that

$$
\tau\beta < \zeta < \tau(2-\beta), \quad t>0, \tag{3.7}
$$

where $\quad \xi(t) = \zeta\mu_1(t)$.

Lemma 3.2.2 *Let (u,z) be a regular solution of problem (3.5). Then the energy functional defined by (3.6) satisfies*

$$
E'(t) \le -\Big(\mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|u'(x,t)\|^2 - \Big(\frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|z(x,1,t)\|^2 \le 0. \tag{3.8}
$$

Proof 3.2.1 *Multiplying the first equation in (3.5) by $u'(x,t)$, integrating over Ω and using Green's identity we obtain*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|u'\|_2^2+k_0\|\nabla u\|_2^2\Big)+\mu_1(t)\|u'\|_2^2+\mu_2(t)\int_\Omega u'z(x,1,t)dx\\&-\alpha\int_0^t g(t-s)\int_\Omega \nabla u(x,s)\nabla u'(x,t)dxds=0.\end{aligned}\tag{3.9}$$

We simplify the last term in (3.9) by applying the lemma 3.2.1, we get

$$\begin{aligned}&-\alpha\int_0^t g(t-s)\int_\Omega \nabla u(x,s)\nabla u'(x,t)dxds=\frac{\alpha}{2}\frac{d}{dt}(g\circ\nabla u)\\&-\frac{\alpha}{2}(g'\circ\nabla u)+\frac{\alpha}{2}g(t)\|\nabla u\|^2-\frac{\alpha}{2}\frac{d}{dt}\int_0^t g(s)ds\|\nabla u\|^2.\end{aligned}\tag{3.10}$$

Replacing (3.10) in (3.9) we arrive at

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{2}\|u'\|_2^2+\Big(\frac{k_0}{2}-\frac{\alpha}{2}\int_0^t g(s)ds\Big)\|\nabla u\|_2^2-\frac{\alpha}{2}(g\circ\nabla u)\Big)=\frac{\alpha}{2}(g'\circ\nabla u)(t)\\&-\frac{1}{2}g(t)\|\nabla u\|_2^2-\mu_1(t)\|u'\|_2^2-\mu_2(t)\int_\Omega z(x,1,s)u'dx.\end{aligned}\tag{3.11}$$

Multiplying the second equation in (3.5) by $\frac{\xi(t)z}{\tau}$, where $\xi(t)$ satisfies (3.7) and integrating over $\Omega\times(0,1)$, we obtain

$$\frac{\xi(t)}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx+\frac{\xi(t)}{2\tau}\int_0^1\int_\Omega\frac{d}{d\rho}z^2(x,\rho,t)dxd\rho,\tag{3.12}$$

which gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx=\frac{\xi'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\\&-\frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)dx+\frac{\xi(t)}{2\tau}\int_\Omega u'^2(x,t)dx=0.\end{aligned}\tag{3.13}$$

A combination of (3.11) and (3.13) leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u'\|_2^2 + (k_0 - \alpha\int_0^t g(s)ds)\|\nabla u\|_2^2\Big)\\
+&\frac{1}{2}\frac{d}{dt}\Big(\alpha(g\circ\nabla u) + \xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\Big)\\
&= \frac{\alpha}{2}(g'\circ\nabla u) - \frac{1}{2}g(s)\|\nabla u\|_2^2 ds - \mu_1(t)\|u_n'\|_2^2 ds - \mu_2(t)\int_\Omega z(x,1,s)u'dx\\
&+ \frac{\xi'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx - \frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)dx + \frac{\xi(t)}{2\tau}\int_\Omega u'^2(x,t)dx.
\end{aligned} \tag{3.14}
$$

Using the definition (3.6) of $E(t)$, we deduce that

$$
\begin{aligned}
E'(t) &= \frac{\alpha}{2}(g'\circ\nabla u) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \mu_1(t)\|u_n'\|_2^2 ds\\
&- \mu_2(t)\int_\Omega z(x,1,s)u'dx + \frac{\xi'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\\
&- \frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)dx + \frac{\xi(t)}{2\tau}\int_\Omega u'^2(x,t)dx.
\end{aligned} \tag{3.15}
$$

Due to Young's inequality we have

$$
\begin{aligned}
E'(t) \le \frac{\alpha}{2}(g'\circ\nabla u) - \frac{1}{2}g(t)\|\nabla u\|_2^2 ds - \Big(\mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|u'(x,t)\|^2\\
- \Big(\frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|z(x,1,t)\|^2
\end{aligned} \tag{3.16}
$$

Using the assumption (3.7) for $\xi(t)$ we see that

$$
C_1 = \mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2} > 0, \quad C_2 = \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2} > 0,
$$

then we easily deduce that

$$
\begin{aligned}
E'(t) \le -\Big(\mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|u'(x,t)\|^2\\
- \Big(\frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|z(x,1,t)\|^2 \le 0.
\end{aligned} \tag{3.17}
$$

This completes the proof of the lemma.

3.3 Global existence

We will use the Faedo-Galerkin method to prove the global existence of solutions. Let $(w_n)_{n\in N}$ be a basis in $H_0^1(\Omega)$ and W_n be the space generated by $w_1, ..., w_n,\ n \in \mathbb{N}$. Now, we define for $1 \le i \le n$ the sequence $\varphi_i(x,\rho)$ as follows $\varphi_i(x,0) = w_i(x)$.Then, we may extend $\varphi_i(x,0)$ by $\varphi_i(x,\rho)$ over $(L^2 \times [0,1])$ and denote V_n to be the space generated by $\varphi_1, ..., \varphi_n,\ n \in \mathbb{N}$. We consider the approximate solution $(u_n(t), z_n(t))$ as follow for any given i

$$u_n(t) = \sum_{i=0}^{n} c_{in}(t) w_i \qquad z_n(t) = \sum_{i=0}^{n} r_{in}(t)\varphi_i,$$

which satisfies

$$\begin{aligned} \int_\Omega u_n''(t) w_i dx - k_0 \int_\Omega \triangle u_n(t) w_i dx + \alpha \int_0^t g(t-s) \int_\Omega \triangle u_n(s) w_i dx ds \\ + \mu_1(t) \int_\Omega u_n'(t) w_i dx + \mu_2(t) \int_\Omega z_n(x,1,t) w_i dx = 0, \end{aligned} \tag{3.18}$$

and

$$\int_\Omega (\tau z_{nt}(x,\rho,t) + z_{n\rho}(x,\rho,t))\varphi_i dx = 0. \tag{3.19}$$

The system is completed by the initials conditions:

$$u_n(0) = \sum_{i=0}^{n} c_{in}(0) w_i \to u_0 \quad \text{in} \quad H_0^1(\Omega) \quad \text{when} \quad n \to \infty$$

$$u_n'(0) = \sum_{i=0}^{n} c_{in}'(0) w_i \to u_1 \quad \text{in} \quad H_0^1(\Omega) \quad \text{when} \quad n \to \infty$$

$$z_n(0) = \sum_{i=0}^{n} r_{in}(0)\varphi_i \to f_0 \quad \text{in} \quad L^2(\Omega \times (0,1)) \quad \text{when} \qquad n \to \infty.$$

Then the problem (3.5) can be reduced to a second-order ODE system and we infer that this problem admits a unique local solution $(u_n(t), z_n(t))$ in $[0, t_n[$ where $0 < t_n < T$. This solution can be extended to $[0; T[,\ 0 < T \le +\infty$ by Zorn lemma. In the next step we shall prove that this solution is global.

1. **First estimate.** Multiplying the equation in (3.18) by $c'_{in}(t)$ and summing with respect to i we obtain

$$
\begin{aligned}
&\int_\Omega u''_n(t)u'_n(t)dx + k_0\int_\Omega \nabla u_n(t)\nabla u'_n(t)dx - \alpha\int_0^t g(t-s)\int_\Omega \nabla u_n(s)\nabla u'_n(t)dxds\\
&+\mu_1(t)\int_\Omega u'^2_n(t)dx + \mu_2(t)\int_\Omega z_n(x,1,t)u'_n(t)dx = 0,
\end{aligned}
$$

then

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u'_n\|_2^2 + \frac{k_0}{2}\|\nabla u_n\|_2^2\right) + \mu_1(t)\|u'_n\|_2^2 + \mu_2(t)\int_\Omega z_n(x,1,t)u'_n(t)dx\\
&-\alpha\int_0^t g(t-s)\int_\Omega u_n(t)\nabla u'_n(t)dxds = 0.
\end{aligned}
\tag{3.20}
$$

We use the lemma 3.2.1 to simplify the last term in (3.20)

$$
\begin{aligned}
&-\alpha\int_0^t g(t-s)\int_\Omega u_n(t)\nabla u'_n(t)dxds = \frac{\alpha}{2}\frac{d}{dt}(g\circ\nabla u_n)(t)\\
&-\frac{\alpha}{2}(g'\circ\nabla u_n)(t) + \frac{\alpha}{2}g(t)\|\nabla u_n(t)\|^2 - \frac{\alpha}{2}\frac{d}{dt}\int_0^t g(s)ds\|\nabla u_n(t)\|^2.
\end{aligned}
\tag{3.21}
$$

Replacing (3.21) in (3.20) and integrating over $(0,t)$ we arrive at

$$
\begin{aligned}
&\frac{1}{2}\|u'_n\|_2^2 + \Big(\frac{k_0}{2} - \frac{\alpha}{2}\int_0^t g(s)ds\Big)\|\nabla u_n\|_2^2 - \frac{\alpha}{2}(g\circ\nabla u_n)(t)\\
&-\frac{\alpha}{2}\int_0^t (g'\circ\nabla u_n)(s)ds + \frac{1}{2}\int_0^t g(s)\|\nabla u_n\|_2^2 ds + \int_0^t \mu_1(s)\|u'_n\|_2^2 ds\\
&+\int_0^t \mu_2(s)\int_\Omega z(x,1,s)u'_n(t)dxds = \frac{1}{2}\|u_{1n}\|^2 + \frac{k_0}{2}\|\nabla u_{0n}\|^2.
\end{aligned}
\tag{3.22}
$$

Multiplying the equation (3.34) by $r_{in}(t)$, summing with respect to i and integrating over $(0,1)$, we obtain

$$
\frac{\xi(t)}{2}\frac{d}{dt}\int_\Omega\int_0^1 z_n^2(x,\rho,t)d\rho dx + \frac{\xi(t)}{2\tau}\int_0^1\int_\Omega \frac{d}{d\rho}z_n^2(x,\rho,t)dxd\rho = 0, \tag{3.23}
$$

which gives

$$
\begin{aligned}
&\frac{1}{2}\Big[\frac{d}{dt}\xi(t)\int_{\Omega}\int_{0}^{1} z_n^2(x,\rho,t)d\rho dx-\xi'(t)\int_{\Omega}\int_{0}^{1} z_n^2(x,\rho,t)d\rho dx\Big]\\
&\qquad+\frac{\xi(t)}{2\tau}\int_{\Omega} z_n^2(x,1,t)dx-\frac{\xi(t)}{2\tau}\int_{\Omega} u_n'^2(x,t)dx=0.
\end{aligned}
\tag{3.24}
$$

Integrating (3.24) over $(0,t)$ we get

$$
\begin{aligned}
&\frac{1}{2}\Big[\xi(t)\int_{\Omega}\int_{0}^{1} z_n^2(x,\rho,t)d\rho dx-\int_{0}^{t}\int_{\Omega}\int_{0}^{1} \xi'(s)z_n^2(x,\rho,s)d\rho dxds\Big]\\
&+\frac{1}{2\tau}\int_{0}^{t}\int_{\Omega}\xi(s)z_n^2(x,1,s)dxds\\
&-\frac{1}{2\tau}\int_{0}^{t}\int_{\Omega}\xi(t)u_n'^2(x,t)dxds=\frac{\xi(0)}{2}\|f_0\|^2.
\end{aligned}
\tag{3.25}
$$

Combining (3.22) and (5.13) we find

$$
\begin{aligned}
&\frac{1}{2}\|u_n'\|_2^2+\Big(\frac{k_0}{2}-\frac{\alpha}{2}\int_{0}^{t} g(s)ds\Big)\|\nabla u_n\|_2^2+\frac{1}{2}\xi(t)\int_{\Omega}\int_{0}^{1} z_n^2(x,\rho,t)d\rho dx\\
&+\frac{\alpha}{2}(g\circ\nabla u_n)(t)-\frac{\alpha}{2}\int_{0}^{t}(g'\circ\nabla u_n)(s)ds+\frac{\alpha}{2}\int_{0}^{t} g(s)\|\nabla u_n\|_2^2ds\\
&+\int_{0}^{t}\mu_1(s)\|u_n'\|_2^2ds+\int_{0}^{t}\mu_2(s)\int_{\Omega} z(x,1,s)u_n'(t)dxds\\
&-\frac{1}{2}\int_{\Omega}\int_{0}^{t}\int_{0}^{1}\xi'(s)z_n^2(x,\rho,s)d\rho dxds+\frac{1}{2\tau}\int_{0}^{t}\int_{\Omega}\xi(s)z_n^2(x,1,s)dxds\\
&-\frac{1}{2\tau}\int_{0}^{t}\int_{\Omega}\xi(t)u_n'^2(x,t)dxds=\frac{1}{2}\|u_{1n}\|^2+\frac{k_0}{2}\|\nabla u_{0n}\|^2+\frac{\xi(0)}{2}\|f_0\|^2.
\end{aligned}
\tag{3.26}
$$

Using Hölder's and Young's inequalities on the eighth term of (3.26) we obtain

$$
\begin{aligned}
\int_{0}^{t}\mu_2(s)\int_{\Omega} z(x,1,s)u_n'(t)dxds\le{}&\frac{1}{2}\int_{0}^{t}\mu_2(s)\int_{\Omega} z^2(x,1,s)dxds\\
&+\frac{1}{2}\int_{0}^{t}\mu_2(s)\int_{\Omega} u_n'^2(t)dxds.
\end{aligned}
\tag{3.27}
$$

Then the equation (3.26) takes the form

$$\begin{aligned}
&E_n(t) - \frac{\alpha}{2}\int_0^t (g' \circ \nabla u_n)(s)ds + \frac{1}{2}\int_0^t g(s)\|\nabla u_n\|_2^2 ds \\
&+ \int_0^t \Big(\mu_1(s) - \frac{\xi(s)}{2\tau} - \frac{\mu_2(s)}{2}\Big)\|u_n'\|_2^2 ds \\
&+ \frac{1}{2}\int_\Omega\int_0^t\int_0^1 \xi'(s) z_n^2(x,\rho,s)d\rho dx ds \\
&+ \int_0^t \Big(\frac{\xi(s)}{2\tau} - \frac{\mu_2(s)}{2}\Big)\int_\Omega z_n^2(x,1,s)dxds \le E_n(0),
\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}
E_n(t) = &\frac{1}{2}\|u_n'\|_2^2 + \Big(\frac{k_0}{2} - \frac{\alpha}{2}\int_0^t g(s)ds\Big)\|\nabla u_n\|_2^2 \\
&+ \frac{\alpha}{2}(g \circ \nabla u_n)(t) + \frac{1}{2}\xi(t)\int_\Omega\int_0^1 z_n^2(x,\rho,t)d\rho dx,
\end{aligned} \tag{3.29}$$

and

$$E_n(0) = \frac{1}{2}\|u_{1n}\|^2 + \frac{k_0}{2}\|\nabla u_{0n}\|^2 + \frac{\xi(0)}{2}\|f_0\|^2. \tag{3.30}$$

Since u_{0n}, u_{1n}, f_0 converge we can find a constant $L_1 > 0$ independent of n such that

$$\begin{aligned}
&\frac{1}{2}\|u_n'\|_2^2 + \Big(\frac{k_0}{2} - \frac{\alpha}{2}\int_0^t g(s)ds\Big)\|\nabla u_n\|_2^2 \\
&+ \frac{\alpha}{2}(g \circ \nabla u_n)(t) + \frac{1}{2}\xi(t)\int_\Omega\int_0^1 z_n^2(x,\rho,t)d\rho dx \le L_1.
\end{aligned} \tag{3.31}$$

So this estimate gives
u_n is bounded in $L^\infty(0,\infty;H_0^1(\Omega))$
u_n' is bounded in $L^\infty(0,\infty;L^2(\Omega))$
z_n is bounded in $L^\infty(0,\infty;L^2(\Omega)\times(0,1))$.

2. **Second estimate.** Differentiating (3.15) with respect to t, we get

$$\begin{aligned}
&\int_\Omega u_n'''(t)w_i dx - k_0\int_\Omega \Delta u_n'(t)w_i dx + \alpha\int_0^t g(t-s)\int_\Omega \Delta u_n'(s)w_i dxds \\
&+\mu_1(t)\int_\Omega u_n''(t)w_i dx + \mu_2(t)\int_\Omega z_n'(x,1,t)w_i dx + \mu_1'(t)\int_\Omega u_n'(t)w_i dx + \mu_2'(t)\int_\Omega z_n(x,1,t)w_i = 0,
\end{aligned} \tag{3.32}$$

Multiplying by $c''_{in}(t)$ and summing with respect to i we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u''_n(t)\|^2 + (k_0 - \alpha\int_0^t g(s)ds)\|\nabla u'_n(t)\| + \frac{\alpha}{2}(g\circ\nabla u')\Big)\\
&-\frac{\alpha}{2}(g'\circ\nabla u')(t) + \frac{\alpha}{2}g(t)\|\nabla u'\|_2^2\\
&+\mu_1(t)\|u''_n(t)\|^2 + \mu'_1(t)\int_\Omega u'_n(t)u''_n(t) + \mu_2(t)\int_\Omega z'_n(x,1,t)u''_n(t)dx\\
&+\mu'_2(t)\int_\Omega z_n(x,1,t)u''_n(t) = 0
\end{aligned}
\tag{3.33}
$$

Differentiating (3.16) with respect to t, we get

$$
\int_\Omega (\tau z''_n(x,\rho,t) + \frac{d}{d\rho}z'_n(x,\rho,t))\varphi_i dx = 0. \tag{3.34}
$$

Multiplying by $r'_{in}(t)$ and summing with respect to i we obtain

$$
\frac{\tau}{2}\|z'_n\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z'_n\|_2^2 = 0 \tag{3.35}
$$

Combination give

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u''_n(t)\|^2 + \mu'_1(t)\|u'_n(t)\|^2 + (k_0 - \alpha\int_0^t g(s)ds)\|\nabla u'_n(t)\|\\
&+\frac{\alpha}{2}(g\circ\nabla u') + \tau\int_0^1 \|z'_n(x,1,t)\|_2^2 d\rho\Big)\\
&-\frac{\alpha}{2}(g'\circ\nabla u')(t) + \frac{1}{2}\|z'_n(x,1,t)\|_2^2 - \frac{1}{2}\|u''_n(t)\|^2 + \frac{1}{2}g(t)\|\nabla u'\|_2^2\\
&+\mu_1(t)\|u''_n(t)\|^2 + \mu_2(t)\int_\Omega z'_n(x,1,t)u''_n(t)dx\\
&+\mu'_2(t)\int_\Omega z_n(x,1,t)u''_n(t) = 0
\end{aligned}
\tag{3.36}
$$

Exploiting the Hölder's, Young's and Poincaré's inequalities and the assumptions $(H1), (H2)$ we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u_n''(t)\|^2+(k_0-\alpha\int_0^t g(s)ds)\|\nabla u_n'(t)\|+\frac{\alpha}{2}(g\circ\nabla u')+\tau\int_0^1\|z_n'(x,1,t)\|_2^2 d\rho\Big)\\
&-\frac{\alpha}{2}(g'\circ\nabla u')(t)+\frac{1}{2}\|z_n'(x,1,t)\|_2^2-\frac{1}{2}\|u_n''(t)\|^2+\frac{1}{2}g(t)\|\nabla u'\|^2+\mu_1(t)\|u_n''(t)\|^2\\
&\leq\frac{1}{2}\mu_1'(t)\|u_n''(t)\|^2+\frac{1}{2}\|u_n'(t)\|^2+\frac{1}{2}\mu_2(t)\|u_n''(t)\|^2+\frac{1}{2}\|z_n'(x,1,t)\|^2+\\
&\frac{1}{2}\mu_2'(t)\|u_n''(t)\|^2+\frac{1}{2}\|z_n(x,1,t)\|^2
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u_n''(t)\|^2+(k_0-\alpha\int_0^t g(s)ds)\|\nabla u_n'(t)\|+\frac{\alpha}{2}(g\circ\nabla u)+\tau\int_0^1\|z_n'(x,1,t)\|_2^2 d\rho\Big)\\
&-\frac{\alpha}{2}(g'\circ\nabla u)(t)+\frac{1}{2}g(t)\|\nabla u'\|^2(s)\\
&\leq C\Big(\|u_n''(t)\|^2+\|z_n'(x,1,t)\|_2^2+\|z_n(x,1,t)\|^2+\|u_n'(t)\|^2\Big)
\end{aligned}
\tag{3.37}
$$

Integrating the last inequality over (0, t) and using (3.6), we get

$$
\begin{aligned}
&\Big(\|u_n''(t)\|^2+(k_0-\alpha\int_0^t g(s)ds)\|\nabla u_n'(t)\|+\frac{\alpha}{2}(g\circ\nabla u)+\tau\int_0^1\|z_n'(x,1,t)\|_2^2 d\rho\Big)\\
&-\frac{\alpha}{2}\int_0^t(g'\circ\nabla u)(s)ds+\frac{1}{2}\int_0^t g(s)\|\nabla u\|^2(s)ds\\
&\leq\Big(\|u_n''(0)\|^2+(k_0-\alpha\int_0^t g(s)ds)\|\nabla u_n'(0)\|+\tau\int_0^1\|z_n'(x,1,0)\|_2^2 d\rho\Big)\\
&+C\int_0^t\Big(\|u_n''(s)\|^2+\|z_n'(x,1,s)\|_2^2+\|z_n(x,1,s)\|^2+\|u_n'(s)\|^2\Big)ds
\end{aligned}
\tag{3.38}
$$

using (H1)and the first estimates, then the formula take the form

$$
\begin{aligned}
&\Big(\|u_n''(t)\|^2+l\|\nabla u_n'(t)\|+\tau\int_0^1\|z_n'(x,1,t)\|_2^2 d\rho\Big)\\
&\leq C'+C\int_0^s\Big(\|u_n''(t)\|^2+2\|z_n'(x,1,t)\|_2^2+\|z_n(x,1,t)\|^2+2\|u_n'(t)\|^2\Big)ds
\end{aligned}
\tag{3.39}
$$

Applying lemma de Gronwal we get

$$\Big(\|u_n''(t)\|^2 + \|\nabla u_n'(t)\| + \tau \int_0^1 \|z_n'(x,1,t)\|_2^2 d\rho\Big) \le L_2 \tag{3.40}$$

where $L_2 = Ce^{C'T}$ is a positive constant independent of $n \in N$ and $t \in [0,T)$. We observe for estimates (3.31) and (3.40) that
u_n is bounded in $L^\infty(0,\infty; H_0^1(\Omega))$, u_n' is bounded in $L^2(0,\infty; H_0^1(\Omega))$
u_n'' is bounded in $L^\infty(0,\infty; L^2(\Omega))$, z_n is bounded in $L^\infty(0,\infty; L^2(\Omega) \times (0,1))$.

