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To my dear husband for supporting me for everything,
To my parents and my sisters,
To my best children.

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Abstract

The research reported in this thesis deals with the problem of fractional stochastic differential equations. We formulated and proved the set of sufficient conditions for the controllability of semilinear fractional stochastic differential systems with nonlocal conditions in Hilbert spaces. We have discussed the existence and uniqueness result for a neutral stochastic fractional differential equations involving nonlocal initial conditions, we also investigated a class of dynamic control systems described by semilinear fractional stochastic differential equations of order $1 < q < 2$ and Sobolev-type fractional functional stochastic integro-differential systems. The main results are obtained by means of the theory of operators semi-group, fractional calculus, fixed point technique and stochastic analysis theory.

The approximate controllability has also been investigated for this class of fractional stochastic functional differential equations. In all the resuts, a new set of sufficient conditions are derived under the assumption that the corresponding linear system is approximately controllable.
As a consequence, some of the above results are extended to study exact controllability.
In this thesis, adequate examples are provided to illustrate the theory.

Lists of Publications

1. Approximate Controllability of Fractional Neutral Stochastic Evolution Equations with Nonlocal Conditions; Toufik Guendouzi and Souad Farahi
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Introduction

Fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics and economics. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in the fluid dynamic traffic model.

Applications of fractional differential equations to different areas were considered by many author sand some basic results on fractional differential equations have been obtained (see for example [28], [55]. Actually, fractional differential equations are considered as an alternative model to integer differential equations. For more details on fractional calculus theory, one can see the monographs of (Abbas and al [3]), (Kilbas and al.[35]), (Lakshmikantham and al.[39]) and(Miller and Ross [51]). Fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attention (see for example [8], [18], [19], [26], [67]).

Stochastic differential equations have attracted great interest due to their applications in various fields of science and engineering. There are many interesting results on the theory and applications of stochastic differential equations,(see[9], [10], [11]) and the references therein). Chang and al [12] investigated the existence of square-mean almost automorphic mild solutions to non-autonomous stochastic differential equations in Hilbert spaces using semigroup theory and fixed point approach. Fu and Liu [21] discussed the existence and uniqueness of square-mean almost automorphic solutions to some linear and nonlinear stochastic differential equations and in which they studied the asymptotic stability of the unique square-mean almost automorphic solution in the square-meansense. Chang et al [11] studied the existence and uniqueness of quadratic mean almost periodic mild solutions for a class of stochastic differential equations in areal separable Hilbert space by employing the contraction mapping principle and an analytic semigroup of linear operators. However, only few papers deal with the existence result for stochastic fractional systems. Cui and Yan [13] studied the existence of mild solutions for a class of fractional neutral stochastic integro-differential equations with infinite delay in Hilbert spaces by means of Sadovskii's fixed-point theorem. The existence of mild solutions for a class of fractional stochastic differential equations with impulses in Hilbert spaces has been established in [26], [59].

On the other hand, the study of controllability plays a major role in the development of modern mathematical control theory. Mainly, the problem of controllability of dynamical systems is widely used in analysis and design of control system. In this way, fractional-order control systems described by fractional-order deterministic and stochastic differential equations are attracting considerable attention in recent years. The fractional-order models need fractional-order controllers for more effective control of dynamical systems [55].Clearly,the use of fractional-order derivatives and integrals in control theory leads to better results than integer-order approaches. The concept of controllability is an important property of a control system which plays an important role in many control problems such as stabilization of unstable systems by feedback control. Therefore, in recent years controllability problems for various types of linear and nonlinear deterministic and stochastic dynamic systems have been studied in many publications [45], [52], [53] Recently, much attention has been paid to establish sufficient conditions for the controllability of linear and nonlinear fractional dynamical systems by several authors, including a recent monograph [34] and various papers [4], [17], [30], [71]. Debbouche and Baleanu [17] established the exact null controllability result for a class of fractional evolution non local integro-differential control system in Banach space via the implicit evolution system. Sakthivel et al [59] studied the controllability for class of fractional neutral control systems governed by abstract nonlinear fractional neutral differential equations. Controllability of fractional evolution systems of Sobolev-type in Banach spaces has been studied by Ahmed [1] and Fečkan et al [19].

Moreover, in general in infinite-dimensional spaces, the concept of exact controllability is usually too strong [46]. Therefore, the class of fractional dynamical systems must be treated by the weaker concept of controllability, namely approximate controllability.

However, the approximate control theory for fractional equations is not yet sufficiently elaborated. More precisely, there are very few contributions regarding the approximate controllability results for fractional deterministic and stochastic dynamical systems in the literature [24], [45], [59], [63], [64].

Sakthivel and Ren [63] obtained approximate controllability results for nonlinear fractional dynamical systems with state dependent delay by using Schauder's fixed point theorem. Sakthivel [64] investigated the problem of approximate controllability for neutral stochastic fractional integro-differential equation with infinite delay in a Hilbert space by means of Krasnoselskii's fixed point theorem. and the work of Mahmudov [46],we offer to study the approximate controllability of a class of fractional functional stochastic integro-differential systems of Sobolev-type via characteristic solutions operators.

This thesis consists of four chapter , The chapter 1 will give definitions and properties of the needed theory. We briefly recall some basic properties of the Brownian motion, then we discuss integration with respect to this process. At the end of this chapter we will present the definitions and properties of semigroups, fractional calculus and the controllability.

In the second chapter, the approximate controllability of neutral stochastic fractional differential equations involving nonlocal initial conditions is studied. By using Sadovskii's fixed point theorem with stochastic analysis theory, we derive a new set of suffcient conditions for the approximate controllability of semilinear fractional stochastic differential equations with nonlocal conditions under the assumption that the corresponding linear system is approximately controllable. Finally, an application to a fractional partial stochastic differential equation with nonlocal initial condition is provided to illustrate the obtained theory.

In the chapter 3, a class of dynamic control systems described by semilinear fractional stochastic differential equations of order $1 < q < 2$ with nonlocal conditions in Hilbert spaces is considered. Using solution operator theory, fractional calculations, fixed-point technique and methods adopted directly from deterministic control problems, a new set of sufficient conditions for nonlocal approximate controllability of semilinear fractional stochastic dynamic systems is formulated and proved by assuming the associated linear system is approximately controllable. As a remark, the conditions for the exact controllability results are obtained. Finally, an example is provided to illustrate the obtained theory.

Finally, we discuss the approximate controllability of Sobolev-type fractional functional stochastic differential systems in Hilbert spaces. Using Schauder fixed point theorem, stochastic analysis the oryand characteristic solutions operators, we derive a new set of sufficient conditions for the approximate controllability of fractional functional Sobolev-type stochastic integro-differential system under the assumption that the corresponding linear system is approximately controllable. Finally, an example is provided to illustrate the obtained theory.

Chapter 1

General Introduction

A family $(X(t), t \geq 0)$ of $\mathbb{R}^d$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *stochastic process*, this process is adapted if all $X(t)$ are $\mathcal{F}_t$-measurable. Denoting $\mathcal{B}$, the Borel σ-field on $[0, \infty)$. The process X is measurable if $(t, \omega) \mapsto X(t, \omega)$ is a $\mathcal{B} \bigotimes \mathcal{F}$-measurable mapping. We say that $(X(t), t \geq 0)$ is *continuous* if the *trajectories* $t \mapsto X(t, \omega)$ are continuous for all $\omega \in \Omega$.

1.1 Brownian motion

1.1.1 Definition of Brownian Motion

Brownian motion is closely linked to the normal distribution. Recall that a random variable X is normally distributed with mean and variance σ^2 if

$$\mathbb{P}\{X > x\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_x^{\infty} e^{-\frac{(u-\mu)^2}{2\sigma^2}} du, \quad \forall x \in \mathbb{R}$$

Definition 1.1.1 *A real-valued stochastic process $\{W(t) : t > 0\}$ is called a (linear) Brownian motion with start in $x \in \mathbb{R}$ if the following holds:*

- $W(0) = x$
- *The process has independent increments, i.e. for all times $0 \leq t_1 \leq t_2 \leq .... \leq t_n$ the increments $W(t_n) - W(t_{n-1}), W(t_{n-1}) - W(t_{n-2}), ..., W(t_2) - W(t_1)$ are independent random variables.*

- *For all* $t > 0$ *and* $h > 0$*, the increments* $W(t+h) - W(t)$ *are normally distributed with expectation zero and variance* h*.*
- *Continuity of paths, the function* $t \to W(t)$ *is continuous.*

We say that $\{W(t) : t > 0\}$ *is a standard Brownian motion if* $x = 0$*.*

Properties 1.1.1 . *Let* $W(t)_{t\in\mathbb{R}_+}$ *be a standard Brownian motion*

1. *Self-similarity. For any* $T > 0$*,* $\{T^{-1/2}W(Tt)\}$ *is Brownian motion.*
2. *Symmetry.* $\{-W(t), t \geq 0\}$ *is also a Brownian motion.*
3. $\{tW(1/t), t > 0\}$ *is also a Brownian motion.*
4. *If* $W(t)$ *is a Brownian motion on* $[0,1]$*, then* $(t+1)W(1/t+1) - W(1)$ *is a Brownian motion on* $[0,\infty)$*.*

Remark 1.1.1 1. *Notice that the natural filtration of the Brownian motion is* $\mathcal{F}_t^W = \sigma(W_s, s \leq t)$*.*

2. *We can define the Brownian motion without the last condition of continuous paths, because with a stochastic process satisfying the second and the third conditions, by applying the Kolmogorov's continuity theorem, there exists a modification of* $(W_t)_{t\in\mathbb{R}_+}$ *which has continuous paths a.s.*
3. *A Brownian motion is also called a Wiener process since, it is the canoncial process defined on the Wiener space.*

1.1.2 Properties of Brownian motion paths

Almost every sample path $W(t), 0 \leq t \leq T$

1. Is a continuous function of t;
2. Is not monotone in any interval, no matter how small the interval is;
3. Is not differentiable at any point.

1.1.3 Variation and quadratic variation

Definition 1.1.2 *The quadratic variation of Brownian motion* $W(t)$ *is defined as*

$$[W,W](t) = [W,W]([0,t]) = \lim_{n\to\infty} \sum_{i=1}^{n} \left| W_{t_i^n} - W_{t_{i-1}^n} \right|^2,$$

where for each $n, \{t_i^n, 0 \le i \le n\}$ *is a partition of* $[0,t]$, *and the limit is taken over all partitions with* $\delta_n = \max_i(t_{i+1}^n - t_i^n) \to 0$ *as* $n \to \infty$, *and in the sense of convergence in probability.*

Theorem 1.1.1 *Quadratic variation of a Brownian motion over* $[0,t]$ *is* t, *that is, the Brownian motion accumulates quadratic variation at rate one per unit time.*

1.1.4 Martingale property for Brownian motion

Definition 1.1.3 *A stochastic process* $\{X(t), t = 0\}$ *is a martingale if for any* t *it is integrable,* $\mathbb{E}|X(t)| < \infty$, *and for any* $s > 0$

$$\mathbb{E}[X(t+s) \setminus \mathcal{F}_t] = X(t) \quad a.s,$$

where $\mathcal{F}_t$ *is the information about the process up to time* t, *that is,* $\{\mathcal{F}_t\}$ *is a collection of* σ*-algebras such that:*

1. $\mathcal{F}_u \subset \mathcal{F}_t$, *if* $u \le t$.

2. $X(t)$ *is* $\mathcal{F}_t$ *mesurable.*

Theorem 1.1.2 *Let* $W(t)$ *be a Brownian motion. Then*

1. $W(t)$ *is a martingale.*

2. $W^2(t) - t$ *is a martingale.*

3. *for any* $u, e^{uW(t)-\frac{u^2}{2}t}$ *is a martingale.*

1.1.5 Brownian motion and a Gaussian process

Definition 1.1.4 *The increments* $W(t_n)-W(t_{n-1}), W(t_{n-1})-W(t_{n-2}), ..., W(t_2)-W(t_1)$ *are independent and normal distributed, as their linear transform, the random variables* $W(t_1), W(t_2), ..., W(t_m)$ *are jointly normally distributed, that is, the infinite dimensional of Brownian motion is multivariate normal. So Brownian motion is a Gaussian process with mean* 0 *and covariance function*

$$\gamma(t,s) = cov(W(t), W(s)) = \mathbb{E}(W(t)W(s).$$

On the other hand, a continuous mean zero Gaussian process with covariance function $\gamma(t,s) = \min(t,s)$ is a Brownian motion.

1.2 Brownian Motion Calculus

Let $W(t)$ be a Brownian motion, together with a filtration $\mathcal{F}_t, t \geq 0$. Our goal to define stochastic integral

$$\int_0^T X(t)dW(t).$$

The integrand $X(t)$ can also be a stochastic process. The integral should be well defined for at least all non-random continuous functions on $[0,T]$. When the integrand is random, we will assume that it is an adapted stochastic process.

1.2.1 Definition of Itô integral

Itô's Integral for simple integrand

Definition 1.2.1 *[22] The integral* $\int_0^T X(t)dW(t)$ *should have the properties:*

- *If* $X(t) = 1$ *then* $\int_0^T X(t)dW(t) = W(t) - W(0).$
- *If* $X(t) = c$ *in* $(a,b] \in [0,T]$ *and zero otherwise, then*

$$\int_0^T X(t)dW(t) = c(W(b) - W(a)).$$

- *For real* α *and* β

$$\int_0^T (\alpha X(t) + \beta Y(t))dW(t) = \alpha \int_0^T X(t)dW(t) + \beta \int_0^T X(t)dW(t).$$

Definition 1.2.2 *[22] A stochastic process* X *is called simple if it is of the form*

$$X(t) = \xi_0 I_0(t) \sum_{i=1}^{n} \xi_{i-1}(W) \mathbf{1}_{(t_{i-1},t_i]}(t),$$

with ξ *is* $\mathcal{F}_i$*- measurable and a partition* $0 = t_0 < t_1 < ... < t_k = T$ *of* $[0, T]$

For a simple process, the Itô integral $\int_0^T X(t)dW(t)$ *is defined as a*

$$\int_0^T X(t)dW(t) = \sum_{i=1}^{n} \xi_{i-1}(W(t_i) - W(t_{i-1})).$$

It is easy to see that the integral is a Gaussian random variable with mean zero and variance

$$Var(\int_0^T X(t)dW(t)) = \sum_{i=1}^{n} \xi_{i-1}^2 Var(W(t_i) - W(t_{i-1})) = \int_0^T X^2(t)d(t).$$

Definition 1.2.3 *(The Itô integral) Let* $X \in \nu(0, T)$*. Then the Itô integral of* X *(from to T) is defined by*

$$\int_0^T X(t)dW(t) = \lim_{n\to\infty} \int_0^T \phi_n(t)dW(t),$$

where ϕ_n *is a sequence of stochastic process such that*

$$\mathbb{E}[\int_0^T (X(t) - \phi_n(t))^2 dt] \to 0, \quad n \to 0.$$

Properties of Itô integral

1. Linearity. If $X(t)$ and $Y(t)$ are simple processes and α and β are constant, then
$$\int_0^T (\alpha X(t) + \beta Y(t))dW(t) = \alpha \int_0^T X(t)dW(t) + \beta \int_0^T X(t)dW(t).$$

2. For all $[a,b] \subset [0,T]$, $\quad \int_0^T I_{[a,b]}(t)dW(t) = W(b) - W(a)$.

3. $\mathbb{E} \int_0^T X(t)dW(t) = 0$.

4. $\mathbb{E}(\int_0^T X(t)dW(t))^2 = \int_0^T \mathbb{E}X^2(t)d(t)$.

5. Let $I(t) = \int_0^T X(t)dB(t)$. Then I(t) is a continuous martingale.

6. The quadratic variation accumulated up to time t by the Itô integral is
$$[I,I](t) = \int_0^T X^2(u)du.$$

7. $\mathbb{E}(\int_0^T X(t)dW(t)\ \mathcal{F}_s)^2 = \int_0^T \mathbb{E}(X^2(t)\ \mathcal{F}_s)d(t), \quad \forall s < t.$

Proof. The proof can be found in [22].

1.3 The Stochastic Integral in Hilbert Space

We fixe two Hilbert spaces $(\mathcal{H}, \langle , \rangle_{\mathcal{H}})$ and $(\mathcal{U}, \langle , \rangle_{\mathcal{U}})$ .This section is devoted to construction of the stochastic Itô integral
$$\int_0^t \Phi(s)dW(s), \quad t \in [0,T],$$
where $W(t)$ is a Wiener process on $\mathcal{U}$ and Φ is process with values that are linear but not necessarily bounded operators from $\mathcal{U}$ to $\mathcal{H}$.

1.3.1 Wiener Processes and Stochastic Integrals

Definition 1.3.1 *A probability measure μ on $(\mathcal{U}, \mathcal{B}(\mathcal{U}))$ is called Gaussian if all bounded linear mappings.*

$$\begin{array}{rcl} v' : \mathcal{U} & \to & \mathbb{R} \\ u & \mapsto & \langle u, v \rangle_{\mathcal{U}}, \quad u \in \mathcal{U} \end{array}$$

have Gaussian laws, i.e. for all $v \in \mathcal{U}$ there exist $m = m(v) \in \mathbb{R}$ and $\sigma = \sigma(v) > 0$ such that

$$\mu(v' \in A) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_A e^{\frac{-(x-m)^2}{2\sigma^2}} dx, \quad A \in \mathcal{B}(\mathcal{U}).$$

Theorem 1.3.1 *A measure μ on $(\mathcal{U}, \mathcal{B}(\mathcal{U}))$ is Gaussian if and only if*

$$\hat{\mu}(u) = \int e^{i<u,v>_U} \mu d(v) = e^{i<m,u>_U - \frac{1}{2}<Qu,u>_U}, \quad u \in U,$$

where $m \in U$ and $Q \in L(U)$ is nonnegative, symmetric, with finite trace.
In this case μ will be denoted by $N(m, Q)$ where m is called mean and Q is called covariance. The measure μ is uniquely determined by m and Q.

Definition 1.3.2 *Let $W(t)$ be an U-valued random process on T. Then $W(t)$ is a Q-Wiener process if:*

i) $\mathbb{E}W(t) - W(s) = 0$ *for all $s, t \in T$;*

ii) $W(t)$ *is continuous in t;*

iii) $\mathbb{E}[W(t) - W(s)]o[W(t) - W(s)]^2 = (t - s)Q$ *for all s, t in T;*
Where Q is a compact, positive, bounded trace classe operator mapping H into self.

iv) $\mathbb{E}\|W(t) - W(s)\|^2 < \infty$ *for all s, t in T;*

v) $W(t_2 - W(t_1)$ *and* $W(s_2 - W(s_1)$ *are independent for all s_1, s_2, t_1, t_2 in T with $s_1 < s_2 \leq t_1 < t_2$.*

We note that the operator Q has countably many eigenvalues λ_i, that $\lambda_i \geq 0$ for all i, that $Tr(Q) = \sum_{i=0}^{\infty} \lambda_i$, and that there is a complete orthonormal basis e_i of H for which $Qe_i = \lambda e_i$.

Proposition 1.3.1 *[36]*(Representation of the Q-Wiener process) *Let $e_k, k \in \mathbb{N}$, be an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues $\lambda_k, k \in \mathbb{N}$. Then a U-valued stochastic process $W(t)$, $t \in [0,T]$, is a Q-Wiener process if and only if*

$$W(t) = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k(t) e_k, \quad t \in [0,T], \tag{1.1}$$

where $\{\beta_k, k \in \mathbb{N}/\lambda_n > 0\}$, are independent real-valued Brownian motions on a probability space $(\Omega, \mathcal{F}, P)$. The series converges in $L^2(\Omega, \mathcal{F}, P, U))$.

Definition 1.3.3 (Hilbert-Schmidt operator) *Suppose $\mathcal{H}$ is a separable Hilbert space, and that $A \in \mathcal{B}(\mathcal{H})$. We say that A is a Hilbert-Schmidt operator if there exists an orthonormal basis $e_k, k \in \mathbb{N}$ such that*

$$\sum_{k \in \mathbb{N}} \|Ae_n\|^2 < \infty.$$

Definition 1.3.4 *If A and B are two Hilbert-Schmidt operators in a Hilbert space $\mathcal{H}$ and let $e_k, k \in \mathbb{N}$ an orthonormal basis in $\mathcal{H}$, and Tr is the trace of a nonnegative self-adjoint operator. the Hilbert-Schmidt inner product can be defined as*

$$< A, B >_{HS} = Tr(A^* B) = \sum_i < Ae_i, Be_i > .$$

Properties 1.3.1 *The class of Hilbert-Schmidt operators is a Hilbert space of the compact operators with the following properties:*

(i) *$HS(H_1, H_2)$ denote the space of Hilbert Schmidt operators from H_1 to H_2;*

(ii) *Every Hilbert-Schmidt operator $A : H \to H$ is compact;*

(iii) *$\|A\|_{HS} = \|A^*\|_{HS}$ and $A \in H \Leftrightarrow A^* \in HS$.*

Proposition 1.3.2 *Let $B, A \in L_2(\mathcal{U}, \mathcal{H})$ and let $e_k, k \in \mathbb{N}$, be an orthonormal basis of $\mathcal{U}$. If we define*

$$\langle A, B\rangle_{L_2} := \sum_{k\in\mathbb{N}} \langle Ae_k, Be_k\rangle.$$

We obtain that $(L_2(\mathcal{U}, \mathcal{H}), \langle\ ,\ \rangle_{L_2})$ is a separable Hilbert space.

If $f_k, k \in \mathbb{N}$, is an orthonormal basis of H we get that $f_j \otimes e_k := f_j\langle e_k, .\rangle_U, j, k \in \mathbb{N}$, is an orthonormal basis of $L_2(\mathcal{U}, \mathcal{H})$.

Definition 1.3.5 (Cylindrical Wiener Process) *Let $Q \in L(U)$ be nonnegative and symmetric. Remember that in the case that Q is to finite trace the Q-Wiener process has the following representation*

$$W(t) = \sum_{k\in\mathbb{N}} \beta_k(t)e_k.$$

Where $e_k \in \mathbb{N}$ is an orthonormal basis of $Q^{\frac{1}{2}}(\mathcal{U}) = \mathcal{U}_0$ and $\beta_k, k \in \mathbb{N}$ is family of independent real bounded Brownian motions. the series converges in $L^2(\Omega, \mathcal{F}, P, \mathcal{U})$. In the case that Q is no longer of finite trace one loses this convergence. Nevertheless it is possible to define the Wiener process under the following assumptions:
There is a further Hilbert space $(U_1, \langle,\rangle_1)$ such that there exists a Hilbert-Schmidt embedding:

$$J : (\mathcal{U}_0, \langle,\rangle_0) \to (\mathcal{U}_1, \langle,\rangle_1).$$

Then the process given by the following proposition is called Cylindrical Q-Wiener process *in U.*

1.3.2 Stochastic Fubini Theorem

We assume that:

1. $(E, \mathcal{E}, \mu)$ is measure space where μ is a finite.

2. $\Phi : \Omega_T \times E \to L_2^0, (t, \omega, x) \mapsto \Phi(t, \omega, x)$ is $\mathcal{P}_T \otimes \mathcal{E}/\mathcal{B}(L_2^0)$-measurable, thus un particular $\Phi(., ., x)$ is a predictable L_2^0-valued process for all $x \in E$

Theorem 1.3.2 (Stochastic Fubini Theorem) *assume 1, 2 and that*

$$\int \|\Phi(.,.,x)\|_T \mu(dx) = \int \mathbb{E}(\int_0^T \|\Phi(t,.,x)\|^2_{L_2^0} dt)^{\frac{1}{2}} \mu(dx) < \infty$$

Then

$$\int_E [\int_0^T \Phi(t,x)dW(t)]\mu(dx) = \int_0^T [\int_E \Phi(t,x)\mu(dx)]dW(t) \quad P-a.s.$$

Proof. We refer the reader to [15].

1.4 Semigroups

The semigroup theory plays a central role and provide a unified and powerful tool for the study of existence uniqueness solutions of ordinary differential equations in abstract spaces. In recent years, the theory of semigroups of bounded linear operators has been extensively applied to study existence problems in differential equations and controllability problems in control theory. Using the method of semigroups, various types of solutions of evolution equations have been discussed in ([54]).

Definition 1.4.1 *[54] Let X be a Banach space. A one parameter family $T(t), 0 \leq t < \infty$ of bounded linear operators from X into X is a semigroup of a bounded linear operators on X if:*

(i) $T(0) = I$, (I is the identity operator on X);

(ii) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$.

A semigroup of a bounded linear operators $T(t)$ is a uniformly continuous if

$$\lim_{t\to 0} \|T(t) - I\| = 0.$$

The linear operator A defined by

$$\mathcal{D}(A) = \{x \in X, \lim_{t\to 0} \frac{T(t)x - x}{t}\} \quad exists.$$

and

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt} |_{t=0}, \qquad x \in \mathcal{D}(A)$$

is the infinitesimal generator of the semigroup $T(t)$, $\mathcal{D}(A)$ is the domain of A.

Definition 1.4.2 *[54] A semigroup $T(t), 0 \leq t < \infty$ of bounded linear operators from X is a strongly continuous semigroup of a bounded linear operators on X if*

$$\lim_{t \to 0} T(t)x = x, \qquad x \in X.$$

A strongly continuous semigroup of a bounded linear operators on X will be called a semigroup of class $\mathcal{C}_0$ or simply a $\mathcal{C}_0$ semigroup.

1.5 The Mild Solutions

Let us consider $(\mathcal{U}, \|.\|_{\mathcal{U}})$ and $(\mathcal{H}, \|.\|_{\mathcal{H}})$ tow separable Hilbert spaces, we take Q-Wiener process $W(t), t \geq 0$, in a probability space $(\Omega, \mathcal{F}, P)$ with a normal filtration $\mathcal{F}_t$. We fix $T > 0$ and consider the following type of stochastic differential equations in $\mathcal{H}$.

$$\begin{cases} dX(t) &= [AX(t) + F(X(t))]dt + B(X(t))dW(t), \quad t \in [0, T] \\ X(0) &= \xi \end{cases} \tag{1.2}$$

Where

1) $A : D(A) \to \mathcal{H}$ is infinitesimal generator of a C_0-semigroup $(S_t)_{t>0}$ of linear operators on $\mathcal{H}$.
2) $F : \mathcal{H} \longrightarrow \mathcal{H}$ is $\mathcal{B}(\mathcal{H})$-measurable.
3) $B : \mathcal{U} \longrightarrow L(\mathcal{U}, \mathcal{H})$.
4) ξ is $\mathcal{H}$-valued, $\mathcal{F}_0$-measurable random variable.

