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Publications

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ABSTRACT

The main objective of this doctoral thesis is the study of solutions for various types of non–autonomous second order differential equation. We give conditions guaranteeing the existence of mild solutions.
One of the main subject of study is prove the existence of mild solution for the second evolution of problem on a bounded interval. Then we studied the problem on the positive real half-axis with local and nonlocal conditions. On the other hand, another important subject of interest is the study of problem with infinite delay. We discuss the existence of mild solutions for the neutral evolution equations of the second order.
Sufficient conditions on the existence of mild solutions for the semilinear evolution inclusion are given.

The technique used is to reduce the study of our problem to the search for a fixed point of a suitably constructed integral operator. Our approach will be based on some fixed point theorems, semi-group theory and noncompact measurement. The latter is often used in several branches of the nonlinear analysis. Especially this technique has proved to be a very useful tool in the existence of solutions of several types of differential and integral equations. All our work has been illustrated with examples, to prove the applicability of our results.

AMS Subject Classification : 34G20, 34G25, 34K20, 34K30.

RÉSUMÉ

L'objectif principal de cette thèse de doctorat est l'étude de solutions pour différents types d'équations différentielles de second ordre non autonome. Nous donnons des conditions garantissant l'existence de solutions faibles.
L'un des principaux sujets d'étude prouve l'existence d'une solution au problème d'évolution de second ordre sur un intervalle borné. On étudie ensuite ce problème sur la demi-droite réelle positive avec des conditions locales et non locales. D'autre part, On étudie aussi ce problème retard infini.
Nous discutons l'existence des solutions faibles pour équations d'évolution du type neutre du second ordre. Des conditions suffisantes sur l'existence de solutions faibles pour l'inclusion de l'évolution semi-linéaire sont données. La technique utilisée est de ramener l'étude de notre problème à la recherche d'un point fixe d'un opérateur intégral convenablement construit. Notre approche sera basée sur quelques théorèmes de point fixe, la théorie des semi-groupes et la mesure de non compacité. Cette dernière est souvent utilisée dans plusieurs branches de l'analyse non linéaire. Spécialement cette technique a prouvé qu'elle est un outil très utile dans l'existence de solutions de plusieurs types d'équations différentielles et intégrales.
Tous nos travaux ont été illustré par des exemples, pour prouver l'applicabilité de nos résultats.
Classification AMS: 34G20, 34G25, 34K20, 34K30.

CONTENTS

INTRODUCTION

In various fields of science and engineering, many problems that are related to linear viscoelasticity, nonlinear elasticity and Newtonian or non-Newtonian fluid mechanics have mathematical models. Popular models essentially fall into two categories: the differential models and the integrodifferential models. A large class of scientific and engineering problems is modelled by partial differential equations, integral equations or coupled ordinary and partial differential equations which can be described as differential equations in infinite dimensional spaces using semigroups. In general functional differential equations or evolution equations serve as an abstract formulation of many partial integrodifferential equations which arise in problems connected with heat-flow in materials with memory and many other physical phenomena. It is well known that the systems described by partial differential equations can be expressed as abstract differential equations [108]. These equations occur in various fields of study and each system can be represented by different forms of differential or integrodifferential equations in Banach spaces. Using the method of semigroups, various solutions of nonlinear and semilinear evolution equations have been discussed by Pazy [108].

The nonlocal Cauchy problem for abstract evolution equations was first investigated by Byszewski and Lakshmikantham [47], by using the Banach fixed point theorem, the authors obtained the existence and uniqueness of mild solutions of nonlocal differential equations. The nonlocal problem was motivated by physical problems. Indeed, it is demonstrated that the nonlocal problems, have better effects in applications than the classical Cauchy problems. For example, it is used to represent mathematical models for evolution of various phenomena, such as non- local neural networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion (see McKibben [96]). For the importance of nonlocal conditions in different fields of applied sciences see [49, 55, 121, 120] and the references cited therein. For example, in [47] the author describes the diffusion phenomenon of a small amount of gas in a transparent tube by using the formula

$$g(u) = \sum_{i=0}^{p} c_j u(ti),$$

where $c_i, i = 0, 1, ..., p$, are given constants and $0 < t_0 < t_1 < \; < t_p < 1$. Early work in this area was made by Byszewski in [44,45,46,47]. Then, Balachandran and his collaborators have considered various classes of nonlinear integrodiferential systems ([22], see also, references cited in [38]). Then, Benchohra and his collaborators considered various classes of problems with the nonlocal conditions in [37,38,35,39,32]. Thenceforth, the study of differential equations with nonlocal initial conditions has been an active topic of research. The interested reader can consult [23,48,62,122].