Applying Dunford Pettis theorem, we deduce that there exists a subsequence (u_i, z_i) of (u_n, z_n) and we can replace the subsequence (u_i, z_i) with the sequence (u_n, z_n) such that

$u_n \rightharpoonup u$ weak star in $L^\infty(0,T; H_0^1(\Omega))$, $u_n' \rightharpoonup u'$ weak star in $L^2(0,T; H_0^1(\Omega)$
$u_n'' \rightharpoonup u''$ weakly in $L^2(0,T; H_0^1(\Omega))$, $z_n \rightharpoonup z$ weak star in $L^2(0,T; L^2(\Omega) \times (0,1))$.

Moreover u_n'' is bounded in $L^2(0,T; H_0^1(\Omega))$. The same method is used to prove that u_n' is bounded in $L^2(0,T; H_0^1(\Omega))$. Consequently u_n' is bounded in $H^1(0,T; H_0^1(\Omega))$. Furthermore, by Aubins-Lions theorem ([47]) there exists a subsequence (u_j) still represented by the same notation such that $u_j' \rightharpoonup u'$ strongly in $L^2(0,T; H_0^1(\Omega))$, which implies that

$$u_j' \rightharpoonup u' \quad \text{a.e. on} \quad \Omega \times (0,T) \quad \text{and} \quad z_j \rightharpoonup z \quad \text{a.e. on} \quad \Omega \times (0,T).$$

And we have for each $w_i \in L^2(\Omega), v_i \in L^2(\Omega)$

$$\int_\Omega \Big(u_j'' - k_0 \Delta u_j + \alpha \int_\Omega g(t-s)\Delta u_j ds + \mu_1 u_j' + \mu_2 z_j\Big) w_i dx$$

$$\to \int_\Omega \Big(u'' - k_0 \Delta u + \alpha \int_\Omega g(t-s)\Delta u ds + \mu_1 u' + \mu_2 z\Big) w_i,$$

and

$$\int_\Omega \tau(z_{jt} + z_{j\rho}) v_i dx \to \int_\Omega \tau(z_t + z_\rho) v_i dx.$$

When $j \to \infty$. Then, problem (3.1) admits a global weak solution u.

3.4 Asymptotic behavior

In this section, we shall investigate the asymptotic behavior of our problem. Our stability result, namely the exponential decay of the energy is obtained by the following theorem.

Theorem 3.4.1 *Let $(u_0, u_1, f_0) \in (H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1)))$ be given. Assume that the assumptions (H1)-(H3) are fulfilled. Then for some positive constants K, k we obtain the following decay property*

$$E(t) \leq E(0)e^{1-k\phi(t)}.$$

Proof. Given $0 \leq S < T < \infty$ arbitrarily. We multiply the first equation of (3.5) by $E^p\phi' u$, $p \in R$ where ϕ is a function will be chosen later satisfying all the hypotheses of Lemma 2.1 and we integrate over $(S,T) \times \Omega$ we obtain

$$\begin{aligned}
0 &= \int_S^T E^p\phi' \int_\Omega uu''(x,t)dxdt - k_0 \int_S^T E^p\phi' \int_\Omega u\Delta u(x,t)dxdt \\
&+ \alpha \int_S^T E^p\phi' \int_\Omega \int_0^t g(t-s)\triangle u(x,s)udsdxdt \\
&+ \int_S^T E^p\phi'\mu_1(t) \int_\Omega uu'(x,t) + E^p\phi'\mu_2(t) \int_\Omega uu'(x,t-\tau)dxdt \\
&= \left[E^p\phi' \int_\Omega uu'dx\right]_S^T - \int_S^T (E^p\phi') \int_\Omega uu'dxdt - \int_S^T E^p\phi' \int_\Omega u'^2 dxdt \qquad (3.41) \\
&+ k_0 \int_S^T E^p\phi' \int_\Omega |\nabla u|^2 dxdt + \alpha \int_S^T E^p\phi'(g \circ \nabla u(x,t))dt \\
&- \frac{\alpha}{2} \int_S^T E^p\phi' \|\nabla u\|^2 \int_0^t g(s)dsdt - \frac{\alpha}{2} \int_S^T E^p\phi' \int_0^t g(s)\|\nabla u\|^2 dsdt \\
&+ \int_S^T E^p\phi'\mu_1(t) \int_\Omega uu'(x,t)dxdt + \int_S^T E^p\phi'\mu_2(t) \int_\Omega uu'(x,t-\tau)dx
\end{aligned}$$

Multiplying the second equation of (3.5) by $E^p\phi'\xi(t)e^{-2\tau\rho}z$ and integrating over $(S,T)\times\Omega\times(0,1)$ we find

$$\begin{aligned}
0 &= \int_S^T\int_0^1 \tau E^p\phi'\xi(t)e^{-2\tau\rho}\int_\Omega zz'dxd\rho dt + \int_S^T E^p\phi'\xi(t)e^{-2\tau\rho}\int_\Omega\int_0^1 zz_\rho d\rho dxdt \\
&= \frac{\tau}{2}\Big[\int_0^1 E^p\phi'\xi(t)e^{-2\tau\rho}\int_\Omega z^2dxd\rho\Big]_S^T - \frac{\tau}{2}\int_S^T\int_0^1\int_\Omega (E^p\phi'\xi(t)e^{-2\tau\rho})'z^2dxd\rho dt \\
&+ \int_S^T E^p\phi'\int_\Omega\int_0^1 \xi(t)\Big(\frac{1}{2}\frac{d}{d\rho}(e^{-2\tau\rho}z^2) + \tau e^{-2\tau\rho}z^2\Big)d\rho dxdt \\
&= \frac{\tau}{2}\Big[\int_0^1 E^p\phi'\xi(t)e^{-2\tau\rho}\int_\Omega z^2dxd\rho\Big]_S^T - \frac{\tau}{2}\int_S^T\int_0^1\int_\Omega (E^p\phi'\xi(t)e^{-2\tau\rho})'z^2dxd\rho dt \\
&+ \tau\int_S^T E^p\phi'\xi(t)\int_\Omega\int_0^1 e^{-2\tau\rho}z^2dxd\rho \\
&+ \frac{1}{2}\int_S^T E^p\phi'\xi(t)\int_\Omega (e^{-2\tau}z^2(x,1,t) - z^2(x,0,t))dxdt.
\end{aligned} \tag{3.42}$$

Summing (3.41) and (3.42) and taking $A = \min(1, \tau e^{-2\tau})$ we get

$$\begin{aligned}
A\int_S^T E^{p+1}\phi'dt &\le -\Big[E^p\phi'\int_\Omega uu'\Big]_S^T + \int_S^T (E^p\phi')'\int_\Omega uu'dxdt \\
&+ \frac{\alpha}{2}\int_S^T E^p\phi'\int_0^t g(s)\|\nabla u\|^2dsdt - \int_S^T E^p\phi'\mu_1(t)\int_\Omega uu'(x,t)dxdt \\
&- \int_S^T E^p\phi'\mu_2(t)\int_\Omega uz(x,1,t)dx - \frac{\tau}{2}\Big[\int_0^1 E^p\phi'\xi(t)e^{-2\tau\rho}\int_\Omega z^2dxd\rho\Big]_S^T \\
&+ \frac{\tau}{2}\int_S^T\int_0^1\int_\Omega e^{-2\tau\rho}(E^p\phi'\xi(t))'z^2dxd\rho dt\frac{3}{2}\int_S^T E^p\phi'\int_\Omega u'^2dxdt \\
&- \frac{1}{2}\int_S^T E^p\phi'\xi(t)\int_\Omega (e^{-2\tau}z^2(x,1,t) - z^2(x,0,t))dxdt
\end{aligned} \tag{3.43}$$

Now assume that ϕ is a strictly increasing concave function. So ϕ' is a bounded function on $\mathbb{R}_+$. Denote λ the maximum of ϕ'. By the Cauchy Schwarz's, Young's and Poincaré's inequalities and the fact that ϕ' is bounded and since E is an increasing function, we have

$$\Big|E^p\phi'\int_\Omega uu'(x,t)dx\Big| \le \lambda c_1E^{p+1}(t), \tag{3.44}$$

where $c_1 = \max\left(1, \frac{C_s^2}{l}\right)$. From (3.44) we deduce the following estimates

$$\Big|\int_S^T (E^p\phi')'\int_\Omega uu'dxdt\Big| = \Big|\int_S^T pE'E^{p-1}\phi'\int_\Omega uu'(x,t)dxdt + \int_S^T E^p\phi''\int_\Omega uu'(x,t)dxdt\Big|$$
$$\leq \lambda c_1 p\int_S^T E^p(-E')dt + c_1E^{p+1}(S)\int_S^T \phi''(t)dt \leq \lambda c_2 E(S)^{p+1}, \tag{3.45}$$

where $c_2 = c_1\max(p,1)$ and

$$\Big|\frac{\alpha}{2}\int_S^T E^p\phi'\int_0^t g(s)\|\nabla u\|^2dsdt\Big| \leq \lambda c_3 E(S)^{p+1}, \tag{3.46}$$

where $c_3 = \frac{(k_0-l)}{l}$. By the hypothesis (H2), Young's and Poincaré's inequalities and (3.44), we have

$$\Big|\int_S^T E^p\phi'\mu_1(t)\int_\Omega uu'(x,t)dxdt\Big| \leq \lambda M\beta E^{p+1} + \lambda\int_S^T E^p(-E')dt$$
$$\leq \lambda c_4 E^{p+1}(S), \tag{3.47}$$

where $c_4 = Mc_1$ and

$$\Big|\int_S^T E^p\phi'\mu_2(t)\int_\Omega uz(x,1,t)dxdt| \leq \lambda c_5 E^{p+1}(S), \tag{3.48}$$

where $c_5 =\ \max\left(\beta M\frac{c_s^2}{l}, 1\right)$ and

$$\frac{\tau}{2}\int_S^T\int_0^1 E^p\phi'\xi(t)e^{-2\tau\rho}\int_\Omega z^2dxd\rho dt \leq \tau\lambda E(^{p+1}S). \tag{3.49}$$

Therefore

$$\frac{\tau}{2}\int_S^T\int_0^1\int_\Omega e^{-2\tau\rho}(E^p\phi'\xi(t))'z^2dxd\rho dt = \frac{\tau}{2}\int_S^T\int_0^1 pE'E^{p-1}\phi'\xi(t)e^{-2\tau\rho}\int_\Omega z^2dxd\rho dt$$
$$+\frac{\tau}{2}\int_S^T\int_0^1\int_\Omega e^{-2\tau\rho}E^p\phi''\xi(t)z^2dxd\rho dt + \frac{\tau}{2}\int_S^T\int_0^1\int_\Omega e^{-2\tau\rho}E^p\phi'\xi'(t)z^2dxd\rho dt \tag{3.50}$$
$$\leq \lambda\tau p\int_S^T E^p(-E')dt + \tau E^{p+1}(s)\int_S^T\phi''(t)dt + \tau\lambda E^{p+1}(s) \leq \lambda c_6 E(S)^{p+1},$$

where $c_6 = \tau \max(1, p)$ and

$$\begin{aligned}\frac{1}{2}\int_S^T E^p\xi(t)\int_\Omega e^{-2\tau}z^2(x,1,t)dxdt &\leq \lambda\int_S^T E^p\xi(t)\int_\Omega z^2(x,1,t)dxdt \\ &\leq \lambda\int_S^T E^p(-E')dt \leq \lambda E(S)^{p+1},\end{aligned} \tag{3.51}$$

and we have

$$\begin{aligned}&\frac{1}{2}\int_S^T E^p\phi'\xi(t)\int_\Omega z^2(x,0,t)dxdt = \frac{1}{2}\int_S^T E^p\phi'\xi(t)\int_\Omega u'^2(x,t)dxdt \\ &\leq \tau\lambda(2-\beta)E(S)^{p+1}\end{aligned} \tag{3.52}$$

$$\frac{3}{2}\int_S^T E^p\phi'\xi(t)\int_\Omega u'^2dxdt \leq 3\lambda E^{p+1}(S). \tag{3.53}$$

From (3.43) and the estimates (3.45), (3.46), (3.48),(3.42), (3.49), (3.52) we obtain

$$\int_S^T E(t)^{p+1}\phi' dt \leq CE(S)^{p+1}, \tag{3.54}$$

where $C = \lambda\max(c_i, 3, \tau(2-\beta)),\ i = 1, ..., 6$. Applying the lemma 2.3.3 we get the decay property. This ends the proof of Theorem 3.4.1.

Chapter 4

Energy decay of solution for viscoelastic wave equations with a dynamic boundary and delay term

4.1 Introduction

In this chapter, we investigate the following wave equation with dynamic boundary conditions and delay term

$$\begin{cases} u_{tt} - \Delta u - \displaystyle\int_0^t g(t-s)\Delta u(s)ds - \delta\Delta u_t = |u|^{p-1}u, & in \quad \Omega \times (0,+\infty), \\ u = 0, & on \quad \Gamma_0 \times (0,+\infty), \\ u_{tt} = -a\Big[\frac{\partial u}{\partial \nu}(x,t) + \delta\frac{\partial u_t}{\partial \nu}(x,t) + \mu_1(t)u_t(x,t) + \mu_2(t)u_t(x,t-\tau)\Big], & \\ on \quad \Gamma_1 \times (0,+\infty), & \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & on \quad \Gamma_1 \times (0,+\infty). \end{cases} \tag{4.1}$$

where $u = u(x,t)$, $t \geq 0$, $x \in \Omega$ and Δ denotes the Laplacian operator with respect to the x variable. Ω is a regular and bounded domain of $\mathbb{R}^N$, $(N \geq 1)$, $\partial\Omega = \Gamma_1 \cup \Gamma_0$, $\Gamma_1 \cap \Gamma_0 = \emptyset$ and $\frac{\partial}{\partial \nu}$ denotes the unit outer normal derivative, μ_1 and μ_2 are functions depend on t. Moreover, $\tau > 0$ represents the delay and u_0, u_1, f_0 are given functions belonging to suitable spaces that will be specified later.

This type of problems arises (for example) in modeling of longitudinal vibrations in a homogeneous bar on which there are viscous effects. The term Δu_t, indicates that the stress is proportional not only to the strain, but also to the strain rate (See [14]).
This type of problem without delay (i.e $\mu_i = 0$), has been considered by many authors during the past decades and many results have been obtained (see [13], [20], [9], [27]) and the references therein.
Such types of boundary conditions are usually called dynamic boundary conditions. They are not only important from the theoretical point of view but also arise in several physical applications. For instance in one space dimension, problem (4.1) can modelize the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end. The dynamic boundary conditions represent the Newton's law for the attached mass (see [13, 1, 20] for more details), which arise when we consider the transverse motion of a flexible membrane whose boundary may be affected by the vibrations only in a region. Also some of them as in problem (4.1) appear when we assume that is an exterior domain of $\mathbb{R}^3$ in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [4] for more details). This type of dynamic boundary conditions is known as acoustic boundary conditions. Among the early results dealing with the dynamic boundary conditions are those of Grobbelaar-Van Dalsen [8,9] in which the author has made contributions to this field. And In [8] the author has introduced a model which describes the damped longitudinal vibrations of a homogeneous flexible horizontal rod of length L when the end $x = 0$ is rigidly fixed while the other end $x = L$ is free to move with an attached load. This yields to a system of two second order equations of the form

$$\begin{cases} u_{tt} - u_{xx} - u_{txx} = 0, \quad x \in (0, L), \quad t > 0, \\ u = 0, \qquad\qquad\qquad\qquad\qquad t > 0, \\ u_{tt}(L, t) = -[u_x + u_{tx}](L, t), \ t > 0, \\ u_t(x, 0) = u_1(x), u_t(L, 0) = \mu, \ x \in (0, L), \\ u(L, 0) = \eta, \quad u_t(L, 0) = \mu. \end{cases} \tag{4.2}$$

By rewriting problem (4.2) within the framework of the abstract theories of the so-called B-evolution theory, the existence of a unique solution in the strong sense has been shown and an exponential decay result was also proved in [9] for a problem related to (4.2), which describes the weakly damped vibrations of an extensible beam (See [9] for more details). Subsequently, Zang and Hu [81], considered the problem

$$
\begin{aligned}
u_{tt} - p(u_x)_{xt} - q(u_x)_x = 0, \quad x \in (0,1), \quad t > 0 \\
p(u_x)_t + q(u_x)(1,t) + ku_{tt}(1,t) = 0, \quad u(0,t) = 0, \quad t \geq 0.
\end{aligned}
$$

By using the Nakao's inequality and under appropriate conditions on p and q, they established both exponential and polynomial decay rates for the energy depending on the form of the terms p and q. It is clear that in the absence of the delay term and for $\mu_1 = 0$, problem (4.2) is the one dimensional model of (4.1). Similarly, and always in the absence of the delay term, Pellicer and Sola-Morales [67] considered the one dimensional problem of (4.1) as an alternative model for the classical spring-mass damper system and by using the dominant eigenvalues method, they proved that their system has the classical second order differential equation

$$
m_1 u''(t) + d_1 u'(t) + k_1 u(t),
$$

as a limit, where the parameters m_1, d_1 and k_1 are determined from the values of the spring-mass damper system. Thus, the asymptotic stability of the model has been determined as a consequence of this limit. But they did not obtain any rate of convergence.(See also [66, 67]) for related results.

It is widely known that delay effects, which arise in many practical problems are source of some instabilities. In this way Datko and al [66] showed that a small delay in a boundary control turns to be a well-behaved hyperbolic system into a wide one which in turn, becomes a source of instability. Nicaise and al [60] studied the following system of a wave equation with a linear boundary term :

$$
\begin{cases}
u_{tt} - \triangle u(x,t) = 0, \quad in \quad \Omega \times (0,+\infty), \\
u(x,t) = 0, \qquad\qquad\qquad\qquad on \quad \Gamma_0 \times (0,+\infty), \\
\frac{\partial u}{\partial v} = \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\sigma), \qquad on \quad \Gamma_1 \times (0,+\infty), \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \\
u_t(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Omega, \quad t \in (0,\tau),
\end{cases} \tag{4.3}
$$

and proved that the energy is exponentially stable, where v is the unit outward normal to $\partial\Omega$, under the condition

$$\mu_2 < \mu_1. \tag{4.4}$$

On the contrary, if (4.4) doesn't hold, there is a sequence of delays for which the corresponding solution of (4.3) will be instable.