Definition 1.5.1 *[16]* (Mild Solution) *An $\mathcal{H}$-valued predictable process $(X_t)_{t\in[0,T]}$ is a called a Mild solution of problem (1.2) if*

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(X(s))ds + \int_0^t S_(t-s)B(X(s))dW(s) \quad P - a.s.$$

1.5.1 Existence, Uniqueness of Mild solution

To get the existence of a mild solution of equation (1.2) on $[0,T]$ we make the following usual assumption H_0: (see [16])

1) $A : D(A) \to \mathcal{H}$ is infinitesimal generator of a C_0-semigroup $(S_t)_{t>0}$ of linear operators on $\mathcal{H}$.

2) $F : \mathcal{H} \longrightarrow \mathcal{H}$ is Lipschitz continuous i.e that there exists a constant $C > 0$ such that

$$\|F(x) - F(y)\| \leq C\|x - y\|, \quad \forall x, y \in \mathcal{H}$$

3) $B : \mathcal{H} \longrightarrow L(\mathcal{U}, \mathcal{H})$ is strongly continuous i.e that the mapping $x \to B(x)u$ is continuous from $\mathcal{H}$ to $\mathcal{H}$ for each $u \in \mathcal{U}$.

4) For all $t \in]0,T]$ and $x \in \mathcal{H}$ we have that
$S(t)B(x) \in L_2(\mathcal{U}, \mathcal{H})$

5) there is a square integrable mapping $K : [0,T] \to [0, \infty[$ such that

$$\begin{array}{rcl} \|S(t)(B(x) - B(y))\| & \leq & K(t)\|x - y\| \\ \|S(t)(B(x)\| & \leq & K(t)(1 + \|x\|), \quad t \in]0,T], \quad x, y \in \mathcal{H} \end{array}$$

1.6 The Fixed Point Theorem

Definition 1.6.1 *[43] Let X be a set and let $T : X \to X$ be a function that maps X into itself. (Such a function is often called an operator, a transformation, or a transform on X, and the notation Tx is often used in place of $T(x)$.) A fixed point of T is an element $x \in X$ for which $T(x) = x$.*

1.6.1 The Banach contraction principle

Definition 1.6.2 *Let (X, d) be a metric space. A contraction of X (also called a contraction mapping on X) is a function $f : X \to X$ that satisfies:*

$$\forall x, x', \quad d(f(x'), f(x)) \leq \beta d(x', x),$$

for some real number $\beta < 1$. Such a β is called a contraction modulus of f.

Theorem 1.6.1 (Banach) *[43] Let f be a contraction on a complete metric space X. Then f has a unique fixed point.*

Proof. The proof can be found in [43].

1.6.2 Schauder's fixed point theorem

Theorem 1.6.2 *[37] Every continuous operator that maps a closed convex subset of a Banach space into a compact subset of itself has at least one fixed point.*

1.7 Fractional Calculus

Fractional calculus is a field of mathematical study that deals with the investigation and applications of derivatives and integrals of non-integer orders. The origin of fractional calculus goes back to times when Newton and Leibniz invented differential and integral calculus. The German mathematician Leibniz in a letter to l'Hospital, has suggested the idea of the fractional derivative of order $\frac{1}{2}$. During the period fractional calculus has drawn the attention of many famous mathematicians such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, and Laurent. In the last three decades, fractional calculus has gained the attention of physicists, mathematicians and engineers and notable contributions have been made to both theory and applications of fractional differential equations. Podlubny (1999) addressed the overview of basic theory of derivatives and integrals of non-integer order, fractional differential equations and the methods of their solutions. Existence and uniqueness results for initial value problems for various fractional differential equations and its applications to real world problems were studied in ([35], [55]).

1.8 Fractional Differential equations

Fractional differential equation is concerned with the notion and methods to solve differential equations involving fractional derivatives of the unknown function. It can be also considered as an alternative model to nonlinear differential equations.
The advantages of fractional derivatives becomes evident in modeling mechanical and electrical properties of real materials, description of rheological properties of rocks and

in various other fields. Fractional integrals and derivatives also appear in the theory of control of dynamical systems, when the controlled system and the controller is described by a fractional differential equation.
The researchers have addressed differential equations of fractional order as a study ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods to derive the solutions. Furthermore, researchers also concentrate on the qualitative behaviors such as the existence, controllability and stability of fractional dynamical systems which are the significant current issues. The theory of fractional differential equations has been extensively studied in ([39], [40], [41]). The class of fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid much attention ([3], [67]). The problem of the existence of solutions for various kind of fractional differential equations have been treated in the literature ([74],[72], [71]).

1.9 Basic Fractional Calculus

Recall the following known definitions. For more details see [55].

1.9.1 Gamma function

Definition 1.9.1 *For any complex number z such as $R(z) > 0$, we define the Gamma function*

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt,$$

this integral converges absolutely on half complex plane or the real part is strictly positive.

The gamma function satisfies the identity

$$\Gamma(z+1) = z\Gamma(z),$$

is demonstrated by integrating by parts

$$\Gamma(z+1) = \int_0^{+\infty} e^{-t} t^{z-1} dt = -e^{-t} t^{-z}|_0^{\infty} + z \int_0^{\infty} e^{-t} t^{z-1} dt.$$

If n is an integer, we get closer and closer as

$$\Gamma(z+n) = z(z+1)...(z+n-1)\Gamma(z)$$

as $\Gamma(1) = 1$, this proves that $\Gamma(n+1) = n!$.

1.9.2 The Fractional Integral

Definition 1.9.2 *[55] The fractional integral of order α with the lower limit 0 for a function f is defined as*

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \alpha > 0. \tag{1.3}$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Proposition 1.9.1 *let $f \in C_0([a,b])$. for α, β complexes such as $Re(\alpha) > 0$ et $Re(\beta) > 0$. We have*

$$I_a^{\alpha}(I_a^{\beta} f) = I_a^{\alpha+\beta} f.$$

and for $Re(\alpha) > 0$ we have

$$\frac{d}{dx} I_a^{\alpha} f = I_a^{\alpha} f.$$

1.9.3 The Fractional Derivative

Definition 1.9.3 *[55] The Rieman-Louiville derivatives of order α with the lower limit 0 for a function $f : [0, \infty)$ can be written as*

$$^{L}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, t > 0, n-1 < \alpha < n. \tag{1.4}$$

Definition 1.9.4 *[55] The Caputo derivatives of order α for a function f can be written as*

$$^{C}D^{\alpha} f(t) = {}^{L}D^{\alpha} \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^k(0) \right), \quad t > 0, 0 \leq n-1 < \alpha < n.$$

If $f(t) \in C^n[0,\infty)$ then

$$\begin{aligned} {}^C D^\alpha f(t) &= \tfrac{1}{\Gamma(n-\alpha)} \int_0^t f^{(n)}(s)(t-s)^{n-\alpha-1} ds \\ &= I^{n-\alpha} f^{(n)}(t), \quad t > 0, n-1 < \alpha < n \end{aligned} \tag{1.5}$$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as:

$$L\{{}^C D^\alpha f(t), s\} = s^\alpha \hat{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0); \quad n-1 < \alpha < n.$$

Proposition 1.9.2 *The derivation operator of Riemann-Liouville D_a^α has the following properties:*

(1) D_a^α is a linear operator;

(2) in general $D_a^\alpha o D_a^\beta \neq D_a^\beta o D_a^\alpha$ and also $\neq D_a^{\alpha+\beta}$;

(3) $D_a^\alpha o L_a^\alpha = id$.

If f is an abstract function with values in $\mathcal{H}$, then the integrals appearing in the above definitions are taken in Bochner's sense [50].

1.10 The controllability

Controllability is one of the important fundamental concept in modern mathematical control theory. Notion of controllability is closely related to the theory of minimal realization and optimal control. there are many different notions of controllability, both for linear and nonlinear dynamical systems ([30],[45],[59]).

Many fundamental problems of control theory such as pole-assignment, stabilisability and optimal control may be solved under assumption that system is controllable. Any control system is said to be controllable if every state corresponding to this process can be affected or controlled in respective time by some control signals.

The problem of controllability is to prove the existence of a control function, which drives the solution of the system from its initial state to a final state, where the initial and final states may vary over the entire space. To be brief, if a dynamic system is controllable, all modes of the system can be excited from the input.

The concept of controllability plays a predominant role in both finite and infinite dimensional spaces of systems represented by ordinary differential equations and partial differential equations. So, it is natural to study this concept for dynamical systems represented by fractional differential equations.

Several literature in this direction so far has been concerned with controllability for nonlinear integer order differential equations in infinite dimensional spaces. But the literature related to the controllability of fractional differential equations in infinite dimensional spaces is very limited and we refer the reader to ([62], [71]).

For infinite dimensional dynamical systems it is necessary to distinguish between the notions of approximate and exact controllability.

- **Exact controllability**: The exact controllability property is the possibility to steer the state of the system from any initial data to any target by choosing the control as a function of time in an appropriate way.

- **Approximate controllability**: The approximate controllability property is the possibility to steer the state of the system from any initial data to a state arbitrarily close to a target by choosing a suitable control.

In other words, approximate controllability gives the possibility of steering the system to the states which form the dense subspace in the state space. The approximate controllability is more appropriate for control systems instead of exact controllability.

It should be noted that it is generally difficult to realize the conditions of exact controllability for infinite dimensional systems and thus the approximate controllability becomes very significant. Approximate controllability of the deterministic and stochastic dynamical control systems in infinite dimensional spaces is well-developed using different kind of approaches, and the details can be found in various papers ([4], [45]).

1.11 Nonlocal Problems

The nonlocal condition, which is a generalization of the classical initial condition, was motivated by physical problems (see [74] and references and therein). In recent years, the interest of physicists in nonlocal field theories has been steadily increasing. The main reason for their development is the expectation that the use of these field theories will lead to much more elegant and effective way to treat problems in particle and high energy physics as it has been possible till now with local field theories. Nonlocal effect may occur in space and time. For example, in the time domain, the extension from local to nonlocal description becomes manifest as a memory effect which roughly states that actual behavior of a given object is not only influenced by the actual state of the system but also by events which happened in the past.

Also, in various real world problems, it is possible to require more measurements at some instances in addition to standard initial data and, therefore, the initial conditions changed to nonlocal conditions ([74], [73]). The nonlocal initial problem was initiated by Byszewski (1991), where the existence and uniqueness of mild, strong and classical solutions of the nonlocal Cauchy problem were discussed. In recent years, the study of differential and integro-differential equations in abstract spaces with nonlocal condition has received a great attention. There exist extensive literatures of differential equations with nonlocal conditions. Several authors ([71]) investigated the existence of solutions of fractional differential equations with nonlocal conditions by using semigroups theorems and fixed point techniques. Debbouche and Baleanu (2012)([17]) studied the controllability of fractional evolution nonlocal impulsive quasi linear delay integro-differential systems.

Chapter 2

Approximate Controllability of Fractional Neutral Stochastic Evolution Equations with Nonlocal Conditions

In this chapter[1] we studies the approximate controllability of semilinear neutral fractional stochastic differential equations with nonlocal conditions in the following form

$$\begin{aligned} {}^cD^\alpha[x(t)+f(t,x(t),x(b_1(t)),....,x(b_m(t)))]+Ax(t) &= Bu(t) \\ \sigma(t,x(t),x(a_1(t)),...,x(a_n(t)))\tfrac{dW(t)}{dt}, \quad t\in J=[0,b]. \\ x(0)+g(x)=x_0. \end{aligned} \tag{2.1}$$

Where ${}^cD^\alpha, 0<\alpha<1$ is understood in the Caputo sense; the state variable $x(.)$ takes value in a real separable Hilbert space $\mathcal{H}; -A:\mathcal{D}(A)\subset\mathcal{H}\to\mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup of a bounded linear operator $\{S(t),t\geq 0\}$ on $\mathcal{H}$; the control function $u(.)$ is given in $L^2(J,\mathcal{U}),\mathcal{U}$ is a Hilbert space; B is a bounded linear operator from $\mathcal{U}$ into $\mathcal{H}; \{W(t),t\geq 0\}$ is a given $\mathcal{K}$ valued Wiener process with a finite trace nuclear covariance operator $Q\geq 0$ defined on a filtered complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P}),\mathcal{K}$ is another separable Hilbert space; f,σ and g are given functions to be specified later.

[1]The chapter is based on the paper [26].

In order, we will introduce a suitable mild solutions and establish set of sufficient conditions for the approximate controllability of the fractional control system (2.1). More precisely, by using the constructive control function, we transfer the controllability problem for semilinear systems into a fixed point problem. In particular, the results on controllability of semilinear fractional systems are derived by assuming the corresponding linear system is controllable. Finally, an example is given to illustrate the obtained theory.

2.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\mathcal{F}_t, t \in J = [0, b]$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets).We consider three real separable Hilbert spaces $\mathcal{H}, \mathcal{K}$ and $\mathcal{U}$, and Q-Wiener process on $(\Omega, \mathcal{F}_b, \mathbb{P})$ with the linear bounded covariance operator Q such that $TrQ < \infty$. We assume that there exists a complete orthonormal system $\{e_n\}_{n\geq 0}$ on $\mathcal{H}$, a bounded sequence of non-negative real numbers λ_n such that $Qe_n = \lambda_n e_n, n = 1, 2, ...$ and a sequence $\{\beta_n\}_{n\geq 1}$ of independent Brownian motions such that

$$< W(t), e >= \sum_{n=1}^{\infty} \sqrt{\lambda_n} < e_n, e > \beta_n(t), \quad e \in \mathcal{K}, t \in J := [0, b]$$

and $\mathcal{F}_t = \mathcal{F}_t^W$ where $\mathcal{F}_t^W$ is the sigma algebra generated by $\{W(s) : 0 \leq s \leq t\}$.
We denote by $L(\mathcal{K}, \mathcal{H})$ the set of all linear bounded operators from $\mathcal{K}$ into $\mathcal{H}$, equipped with the usual operator norm $\|.\|$.Let $L_2^0 = L_2(Q^{\frac{1}{2}}\mathcal{K}; \mathcal{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathcal{K}$ into $\mathcal{H}$ with the inner product $< \psi, \pi >_{L_2^0} = Tr[\psi Q \pi^\star]$. We assume without loss of generality that $0 \in \rho(A)$, the resolvent set of A, and the semigroup $S(\cdot)$ is uniformly bounded. This means that there exists a $M \geq 1$ such that $\|S(t)\| \leq M$ for every $t \geq 0$. Then, for $0 < \beta \leq 1$, it is possible to define the fractional power A^β as a closed linear operator on its domain $\mathcal{D}(A^\beta)$ with inverse $A^{-\beta}$.

We will introduce the following basic properties of A^β.

Lemma 2.1.1 *[54]*

1. *$\mathcal{H}_\beta = \mathcal{D}(A^\beta)$ is a Hilbert space with the norm $\|x\|_\beta = \|A^\beta x\|, x \in \mathcal{H}_\beta$.*

2. $S(t) : \mathcal{H} \to \mathcal{H}_\beta$ *for each* $t > 0$ *and* $A^\beta S(t)x = S(t)A^\beta x$ *for each* $x \in \mathcal{H}_\beta$ *and* $t \geq 0$.

3. *For every* $t > 0, A^\beta S(t)$ *is bounded on* $\mathcal{H}$ *and there exists a positive constant* C_β *such that*

$$\|A^\beta S(t)\| \leq \frac{C_\beta}{t^\beta}, t > 0.$$

4. *If* $0 < \gamma < \beta \leq 1$ *then* $\mathcal{D}(A^\beta) \hookrightarrow \mathcal{D}(A^\gamma)$ *and the embedding is compact whenever the resolvent operator of* A *is compact.*

Now, we introduce the concept of mild solution.

Definition 2.1.1 *[74]: A stochastic process* $x : J \to \mathcal{H}$ *is said to be a mild solution of the nonlocal system (2.1) if*

i) $x(t)$ *is measurable and* $\mathcal{F}_t$*-adapted,*

ii) $x(t)$ *is continuous on* J *almost surely and for each* $s \in [0, b)$*, the function* $(t - s)^{\alpha-1}AT_\alpha(t - s)f(s, x(s), x(b_1(s)), ..., x(b_m(s)))$ *is integrable on* $[0, b)$*.*
and the following integral equation is verified:

$$\begin{aligned} x(t) \;=\; & S_\alpha[x_0 + f(0, x(0), x(b_1(0)), ..., x(b_m(s))) - g(x)] \\ & -f(t, x(t), x(b_1(t)), ..., x(b_m(t))) \\ & - \int_0^t (t - s)^{\alpha-1}AT_\alpha(t - s)f(s, x(s), x(b_1(s)), ..., x(b_m(s)))ds \\ & + \int_0^t (t - s)^{\alpha-1}T_\alpha(t - s)Bu(s)ds \\ & + \int_0^t (t - s)^{\alpha-1}AT_\alpha(t - s)\sigma(s, x(s), x(a_1(s)), ..., x(a_m(s)))dW(s) \\ & 0 \leq t \leq b. \end{aligned} \tag{2.2}$$

Where

$$S_\alpha(t) = \int_0^\infty \eta_\alpha(\theta)S(t^\alpha\theta)xd\theta, \quad T_\alpha(t) = \alpha\int_0^\infty \theta\eta_\alpha(\theta)S(t^\alpha\theta)xd\theta,$$

with η_α *is a probability density function defined on* $(0,\infty)$*, that is* $\eta_\alpha(\theta) \geq 0, \theta \in (0,\infty)$ *and* $\int_0^\infty \eta_\alpha(\theta)d\theta = 1$

remark: $\int_0^\infty \theta\eta_\alpha(\theta)d\theta = \frac{1}{\Gamma(1+\alpha)}$.

Lemma 2.1.2 *[74] The operator* $S_\alpha(t)$ *and* $T_\alpha(t)$ *have the following properties:*

i) *For any fixed* $x \in \mathcal{H}, \|S_\alpha(t)x\| \leq M\|x\|, \|T_\alpha(t)x\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\|$;

ii) $\{S_\alpha(t), t \geq 0\}$ *and* $\{T_\alpha(t), t \geq 0\}$ *are strongly continuous;*

iii) *For every* $t > 0, S_\alpha(t)$ *and* $T_\alpha(t)$ *are also compact operators;*

iv) *For any* $x \in \mathcal{H}, \beta, \delta \in (0,1)$*, we have* $AT_\alpha(t)x = A^{1-\beta}T_\alpha(t)A^\beta x$ *and*

$$\|A^\delta T_\alpha(t)\| \leq \frac{\alpha C_\delta \Gamma(2-\delta)}{t^{\alpha\delta}\Gamma(1+\alpha(1-\delta))}, t \in (0,b].$$

At the end of this section, we recall the fixed point theorem of Sadovskii [58]which is used to establish the existence of the mild solution to the fractional control system (2.1).

Theorem 2.1.1 (Sadovskii's fixed-point theorem) . *Let* ϕ *be a condensing operator on a Banach space* X*, that is,* ϕ *is continuous and takes bounded sets into bounded sets, and* $\mu(\phi(N)) \leq \mu(N)$ *for every bounded set* N *of* X *with* $\mu(N) > 0$*. If* $\phi(\Upsilon) \subset \Upsilon$ *for a convex, closed and bounded set*Υ *of* X*, the* ϕ *has a fixed point in* X *(where* $\mu(.)$ *denotes Kuratowski's measure of noncompactness).*

2.2 The Main Results

In this section, we shall formulate and prove sufficient conditions for the approximate controllability of the fractional control system (2.1). We first prove the existence of solutions for fractional control system. Then, we show that under certain assumptions, the approximate controllability of semilinear control system (2.1) is implied by the approximate controllability of the associated linear system.

Definition 2.2.1 *[45] Let $x(b, u)$ be the state value of (2.1) at the terminal time b corresponding to the control u. Introduce the set*

$$\mathcal{R}(b) = \{x(b, u) : u(.) \in L^2(J, \mathcal{U})\}.$$

which is called the reachable set of (2.1) at the terminal time b and its closure in $\mathcal{H}$ is denoted by $\overline{\mathcal{R}(b)}$.The system (2.1) is said to be approximately controllable on the interval J if $\overline{\mathcal{R}(b)} = \mathcal{H}$.

In order to study the approximate controllability for the fractional control system (2.1), we introduce the approximate controllability of its linear part

$$\begin{array}{lcl} D^\alpha x(t) & = & Ax(t) + (Bu)(t), \quad t \in [0, b] \\ x(0) & = & x_0 \end{array} \tag{2.3}$$

Let us now introduce the following operators. Define the operator $\Gamma_0^b : \mathcal{H} \to \mathcal{H}$ associated with (2.3) as

$$\Gamma_0^b = \int_0^b T_\alpha(b - s)BB^*T_\alpha^*(b - s)ds.$$

$$R(k, \Gamma_0^b) = (kI + \Gamma_0^b)^{-1}.$$

where B^* denotes the adjoint of B and $T_\alpha^*(t)$ is the adjoint of $T_\alpha(t)$. It is straightforward that the operator Γ_0^b is a linear bounded operator.

In order to establish the result, we need the following assumptions.
(**A1**) The semigroup $T_\alpha(t)$ is a compact operator.

(**A2**) $f : J \times \mathcal{H}^{m+1} \to \mathcal{H}$ is a continuous function, and there exists a constant $\beta \in (0,1)$ and $M_1, M_2 > 0$ such that the function $A^\beta f$ satisfies the Lipschitz condition:

$$\mathbb{E}\|A^\beta f(s_1, x_0, x_1, ..., x_m) - A^\beta f(s_2, y_0, y_1......, y_m\|^2_{\mathcal{H}} \leq M_1(|s_1 - s_2|^2 + \max_{i=0,1,...,m} \mathbb{E}\|x_i - y_i\|^2_{\mathcal{H}}),$$

for $0 \leq s_1, s_2 \leq b, x_i, y_i \in \mathcal{H}, i = 0, 1, ..., m$ and the inequality

$$\mathbb{E}\|A^\beta f(t, x_0, x_1, ..., x_m)\|^2_{\mathcal{H}} \leq M_2(1 + \max_{i=0,1,...,m} \mathbb{E}\|x_i\|^2_{\mathcal{H}}),$$

holds for $(t, x_0, x_1, ..., x_m) \in J \times \mathcal{H}^{m+1}$.
(**A3**) The function $\sigma : J \times \mathcal{H}^{n+1} \to L_2^0$ satisfies the following conditions:

i) for each $t \in J$ the function $\sigma(t,.) : \mathcal{H}^{n+1} \to L_2^0$ is continuous and for each $(x_0, x_1, ..., x_n) \in \mathcal{H}^{n+1}$ the function $\sigma(., x_0, x_1, ..., x_n) : J \to L_2^0$ is strongly measurable;

ii) for each positive integer q, there exists $\mu_q \in\in L^1(J, \mathbb{R}^+)$ such that

$$\sup_{\|x_0\|^2,...,\|x_n\|^2 \leqslant q} \mathbb{E}\|\sigma(t, x_0, X_1, ..., x_n)\|^2_{L_2^0} \leq \mu_q(t)$$

the function $s \to (t-s)^{2\alpha-2}\mu_q(s) \in L^1([0,t], \mathbb{R}^+)$ and there exists a $\Lambda > 0$ such that.

$$\liminf_{q\to\infty} \frac{\displaystyle\int_0^t (t-s)^{2\alpha-2}\mu_q(s)sd}{q} = \Lambda < \infty, \quad t \in [0,b].$$

(**A4**) $a_i, b_j \in \mathcal{C}(J,J), i = 1, 2, ..., n; j = 1, 2, ..., m; g \in \mathcal{C}(E, \mathcal{H})$ here and hereafter $E = \mathcal{C}(J, H)$, and g satisfies the following conditions:

i) There exists a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\mathbb{E}\|g(x)\|^2_{\mathcal{H}} \leq \psi(\mathbb{E}\|x\|^2_{\mathcal{H}}) \quad \text{and} \quad \liminf_{q\to\infty} \frac{\psi(q)}{q} = \Xi < \infty.$$

ii) g is completely continuous map.

The following lemma is required to define the control function.

Lemma 2.2.1 *[45] For any* $\hat{x}_b \in L^2(\mathcal{F}_b, \mathcal{H})$ *there exists* $\hat{\phi} \in L^2_{\mathcal{F}}(\Omega, L^2(J, L_2^0))$. *such that* $\hat{x}_b = \mathbb{E}\hat{x}_b + \int_o^b \hat{\phi}(s)W(s)$.

Now for any $k > 0$ *and* $\hat{x}_b \in L^2(\mathcal{F}_b, \mathcal{H})$*, we define the control function.*

$$\begin{aligned} u^k(t) &= B^*T^*(b-t)(kI+\Gamma_0^b)^{-1}\Big\{\mathbb{E}\hat{x}_b + \int_o^b \hat{\phi}(s)dW(s) - S_\alpha(b) \\ &\quad \times[x_0 + f(0,x(0)), x(b_1(0)), ..., x(b_m(0)) - g(x)] \\ &\quad + f(b, x(b), x(b_1(b)), ..., x(b_m(b))\Big\} \\ &\quad + B^*T^*(b-t)\int_0^b (kI+\Gamma_0^b)^{-1}(b-s)^{\alpha-1}AT_\alpha(b-s) \\ &\quad \times f(s, x(s)), x(b_1(s)), ..., x(b_m(s)))ds \\ &\quad - B^*T^*(b-t)\int_0^b (kI+\Gamma_0^b)^{-1}(b-s)^{\alpha-1}T_\alpha(b-s) \\ &\quad \times \sigma(s, x(s)), x(a_1(s)), ..., x(a_n(s)))dW(s). \end{aligned}$$

2.2.1 Existence of mild solution

Theorem 2.2.1 *Assume that* (**A1**) - (**A4**) *hold and* $x_0 \in \mathcal{H}$. *Then for each* $k > 0$*, the nonlocal problem (1) has a mild solution on* $[0, b]$ *provided that:*

$$\Big[M^2M_0^2M_2 + M^2\Xi + M_0^2M_2\frac{C_{1-\beta}^2\Gamma^2(1+\beta)b^{2\alpha\beta}}{\Gamma^2(1+\alpha\beta)\beta^2}M_2 + \Big(\frac{\alpha M}{\Gamma(\alpha+1)}\Big)^2\Lambda\Big] \times \Big[7 + \frac{49}{k^2}\Big(\frac{\alpha M M_\beta}{\Gamma(\alpha+1)}\Big)\frac{b^{2\alpha}}{\alpha^2}\Big] < 1. \tag{2.4}$$

and

$$M_1[(M^2+1)M_0^2 + \frac{C_{1-\beta}^2\Gamma^2(1+\beta)b^{2\alpha\beta}}{\Gamma^2(1+\alpha\beta)\beta^2}] < 1. \tag{2.5}$$

where $M_B = \|B\|$ *and* $M_0 = \|A^{-\beta}\|$.