In recent years we see an increasing interest in infinite delay equations. The main reason is that equations of this type become more and more important for different applications. When the delay is infinite, the notion of the *phase space* $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theories. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato in [74], see also the books by Ahmed [8], Corduneanu and Lakshmikantham [50], Kappel and Schappacher [87]. For detailed discussion and applications on this topic, we refer the reader to the book by Hale and Verduyn Lunel [76], Hino et al. [85], Wu [117] and Baghli and Benchohra [16] and the references therein.

Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last few decades. The literature related to ordinary neutral functional differential equations is very extensive. The work in partial neutral functional differential equations with infinite delay was initiated by Hernández and Henríquez. First order partial neutral functional differential equations have been studied by different authors. The reader can consult Hale [75,73,76] and Wu [119] for systems with finite delay and Hernández et al. [80,81,82] for the unbounded delay case. Hernández [79] established the existence results for partial neutral functional differential equations with nonlocal conditions. An extensive theory for ordinary neutral functional differential equations which the includes qualitative behavior of, classes of such equations and applications to biological and engineering processes. Several authors have studied the existence of solutions of neutral functional differential equations in Banach space [35,39,64,65,81,79,82]. In the literature, there are many papers studying the problems of neutral differential equations using different methods. Among them, the fixed point method combined by semigroup theory in Fréchet space, see for example Baghli and Benchohra [16,17,18]. Lee et al. [93] and Benchohra et al. [33] discussed the existence of mild solutions to second-order neutral differential inclusions. See also Hernández and Mckibben [83] and the references therein.

The theory of differential inclusions in Banach spaces has been developing fast because of the possibility of extensive practical applications, Benchohra et al. [35] discussed the existence of mild solutions to second-order neutral differential inclusions, many authors have taken a growing interest in the investigation on this subject, such as [8, 10, 43, 54, 56, 60, 86, 111, 112, 117] and the references therein. This type of equations has received much attention in recent years [1].

Measures of noncompactness are very useful tools in functional analysis, for instance, in metric fixed piont theory and in the theory of operator equations in Banach spaces. They are also used in the studies of functional equations, ordinary and partial differential equations, fractional partial differential equations, integral and integro–differential equations, optimal control theory, and in the characterizations of compact operators between Banach spaces. We give now a list of three important examples of measures of noncompactness which arise over and over in applications. The first example, is the *Kuratowski measure of noncompactness (or set measure of noncompactness)*

$$\alpha(D) = \inf\{r > 0 : D \text{ has a finite cover by sets of diameter} \leq r\},$$

was defined and studied by Kuratowski [90] in 1930. In 1955, G. Darbo [51] used the function α to prove his fixed point theorem. Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem, and includes the existence part of Banach's fixed point theorem.

The second measures of noncompactness were introduced by Goldenštein, Goh'berg and Markus [66], *Hausdorff measure of noncompactness (or ball measure of noncompactness)*

$$\chi(D) = \inf\{\varepsilon > 0 : D \text{ has a finite cover by balls of radius} \leq \varepsilon\},$$

which was later studied by Goldenštein and Markus [67] in 1968, and the third one is the *Istrătesku measure of noncompactness (or lattice measure of noncompactness)* [106] in 1972.

$$I(D) = \inf\{\varepsilon > 0 : D \text{ contains no infinite } \varepsilon\text{-discrete set in } D\}.$$

The relation between this measures are given by the following inequalities, which are obtained by Danes [52]

$$\chi(D) \leq I(D) \leq \alpha(D) \leq 2\chi(D)$$

for any bounded set $D \subset E$. Apparently Goldenštein, Goh'berg and Markus were not aware of Kuratowski's and Darbo's work. It is surprising that Darbo's theorem was almost never noticed and applied until in the 1970's mathematicians working in functional analysis, operator theory and differential equations started to apply a Darbo's theorem and developed the theory connected with measures of noncompactness.

The use of these measures is discussed, for instance, in the monographs [11, 26, 28, 107,91,94,109,15], Ph.D. thesis [12,14,59,95,101,114] and expository papers [71,115]. Applications of measures of noncompactness to differential and integral equations can be found, for instance, in [7, 27, 29, 30, 98, 102], to fractional partial differential equations in [2]. Measures of noncompactness on Fréchet spaces and applications to an infinite system of functional–integral functions can be found in [103]. Additional related results can be found in the references of the mentioned monographs and papers.