The problem (4.3) with time varying delay term has been studied by Nicaise and al. We refer the readers to ([60], [62]).

Recently, inspired by the works of Al and Nicaise [60], Sthéphane Gherbi and Said El Houari [29] considered the following problem in bounded domain

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t = 0, \quad in \quad \Omega \times (0, +\infty), \\ u = 0, \qquad\qquad\qquad\qquad\qquad\qquad on \quad \Gamma_0 \times (0, +\infty), \\ u_{tt} = -a\Big[\frac{\partial u}{\partial v}(x,t) + \alpha \frac{\partial u_t}{\partial v}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau)\Big], \\ \quad on \quad \Gamma_1 \times (0, +\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x, t-\tau(t)) = f_0(x, t-\tau), \qquad on \quad \Gamma_1 \times (0, +\infty), \end{cases} \tag{4.5}$$

and obtained several results concerning global existence and exponential decay rates for various sign of μ_1, μ_2.

Motivated by the previous works, in the present chapter we investigate problem (4.1) in which we generalize the results obtained in [5] by supposing new conditions with which the global existence and stability results are assured. The stable set is used to prove the existence result and Nakao's technique to establish energy decay rates.

The content of this chapter is organized as follows: In Section 2, we provide assumptions that will be used later. In Section 3, we state and prove the global existence result. In Section 4, the stability result given in Theorem 4.3.2 will be proved.

4.2 Preliminary Results

In this section, we present some material in the proof of our main result. we denote

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega)/u_{\Gamma_0} = 0\}$$

, we set γ_1 the trace operator from $H^1_{\Gamma_0}(\Omega)$ on $L^2(\Gamma_1)$ we denote by B the norm of γ_1 and we have

$$\forall u \in H^1_{\Gamma_0}, \|u\|_{2,\Gamma_1} \leq B\|\nabla u\|^2$$

We assume

(A_1) $\mu_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing function of class $C^1(\mathbb{R}_+)$ satisfying

$$\left|\frac{\mu_1^{'}(t)}{\mu_1(t)}\right| \leq M, \tag{4.6}$$

(A_2) $\mu_2 : \mathbb{R}_+ \to \mathbb{R}$ is a function of class $C^1(\mathbb{R}_+)$, which is not necessarily positive or monotone, such that

$$|\mu_2(t)| \leq \beta\mu_1(t), \tag{4.7}$$

$$|\mu_2^{'}(t)| \leq \bar{M}\mu_1(t), \tag{4.8}$$

for some $0 < \beta < 1$ and $\bar{M} > 0$. For the relaxation function g we assume
(A_3) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^1 function satisfying

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds = l < 1,$$

(A_4) There exists a nonincreasing differentiable function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$g'(t) \leq -\eta(t)g(t),$$

(A_5) We suppose therefore

$$2 \leq p \leq \frac{2n-2}{n-2} \quad if \quad n \geq 3; \quad p > 2, \quad if \quad n = 1, 2. \tag{4.9}$$

Now we choose $\tilde{\zeta}$ such that

$$\tau\beta < \tilde{\zeta} < \tau(2-\beta). \tag{4.10}$$

Lemma 4.2.1 *(Sobolev-Poincaré's inequality). Let $2 \leq m \leq \frac{2n}{n-2}$. The inequality*

$$\|u\,\|_m \leq c_s\|\nabla u\|_2 \quad for \quad u \in H^1_0(\Omega),$$

holds with some positive constant c_s.

Lemma 4.2.2 [4] *For any $g \in C^1$ and $\phi \in H^1(0,T)$, we have*

$$\int_0^t \int_\Omega g(t-s)\varphi\varphi_t dxds = -\frac{1}{2}\frac{d}{dt}\left((go\varphi)(t) + \int_0^t g(s)ds\|\varphi\|_2^2\right) - g(t)\|\varphi\|_2^2 + (g'o\varphi)(t),$$

where

$$(go\varphi)(t) = \int_0^t g(t-s)\int_\Omega |\varphi(s)-\varphi(t)|^2 dxds.$$

Lemma 4.2.3 [4] *For $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^2 dx \le (1-l)c_s^2(go\nabla u)(t), \tag{4.11}$$

where c_s^2 is the Poincaré's constant and l is given in (A_3) and

$$(go\nabla u)(t) = \int_0^t g(t-s)\int_\Omega |\nabla u(s) - \nabla u(t)|^2 dxds.$$

Lemma 4.2.4 [58] *Let ϕ be a nonincreasing and nonnegative function on $[0,T]$, $T > 1$, such that*

$$\phi(t)^{1+r} \le \omega_0(\phi(t) - \phi(t+1)), \quad on \quad [0,T],$$

where $\omega_0 > 1$ and $r \ge 0$. Then we have, for all $t \in [0,T]$
(i) if $r = 0$, then

$$\phi(t) \le \phi(0)e^{-\omega_1[t-1]^+},$$

(ii) if $r > 0$, then

$$\phi(t) \le \left(\phi(0)^{-r} + \omega_0^{-1}r[t-1]^+\right)^{\frac{-1}{r}},$$

where $\omega_1 = \ln\left(\frac{\omega_0}{\omega_0 - 1}\right)$ and $[t-1]^+ = max(t-1,0)$.

4.3 Global existence and energy decay

We introduce the new variable z as in [60],

$$z(x,k,t)=u_t(x,t-\tau k),\quad x\in\Gamma_1,\quad k\in(0,1),$$

which implies that

$$\tau z_t(x,k,t)+z_k(x,k,t)=0,\quad in\quad \Gamma_1\times(0,1)\times(0,\infty).$$

Therefore, problem (4.1) can be transformed as follows

$$\begin{cases} u_{tt}-\Delta u-\delta\Delta u_t+\displaystyle\int_0^t g(t-s)\Delta u(s)ds=|u|^{p-1}u, \quad in \quad \Omega\times(0,\infty), \\ \\ u_{tt}=-a\Big[\frac{\partial u}{\partial v}(x,t)+\alpha\frac{\partial u_t}{\partial v}(x,t)+\mu_1(t)u_t(x,t)+\mu_2(t)z(x,1,t)\Big], \\ on \quad \Gamma_1\times(0,+\infty), \\ \\ \tau z_t(x,k,t)+z_k(x,k,t)=0, \qquad in \quad \Gamma_1\times(0,1)\times(0,\infty), \\ z(x,k,0)=f_0(x,-\tau k), \qquad x\in\Gamma_1, \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), \qquad x\in\Omega, \\ u(x,t)=0, \quad x\in\Gamma_0, \qquad t\geq 0. \end{cases} \tag{4.12}$$

Remark 4.3.1 *For seeking of simplicity, we take $a=1$ in (4.12).*

Now inspired by [73, 81], we define the energy functional related with problem (12) by

$$\begin{aligned} E(t)=&\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|u_t\|_{2.\Gamma_1}^2+\frac{1}{2}(go\nabla u)(t)+\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\ &-\frac{1}{p+1}\|u\|_{p+1}^{p+1}+\frac{\zeta(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(x,k,s)dkd\gamma, \end{aligned} \tag{4.13}$$

where

$$\zeta(t)=\tilde{\zeta}\mu_1(t).$$

Lemma 4.3.1 *Let (u,z) be the solution of (4.12) then, the energy equation satisfies*

$$\begin{aligned} E'(t) \leq \frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 - \mu_1(t)\left(1 - \frac{\tilde{\zeta}}{2\tau} - \frac{\beta}{2}\right)\|u_t(t)\|_{2.\Gamma_1}^2 \\ - \mu_1(t)\left(\frac{\tilde{\zeta}}{2\tau} - \frac{\beta}{2}\right)\|z(x,1,t)\|_{2.\Gamma_1}^2 - \delta\|\nabla u_t(t)\|_2^2 \leq 0 \end{aligned} \tag{4.14}$$

Proof 4.3.1 *By multiplying the first and second equation in $(4,12)$ by $u_t(t)$, integrating the first equation over Ω and the second equation over Γ_1, using Green's formula, we get*

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_{2.\Gamma_1}^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}\right] \\ &- \delta\|\nabla u_t\|_2^2 + \mu_1(t)\|u_t(t)\|_{2.\Gamma_1}^2 - \int_\Omega\int_0^t g(t-s)\nabla u(s)\nabla u_t(t)dsdx \\ &+ \int_{\Gamma_1}\mu_2(t)z(\gamma,1,t)u_t(t)d\gamma = 0. \end{aligned} \tag{4.15}$$

As in [5] we multiply the third equation in (4.12) by $\zeta(t)z$ and integrate over $\Gamma_1 \times (0,1)$ to obtain

$$\zeta(t)\tau\int_{\Gamma_1}\int_0^1 z_t z(\gamma,k,t)dkd\gamma + \zeta(t)\int_{\Gamma_1}\int_0^1 z_k z(x,k,t)dkd\gamma = 0, \tag{4.16}$$

this yields

$$\frac{\zeta(t)\tau}{2}\frac{d}{dt}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma + \frac{\zeta(t)}{2}\int_{\Gamma_1}\int_0^1 \frac{\partial}{\partial k}z^2(\gamma,k,t)dkd\gamma = 0, \tag{4.17}$$

then

$$\begin{aligned} &\frac{\tau}{2}\left[\frac{d}{dt}\left(\zeta(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma\right) - \zeta'(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma\right] \\ &+ \frac{\zeta(t)}{2}\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma - \frac{\zeta(t)}{2}\int_{\Gamma_1} z^2(x,0,t)d\gamma = 0, \end{aligned} \tag{4.18}$$

consequently

$$\begin{aligned} &\frac{\tau}{2}\frac{d}{dt}\left(\zeta(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma\right) \\ &= \frac{\tau}{2}\zeta'(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma - \frac{\zeta(t)}{2}\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma \\ &+ \frac{\zeta(t)}{2}\int_{\Gamma_1} z^2(x,0,t)d\gamma, \end{aligned} \tag{4.19}$$

finally from (4.15) and (4.18), we get

$$
\begin{aligned}
&E(t) + \left(\mu_1(t) - \frac{\zeta(t)}{2\tau}\right) \|u_t(t)\|_{2.\Gamma_1}^2 + \mu_2(t) \int_{\Gamma_1} z^2(\gamma, 1, t) u_t(\gamma, t) d\gamma \\
&- \frac{\zeta'(t)}{2} \int_{\Gamma_1} \int_0^1 z^2(\gamma, k, t) dk d\gamma + \frac{\zeta(t)}{2\tau} \int_{\Gamma_1} z^2(\gamma, 1, t) d\gamma \\
&- \frac{1}{2} \int_0^t (g' o \nabla u)(s) ds + \frac{1}{2} \int_0^t g(s) \|\nabla u(s)\|_2^2 ds = 0.
\end{aligned}
\tag{4.20}
$$

Due to Young's inequality, we have

$$
\int_{\Gamma_1} z(\gamma, 1, t) u_t(\gamma, t) d\gamma \leq \frac{1}{2} \|u_t(t)\|_{2.\Gamma_1}^2 + \frac{1}{2} \int_{\Gamma_1} z^2(\gamma, 1, t) d\gamma. \tag{4.21}
$$

Noting that $\zeta'(t) \leq 0$. Inserting (4.21) into (4.20) and deriving it, we get the desired result.

Now we are in position to state the local existence result to problem (12), which can be established by combining arguments of [(21), (66)].

Theorem 4.3.1 *Let $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$ and $f_0 \in L^2(\Gamma_1 \times (0,1))$ be given. Suppose that $(A_1) - (A_5)$ hold. Then the problem (4.12) admits a unique weak solution (u, z) satisfying*

$$
u \in L^\infty((0,T); H^1_{\Gamma_0}(\Omega)), \quad u_t \in L^\infty((0,T); H^1_{\Gamma_0}(\Omega)) \cap L^\infty((0,T); L^2(\Gamma_1)),
$$

$$
u_{tt} \in L^\infty((0,T); L^2(\Omega)) \cap L^\infty((0,T); L^2(\Gamma_1)).
$$

Now we will prove that the solution abstained above is global and bounded in time. For this purpose let us define

$$
\begin{aligned}
I(t) = &\left(1 - \int_0^t g(s) ds\right) \|\nabla u\|_2^2 + (g o \nabla u)(t) - \|u\|_{p+1}^{p+1} \\
&+ \zeta(t) \int_{\Gamma_1} \int_0^1 z^2(\gamma, k, s) dk d\gamma,
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
J(t) = &\frac{1}{2} (g o \nabla u)(t) + \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla u\|_2^2 - \frac{1}{p+1} \|u\|_{p+1}^{p+1} \\
&+ \frac{\zeta(t)}{2} \int_{\Gamma_1} \int_0^1 z^2(\gamma, k, s) dk d\gamma,
\end{aligned}
\tag{4.23}
$$

where

$$
E(t) = J(t) + \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} \|u_t\|_{2.\Gamma_1}^2. \tag{4.24}
$$

Lemma 4.3.2 *Suppose that $(A_3)-(A_4)$ hold. Let (u,z) be the solution of the problem (4.12). Assume further that $I(0)>0$ and*

$$\alpha = \frac{c_s^{p+1}}{l}\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}} < 1. \tag{4.25}$$

Then $I(t)>0$ for all $t \geq 0$.

Proof 4.3.2 *Since $I(0)>0$, then there exists (by continuity of $u(t)$) $T^* < T$ such that*

$$I(t) \geq 0, \tag{4.26}$$

for all $t \in [0,T^*]$. From (4.22) and (4.23) we have

$$\begin{aligned}
J(t) &\geq \frac{p-1}{2(p+1)}\left[\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + go\nabla u(t)\right] \\
&+ \frac{(p-1)}{2(p+1)}\left[\zeta(t)\int_0^1\int_{\Gamma_1} z^2(x,k,t)dkd\gamma\right] + \frac{1}{p+1}I(t) \\
&\geq \frac{p-1}{2(p+1)}\left[\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + (go\nabla u)(t)\right] \\
&+ \frac{(p-1)}{2(p+1)}\left[\zeta(t)\int_0^1\int_{\Gamma_1} z^2(\gamma,k,t)dkd\gamma\right] \geq \frac{p-1}{2(p+1)}l\|\nabla u\|_2^2.
\end{aligned} \tag{4.27}$$

Thus by (4.24) and (4.25) we deduce $\forall t \in [0,T^*]$

$$l\|\nabla u\|_2^2 \leq \left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 \leq \frac{2(p+1)}{(p-1)}E(t) \leq \frac{2(p+1)}{(p-1)}E(0). \tag{4.28}$$

Exploiting Lemma 4.2.1 and formula (4.28), we obtain

$$\begin{aligned}
\|u\|_{p+1}^{p+1} &\leq c_s^{p+1}\|\nabla u\|_2^{p+1} \leq \frac{c_s^{p+1}}{l}\|\nabla u\|_2^{p-1}\|\nabla u\|_2^2 \leq \frac{c_s^{p+1}}{l}\left(\frac{2(p+1)}{(p-1)l}E(0)\right)^{\frac{p-1}{2}} l\|\nabla u\|_2^2 \\
&= \alpha l\|\nabla u\|_2^2 < \left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2, \ \forall t \in [0,T^*].
\end{aligned} \tag{4.29}$$

Hence $\forall t \in [0,T^*]$, we have

$$I(t) = \left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + (go\nabla u)(t) - \|u\|_{p+1}^{p+1} + \zeta(t)\int_0^1\int_{\Gamma_1} z^2(\gamma,k,t)dkd\gamma > 0.$$

Repeating this procedure and using the fact that

$$\lim_{t\to T^*} \frac{c_s^{p+1}}{l}\left(\frac{2(p+1)}{2l(p-1)}E(u(t))\right)^{\frac{p-1}{2}} \leq \alpha < 1,$$

we can take $T^* = T$. This completes the proof.

Theorem 4.3.2 *Let $(A_3)-(A_5)$ hold. Let $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$, $f_0 \in L^2(\Gamma_1 \times (0,1))$ be given. Then the solution of the problem* (4.12) *is global and bounded in time. Furthermore, there exists $\theta > 0$, such that*

$$\theta > \frac{3-3l}{l}, \tag{4.30}$$

and we have the following decay estimate:

$$E(t) \leq E(0)e^{-\sigma t}, \quad \forall t \geq 0, \quad \sigma = \ln\left(\frac{c_{12}}{c_{12}-1}\right),$$

where c_{12} is a positive constant.

Proof First, we prove $T = \infty$. It is sufficient to show that $l\|\nabla u\|_2^2$ is bounded independently of t. From (4.24) we have

$$E(0) \geq E(t) = \frac{1}{2}\|u_t\|_2^2 + \|u_t\|_{2.\Gamma_1}^2 + J(t) \geq \frac{1}{2}\|u_t\|_2^2 + \left(\frac{p-1}{2(p+1)}\right) l\|\nabla u\|_2^2 \geq l\|\nabla u\|_2^2.$$

Therefore $l\|\nabla u\|_2^2 \leq \rho E(0)$, where ρ is a positive constant which depends only on p, thus we obtain the global existence result. From now and on, we focus our attention to the decay rate of the solutions to problem (4.12). In order to do so, we will derive the decay rate of the energy function for problem (4.12) by Nakao's method, as in [58]. For this aim, we have to show that the energy function defined by (4.24) satisfies the hypotheses of lemma 4.2.4. By integrating (4.14) over $[t, t+1]$, we have

$$E(t) - E(t+1) = D(t)^2, \tag{4.31}$$

where

$$\begin{aligned} D(t)^2 &= c_1 \int_t^{t+1} \mu_1(s)\|u_t\|_{2.\Gamma_1}^2 ds + c_2 \int_t^{t+1} \mu_2(s) \int_{\Gamma_1} z^2(\gamma, 1, s) d\gamma ds \\ &- \int_t^{t+1} \frac{1}{2}(g' o \nabla u)(s) ds + \int_t^{t+1} \frac{1}{2} g(s)\|\nabla u(s)\|_2^2 ds + c_3 \int_t^{t+1} \|\nabla u_t\|_2^2 ds. \end{aligned} \tag{4.32}$$

We observe that

$$\begin{aligned}&\int_t^{t+1}\int_{\Gamma_1}\mu_1(s)|u_t|^2 d\gamma ds+\int_t^{t+1}\mu_2(s)\int_{\Gamma_1}|z(\gamma,1,s)|^2 d\gamma ds+\int_t^{t+1}\|\nabla u_t\|_2^2 ds\\&\leq CD(t)^2,\end{aligned} \tag{4.33}$$

where $C=\max\{c_1,c_2,c_3\}$. Applying the mean value, there exist $t_1\in[t,t+\frac{1}{4}]$ and $t_2\in[t+\frac{3}{4},t+1]$ such that for $i=1,2$, we get

$$\mu_1(t_i)\|u_t(t_i)\|_{2.\Gamma_1}^2+\mu_2(t_i)\|z(\gamma,1,t_i)\|_{2.\Gamma_1}^2+\|\nabla u_t(t_i)\|_2^2\leq CD(t)^2. \tag{4.34}$$

Multiplying the first equation in (4.12) by u and integrating over $\Omega\times[t_1,t_2]$, multiplying the second equation in (4.12) by u and integrating over $\Gamma_1\times[t_1,t_2]$, adding and subtracting the following term $\int_0^1\int_{\Gamma_1}\zeta(t)z^2(\gamma,k,t)dkd\gamma$, we obtain

$$\begin{aligned}&\int_{t_1}^{t_2}I(t)dt\leq\sum_{i=1}^{2}\|u_t(t_i)\|_2\|u(t_i)\|_2+\int_{t_1}^{t_2}\|u_t\|_2^2dt+\int_{t_1}^{t_2}\int_\Omega\nabla u_t\nabla udtdx\\&+\int_{t_1}^{t_2}(go\nabla u)(t)dt+\int_{t_1}^{t_2}\int_\Omega\int_0^t g(t-s)\nabla u(t)(\nabla u(s)-\nabla u(t))dsdxdt\\&+\int_{t_1}^{t_2}\zeta(t)\int_0^1\int_{\Gamma_1}z^2(x,k,t)dkd\gamma dt-\int_{\Gamma_1}\int_{t_1}^{t_2}\mu_2(t)z(x,1,t)udtd\gamma\\&-\int_{t_1}^{t_2}\mu_1(t)\int_{\Gamma_1}u_tud\gamma dt.\end{aligned} \tag{4.35}$$