Proof. For the sake of brevity, we rewrite that

$$\begin{array}{rcl} (t, x(t), x(b_1(t)), ..., x(b_m(t))) & = & (t, v(t)) \\ (t, x(t), x(a_1(t)), ..., x(a_n(t))) & = & (t, \tilde{v}(t)). \end{array}$$

Let $x \in E = \mathcal{C}(J, \mathcal{H})$ and. $\tilde{E}$ be the classes of all stochastic processes $x \in E$ which are measurable and $\mathcal{F}_t$ -adapted such that $\|x\|_{\tilde{E}} = \sup_{t \in J}(\mathbb{E}\|x(t)\|^2_{\mathcal{H}})^{\frac{1}{2}} < \infty$
Define the operator Φ on $\tilde{E}$ by

$$\begin{array}{rcl} \Phi x(t) & = & S_\alpha(t)[x_0 + f(0, v(0)) - g(x)] - f(t, v(t)) \\ & & - \displaystyle\int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) f(s, v(s)) ds \\ & & + \displaystyle\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) B u^k(s) ds \\ & & + \displaystyle\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \sigma(s\tilde{v}(s)) dW(s), t \in J. \end{array}$$

For each positive integer q, let $B_q = x \in \tilde{E} : \mathbb{E}\|x(t)\|^2_{\mathcal{H}} \leq q$ the set B_q is clearly a bounded closed convex set in $\tilde{E}$.
From Lemma (4.1.1), Holder's inequality and assumption (A2), we derive that

$$\begin{array}{rl} & \mathbb{E}\Big\| \displaystyle\int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) f(s, v(s)) ds \Big\|^2_{\mathcal{H}} \\ \leq & \mathbb{E}\Big[\displaystyle\int_0^t \|(t-s)^{\alpha-1} A^{1-\beta} T_\alpha(t-s) A^\beta f(s, v(s))\|_{\mathcal{H}} ds \Big]^2 \\ \leq & \frac{\alpha^2 C^2_{1-\beta} \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \mathbb{E}\Big[\displaystyle\int_0^t \|(t-s)^{\alpha\beta-1} A^\beta f(s, v(s)) ds\|_{\mathcal{H}} ds \Big]^2 \\ \leq & \frac{\alpha^2 C^2_{1-\beta} \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \frac{b^{\alpha\beta}}{\alpha\beta} \displaystyle\int_0^t (t-s)^{\alpha\beta-1} \mathbb{E}\|A^\beta f(s, v(s)) ds\|^2_{\mathcal{H}} ds \\ \leq & \frac{\alpha^2 C^2_{1-\beta} \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \frac{b^{\alpha\beta}}{\alpha\beta} \displaystyle\int_0^t (t-s)^{\alpha\beta-1} M_2 \Big(1 + \max_{i=1,..,m} \mathbb{E}\|x_i(s)\|^2_{\mathcal{H}} \Big) ds \\ \leq & \frac{C^2_{1-\beta} \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \frac{b^{2\alpha\beta}}{\beta^2} M_2(1+q). \end{array} \quad (2.6)$$

which deduces that $(t-s)^{\alpha-1}AT_\alpha(t-s)f(s,v(s))$ is integrable on J, by Bochner's theorem [50], so Φ is well defined on B_q.

Similarly, from (**A2**), we derive that

$$\begin{aligned}
& U\mathbb{E}\| \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\sigma(s,\tilde{v}(s))dW(s)\|^2_{\mathcal{H}} \\
\leq & \int_0^t \|(t-s)^{\alpha-1}T_\alpha(t-s)\|^2\mathbb{E}\|\sigma(s,\tilde{v}(s))\|^2_{L_2^0}ds \\
\leq & (\tfrac{\alpha M}{\Gamma(\alpha+1)})^2 \int_0^t (t-s)^{2(\alpha-1)}\mathbb{E}\|\sigma(s,\tilde{v}(s))\|^2_{L_2^0}ds \\
\leq & (\tfrac{\alpha M}{\Gamma(\alpha+1)})^2 \int_0^t (t-s)^{2(\alpha-1)}\mu_q(s)ds.
\end{aligned} \tag{2.7}$$

Further, by using (**A2**) – (**A4**), Lemma (4.1.1) and the estimates (2.6), (2.7), we get

$$\begin{aligned}
\mathbb{E}\|u^k(s)\|^2 \leq & \tfrac{1}{k^2}M_B^2\Big(\tfrac{\alpha M}{\Gamma(\alpha+1)}\Big)^2\Big\{7\Big\|\mathbb{E}\hat{x}_b + \int_0^b \hat{\phi}(s)d\omega(s)\Big\|^2 + 7\mathbb{E}\|S_\alpha(b)x_0\|^2 \\
& +7\mathbb{E}\|S_\alpha(b)f(0,v(0))\|^2 + 6\mathbb{E}\|S_\alpha(b)g(x)\|^2 + 7\mathbb{E}\|f(b,v(b))\|^2 \\
& +7\mathbb{E}\Big\|\int_0^b (b-s)^{\alpha-1}AT_\alpha(b-s)f(s,v(s))ds\Big\|^2_{\mathcal{H}} \\
& +7\mathbb{E}\Big\|\int_0^b (b-s)^{\alpha-1}T_\alpha(b-s)\sigma(s,\tilde{v}(s))dW(s)\Big\|^2_{\mathcal{H}}\Big\} \\
\leq & \tfrac{7}{k^2}\Big(\tfrac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^2\Big\{2\|\mathbb{E}\hat{x}_b\|^2 + 2\int_0^b \mathbb{E}\|\hat{\phi}(s)\|^2ds + M^2\mathbb{E}\|x_0\|^2 \\
& +M^2M_0^2M_2(1+q) + M^2\psi(q) \\
& +M_0^2M_2(1+q) + \tfrac{C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}\tfrac{b^{2\alpha\beta}}{\beta^2}M_2(1+q) \\
& +\Big(\tfrac{\alpha M}{\Gamma(\alpha+1)}\Big)^2\int_0^t (t-s)^{2(\alpha-1)}\mu_q(s)ds\Big\}.
\end{aligned}$$

Now, we have

$$\mathbb{E}\Big\| \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) B_u^k(s) ds \Big\|_{\mathcal{H}}^2 \leq \Big(\frac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^2 \frac{b^{2\alpha}}{\alpha^2} \times \frac{7}{k^2} \Big(\frac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^2 M_C. \quad (2.8)$$

where

$$\begin{aligned} M_C &= \Big\{ 2\|\mathbb{E}\hat{x}_b\|^2 + 2\int_0^b \mathbb{E}\|\hat{\phi}(s)\|^2 ds + M^2 \mathbb{E}\|x_0\|^2 + M^2 M_0^2 M_2(1+q) \\ &\quad + M^2\psi(q) + M_0^2 M_2(1+q) + \tfrac{C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \tfrac{b^{2\alpha\beta}}{\beta^2} M_2(1+q) \\ &\quad \Big(\tfrac{\alpha M}{\Gamma(\alpha+1)}\Big)^2 \int_0^t (t-s)^{2(\alpha-1)} \mu_q(s) ds \Big\}. \end{aligned}$$

We claim that there exists a positive number q such that $\Phi B_q \subset B_q$. If it is not true, then for each positive number q, there is a function $x_q(.) \in B_q$ but $\Phi x_q \in B_q$ that is $\mathbb{E}\|\Phi x_q(t)\|_{\mathcal{H}}^2 > q$ for some $t = t(q) \in J$. However, from assumptions $(\mathbf{A2}) - (\mathbf{A4})$ and equations (2.6),(2.8)
we have:

$$\begin{aligned} q &\leq \mathbb{E}\|\Phi x_q(t)\|_{\mathcal{H}}^2 \\ &\leq 7M^2 \mathbb{E}\|x_0\|^2 + 7M^2 M_0^2 M_2(1+q) + 7M^2 \psi(q) + 7M_0^2 M_2(1+q) \\ &\quad + 7\tfrac{C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \tfrac{b^{2\alpha\beta}}{\beta^2} M_2(1+q) + \Big(\tfrac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^2 \tfrac{b^{2\alpha}}{\alpha^2} \\ &\quad \times \tfrac{7}{k^2} \Big(\tfrac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^2 M_C + 7\Big(\tfrac{\alpha M}{\Gamma(\alpha+1)}\Big)^2 \int_0^t (t-s)^{2(\alpha-1)} \mu_q(s) ds. \end{aligned} \quad (2.9)$$

Dividing both sides of (2.9) by q and taking $q \to \infty$, we obtain that

$$\Big[M^2 M_0^2 M_2 + M^2 \Xi + M_0^2 M_2 + \frac{C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \frac{b^{2\alpha\beta}}{\beta^2} M_2 + (\frac{\alpha M}{\Gamma(\alpha+1)})^2 \Lambda \Big] \times \Big[7 + \frac{49}{k^2} (\frac{\alpha M M_B}{\Gamma(\alpha+1)})^4 \frac{b^{2\alpha}}{\alpha^2} \Big] \geq 1.$$

This contradicts (2.4). Thus for $k > 0$, for some positive number $q, \Phi B_q \subseteq B_q$.

Next, we will show that the operator Φ has a fixed point on B_q, which implies that equation (2.1) has a mild solution.We decompose Φ as $\Phi = \Phi_1 + \Phi_2$, where Φ_1 and Φ_2

are defined on B_q by:

$$(\Phi_1 x)(t) = S_\alpha(t)f(0, v(0)) - f(t, v(t)) - \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) f(s, v(s))ds.$$

and

$$(\Phi_2 x)(t) = S_\alpha(t)[x_0 - g(x)] + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) B u^k(s)ds + \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s)\sigma(s, \tilde{v}(s))dW(s)$$

for $t \in J$ we will show that Φ_1 is a contractive mapping while Φ_2 is compact.

Let $x_1, x_2 \in B_q, t \in J$. By assumption (**A2**) and (2.5), we have:

$$\begin{aligned}
&\mathbb{E}\|(\Phi_1 x_1)(t) - (\Phi_1 x_2)(t)\|^2_{\mathcal{H}} \\
&\leq 3\mathbb{E}\|S_\alpha(t)(f(0, v_1(0)) - f(0, v_2(0)))\|^2_{\mathcal{H}} + 3\mathbb{E}\|(f(t, v_1(t)) - f(t, v_2(t)))\|^2_{\mathcal{H}} \\
&\quad + \mathbb{E}\left\| \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s)(f(s, v_1(s)) - f(s, v_2(s))) \right\|^2_{\mathcal{H}} \\
&\leq (M^2+1) M_0^2 M_1 \sup_{0 \leq s \leq b} \mathbb{E}\|x_1(s) - x_2(s)\|^2_{\mathcal{H}} \\
&\quad + \tfrac{C^2_{1-\beta}\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \tfrac{b^{2\alpha\beta}}{\beta^2} M_1 \sup_{0 \leq s \leq b} \mathbb{E}\|x_1(s) - x_2(s)\|^2_{\mathcal{H}}.
\end{aligned}$$

Thus

$$\|(\Phi_1 x_1)(t) - (\Phi_1 x_2)(t)\|^2_{\tilde{E}} \leq L_0 \|x_1 - x_2\|^2_{\tilde{E}}$$

Where

$$L_0 = M_1 \Big[(M^2+1) M_0^2 + \frac{C^2_{1-\beta}\Gamma^2(1+\beta) b^{2\alpha\beta}}{\beta^2 \Gamma^2(1+\alpha\beta)} \Big] < 1.$$

so Φ_1 is contraction.

Now, to prove that Φ_2 is compact, firstly we prove that Φ_2 is continuous on B_q. Let $x_n \subseteq B_q$, with $x_n \to x$ in B_q and rewrite $u^k(t, x) = u^q(t)$ the control function defined above. Then for each $s \in J, v_n(s) \to v(s), \tilde{v}_n(s) \to \tilde{v}(s)$, and from the assumptions on f, σ we have $f(s, v_n(s)) \to f(s, v(s))$ and $\sigma(s, \tilde{v}_n(s)) \to \sigma(s, \tilde{v}(s))$, as $n \to \infty$. By the dominated convergence theorem, we have

$$\begin{aligned} \|\Phi_2 x_n - \Phi_2 x\|^2_{\tilde{E}} &= \sup_{0\le s\le b} \mathbb{E}\|S_\alpha(t)[g(x) - g(x_n] \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) B[u^k(s,x_n) - u^k(s,x)]ds \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[\sigma(s,\tilde{v}_n(s)) - \sigma(s,\tilde{v}(s))]dW(s)\|^2_{\mathcal{H}} \to 0. \end{aligned}$$

as $n \to \infty$ that is continuous.

Next, we prove that the set $\Phi_2 x : x \in B_q$ is an equicontinuous family of functions on J. Let $0 < \epsilon < t < b$ and $\delta > 0$ be such that $\|T_\alpha(s_1) - T\alpha(s_2) < \epsilon\|$ for every $s_1, s_2 \in J$, with $|s_1 - s_2| < \delta$. For any $x \in B_q$ and $0 < t_1 < t_2 \in J$, we get:

$$\begin{aligned} &\mathbb{E}\|(\Phi_2 x)(t_2) - (\Phi_2 x)(t_1)\|^2_{\mathcal{H}} \\ \le\ & 7\|S_\alpha(t_2) - S_\alpha(t_1)\|^2_{\mathcal{H}} \mathbb{E}\|x_0 - g(x)\|^2_{\mathcal{H}} \\ &+7\mathbb{E}\left\|\int_0^{t_1} (t_1-s)^{\alpha-1}[T_\alpha(t_2-s) - T_\alpha(t_1-s)]Bu^k(s)ds\right\|^2_{\mathcal{H}} \\ &+7\mathbb{E}\left\|\int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]T_\alpha(t_2-s)Bu^k(s)ds\right\|^2_{\mathcal{H}} \\ &+7\mathbb{E}\left\|\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}T_\alpha(t_2-s)Bu^k(s)ds\right\|^2_{\mathcal{H}} \\ &+7\mathbb{E}\left\|\int_0^{t_1} [(t_1-s)^{\alpha-1}[T_\alpha(t_2-s) - T_\alpha(t_1-s)]\sigma(s,\tilde{v}(s))dW(s)\right\|^2_{\mathcal{H}} \\ &+7\mathbb{E}\left\|\int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]T_\alpha(t_2-s)\sigma(s,\tilde{v}(s)))dW(s)\right\|^2_{\mathcal{H}} \\ &+7\mathbb{E}\left\|\int_{t^1}^{t_2} (t_2-s)^{\alpha-1}T_\alpha(t_2-s)\sigma(s,\tilde{v}(s)))dW(s)\right\|^2_{\mathcal{H}}. \end{aligned}$$

Therefore

$$
\begin{aligned}
& \mathbb{E}\|(\Phi_2 x)(t_2) - (\Phi_2 x)(t_1)\|^2_{\mathcal{H}} \\
\leq\ & 7\|S_\alpha(t_2) - S_\alpha(t_1)\|^2_{\mathcal{H}}\mathbb{E}\|x_0 - g(x)\|^2_{\mathcal{H}} \\
& +7\epsilon^2 M_B^2 \tfrac{t_\alpha^1}{\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}\mathbb{E}\|u^k(s)\|^2 ds \\
& +7\Big(\tfrac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^2\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds \\
& \times[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathbb{E}\|u^k(s)\|^2 ds \\
& +7\Big(\tfrac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^2\tfrac{(t_2-t_1)^\alpha}{\alpha}\int_{t_1}^{t_2}[(t_2-s)^{\alpha-1}\mathbb{E}\|u^k(s)\|^2 ds \\
& +7\epsilon^2\tfrac{t_\alpha^1}{\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}\mu_q(s)ds \\
& +7(\tfrac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^2\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds \\
& \times[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mu_q(s)ds \\
& +7\Big(\tfrac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^2\tfrac{(t_2-t_1)^\alpha}{\alpha}\int_{t_1}^{t_2}[(t_2-s)^{\alpha-1}\mu_q(s)ds.
\end{aligned}
$$

Observe that the right-hand side of the above inequality tends to zero independently of $x \in B_q$ as $t_2 \to t_1$, with ϵ sufficiently small since the compactness of $S_\alpha(t)$ for$t > 0$ (see [54]) implies the continuity of $S_\alpha(t)$ in t in the uniform operator topology. Similarly, using the compactness of the set $g(B_q)$ we can prove that the function $\Phi_2 x, x \in B_q$ are equicontinuous at $t = 0$. Hence, the set $\Phi_2 x : x \in B_q$ is an equicontinuous family of functions on J.

Now, we prove that $V(t) = \{(\Phi_2 x)(t) : x \in Bq\}$ is relatively compact in $\mathcal{H}$. Obviously, by assumption $(\mathbf{A4}), \mathbf{V}(\mathbf{0})$ is relatively compact in $\mathcal{H}$.

Let $0 < t \leq b$ be fixed and let ϵ be a given real number such that $0 < \epsilon < t$. We define an operator $\Phi_2^{\epsilon,\delta}$ on B_q by

$$
\begin{aligned}
(\Phi_2^{\epsilon,\delta}x)(t) &= \int_\delta^\infty \eta_\alpha(\theta)S(t^\alpha\theta)[x_0-g(x)]d\theta \\
&\quad +\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)T((t-s)^\alpha\theta)B^k(s)d\theta ds \\
&\quad +\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)T((t-s)^\alpha\theta)\sigma(s,\tilde{v}(s))d\theta dWs \\
&= S(\epsilon^\alpha\delta)\int_\delta^\infty \eta_\alpha(\theta)S(t^\alpha\theta-\epsilon^\alpha\delta)[x_0-g(x)]d\theta \\
&\quad +\alpha S(\epsilon^\alpha\delta)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta-\epsilon^\alpha\delta)B^k(s)d\theta ds \\
&\quad +\alpha S(\epsilon^\alpha\delta)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta-\epsilon^\alpha\delta) \\
&\quad \times\sigma(s,\tilde{v}(s))d\theta dWs.
\end{aligned}
$$

Then from the compactness of $S(\epsilon^\alpha\delta), \epsilon^\alpha\delta > 0$, the set $V^{\epsilon,\delta}(t) = \{(\Phi_2^{\epsilon,\delta}x)(t) : x \in B_q\}$ is relatively compact in $\mathcal{H}$, for every $\epsilon, 0 < \epsilon < t$ and all $\delta > 0$.

Moreover, for every $x \in B_q$, we have

$$
\begin{aligned}
&\mathbb{E}\|(\Phi_2 x)(t)(\Phi_2^{\epsilon,\delta}x)(t)\|^2 \\
\leq\ & 5\mathbb{E}\left\|\int_0^\delta \eta_\alpha(\theta)S(t^\alpha\theta)[x_0-g(x)]d\theta\right\|^2. \\
&+5\alpha^2\mathbb{E}\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)T((t-s)^\alpha\theta)Bu^k(s)d\theta ds\right\|^2 \\
&+5\alpha^2\mathbb{E}\left\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)T((t-s)^\alpha\theta)Bu^k(s)d\theta ds\right\|^2 \\
&+5\alpha^2\mathbb{E}\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)T((t-s)^\alpha\theta)\sigma(s,\tilde{v}(s))d\theta dW(s)\right\|^2 \\
&+5\alpha^2\mathbb{E}\left\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)T((t-s)^\alpha\theta)\sigma(s,\tilde{v}(s))d\theta dW(s)\right\|^2
\end{aligned}
$$

$$\leq \quad 5M^2b^\alpha[\mathbb{E}\|x_0\|^2+\psi(q)](\int_0^\delta \eta_\alpha(\theta)d\theta)^2$$

$$+5\alpha M^2b^\alpha\int_0^t (t-s)^{\alpha-1}\Big[M_B^2\frac{7}{k^2}\Big(\frac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^2 M_C+\mu_q(s)ds\Big]$$

$$\times\Big(\int_0^\delta \eta_\alpha(\theta)d\theta\Big)^2$$

$$+\tfrac{5\alpha M^2\epsilon^\alpha}{\Gamma^2(\alpha+1)}\int_{t-\epsilon}^t (t-s)^{\alpha-1}\Big[M_B^2\frac{7}{k^2}\Big(\frac{\alpha M M_B}{\Gamma(\alpha+1)}\Big)^2 M_C+\mu_q(s)\Big]ds$$

$$\rightarrow \quad 0 \quad as \quad \epsilon,\delta\rightarrow 0^+.$$

Therefore, there are relative compact sets arbitrary close to the set $V(t), t>0$. Hence, the set $V(t), t>0$ is also relatively compact in $\mathcal{H}$.

Thus, by Arzelà-Ascoli theorem Φ_2 is a compact operator. Finally, we conclude that $\Phi = \Phi_1+\Phi_2$ is a condensing map on B_q, and by the fixed-point theorem of Sadovskii there exists a fixed point $x(.)$ for Φ on B_q. Therefore, the nonlocal problem (2.1) has a mild solution, and the proof is completed. □

2.2.2 Approximate Controllability

Now, to prove the approximate controllability result, the following additional assumptions are required.

(**A5**) The linear fractional control system (2.4) is approximately controllable on $[0,b]$.

(**A6**) The functions $f: J\times\mathcal{H}^{m+1}\rightarrow\mathcal{H}$ and $\sigma: J\times\mathcal{H}^{n+1}\rightarrow L_2^0$ are uniformly bounded.

Remark. In view of [45], the assumption **(A5)** is equivalent to $kR(k,\Gamma_0^b)=k(kI+\Gamma_0^b)^{-1}$ as $k\rightarrow 0^+$ in the strong operator topology.

Theorem 2.2.2 *Assume that the assumptions of Theorem (2.2.1) hold and in addition,* ***(A5)*** *and* ***(A6)*** *are satisfied. Then, the fractional control system (2.1) is approximately controllable on* $[0,b]$.

Proof. Let $x_k(.)\in B_q$ be a fixed point of the $\Phi=\Phi_1+\Phi_2$. By Theorem (2.2.1), any fixed point of $\Phi=\Phi_1+\Phi_2$ is a mild solution of (2.1) on $[0,b]$ under the control.

$$
\begin{aligned}
u^k(t) \;=\;& B^*T^*_\alpha(b-t)(kI+\Gamma_0^b)^{-1}\Big\{\mathbb{E}\hat{x}_b+\int_0^b\hat{\phi}(s)dW(s)-S_\alpha(b)[x_0\\
&+f(0,x_k(0),x_k(b_1(0)),....,x_k(b_m(0)))-g(x_k)]\\
&+f(b,x_k(b),x_k(b_1(b)),....,x_k(b_m(b)))\Big\}\\
&+B^*T^*_\alpha(b-t)\int_0^b(kI+\Gamma_s^b)^{-1}(b-s)^{\alpha-1}AT_\alpha(b-s)\\
&f(s,x_k(s),x_k(b_1(s)),....,x_k(b_m(s)))ds\\
&-B^*T^*_\alpha(b-t)\int_0^b(kI+\Gamma_s^b)^{-1}(b-s)^{\alpha-1}T_\alpha(b-s)\\
&\sigma(s,x_k(s),x_k(a_1(s)),....,x_k(a_n(s)))dW(s).
\end{aligned}
$$

By using the stochastic Fubini theorem, it is easy to see that

$$
\begin{aligned}
x_k(b) \;=\;& \hat{x}_b-(kI+\Gamma_0^b)^{-1}\Big[\mathbb{E}\hat{x}_b+\int_0^b\hat{\phi}(s)dW(s)\\
&-S_\alpha(b)[x_0f(0,x_k(0),x_k(b_1(0)),....,x_k(b_m(0)))-g(x_k)]\\
&-f(b,x_k(b),x_k(b_1(b)),....,x_k(b_m(b)))\Big]\\
&-k\int_0^b(kI+\Gamma_s^b)^{-1}(b-s)^{\alpha-1}AT_\alpha(b-s)f(s,x_k(s),x_k(b_1(s)),....,x_k(b_m(s)))ds\\
&+k\int_0^b(kI+\Gamma_s^b)^{-1}(b-s)^{\alpha-1}T_\alpha(b-s)\sigma(s,x_k(s),x_k(a_1(s)),....,x_k(a_n(s)))dW(s).
\end{aligned}
\tag{2.10}
$$

Moreover, by the assumption **(A6)**, there exists $N_1, N_2 > 0$ such that

$$\|A^\beta f(s,x_k(s),x_k(b_1(s)),....,x_k(b_m(s)))\|^2 \leq N_2.$$

and

$$\|\sigma(s,x_k(s),x_k(a_1(s)),....,x_k(a_n(s)))\|^2 \leq N_2$$

and consequently, there is a sequence still denoted by
$\{A^{\beta}f(s,x_k(s),x_k(b_1(s)),....,x_k(b_m(s))),\sigma(s,x_k(s),x_k(a_1(s)),....,x_k(a_n(s)))\}$
weakly converges to say $f(s),\sigma(s)$. Thus, from the equation (2.10), we have

$$
\begin{aligned}
&\mathbb{E}\|x_k(b)-\hat{x}_b\|^2\\
\leq\ & 7\|k(kI+\Gamma_0^b)^{-1}(\mathbb{E}\hat{x}_b - S_\alpha(b)[x_0\\
&+f(0,x_k(0),x_k(b_1(0)),....,x_k(b_m(0)))-g(x_k)])\|^2\\
&+7\mathbb{E}\left(\int_0^b \|k(kI+\Gamma_0^b)^{-1}\hat{\phi}(s)\|^2_{L_2^0}\right)\\
&+7\mathbb{E}\|k(kI+\Gamma_0^b)^{-1}f(b,x_k(b),x_k(b_1(b)),....,x_k(b_m(b)))\|^2\\
&+7\mathbb{E}\Big(\int_0^b (b-s)\alpha-1\|k(kI+\Gamma_0^b)^{-1}AT_\alpha(b-s)\\
&\times[f(s,x_k(s),x_k(b_1(s)),....,x_k(b_m(s)))-f(s)]\|ds\Big)^2\\
&+7\mathbb{E}\left(\int_0^b (b-s)\alpha-1\|k(kI+\Gamma_0^b)^{-1}AT_\alpha(b-s)f(s)\|ds\right)^2\\
&+7\mathbb{E}\Big(\int_0^b (b-s)\alpha-1\|k(kI+\Gamma_0^b)^{-1}T_\alpha(b-s)[\sigma(s,x_k(s),x_k(a_1(s))]\\
&...,x_k(a_n(s))-\sigma(s)]\|^2_{L_2^0}ds\Big)\\
&+7\mathbb{E}\left(\int_0^b (b-s)\alpha-1\|k(kI+\Gamma_0^b)^{-1}T_\alpha(b-s)\sigma(s)\|^2_{L_2^0}ds\right).
\end{aligned}
$$

On the other hand, by Remark (2.2.2), it can be seen that approximate controllability of (2.4) is equivalent to convergence of the operator $kR(k,\Gamma_0^b)$ to zero operator in the strong operator topology, as $k\to 0^+$, and moreover $\|k(kI+\Gamma_s^b)^{-1}\|\leq 1$. Thus, it follows from the Lebesgue dominated convergence theorem, the compactness of $T_\alpha(t)$ and Remark (2.2.2) that $\mathbb{E}\|x_n(b)-\hat{x}_b\|^2\to 0$ as $k\to 0^+$. This proves the approximate controllability of (2.1). Hence the proof is complete.

2.3 An Example

Consider the fractional neutral stochastic partial differential equation of the form

$$\begin{aligned}
{}^cD_t^\alpha\left[x(t,z)+\int_0^\pi (a(z,y))x(t,y)dy\right] &= \tfrac{\partial^2}{\partial z^2}x(t,z)+\mu(t,z)+h(t,x(t,z))\tfrac{d\beta(t)}{dt},\\
0\le t\le b,\quad 0\le z\le\pi, & \\
x(t,0)=x(t,\pi)=0,\quad 0\le t\le b & \\
x(0,z)+\sum_{i=1}^{p}\int_0^\pi k(z,y)x(t_i,y)dy &= x_0(z),\quad 0\le z\le\pi
\end{aligned} \tag{2.11}$$

where p is a positive integer, $b\le\pi, 0<t_0<t_1,....,<t_p<b, x_0(z)\in\mathcal{H}=L^2([0,\pi])$, $K(z,y)\in L^2([0,\pi]\times[0,\pi], {}^cD_t^\alpha$ is the Caputo fractional derivative of order $0<\alpha<1$ and $\beta(t)$ denotes a standard cylindrical Wiener process in $\mathcal{H}$ defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P})$.

To write the above system (2.11) into the abstract form (2.1), we choose the space $\mathcal{U}=\mathcal{H}=\mathcal{K}=L^2([0,\pi])$. Define the operator $A:\mathcal{D}(A)\subset\mathcal{H}\to\mathcal{H}$ by $A_y=y''$ with the domain
$D(A)=\{y\in\mathcal{H}; y,y'\}$ are absolutely continuous, $y''\in\mathcal{H}$ and $y(0)=y(\pi)=0$.
Then $-A$ is the infinitesimal generator of an analytic semigroup $\{S_\alpha(t)\}, t\ge 0$ in $\mathcal{H}$ which is compact. Furthermore, $-A$ has a discrete spectrum with eigenvalues of the form $-n^2, n=0,1,2...$ and corresponding normalized eigenfunctions are given by $x_n(z)=\sqrt{\frac{2}{\pi}}\sin(nz)$. We also use the following properties:

i) if $y\in\mathcal{D}(A)$,then $Ay=\sum_{n=1}^\infty n^2\langle y,x_n\rangle x_n$.

ii) For each $y\in\mathcal{H}, A^{\frac{-1}{2}y}=\sum_{n=1}^\infty \frac{1}{n}\langle y,x_n\rangle x_n$. In particular, $\|A^{-\frac{1}{2}}\|=1$.

iii) The operator$A^{\frac{1}{2}}$ is given by $A^{\frac{1}{2}}y=\sum_{n=1}^\infty n\langle y,x_n\rangle x_n$ on the space $\mathcal{D}(A^{\frac{-1}{2}})=y(.)\in\mathcal{H},\sum_{n=1}^\infty n\langle y,x_n\rangle x_n\in\mathcal{H}$.

Define an infinite-dimensional space $\mathcal{U}$ by $\mathcal{U} = \{u \backslash u = \sum_{n=2}^{\infty} u_n v_n\}$ with $\{\sum_{n=2}^{\infty} \mathcal{U}_n^2 < \infty\}$ for each $v \in \mathcal{H}$. The norm in $\mathcal{U}$ is defined by $\|u\|_{\mathcal{U}}^2 = \sum_{n=2}^{\infty} \mathcal{U}_n^2$. Now, define a continuous linear mapping B from $\mathcal{H}$ into $\mathcal{H}$ as $Bu = 2u_2 v_1 + \sum_{n=2}^{\infty} u_n v_n$ for $u = \sum_{n=2}^{\infty} u_n v_n \in \mathcal{U}$. The system (2.11) can be reformulated as the following nonlocal problem in $\mathcal{H}$:

$$
\begin{aligned}
&{}^cD^{\alpha}[x(t) + f(t, x(t), x(b_1(t)), ..., x(b_m(t)))] + Ax(t) \quad = \quad Bu(t) \\
&\sigma(t, x(t), x(a_1(t)), ..., x(a_n(t)))\tfrac{d\omega(t)}{dt}, t \in J = [0, b], \\
&x(0) + g(x) = x_0.
\end{aligned}
$$

Where $x(t) = x(t, .)$ that is $(x(t))(z) = x(t, z), t \in [0, b], z \in [0, \pi]$; the bounded linear operator $B : \mathcal{U} \to \mathcal{H}$ is defined by $Bu(t)(z) = \mu(t, z), u \in \mathcal{H}$; the function $f : [0, b] \times \mathcal{H} \to \mathcal{H}$ is given by $(f(t, \varphi))(z) = \int_0^{\pi} a(z, y)\varphi(y)dy$ hold for $(\varphi, t) \in [a, b] \times \mathcal{H} \to \mathcal{H}$ and $z \in [0, \pi]$; the function $\sigma : [0, b] \times \mathcal{H} \to L_2^0$ is given by $(\sigma(t, \varphi))(z) = h(t, x(t, z))$ hold for $(\varphi, t) \in [a, b] \times \mathcal{H} \to L_2^0$ and $z \in [0, \pi]; g : E \to \mathcal{H}$ is given by $g(x) = \sum_{i=0}^{p} K_g(x)(t_i)$ where $K_g(x)(z) = \int_0^{\pi} k(z, y)x(y)dy$, for $z \in [0, b]$.

We can take $\alpha = \frac{1}{2}$ and $\sigma(t, x) = \frac{1}{t^{\frac{1}{3}}} \sin x$, then (**A3**) is satisfied. Furthermore, assume that the function $\psi(\|x\|^2) = N_3\|x\|^2$, where $N_3 = (p + 1)[\int_0^{\pi} \int_0^{\pi} k^2(z, y)dydz]^{\frac{1}{2}}$. Then (**A4**) is satisfied (noting that $K_g : \mathcal{H} \to \mathcal{H}$ is completely continuous). Moreover, we assume the following conditions hold:

i) The function $a(z, y); z, y \in [0, \pi]$ is measurable and $\int_0^{\pi} \int_0^{\pi} a^2(z, y)dydz < \infty$.

ii) The function $\partial_z a(z, y)$ is measurable, $a(0, y) = a(\pi, y) = 0$ and let $N_4 = \Big[\int_0^{\pi} \int_0^{\pi} (\partial_z a(z, y))^2\Big]^2 < \infty$.

Therefore, the assumptions **(A1)-(A4)** are all satisfied. Hence, according to Theorem (2.2.1) system (2.11) has a mild solution provided that (2.5) and (2.6) hold. On the other hand, it can be easily seen that the deterministic linear fractional control system corresponding to (2.11) is approximately controllable on $[0, \pi]$ (see [45]). Also, all the conditions of Theorem (2.2.1) are satisfied. Thus by Theorem (2.2.1), fractional stochastic control system (2.11) is approximately controllable on $[0, \pi]$.

Chapter 3

Approximate Controllability of Semilinear Fractional Stochastic Dynamic Systems with Nonlocal Conditions in Hilbert Spaces

In this chapter[1] we studies the approximate controllability for a class of fractional semilinear stochastic dynamic systems of the form

$$
\begin{aligned}
{}^{c}D_t^{\alpha}x(t) + Ax(t) &= f(t, x(t), x(b_1(t)), ..., x(b_m(t))) + Bu(t) \\
&+\sigma(t, x(t), x(a_1(t)), .., x(a_n(t)))\frac{dW(t)}{dt}, \quad t \in J := [0, b] \qquad (3.1) \\
x(0) + g(x) &= x_0 \in \mathcal{H}, \quad x'(0) + h(x) = x_1 \in \mathcal{H}.
\end{aligned}
$$

where $1 < \alpha < 2, {}^{c}D_t^{\alpha}$ denotes the Caputo fractional derivative operator of order α and $x(.)$ takes its values in the separable Hilbert space $\mathcal{H}$. Then, we will introduce a suitable mild solutions and establish set of suficient conditions for the approximate controllability of fractional dynamic system (3.1). More precisely, using some constructive control function, we transfer the controllability problem for semilinear dynamic systems into a fixed-point problem. Further, as a remark, exact controllability of the considered systems is discussed. In particular, the results on controllability of nonlinear fractional dynamic systems are derived by assuming the corresponding linear system is controllable. Finally, an example is given to illustrate the obtained theory.

[1]The chapter is based on the paper [26].

3.1 Preliminaries and Basic Properties

In this chapter, we provide definitions, lemmas and notations necessary to establish our main results. we use the following notations.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\mathcal{F}_t, t \in J = [0, b]$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets).We consider three real separable Hilbert spaces $\mathcal{H}, \mathcal{K}$ and $\mathcal{U}$, and Q-Wiener process on $(\Omega, \mathcal{F}_b, \mathbb{P})$ with the linear bounded covariance operator Q such that $TrQ < \infty$. We assume that there exists a complete orthonormal system $\{e_n\}_{n\geq 0}$ on $\mathcal{H}$, a bounded sequence of non-negative real numbers λ_n such that $Qe_n = \lambda_n e_n, n = 1, 2, ...$ and a sequence $\{\beta_n\}_{n\geq 1}$ of independent Brownian motions such that

$$< W(t), e >= \sum_{n=1}^{\infty} \sqrt{\lambda_n} < e_n, e > \beta_n(t), \quad e \in \mathcal{K}, t \in J := [0, b]$$

and $\mathcal{F}_t = \mathcal{F}_t^W$ where $\mathcal{F}_t^W$ is the sigma algebra generated by $\{W(s) : 0 \leq s \leq t\}$.

Let $L_2^0 = L_2(Q^{\frac{1}{2}}\mathcal{K}; \mathcal{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathcal{K}$ into $\mathcal{H}$ with the inner product $< \psi, \pi >_{L_2^0}= Tr[\psi Q \pi^\star]$.

Let $L^2(\mathcal{F}_b, \mathcal{H})$ be the Banach space of all $\mathcal{F}_b$-measurable square integrable random variables with values in the Hilbert space $\mathcal{H}$. Let $\mathbb{E}(.)$ denote the expectation with respect to the measure $\mathbb{P}$. Let $\mathcal{C}(J; L^2(\mathcal{F}, \mathcal{H}))$ be the Banach space of continuous maps from J into $L^2(\mathcal{F}, \mathcal{H})$ satisfying $\sup_{t\in J} \mathbb{E}\|x(t)\|^2 < \infty$. Let $H_2 = H_2(J, X)$ is a closed subspace of $\mathcal{C}(J; L^2(\mathcal{F}, \mathcal{H}))$ consisting of measurable and $\mathcal{F}_t$-adapted $\mathcal{H}$-valued process $x \in \mathcal{C}(J; L^2(\mathcal{F}, \mathcal{H}))$ endowed with the norm $\|x\|_{H_2} = (\sup_{t\in J} \mathbb{E}\|x(t)\|^2_{\mathcal{H}})^{\frac{1}{2}}$.

Consider the nonlinear fractional stochastic control system (3.1), where $x(.)$ takes its values in the separable Hilbert space $\mathcal{H}; -A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is a sectorial operator of type $(M, \theta, \alpha, \omega)$ on $\mathcal{H}$; the control function $u(.)$ is given in $L^2_{\mathcal{F}}(J,\mathcal{U})$ of admissible control functions, and $\mathcal{U}$ is a Hilbert space.B is a bounded linear operator from $\mathcal{U}$ into $\mathcal{H}; f : J \times \mathcal{H}^{m+1} \to \mathcal{H}, \sigma : J \times \mathcal{H}^{n+1} \to L_2^0$ are continuous, x_0 is $\mathcal{F}_0$-mesurable $\mathcal{H}$-valued random variables independent of $W; a_i, b_j \in \mathcal{C}(J, J); i = 1, 2, ..., n; j = 1, 2, ..., m$
and the nonlocal terms g, h are given functions to be specified later.

Definition 3.1.1 *[67]. Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X. A is said to be sectorial operator of type $(M, \theta, \alpha, \omega)$ if there are constants $\omega \in \mathbb{R}, 0 < \theta < \frac{\pi}{2}, M > 0$, such that the following two conditions are satisfied:*

1. *The α-resolvent of A exists outside the sector $\omega + S_\theta = \{\omega + \lambda^\alpha : \lambda \in \mathbb{C}, \| \arg(-\lambda^\alpha)\| < \theta\}$.*
2. *$\|R(\lambda, A)\| = \|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda-\omega|}, \lambda \in \Psi_{\theta,\omega}$.*

If A is a sectorial operator of type $(M, \theta, \alpha, \omega)$, then it is not difficult to see that A is the infinitesimal generator of a α-resolvent operator family $T_\alpha(t), t \geq 0$ in a Banach space X, where $T_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} R(\lambda^\alpha, A) d\lambda$ (for more details, see [67]).

Lemma 3.1.1 *[67] If f satisfies the uniform Hölder condition with the exponent $\beta \in (0, 1]$ and A is a sectorial operator of type $(M, \theta, \alpha, \omega)$, then the unique solution of the Cauchy problem.*

$$\begin{aligned} D^\alpha x(t) &= Ax(t) + f(t) \quad t \in J := [0, b], \quad 1 < \alpha < 2 \\ x(0) &= x_0 \in X, \quad x\prime(0) = x_1 \in X. \end{aligned} \tag{3.2}$$

is given by

$$x(t) = S_\alpha(t)x_0 + K_\alpha(t)x_1 + \int_0^t T_\alpha(t-s)f(s)ds, \tag{3.3}$$

where

$$S_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda, \quad K_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{\alpha-2} R(\lambda^\alpha, A) d\lambda,$$

$$T_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} R(\lambda^\alpha, A) d\lambda$$

where c is a suitable path such that:$\lambda^\alpha \in \omega + S_\theta, \lambda \in \mathbb{C}$.

Now, we present the defn of mild solutions for the system (3.1) based on the paper [59],[67].

Definition 3.1.2 *A continuous stochastic process $x \in H_2(J, \mathcal{H})$ is called a mild solution of the system(3.1) if for each $u \in L^2_{\mathcal{F}}(J, \mathcal{U})$ and the following conditions hold:*

1. *$x(t)$ is mesurable and $\mathcal{F}_t$-adapted.*

2. *$x(0) + g(x) = x_0$ and $x'(0) + h(x) = x_1$.*

3. *x satisfies the following equation:*

$$\begin{aligned} x(t) &= S_\alpha(t)(x_0 - g(x)) + K_\alpha(t)(x_1 - h(x)) \\ &+ \int_0^t T_\alpha(t-s)[Bu(s) + f(s, x(s), x(b_1(s)), ..., x(b_m(s)))]ds \\ &+ \int_0^t T_\alpha(t-s) f(s, x(s), x(a_1(s)), ..., x(a_n(s)))dW(s). \end{aligned}$$

3.2 The Main Results

In this section, we shall formulate and prove sufficient conditions for the approximate controllability of the system (3.1). To do this, we first prove the existence of a family of solutions to problem (3.1) with control function using a fixed-point theorem. Then, we show that under certain assumptions, the approximate controllability of (3.1) is implied by the approximate controllability of the corresponding linear system. In particular, we formulate and prove conditions for approximate controllability for the semilinear fractional stochastic control systems with nonlocal conditions.

Definition 3.2.1 *Let $x(b, u)$ be the state value of (3.1) at the terminal time b corresponding to the control u. Introduce the set*

$$\mathcal{R}(b) = \{x(b, u) : u(.) \in \mathcal{L}^2(J, U)\}.$$

Which is called the reachable set of (3.1) at the terminal time b and its closure in$\mathcal{H}$ is denoted by $\overline{\mathcal{R}(b)}$.The system (3.1) is said to be approximately controllable on the interval J if $\mathcal{R}(b) = \mathcal{H}$.

In order to study the approximate controllability for the fractional control system (3.1), we introduce the approximate controllability of its linear part;

$$\begin{aligned} D^{\alpha}x(t) &= Ax(t) + (Bu)(t), \quad t \in [0, b] \\ x(0) &= x_0, \quad x'(0) = x_1. \end{aligned} \tag{3.4}$$

Let us now introduce the following operators. Define the operator $\Gamma_0^b : \mathcal{H} \to \mathcal{H}$ associated with (3.4) as

$$\Gamma_0^b = \int_0^b T_{\alpha}(b-s)BB^*T_{\alpha}^*(b-s)ds$$

$$R(k, \Gamma_0^b) = (kI + \Gamma_0^b)^{-1},$$

where B^* denotes the adjoint of B and T_{α}^* is the adjoint of T_{α}. It is straight-forward that the operator Γ_0^b is a linear bounded operator.
To establish the result, we need the following assumptions:

(**A1**) The operators $S_{\alpha}(t), K_{\alpha}(t), T_{\alpha}(t)$ generated by A are compact in $\overline{\mathcal{D}(A)}$ when $t > 0$ such that

$$\sup_{0\leq t\leq b} \|S_{\alpha}(t)\| \leq \widehat{M}, \quad \sup_{0\leq t\leq b} \|K_{\alpha}(t)\| \leq \widehat{M}, \quad \sup_{0\leq t\leq b} \|T_{\alpha}(t)\| \leq \widehat{M}.$$

(**A2**) The functions $f : J \times \mathcal{H}^{m+1} \to \mathcal{H}$ and $\sigma : J \times \mathcal{H}^{n+1} \to L_2^0$ satisfy the following conditions:

i . For each $t \in J$ the functions $f(t, .) : \mathcal{H}^{m+1} \to \mathcal{H}$ and $\sigma(t, .) : \mathcal{H}^{n+1} \to L_2^0$ are continuous and for each $(x_0, x_1, ..., x_m) \in \mathcal{H}^{m+1}$, $(x_0, x_1, ..., x_n) \in \mathcal{H}^{n+1}$ the functions $f(., x_0, x_1, ..., x_m) : J \to \mathcal{H}$ and $\sigma(., x_0, x_1, ..., x_n) : J \to\to L_2^0$ are strongly $\mathcal{F}_t$ measurable;

ii . For each positive integer r, there exists $\mu_r \in L^1(J, \mathbb{R}^+)$ such that

$$\sup_{\|x_0\|^2,...,\|x_m\|^2 \leqslant r} \mathbb{E}\|f(t, x_0, X_1, ..., x_m)\|_{\mathcal{H}}^2 \leq \mu_r(t)$$

$$\sup_{\|x_0\|^2,...,\|x_n\|^2 \leqslant r} \mathbb{E}\|\sigma(t, x_0, X_1, ..., x_n)\|_{L_2^0}^2 \leq \mu_r(t).$$

(**A3**) The functions $g, h : \mathcal{H} \to \overline{\mathcal{D}(A)}$ are continuous and there exist constants β_1, β_2 such that for $x \in \mathcal{H}$.

$$\|g(x)\|^2 \leq \beta_1, \quad \|h(x)\|^2 \leq \beta_2$$

(**A4**) $a_i, b_j \in \mathcal{C}(J, J), i = 1, 2, ..., n; j = 1, 2, ..., m$ and there exist continuous functions $m_f : J \to \mathbb{R}$ and $m_\sigma : J \to \mathbb{R}$ such that:

$$\mathbb{E}\|f(t, x(t), x(b_1(t)), ..., x(b_m(t)))\|^2_{\mathcal{H}} \leq m_f(t)\varphi(\mathbb{E}\|x\|^2_{\mathcal{H}}), \quad \forall t \in J, x \in \mathcal{H},$$

and

$$\mathbb{E}\|\sigma(t, x(t), x(a_1(t)), ..., x(a_n(t)))\|^2_{L_2^0} \leq m_\sigma(t)\varphi(\mathbb{E}\|x\|^2_{\mathcal{H}}), \quad \forall t \in J, x \in \mathcal{H}.$$

Where $\varphi : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function with:

$$\int_0^b m(s)ds \leq \int_\gamma^\infty \frac{ds}{2\varphi(s)}.$$

Where $\gamma = 10\widehat{M^2}[\mathbb{E}\|x_0\|^2_{\mathcal{H}} + \mathbb{E}\|x_1\|^2_{\mathcal{H}} + (\beta_1 + \beta_2)] + 5M_B^4\widehat{M^4}b^2\frac{7}{k^2}M_c$.
and
$m(t) = \max\Big\{m_f(t)\Big(5b\widehat{M^2} + 5M_B^4\hat{M}^4b^2\frac{7}{k^2}b^2\widehat{M^2}m_\sigma(t)(5\widehat{M^2} + 5M_B^4\widehat{M^4}b^2\frac{7}{k^2}b\widehat{M_2})\}$.

(**A5**) The sets $\{\omega_0 - g(\omega), \omega \in B_r\}$ and $\{\omega_1 - h(\omega), \omega \in B_r\}$ where $B_r = \{\omega \in H_2 : \mathbb{E}\|\omega\|^2_{\mathcal{H}} \leq r\}$ are precompact in $\mathcal{H}$.
The following lemma is required to define the control function.

Lemma 3.2.1 *For any $\hat{x}_b \in L^2(\mathcal{F}_b, \mathcal{H})$ there exists $\hat{\phi} \in L^2_{\mathcal{F}}(\Omega, L^2(J, L_2^0))$ such that $\hat{x}_b = \mathbb{E}\hat{x}_b + \int_o^b \hat{\phi}(s)dW(s)$.*

Now for any $k > 0$ and $\hat{x}_b \in L^2(\mathcal{F}_b, \mathcal{H})$, we define the control function.

$$\begin{aligned}
u^k(t) &= B^*T^*(b-t)(kI+\Gamma_0^b)^{-1} \\
&\times\Big\{\mathbb{E}\hat{x}_b + \int_o^b \hat{\phi}(s)dW(s) - S_\alpha(b)[x_0 - g(x)] - K_\alpha(b)[x_1 - h(x)]\Big\} \\
&-B^*T^*(b-t)\int_0^b (kI+\Gamma_0^b)^{-1}T_\alpha(b-s) \\
&\times f(s, x(s)), x(b_1(s)), ..., x(b_m(s)))ds \\
&-B^*T^*(b-t)\int_0^b (kI+\Gamma_0^b)^{-1}T_\alpha(b-s) \\
&\times\sigma(t, x(t)), x(a_1(t)), ..., x(a_n(t)))dW(t).
\end{aligned}$$

3.2.1 The existence of a mild solution

Before stating the theorem on the existence of a mild solution, we recall the following fixed-point theorem which is used to establish the existence of the mild solution to the system (3.1).