Since

$$\begin{aligned}&\int_\Omega\int_0^t g(t-s)\nabla u(t)(\nabla u(s)-\nabla u(t))dsdx\\&=\frac{1}{2}\left[\int_0^t g(t-s)(\|\nabla u(s)\|_2^2+\|\nabla u(t)\|_2^2)ds-\int_0^t g(t-s)(\|\nabla u(s)\|_2^2+\|\nabla u(t)\|_2^2)ds\right]\\&-\int_\Omega\int_0^t g(s)|\nabla u(t)|^2dsdx,=-\frac{1}{2}\int_\Omega\int_0^t g(s)|\nabla u(s)|^2dsdx\\&+\frac{1}{2}\int_0^t g(t-s)ds\|\nabla u(s)\|_2^2ds-\frac{1}{2}(go\nabla u)(t).\end{aligned} \tag{4.36}$$

Then (4.35) takes the form

$$\begin{aligned}
&\int_{t_1}^{t_2} I(t)dt \le \sum_{i=1}^{2} \|u_t(t_i)\|_2\|u(t_i)\|_2 + \int_{t_1}^{t_2} \|u_t\|_2^2 dt + \frac{1}{2}\int_{t_1}^{t_2} (go\nabla u)(t)dt \\
&+ \int_{t_1}^{t_2}\int_\Omega \nabla u_t \nabla u dt dx - \frac{1}{2}\int_\Omega \int_0^t g(s)|\nabla u(s)|^2 ds dx + \frac{1}{2}\int_0^t g(t-s)ds\|\nabla u(s)\|_2^2 ds \\
&+ \int_{t_1}^{t_2}\int_0^1 \zeta(t)\int_{\Gamma_1} z^2(x,k,t)dk d\gamma dt - \int_{\Gamma_1}\int_{t_1}^{t_2} \mu_2(t)z(x,1,t)u dt d\gamma \\
&- \int_{t_1}^{t_2} \mu_1(t)\int_{\Gamma_1} u_t u d\gamma dt.
\end{aligned} \tag{4.37}$$

Now we will estimate the right hand side of (4.37). First by (4.36) and lemma 4.2.1, we have

$$\begin{aligned}
\|u_t(t_i)\|_2\|u(t_i)\|_2 &\le c_s C^{\frac{1}{2}} D(t) \sup_{t_1\le s\le t_2} \|\nabla u(s)\|_2 \\
&\le D(t) c_s C^{\frac{1}{2}} \left(\frac{2(p+1)}{p-1}\right)^{\frac{1}{2}} \sup_{t_1\le s\le t_2} E(s)^{\frac{1}{2}} \\
&\le D(t) c_s C^{\frac{1}{2}} \left(\frac{2(p+1)}{p-1}\right)^{\frac{1}{2}} E(t)^{\frac{1}{2}}.
\end{aligned} \tag{4.38}$$

As in [74], by employing Young's inequality for convolution $\|\varphi * \phi\| \le \|\varphi\|\|\phi\|$ and noting that

$$l\|\nabla u(t)\|_2^2 \le \frac{1}{\theta} I(t). \tag{4.39}$$

Then we have

$$\begin{aligned}
&\int_0^t \int_{t_1}^{t_2} g(t-s)\|\nabla u(s)\|_2^2 ds dt \le \int_{t_1}^{t_2} g(t)dt \int_{t_1}^{t_2} \|\nabla u(t)\|_2^2 dt \\
&\le (1-l)\int_{t_1}^{t_2} \|\nabla u(t)\|_2^2 dt \le \frac{1-l}{l\theta}\int_{t_1}^{t_2} I(t)dt.
\end{aligned} \tag{4.40}$$

Exploiting (4.36) to obtain

$$\begin{aligned}
&\frac{1}{2}\int_{t_1}^{t_2} (go\nabla u)(t)dt = \frac{1}{2}\int_0^t \int_{t_1}^{t_2} g(t-s)\|\nabla u(s) - \nabla u(t)\| ds dt \\
&\le \int_0^t g(t-s)\int_{t_1}^{t_2} (\|\nabla u(t)\|^2 + \|\nabla u(s)\|^2) dt ds \\
&\le \frac{2(1-l)}{l\theta}\int_{t_1}^{t_2} I(t)dt.
\end{aligned} \tag{4.41}$$

Multiplying the second equation in (4.12) by ζz and integrating the result over $\Gamma_1 \times (0,1)$, we obtain

$$\begin{aligned}
&\frac{\tau}{2}\frac{d}{dt}\left(\zeta(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma\right) = \frac{\tau}{2}\zeta^{'}(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma \\
&-\frac{\zeta(t)}{2}\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma + \frac{\zeta(t)}{2}\int_{\Gamma_1} z^2(x,0,t)d\gamma.
\end{aligned} \tag{4.42}$$

Recalling that $\zeta^{'}(t) \leq 0$, we have

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\Gamma_1}\zeta(t)\int_0^1 z^2(x,k,t)dkd\gamma dt \leq \int_{t_1}^{t_2}\int_{t_1}^{t_2}\frac{\tilde{\zeta}\mu_1(s)}{2\tau}\|u_t(s)\|^2_{2.\Gamma_1}dsd\nu \\
&\leq c\left(\int_{t_1}^{t_2}d\nu\right)\left(\int_{t_1}^{t_2}\mu_1(s)\|u_t(s)\|^2_{2.\Gamma_1}ds\right) \leq \frac{C}{\tau}(t_2-t_1)D(t)^2.
\end{aligned} \tag{4.43}$$

Using Sobolev's inequality, also we have

$$\begin{aligned}
&\left|\int_{t_1}^{t_2}\mu_2(s)z(x,1,t)udxdt\right| \leq \int_{t_1}^{t_2}\mu_2(s)\|z(x,1,t)\|_2\|u\|_2 dt \\
&\leq c_s\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\int_{t_1}^{t_2}\mu_2(s)\|z(x,1,t)\|_2 dt \\
&\leq c_s C^{\frac{1}{2}}\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}}D(t),
\end{aligned} \tag{4.44}$$

and

$$\int_{t_1}^{t_2}\|u_t\|_2^2 dt \leq c_s^2\int_{t_1}^{t_2}\|\nabla u_t\|_2^2 dt, \leq c_s^2 C D(t)^2, \tag{4.45}$$

$$\begin{aligned}
\left|\int_{t_1}^{t_2}\int_\Omega \nabla u_t \nabla u dt dx\right| &\leq \int_{t_1}^{t_2}\|\nabla u_t\|_2\|\nabla u\|_2 dt \\
&\leq \left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\int_{t_1}^{t_2}\|\nabla u_t\|_2 \\
&\leq C\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}}D(t),
\end{aligned} \tag{4.46}$$

also we have

$$\begin{aligned}
&\left|\int_{t_1}^{t_2}\int_{\Gamma_1}\mu_1(s)u_t u d\gamma dt\right| \leq \int_{t_1}^{t_2}\mu_1(s)\|u_t\|_{2.\Gamma_1}\|\nabla u\|_{2.\Gamma_1}dt \\
&\leq c_s B\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\int_{t_1}^{t_2}\mu_1(s)\|u_t\|_{2.\Gamma_1}dt \\
&\leq c_s c(\Gamma_1)B\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}}D(t),
\end{aligned} \tag{4.47}$$

therefore, from (4.38) − (4.47) we deduce

$$\begin{aligned}
&\int_{t_1}^{t_2}I(t)dt \leq C^{\frac{1}{2}}\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}}[c_s(B+3)+1]E(t)^{\frac{1}{2}}D(t) \\
&+\left(3\frac{(1-l)}{l\theta}\right)\int_{t_1}^{t_2}I(t)dt+\left(\frac{3}{4\tau}C+c_s^2C\right)D(t)^2.
\end{aligned} \tag{4.48}$$

Then, rewriting (4.48), we get

$$c_5\int_{t_1}^{t_2}I(t)dt \leq c_4 D(t)^2 + c_3 E(t)^{\frac{1}{2}}D(t), \tag{4.49}$$

with

$$c_5=\left[1-\frac{(3-3l)}{l\theta}\right], c_3 = C^{\frac{1}{2}}\left(\frac{2(p+1)}{l(p-1)}\right)^{\frac{1}{2}}[c_s(B+3)+1], c_4 = C\frac{3}{4\tau}+c_s^2C.$$

From the condition (4.30) and observing that is equivalent to $c_5 > 0$, thus

$$\int_{t_1}^{t_2}I(t)dt \leq c_7\left[D(t)^2+E(t)^{\frac{1}{2}}D(t)\right], \tag{4.50}$$

where $c_7 = \frac{max(c_3,c_4)}{c_5}$. On the other hand, from the definition of $E(t)$ and by (4.22) and (4.24) we have

$$\begin{aligned}
\int_{t_1}^{t_2} E(t)dt &\leq \frac{p-1}{2(p+1)} \int_{t_1}^{t_2} \left[(go\nabla u)(t) + \left(1 - \int_0^t g(s)ds\right) \|\nabla u\|_2^2 \right] \\
&+ \frac{p-1}{2(p+1)} \int_{t_1}^{t_2} \int_{\Gamma_1} \int_0^1 z^2(x,k,s) dk d\gamma dt \\
&+ \frac{1}{p+1} \int_{t_1}^{t_2} I(t)dt + \int_{t_1}^{t_2} \frac{1}{2}\|u_t\|_2^2 dt + \int_{t_1}^{t_2} \frac{1}{2}\|u_t\|_{2.\Gamma_1}^2 dt. \\
&\leq c_7 \frac{p-1}{2(p+1)} \left[\frac{1}{\theta} + \frac{2-2l}{2\theta} + \frac{1}{p+1} \right] (D(t)^2 + E(t)^{\frac{1}{2}} D(t)) \\
&+ \left[c_s^2 C(1+B) + C\frac{3}{4\tau m} \right] D(t)^2, \\
&\leq \left[c_8 D(t)^2 + c_9 E(t)^{\frac{1}{2}} D(t) \right] \leq c_{10} \left[D(t)^2 + E(t)^{\frac{1}{2}} D(t) \right],
\end{aligned} \tag{4.51}$$

where $c_8 = c_7 \frac{p-1}{2(p+1)} \left[\frac{2-2l}{\theta} + \frac{1}{\theta} + \frac{1}{p+1} + c_s^2 C(1+B) + \frac{3}{4\tau m} C \right]$,
$c_9 = c_7 \frac{p-1}{2(p+1)} \left[\frac{1-l}{2\theta} + \frac{1}{\theta} + \frac{1}{p+1} \right]$, $c_{10} = max(c_8, c_9)$.
Moreover, integrating (4.14) over (t, t_2) and using (4.51) and the fact that $E(t_2) \leq 2 \displaystyle\int_{t_1}^{t_2} E(t)dt$, due to $t_2 - t_1 \geq \frac{1}{2}$, we obtain

$$\begin{aligned}
E(t) &= E(t_2) + \int_t^{t_2} \frac{1}{2}(g' o\nabla u)(t)dt + \int_t^{t_2} \frac{1}{2} g(t)\|\nabla u(t)\|_2^2 dt \\
&+ \int_t^{t_2} \mu_1(t) \left(1 - \frac{\tilde{\zeta}}{2\tau} - \frac{\beta}{2}\right) \|u_t(t)\|_{2.\Gamma_1}^2 dt \\
&+ \int_t^{t_2} \mu_1(t) \left(\frac{\tilde{\zeta}}{2\tau} - \frac{\beta}{2}\right) \|z(x,1,t)\|_{2.\Gamma_1}^2 dt + \delta \int_t^{t_2} \|\nabla u_t(t)\|_2^2 dt \\
&\leq 2 \int_{t_1}^{t_2} E(t)dt,
\end{aligned} \tag{4.52}$$

exploiting (4.51) we arrive at

$$E(t) \leq c_{11} \left[D(t)^2 + E(t)^{\frac{1}{2}} D(t) \right]. \tag{4.53}$$

Then a simple application of Young's inequality gives, for all $t \geq 0$

$$E(t) \leq c_{12} D(t)^2, \tag{4.54}$$

where c_{11},c_{12} are some positive constants. Therefore, from (4.54), we get

$$E(t) \leq c_{12}[E(t) - E(t+1)],$$

here we choose $c_{12} > 1$. Thus by lemma 4.2.4, we obtain

$$E(t) \leq E(0)e^{-\sigma t} \quad for \quad t \geq 0,$$

with $\sigma = ln\left(\frac{c_{12}}{c_{12}-1}\right)$.

Chapter 5

Global existence and asymptotic behavior of a plate equations with a constant delay term and logarithmic nonlinearities

5.1 Introduction

In this work, we consider the following Cauchy problem with logarithmic nonlinearity

$$\begin{cases} u_{tt}(x,t) - \Delta^2 u + \phi(x)\left(\alpha\Delta^2 u - \displaystyle\int_0^t g(t-s)\Delta^2 u(x,s)ds\right) & \\ +\mu_1(t)u_t(x,t) + \mu_2(t)u_t(x,t-\tau) = u\ln|u|^k & \text{in } \mathbb{R}^n\times]0,+\infty[, \\ u(x,t) = 0, & \text{on } \partial\mathbb{R}^n\times]0,+\infty[, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \mathbb{R}^n, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \text{in } \mathbb{R}^n\times]0,t[. \end{cases} \tag{5.1}$$

Where $n \geq 1$, $\phi(x) > 0$ and $(\phi(x))^{-1} = \rho(x)$ such that ρ is a function that will be defined later. The initial datum u_0, u_1, f_0 are given functions belonging to suitable spaces that will be specified later. μ_1, μ_2 are real functions and g is a positive non-increasing function defined on $\mathbb{R}^+$. Moreover $\tau > 0$ represents the time delay term.

It is well known that the logarithmic nonlinearity is distinguished by several interesting physical properties. In recent years, there has been a growing interest in the viscoelastic wave equation, its properties and variants of the problem can be found for example in ([17],[35],[50],[52]). The plate equation in $\mathbb{R}^n$ has been studied by many authors and some results have been proved (see for instance [2],[37],[40]) and the references therein.

The author in [39] looked into a linear Cauchy viscoelastic problem with density. He obtained the exponential and polynomial rates by using the spaces weighted by density to compensate for the lack of Poincare's inequality.

In the case of delay term, Nicaise, Valein and Pignotti [60] proved an exponential stability result under the condition $\mu_1 < \sqrt{1-d}\mu_1$ where d is a constant such that $\tau'(t) \leq d < 1$. After that, Serge Nicaise, Cristina Pignotti and Julie Valein considered the following problem

$$\begin{cases} u_{tt}(x,t) - \Delta u = 0 & \text{in } \Omega \times [0,+\infty[\\ u(x,t) = 0 & \text{on } \Gamma_D \times [0,+\infty[\\ \dfrac{\partial u_t}{\partial \nu} = \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau(t)) & \text{on } \Gamma_N \times [0,+\infty[\\ u(x,0) = u_0, u'(x,0) = u_1 & \text{in } \Omega \\ u_t(x,t) = f_0(x,t-\tau(t)) & \text{on } \Gamma_N \times (0,\tau(0)) \end{cases} \tag{5.2}$$

where they extend the last result to general space dimension under some hypothesis. A. Benaissa, A. Benguessoum and S. A. Messaoudi in considered the wave equation with a weak internal constant delay term. Keltoum Bouhali and Fatheh Ellaggoune in [11], studied in any spaces dimension, a general decay rate of solutions of viscoelastic wave equations with logarithmic nonlinearities. Furthermore, they established, under convenient hypotheses on g and the initial data, the existence of weak solution associated to this equation.

The content of this chapter is organized as follows. In Section 2, we provide assumptions that will be used later, state and prove the existence result. In Section 3, we prove energy decay result of our problem.

5.2 Preliminary Results

We first recall some basic definitions and abstract results on weighted spaces. We define the function spaces of our problem and its norm as follows

$$\mathcal{D}^{2,2}(\mathbb{R}^n) = \left\{ f \in L^{2n/n-4}(\mathbb{R}^n) / \quad \Delta_x f \in L^2(\mathbb{R}^n) \right\}$$

and $\mathcal{D}^{2,2}(\mathbb{R}^n)$ can be embedded continuously in $L^{2n/n-4}(\mathbb{R}^n)$. The space $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$

$$\|f\|_{L^q_\rho} = \left(\int_{\mathbb{R}^n} \rho|f|^q dx\right)^{\frac{1}{q}}$$

Remark 5.2.1 *The space $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space.*

In the following, we will give sufficient conditions and assumptions that guarantee the global existence of the problem 5.1.
$(H1)$ g is a positive bounded function satisfying:

$$\alpha - \int_0^\infty g(s)ds = l > 0, \quad \alpha > 0, \tag{5.3}$$

and there exists a positive non-increasing function $H \in C^2(\mathbb{R}^+)$ such that, for $t \geq 0$, we have

$$g'(t) \leq -H(t)g(t), \quad H(0) = 0. \tag{5.4}$$

where H is linear or strictly increasing and strictly convex function on $(0; r]; r < 1$.
$(H2)$ According to results in [18],we obtain

1. We can deduce that there exists $t_1 > 0$ large enough such that:

2. $\forall t \geq t_1$, we have $\lim\limits_{t\to\infty} g(s) = 0$ so $\lim\limits_{t\to\infty} g'(s) = 0$ and $g(t_1) > 0$.
   Then
   $$\max\{g(s), -g'(s)\} < \min\{r, H(s), H_0(s)\},$$
   where $H_0(t) = H(D(t))$, and D is a positive C^1 function, with $D(0) = 0$, for which H_0 is strictly increasing and strictly convex function on $(0; r]$ and
   $$\int_0^\infty g(s)H_0(-g'(s))ds < \infty$$

3. For $0 \leq t \leq t_1$ we have $g(0) \leq g(t) \leq g(t_1)$, ($g$ is non-increasing).
   Since H is a positive continuous function, then
   $$g'(t) < H(g(t)) \leq -kg(t), \quad k > 0$$

1. Let ${H_0}^*$ be the convex conjugate of H_0 in the sense of Young (see [2]), then

$$
{H_0}^* = s(H_0')^{-1}(s) - H_0((H_0')^{-1}(s)), \quad s \in (0, (H_0'(r))
$$

and satisfies the following Young's inequality

$$
AB \leq {H_0}^*(A) + H_0(B), \quad A \in (0, (H_0'(r)), \quad B \in (0; r]
$$

$(H3)$

1. μ_1 is a positive function of class C^1 satisfying:

$$
\mu_1(t) \leq M, \qquad M > 0. \tag{5.5}
$$

2. μ_2 is a real function of class C^1 such that:

$$
\mu_2(t) \leq \beta\mu_1(t), \quad 0 < \beta < 1. \tag{5.6}
$$

$(H4)$ The function $\rho : \mathbb{R}^n \to \mathbb{R}^n_+$ satisfies $\rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$
with $\gamma \in (0, 1)$ and $\rho \in L^s(\mathbb{R}^n)$,where $s = \frac{2n}{2n-qn+4q}$.
We also need the following technical Lemmas in the course of our investigation.
Let λ_1 be the first eigenvalue of the spectral Dirichlet problem

$$
\Delta u = \lambda_1 u \quad \text{in } \mathbb{R}^n, \qquad u = \frac{\partial u}{\partial \eta} = 0 \quad \text{in } \partial\mathbb{R}^n,
$$

$$
\|\nabla u\|_2 \leq \omega \|\Delta u\|_2,
$$

where $\omega = \frac{1}{\sqrt{\lambda_1}}$.

Lemma 5.2.1 [37] *Assume that the function ρ satisfies the assumption $(H4)$, then for any $u \in \mathcal{D}^{2,2}(\mathbb{R}^n)$ we have*

$$
\|u\|_{L^2_\rho(\mathbb{R}^n)} \leq C_0 \|\Delta u\|_{L^2(\mathbb{R}^n)},
$$

where $C_0 = \|\rho\|_{L^s_\rho}$, with $s = \frac{2n}{2n-qn+4q}$ and $2 \leq q \leq \frac{2n}{n-4}$.

Lemma 5.2.2 *For any $g \in C^1$ and $\varphi \in H_0^1(0,T)$, we have*

$$-2\int_0^t \int_{\mathbb{R}^n} g(t-s)\varphi \ \varphi_t dxds = \frac{d}{dt}\Big((g\circ\varphi)(t) - \int_0^t g(s)ds\|\varphi\|_2^2\Big) + g(t)\|\varphi\|_2^2 - (g'\circ\varphi)(t). \tag{5.7}$$

where

$$(g\circ\varphi)(t) = \int_0^t g(t-s)\int_{\mathbb{R}^n} |(\phi(s)-\varphi(t))|^2 dxds.$$

Lemma 5.2.3 [19] *Let $u \in \mathcal{D}^{2,2}(\mathbb{R}^n)$ and $c_1, c_2 > 0$ be two numbers. Then*

$$2\int_{\mathbb{R}^n} \rho(x)|u|^2 \ln\Big(\frac{|u|}{\|u\|^2_{L^2_\rho}}\Big)dx + n(1+c_1)\|u\|^2_{L^2_\rho} \leq c_2\frac{\|\rho\|^2_{L^2}}{\pi}\|\nabla u\|_2^2. \tag{5.8}$$

If u is a solution of the problem 5.1,and $v \in \mathcal{D}^{2,2}(\mathbb{R}^n)$ then

$$\begin{aligned}\int_{\mathbb{R}^n} \rho(x)|u|\ln|u|^k v dx &= \int_{\mathbb{R}^n} \rho(x)u_{tt} v dx - \int_{\mathbb{R}^n} \rho(x)\Delta u \Delta v dx \\ &+ \int_{\mathbb{R}^n} \alpha \Delta u \Delta v dx - \int_{\mathbb{R}^n}\int_0^t g(t-s)\Delta u(x,s)\Delta v ds dx \\ &+\mu_1 \int_{\mathbb{R}^n} (t)\rho(x)u_t v dx + \mu_2(t)\int_{\mathbb{R}^n} \rho(x)u_t(x,t-\tau)v dx\end{aligned} \tag{5.9}$$

Lemma 5.2.4 (16) *Let $u \in \mathcal{D}^{2,2}(\mathbb{R}^n)$, then we have*

$$\Big(\int_0^t g(t-s)(u(s)-u(t))ds\Big)^2 \leq \Big(\int_0^t |g(s)|^{2(1-\theta)}ds\Big)\Big(\int_0^t |g(t-s)|^{2\theta}|(u(s)-u(t))|^2 ds\Big). \tag{5.10}$$

Like in ([59]) we introduce the auxiliary unknown

$$z(x,\gamma,t) = u_t(x, t-\tau\gamma), \ x \in \mathbb{R}^n, \ \gamma \in (0,1), \ t > 0.$$

Then, we have

$$\tau z_t(x,\gamma,t) + z_\gamma(x,\gamma,t) = 0.$$

Therefore the problem 5.1 takes the form

$$\begin{cases} u_{tt}(x,t) - \Delta^2 u + \phi(x)\left(\alpha\Delta^2 u - \displaystyle\int_0^t g(t-s)\Delta^2 u(x,s)ds\right) & \\ +\mu_1(t)u_t(x,t) + \mu_2(t)z(x,1,t) = u\ln|u|^k, & \text{in } \mathbb{R}^n\times]0,+\infty[\\ \tau z_t(x,\gamma,t) + z_\rho(x,\gamma,t) = 0 & \text{in } \mathbb{R}^n\times]0,+\infty[\\ u(x,t) = 0, & \text{on } \partial\mathbb{R}^n\times]0,+\infty[\\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \mathbb{R}^n \\ z(x,\gamma,0) = f_0(x,-\tau\gamma), & \text{in } \mathbb{R}^n\times]0,t[. \end{cases} \tag{5.11}$$

First we define the energy of solution by

$$\begin{aligned} E(t) = \frac{1}{2}\|u_t\|^2_{L^2_\rho} + \frac{1}{2}\|\Delta u\|^2_{L^2_\rho} + \Big(\frac{\alpha}{2} - \frac{1}{2}\int_0^t g(s)ds\Big)\|\Delta u\|_2^2 \\ +\frac{1}{2}(g\circ\Delta u)(t) - \frac{k}{2}\int_{\mathbb{R}^n}\rho(x)u^2 ln|u|dx + \frac{k}{4}\|u\|^2_{L^2_\rho} \\ +\frac{1}{2}\xi(t)\int_{\mathbb{R}^n}\int_0^1 \rho(x)z^2(x,\gamma,t)d\gamma dx. \end{aligned} \tag{5.12}$$

Where ξ is non-increasing function such that

$$\tau\beta < \overline{\xi} < \tau(2-\beta), \quad t>0, \quad \xi(t) = \overline{\xi}\mu_1(t). \tag{5.13}$$

Lemma 5.2.5 *Let (u,z) be a solution of the problem 5.11. Then, the energy functional defined by 5.12 satisfies*

$$\begin{aligned} E'(t) \le \frac{1}{2}(g'\circ\Delta u) - \frac{1}{2}g(t)\|\Delta u\|_2^2 - \Big(\mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|u_t\|^2_{L^2_\rho} \\ -\Big(\frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|z(x,1,t)\|^2_{L^2_\rho} \le 0. \end{aligned} \tag{5.14}$$

Proof. Multiplying the first equation in 5.11 by $\rho(x)u_t$, integrating over $\mathbb{R}^n$ and using Green's identity, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|u_t\|^2_{L^2_\rho} - \|\Delta u\|^2_{L^2_\rho} + \alpha\|\Delta u\|^2_2 + \frac{k}{2}\|u\|^2_{L^2_\rho} - \int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k dx\Big)\\ +\mu_1(t)\|u_t\|^2_{L^2_\rho} + \mu_2(t)\int_{\mathbb{R}^n}\rho(x)u_t z(x,1,t)dx\\ -\int_0^t g(t-s)\int_{\mathbb{R}^n}\Delta u(x,s)\Delta u_t(x,t)dxds = 0\end{aligned} \tag{5.15}$$

We simplify the last term by using lemma 5.2.2, we get

$$\begin{aligned}-\int_0^t g(t-s)\int_{\mathbb{R}^n}\Delta u(x,s)\Delta u_t(x,t)dxds = \frac{1}{2}\frac{d}{dt}(g\circ\Delta u)\\ -\frac{1}{2}(g'\circ\Delta u) + \frac{1}{2}g(t)\|\Delta u\|^2_2 - \frac{1}{2}\frac{d}{dt}\int_0^t g(s)ds\|\Delta u\|^2_2\end{aligned} \tag{5.16}$$

Replacing 5.16 in 5.15 we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|u_t\|^2_{L^2_\rho} - \|\Delta u\|^2_{L^2_\rho} + \left(\alpha - \int_0^t g(s)ds\right)\|\Delta u\|^2_2\right)\\ +\frac{1}{2}\frac{d}{dt}\left(\frac{k}{2}\|u\|^2_{L^2_\rho} - \int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k + g\circ\Delta u\right)\\ +\mu_1(t)\|u_t\|^2_{L^2_\rho} + \mu_2(t)\int_{\mathbb{R}^n}\rho(x)u_t z(x,1,t)dx\\ -\frac{1}{2}(g'\circ\Delta u) + \frac{1}{2}g(t)\|\Delta u\|^2_2 = 0.\end{aligned} \tag{5.17}$$

Multiplying the second equation in 5.11 by $\frac{1}{\tau}\xi(t)\rho(x)z$, where $\xi(t)$ satisfying 5.13 and integrating over $\mathbb{R}^n\times(0,1)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\xi(t)\int_{\mathbb{R}^n}\int_0^1\rho(x)z^2(x,\gamma,t)d\gamma dx - \frac{\xi'(t)}{2}\int_{\mathbb{R}^n}\int_0^1\rho(x)z^2(x,\gamma,t)d\gamma dx\\ +\frac{\xi(t)}{2\tau}\|z^2(x,1,t)\|^2_{L^2_\rho} - \frac{\xi(t)}{2\tau}\|u_t\|^2_{L^2_\rho} = 0.\end{aligned} \tag{5.18}$$

Combination of 5.17 and 5.18, by recalling at the definition of $E(t)$, we deduce that

$$\begin{aligned}
E'(t) + \mu_1(t)\|u_t\|^2_{L^2_\rho} + \mu_2(t)\int_{\mathbb{R}^n} \rho(x)u_t z(x,1,t)dx \\
-\frac{1}{2}(g' \circ \Delta u) + \frac{1}{2}g(t)\|\Delta u\|_2^2 \\
-\frac{\xi'(t)}{2}\int_{\mathbb{R}^n}\int_0^1 \rho(x)z^2(x,\gamma,t)d\gamma dx \\
+\frac{\xi(t)}{2\tau}\|z^2(x,1,t)\|^2_{L^2_\rho} - \frac{\xi(t)}{2\tau}\|u_t\|^2_{L^2_\rho} = 0,
\end{aligned} \tag{5.19}$$

then

$$\begin{aligned}
E'(t) = -\Big(\mu_1(t) - \frac{\xi(t)}{2\tau}\Big)\|u_t\|^2_{L^2_\rho} - \mu_2(t)\int_{\mathbb{R}^n} \rho(x)u_t z(x,1,t)dx \\
+\frac{1}{2}(g' \circ \Delta u) - \frac{1}{2}g(t)\|\Delta u\|_2^2 \\
+\frac{\xi'(t)}{2}\int_{\mathbb{R}^n}\int_0^1 \rho(x)z^2(x,\gamma,t)d\gamma dx - \frac{\xi(t)}{2\tau}\|z^2(x,1,t)\|^2_{L^2_\rho}.
\end{aligned} \tag{5.20}$$

Due to Young's inequality and using the assumptions for $\xi(t)$ and g, we obtain

$$\begin{aligned}
E'(t) \le \frac{1}{2}(g' \circ \Delta u) - \frac{1}{2}g(t)\|\Delta u\|_2^2 - \Big(\mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|u_t\|^2_{L^2_\rho} \\
-\Big(\frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2}\Big)\|z(x,1,t)\|^2_{L^2_\rho} \le 0,
\end{aligned} \tag{5.21}$$

where

$$C_1 = \mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2} > 0; \quad C_2 = \frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2} > 0.$$

This complete the proof of energy decay.

5.3 Global existence

According to logarithmic Sobolev inequality and by using Galerkin's method combined with compact theorem, similar to the proof in ([35], [19]), we have the following result

Theorem 5.3.1 *(Local existence) Let* $(u_0, u_1, f_0) \in \mathcal{D}^{2,2}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n) \times L^2(\mathbb{R}^n \times (0,1))$ *be given. Assume that* g *satisfies* $(H1)$ *and* μ_1, μ_2 *satisfy* $(H3)$*. Then the problem 5.11 admits a unique local solution* (u, z) *satisfying:*

$$u \in C([0,T); \mathcal{D}^{2,2}(\mathbb{R}^n))$$

$$u' \in C([0,T); L^2_\rho(\mathbb{R}^n))$$

$$z \in C([0,T); L^2(\mathbb{R}^n \times (0,1)).$$

Now, we introduce the two functionals as follow

$$\begin{aligned} J(t) = & \frac{1}{2}\|\Delta u\|^2_{L^2_\rho} + \Big(\frac{\alpha}{2} - \frac{1}{2}\int_0^t g(s)ds\Big)\|\Delta u\|_2^2 \\ & + \frac{1}{2}(g \circ \Delta u)(t) - \frac{k}{2}\int_{\mathbb{R}^n} \rho(x) u^2 ln|u| dx \\ & + \frac{1}{2}\xi(t)\int_0^1 \|z^2(x,\gamma,t)\|^2_{L^2_\rho} d\gamma dx + \frac{k}{4}\|u\|^2_{L^2_\rho}, \end{aligned} \tag{5.22}$$

and

$$I(t) = 2J(t) - \frac{k}{2}\|u\|^2_{L^2_\rho}.$$

As in ([49]) to establish the corresponding method of potential wells which is related to the logarithmic nonlinear term, we introduce the stable set as follows

$$W = \Big\{u \in \mathcal{D}^{2,2}(\mathbb{R}^n - \{0\}) : I(t) > 0,\ J(t) < d\Big\} \cup \{0\}$$

Where d is the mountain pass level defined by

$$d = \inf\{\sup J(\mu u)\}$$

With $\mu \geq 0, u \in \mathcal{D}^{2,2}(\mathbb{R}^n - \{0\})$. Also, by introducing the set called "Nehari manifold"

$$\mathcal{N} = \Big\{u \in \mathcal{D}^{2,2}(\mathbb{R}^n) - \{0\}, I(t) = 0\Big\}$$

Similar to results in ([87]), it is readily seen that the potential depth d is characterized by

$$d = \inf_{u \in \mathcal{N}} J(t).$$

This characterization of d shows that

$$dist(0, \mathcal{N}) = \min_{u \in \mathcal{N}} \|u\|_{\mathcal{D}^{2,2}(\mathbb{R}^n)}.$$

By the fact of $E'(t) < 0$, we will prove the invariance of the set W. That is if for some $t_0 > 0$ if $u(t_0) \in W$, then $u(t) \in W$, $\forall t \geq t_0$. Now we give the existence Lemma of the potential depth (see Lemma 2.4 in [19]).

Lemma 5.3.1 *d is positive constant.*

Lemma 5.3.2 *Let $u \in \mathcal{D}^{2,2}(\mathbb{R}^n)$, and $\eta = \exp(\frac{n}{2}(1 + c_1))$. If $\|u\|^2_{L^2(\rho)} < \eta$ then $I(t) > 0$. If $I(t) = 0$, $\|u\|^2_{L^2(\rho)} \neq 0$ then $\|u\|^2_{L^2(\rho)} > \eta$.*

Proof. By the lemma 5.2.3 we have

$$\begin{aligned} J(t) &= \|\Delta u\|^2_{L^2_\rho} + \Big(\alpha - \int_0^t g(s)ds\Big)\|\Delta u\|_2^2 \\ &+(g \circ \Delta u)(t) - k\int_{\mathbb{R}^n} \rho(x)u^2 ln|u|^2 dx \\ +\xi(t)\int_{\mathbb{R}^n}\int_0^1 & z^2(x,\gamma,t)d\gamma dx \geq (l - k\omega c_2\frac{\|\rho\|^2_{L^2}}{2\pi})\|\Delta u\|_2^2 \\ &+k(\frac{n}{2}(1+c_1) - \ln\|u\|^2_{L^2_\rho})\|u\|^2_{L^2_\rho} + \frac{k}{4}\|u\|^2_{L^2_\rho}. \end{aligned} \tag{5.23}$$

Choosing $c_2 < \frac{\pi}{\omega k \|\rho\|^2_{L^2_\rho}}$, then

$$I(t) \geq k(\frac{n}{2}(1+c_1) - \ln\|u\|^2_{L^2_\rho})\|u\|^2_{L^2_\rho}. \tag{5.24}$$

Therefore if $\|u\|^2_{L^2_\rho} < \eta$ then $I(t) > 0$.
If $I(t) = 0$, $\|u\|^2_{L^2_\rho} \neq 0$ we have $\|u\|^2_{L^2_\rho} > \eta$.

Theorem 5.3.2 *(Global Existence) Let $u_0 \in \mathcal{D}^{2,2}(\mathbb{R}^n), u_1(x) \in L^2_\rho(\mathbb{R}^n)$ and $0 < E(0) < d, I(0) > 0$. Then, under hypothesis $(H1)$ and conditions of the function ρ, the problem 5.11 has a global solution in time.*

Proof. From the definition of energy for the weak solution and since E is increasing, we have

$$\frac{1}{2}\|u_t\|^2_{L^2_\rho} + J(t) \leq \frac{1}{2}\|u_1\|^2_{L^2_\rho} + J(0), \quad \forall t \in [0, T_{max}),$$

where T_{max} is the maximal existence time of weak solution of u. Then, by the definition of the stable set and using Lemma 5.3.2, we have $u \in W, \forall t \in [0, T_{max})$.

5.4 Asymptotic behavior

We apply the multiplier techniques and we introduce an appropriate Lyapunov functional to obtain the asymptotic behavior. For this purpose, we introduce the so called Liapunov functional L defined by

$$L(t) = \xi_1 E(t) + \psi(t) + \xi_2\varphi(t) + \varepsilon_1\phi(t), \quad \xi_1 > 1,\ \xi_2 > 1,\ \varepsilon_1 > 0, \tag{5.25}$$

where

$$\begin{aligned} \psi(t) &= \int_{\mathbb{R}^n} \rho(x)u_t u dx, \\ \varphi(t) &= -\int_{\mathbb{R}^n} \rho(x)u_t \int_0^t g(t-s)(u(t)-u(s))dsdx, \\ \phi(t) &= -\xi(t)\tau\int_{\mathbb{R}^n} \rho(x)\int_0^1 e^{-2\tau\gamma}u_t^2(x, t-\tau\gamma)d\gamma dx. \end{aligned} \tag{5.26}$$

Now we present some lemmas to get the asymptotic behavior of solutions.

Lemma 5.4.1 *Suppose that $(H1)-(H4)$ hold and let $(u_0, u_1) \in \mathcal{D}^{2,2}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)$ be given. If (u, z) is the solution of 5.11, then the derivative of the functional ψ satisfies the following inequality for $\delta > 0$.*

$$\begin{aligned} \psi'(t) \leq{}& (1 + \frac{1}{2}\mu_1(t))\|u_t\|^2_{L^2_\rho} + (\alpha - l)c_\delta(g \circ \Delta u)(t) \\ &+(\delta - l + \|\Delta u\|_\infty)\|\Delta u\|_2^2 + \frac{1}{2}\mu_2(t)\|z(x,1,t)\|^2_{L^2_\rho} \\ &+\|\rho\|^2_{L^2_\rho}\Big(\frac{k\omega c_2}{2\pi} + k\ln\|u\|^2_{L^2_\rho} - \frac{kn}{2}(1+c_1) + \frac{1}{2}[\mu_1(t) + \mu_2(t)]\Big)\|\Delta u\|_2^2. \end{aligned} \tag{5.27}$$

Proof 5.4.1 *By using the first equation in 5.11, we have*

$$\psi'(t) = \int_{\mathbb{R}^n} \rho(x)u_{tt}udx + \int_{\mathbb{R}^n} \rho(x)|u_t|^2dx, \tag{5.28}$$

$$\psi'(t) = \int_{\mathbb{R}^n} \rho(x)|u_t|^2 dx + \int_{\mathbb{R}^n} \rho(x)\Delta u^2 dx - \alpha \int_{\mathbb{R}^n} \Delta u^2 dx$$
$$+ \int_{\mathbb{R}^n} \Delta u_t \int_0^t g(t-s)\Delta u(x,s) ds dx - \mu_1(t) \int_{\mathbb{R}^n} \rho(x) u_t u(x,t) dx \tag{5.29}$$
$$-\mu_2(t) \int_{\mathbb{R}^n} \rho(x) u z(x,1,t) dx + \int_{\mathbb{R}^n} \rho(x) u^2 ln|u|^k dx.$$

We now estimate the right hand side of 5.29 and applying lemma 5.2.2, we have the estimates as follows

$$\begin{aligned}
\int_{\mathbb{R}^n} \Delta u \int_0^t g(t-s)\Delta u(s) ds dx &= \int_0^t g(s) ds \|\Delta u\|_2^2 \\
&+ \int_{\mathbb{R}^n} \Delta u \int_0^t g(t-s)(\Delta u(s) - \Delta u(t)) ds dx \\
&\le (\delta + \alpha - l)\|\Delta u\|_2^2 + c_\delta \int_{\mathbb{R}^n} \Big(\int_0^t g(t-s)|\Delta u(s) - \Delta u(t)| ds \Big)^2 dx \\
&\le (\delta + \alpha - l)\|\Delta u\|_2^2 + (\alpha - l) c_\delta (g \circ \Delta u)(t).
\end{aligned} \tag{5.30}$$

By using Young's inequality, Sobolevs̀ inequality and Lemma 2.3 we have

$$\begin{aligned}
&-\mu_1(t) \int_{\mathbb{R}^n} \rho(x) u_t u(x,t) dx - \mu_2(t) \int_{\mathbb{R}^n} \rho(x) u z(x,1,t) dx \le \\
&\frac{1}{2}\Big(\mu_1(t) + \mu_2(t)\Big)\|\rho\|_{L^2_\rho}^2 \|\Delta u\|_2^2 + \frac{1}{2}\mu_1(t)\|u_t\|_{L^2_\rho}^2 + \frac{1}{2}\mu_2(t)\|z(x,1,t)\|_{L^2_\rho}^2,
\end{aligned} \tag{5.31}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^n} \rho(x) u^2 \ln|u|^k dx &= k \int_{\mathbb{R}^n} \rho(x) u^2 \Big(\ln \frac{u}{\|u\|_{L^2_\rho}^2} + \ln \|u\|_{L^2_\rho}^2 \Big) dx \\
&\le \frac{k\omega c_2}{2\pi}\|\rho\|_{L^2_\rho}^2 \|\Delta u\|_2^2 + k\Big[\ln \|u\|_{L^2_\rho}^2 - \frac{n}{2}(1 + c_1)\Big]\|u\|_{L^2_\rho}^2 \\
&\le \Big(\frac{k\omega c_2}{2\pi}\|\rho\|_{L^2_\rho}^2 \|\Delta u\|_2^2 + k\Big[\ln \|u\|_{L^2_\rho}^2 - \frac{n}{2}(1 + c_1)\Big]\Big)\|\rho\|_{L^2_\rho}^2 \|\Delta u\|_2^2.
\end{aligned}$$