Lemma 3.2.2 *(Schaefer's fixed-point theorem). Let Y be a closed convex subset of a Banach space X such that $0 \in Y$. Let $\Phi : Y \to Y$ be a completely continuous map. If the set $U := \{x \in Y : \lambda x = \Phi x\}$ for some $\lambda > 1$ is bounded, then Φ has a fixed point.*

Theorem 3.2.1 *Assume that **(A1)-(A5)** hold, then for each $k > 0$, the fractional stochastic control system (3.1) has at least one mild solution on J.*

Proof. We transform the problem (3.1) into a fixed-point problem. Consider the map $\Phi_k : H_2 \to H_2$ defined by

$$\begin{aligned}
(\Phi_k x)(t) &= S_\alpha(t)[x_0 - g(x)] + K_\alpha(t)[x_1 - h(x)] + \int_o^t T_\alpha(t-s)Bu^k(s)ds \\
&+ \int_o^t T_\alpha(t-s)f(s, x(s)), x(b_1(s)), ..., x(b_m(s)))ds
\end{aligned}$$

$$+\int_o^t T_\alpha(t-s)\sigma(s,x(s)),x(a_1(s)),....,x(a_n(s)))dW(s), t\in J$$

We shall prove that the operator Φ_k is completely continuous operator. For the sake of brevity, we rewrite that

$(t,x(t),x(b_1(t)),....,x(b_m(t))=(t,y(t)), \quad (t,x(t),x(a_1(t)),....,x(a_n(t))=(t,z(t))$

Let $B_r=\{x\in H_2:\mathbb{E}\|x(t)\|^2_{\mathcal{H}}\le r\}$ for some $r\le 1$. We first show that Φ_k maps B_r into an equicontinuous family. Let $x\in B_r, t_1,t_2\in J$ and $\epsilon>0$. Then, if $0<\epsilon<t_1<t_2\le b$.

$$\begin{aligned}
&\mathbb{E}\|(\Phi_k x)(t_1)-(\Phi_k x)(t_2)\|^2_{\mathcal{H}}\\
\le\ & 10\|S_\alpha(t_1)-S_\alpha(t_2)\|^2(\mathbb{E}\|x_0\|^2_{\mathcal{H}}+\mathbb{E}\|g(x)\|^2_{\mathcal{H}})\\
&+10\|K_\alpha(t_1)-K_\alpha(t_2)\|^2(\mathbb{E}\|x_1\|^2_{\mathcal{H}}+\mathbb{E}\|h(x)\|^2_{\mathcal{H}})\\
&+15\mathbb{E}\|\int_o^{t_1-\epsilon}[T_\alpha(t_1-\tau)-T_\alpha(t_2-\tau)]f(\tau,y(\tau))d\tau\|^2_{\mathcal{H}}\\
&+15\mathbb{E}\|\int_{t_1}^{t_2}T_\alpha(t_2-\tau)f(\tau,y(\tau))d\tau\|^2_{\mathcal{H}}+15\mathbb{E}\|\int_{t_1-\epsilon}^{t_1}[T_\alpha(t_1-\tau)\\
&-T_\alpha(t_2-\tau)f(\tau,y(\tau))d\tau\|^2_{\mathcal{H}}\\
&-15\mathbb{E}\|\int_o^{t_1-\epsilon}[T_\alpha(t_1-\tau)-T_\alpha(t_2-\tau)]Bu^k d\tau\|^2_{\mathcal{H}}\\
&+15\mathbb{E}\|\int_{t_1}^{t_2}T_\alpha(t_2-\tau)Bu^k d\tau\|^2_{\mathcal{H}}\\
&+15\mathbb{E}\|\int_{t_1-\epsilon}^{t_1}[T_\alpha(t_1-\tau)-T_\alpha(t_2-\tau)]Bu^k d\tau\|^2_{\mathcal{H}}\\
&+15\mathbb{E}\|\int_o^{t_1-\epsilon}[T_\alpha(t_1-\tau)-T_\alpha(t_2-\tau)]\sigma(\tau,z(\tau))dW(\tau)\|^2_{\mathcal{H}}\\
&+15\mathbb{E}\|\int_{t_1}^{t_2}T_\alpha(t_2-\tau)\sigma(\tau,z(\tau))dW(\tau)\|^2_{\mathcal{H}}\\
&+15\mathbb{E}\|\int_{t_1-\epsilon}^{t_1}[T_\alpha(t_1-\tau)-T_\alpha(t_2-\tau)]\sigma(\tau,z(\tau))dW(\tau)\|^2_{\mathcal{H}}.
\end{aligned}$$

We have

$$
\begin{aligned}
&\mathbb{E}\|u^k(s)\|^2 \\
\leq\ & \tfrac{1}{k^2}M_B^2\widehat{M^2}\Big[7\|\mathbb{E}\hat{x}_b + \int_0^b \hat{\phi}(s)dW(s)\|^2 + 7\mathbb{E}\|S_\alpha(b)x_0\|^2 + 7\mathbb{E}\|S_\alpha(b)g(x)\|^2 \\
& +7\mathbb{E}\|K_\alpha(b)x_1\|^2 + 7\mathbb{E}\|K_\alpha(b)h(x)\|^2 + 7\mathbb{E}\|\int_o^b T_\alpha(b-s)f(s,y(s))ds\|^2 \\
& +7\mathbb{E}\|\int_o^b T_\alpha(b-s)\sigma(s,z(s))dW(s)\|^2\Big]. \\
\leq\ & \tfrac{7}{k^2}M_B^2\widehat{M^2}\Big[2\|\mathbb{E}\hat{x}_b\|^2 + 2\int_0^b \mathbb{E}\|\hat{\phi}(s)\|^2 ds + 2\widehat{M^2}r + \widehat{M^2}(\beta_1+\beta_2) \\
& +2\int_o^b \|T_\alpha(b-s)\|^2\mu_r(s)ds\Big].
\end{aligned}
$$

where $M_B = \|B\|$ Therefore

$$
\begin{aligned}
&\|(\Phi_k x)(t_1) - (\Phi_k x)(t_2)\|_{H_2}^2 \\
\leq\ & 10(r+\beta_1)\|S_\alpha(t_1) - S_\alpha(t_2)\|^2 + 10(r+\beta_2)\|K_\alpha(t_1) - K_\alpha(t_2)\|^2 \\
& +15b\|\int_o^{t_1-\epsilon} \|T_\alpha(t_1-\tau) - T_\alpha(t_2-\tau)\|^2\mu_r(\tau)d\tau \\
& +15b\|\int_{t_1}^{t_2} \|T_\alpha(t_2-\tau)\|^2\mu_r(\tau)d\tau \\
& +15b\|\int_{t_1-\epsilon}^{t_1} \|T_\alpha(t_1-\tau) - T_\alpha(t_2-\tau)\|^2\mu_r(\tau)d\tau \\
& +15M_B^2\widehat{M^2}(t_2-t_1)\int_{t_1}^{t_2} \|u^k(\tau)\|^2 d\tau \\
& +15M_B^2\|\int_{t_1-\epsilon}^{t_1} \|T_\alpha(t_1-\tau) - T_\alpha(t_2-\tau)\|^2\|u^k(\tau)\|^2 d\tau \\
& +15M_B^2\|\int_0^{t_1-\epsilon} T_\alpha(t_1-\tau) - T_\alpha(t_2-\tau)\|^2\|u^k(\tau)\|^2 d\tau
\end{aligned}
$$

$$+15\| \int_0^{t_1-\epsilon} \|T_\alpha(t_1-\tau) - T_\alpha(t_2-\tau)\|^2 \mu_r(\tau) d\tau$$

$$+15\| \int_{t_1}^{t_2} \|T_\alpha(t_2-\tau)\|^2 \mu_r(\tau) d\tau$$

$$+15\| \int_{t_1-\epsilon}^{t_1} \|T_\alpha(t_1-\tau) - T_\alpha(t_2-\tau)\|^2 \mu_r(\tau) d\tau.$$

The right-hand side of the above inequality is independent of $x \in B_r$ and tends to zero as $t_2 - t_1 \to 0$ and ϵ sufficiently small, since the compactness of $S_\alpha(t), K_\alpha(t), T_\alpha(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus, Φ_k maps B_r into an equicontinuous family of functions. It is easy to see that the family B_r is uniformly bounded.

Next, we show that $\overline{\Phi_k B_r}$ is compact. Since we have shown that $\Phi_k B_r$ is an equicontinuous family, it suffices by Arzela-Ascoli theorem to show that Φ_k maps B_r into a precompact set in $\mathcal{H}$.

Let $0 < t < b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_r$, we define

$$\begin{aligned}
(\Phi_k^\epsilon x)(t) &= S_\alpha(t)(x_0 - g(x)) + K_\alpha(t)(x_1 - h(x)) + \int_o^{t-\epsilon} T_\alpha(t-s) B u^k(s) ds \\
&+ \int_o^{t-\epsilon} T_\alpha(t-s) f(s, x(s), x(b_1(s)), ..., x(b_m(s))) ds \\
&+ \int_o^{t-\epsilon} T_\alpha(t-s) \sigma(s, x(s), x(a_1(s)), ..., x(a_n(s))) dW(s) \\
&= S_\alpha(t)(x_0 - g(x)) + K_\alpha(t)(x_1 - h(x)) \\
&+ T_\alpha(\epsilon) \int_o^{t-\epsilon} T_\alpha(t-s-\epsilon) B u^k(s) ds \\
&+ T_\alpha(\epsilon) \int_o^{t-\epsilon} T_\alpha(t-s-\epsilon) f(s, x(s), x(b_1(s)), ..., x(b_m(s))) ds \\
&+ T_\alpha(\epsilon) \int_o^{t-\epsilon} T_\alpha(t-s-\epsilon) \sigma(s, x(s), x(a_1(s)), ..., x(a_n(s))) dW(s).
\end{aligned}$$

Since $S_\alpha(t), K_\alpha(t), T_\alpha(t)$ are compact operators, the set $\{(\Phi_k^\epsilon x)(t) : x \in B_r\}$ is precompact in $\mathcal{H}$, for every $\epsilon, 0 < \epsilon < t$ and every $k > 0$. Moreover, forevery $x \in B_r$ we have:

$$
\begin{aligned}
&\mathbb{E}\|(\Phi_k x)(t) - (\Phi_k^\epsilon x)(t)\|_{\mathcal{H}}^2 \\
\leq\ & 3\mathbb{E}\|\int_{t-\epsilon}^{t} T_\alpha(t-s)Bu^k(s)ds\|_{\mathcal{H}}^2 + 3\mathbb{E}\|\int_{t-\epsilon}^{t} T_\alpha(t-s) \\
&\times f(s, x(s), x(b_1(s)), ..., x(b_m(s)))ds\|_{\mathcal{H}}^2 \\
&+3\mathbb{E}\|\int_{t-\epsilon}^{t} T_\alpha(t-s)\sigma(s, x(s), x(a_1(s)), ..., x(a_n(s)))dW(s)\|_{\mathcal{H}}^2 \\
\leq\ & 3M_B^2 \int_{t-\epsilon}^{t} \|T_\alpha(t-s)\|ds \int_{t-\epsilon}^{t} \|T_\alpha(t-s)\|\mathbb{E}\|u^k(s)\|_{\mathcal{H}}^2 ds \\
&+3\int_{t-\epsilon}^{t} \|T_\alpha(t-s)\|ds \int_{t-\epsilon}^{t} \|T_\alpha(t-s)\|\mathbb{E}\|f(s, y(s))\|_{\mathcal{H}}^2 ds \\
&+3\int_{t-\epsilon}^{t} \|T_\alpha(t-s)\|^2\|\mathbb{E}\|\sigma(s, z(s))\|_{L_2^0}^2 ds \\
\leq\ & 3M_B^4\widehat{M}^4 b^2 \tfrac{7}{k^2}(M_C + 2\widehat{M}^2 \int_{t-\epsilon}^{t} \mu_r(s)ds) + 3\widehat{M}^2(b^2+1)\int_{t-\epsilon}^{t} \mu_r(s)ds).
\end{aligned}
$$

Therefore, there are precompact sets arbitrary close to the set $\{(\Phi_k^\epsilon x)(t) : x \in B_r\}$. Hence, from assumption (**A5**), the set $\{(\Phi_k^\epsilon x)(t) : x \in B_r\}$. is precompact in $\mathcal{H}$ (see [54]). Next, we show that $\Phi_k : H_2 \to H_2$ is continuous. Let $\{x_n\}_{n=0}^\infty$ be a sequence in H_2 such that $x_n \to x$ in H_2. Then, there is an integer q such that $\|u_n(t)\|^2$ for all n and $t \in J$, so $u_n \in B_q$ and $u \in B_q$. By assumption (**A2**), we have
$f(t, y_n(t)) \to f(t, y(t))$ and $\sigma(t, z_n(t)) \to \sigma(t, z(t))$ for each $t \in J$.
Since
$\|f(t, y_n(t)) - f(t, y(t))\|_{\mathcal{H}}^2 \leq 2\nu_q(t)$; and $\|\sigma(t, z_n(t)) - \sigma(t, z(t))\|_{L_2^0}^2 \leq 2\nu_q(t)$.
we have by dominated convergence

$$\begin{aligned}
\mathbb{E}\|(\Phi_k x)(t) - (\Phi_k^{\epsilon} x)(t)\|^2_{\mathcal{H}} &\leq 3M_B^4\widehat{M}^4 b\tfrac{7}{k^2} \times \Big[\mathbb{E}\|\int_0^b T_\alpha(b-s)[f(s,y_n(s)-f(s,y(s)))]ds\|^2_{\mathcal{H}} \\
&+\mathbb{E}\|\int_0^b T_\alpha(b-s)[\sigma(s,z_n(s))-\sigma(s,z(s))]dW(s)\|^2_{\mathcal{H}}\Big] \\
&+3\mathbb{E}\|\int_0^b T_\alpha(b-s)[f(s,y_n(s)-f(s,y(s)))]ds\|^2_{\mathcal{H}} \\
&+3\mathbb{E}\|\int_0^b T_\alpha(b-s)[\sigma(s,z_n(s))-\sigma(s,z(s))]dW(s)\|^2_{\mathcal{H}} \\
&\leq 3M_B^4\widehat{M}^4 b\tfrac{7}{k^2}\Big[b\widehat{M}^2\int_0^b \mathbb{E}\|f(s,y_n(s)-f(s,y(s)))\|^2_{\mathcal{H}} \\
&+\widehat{M}^2\int_0^b \mathbb{E}\|\sigma(s,z_n(s)-\sigma(s,z(s)))\|^2_{L_2^0}\Big] \\
&+3b\widehat{M}^2\int_0^b \mathbb{E}\|f(s,y_n(s)-f(s,y(s)))\|^2_{\mathcal{H}} \\
&+3\widehat{M}^2\int_0^b \mathbb{E}\|\sigma(s,z_n(s)-\sigma(s,z(s)))\|^2_{L_2^0} \to 0.
\end{aligned}$$

Hence, $\|(\Phi_k u_n)(t) - (\Phi_k u)(t)\|^2_{H_2} \to 0$ Thus, Φ_k is continuous.This completes the proof that Φ_k is completely continuous.

Now, we prove that the set $U := \{x \in H_2 : \lambda x = \Phi_k x \text{ for some } \lambda > 1\}$ is bounded.

Let $x \in U$. Then $\lambda X = \Phi_k x$ for some $\lambda > 1$ and $k > 0$ Then

$$\begin{aligned}
x(t) &= \lambda^{-1}S_\alpha(x_0 - g(x)) + \lambda^{-1}K_\alpha(x_1 - h(x)) + \lambda^{-1}\int_0^t T_\alpha(t-s)Bu^k(s)ds \\
&+\lambda^{-1}\int_0^t T_\alpha(t-s)f(s,y(s))ds \\
&+\lambda^{-1}\int_0^t T_\alpha(t-s)\sigma(s,z(s))dW(s), \quad t \in J.
\end{aligned}$$

We have

$$\begin{aligned}
\mathbb{E}\|x(t)\|^2_{\mathcal{H}} &\leq 10\|S_\alpha(t)\|^2(\mathbb{E}\|x_0\|^2_{\mathcal{H}} + \mathbb{E}\|g(x)\|^2_{\mathcal{H}} \\
&+10\|K_\alpha(t)\|^2(\mathbb{E}\|x_1\|^2_{\mathcal{H}} + \mathbb{E}\|h(x)\|^2_{\mathcal{H}} \\
&+5M_B^2\int_0^t \|T_\alpha(t-s)\|ds\int_0^t \|T_\alpha(t-s)\|ds\mathbb{E}\|u^k(s)\|^2 ds \\
&+5\int_0^t \|T_\alpha(t-s)\|ds\int_0^t \|T_\alpha(t-s)\|ds\mathbb{E}\|f(s,y(s))\|^2_{\mathcal{H}} ds \\
&+5\int_0^t \|T_\alpha(t-s)\|^2 ds\mathbb{E}\|\sigma(s,z(s))\|^2_{L_2^0} \\
&\leq 10\widehat{M}^2\mathbb{E}\|x_0\|^2_{\mathcal{H}} + 10\widehat{M}^2\beta_1 + 10\widehat{M}^2\mathbb{E}\|x_1\|^2_{\mathcal{H}}10\widehat{M}^2\beta_2 + 5M_B^4\widehat{M}^4 b\tfrac{7}{k^2} \\
&\times\left[M_C + b\widehat{M}^2\int_0^b m_f(s)\varphi(\mathbb{E}\|x(s)\|^2_{\mathcal{H}})ds + \widehat{M}^2\int_0^b m_\sigma(s)\varphi(\mathbb{E}\|x(s)\|^2_{\mathcal{H}})ds\right] \\
&+5b\widehat{M}^2\int_0^b m_f(s)\varphi(\mathbb{E}\|x(s)\|^2_{\mathcal{H}})ds + 5\widehat{M}^2\int_0^b m_\sigma(s)\varphi(\mathbb{E}\|x(s)\|^2_{\mathcal{H}})ds.
\end{aligned}$$

Consider the function $\eta(t)$ defined by $\eta(t) = \sup\{\mathbb{E}\|x(s)\|^2_{\mathcal{H}}), 0 \leq s \leq b \leq\}, t \in J$ we have:

$$\begin{aligned}
\eta(t) &\leq 10\widehat{M}^2[\mathbb{E}\|x_0\|^2_{\mathcal{H}} + \beta_1] + 10\widehat{M}^2[\mathbb{E}\|x_1\|^2_{\mathcal{H}} + \beta_2] \\
&+5M_B^4\widehat{M}^4 b\tfrac{7}{k^2}\left[M_C + b^2\widehat{M}^2\int_0^b m_f(s)\varphi(\eta(s))ds + \widehat{M}^2 b\int_0^b m_\sigma(s)\varphi(\eta(s))ds\right] \\
&+5b\widehat{M}^4\int_0^b m_f(s)\varphi(\eta(s))ds + 5\widehat{M}^4\int_0^b m_\sigma(s)\varphi(\eta(s))ds.
\end{aligned}$$

Denote by $\eta(t)$ the right-hand side of the last inequality, we have

$$v(0) = \gamma = 10\widehat{M}^2[\mathbb{E}\|x_0\|^2_{\mathcal{H}}+\beta_1]+10\widehat{M}^2[\mathbb{E}\|x_1\|^2_{\mathcal{H}}+\beta_2]+5M_B^4\widehat{M}^4 b\frac{7}{k^2}M_C, \quad \eta(t) \leq v(t), \quad t \in J.$$

Moreover

$$v\prime(t) = 5M_B^4\widehat{M}^4 b\tfrac{7}{k^2}\left[b^2\widehat{M}^2 m_f(t)\varphi(\eta(t)) + \widehat{M}^2 b m_\sigma(t)\varphi(\eta(t))\right]$$

$$
\begin{aligned}
& +5b\widehat{M}^4 m_f(t)\varphi(\eta(t)) + 5\widehat{M}^4 m_\sigma(t)\varphi(\eta(t)) \\
\leq\ & 5M_B^4\widehat{M}^4 b\tfrac{7}{k^2}\left[b^2\widehat{M}^2 m_f(t)\varphi(v(t)) + \widehat{M}^2 b m_\sigma(t)\varphi(v(t))\right] \\
& +5b\widehat{M}^4 m_f(t)\varphi(v(t)) + 5\widehat{M}^4 m_\sigma(t)\varphi(v(t)) \\
=\ & m_f(t)\varphi(v(t))\left(5b\widehat{M}^4 + 5M_B^4\widehat{M}^4 b^2\tfrac{7}{k^2}b^2\widehat{M}^2\right) \\
& +m_\sigma(t)\varphi(v(t))\left(5\widehat{M}^4 + 5M_B^4\widehat{M}^4 b^2\tfrac{7}{k^2}b\widehat{M}^2\right) \\
\leq\ & m(t)\left(\varphi(v(t)) + \varphi(v(t))\right) = 2m(t)\varphi(v(t)) \quad t \in J.
\end{aligned}
$$

This implies

$$
\int_{v(0)}^{v(t)} \frac{ds}{2\varphi(s)} \leq \int_0^b m(s)ds < \int_\gamma^\infty \frac{ds}{2\varphi(s)}, \quad t \in J.
$$

This inequality implies that there is a constant k such that $v(t) \leq k, t \in J$ and, hence, $\eta(t) \leq k$. Furthermore, we get $\|x(t)\|^2 \leq \eta(t) \leq k, t \in J$. By the Schaefer's fixed-point theorem (Lemma 3.2.2), we deduce that Φ_k has a fixed point on J which is a solution to (3.1). This completes the proof of the theorem □

3.2.2 The approximate controllability

To prove the approximate controllability result, the following additional assumption is required:

(**A6**) The linear system (3.4) is approximately controllable.

(**A7**) The functions $f : J \times \mathcal{H}^{m+1} \to \mathcal{H}$ and $\sigma : J \times \mathcal{H}^{n+1} \to L_2^0$ are bounded.

Remark: (**A6**) is equivalent to $kR(k, \Gamma_0^b) = k(kI + \Gamma_0^b)^{-1}$ as $k \to 0^+$ in the strong operator topology ([45]).

Theorem 3.2.2 *Assume that the assumptions of Theorem (3.2.1) hold and in addition, **(A6)** and **(A7)** are satisfied. Then, the fractional control system (3.1) is approximately controllable on J.*

Proof. Let $x_k(.) \in B_q$ be a fixed point of the Φ_k Using the stochastic Fubini theorem, it is easy to see that

$$\begin{aligned} x_k(b) &= \hat{x}_b - (kI+\Gamma_0^b)^{-1}\Big[\mathbb{E}\hat{x}_b + \int_0^b \hat{\phi}(s)dW(s) - S_\alpha(b)(x_0 - g(x^k)) \\ & -K_\alpha(b)(x_1 - h(x^k))\Big] \\ & +k\int_0^b (kI+\Gamma_s^b)^{-1}T_\alpha(b-s)f(s, x_k(s), x_k(b_1(s)), ..., x_k(b_m(s)))ds \\ & +k\int_0^b (kI+\Gamma_s^b)^{-1}T_\alpha(b-s)\sigma(s, x_k(s), x_k(a_1(s)), ..., x_k(a_n(s)))dW(s). \end{aligned} \tag{3.5}$$

Moreover, by the assumption **(A7)**, there exists $N_1 > 0$ such that

$$\|f(s, x_k(s), x_k(b_1(s)), ..., x_k(b_m(s)))\|^2 + \|\sigma(s, x_k(s), x_k(a_1(s)), ..., x_k(a_n(s)))\|^2 \leq N_1$$

and consequently, there is a sequence still denoted by $\{f(s, x_k(s), x_k(b_1(s)), ..., x_k(b_m(s))), \sigma(s, x_k(s), x_k(a_1(s)), ..., x_k(a_n(s)))\}$. weakly converges to say $f(s), \sigma(s)$. Thus, from the equation (3.5), we have

$$\begin{aligned} & \mathbb{E}\|x_k(b) - \hat{x}_b\|^2 \\ \leq\ & 6\|k(kI+\Gamma_0^b)^{-1}(\mathbb{E}\hat{x}_b - S_\alpha(b)(x_0 - g(x_k)) - K_\alpha(b)(x_1 - h(x^k))]\|^2 \\ & +6\mathbb{E}\left(\int_0^b \|k(kI+\Gamma_0^b)^{-1}\hat{\phi}(s)\|_{L_2^0}^2 ds\right) \\ & +6\mathbb{E}\left(\int_0^b \|k(kI+\Gamma_0^b)^{-1}\|\|T_\alpha(b-s)[f(s, y_k(s)) - f(s)]\|\right)^2 \\ & +6\mathbb{E}\left(\int_0^b \|k(kI+\Gamma_0^b)^{-1}T_\alpha(b-s)f(s)\|\right)^2 \\ & +6\mathbb{E}\left(\int_0^b \|k(kI+\Gamma_0^b)^{-1}\|\|T_\alpha(b-s)[\sigma(s, z_k(s)) - \sigma(s)]\|_{L_2^0}^2\right) \\ & +6\mathbb{E}\left(\int_0^b \|k(kI+\Gamma_0^b)^{-1}T_\alpha(b-s)\sigma(s)\|_{L_2^0}^2\right). \end{aligned}$$

On the other hand, by assumption **(A6)**, for all $0 \leq s \leq b$ the operator $kR(k,\Gamma_0^b)$to zero operator in the strong operator topology, as $k \to 0^+$, and moreover $\|k(kI+\Gamma_s^b)^{-1}\| \leq 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of $T_\alpha(t)$ we obtain $\mathbb{E}\|x_n(b)-\hat{x}_b\|^2 \to 0$ as $k \to 0^+$. This gives the approximate controllability of (3.1). □

3.2.3 The Exact Controllability

Remark: The stochastic control system (3.1) is said to be exactly controllable on J if $\mathcal{R}(b)=\mathcal{H}$. Assume that the linear fractional stochastic control system

$$\begin{aligned} D^\alpha x(t) &= Ax(t)+(Bu)(t)+\sigma(t)\tfrac{dW(t)}{dt} \quad t\in[0,b] \\ x(0) &= x_0, \quad x'(0)=x_1 \end{aligned} \tag{3.6}$$

is exactly controllable.

Now, we introduce the controllability operator associated with (3.6) as $\Gamma_0^b = \int_0^b T_\alpha(b-s)BB^*T_\alpha^*(b-s)\mathbb{E}\{.\backslash\mathcal{F}_t\}ds$ It should be mentioned that the linear fractional stochastic system (3.6) is exactly controllable if and only if there exists a $\gamma>0$ such that $\mathbb{E}(\Gamma_0^b x,x)\geq\gamma\mathbb{E}\|x\|^2$, for all $x\in\mathcal{H}$ and consequently $\|(\Gamma_0^b)^{-1}\|\leq\frac{1}{\gamma}$.