By combining the last inequalities, we arrive at

$$\begin{aligned}
\psi'(t) &\le (1 + \frac{1}{2}\mu_1(t))\|u_t\|_{L^2_\rho}^2 + (\alpha - l)c_\delta (g \circ \Delta u)(t) \\
&+(\delta - l + \|\rho\|_\infty)\|\Delta u\|_2^2 + \frac{1}{2}\mu_2(t)\|z(x,1,t)\|_{L^2_\rho}^2 \\
&+\|\rho\|_{L^2_\rho}^2 \Big(\frac{k\omega c_2}{2\pi} + k\ln\|u\|_{L^2_\rho}^2 - \frac{kn}{2}(1 + c_1) + \frac{1}{2}[\mu_1(t) + \mu_2(t)]\Big)\|\Delta u\|_2^2.
\end{aligned} \tag{5.32}$$

Lemma 5.4.2 *Suppose that (H3) is fulfilled and let $(u_0, u_1) \in \mathcal{D}^{2,2}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)$ be given.*

If (u,z) is the solution of 5.11,then the derivative of the functional φ satisfies the following inequality for some $\delta > 0$.

$$\begin{aligned}
&\varphi'(t) \leq \Big[\delta l + \alpha\delta\|\rho\|_\infty + k\Big(\frac{\delta c_2}{2\pi} + \ln\|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2}\Big)\|\rho\|^2_{L^2_\rho}\Big]\|\Delta u\|^2_2 \\
&+ \Big(1 + c_\delta l + c_\delta\|\rho\|_\infty + (1 + \frac{kc_\delta\omega c_2}{2\pi})\|\rho\|^2_{L^2_\rho}\Big)(\alpha - l)(g\circ\Delta u) \\
&- c_\delta\|\rho\|^2_{L^2_\rho}(g'\circ\Delta u) + \Big(\delta + \frac{1}{2}\mu_1(t) - \int_0^t g(s)\Big)\|u_t\|^2_{L^2_\rho} + \frac{1}{2}\mu_2(t)\|z(x,1,t)\|^2_{L^2_\rho}.
\end{aligned} \tag{5.33}$$

Proof. *Taking the derivative of φ, we obtain easily*

$$\begin{aligned}
&\varphi'(t) = -\int_{\mathbb{R}^n}\rho(x)u_{tt}\int_0^t g(t-s)(u(t)-u(s))dsdx \\
&- \int_{\mathbb{R}^n}\rho(x)u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx \\
&- \int_0^t g(s)ds\|u_t\|^2_{L^2_\rho} = -\int_{\mathbb{R}^n}\rho(x)\Delta u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx \\
&+ \alpha\int_{\mathbb{R}^n}\Delta u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx \\
&- \int_{\mathbb{R}^n}\Big(\int_0^t g(t-s)\Delta u(s)ds\Big)\Big(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\Big)dx \\
&+ \mu_1(t)\int_{\mathbb{R}^n}\rho(x)u_t\int_0^t g(t-s)(u(t)-u(s))dsdx \\
&+ \mu_2(t)\int_{\mathbb{R}^n}\rho(x)z(x,\rho,t)\int_0^t g(t-s)(u(t)-u(s))dsdx \\
&- \int_{\mathbb{R}^n}\rho(x)u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx \\
&- \int_{\mathbb{R}^n}\rho(x)u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx - \int_0^t g(s)ds\|u_t\|^2_{L^2_\rho},
\end{aligned} \tag{5.34}$$

then

$$\begin{aligned}
\varphi'(t) =& -\int_{\mathbb{R}^n} \rho(x)\Delta u \int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx \\
&+ \Big(\alpha - \int_0^t g(s)ds\Big) \int_{\mathbb{R}^n} \Delta u \int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx \\
&+ \int_{\mathbb{R}^n} \Big(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\Big)^2 dx \\
&+ \mu_1(t) \int_{\mathbb{R}^n} \rho(x)u_t \int_0^t g(t-s)(u(t)-u(s))dsdx \\
&+ \mu_2(t) \int_{\mathbb{R}^n} \rho(x)z(x,\rho,t) \int_0^t g(t-s)(u(t)-u(s))dsdx \\
&- \int_{\mathbb{R}^n} \rho(x)u\ln|u|^k \int_0^t g(t-s)(u(t)-u(s))dsdx \\
&- \int_{\mathbb{R}^n} \rho(x)u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx - \int_0^t g(s)ds\|u_t\|^2_{L^2_\rho}.
\end{aligned} \tag{5.35}$$

By Holder's and Young's inequalities and Sobolev Poincare's inequality, we estimate

$$\begin{aligned}
&\Big(\alpha - \int_0^t g(s)ds\Big) \int_{\mathbb{R}^n} \Delta u \int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx \\
&+ \int_{\mathbb{R}^n} \Big(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\Big)^2 dx \le l\delta\|\Delta u\|_2^2 \\
&+ (c_\delta l + 1)(\alpha - l)(g\circ\Delta u).
\end{aligned} \tag{5.36}$$

And

$$\begin{aligned}
&-\alpha \int_{\mathbb{R}^n} \rho(x)\Delta u \int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx \\
&\le \alpha\delta\|\rho\|_\infty\|\Delta u\|_2^2 + (\alpha-l)c_\delta\|\rho\|_\infty(g\circ\Delta u),
\end{aligned} \tag{5.37}$$

$$\begin{aligned}
&-\mu_1(t) \int_{\mathbb{R}^n} \rho(x)u_t \int_0^t g(t-s)(u(t)-u(s))dsdx \\
&-\mu_2(t) \int_{\mathbb{R}^n} \rho(x)z(x,1,t) \int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\le \frac{1}{2}\mu_1(t)\|u_t\|^2_{L^2_\rho} + \frac{1}{2}\mu_2(t)\|z(x,1,t)\|^2_{L^2_\rho} + \|\rho\|^2_{L^2_\rho}(\alpha-l)(g\circ\Delta u),
\end{aligned} \tag{5.38}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^n}\rho(x)u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx\\
&\le \delta\|u_t\|^2_{L^2_\rho}-c_\delta\|\int_0^t -g'(t-s)(u(t)-u(s))ds\|^2_{L^2_\rho}\\
&\le \delta\|u_t\|^2_{L^2_\rho}-c_\delta\|\rho\|^2_{L^2_\rho}(g'\circ\Delta u).
\end{aligned}
\tag{5.39}
$$

Using Poincaré-Sobolev inequality and lemma 5.2.3 and conditions in Lemma 5.3.2, we have

$$
\begin{aligned}
&-\int_{\mathbb{R}^n}\rho(x)u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&\le \int_{\mathbb{R}^n}\rho(x)u\Big(\ln\frac{|u|^k}{\|u\|^2_{L^2_\rho}}+\ln\|u\|^2_{L^2_\rho}\Big)\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&\le k\Big(\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|u\|^2_{L^2_\rho}\\
&+\frac{kc_2}{2\pi}\Big\|u\int_0^t g(t-s)(u(t)-u(s))ds\Big\|^2_{L^2_\rho}\\
&\le k\Big(\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|\rho\|^2_{L^2_\rho}\|\Delta u\|^2_2\\
&+\frac{k\omega c_2}{2\pi}\|\rho\|^2_{L^2_\rho}\Big\|\Delta u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\Big\|^2_2\\
&\le k\Big(\frac{\delta c_2}{2\pi}+\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|\rho\|^2_{L^2_\rho}\|\Delta u\|^2_2+\frac{kc_\delta\omega c_2}{2\pi}\|\rho\|^2_{L^2_\rho}(\alpha-l)(g\circ\Delta u).
\end{aligned}
\tag{5.40}
$$

Combining these estimates we arrive at

$$
\begin{aligned}
&\varphi'(t)\le\Big[\delta l+\alpha\delta\|\rho\|_\infty k\Big(\frac{\delta c_2}{2\pi}+\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|\rho\|^2_{L^2_\rho}\Big]\|\Delta u\|^2_2\\
&+\Big(1+c_\delta l+c_\delta\|\rho\|_\infty+(1+\frac{kc_\delta\omega c_2}{2\pi})\|\rho\|^2_{L^2_\rho}\Big)(\alpha-l)(g\circ\Delta u)\\
&-c_\delta\|\rho\|^2_{L^2_\rho}(g'\circ\Delta u)+\Big(\delta+\frac{1}{2}\mu_1(t)-\int_0^t g(s)\Big)\|u_t\|^2_{L^2_\rho}+\frac{1}{2}\mu_2(t)\|z(x,1,t)\|^2_{L^2_\rho}.
\end{aligned}
\tag{5.41}
$$

This completes the proof.

Lemma 5.4.3 *Suppose that $(H1),(H2)$ hold and let $(u_0,u_1)\in\mathcal{D}^{2,2}(\mathbb{R}^n)\times L^2_\rho(\mathbb{R}^n)$ be given. If (u,z) is the solution of 5.11 then the derivative of the functional $\phi(t)$ satisfies the following inequality*

$$\phi'(t)\leq\tau(\xi'(t)-2\xi(t))\int_0^1\|z(x,\gamma,t)\|^2_{L^2_\rho}d\gamma+\xi(t)e^{-2\tau}\|z(x,1,t)\|^2_{L^2_\rho}-\xi(t)\|u_t\|^2_{L^2_\rho}\tag{5.42}$$

Proof. Differentiating $\phi(t)$, we get

$$\begin{aligned}\phi'(t)&=-\xi'(t)\tau\int_{\mathbb{R}^n}\rho(x)\int_0^1e^{-2\tau\gamma}z^2(x,\gamma,t)dxd\gamma\\-2\xi(t)\tau\int_{\mathbb{R}^n}\rho(x)&\int_0^1e^{-2\tau\gamma}z(x,\gamma,t)z_t(x,\gamma,t)dxd\gamma.\end{aligned}\tag{5.43}$$

Using the second equality in 5.2 we obtain

$$\begin{aligned}\phi'(t)&=\Big(\frac{\xi'(t)}{\xi(t)}-2\Big)\phi(t)+\xi(t)e^{-2\tau}\|z(x,1,t)\|^2_{L^2_\rho}-\xi(t)\|u_t\|^2_{L^2_\rho}\\&\leq\Big(\xi'(t)+2\xi(t)\Big)\tau\int_0^1\|z(x,\gamma,t)\|^2_{L^2_\rho}d\gamma\\&\quad+\xi(t)e^{-2\tau}\|z(x,1,t)\|^2_{L^2_\rho}-\xi(t)\|u_t\|^2_{L^2_\rho}.\end{aligned}\tag{5.44}$$

Lemma 5.4.4 *If the functional L satisfies 5.25 then there exists two constants α_1 and α_2 such that*

$$\alpha_1E(t)\leq L(t)\leq\alpha_2E(t).\tag{5.45}$$

Proof. Using the Cauchy Schwartz and Young's inequalities, Poincaré-Sobolev inequality and lemma 5.2.4, we obtain

$$\begin{aligned}&|L(t)-\xi_1E(t)|\leq(\frac{1}{2}+\frac{\xi_2}{2}-\varepsilon_1\xi(t))\|u_t\|^2_{L^2_\rho}+\frac{1}{2}\|\rho\|^2_{L^2_\rho}\|\Delta u\|^2_2\\&+\frac{\xi_2}{2}\|\rho\|^2_{L^2_\rho}(\alpha-l)(g\circ\Delta u)\varepsilon_1\tau(\xi'(t)+2\xi(t)\Big)\int_0^1\|z(x,\gamma,t)\|^2_{L^2_\rho}d\gamma\\&+\varepsilon_1\xi(t)e^{-2\tau}\|z(x,1,t)\|^2_{L^2_\rho}\leq cE(t).\end{aligned}\tag{5.46}$$

Choosing ε_1 small enough such that

$$|L(t)-\xi_1E(t)|\leq cE(t).\tag{5.47}$$

Then we can choose ξ_1 such that

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t). \tag{5.48}$$

Lemma 5.4.5 *For all $t \geq t_1 > 0$, we have*

$$\int_{t_1}^{t} (g \circ \Delta u)(s) ds \leq H_0^{-1} \left(\int_{t_1}^{t} H_0(-g'(s)) g(s) \int_{\mathbb{R}^n} g(s) |\Delta u(t) - \Delta u(t-s)|^2 dx ds \right),$$

where H_0 is introduced in (H2).

Proof. By properties of E' and by $(H4)$ we have for $t \geq t_1$

$$\begin{aligned}\int_{\mathbb{R}^n} \int_0^{t_1} g(t-s) |\Delta u(t) - \Delta u(s)|^2 ds dx &\leq -\frac{1}{k} \int_{\mathbb{R}^n} \int_0^{t_1} g(t-s) |\Delta u(t) - \Delta u(s)|^2 ds dx \\ &\leq -cE'(t).\end{aligned}$$

We define now

$$\chi(t) = \int_{t_1}^{t} H_0(-g'(s))(g \circ \Delta u)(t) ds.$$

Since $\displaystyle\int_0^{\infty} H_0(-g'(s)) g(s) ds < +\infty$, we have

$$\begin{aligned}\chi(t) &= \int_{t_1}^{t} H_0(-g'(s)) \int_{\mathbb{R}^n} g(s) |\Delta u(s) - \Delta u(t)|^2 dx ds \\ &\leq 2 \int_{t_1}^{t} H_0(-g'(s)) g(s) (\|\Delta u(s)\|_2^2 - \|\Delta u(t)\|_2^2) dx ds \\ &\leq cE(0) \int_{t_1}^{t} H_0(-g'(s)) g(s) < 1.\end{aligned} \tag{5.49}$$

We define again a new functional $\lambda(t)$ related with $\chi(t)$ as

$$\lambda(t) = -\int_{t_1}^{t} H_0(-g'(s)) g'(s) \int_{\mathbb{R}^n} g(s) |\Delta u(t) - \Delta u(t-s)|^2 dx ds. \tag{5.50}$$

From $(H1)-(H2)$ and for some positive constant k_0, we conclude for all $t \geq t_1$

$$\begin{aligned}\lambda(t) &\leq -k_0 \int (g'(s)) \int_{\mathbb{R}^n} |\Delta u(t) - \Delta u(t-s)|^2 dxds \\ &\leq -k_0 \int_{t_1}^{t} (g'(s)) \int_{\mathbb{R}^n} |\Delta u(t)|^2 + |\Delta u(t-s)|^2 dxds \\ &\leq -cE(0) \int_{t_1}^{t} g'(s)ds \leq -cE(0)g(t_1) < \min\{r, H(r), H_0(r)\}.\end{aligned}$$

Using the properties of $H_0(\theta x) \leq \theta H_0(x)$ and hypothesis in $(H2)$, 5.50, 5.49 and Jensen's inequality we get

$$\begin{aligned}\lambda(t) &= \frac{1}{\chi(t)} \int_{t_1}^{t} H_0\Big(H_0^{-1}(-g'(s))\Big)\chi(t)H_0(-g'(s))g'(s) \int_{\mathbb{R}^n} g(s)|\Delta u(t) - \Delta u(t-s)|^2 dxds \\ &\geq H_0 \int_{t_1}^{t} \int_{\mathbb{R}^n} g(s)|\Delta u(t) - \Delta u(t-s)|^2 dxds.\end{aligned}$$

Which implies

$$\int_{t_1}^{t} \int_{\mathbb{R}^n} g(s)|\Delta u(t) - \Delta u(t-s)|^2 dxds \leq H_0^{-1}(\lambda(t)).$$

This ends the proof.

Theorem 5.4.1 *Let $(u_0, u_1) \in \mathcal{D}^{2,2}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)$ be given. Assume that g and ξ satisfy (H1) and 5.13. Then, for each $t_0 > 0$, there exist positive constants n_1, n_2, n_3, n_4 and k such that, for any solution of the problem 5.1, the energy satisfies*

$$E(t) \leq n_3 H_1^{-1}(n_1 + n_2), \quad \forall t \geq 0, \tag{5.51}$$

where

$$H_1(t) = \int_t^1 (sH_0'(n_4 s))^{-1} ds.$$

Proof. From the definition of $L(t)$ we obtain

$$L'(t) = \xi_1 E'(t) + \psi'(t) + \xi_2 \varphi'(t) + \varepsilon_1 \phi(t). \tag{5.52}$$

Then

$$\begin{aligned} L'(t) \leq &-m_0 \|u_t\|^2_{L^2_\rho} - M_1 \|\Delta u\|_2^2 + M_2 (g \circ \Delta u) \\ &+\Big(\frac{\xi_1}{2} - c_\delta \xi_2 \|\rho\|^2_{L^2_\rho}\Big)(g' \circ \Delta u) - (\frac{1+\xi_2}{2} + \varepsilon_1 \xi(t) e^{-2\tau})\mu_2(t)\|z(x,1,t)\|^2_{L^2_\rho} \\ &+\frac{\xi(t)}{2\tau}\|z(x,1,t)\|^2_{L^2_\rho} - \mu_2(t)\int_{\mathbb{R}^n} \rho(x) u_t z(x,1,t)dx \\ &+\frac{\xi'(t)}{2}\int_{\mathbb{R}^n}\int_0^1 \rho(x) z^2(x,\gamma,t) d\gamma dx, \end{aligned} \tag{5.53}$$

hence

$$\begin{aligned} L'(t) \leq &-M_0 \|u_t\|^2_{L^2_\rho} - M_1 \|\Delta u\|_2^2 + M_2 (g \circ \Delta u) \\ &+\Big(\frac{1}{2} - c_\delta \xi_2 \|\rho\|^2_{L^2_\rho}\Big)(\alpha - l)(g' \circ \Delta u) - M_3 \|z(x,1,t)\|^2_{L^2_\rho} \\ &+\Big(\varepsilon_1(\xi'(t) + 2\xi(t))\tau + \frac{\xi'(t)}{2}\Big)\int_{\mathbb{R}^n}\int_0^1 \rho(x) z^2(x,\gamma,t) d\gamma dx, \end{aligned} \tag{5.54}$$

where

$$M_0 = \Big(\xi_1\Big(\frac{\mu_2(t)}{2} + \varepsilon_1 \xi(t) - \mu_1(t) - \frac{\xi(t)}{2\tau}\Big) - 1 + \frac{\mu_1(t)}{2} + \xi_2\Big(\delta - \int_0^{t_1} g(s)ds + \frac{\mu_1(t)}{2}\Big)\Big),$$

$$\begin{aligned} M_1 = &\frac{\xi_1}{2} g(t) - \Big(\delta - l + \alpha\delta\|\rho\|_\infty + \frac{k\omega c_2}{2\pi} + k \ln \|u\|^2_{L^2_\rho} - \frac{kn}{2}(1 + c_1) \\ &+\frac{1}{2}(\mu_1(t) + \mu_2(t))\Big) - \xi_2\Big(\delta l + \|\rho\|_\infty + k\frac{\delta\omega c_2}{2\pi} + k \ln \|u\|^2_{L^2_\rho} - \frac{kn(1+c_1)}{2}\Big)\|\rho\|^2_{L^2_\rho} - \varepsilon_1 \xi\Big(t\Big). \end{aligned}$$

$$M_2 = \xi_2\Big(1 + c_\delta l + c_\delta \|\rho\|_\infty + (1 + \frac{k c_\delta \omega c_2}{2\pi})\|\rho\|^2_{L^2_\rho}\Big)(\alpha - l) + (\alpha - l)c_\delta) > 0.$$

$$M_3 = (\frac{\xi(t)}{2\tau} - \frac{\mu_2(t)}{2} + \frac{\mu_2(t)\xi_2}{2} - \varepsilon_1 \xi(t) e^{-2\tau}).$$

At this point, we choose δ so small such that $\xi_1 > 2c_\delta \|\rho\|^2_{L^2_\rho} \xi_2$.
Whence δ is fixed, we can choose ξ_2 such that $M_0 > 0$

$$M_0 = \Big(\xi_1\Big(\frac{\mu_2(t)}{2} - \mu_1(t) - \frac{\xi(t)}{2\tau}\Big) + \varepsilon_1\xi(t) - 1 + \frac{\mu_1(t)}{2}$$
$$+\xi_2\Big(\delta - \int_0^{t_1} g(s)ds + \frac{\mu_1(t)}{2}\Big)\Big) \geq \xi_2\Big(\delta - \int_0^{t} g(s)ds\Big),$$

then for $t > t_1$ we can choose

$$\xi_2 > \Big(\int_0^{t_1} g(s)ds - \delta\Big)^{-1}.$$

Now choosing ε_1 small enough such that $M_3 > 0$. After this conditions we estimate that

$$L'(t) \leq M_2(g \circ \Delta u) - cE'(t).$$

Now we set $F(t) = L(t) + cE(t)$, which is equivalent to $E(t)$. Then

$$F'(t) \leq -cE(t) + c\int_{\mathbb{R}^n}\int_{t_1}^{t} g(t-s)|\Delta u(t) - \Delta u(s)|^2 dsdx, \forall t > t_1.$$

Using Lemma 5.4.5, we obtain

$$F'(t) = L'(t) + cE'(t) \leq -cE(t) + cH_0^{-1}(\lambda(t)).$$

Now, we will use the following steps in ([56]) and using the fact that $E' < 0, H' > 0, H'' > 0$ on $(0; r]$ to define the functional

$$F_1(t) = H_0'\Big(\alpha_0\frac{E(t)}{E(0)}\Big)F(t) + cE(t), \alpha_0 < r, c > 0.$$

Where $F_1(t) \sim E(t)$ and

$$F_1'(t) = \alpha_0\frac{E'(t)}{E(0)}H_0''\Big(\alpha_0\frac{E(t)}{E(0)}\Big)F(t) + H_0'\Big(\alpha_0\frac{E(t)}{E(0)}\Big) + cE'(t)$$
$$\leq -cE(t)H_0'\Big(\alpha_0\frac{E(t)}{E(0)}\Big)F(t) + cH_0'\Big(\alpha_0\frac{E(t)}{E(0)}\Big)H^{-1}(\lambda(t)) + cE'(t).$$

Let H_0^* given in $(H2)$ and using Young's inequality in $(H2)$ with $A = H'\Big(\alpha_0 \frac{E(t)}{E(0)}\Big)$, $B = H_0^{-1}(\lambda(t))$ to get

$$\begin{aligned} F_1'(t) &\leq -cE(t)H_0'\Big(\alpha_0 \frac{E(t)}{E(0)}\Big) + cH_0^*\Big(H_0'\Big(\alpha_0 \frac{E(t)}{E(0)}\Big)\Big) + c(\lambda(t)) + cE'(t) \\ &\leq -cE(t)H_0'\Big(\alpha_0 \frac{E(t)}{E(0)}\Big) + c\alpha_0 \frac{E(t)}{E(0)} H_0'\Big(\alpha_0 \frac{E(t)}{E(0)}\Big) - c'E'(t) + cE'(t) \end{aligned}$$

Choosing α_0, c, c', such that for all $t \geq t_1$ we have

$$F_1'(t) \leq -k\frac{E(t)}{E(0)} H_0'\Big(\alpha_0 \frac{E(t)}{E(0)}\Big) = -kH_2 \frac{E(t)}{E(0)},$$

where $H_2(t) = H_0'(\alpha_0 t)$. By using the strict convexity of H_0 on $(0; r]$, to find that H_2', H_2 are strict positives function on $(0; 1]$, then

$$R(t) = \gamma \frac{k_1 F_1(t)}{E(0)} \sim E(t), \gamma \in (0, 1), \tag{5.55}$$

and

$$R'(t) \leq -\gamma k_0 H_2(R(t)), k_0 \in (0, +\infty), t \geq t_1.$$

hence, a simple integration gives

$$R(t) \leq H_1^{-1}(n_1 t + n_2), n_1, n_2 \in (0, +\infty), t \geq t_1$$

here $H_1(t) = \displaystyle\int_t^1 H^{-1}(s)ds$. From 5.55, for a positive constant n_3, we have

$$E(t) \leq n_3 H_1^{-1}(n_1 t + n_2), \quad n_1, n_2 \in (0, +\infty), \quad t \geq t_1.$$

The fact that H_1 is strictly decreasing function on $(0; 1]$ and due to properties of $H2$, we have: $\lim_{t \to 0} H_1(t) = +\infty$. Then

$$E(t) \leq n_3 H_1^{-1}(n_1 t + n_2), \quad n_1, n_2 \in (0, +\infty), \quad t \geq 0.$$

This completes the proof of the Theorem.

Remark 5.4.1 *Noting that, we have obtained all results without any conditions on the exponent k in the logarithmic nonlinearities.*

Chapter 6

General decay result for the wave equation with distributed delay

We consider the nonlinear wave equation with distributed delay

$$\left(|u_t|^{\gamma-2}u_t\right)_t - Lu - \int_0^t g(t-s)Lu(s)ds + \mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s)u_t(x,t-s)ds = 0,$$

in a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$. By using an appropriate Lyapunov functional, we study the asymptotic behavior of solutions. Moreover, we extend and improve the previous results in the literature.

6.1 Introduction

In this chapter, we investigate the decay properties of solutions for the initial boundary value problem of a nonlinear wave equation of the form

$$\begin{cases} (|u_t|^{\gamma-2}u_t)_t - Lu - \displaystyle\int_0^t g(t-s)Lu(s)ds + \mu_1 u_t(x,t) & \\ + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s)u_t(x,t-s)ds = 0, & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0, & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \\ u_t(x,-t) = f_0(x,t), & \text{in } \Omega\times]0,\tau_2[, \end{cases} \tag{6.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$. The constant τ_1 is nonnegative such that $\tau_1 < \tau_2$ and $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function. The initial datum (u_0, u_1, f_0) belong to a suitable function space, where $Lu = -div(A\nabla u) = -\sum_{i,j=1}^{N}\left(\frac{\partial u}{\partial x_i}a_{i,j}(x)\frac{\partial u}{\partial x_j}\right)$ and $A = (a_{i,j}(x))$ is a matrix that will be specified later. In absence of delay ($\mu_2 = 0$), the problem of existence and energy decay have been extensively studied by several authors (see [6, 18, 22, 44, 45, 57, 63]) and many energy estimates have been derived for arbitrary growing feedbacks. The delay term is often used in process control systems (see [61])

Recently, much attention has been focused on the study of the control of PDEs with time delay effects (see for example [62,85,62]) and the references therein. In [73], the authors showed that a small delay in a boundary control becomes a source of instability. However, sometimes it also can improve the performance of the systems (see [62]). Additional terms are needed to stabilize hyperbolic systems involving input delay terms (see [70,72,79,34]). For instance in 70 the authors studied the wave equation with linear internal damping term with constant delay. They determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (6.1) will be instable if $\mu_2 \geq \mu_1$. The main approach used in [70], is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay are acting in the boundary. We also recall the result by Xu, Yung and Li [79], where the authors proved a result similar to the one in [63] for the one-space dimension by adopting the spectral analysis approach. The case of time varying delay in the wave equation has been studied recently by Nicaise, Valein and Fridman 63 and proved an exponential stability result under the condition

$$\mu_2 < \sqrt{1-d}\mu_1,$$

where the constant d satisfies

$$\tau'(t) \leq d < 1, \quad \forall t > 0.$$

In 60 Nicaise, Pignotti and Valein extended the above result to higher-space dimension and established an exponential decay.

Motivated by the previous works, our purpose in this chapter is to give an energy decay estimate of the solution to problem (6.1) for linear damping and in the presence of distributed delay term. We use for this goal a suitable energy and Lyapunov functionals.

6.2 Preliminary Results

In this section, we present some material for the proof of our results. Let $H(L,\Omega) = \{\nu \in H^1(\Omega)/L\nu \in L^2(\Omega)\}$ be the Hilbert space equipped with the norm

$$\|\nu\|_{H(L,\Omega)} = \left(\|\nu\|^2_{H^1(\Omega)} + \|L\nu\|_2^2\right)^{\frac{1}{2}},$$

where $H^1(\Omega)$ is the real Sobolev space of first order, $\|\nu\|_2^2$ is a L^2-norm and $\langle .,. \rangle$ is the scalar product in L^2 i.e. $\langle \nu, \vartheta \rangle = \int_\Omega \nu(x,t).\vartheta(x,t)dx$. We define in $H^1(\Omega)$ the inner product and norm by

$$\langle \nu, \vartheta \rangle = \sum_{i=1}^{N} \int_\Omega \frac{\partial \nu}{\partial x_i}\frac{\partial \vartheta}{\partial x_i} dx, \qquad \|\nu\|^2 = \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial \nu}{\partial x_i}\right|^2 dx.$$

And we define

$$a(u(t), \upsilon(t)) = \sum_{i,j=1}^{N} a_{i,j}(x)\frac{\partial u(t)}{\partial x_j}\frac{\partial \upsilon(t)}{\partial x_i} dx = \int_\Omega A\nabla u(t).\nabla \upsilon(t) dx,$$

(A_1) : The matrix $A = (a_{i,j}(x))$, where $a_{i,j} \in C^1(\overline{\Omega})$, is symmetric and there exists a constant $a_{01} > 0$ such that for all $x \in \overline{\Omega}$ and $\delta = (\delta_1, \delta_2, \ldots.., \delta_N) \in \mathbb{R}^N$, we have

$$\sum_{i,j=1}^{N} a_{i,j}(x)\delta_j\delta_i \geq a_{01}|\delta|^2, \tag{6.2}$$

And

$$a_{11} = \max_{1\leq j\leq n}\left(\sum_{i=1}^{N} \|a_{i,j}\|_\infty^2\right)$$

(A_2) : $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function satisfying

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds = l > 0,$$

and there exists a non-increasing differentiable function : $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g'(t) \leq -\sigma(t)g(t)$. We now state some Lemmas needed later.

Lemma 6.2.1 (Sobolev-Poincaré's inequality) *Let q be a number with $2 \leq q < +\infty$ $(n = 1, 2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$. Then there exists a constant $c_* = c_*(\Omega, q)$ such that*

$$\|u\|_q \leq c_* \|\nabla u\|_2 \quad \textit{for} \quad u \in H_0^1(\Omega).$$

Like in [19] we introduce the auxiliary unknown

$$z(x,\rho,s,t) = u_t(x, t-\rho s), \quad (x,\rho,s,t) \in \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty). \tag{6.3}$$

Then, we have

$$sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0 \quad \text{in} \quad \Omega \times (0,1) \times (\tau_1,\tau_2) \times]0,+\infty[,$$

Therefore, problem (6.1) is equivalent to

$$\begin{cases} (|u_t|^{\gamma-2}u_t)_t - Lu - \displaystyle\int_0^t g(t-s)Lu(s)ds + \mu_1 u_t & \\ + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)ds = 0, & \text{in } \Omega \times]0,+\infty[, \\ sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0 & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2) \times]0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu \upsilon = 0, & \text{on } \Gamma \times]0,+\infty[, \\ u = \nu = 0, & \text{in } \Gamma \times]0,+\infty[, \\ z(x,\rho,s,0) = f_0(x,\rho s), & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2). \end{cases} \tag{6.4}$$

Let ξ be a positive constant such that

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds + \frac{\xi(\tau_2 - \tau_1)}{2} < \mu_1. \tag{6.5}$$

By combining the arguments of [7,6,73], we recall the existence result in [10].

Theorem 6.2.1 *Let $u_0 \in H_0^1(\Omega) \cap H^2(\Omega), u_1 \in H_0^1(\Omega)$ and $f_0 \in (L^2(\Omega \times (0,1) \times (\tau_1, \tau_2))$ such that the compatibility condition $f_0(.,0) = u_1$ is fulfilled. Assume that the hypotheses $(A_1) - (A_2)$ hold. Then there exists a unique weak solution u of* (6.4) *such that*

$$u(t) \in C\left([0, +\infty]; H_0^1(\Omega)\right) \cap C^1\left([0, +\infty]; L^2(\Omega)\right).$$

Now, we define the energy associated to the solution of the problem (6.4) by

$$\begin{aligned} E(t) = & \frac{\gamma - 1}{\gamma}\|u_t(t)\|_\gamma^\gamma + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) a(u(t), u(t)) + (g \circ u)(t) \\ & + \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(|\mu_2(s)| + \xi) z^2(x, \rho, s, t) ds d\rho dx. \end{aligned} \tag{6.6}$$

Lemma 6.2.2 *Let (u, z) be a solution of the problem (6.4). Then, the energy functional defined by (6.6) satisfies*

$$\begin{aligned} E'(t) \leq & -\left[\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds - \frac{\xi(\tau_2 - \tau_1)}{2}\right] \int_\Omega u_t^2 dx - m \int_\Omega \int_{\tau_1}^{\tau_2} z^2(x, 1, s, t) ds dx \\ & + \frac{1}{2}(g' o u)(t) - \frac{1}{2} g(t) a(u(t), u(t)) \leq 0. \end{aligned} \tag{6.7}$$

Proof. Multiplying the first equation in (6.4) by u_t, integrating over Ω and using integration by parts, we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left(\|u_t\|_\gamma^\gamma + a(u(t), u(t))\right) + \mu_1 \|u_t\|_2^2 \, dx + \int_{\tau_1}^{\tau_2} \mu_2(s) \int_\Omega z(x, 1, s, t) u_t(x, t) ds dx \\ & - \int_0^t g(t - s) \int_\Omega A \nabla \upsilon(s) \nabla \upsilon_t(t) dx ds = 0. \end{aligned} \tag{6.8}$$

Where

$$a(\psi(t), \phi(t)) = \sum_{i,j=1}^N \int_\Omega a_{i,j}(x) \frac{\partial \psi(t)}{\partial x_j} \frac{\partial \phi(t)}{\partial x_i} dx = \int_\Omega A \nabla \psi(t) \phi(t) dx.$$

By using hypothesis (A_1), we verify that the bilinear forms $a(.,.) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ are symmetric and continuous. On the other hand, from (6.2) for $\delta = \nabla \psi$, we get

$$a(\psi(t), \psi(t)) \geq a_{01} \int_\Omega \sum_{i,j=1}^N \left|\frac{\partial \psi}{\partial x_i}\right|^2 dx = a_{01} \|\nabla \psi(t)\|_2^2, \tag{6.9}$$

which implies that $a(.,.)$, are coercive. Note that

$$a(u(t), u_t(t)) = \frac{1}{2}\frac{d}{dt}a(u(t), u(t)). \tag{6.10}$$

Following the same technique as in [73], we obtain

$$\begin{aligned}
&\int_0^t g(t-s)\int_\Omega A\nabla u(s)\nabla u_t(t)dxds = \sum_{i,j=1}^N \int_0^t \int_\Omega g(t-s)a_{i,j}(x)\frac{\partial u(s)}{\partial x_j}\frac{\partial u_t(t)}{\partial x_i}dxds \\
&= \sum_{i,j=1}^N \int_0^t \int_\Omega g(t-s)a_{i,j}(x)\frac{\partial u(t)}{\partial x_i}\frac{\partial u_t(t)}{\partial x_i}dxds \\
&- \sum_{i,j=1}^N \int_0^t \int_\Omega g(t-s)a_{i,j}(x)\left(\frac{\partial u(t)}{\partial x_i} - \frac{\partial u_t(s)}{\partial x_j}\right)\frac{\partial u_t(t)}{\partial x_i}dxds \\
&= \frac{1}{2}\int_0^t g(t-s)\left(\frac{d}{dt}a(u(t), u(t))ds\right) - \frac{1}{2}\int_0^t g(t-s)\left(\frac{d}{dt}a(u(t)-u(s), u(t)-u(s))ds\right) \\
&=\frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t-s)a(u(t), u(t)\right)ds - \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t-s)a(u(t)-u(s), u(t)-u(s))\right)ds \\
&- \frac{1}{2}g(t)a(u(t), u(t)) + \frac{1}{2}\int_0^t g'(t-s)a(u(t)-u(s), u(t)-u(s))ds \\
&= -\frac{1}{2}\frac{d}{dt}(g\circ u)(t) + \frac{1}{2}(g'\circ u)(t) + \frac{1}{2}\frac{d}{dt}\left[a(u(t), u(t))\int_0^t g(s)ds\right] - \frac{1}{2}g(t)a(u(t), u(t)),
\end{aligned} \tag{6.11}$$

Multiplying the second equation in (6.4) by $(\mu_2(s)+\xi)z$ and integrating over $\Omega\times(0,1)\times(\tau_1,\tau_2)$ with respect to ρ, x and s, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\mu_2(s)+\xi)z^2(x,\rho,s,t)dsd\rho dx \\
&= -\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}(\mu_2(s)+\xi)zz_\rho(x,\rho,s,t)dsd\rho dx \\
&= -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}(\mu_2(s)+\xi)\frac{\partial}{\partial\rho}z^2(x,\rho,s,t)dsd\rho dx \\
&= \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}(\mu_2(s)+\xi)(z^2(x,0,s,t)-z^2(x,1,s,t))dsdx \\
&\le \frac{1}{2}\left[\xi(\tau_2-\tau_1)+\int_{\tau_1}^{\tau_2}\mu_2(s)ds\right]\int_\Omega u_t^2dx \\
&- \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}(\mu_2(s)+\xi)z^2(x,1,s,t)dsdx.
\end{aligned} \tag{6.12}$$

From (6.8), (6.11) and (6.13), we obtain

$$\begin{aligned} E'(t) \leq & -\left[\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|ds - \frac{\xi(\tau_2-\tau_1)}{2}\right] \int_\Omega u_t^2 dx - m\int_\Omega \int_{\tau_1}^{\tau_2} z^2(x,1,s,t)dsdx \\ & + \frac{1}{2}(g'ou)(t) - \frac{1}{2}g(t)a(u(t),u(t)). \end{aligned} \tag{6.13}$$

In this way the proof of Lemma 6.2.2 is completed.

6.3 Asymptotic Behavior

In this section, we prove the energy decay result by constructing a suitable Lyapunov functional. We denote by c various positive constants which may be different at different occurrences. Now we define the following functional

$$L(t) = ME(t) + \epsilon\phi(t) + \epsilon\varphi(t) + \epsilon I(t), \tag{6.14}$$

where

$$\phi(t) = \int_\Omega u|u_t|^{\gamma-2}u_t dx, \tag{6.15}$$

$$\varphi(t) = -\int_\Omega |u_t|^{\gamma-2}u_t \int_0^t g(t-s)(u(t)-u(s))dsdx, \tag{6.16}$$

and

$$I(t)=\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}(|k(s)|+\xi)z^2(x,\rho,s,t)dsd\rho dx. \tag{6.17}$$

We need also the following lemmas

Lemma 6.3.1 *Let (u,z) be a solution of problem (6.4), then there exists two positive constants λ_1, λ_2 such that*

$$\lambda_1 E(t)\leq L(t)\leq \lambda_2 E(t), \quad t\geq 0, \tag{6.18}$$

for M sufficiently large .

Proof. By applying the Hölder inequality and Young's inequality and Lemma 6.4 we easily see that

$$\begin{aligned}\phi(t)|\int_\Omega u|u_t|^{\gamma-2}u_t dx| &\leq C_\epsilon\int_\Omega |u|^\gamma dx+\epsilon\int_\Omega |u_t|^l dx\leq C_\epsilon\|\nabla u\|_2^\gamma+\epsilon\|u_t\|_\gamma^\gamma\\ &\leq C_\epsilon E^{\frac{\gamma}{2}}(t)+c\epsilon E(t)\leq \frac{C_\epsilon}{a_{01}}E^{\frac{\gamma-2}{2}}(0)E(t)+c\epsilon E(t),\end{aligned} \tag{6.19}$$

then

$$\begin{aligned}&\phi(t)\\ &\qquad\geq -C_\epsilon E^{\frac{\gamma}{2}}(t)-c\epsilon E(t)\geq -\frac{C_\epsilon}{a_{01}}E^{\frac{\gamma-2}{2}}(0)E(t)-c\epsilon E(t),\end{aligned} \tag{6.20}$$

and

$$\begin{aligned}\varphi(t)&=\left|-\int_\Omega |u_t|^{\gamma-2}u_t\int_0^t g(t-s)(u(t)-u(s))dsdx\right|\\ &\leq \frac{1}{2}\|u_t\|_\gamma^\gamma+\frac{1}{2}\int_\Omega\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^2dx\\ &\leq \frac{1}{2}\left(\|u_t\|_\gamma^\gamma+(1-l)c_s^2\int_0^t g(t-s)a(u(t)-u(s),u(t)-u(s))ds\right)\\ &\leq \frac{1}{2}\left(\|u_t\|_\gamma^\gamma+(1-l)\left(\frac{\beta E(0)}{l}\right)c_s^2(gou)(t)\right),\end{aligned} \tag{6.21}$$

it follows from (6.18) that $\forall c>0$, we have

$$\begin{aligned}|I(t)|&=\left|\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}(|k(s)|+\xi)z^2(x,\rho,s,t)dsd\rho dx\right|\\ &\leq c\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(|k(s)|+\xi)z^2(x,\rho,s,t)dsd\rho dx.\end{aligned} \tag{6.22}$$

Hence, combining (6.20)-(6.22). This yields

$$|L(t) - ME(t)| = \epsilon\phi(t) + \varphi(t) + \epsilon I(t) \leq \frac{C_\epsilon}{a_{01}} E^{\frac{\gamma-2}{2}}(0)E(t) + c\epsilon E(t)\epsilon \|u_t\|_\gamma^\gamma$$
$$+ \epsilon(1-l)\left(\frac{\beta E(0)}{l}\right) c_s^2(gou)(t) + c\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(|k(s)| + \xi)z^2(x,\rho,s,t)dsd\rho dx. \tag{6.23}$$

Finally, we get

$$|L(t) - ME(t)| \leq c_5 E(t), \tag{6.24}$$

where $c_5 = max(c_1, c_2, c_3, c_4)$. Thus, from the definition of $E(t)$ and selecting M sufficiently large, we find

$$\beta_2 E(t) \leq L(t) \leq \beta_1 E(t), \tag{6.25}$$

such that $\beta_1 = (M - \epsilon c_5)$, $\beta_2 = (M + \epsilon c_5)$. This completes the proof.