To prove the exact controllability result, we assume the following assumptions:
(A8) $f: J\times\mathcal{H}^{m+1}\to\mathcal{H}, \sigma: J\times\mathcal{H}^{n+1}\to L_2^0$ are continuous and there exist constants L_f and L_σ such that

$$\mathbb{E}\|f(t_1,x_0,x_1,...,x_m)-\mathbb{E}\|f(t_2,y_0,y_1,...,y_m)\|_{\mathcal{H}}^2 \leq L_f\Big(|t_1-t_2|+\max_{i=0,1,\ldots,m}\mathbb{E}\|x_i-y_i\|_{\mathcal{H}}^2\Big),$$

$$\mathbb{E}\|\sigma(t_1,x_0,x_1,...,x_n)-\mathbb{E}\|\sigma(t_2,y_0,y_1,...,y_n)\|_{L_2^0}^2 \leq L_\sigma\Big(|t_1-t_2|+\max_{j=0,1,\ldots,n}\mathbb{E}\|x_i-y_i\|_{\mathcal{H}}^2\Big)$$

for all $0\leq t_1,t_2\leq b,(x_i,y_i),(x_j,y_j)\in\mathcal{H}\times\mathcal{H}, i=0,1,...,m; j=0,1,...,n.$
(A9) There exist constants $\beta_1,\beta_2>0$ such that

$$\begin{aligned} \mathbb{E}\|g(x)-g(y)\|_{\mathcal{H}}^2 &\leq \beta_1\mathbb{E}\|x-y\|_{\mathcal{H}}^2 \\ \mathbb{E}\|h(x)-h(y)\|_{\mathcal{H}}^2 &\leq \beta_2\mathbb{E}\|x-y\|_{\mathcal{H}}^2, \forall x,y \in \mathcal{H}. \end{aligned}$$

Theorem 3.2.3 *Assume that **(A1)**, **(A8)** and **(A9)** hold. If the linear stochastic system associated with the system (3.1) is exactly controllable on all $[0,t], t>0$, then the semilinear fractional stochastic control system (3.1) is exactly controllable on J provided that*

$$5\Lambda\widehat{M}^2(1+\Lambda\widehat{M}^2\frac{1}{\gamma^2})\left[(\beta_1+\beta_2)+b^2(m+1)L_f+b(n+1)L_\sigma\right] \leq 1. \tag{3.7}$$

proof. Define the operator $\Psi : H_2 \to H_2$ by

$$\begin{aligned} (\Psi x)(t) &= S_\alpha(t)[x_0-g(x)]+K_\alpha(t)[x_1-h(x)]+\int_o^t T_\alpha(t-s)Bu(s,x)ds \\ &+\int_o^t T_\alpha(t-s)f(s,x(s)),x(b_1(s)),....,x(b_m(s)))ds \\ &+\int_o^t T_\alpha(t-s)\sigma(s,x(s)),x(a_1(s)),....,x(a_n(s)))dW(s), t\in J \end{aligned}$$

Where

$$\begin{aligned} u(t,x) &= B^*T_\alpha^*(b-t)\mathbb{E}\Big\{(\Gamma_0^b)^{-1}\Big(\hat{x}_b - S_\alpha(b)[x_0-g(x)]-K_\alpha(b)[x_1-h(x)] \\ &-\int_o^t T_\alpha(b-s)f(s,x(s)),x(b_1(s)),....,x(b_m(s)))ds \\ &-\int_o^t T_\alpha(b-s)\sigma(t,x(s)),x(a_1(s)),....,x(a_n(s)))dW(s)\Big)\backslash\mathcal{F}_t\Big\}. \end{aligned} \tag{3.8}$$

Note that, the control (3.8) transfers the system (3.1) from the initial state x_0 to the final state $x(b)=\hat{x}_b$ provided that the operator Ψ has a fixed point. To prove the exact controllability result, it is enough to show that the operator Ψ has a fixed point in H_2. To do this, we can employ the Banach contraction principle.
First, It can be seen that Ψ maps H_2 into itself. Let us show that Φ is a contraction on H_2. For $t\in J$, it follows from the assumptions **(A1)**, **(A8)** and **(A9)** that

$$
\begin{aligned}
& \mathbb{E}\|(\Psi x)(t)-(\Psi y)(t)\|_{\mathcal{H}}^2 \\
\leq\ & 5\|S_\alpha(t)\|^2\mathbb{E}\|g(x)-g(y)\|_{\mathcal{H}}^2+5\|K_\alpha(t)\|^2 \\
& \mathbb{E}\|h(x)-h(y)\|_{\mathcal{H}}^2+5\mathbb{E}\|\int_o^t T_\alpha(b-s)B(u(s,x)-u(s,y))ds\|^2 \\
& +5\mathbb{E}\|\int_o^t T_\alpha(b-s)[f(s,x(s)),x(b_1(s)),....,x(b_m(s))) \\
& -f(s,y(s)),y(b_1(s)),....,y(b_m(s)))]ds\|_{\mathcal{H}}^2 \\
& +5\mathbb{E}\|\int_o^t T_\alpha(b-s)[\sigma(s,x(s)),x(a_1(s)),....,x(a_n(s))) \\
& -\sigma(s,y(s)),y(a_1(s)),....,y(a_n(s)))]dW(s)\|_{\mathcal{H}}^2.
\end{aligned}
$$

We have

$$
\begin{aligned}
& \int_o^t T_\alpha(b-s)B(u(s,x)-u(s,y))ds \\
=\ & \Gamma_0^b T_\alpha^*(b-t)(\Gamma_0^b)^{-1} \\
& \times\Big(S_\alpha(t)(g(x)-g(y))+K_\alpha(t)(h(x)-h(y)) \\
& +\int_o^t T_\alpha(b-s)(f(s,y(s)),y(b_1(s)),....,y(b_m(s))) \\
& -f(s,x(s)),x(b_1(s)),....,x(b_m(s))))ds \\
& +\int_o^t T_\alpha(b-s)(\sigma(s,x(s)),x(a_1(s)),....,x(a_n(s))) \\
& -\sigma(s,x(s)),x(a_1(s)),....,x(a_n(s))))dW(s)\Big).
\end{aligned}
$$

Then

$$
\begin{aligned}
& \mathbb{E}\|(\Psi x)(t)-(\Psi y)(t)\|_{\mathcal{H}}^2 \\
\leq\ & 5\widehat{M}^2(\beta_1+\beta_2)\mathbb{E}\|x-y\|_{\mathcal{H}}^2+5\Lambda\widehat{M}^2\tfrac{1}{\gamma^2}\Big(\widehat{M}^2(\beta_1+\beta_2)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\|x-y\|_{\mathcal{H}}^2 + b\widehat{M}^2\int_0^t L_f \max_{i=0,1,..,m} \mathbb{E}\|x_i-y_i\|_{\mathcal{H}}^2 ds \\
&+\widehat{M}^2\int_0^t L_\sigma \max_{j=0,1,..,m} \mathbb{E}\|x_j-y_j\|_{\mathcal{H}}^2 ds \\
&+5b\widehat{M}^2\int_0^t L_f \max_{i=0,1,..,m} \mathbb{E}\|x_i-y_i\|_{\mathcal{H}}^2 ds \\
&+5\widehat{M}^2\int_0^t L_\sigma \max_{j=0,1,..,m} \mathbb{E}\|x_j-y_j\|_{\mathcal{H}}^2 ds \\
\leq\ & 5\widehat{M}^2(\beta_1+\beta_2)\Big(1+\Lambda\widehat{M}^2\tfrac{1}{\gamma^2}\Big)\mathbb{E}\|x-y\|_{\mathcal{H}}^2 \\
&+5\Big(1+\Lambda\widehat{M}^2\tfrac{1}{\gamma^2}\Big)b^2\widehat{M}^2(m+1)L_f \\
&\mathbb{E}\|x-y\|_{\mathcal{H}}^2 + 5\Big(1+\Lambda\widehat{M}^2\tfrac{1}{\gamma^2}\Big)b\widehat{M}^2(n+1)L_\sigma\mathbb{E}\|x-y\|_{\mathcal{H}}^2 \\
=\ & 5\Lambda\widehat{M}^2\Big(1+\Lambda\widehat{M}^2\tfrac{1}{\gamma^2}\Big)\left[(\beta_1+\beta_2)b^2(m+1)L_f + b(n+1)L_\sigma\right]\mathbb{E}\|x-y\|_{\mathcal{H}}.
\end{aligned}
$$

Where $\Lambda = \max\{\|\Gamma_s^b : 0 \leq s \leq b\|^2\}$

Hence by the condition (3.7), Ψ is a contraction mapping. Therefore, by the Banach contraction principle Ψ has a unique fixed point.Further $x(b) = \hat{x}_b$. Thus, the system (3.1) is exactly controllable on $[0, b]$.

3.3 An example

Consider the following fractional stochastic control system of the form

$$
\begin{aligned}
&{}^cD_t^\alpha x(t,z) = \frac{\partial^2}{\partial z^2}x(t,z) + \mu(t,z) + \varphi(t,x(t,z)) + \psi(t,x(t,z))\frac{d\beta(t)}{dt} \\
&0 \le t \le b, \quad 0 \le z \le \pi, x(t,0) = x(t,\pi) = 0, \quad x'(t,0) = x'(t,\pi) = 0 \\
&x(0,z) + \sum_{i=0}^{p}\int_0^\pi H(z,y)x(t_i,y)dy = x_0(z) \\
&x'(0,z) + \sum_{i=0}^{p}\int_0^\pi K(z,y)x(t_i,y)dy = x_1(z), \quad 0 \le z \le \pi.
\end{aligned} \tag{3.9}
$$

Where p is a positive integer, $b \le \pi, 0 < t_0 < t_1, ..., < t_p < b, x_1(z) \in \mathcal{H} = L^2([0,\pi]), H(z,y), K(z,y) \in L^2([0,\pi] \times [0,\pi], \mu : [0,b] \times [0,\pi] \to [0,\pi]$ is continuous in $t, {}^cD_t^\alpha$ is the Caputo fractional derivative of order $1 < \alpha < 2$ and $\beta(t)$ is a two sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
To write the above system (3.9) into the abstract form (3.7), we choose the space $\mathcal{U} = \mathcal{H} = \mathcal{K} = L^2([0,\pi])$. Define the operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ by $A_y = y''$ with the domain

$D(A) = \{y \in \mathcal{H}; y, y'\}$ are absolutely continuous, $y' \in \mathcal{H}$ and $y(0) = y(\pi) = 0$. Then A is densely defined in $\mathcal{H}$ and it is the infinitesimal generator of a resolvent family $\{S_\alpha(t), t \ge 0\}$. Furthermore, $-A$ has a discrete spectrum with eigenvalues of the form $-n^2, n = 0, 1, 2...$ and corresponding normalized eigenfunctions are given by $z_n(x) = \sqrt{\frac{2}{\pi}}\sin(nz)$. In addition $\{z_n, n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}$.

$$
T(t)(y) = \sum_{n=1}^{\infty} e^{-n^2 t}(y, y_n)y_n, \quad y \in \mathcal{H}, t > 0
$$

Let $x(t)(z) = x(t,z)$ and define the bounded linear operator $B : \mathcal{U} \to \mathcal{H}$ by $Bu(t)(z) = \mu(t,z), 0 \le z \le \pi, u \in \mathcal{H}$; Also, define the function $f : J \times \mathcal{H}^{m+1} \to \mathcal{H}, \sigma : J \times \mathcal{H}^{n+1} \to$

$L_2^0, g : \mathcal{H} \to \mathcal{H}$ and $h : \mathcal{H} \to \mathcal{H}$ by $f(t, x(t))(z) = \varphi(t, x(z)), \sigma(t, x(t))(z) = \psi(t, x(z))$ and $g(x) = \sum_{i=0}^{p} H(x)(t_i), h(x) = \sum_{i=0}^{p} K(x)(t_i)$ where $H(x)(z) = \int_0^\pi H(z, y)x(y)dy$, and $K(x)(z) = \int_0^\pi K(z, y)x(y)dy$ for $z \in [0, b]$

On the other hand, the linear fractional stochastic system corresponding to (3.9) is approximately controllable. Thus, with the above choices of A, B, f and σ, system (3.9) can be rewritten into the abstract form of (3.1). Thus, all conditions of Theorem (3.2.2) are satisfied. Hence, by Theorem (3.2.2) the fractional control system (3.9) is approximately controllable on $[0, b]$.

Chapter 4

Approximate controllability of sobolev-type fractional functional stochastic integro-differential systems

In this chapter[1] we studies the approximate controllability of sobolev-type fractional functional stochastic integro-differential systems in the following form

$$\begin{aligned} {}^cD_t^\alpha(Ex(t)) + Ax(t) &= Bu(t) + f(t, x_t) + \int_0^t \sigma\Big(t, s, x_s, \int_0^s H(s, \tau, x_\tau)d\tau\Big)dW(s), \\ t \in J &:= [0, b], \\ x(t) &= \phi(t), \qquad -r \leq t \leq 0. \end{aligned} \tag{4.1}$$

4.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\mathcal{F}_t, t \in J = [0, b]$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). and $\mathbb{E}(.)$ denotes the expectation with respect to the measure $\mathbb{P}$, We consider three real separable Hilbert spaces $\mathcal{H}, \mathcal{K}$ and $\mathcal{Z}$ with inner products $(.,.)_\mathcal{H}, (.,.)_\mathcal{K}$ and $(.,.)_\mathcal{Z}$, respectively and norms $\|.\|_\mathcal{H}; \|.\|_\mathcal{K}; \|.\|_\mathcal{Z}$. Let $W = (W_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with the linear bounded covariance operator Q such that $TrQ < \infty$. We assume that there exists a complete orthonormal system $\{e_n\}_{n\geq 0}$ on $\mathcal{H}$, a bounded

[1]The chapter is based on the paper [26].

sequence of non-negative real numbers λ_n such that $Qe_n = \lambda_n e_n, n = 1, 2, ...$ and a sequence $\{\beta_n\}_{n\geq 1}$ of independent Brownian motions such that

$$< W(t), e >= \sum_{n=1}^{\infty} \sqrt{\lambda_n} < e_n, e > \beta_n(t), \quad e \in \mathcal{K}, t \in J := [0, b]$$

and $\mathcal{F}_t = \mathcal{F}_t^W$ where $\mathcal{F}_t^W$ is the sigma algebra generated by $\{W(s) : 0 \leq s \leq t\}$.

We denote by $L(\mathcal{K}, \mathcal{Z})$ the set of all linear bounded operators from $\mathcal{K}$ into $\mathcal{Z}$, equipped with the usual operator norm $\|.\|$.Let $L_2^0 = L_2(Q^{\frac{1}{2}}\mathcal{K}; \mathcal{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathcal{K}$ into $\mathcal{H}$ with the inner product $< \psi, \pi >_{L_2^0} = Tr[\psi Q \pi^\star]$.

Let $L^2(\mathcal{F}_b, \mathcal{H})$ be the Banach space of all $\mathcal{F}_b$-measurable square integrable random variables with values in the Hilbert space $\mathcal{H}$. Let $\mathcal{C}([0, b], L^2(\mathcal{F}, \mathcal{H}))$ be the Banach space of continuous maps from $[0, b]$ into $L^2(\mathcal{F}, \mathcal{H})$ satisfying $\sup_{t\in J} \mathbb{E}\|x(t)\|^2 < \infty$.

Let $\mathcal{C}_0 := \mathcal{C}_0([0, b]; L^2(\mathcal{F}, \mathcal{H}))$ be a closed subspace of $\mathcal{C}([0, b], L^2(\mathcal{F}, \mathcal{H}))$ consisting of measurable and $\mathcal{F}_t$-adapted $\mathcal{H}$-valued process $x \in \mathcal{C}([0, b]; L^2(\mathcal{F}, \mathcal{H}))$ endowed with the norm $\|x\|_{\mathcal{C}_O} = (\sup_{t\in J} \mathbb{E}\|x(t)\|_{\mathcal{H}}^2)^{\frac{1}{2}}$.

Consider the sobolev-type fractional functional stochastic integro-differential system (4.1), Where $x(.)$ takes value in the Hilbert space $\mathcal{H}$; the fractional derivative ${}^cD^\alpha, 0 < \alpha < 1$ is understood in the Caputo sense; $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{Z}$ and $E : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{Z}; W = \{W(t), t \geq 0\}$ is a given $\mathcal{K}$-Valued Wiener process with a finite trace nuclear operator $Q \geq 0$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$; the control function $u(.)$ is given in $L^2(J, \mathcal{U})$ of admissible control functions, $\mathcal{U}$ is a Hilbert space; B is a bounded linear operator from $\mathcal{U}$ into $\mathcal{Z}$; $f : J \times \mathcal{C}_r \to \mathcal{Z}, H : J \times J \times \mathcal{C}_r \to \mathcal{H}$ and $\sigma : J \times J \times \times \mathcal{C}_r \times \mathcal{H} \to L_2^0$ with $\mathcal{C}_r := \mathcal{C}_r([-r, 0], \mathcal{H})$ will de specified later; $x : J^* := [-r, b] \to \mathcal{H}$ is continuous; x_t is the element of $\mathcal{C}_r$ defined by $x_t(s) = x(t + s), -r \leq s \leq 0$, the domain $\mathcal{D}(E)$ of E becomes a Hilbert space with the norm $\|x\| = \|Ex\|_{\mathcal{Z}}, x \in \mathcal{D}(E)$ and $\phi \in \mathcal{C}_r(E) = \mathcal{C}_r([-r, 0], \mathcal{D}(E))$.

Now, we introduce the following hypotheses on the operators A and E.

(**H1**): A and E are linear operators, and A is closed.

(**H2**): $\mathcal{D}(E) \subset \mathcal{D}(A)$ and E is bijective.
(**H3**): $E^{-1} : \mathcal{Z} \to \mathcal{D}(E) \subset \mathcal{H}$ is compact (which implies that E^{-1} is bounded).

The hypothesis (**H3**) implies that E is closed since the fact: E^{-1} is closed and injective, then its inverse is also closed. It comes from (**H1**)-(**H3**) and the closed graph theorem, we obtain the boundedness of the linear operator $-AE^{-1} : \mathcal{Z} \to \mathcal{Z}$.
Consequently, $-AE^{-1}$ generates a semigroup $\{T(t), t \geq 0\}$ in $\mathcal{Z}$ which means that there exists $M > 1$ such that $\sup_{t\in J} \|T(t)\| \leq M$.

According to definitions (1.9.2) and (1.9.4), it is suitable to rewrite the system (4.1) in the equivalent fractional integral equation

$$
\begin{aligned}
Ex(t) &= E\phi(0) + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[-Ax(s) + f(s, x_s) + Bu(s)]ds \\
&\quad + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\int_0^s \sigma\Big(s, \tau, x_\tau, R(\tau)\Big) dW(\tau) \right] ds, \quad t \in J := [0, b], \\
x(t) &= \phi(t), \quad -r \leq t \leq 0,
\end{aligned}
\tag{4.2}
$$

Where $R(\tau) = \int_0^\tau H(\tau, \upsilon, x_\upsilon)d\upsilon$, provided that the integral in (4.2) exists.

If the Formula (4.2) holds, then we have (see [1], [19])

$$
\begin{aligned}
x(t) &= \mathcal{T}_E(t)E\phi(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_E(t-s)f(s, x_s)ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_E(t-s)Bu(s)ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_E(t-s) \left[\int_0^s \sigma\Big(s, \tau, x_\tau, R(\tau)\Big) dW(\tau) \right] ds, \\
&\quad t \in J := [0, b], \\
x(t) &= \phi(t), \quad -r \leq t \leq 0.
\end{aligned}
\tag{4.3}
$$

Here $\mathcal{T}_E(.)$ and $\mathcal{S}_E(.)$ are called characteristic solution operators and given by

$$\begin{aligned} \mathcal{T}_E(t) &:= \int_0^\infty E^{-1}\xi_\alpha(\theta)T(t^\alpha\theta)d\theta \\ \mathcal{S}_E(t) &:= \alpha\int_0^\infty E^{-1}\theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta \end{aligned} \tag{4.4}$$

Where

$$\begin{aligned} \xi_\alpha(\theta) &:= \tfrac{1}{\alpha}\theta^{-(1+\frac{1}{q})}\varpi_\alpha(\theta^{-\frac{1}{q}}) \geq 0, \\ \varpi_\alpha(\theta) &:= \tfrac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha). \end{aligned}$$

ξ_α is a probability density defined on $]0,\infty[$.

Remark: When $E = I, I : \mathcal{Z} \to \mathcal{Z}$ is the identity operator, we have

$$\mathcal{T}_I(t) := \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \quad \mathcal{S}_I(t) := \alpha\int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta,$$

where $\displaystyle\int_0^\infty \theta\xi_\alpha(\theta)d\theta = \frac{1}{\Gamma(1+\alpha)}$.

Definition 4.1.1 *A stochastic process $x \in \mathcal{C}(J^*, \mathcal{H})$ is a mild solution of (4.1) if or each $u \in L^2_{\mathcal{F}}(J, \mathcal{U})$ and $\phi \in \mathcal{C}_r(E)$, it satisfies the following integral equation,*

$$\begin{aligned} x(t) &= \mathcal{T}_E(t)E\phi(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_E(t-s)f(s,x_s)ds + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_E(t-s)Bu(s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_E(t-s)\left[\int_0^s \sigma\Big(s,\tau,x_\tau,R(\tau)\Big)dW(\tau)\right]ds, \quad t \in J. \end{aligned}$$

Where $\mathcal{T}_E(.)$ and $\mathcal{S}_E(.)$ are defined as in (4.4)

The following properties of $\mathcal{T}_E(.)$ and $\mathcal{S}_E(.)$ appeared and proved in [20] are useful.

Lemma 4.1.1 *Assume (**H1**)-(**H3**) hold, then:*

(i) *For any fixed $t \geq 0, \mathcal{T}_E(t)$ and $\mathcal{S}_E(t)$ are linear and bounded operators, i.e; for any $x \in \mathcal{H}$*

$$\|\mathcal{T}_E(t)x\| \leq M\|E^{-1}\|\|x\| \quad , \quad \|\mathcal{S}_E(t)x\| \leq \frac{M\|E^{-1}\|}{\Gamma(\alpha)}\|x\|.$$

(ii) $\{\mathcal{T}_E(t), t \geq 0\}$ *and* $\{\mathcal{S}_E(t), t \geq 0\}$ *are compact.*

4.2 The Main Resuls

In this chapter, we shall formulate and prove sufficient conditions for the approximate controllability of the system (4.1). To do this, we first prove the existence of solutions for fractional control system. Then, we show that under certain assumptions, the approximate controllability of (4.1) is implied by the approximate controllability of the associated linear system.

Definition 4.2.1 *Let $x_b(\phi, u)$ be the state value of (4.1) at the terminal time b corresponding to the control u and the initial value ϕ . Introduce the set*

$$\mathcal{R}(b, \phi) = \{x_b(\phi, u)(0) : u(.) \in \mathcal{L}^2(J, \mathcal{U})\}$$

which is called the reachable set of (4.1) at the terminal time b and its closure in $\mathcal{H}$ is denoted by $\overline{\mathcal{R}(b, \phi)}$. The system (4.1) is said to be approximately controllable on the interval J if $\overline{\mathcal{R}(b, \phi)} = \mathcal{H}$; that is, given an arbitrary $\epsilon > 0$, it is possible to steer from the point $\phi(0)$ to within a distance ϵ from all points in the state space $\mathcal{H}$ at the time b.

to study the approximate controllability for the fractional control system (4.1), we introduce the approximate controllability of its linear part.

$$\begin{aligned} D_t^\alpha Ex(t) &= Ax(t) + (Bu)(t), \quad t \in [0, b] \\ x(0) &= \phi(0) \end{aligned} \tag{4.5}$$

It is convenient at the point to introduce the controllability and resolvent operators associated with (4.5) as.

$$\begin{aligned} L_0^b &= \int_0^t (t-s)^{\alpha-1}\mathcal{S}_E(t-s)Bu(s)ds : L^2(J,\mathcal{U}) \to \mathcal{D}(E) \\ \Gamma_0^b &= L_0^b(L_0^b)^* = \int_0^b (b-s)^{2(\alpha-1)}\mathcal{S}_E(b-s)BB^*\mathcal{S}_E^*(b-s)ds : \mathcal{D}(E) \to \mathcal{D}(E). \end{aligned} \tag{4.6}$$

respectively,where B^* denotes the adjoint of B and $\mathcal{S}_E^*(t)$ is the adjoint of $\mathcal{S}_E(t)$. It is straightforward that the operator Γ_0^b is a linear bounded operator for $\frac{1}{2} < \alpha \le 1$.
For more details, one con see [45],[14].

To establish the existence result, we need the following assumptions.
(**H4**)The function f satisfies the following tow conditions:

(i) For each $x \in \mathcal{C}_r$, the function $f(.,x) : J \to \mathcal{Z}$ is strongly measurable, and for each $t \in J$, the function $f(t,.) : \mathcal{C}_r \to \mathcal{Z}$ is continuous.

(ii) There is a positive integrable function $n \in L^1([0,b])$ and a continuous nondecreasing function $\Xi_f : [0,\infty) \to (0,\infty)$ such that for every $(t,x) \in J \times \mathcal{C}_r$, we have

$$\mathbb{E}\|f(t,x)\|^2 \le n(t)\Xi_f(\|x\|^2), \quad \liminf_{k\to\infty} \frac{\Xi_f(k)}{k} = \Lambda_f < \infty.$$

(**H5**): For each $(t,s) \in J \times J$, the function $H(t,s,.) : \mathcal{C}_r \to \mathcal{H}$ is continuous, and for each $x \in \mathcal{C}_r$, the function $H(.,.,x) : J \times J \to \mathcal{H}$ is strongly measurable.
(**H6**): The function σ satisfies the following tow conditions:

(i) For each $(t,s,x) \in J \times J \times \mathcal{C}_r$, the function $\sigma(t,s,.,.) : \mathcal{C}_r \times \mathcal{H} \to L_2^0$ is continuous, and for each $x \in \mathcal{C}_r, y \in \mathcal{H}$, the function $\sigma(.,x,y) : J \times J \to L_2^0$ is strongly measurable.

(ii) There is a positive integrable function $m \in L^1([0,b])$ and a continuous nondecreasing function $\Xi_\sigma : [0,\infty) \to (0,\infty)$ such that for every $(t,s,x,H) \in J \times J \times \mathcal{C}_r \times \mathcal{H}$, we have

$$\int_0^t \mathbb{E}\Big\|\sigma(t,s,x,\int_0^s H(s,\tau,x)d\tau)\Big\|_{L_2^0}^2 \le m(t)\Xi_\sigma(\|x\|^2), \quad \liminf_{k\to\infty} \frac{\Xi_\sigma(k)}{k} = \Lambda_\sigma < \infty.$$

The following lemma is required to define the control function. For more details see [46],[48]

Lemma 4.2.1 *[46],[48] For any* $\hat{x}_b \in L^2(\mathcal{F}_b, \mathcal{D}(E))$ *there exists* $\hat{\phi} \in L^2_{\mathcal{F}}(\Omega, L^2(J, L_2^0))$ *such that* $\hat{x}_b = \mathbb{E}\hat{x}_b + \int_o^b \hat{\phi}(s)dW(s)$

Now for any $k > 0$ *and* $\hat{x}_b \in L^2(\mathcal{F}_b, \mathcal{D}(E))$ *, we define the control function.*

$$\begin{aligned} u_\epsilon(t,x) &= (b-t)^{\alpha-1}B^*\mathcal{S}_E^*(b-t)(\epsilon I + \Gamma_0^b)^{-1}\Big\{\mathbb{E}\hat{x}_b + \int_o^b \hat{\phi}(s)dW(s) - \mathcal{T}_E(b)E\phi(0)\Big\} \\ &\quad -(b-t)^{\alpha-1}B^*\mathcal{S}_E^*(b-t)\int_0^b (\epsilon I + \Gamma_0^b)^{-1}(b-t)^{\alpha-1}\mathcal{S}_E(b-s)f(s,x(s))ds \\ &\quad -(b-t)^{\alpha-1}B^*\mathcal{S}_E^*(b-t)\int_0^b (\epsilon I + \Gamma_0^b)^{-1}(b-t)^{\alpha-1}\mathcal{S}_E(b-s) \\ &\quad \times\Big\{\int_o^s \sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\Big\}ds. \end{aligned} \tag{4.7}$$

4.2.1 Existence theorem

Let us now explain and prove the following theorem about the existence of solution for the fractional system (4.1)

Theorem 4.2.1 *Assume that **(H1)**-**(H6)** hold, then for each* $\epsilon > 0$*, the system (4.1) has a mild solution on J provided that.*

$$\frac{M^2\|E^{-1}\|^2 b^{2\alpha}}{\alpha^2\Gamma^2(\alpha)}\Big(4 + \|B\|^2\frac{16}{\epsilon^2}\|B\|^2\frac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}b^{\alpha-1}\frac{M^2\|E^{-1}\|^2 b^{2\alpha}}{\alpha^2\Gamma^2(\alpha)}\Big)$$

$$\times\left[\Lambda_f \sup_{s\in J} n(s) + \Lambda_\sigma \sup_{s\in J} m(s)\right] < 1$$

Proof: For all $\epsilon > 0$, consider the operator $\mathcal{P}_\epsilon : \mathcal{C}(J^*, \mathcal{H}) \to \mathcal{C}(J^*, \mathcal{H})$ defined by

$$\begin{aligned}(\mathcal{P}_\epsilon x)(t) &= \mathcal{T}_E(t)E\phi(0) + \int_o^t (t-s)^{\alpha-1}\mathcal{S}_E(t-s)Bu_\epsilon(s,x)ds \\ &\quad + \int_o^t (t-s)^{\alpha-1}\mathcal{S}_E(t-s)f(s,x_s)ds \\ &\quad + \int_o^t (t-s)^{\alpha-1}\mathcal{S}_E(t-s)\Big\{\int_o^s \sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\Big\}ds, \quad t \in J \\ (\mathcal{P}_\epsilon x)(t) &= \phi(t), \quad -r \le t \le 0.\end{aligned}$$

We shall show that for all $\epsilon > 0$, the operator $\mathcal{P}_\epsilon$ has a fixed point, which is then a mild solution for the system (4.1). To prove this we will employ the schauder fixed point theorem.