Lemma 6.3.2 *Let (u, z) be the solution of (6.4), then it holds for any $\forall\delta > 0$*

$$\frac{d}{dt}\phi(t) \leq \left\{\frac{\mu a_{11}}{a_{01}} + 2\mu_1\delta c_s^2 - l\right\} a(u(t), u(t)) + \frac{N}{4a_{01}\mu}(1-l)(gou)(t)$$
$$+ \frac{\mu_1}{4\delta}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\|z(x,1,s,t)\|_2^2 ds + \|u_t\|_\gamma^\gamma + \frac{\mu_1}{4\delta}\|u_t\|_2^2. \tag{6.26}$$

Proof. We take the derivative of $\phi(t)$. It follows from (6.17) that

$$\frac{d}{dt}\phi(t) = \int_\Omega (|u_t|^{\gamma-2}u_t)_t u dx + \|u_t\|_\gamma^\gamma, \tag{6.27}$$

using the problem (6.4), then we have

$$\frac{d}{dt}\phi(t) = \|u_t\|_\gamma^\gamma - a(u(t), u(t)) + \int_\Omega \int_0^t g(t-s)A\nabla u(s)\nabla u(t)dsdx$$
$$-\mu_1 \int_\Omega u_t u dx - \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s)z(x,1,s,t)\ u\ dsdx, \tag{6.28}$$

following the same idea in [10], yields

$$
\begin{aligned}
&\int_\Omega A\int_0^t g(t-s)(\nabla u(t)\nabla u(s)dsdx\\
&=\sum_{i,j=1}^N\int_0^t g(t-s)\int_\Omega a_{ij}(x)\frac{\partial u(t)}{\partial x_j}\left(\frac{\partial u(s)}{\partial x_i}-\frac{\partial u(t)}{\partial x_i}+\frac{\partial u(t)}{\partial x_i}\right)dxds\\
&=\sum_{i,j=1}^N\int_\Omega\int_0^t g(t-s)a_{ij}(x)\frac{\partial u(t)}{\partial x_j}\frac{\partial u(t)}{\partial x_i}dsdx\\
&+\sum_{i,j=1}^N\int_\Omega\int_0^t \left(g(t-s)a_{ij}(x)\frac{\partial u(t)}{\partial x_j}\left(\frac{\partial u(s)}{\partial x_i}-\frac{\partial u(t)}{\partial x_i}\right)\right)dsdx \qquad (6.29)\\
&\leq (1-l)a(u(t),u(t))+\mu\sum_{i,j=1}^N\int_\Omega\left(a_{ij}(x)\frac{\partial u(s)}{\partial x_j}ds\right)^2dx\\
&+\frac{1}{4\mu}\sum_{i,j=1}^N\int_\Omega\left(\int_0^t g(t-s)\left(\frac{\partial u(s)}{\partial x_i}-\frac{\partial u(t)}{\partial x_i}\right)ds\right)^2dx\\
&\leq\left[(1-l)+\frac{\mu a_{11}}{a_{01}}\right]a(u(t),u(t))+\frac{N}{4a_{01}\mu}(1-l)(gou)(t),
\end{aligned}
$$

for the forth and fifth term in (6.29), we use Holder and Young's inequalities, then for any $\delta>0$, we get

$$\left|\int_\Omega u_t u dx\right|\leq \delta c_s^2 a(u(t),u(t))+\frac{1}{4\delta}\|u_t\|_2^2, \qquad (6.30)$$

and

$$
\begin{aligned}
&\left|\int_\Omega\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)udx\right|\leq\delta c_s^2 a(u(t),u(t))\\
&+\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx,
\end{aligned} \qquad (6.31)
$$

hence

$$\left|\int_{\tau_1}^{\tau_2}\int_\Omega\mu_2(s)z(x,1,s,t)udxds\right|\leq\delta c_s^2 a(u(t),u(t))+\frac{\mu_1}{4\delta}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\|z(x,1,s,t)\|_2^2ds. \quad (6.32)$$

Using (6.31)-(6.33), then (6.30) becomes

$$
\begin{aligned}
&\frac{d}{dt}\phi(t) \leq \left\{\frac{\mu a_{11}}{a_{01}} + (\mu_1+\mu_2)\delta c_s^2 - l\right\} a(u(t),u(t)) + \frac{N}{4a_{01}\mu}(1-l)(gou)(t) \\
&+ \frac{\mu_1}{4\delta}\int_{\tau_1}^{\tau_2} |\mu_2(s)| \|z(x,1,s,t)\|_2^2 ds + \|u_t\|_\gamma^\gamma + \frac{\mu_1}{4\delta}\|u_t\|_2^2.
\end{aligned} \tag{6.33}
$$

This completes the proof.

Lemma 6.3.3 *Let (u,z) be the solution of (2.3), then $\varphi(t)$ satisfies for any $\forall\ \delta > 0$*

$$
\begin{aligned}
&\varphi'(t) \leq \left\{\frac{\beta}{a_{01}} + \frac{a_{11}\beta(1-l)^2}{a_{01}}\right\} a(u(t),u(t)) - (g_0-\delta)\|u_t\|_\gamma^\gamma \\
&+ \mu_1\|u_t\|_2^2 + \frac{1}{4\delta}c_s^2\mu_1 \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t) ds dx + \frac{g(0)c_s^2}{4\delta}(-g'ou)(t) \\
&+ \left\{(1-l)\left\{\frac{1}{4a_{01}\beta} + \frac{1}{a_{01}}\left(2\beta a_{11} + \frac{N}{4\beta}\right) + \frac{2\mu_1 c_s^2}{4\delta}\right\}\right\}(gou)(t).
\end{aligned} \tag{6.34}
$$

Proof. Now Taking the derivatives of $\varphi(t)$ and using the problem (6.4), we obtain

$$
\begin{aligned}
\frac{d\varphi(t)}{dt} &= -\int_\Omega (|u_t|^{\gamma-2}u_t)_t \int_0^t g(t-s)(u(t)-u(s)) ds dx \\
&- \int_\Omega |u_t|^{\gamma-2}u_t \int_0^t g'(t-s)(u(t)-u(s)) ds dx - \left(\int_0^t g(s) ds\right) \int_\Omega u_t^\gamma dx \\
&= \sum_{i,j=1}^N \int_\Omega a_{ij}(x) \frac{\partial u(t)}{\partial x_j} \left(\int_0^t g(t-s)\left(\frac{\partial u(t)}{\partial x_i} - \frac{\partial u(s)}{\partial x_i}\right) ds\right) dx \\
&- \sum_{i,j=1}^N \int_\Omega \left(\int_0^t g(t-s)\frac{\partial u(s)}{\partial x_i} ds\right)\left(\int_0^t g(t-s)\left(\frac{\partial u(t)}{\partial x_i} - \frac{\partial u(s)}{\partial x_i}\right) ds\right) dx \\
&+ \int_\Omega \mu_1 u(t) \int_0^t g(t-s)(u(t)-u(s)) ds dx \\
&+ \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) z_1(x,1,s,t) \int_0^t g(t-s)(u(t)-u(s)) ds ds dx \\
&- \int_\Omega |u_t|^{\gamma-2}u_t \int_0^t g'(t-s)(u(t)-u(s)) ds dx - \left(\int_0^t g(s) ds\right) \int_\Omega u_t^\gamma dx.
\end{aligned} \tag{6.35}
$$

Using Young's inequality and A_2, we infer

$$\begin{aligned}&\sum_{i,j=1}^{N}\int_{\Omega} a_{ij}(x)\frac{\partial u(t)}{\partial x_j}\left(\int_0^t g(t-s)\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u(s)}{\partial x_i}\right)ds\right)dx\\ &\leq \frac{\beta}{a_{01}}a(u(t),u(t))+\frac{(1-l)}{4a_{01}\beta}(gou)(t),\end{aligned} \tag{6.36}$$

Using (6,29)

$$\begin{aligned}&\left|\sum_{i,j=1}^{N}\int_{\Omega}\left(\int_0^t g(t-s)\frac{\partial u(s)}{\partial x_i}ds\right)\left(\int_0^t g(t-s)\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u(s)}{\partial x_i}\right)ds\right)dx\right|\\ &\leq \beta\sum_{i,j=1}^{N}\int_{\Omega}\left(\int_0^t g(t-s)\frac{\partial u(s)}{\partial x_i}ds\right)^2dx\\ &+\frac{1}{4\beta}\sum_{i,j=1}^{N}\int_{\Omega}\left(\int_0^t g(t-s)\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u(s)}{\partial x_i}\right)ds\right)^2dx\\ &\leq \frac{(1-l)}{a_{01}}\left[2\beta a_{11}+\frac{N}{4\beta}\right](g\circ u)(t)+\frac{2a_{11}\beta}{a_{01}}(1-l)^2a(u(t),u(t)).\end{aligned} \tag{6.37}$$

Next we will estimate the right hand side of (6.36). Applying the Hölder inequality and Young's inequality and the assumptions $(A_1)-(A_2)$, we have for any $t_0>0$

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0, \quad \forall t\geq t_0, \tag{6.38}$$

Invoking (6.38), we get the following estimates

$$\begin{aligned}&\int_{\Omega}|u_t|^{\gamma-2}u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx-\left(\int_0^t g(s)ds\right)\int_{\Omega}u_t^{\gamma}dx\\ &\leq \delta\|u_t\|_{\gamma}^{\gamma}+\frac{g(0)c_s^2}{4\delta}(-g'ou)(t)-g_0\|u_t\|_{\gamma}^{\gamma},\end{aligned} \tag{6.39}$$

$$\begin{aligned}&\left|-\int_{\Omega}\mu_1 u_t\int_0^t g(t-s)(u(t)-u(s))dsdx\right|\\ &\leq \mu_1\|u_t\|_2^2+\frac{\mu_1(1-l)c_s^2}{4\delta}(gou)(t),\end{aligned} \tag{6.40}$$

and

$$
\begin{aligned}
&\left| -\int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) \int_0^t g(t-s)(u(t)-u(s)) ds dx \right| \\
&\leq \mu_1 \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t) ds dx + \frac{\mu_1(1-l)c_s^2}{4\delta}(gou)(t).
\end{aligned}
\tag{6.41}
$$

A substitution of (6.36)-(6.41) into (6.35) yields

$$
\begin{aligned}
\varphi'(t) \leq &\left\{ \frac{\beta}{a_{01}} + \frac{a_{11}\beta(1-l)^2}{a_{01}} \right\} a(u(t),u(t)) - (g_0-\delta)\|u_t\|_\gamma^\gamma \\
&+ \mu_1 \|u_t\|_2^2 + \frac{1}{4\delta} c_s^2 \mu_1 \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t) ds dx + \frac{g(0)c_s^2}{4\delta}(-g' ou)(t) \\
&+ \left\{ (1-l) \left\{ \frac{1}{4a_{01}\beta} + \frac{1}{a_{01}} \left(2\beta a_{11} + \frac{N}{4\beta} \right) + \frac{2\mu_1 c_s^2}{4\delta} \right\} \right\} (gou)(t).
\end{aligned}
\tag{6.42}
$$

This completes the proof.

Lemma 6.3.4 *Let I the functional defined by (6.17), then it holds*

$$
\frac{d}{dt} I(t) \leq c \int_\Omega u_t^2 dx - \gamma_0 \int_\Omega \int_{\tau_1}^{\tau_2} s(|\mu_2(s)| + \xi) z^2(x,s,\rho,t) ds dx \tag{6.43}
$$

where τ_0, τ_2 are some positive constants.

Proof. Differentiating (6.17) with respect to t and using the second equation in (6.4), we have

$$
\begin{aligned}
I_1'(t) &= -2 \int_\Omega \int_{\tau_1}^{\tau_2} (|\mu_2(s)| + \xi) \int_0^1 e^{-s\rho} z z_\rho d\rho ds dx \\
&= -\int_\Omega \int_{\tau_1}^{\tau_2} (|\mu_2(s)| + \xi) \int_0^1 e^{-s\rho} z^2 d\rho ds dx \\
&- \int_\Omega \int_{\tau_1}^{\tau_2} (|\mu_2(s)| + \xi) \left[e^{-s} z^2(x,1,s,t) - z^2(x,0,s,t) + s \int_0^1 e^{-s\rho} z^2 d\rho \right] ds dx \\
&\leq c \int_\Omega u_t^2 dx - \gamma_0 \int_\Omega \int_{\tau_1}^{\tau_2} s(|\mu_2(s)| + \xi) z^2(x,s,\rho,t) ds dx.
\end{aligned}
\tag{6.44}
$$

The proof is hence completed.

Theorem 6.3.1 *Assume that assumptions $(A_1)-(A_2)$ are fulfilled. Let $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $f_0 \in \left(L^2(\Omega \times (0,1) \times (\tau_1,\tau_2)\right)$ be given. Then, we have the following decay estimates:*

$$E(t) \leq Ke^{-\alpha \int_{t_0}^{t} \sigma(s)ds}, \quad for \quad t \geq t_0. \tag{6.45}$$

Proof. Since g is positive for any $t_0 > 0$, we have

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 \quad \forall t \geq t_0.$$

Hence we conclude from Lemma 6.2.1, Lemma 6.3.2, Lemma 6.3.3 and Lemma 6.3.4 that

$$\begin{aligned} \frac{dL(t)}{dt} &\leq \epsilon \left\{(1-l)\left\{\frac{N}{4a_{01}\mu} + \frac{1}{4a_{01}\beta} + \frac{1}{a_{01}}\left(2\beta a_{11} + \frac{N}{4\beta}\right) + \frac{2\mu_1 c_s^2}{4\delta}\right\}\right\}(g \circ u)(t) \\ &+ \epsilon\left\{\frac{\mu_1}{4\delta} + \mu_1\right\}\|u_t\|_2^2 - \epsilon\left\{1 - \frac{\mu a_{11}}{a_{01}} - 2\mu_1\delta c_s^2 - \frac{\beta}{a_{01}} - \frac{a_{11}\beta(1-l)^2}{a_{01}}\right\}a(u(t),u(t)) \\ &- \left[mM - \left(\frac{\mu_1}{4\delta} - \frac{c_s^2\mu_1}{4\delta}\mu_2\right)(s) + \gamma_0(s(|\mu_2(s)| + \xi))\right]\int_{\tau_1}^{\tau_2}\|z(x,s,\rho,t)\|_2^2 ds \\ &- 2\epsilon I(t) + \left\{\frac{M}{2} - \frac{\epsilon g(0)c_s^2}{4\delta}\right\}(g' ou)(t) - \epsilon(g_0 - \delta - 1)\|u_t\|_\gamma^\gamma. \end{aligned} \tag{6.46}$$

Choosing carefully ϵ sufficiently small and M sufficiently large such that

$$\left\{(1-l)\left\{\frac{N}{4a_{01}\mu} + \frac{1}{4a_{01}\beta} + \frac{1}{a_{01}}\left(2\beta a_{11} + \frac{N}{4\beta}\right) + \frac{2\mu_1 c_s^2}{4\delta}\right\}\right\} = \eta_0 > 0,$$

$$\left\{1 - \frac{\mu a_{11}}{a_{01}} - 2\mu_1\alpha c_s^2 - \frac{\beta}{a_{01}} - \frac{a_{11}\beta(1-l)^2}{a_{01}}\right\} = \eta_1 > 0,$$

$$(g_0 - \alpha - 1) = \eta_2 > 0, \quad \left[mM - \left(\frac{\mu_1}{4\delta} - \frac{c_s^2\mu_1}{4\delta}\mu_2(s)\right) + \gamma_0(s(|\mu_2(s)| + \xi))\right] = \eta_3 > 0,$$

then (6.46) takes the form

$$\frac{dL(t)}{dt} \leq -\theta\epsilon E(t) + \epsilon\frac{\eta_1}{2}(gou)(t), \tag{6.47}$$

where θ is positive constant. Setting

$$\lambda_1 = \frac{\theta\epsilon}{\beta_2},\ \lambda_2 = \frac{\eta_1\epsilon}{2},\ \lambda_3 = \epsilon c,$$

the last inequality becomes

$$\frac{dL(t)}{dt} \leq -\lambda_1 E(t) + \lambda_2 (gou)(t), \tag{6.48}$$

multiplying (6.48) by $\sigma(t)$, we get

$$\begin{aligned} &\sigma(t)\frac{dL(t)}{dt} \leq -\lambda_1\sigma(t)E(t) + \lambda_2\sigma(t)(gou)(t) \\ &\leq -\lambda_1\sigma(t)E(t) - \lambda_2\sigma(t)(g^{'}ou)(t) \leq -\lambda_1\sigma(t)E(t) - cE^{'}(t). \end{aligned} \tag{6.49}$$

Let

$$H(t) = \sigma(t)L(t) + 2\lambda E(t).$$

At last, we can easily see that $H(t)$ is equivalent to $E(t)$. Now, subtracting and adding $\sigma^{'}(t)F(t)$ in the right hand side of (6.49), using the fact that $\sigma^{'}(t) \leq 0$ and (A_1), then $\forall t \geq t_0$, we obtain

$$\frac{dH(t)}{dt} \leq \sigma^{'}(t)L(t) - \rho_1\sigma(t)E(t) \leq -\rho_1\sigma(t)E(t) \leq -\rho_3\sigma(t)H(t). \tag{6.50}$$

Integrating this over (t_0, t), we conclude that

$$H(t) \leq H(t_0)e^{-\rho_3 \displaystyle\int_{t_0}^{t} \sigma(s)ds} \quad for \quad t \geq t_0. \tag{6.51}$$

Where ρ_3 is a positive constant. Finally, we get

$$E(t) \leq Ke^{-\alpha \int_{t_0}^{t} \sigma(s)ds} \quad for \quad t \geq t_0. \tag{6.52}$$

where α and K are some positive constants. This completes the proof.

Remark 6.3.1 *We illustrate the energy decay rate given by Theorem 6.3.1 through the following examples which are introduced in [55,73].*

1. *If* $g(t) = \frac{a}{(1+t)^\nu}$*, for* $a > 0$ *and* $\nu > 1$*, then* $\sigma(t) = \frac{\nu}{(1+t)}$ *satisfies the condition* (A_2)*. Thus* (6.52) *gives the estimate*

$$E(t) \leq K(1+t)^{-\alpha} \text{ where } \alpha > 0,$$

2. *If* $g(t) = ae^{-b(1+t)^\nu}$*, for* $a, b > 0$ *and* $0 < \nu \leq 1$*, then* $\sigma(t) = b\nu(1+t)^{\nu-1}$ *satisfies the condition* (A_2)*. Thus* (6.52) *gives the estimate*

$$E(t) \leq Ke^{-\alpha(1+t)^\nu},$$

3. *If* $g(t) = ae^{-b\ln^\nu(1+t)}$*, for* $a, b > 0$ *and* $\nu > 1$*, then* $\sigma(t) = \frac{b\nu\ln^{\nu-1}(1+t)}{1+t}$ *satisfies the condition* (A_2)*. Thus* (6.52) *gives the estimate*

$$E(t) \leq Ke^{-\alpha\ln^\nu(1+t)},$$

4. *If* $g(t)\frac{a}{(1+t)\ln\nu(1+t)}$*, for* $a > 0$ *and* $\nu > 1$*, then* $\sigma(t) = \frac{\ln(1+t)+\nu}{(1+t)\ln^\nu(1+t)}$ *satisfies the condition* (A_2)*. Thus* (6.52) *gives the estimate*

$$E(t) \leq K((1+t)\ln^\nu(1+t))^{-\alpha}.$$

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