For each positive number q, define $B_q := \{x \in \mathcal{C}(j^*, \mathcal{H}) : \|x(t)\|^2 \le q, t \in J^*\}$. Then, for each q, B_r is clearly a bounded closed convex subset in $\mathcal{C}(j^*, \mathcal{H})$. The proof will be given in several steps.

step 1. We show that there exists a positive number $q := q(\epsilon)$ such that $\mathcal{P}_\epsilon(B_q) \subset B_q$. If it is not true, then for each positive number q, there exists a function $z^q(.) \in B_q$, but $\mathcal{P}_\epsilon(z^q) \in B_q$, that is, $\mathbb{E}\|(\mathcal{P}_\epsilon z^q)(t)\|^2 > q$ for some $t = t(q) \in J$. One can show that

$$\begin{aligned} q \le\ & \mathbb{E}\|(\mathcal{P}_\epsilon z^q)(t)\|^2 \le 4\mathbb{E}\|\mathcal{T}_E(t)E\phi(0)\|^2 + 4\mathbb{E}\Big\|\int_o^t (t-s)^{\alpha-1}\mathcal{S}_E(t-s)Bu_\epsilon(s,x)ds\Big\|^2 \\ & +4\mathbb{E}\Big\|\int_o^t (t-s)^{\alpha-1}\mathcal{S}_E(t-s)f(s,x_s)ds\Big\|^2 \\ & +4\mathbb{E}\Big\|\int_o^t (t-s)^{\alpha-1}\mathcal{S}_E(t-s)\Big\{\int_o^s \sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\Big\}ds\Big\|^2 =: 4\sum_{i=1}^4 I_i \end{aligned} \tag{4.8}$$

Let us now estimate each term above I_i,i=1,...,4; We have

$$I_1 \le M^2\|E^{-1}\|^2\mathbb{E}\|E\phi(0)\|^2. \tag{4.9}$$

Using the assuption (**H4**) and lemma 4.1.1, we have

$$\begin{aligned} I_2 &\leq \mathbb{E}\left[\int_o^t \|(t-s)^{\alpha-1}\mathcal{S}_E(t-s)f(s,x_s)\|ds\right]^2 \\ &\leq \tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\int_o^t (t-s)^{\alpha-1}\int_o^t (t-s)^{\alpha-1}\mathbb{E}\|f(s,x_s)\|^2 ds \\ &\leq \tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{b^\alpha}{\alpha}\int_o^t (t-s)^{\alpha-1}\mathbb{E}\|f(s,x_s)\|^2 ds \\ &\leq \tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{b^\alpha}{\alpha}\int_o^t (t-s)^{\alpha-1}n(s)\Xi_f(\|x(s)\|^2)ds \\ &\leq \tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{b^{2\alpha}}{\alpha^2}\Xi_f(q)\sup_{s\in J} n(s). \end{aligned} \tag{4.10}$$

A similar argument involves Bukholder-David-Gundy's inequality and assumptions (**H5**),(**H6**); we obtain

$$\begin{aligned} I_3 &\leq \mathbb{E}\left[\int_o^t \|(t-s)^{\alpha-1}\mathcal{S}_E(t-s)\Big\{\int_o^s \sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\Big\}\|ds\right]^2 \\ &\leq \tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{b^\alpha}{\alpha}\int_o^t (t-s)^{\alpha-1}\mathbb{E}\|\sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\|^2 ds \\ &\leq \tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{b^\alpha}{\alpha}ds\int_o^t (t-s)^{\alpha-1}\Big(\int_o^t \mathbb{E}\|\sigma(s,\tau,x_\tau,R(\tau))\|^2_{L_2^0}d\tau\Big)ds \\ &\leq \tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{b^\alpha}{\alpha}\int_o^t (t-s)^{\alpha-1}m(s)\Xi_\sigma(\|x(s)\|^2)ds \\ &\leq \tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{b^{2\alpha}}{\alpha^2}\Xi_\sigma(q)\sup_{s\in J} m(s). \end{aligned} \tag{4.11}$$

$$\begin{aligned} I_4 &\leq \mathbb{E}\left[\int_o^t \|(t-s)^{\alpha-1}\mathcal{S}_E(t-s)Bu_\epsilon(s,x)\|ds\right]^2 \\ &\leq \tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\|B\|^2\int_o^t (t-s)^{\alpha-1}ds\int_o^t (t-s)^{\alpha-1}\mathbb{E}\|u_\epsilon(s,x)\|^2 ds \end{aligned}$$

$$\leq \quad \tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\|B\|^2\tfrac{b^\alpha}{\alpha}\int_o^t (t-s)^{\alpha-1}ds\mathbb{E}\|u_\epsilon(s,x)\|^2 ds.$$

Further, by assumptions (**H4**)-(**H6**) and the Hölder inequality, we get

$$\begin{aligned}
\mathbb{E}\|u_\epsilon(s,x)\|^2 &\leq \tfrac{1}{\epsilon^2}\|B\|^2\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}(b-t)^{\alpha-1}\Big[4\Big\|\mathbb{E}\hat{x}_b+\int_o^b \hat{\phi}(s)dW(s)\Big\|^2 \\
&+4\mathbb{E}\|\mathcal{T}_E(b)E\phi(0)\|^2+4\mathbb{E}\Big\|\int_o^b \|(b-s)^{\alpha-1}\mathcal{S}_E(b-s)f(s,x_s)ds\Big\|^2 \\
&+4\mathbb{E}\Big\|\int_o^b \|(b-s)^{\alpha-1}\mathcal{S}_E(b-s)\Big\{\int_o^t \mathbb{E}\|\sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\Big\}ds\Big\|^2\Big] \\
&\leq \tfrac{4}{\epsilon^2}\|B\|^2\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}(b-t)^{\alpha-1}\Big[2\|\mathbb{E}\hat{x}_b+\int_o^b \mathbb{E}\|\hat{\phi}(s)\|^2 ds \\
&+M^2\|E^{-1}\|^2\mathbb{E}\|E\phi(0)\|^2+\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{b^{2\alpha}}{\alpha^2}\Xi_f(q)\sup_{s\in J} n(s). \\
&\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{b^{2\alpha}}{\alpha^2}\Xi_\sigma(q)\sup_{s\in J} m(s)\Big].
\end{aligned}$$

Therefore,

$$I_4 \leq \frac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\|B\|^2\frac{b^{2\alpha}}{\alpha^2}\times\frac{4}{\epsilon^2}\|B\|^2\frac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}b^{\alpha-1}M_1. \tag{4.12}$$

Where

$$\begin{aligned}
M_1 &= \Big[2\|\mathbb{E}\hat{x}_b\|^2+\int_o^b \mathbb{E}\|\hat{\phi}(s)\|^2 ds+M^2\|E^{-1}\|^2\mathbb{E}\|E\phi(0)\|^2 \\
&+\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{b^{2\alpha}}{\alpha^2}\Xi_f(q)\sup_{s\in J} n(s)+\frac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\frac{b^{2\alpha}}{\alpha^2}\Xi_\sigma(q)\sup_{s\in J} m(s)\Big].
\end{aligned}$$

Combining these estimates (4.8)-(4.12) yeilds

$$\begin{aligned} q &\le \mathbb{E}\|(\mathcal{P}_\epsilon z^q)(t)\|^2 \\ &\le M_2 + 4\tfrac{M^2\|E^{-1}\|^2 b^{2\alpha}}{\alpha^2\Gamma^2(\alpha)}\Big[\Xi_f(q)\sup_{s\in J} n(s) + \Xi_\sigma(q)\sup_{s\in J} m(s)\Big] \\ &\quad + \tfrac{M^2\|E^{-1}\|^2 b^{2\alpha}}{\alpha^2\Gamma^2(\alpha)}\|B\|^2\tfrac{16}{\epsilon^2}\|B\|^2\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}b^{\alpha-1}\tfrac{M^2\|E^{-1}\|^2 b^{2\alpha}}{\alpha^2\Gamma^2(\alpha)} \\ &\quad \times\Big[\Xi_f(q)\sup_{s\in J} n(s) + \Xi_\sigma(q)\sup_{s\in J} m(s)\Big]. \end{aligned} \tag{4.13}$$

Where

$$\begin{aligned} M_2 &= 4M^2\|E^{-1}\|^2\mathbb{E}\|E\phi(0)\|^2 + \tfrac{M^2\|E^{-1}\|^2 b^{2\alpha}}{\Gamma^2(\alpha)\alpha^2}\|B\|^2\tfrac{16}{\epsilon^2}\|B\|^2\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}b^{\alpha-1} \\ &\quad\times\Big[2\|\mathbb{E}\hat{x}_b + \int_o^b \mathbb{E}\|\hat{\phi}(s)\|^2 ds + M^2\|E^{-1}\|^2\mathbb{E}\|E\phi(0)\|^2\Big]. \end{aligned}$$

Dividing both sides of (4.13) by q and taking $q\to\infty$, we obtain that

$$\frac{M^2\|E^{-1}\|^2 b^{2\alpha}}{\alpha^2\Gamma^2(\alpha)}\Big(4+\|B\|^2\frac{16}{\epsilon^2}\|B\|^2\frac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}b^{\alpha-1}\frac{M^2\|E^{-1}\|^2 b^{2\alpha}}{\alpha^2\Gamma^2(\alpha)}\Big)\times\Big[\Lambda_f(q)\sup_{s\in J} n(s)+\Lambda_\sigma(q)\sup_{s\in J} m(s)\Big] \ge 1$$

Which is a contradiction to our assumption. Thus, for $\epsilon > 0$, for some positive number $q, \mathcal{P}_\epsilon(B_q) \subset B_q$.

step 2: We show the set $\mathcal{P}_\epsilon B_q = \{\mathcal{P}_\epsilon x : x \in B_q\}$ is an equicontinuous family of function. Let $0 < \eta < t < b$ and $\delta > 0$ such that $\|\mathcal{S}_E(s_1) - \mathcal{S}_E(s_2)\| < \eta$, for every $s_1, s_2 \in J$ with $|s_1, s_2| < \delta$. For $x \in B_q, 0 < |h| < \delta, t+h \in J$, we have

$$\begin{aligned} &\mathbb{E}\|(\mathcal{P}_\epsilon x)(t+h) - (\mathcal{P}_\epsilon x)(t)\|^2 \\ \le\ & 10\|\mathcal{T}_E(t+h) - \mathcal{T}_E(t)\|^2\mathbb{E}\|E\phi(0)\|^2 \\ &+10\mathbb{E}\Big\|\int_0^t \Big((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\Big)\mathcal{S}_E(t+h-s)Bu_\epsilon(s,x)ds\Big\|^2 \\ &+10\mathbb{E}\Big\|\int_t^{t+h}(t+h-s)^{\alpha-1}\mathcal{S}_E(t+h-s)Bu_\epsilon(s,x)ds\Big\|^2 \\ &+10\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\Big(\mathcal{S}_E(t+h-s) - \mathcal{S}_E(t-s)\Big)Bu_\epsilon(s,x)ds\Big\|^2 \end{aligned}$$

$$+10\mathbb{E}\Big\|\int_0^t \Big((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\Big)\mathcal{S}_E(t+h-s)f(s,x_s)ds\Big\|^2$$

$$+10\mathbb{E}\Big\|\int_t^{t+h}(t+h-s)^{\alpha-1}\mathcal{S}_E(t+h-s)f(s,x_s)ds\Big\|^2$$

$$+10\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\Big(\mathcal{S}_E(t+h-s)-\mathcal{S}_E(t-s)\Big)f(s,x_s)ds\Big\|^2$$

$$+10\mathbb{E}\Big\|\int_t^{t+h}(t+h-s)^{\alpha-1}\mathcal{S}_E(t+h-s)\left\{\int_0^s \sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\right\}ds\Big\|^2$$

$$+10\mathbb{E}\Big\|\int_0^t \Big((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\Big)\mathcal{S}_E(t+h-s)\left\{\int_0^s \sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\right\}ds\Big\|^2$$

$$+10\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\Big(\mathcal{S}_E(t+h-s)-\mathcal{S}_E(t-s)\Big)\left\{\int_0^s \sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\right\}ds\Big\|^2.$$

Applying Lemma 4.1.1 and the Hölder inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{P}_\epsilon x)(t+h)-(\mathcal{P}_\epsilon x)(t)\|^2\\
\leq\ & 10\|\mathcal{T}_E(t+h)-\mathcal{T}_E(t)\|^2\mathbb{E}\|E\phi(0)\|^2\\
&+\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\|B\|^2\int_0^t (t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}ds\\
&\times\int_0^t \Big((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\Big)\mathbb{E}\|u_\epsilon(s,x)\|^2ds\\
&+10\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\|B\|^2\tfrac{h^\alpha}{\alpha}\int_t^{t+h}(t+h-s)^{\alpha-1}\mathbb{E}\|u_\epsilon(s,x)\|^2ds\\
&+10\eta^2\tfrac{b^\alpha}{\alpha}\|B\|^2\int_0^t (t-s)^{\alpha-1}\mathbb{E}\|u_\epsilon(s,x)\|^2ds\\
&+10\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\int_0^t (t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}ds\\
&\times\int_0^t \Big((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\Big)\mathbb{E}\|f(s,x_s)\|^2ds
\end{aligned}$$

$$+10\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{h^\alpha}{\alpha}\int_t^{t+h}(t+h-s)^{\alpha-1}\mathbb{E}\|f(s,x_s)\|^2ds$$

$$+10\eta^2\tfrac{b^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}\mathbb{E}\|f(s,x_s)\|^2ds$$

$$+10\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\int_0^t(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}ds$$

$$\times\int_0^t\Big((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\Big)\mathbb{E}\|\int_0^s\sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\|^2ds$$

$$+10\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{h^\alpha}{\alpha}\int_t^{t+h}(t+h-s)^{\alpha-1}\mathbb{E}\|\int_0^s\sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\|^2ds$$

$$+10\eta^2\tfrac{b^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}\mathbb{E}\|\int_0^s\sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\|^2ds$$

By assumptions (**H4**)-(**H6**), we have

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{P}_\epsilon x)(t+h)-(\mathcal{P}_\epsilon x)(t)\|^2\\
\leq\;& 10\|\mathcal{T}_E(t+h)-\mathcal{T}_E(t)\|^2\mathbb{E}\|E\phi(0)\|^2\\
&+\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\|B\|^2\int_0^t(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}ds\\
&\times\int_0^t\Big((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\Big)\mathbb{E}\|u_\epsilon(s,x)\|^2ds\\
&+10\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\|B\|^2\tfrac{h^\alpha}{\alpha}\int_t^{t+h}(t+h-s)^{\alpha-1}\mathbb{E}\|u_\epsilon(s,x)\|^2ds\\
&+10\eta^2\tfrac{b^\alpha}{\alpha}\|B\|^2\int_0^t(t-s)^{\alpha-1}\mathbb{E}\|u_\epsilon(s,x)\|^2ds\\
&+10\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\Xi_f(q)\int_0^t(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}ds\\
&\times\int_0^t\Big((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\Big)n(s)ds
\end{aligned}$$

$$+10\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{h^\alpha}{\alpha}\Xi_f(q)\int_t^{t+h}(t+h-s)^{\alpha-1}n(s)ds$$

$$+10\eta^2\tfrac{h^\alpha}{\alpha}\Xi_f(q)\int_0^t(t-s)^{\alpha-1}n(s)ds$$

$$+10\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\int_0^t(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}ds$$

$$\times\int_0^t\Big((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\Big)m(s)\Xi_\sigma(q)ds$$

$$+10\tfrac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}\tfrac{h^\alpha}{\alpha}\Xi_f(q)\int_t^{t+h}(t+h-s)^{\alpha-1}m(s)\Xi_\sigma(q)ds$$

$$+10\eta^2\tfrac{h^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}m(s)\Xi_\sigma(q)ds.$$

Therefore,for ϵ sufficiently small,the right-hand side of the above inequality tends to zero as $h\to 0$. On the other hand, the compactness of $\mathcal{T}_E$ and $\mathcal{S}_E$,(Lemma 4.1.1 implies the continuity in the uniform operator topology. Thus,the set $\mathcal{P}_\epsilon B_q$ is equicontinuous.

Step 3. The operator $\mathcal{P}_\epsilon$ maps B_q into a precompact set in B_q. To prove this, we first show that the set $V_q(t)=\{(\mathcal{P}_\epsilon x)(t): x\in B_q\}$ is precompact in $\mathcal{H}$, for every $t\in J$.This is trivial for $t\in[-r,0]$, since $V_q(t)=\{\phi(t)\}$. Let $0<t\le b$ be fixed and η be real number satisfying $0<\eta<t$. For $\delta>0$, define an operator $\mathcal{P}_\epsilon^{\eta,\delta}$ on B_q by

$$\begin{aligned}(\mathcal{P}_\epsilon x)(t) \;=\; & \int_\delta^\infty \xi_\alpha(\theta)E^{-1}T(t^\alpha\theta)E\phi(0)d\theta\\ & +\alpha\int_0^{t-\eta}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\Big((t-s)^\alpha\theta\Big)Bu_\epsilon(s,x)d\theta ds\\ & +\alpha\int_0^{t-\eta}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\Big((t-s)^\alpha\theta\Big)f(s,x_s)d\theta ds\\ & +\alpha\int_0^{t-\eta}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\Big((t-s)^\alpha\theta\Big)\\ & \times\Big\{\int_0^s\sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\Big\}\,d\theta ds\end{aligned}$$

$$
\begin{aligned}
&= T(\eta^\alpha\delta)\int_\delta^\infty \xi_\alpha(\theta)E^{-1}T(t^\alpha\theta-\eta^\alpha\delta)E\phi(0)d\theta \\
&+T(\eta^\alpha\delta)\alpha\int_0^{t-\eta}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\Big((t-s)^\alpha\theta-\eta^\alpha\delta\Big)Bu_\epsilon(s,x)d\theta ds \\
&+T(\eta^\alpha\delta)\alpha\int_0^{t-\eta}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\Big((t-s)^\alpha\theta-\eta^\alpha\delta\Big)f(s,x)d\theta ds \\
&+T(\eta^\alpha\delta)\alpha\int_0^{t-\eta}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\Big((t-s)^\alpha\theta-\eta^\alpha\delta\Big) \\
&\times\left\{\int_0^s \sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\right\}d\theta ds.
\end{aligned}
$$

Since E^{-1} is compact operator, the set $\{(\mathcal{P}_\epsilon^{\eta,\delta}x)(t) : x \in B_q\}$ is precompact in $\mathcal{H}$, for every $0<\eta<t, \delta>0$.
Moreover, for each $x\in B_q$, We have:

$$
\begin{aligned}
&\mathbb{E}\|(\mathcal{P}_\epsilon x)(t)-(\mathcal{P}_\epsilon^{\eta,\delta}x)(t)\|^2 \\
\leq\ & 7\mathbb{E}\Big\|\int_0^\delta \xi_\alpha(\theta)E^{-1}T(t^\alpha\theta)E\phi(0)d\theta\Big\|^2 \\
&+7\alpha^2\mathbb{E}\Big\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\Big((t-s)^\alpha\theta\Big)Bu_\epsilon(s,x)d\theta ds\Big\|^2 \\
&+7\alpha^2\mathbb{E}\Big\|\int_{t-\eta}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\Big((t-s)^\alpha\theta\Big)Bu_\epsilon(s,x)d\theta ds\Big\|^2 \\
&+7\alpha^2\mathbb{E}\Big\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\Big((t-s)^\alpha\theta\Big)f(s,x_s)d\theta ds\Big\|^2 \\
&+7\alpha^2\mathbb{E}\Big\|\int_{t-\eta}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\Big((t-s)^\alpha\theta\Big)f(s,x_s)d\theta ds\Big\|^2 \\
&+7\alpha^2\mathbb{E}\Big\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\Big((t-s)^\alpha\theta\Big) \\
&\times\left\{\int_0^s \sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\right\}d\theta ds\Big\|^2
\end{aligned}
\tag{4.14}
$$

$$+7\alpha^2\mathbb{E}\Big\|\int_{t-\eta}^{t}\int_{\delta}^{\infty}\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\Big((t-s)^\alpha\theta\Big)$$

$$\times\Big\{\int_0^s\sigma(s,\tau,x_\tau,R(\tau))dW(\tau)\Big\}d\theta ds\Big\|^2=\sum_{i=1}^{7}J_i.$$

A similar argument as before can show that

$$J_1\le 7M^2\|E^{-1}\|^2\mathbb{E}\|E\phi(0)\|^2\int_0^\delta\xi_\alpha(\theta)d\theta, \tag{4.15}$$

$$\begin{aligned}
J_2 &\le 7\alpha^2\mathbb{E}\Big[\int_0^t\int_0^\delta\|\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\Big((t-s)^\alpha\theta\Big)Bu_\epsilon(s,x)d\theta\|ds\Big]^2\\
&\le 7\alpha^2\|B\|^2M^2\|E^{-1}\|^2\int_0^t(t-s)^{\alpha-1}ds\int_0^t(t-s)^{\alpha-1}\mathbb{E}\|u_\epsilon(s,x)\|^2ds\Big(\int_0^\delta\theta\xi_\alpha(\theta)d\theta\Big)^2\\
&\le 7\alpha^2\|B\|^2M^2\|E^{-1}\|^2b^\alpha\int_0^t(t-s)^{\alpha-1}\Big(\frac{4}{\epsilon^2}\|B\|^2\frac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}b^{\alpha-1}M_1\Big)ds\Big(\int_0^\delta\theta\xi_\alpha(\theta)d\theta\Big)^2.
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
J_3 &\le 7\alpha^2\|B\|^2M^2\|E^{-1}\|^2\int_{t-\eta}^t(t-s)^{\alpha-1}ds\int_{t-\eta}^t(t-s)^{\alpha-1}\mathbb{E}\|u_\epsilon(s,x)\|^2ds\Big(\int_0^\delta\theta\xi_\alpha(\theta)d\theta\Big)^2\\
&\le 7\alpha^2\|B\|^2M^2\|E^{-1}\|^2\eta^\alpha\int_{t-\eta}^t(t-s)^{\alpha-1}\Big(\frac{4}{\epsilon^2}\|B\|^2\frac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}b^{\alpha-1}M_1\Big)ds\Big(\int_0^\infty\theta\xi_\alpha(\theta)d\theta\Big)^2\\
&\le \tfrac{7\alpha^2\|B\|^2M^2\|E^{-1}\|^2\eta^\alpha}{\Gamma^2(1+\alpha)}\int_{t-\eta}^t(t-s)^{\alpha-1}\Big(\frac{4}{\epsilon^2}\|B\|^2\frac{M^2\|E^{-1}\|^2}{\Gamma^2(\alpha)}b^{\alpha-1}M_1\Big)ds.
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
J_4 &\le 7\alpha^2M^2\|E^{-1}\|^2\int_0^t(t-s)^{\alpha-1}ds\int_0^t(t-s)^{\alpha-1}\mathbb{E}\|f(s,x_s)\|^2ds\Big(\int_0^\delta\theta\xi_\alpha(\theta)d\theta\Big)^2\\
&\le 7\alpha M^2\|E^{-1}\|^2b^\alpha\Xi_f(q)\int_0^t(t-s)^{\alpha-1}n(s)ds\Big(\int_0^\delta\theta\xi_\alpha(\theta)d\theta\Big)^2.
\end{aligned} \tag{4.18}$$

and

$$\begin{aligned} J_5 &\leq 7\alpha^2 M^2 \|E^{-1}\|^2 \int_{t-\eta}^{t} (t-s)^{\alpha-1} ds \int_{t-\eta}^{t} (t-s)^{\alpha-1} \mathbb{E}\|f(s,x_s)\|^2 ds \Big(\int_0^{\delta} \theta \xi_\alpha(\theta) d\theta \Big)^2 \\ &\leq 7\alpha M^2 \|E^{-1}\|^2 \eta^\alpha \Xi_f(q) \int_{t-\eta}^{t} (t-s)^{\alpha-1} n(s) ds \Big(\int_0^{\infty} \theta \xi_\alpha(\theta) d\theta \Big)^2 \\ &\leq \tfrac{7\alpha M^2 \|E^{-1}\|^2 \eta^\alpha \Xi_f(q)}{\Gamma^2(1+\alpha)} \int_{t-\eta}^{t} (t-s)^{\alpha-1} n(s) ds. \end{aligned} \tag{4.19}$$

Similarly, employing Burkhölder-Davis-Gundy's inequality, we further derive that

$$\begin{aligned} J_6 &\leq 7\alpha M^2 \|E^{-1}\|^2 b^\alpha \int_0^t (t-s)^{\alpha-1} \mathbb{E}\| \int_0^s \sigma(s,\tau,x_\tau,R(\tau))\|^2 ds \Big(\int_0^{\infty} \theta \xi_\alpha(\theta) d\theta \Big)^2 \\ &\leq 7\alpha M^2 \|E^{-1}\|^2 b^\alpha \Xi_\sigma(q) \int_0^t (t-s)^{\alpha-1} m(s) ds \Big(\int_0^{\infty} \theta \xi_\alpha(\theta) d\theta \Big)^2. \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} J_7 &\leq 7\alpha M^2 \|E^{-1}\|^2 \eta^\alpha \int_{t-\eta}^{t} (t-s)^{\alpha-1} \mathbb{E}\| \int_0^s \sigma(s,\tau,x_\tau,R(\tau))\|^2 ds \Big(\int_0^{\infty} \theta \xi_\alpha(\theta) d\theta \Big)^2 \\ &\leq \tfrac{7\alpha M^2 \|E^{-1}\|^2 \eta^\alpha}{\Gamma^2(1+\alpha)} \Xi_\sigma(q) \int_{t-\eta}^{t} (t-s)^{\alpha-1} m(s) ds. \end{aligned} \tag{4.21}$$

Recalling (4.14), from (4.15)-(4.21), we see that for each $x \in B_q$,

$$\mathbb{E}\|(\mathcal{P}_\epsilon x)(t) - (\mathcal{P}_\epsilon^{\eta,\delta} x)(t)\|^2 \to 0 \quad as \quad \eta \to 0^+, \delta \to 0^+.$$

Therefore, there are relatively compact sets arbitrary close to the set $\{(\mathcal{P}_\epsilon x)(t) : x \in B_q\}$; hence, the set $\{(\mathcal{P}_\epsilon x)(t) : x \in B_q\}$ is also precompact in B_q.

Finally, combining *Step 1* to *Step 3* with Arzela-Ascoli theorem, we conclude that for all $\epsilon > 0, \mathcal{P}_\epsilon B_q$ is precompact in $\mathcal{C}(J^*, \mathcal{H})$.Hence,$\mathcal{P}_\epsilon$ is a completely continuous operator on $\mathcal{C}(J^*, \mathcal{H})$.From the Schauder fixed point theorem, $\mathcal{P}_\epsilon$ has a fixed point in B_q.Any fixed point of $\mathcal{P}_\epsilon$ is a mild solution of (4.1) on J satisfying $(\mathcal{P}_\epsilon x)(t) = x(t) \in \mathcal{H}$.

4.2.2 Approximate controllability result

Further, to prove the approximate controllability result, the following additional assumption is required;

(**H7**) The linear system (4.5) is approximately controllable.

(**H8**) The functions $f(t,x) : J \times \mathcal{C}_r \to \mathcal{Z}$ and $\sigma(t,s,x,y) : J \times J \times \mathcal{C}_r \times \mathcal{H} \to L_2^0$ are bounded for all $t,s \in J, x \in \mathcal{C}_r$ and $y \in \mathcal{H}$

Remark. In view of [45], the assumption **(H7)** is equivalent to $\epsilon R(\epsilon, \Gamma_0^b) = \epsilon(\epsilon I + \Gamma_0^b)^{-1} \to 0$ as $\epsilon \to 0$ in the strong operator.

Theorem 4.2.2 *Assume that the assumptions of Theorem (4.2.1) hold and in addition,* ***(H7)*** *and* ***(H8)*** *are satisfied. Then, the fractional control system (4.1) is approximately controllable on J.*

Proof:Let $x^\epsilon \in B_q$ be a fixed point of the operator $\mathcal{P}_\epsilon$. Using the stochastic fubini theorem, it is easy to see that

$$\begin{aligned} x^\epsilon(b) &= \hat{x}_b - \epsilon(\epsilon I + \Gamma_0^b)^{-1}\left[\mathbb{E}\hat{x}_b + \int_0^b \hat{\phi}(s)ds - \mathcal{T}_E(b)E\phi(0)\right] \\ &\quad + \epsilon\int_0^b (\epsilon I + \Gamma_s^b)^{-1}(b-s)^{\alpha-1}\mathcal{S}_E(b-s)f(s,x_s^\epsilon)ds \\ &\quad + \epsilon\int_0^b (\epsilon I + \Gamma_s^b)^{-1}(b-s)^{\alpha-1}\mathcal{S}_E(b-s) \\ &\quad \times\left\{\int_0^s \sigma\Big(s,\tau,x_\tau^\epsilon, \int_0^\tau H(\tau,\nu,x_\nu^\epsilon)d\nu\Big)dW(\tau)\right\}ds. \end{aligned}$$

Moreover, by the assumption (**H8**), there exists $N_1 > 0$ and $N_2 > 0$ such that $\|f(s,x_s^\epsilon)\|^2 \leq N_1, \|\sigma(s,\tau,x_\tau^\epsilon, \int_0^\tau H(\tau,\nu,x_\nu^\epsilon)d\nu\|^2 \leq N_2$, and consequently, there is a sequence still denoted by $\{f(s,x_\tau^\epsilon), \sigma(s,\tau,x_\tau^\epsilon, \int_0^\tau H(\tau,\nu,x_\nu^\epsilon)d\nu)\}$ weakly converges to say $\{f(s), \sigma(s,\tau, \int_0^\tau H(\tau,\nu,x_\nu^\epsilon)d\nu)\}$. Thus, from the above aquation, we have

$$
\begin{aligned}
\mathbb{E}\|x^{\epsilon}(b)-\hat{x}_b\|^2 \;\leq\;& 6\|\epsilon(\epsilon I+\Gamma_0^b)^{-1}[\mathbb{E}\hat{x}_b-\mathcal{T}_E(b)E\phi(0)]\|^2\\
&+6\mathbb{E}\Big(\int_0^b \|\epsilon(\epsilon I+\Gamma_0^b)^{-1}\hat{\phi}(s)ds\|^2_{L_2^0}ds\Big)\\
&+6\mathbb{E}\Big(\int_0^b (b-s)^{\alpha-1}\|\epsilon(\epsilon I+\Gamma_0^b)^{-1}\mathcal{S}_E(b-s)[f(s,x_s^{\epsilon})-f(s)]\|ds\Big)^2\\
&+6\mathbb{E}\Big(\int_0^b (b-s)^{\alpha-1}\|\epsilon(\epsilon I+\Gamma_0^b)^{-1}\mathcal{S}_E(b-s)f(s)\|ds\Big)^2\\
&+6\mathbb{E}\Big(\int_0^b (b-s)^{\alpha-1}\|\epsilon(\epsilon I+\Gamma_0^b)^{-1}\mathcal{S}_E(b-s)\\
&\times\Big[\int_0^s[\sigma\Big(s,\tau,x_\tau^{\epsilon},\int_0^\tau H(\tau,\nu,x_\nu^{\epsilon})d\nu\Big)-\sigma\Big(s,\tau,\int_0^\tau H(\tau,\nu)d\nu\Big)]dW(\tau)\Big]\|ds\Big)^2\\
&+6\mathbb{E}\Big(\int_0^b (b-s)^{\alpha-1}\|\epsilon(\epsilon I+\Gamma_0^b)^{-1}\mathcal{S}_E(b-s)\\
&\times\Big[\int_0^s[\sigma\Big(s,\tau,\int_0^\tau H(\tau,\nu)d\nu\Big)dW(\tau)\Big]\|ds\Big)^2\\
&+6\mathbb{E}\Big(\int_0^b (b-s)^{\alpha-1}\|\epsilon(\epsilon I+\Gamma_0^b)^{-1}\mathcal{S}_E(b-s)\\
&\times\Big[\int_0^s[\sigma\Big(s,\tau,\int_0^\tau H(\tau,\nu)d\nu\Big)dW(\tau)\Big]\|ds\Big)^2.
\end{aligned}
$$

On the other hand, by assumption(**H7**) for all $0\leq s\leq b$, the operator is $\epsilon(\epsilon I+\Gamma_0^b)^{-1}$ strongly as $\epsilon\to 0^+$, and moreover $\|\epsilon(\epsilon I+\Gamma_0^b)^{-1}\|\leq 1$. Thus, by the Lebesgue's dominated convergence theorem and the compactness of $\mathcal{S}_E(t)$, we obtain $\mathbb{E}\|x^{\epsilon}(b)-\hat{x}_b\|^2\to 0$ as $\epsilon\to 0^+$. This prove the approximate controllability of (4.1)

Remark: theorem (4.2.2) assume that the operator E^{-1} is compact and consequently $\mathcal{S}_E(t)$ is compact (lemme(4.1.1)). Therefore, the associated linear control system (4.5) is not exactly controllable. Thus, Theorem (4.2.1) has no analog for the concept of exact controllability.

4.3 An example

In this chapter, we consider an example to illustrate our main theorem. We consider the following fractional stochastic integro-partial differential equation in the form

$$\begin{aligned}
{}^{c}\partial_t^{\alpha}\Big(z(t,x)-z_{xx}(t,x)\Big)-z_{x,x}(t,x) &= Bu+\mu_1\Big(t,z_{xx}(t-r,x)\Big)\\
&\quad+\int_0^t \mu_3\Big(t,s,z_{xx}(s-r,x),\\
&\quad\int_0^s \mu_2\Big(s,\tau,z_{xx}(\tau-r,x)\Big)d\tau\Big)d\beta(s) \qquad (4.22)\\
&\quad 0\le x\le\pi, \tau>0, t\in J=[0,1]\\
z(t,0)=z(t,\pi)=0, &\quad t\ge 0\\
z(t,x)=\phi(t,x), &\quad 0\le x\le\pi,\quad -1\le t\le 0,
\end{aligned}$$

where $\beta(t)$ is a standard cylindrical Wiener process in $\mathcal{H}$ defined on a stochastic space $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\})$ and ${}^{c}\partial_t^{\alpha}$ is the Caputo fractional partial derivative of order $0<\alpha<1$.

Take $\mathcal{H}=\mathcal{Z}=\mathcal{U}=L^2([0,\pi])$ and define the operator $A:\mathcal{D}(A)\subset\mathcal{H}\to\mathcal{Z}$ and $E:\mathcal{D}(A)\subset\mathcal{H}\to\mathcal{Z}$ by $Az=-z_{xx}$ and $Ez=z-z_{xx}$, where each domain $\mathcal{D}(A)$ and $\mathcal{D}(E)$ is given by
$\{z\in\mathcal{H}; z, z_x$ are absolutely continuous, $z_{xx}\in\mathcal{H}$ and $z(0)=z(\pi)=0\}$.

Then A and E can be written, respectively, as [42],

$$Az=\sum_{n=1}^{\infty}n^2(z,z_n)z_n,\quad z\in\mathcal{D}(A) \text{ and } Ez=\sum_{n=1}^{\infty}(1+n^2)(z,z_n)z_n,\quad z\in\mathcal{D}(E).$$

Where $z_n(x)=\sqrt{\frac{2}{\pi}}\sin(nz), n=1,2...$ is the orthonormal set of eigenvectors of A and (z,z_n) is the L^2 inner product. Moreover, for any $z\in\mathcal{H}$, we get

$$E^{-1}z=\sum_{n=1}^{\infty}\frac{1}{1+n^2}(z,z_n)z_n,\quad -AE^{-1}z=\sum_{n=1}^{\infty}\frac{-n^2}{1+n^2}(z,z_n)z_n,$$

and

$$T(t)z=\sum_{n=1}^{\infty}e^{\frac{-n^2}{1+n^2}}(z,z_n)z_n.$$

Define an infinite-dimensional space $\mathcal{U}$ by $\mathcal{U} = \{u\ u = \sum_{n=2}^{\infty} u_n z_n\}$ with $\{\sum_{n=2}^{\infty} \mathcal{U}_n^2 < \infty\}$. The norm in $\mathcal{U}$ is defined by $\|u\|_{\mathcal{U}} = \sum_{n=2}^{\infty} \mathcal{U}_n^2$. Now, define a continuous linear mapping B from $\mathcal{U}$ into $\mathcal{Z}$ as $Bu = 2u_2 z_1 + \sum_{n=2}^{\infty} u_n z_n$ for $u = \sum_{n=2}^{\infty} u_n z_n \in \mathcal{U}$.
We assume that

(i) The operator $B : \mathcal{U} \to \mathcal{Z}$ with $\mathcal{U} \subset J$, is a bounded linear operator .

(ii) The nonlinear operator $\mu_1 : [0,1] \times \mathcal{H} \to \mathcal{Z}$ satisfies the following conditions:

- **(a)** For each $t \in J, \mu_1(t,z)$ is continuous.
- **(b)** For each $z \in \mathcal{H}, \mu_1(t,z)$ is measurable.
- **(c)** There is a constant $\nu(0 < \nu < 1)$ and a positive integrable function $\gamma \in L^1([0,1])$ such that for all $(t,z) \in [0,1] \times \mathcal{H}$

$$\|\mu_1(t,z)\|_{\mathcal{C}_0} \leq \gamma(t)\|z\|_{\mathcal{H}}^{\nu}.$$

(iii) The nonlinear operator $\mu_2 : J \times J \times \mathcal{H} \to \mathcal{H}$ satisfies the following conditions:

- **(a)** For each $(t,s) \in J \times J, \mu_2(t,s,z)$ is continuous.
- **(b)** For each $z \in \mathcal{H}, \mu_2(t,s,z)$ is measurable.

(iv) The nonlinear operator $\mu_3 : J \times J \times \mathcal{H} \times \mathcal{H} \to L(\mathcal{K}, \mathcal{Z})$ satisfies the following conditions:

- **(a)** For each $(t,s,z) \in J \times J \times \mathcal{H}, \mu_3(t,s,z)$ is continuous.
- **(b)** For each $z \in \mathcal{H}, \mu_3(t,s,z)$ is measurable.
- **(c)** There is a constant $\nu(0 < \nu < 1)$ and a positive integrable function $\tilde{\gamma} \in L^1([0,1])$ such that for all $(t,s,z,y) \in J \times J \times \mathcal{H} \times \mathcal{H}$

$$\int_0^t \|\mu_3(t,s,z, \int_0^s \mu_2(s,\tau,z)d\tau)\|ds \leq \tilde{\gamma}(t)\|z\|_{\mathcal{H}}^{\nu}.$$

Define an operator $f : [0,1] \times \mathcal{C}([-1,0],\mathcal{H}) \to \mathcal{Z}$ by $f(t,z)(x) = \mu_1(t, z_{xx}(-r)(x))$, and let $H(t,s,z)(x) = \mu_2(t,s,z_{xx}(-r)(x)), (t,s,z) \in [0,1] \times [0,1] \times \mathcal{C}([-1,0],\mathcal{H})$,

$$\sigma\Big(t,s,z,\int_0^s H(s,\tau,z)d\tau\Big)(x) = \mu_3\Big(t,s,z_{xx},\int_0^s \mu_2(s,\tau,z_{xx}(-r)(x))d\tau\Big), \quad x \in [0,\pi]$$

On the other hand, the linear system corresponding to (4.22) is approximately controllable (but not exactly controllable). Thus, with the above choices of A, E, B, μ_1, μ_2 and μ_3, the problem (4.22) can be formulated abstractly as

$$\begin{aligned} {}^C D^\alpha\Big(Ez(t)\Big) + Az(t) &= Bu(t) + f(t,z_t) + \int_0^t \sigma\Big(t,s,z,\int_0^s H(s,\tau,z)d\tau\Big)dW(x), \\ & t \in J, z(t) = \phi(t). \end{aligned}$$

Also, all the conditions of Theorem 4.2.2 are satisfied. Hence, by Theorem 4.2.2 the fractional control system (4.22) is approximately controllable on $J := [0,1]$.

Conclusion

We are focused on establishing the approximate controllability result for a class of fractional stochastic differential systems involving the Caputo fractional derivative in Hilbert spaces.

By employing fractional calculus, fixed-point technique and solution operator theory, sufficient conditions for the approximate controllability of semilinear fractional stochastic dynamic system are formulated and proved under the assumption that the associated linear system is approximately controllable.

Our further work will be devoted to study approximate controllability of the above problems via characteristic solution operators with the help of the theory of propagation family and the techniques of the measure of noncompactness.

Bibliography

[1] Ahmed .H.M: Fractional neutral evolution equations with nonlocal conditions. Adv. Differ. Equ. 2013, 117 (2013)

[2] Ahmed .H.M: Controllability for Sobolev-type fractional integro-differential systems in a Banach space. Adv. Differ. Equ. 2012, 167 (2012)

[3] Abbas.S, Benchohra.M, N'Guérékata.G.M : Topics in Fractional Differential Equations. Developments in Mathematics, 27. Springer, New York (2012)

[4] Balachandran.K, Govindaraj.V, Rodrïguez-Germa.L, Trujillo.J.J: Controllability results for nonlinear fractional-order dynamical systems. J. Optim. Theory Appl. 156, 33-44 (2013).

[5] Balachandran.J, Dauer .J.P, Controllability of nonlinear systems via fixed point theorems, *Journal of Optimization Theory and Applications*, 53(1987), 345-352.

[6] Balachandran K, Kim J.H, Karthikeyan S, Complete controllability of stochastic integrodifferential systems, *Dynamic Systems and Applications*, 17(2008), 43-52.

[7] Balachandran K, Karthikeyan S, Controllability of stochastic integrodifferential systems, *International Journal of Control*, 80(2007), 486-491.

[8] Benchora M,Henderson J,Ntouyas S.K, A.Ouahab: Existence results for fractional order functional differential equations with infinite delay. J. Math. Anal. Appl. 338, 1340-1350 (2008)

[9] Cao J,Yang Q, Z.Huang,Liu Q: Asymptotically almost periodic solutions of stochastic functional differential equations. Appl. Math. Comput. 218, 1499-1511 (2011)

[10] Cao J., Q.Yang, Z. Huang: On almost periodic mild solutions for stochastic functional differential equations. Nonlinear Anal. TMA. 13, 275-286 (2012).

[11] Chang Y.K., Z.H.Zhao, G.M.N'Guérékata, R.Ma : Stepanov-like almost automorphy for stochastic processes and applications to stochastic differential equations. Nonlinear Anal. RWA. 12, 1130-1139 (2011)

[12] Chang Y.-K., R.Ma, Z.-H.Zhao: Almost periodic solutions to a stochastic differential equation in Hilbert spaces. Results Math. (2012). doi:10.1007/ s00025-011-0207-9.

[13] Cui J., L.Yan,: Existence result for fractional neutral stochastic integrodifferential equations with infinite delay. J. Phys. A 44, 335201 (2011).

[14] Carimichel N., M.D.Quinn: Fixed point methods in nonlinear control. In: Lecture Notes in Control and Information Society, vol. 75. Springer, Berlin (1984).

[15] Da Prato,G Zabczyk,J. stochastic equations in infinite dimensions. Cambridge University press 1992.

[16] Da Prato,G Zabczyk,J. Ergodicity of infinite dimensionnel system. Cambridge University press 1996.

[17] Debbouche A., D.Baleanu: Exact null controllability for fractional nonlocal integrodifferential equations via implicit evolution system. J. Appl. Math. 2012, 1-17 (2012).

[18] Dabas J., A.Chauhan, M.Kumar: Existence of the mild solutions for impulsive fractional equations with infinite delay. Int. J. Differ. Equ. 2011, 1-20 (2011). Art ID 793023

[19] Fečkan M., Y.Zhou, J.R.Wang,: On the concept and existence of solution for impulsive fractional differential equations. Commun. Nonlinear Sci. 17, 3050-3060 (2012)

[20] Fečkan M., J.R.Wang, YZhou,: Controllability of fractional functional evolution equations of Sobolev-type via characteristic solution operators. J. Optim. Theory Appl. 156, 79-95 (2013)

[21] Fu M.M., Z.X.Liu: Square-meanal most automorphic solutions for some stochastic differential equations. Proc. Am. Math. Soc. 138(10), 3689-3701 (2010)

[22] Fima C Klebaner, Introduction to Stochastic Calculus with Applications, Imperial College Press, 1998.

[23] Guendouzi T., K.Mehdi: Existence of mild solutions for impulsive fractional stochastic equations with infinite delay. Malaya J. Matematik 4(1), 30-43 (2013)

[24] Guendouzi T.: Existence and controllability of fractional-order impulsive stochastic system with infinite delay. Discuss. Math. Differ. Incl. Control Optim. 33(1), 65-87 (2013)

[25] Guendouzi T., F.souad : Approximate Controllability of Semilinear Fractional Stochastic Dynamic Systems with Nonlocal Conditions in Hilbert Spaces 13(2016), Springer Basel

[26] Guendouzi T., F.souad : Approximate Controllability of Sobolev-type fractional functional stochastic integro-differential systems; Sociedad Matemática Mexicana 2015

[27] Guendouzi T., F.souad : Approximate Controllability of Fractional Neutral Stochastic Evolution Equations with Nonlocal Conditions; 65(2014), Springer Basel

[28] Hilfer R.: Applications of Fractional Calculus in Physics. World Scientific, River Edge (2000)

[29] Klamka J.: Local controllability of fractional discrete-time semilinear systems. Acta Mech Et Autom 15, 55-68 (2011)

[30] Klamka J., Schauder's fixed point theorem in nonlinear controllability problems, *Control and Cybernetics*, 29(2000), 153-165.

[31] Klamka J., Stochastic controllability of systems with multiple delays in control, *Int. J. Appl. Math. Comput. Sci*, 19(1)(2009), 39-47.

[32] Klamka J., Constrained exact controllability of semilinear systems, *Systems and Control Letters*, 47(2002), 139-147.

[33] Klamka J., Constrained approximate controllability, *IEEE Transactions on Automatic Control*, 45(2000), 1745-1749.

[34] Kaczorek T.: Selected Problems of Fractional Systems Theory. Springer, Berlin (2011)

[35] Kilbas A.A., H.M.Srivastava, J.J.Trujillo :Theory and Applications of Fractional Differential Equations. Elsevier Science B.V, Amsterdam (2006)

[36] Knoche, C., Frieler, K. Solution of stochastic differential equations in infinite dimensional hilbert spaces and their dependence on initial data, Fakultät für Mathematik universität Bielefeld, Juni (2001).

[37] Kumlin, P.(2004). A note of fixed point theory, functional Analysis Lecture, Mathematic, Chalmers, GU, Gothumberg.

[38] Kumar S., N.Sukavanam, Approximate controllability of fractional order semilinear systems with bounded delay. J. Differ. Equ. 251, 6163-6174, (2012)

[39] Lakshmikantham V., S.Leela, J.Vasundhara Devi: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge (2009)

[40] Lakshmikantham, V. Theory of fractional functional differential equation, Nonlinear Anal. **69** (2008), 3337-3343.

[41] Lakshmikantham, V. and Devi, J. V. Theory of fractional differential equations in a Banach space, Eur. J. Pure Appl. Math. **1** (1) (2008), 38-45.

[42] Lightboure J.H., S.M.Rankin : A partial functional differential equation of Sobolev type. J. Math. Anal. Appl. 93, 328-337 (1983)

[43] Ludovic, Dan Lemle. Une étude comparative concernat les semi-groupes de classe C0 et les semi groupes intégrés. Lecturas Mathematicas Volumen 26 (2005), pages 27-88.

[44] Machado J.T., V.Kiryakova, F.Mainardi: Recent history of fractional calculus. Commun. Nonlinear Sci. Numer. Simul. 16, 1140-1153 (2011)

[45] Mahmudov N.I., S.Zorlu, Controllability of nonlinear stochastic systems. Int. J. Control 76, 95-104 (2003)

[46] Mahmudov N.I.: Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces. SIAM J. Control Optim. 42, 1604-1622 (2003)

[47] Mahmudov N.I., Approximate controllability of fractional Sobolev-type evolution equations in Banach spaces. Abstr. Appl. Anal. 2013, 1-9 (2013). Art ID 502839

[48] Mahmudov N.I.: Controllability of linear stochastic systems in Hilbert spaces. J. Math. Anal. Appl. 259(1), 64-82 (2001)

[49] Mandelbrot B.; J.W.van Ness (1968), "Fractional Brownian motions, fractional noises and applications", SIAM Review 10 (4): 422-437, doi:10.1137/1010093, JSTOR 2027184.

[50] Marle C.M. : Measures et Probabilités. Hermann, Paris (1974)

[51] Miller K.S., B.Ross : An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)

[52] Muthukumar P., P.Balasubramaniam : Approximate controllability of mixed stochastic Volterra-Fredholm type integro-dfferential systems in Hilbert space. J. Frankl. Inst. 345, 2911-2922 (2011)

[53] Muthukumar P., P.Balasubramaniam : Approximate controllability of secondorder damped McKean-Vlasov stochastic evolution equations. Comput. Math. Appl. 60, 2788-2796 (2010)

[54] Pazy A. : Semigroups of linear operators and application to partial differential equations. Springer, New York (1983)

[55] Podlubny I.: Fractional Differential Equations. Academic Press, San Diego (1999)

[56] Podlubny I. :Fractional-order systems and $PI^{\lambda}D^{\mu}$ controller.IEEETrans.Auto.Control.44,208-214 (1999)

[57] Ren Y., D.D.Sun : Second-order neutral stochastic evolution equations with infinite delay under carathodory conditions. J. Optim. Theory Appl. 147, 569-582 (2010)

[58] Sadovskii B.N. : On a fixed point principle. Funct. Anal. Appl. 1, 74-76 (1967)

[59] Sakthivel R., R.Ganesh, N.I.Mahmudov : Approximate controllability of fractional functional equations with infinite delay. Topol. Methods Nonlinear Anal. 43, 345-364 (2014)

[60] Sakthivel R., R.Ganesh, Y.Ren, S.M.Anthoni: Approximate controllability of nonlinear fractional dynamical systems. Commun. Nonlinear Sci. Numer. Simul. (2013). doi:10.1016/j.cnsns.2013.05.015

[61] Sakthivel R., P.Revathi, Y.Ren : Existence of solutions for nonlinear fractional stochastic differential equations. Nonlinear Anal. TMA 81, 70-86 (2013)

[62] Sakthivel R., S.Suganya, S.M.Anthoni : Approximate controllability of fractional stochastic evolution equations. Comput. Math. Appl. 63, 660-668 (2012)

[63] Sakthivel R., Y. Ren: Approximate controllability of fractional differential equations with state-dependent delay. Results Math. (2012). doi:10.1007/ s00025-012-0245-y

[64] Sakthivel R., R.Ganesh, S.Suganya: Approximate controllability of fractional neutral stochastic system with infinite delay. R. Math. Phys. 70, 291-311 (2012)

[65] Sakthivel R., Y.Ren, N.I.Mahmudov : On the approximate controllability of semilinear fractional differential systems. Comput. Math. Appl. 62, 1451-1459 (2011)

[66] Samko.S.G, Kilbas.A.A, Marichev O.I. : Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach, Yverdon (1993)

[67] Shu .XB, Wang. Q. : The existence and uniqueness of mild solutions for fractional differential equations with nonlocal conditions of order $1 < \alpha < 2$. Comput. Math. Appl. 64, 2100-2110 (2012)

[68] Steven E. Shreve, Stochastic Calculus for Finance II: Continuous-Time Models, Springer, 2004.

[69] Shu .X.B, Lai.Y, Chen .Y : The existence of mild solutions for impulsive fractional partial differential equations. Nonlinear Anal. TMA 74, 2003-2011 (2011)

[70] Sukavanam .N, S.Kumar: Approximate controllability of fractional order semilinear delay systems. J. Optim. Theory Appl. 151, 373-384 (2011)

[71] Wang .J.R, Fan. Z, Zhou. Y. : Nonlocal controllability of semilinear dynamic systems with fractional derivative in Banach spaces. J. Optim. Theory Appl. 154(1), 292-302 (2012)

[72] Yan.Z : Controllability of fractional-order partial neutral functional integro differential inclusions with infinite delay. J. Frankl. Inst. 348, 2156-2173 (2011)

[73] Zhang.X, Huang.X, Liu.Z : The existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay. Nonlinear Anal. Hybrid Syst. 4, 775-781 (2010)

[74] Zhou.Y ,Jiao.F, Li.J : Existence of mild solutions for fractional neutral evolution equations. Comput. Math. Appl. 59, 1063-1077 (2010)