This thesis is devoted to various semilinear differential equations of non-autonomous second order. Specifically, we are interested in following problem type

$$y''(t) - A(t)y(t) = f(t, y(t)), \tag{1}$$

there is many results concerning the second-order differential equations, see for example Fattorini [63], Travis and Webb [113], Baliki and Benchohra [24, 25] R. Henríquez [78] and H. R. Henríquez *et al.* [84]. Among useful for the study of abstract second order equations is the existence of an evolution system $U(t; s)$ for the homogeneous equation

$$y''(t) = A(t)y(t), \text{ for } t \geq 0. \tag{2}$$

For this purpose there are many techniques to show the existence of $U(t; s)$ which has been developed by Kozak [89].
In what follows, we will give a brief description of each Chapter of this thesis.
Chapter 1 contains notation and preliminary results, definitions, theorems and other auxiliary results which will be needed in this thesis, in the first section we give some generalities, in **Section 2** we present some properties of Measures of noncompactness, in the third **Section 3** we give some properties evolution systems, in **Section 4** we present some properties of phase spaces, in **Section 5** we give some properties of set-valued maps and in the last section we cite some fixed point theorems.
In **Chapter 2,** we consider the existence of second order evolution equation, defined on a bounded interval J

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in J = [0; T],$$

$$y(0) = y_0,\ y'(0) = y_1.$$

In **chapter 3**, we establish the existence of second order evolution equation in Banach spaces, defined on a semi infinite interval $\bar{J}$ with local conditions

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in \bar{J} = [0, \infty), ,$$

$$y(0) = y_0,\ y'(0) = y_1.$$

In **chapter 4**, we prove results on the existence of mild solutions of second order evolution equation in Fréchet spaces, defined on a semi infinite interval $\bar{J}$ with nonlocal conditions

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in \bar{J} = [0, \infty).$$
$$y(0) = g(y),\ y'(0) = h(y).$$

In Chapter 5, we study the existence of mild solutions of second order evolution equation with infinite delay of the form

$$\begin{array}{rcl} y''(t) - A(t)y(t) & = & f(t, y_t), t \in J = [0; T] \\ y(t) & = & \phi(t), t \in (-\infty, 0], \\ y'(0) & = & \tilde{y}. \end{array}$$

Chapter 6 is devoted to study the existence of mild solutions of second order neutral evolution equation

$$\begin{array}{rcl} \dfrac{d}{dt}[y'(t) - g(t, y_t)] - A(t)y(t) & = & f(t, y_t), t \in J = [0; T] \\ y(t) & = & \phi(t), t \in (-\infty, 0], \\ y'(0) & = & \tilde{y}. \end{array}$$

Finally, In **Chapter 7,** we discuss the existence of mild solution of second order evolution inclusion

$$y''(t) - A(t)y(t) \in F(t, y(t)),\ t \in J = [0; T],$$
$$y(0) = y_0,\ y'(0) = y_1,$$

CHAPTER 1

CHAPTER 1. PRELIMINARIES

This chapter, we collect some notations, definitions, theorems, lemmas and facts concerning measures of noncompactness, phase spaces and multivalued analysis and other auxiliary results which will be needed in the sequel.

1.1 Generalities

Let E be a Banach space with the norms $|\cdot|$.
Denote by

- $C(J,E)$ the Banach space of continuous functions y mapping $J=[0;T]$ into E with the usual supremum norm

$$\|y\| = \sup_{t\in J} |y(t)|,$$

- $BC(\bar{J},E)$ the Banach space of all bounded and continuous functions y mapping $\bar{J}:=[0,\infty)$ into E with the usual supremum norm

$$\|y\| = \sup_{t\in \bar{J}} |y(t)|.$$

- $C(\bar{J},E)$ the Fréchet space of all continuous functions y mapping $\bar{J}$ into E equipped with the family of seminorms

$$\|y\|_T = \sup\{|y(t)| : t\in[0;T], T>0\}.$$

- $L^\infty(J,[0,\infty))$ the Banach space of essentially bounded measurable function $\gamma : J\to[0,\infty)$ with the norm

$$\|\gamma\|_\infty = \inf\left\{c>0 : |\gamma(t)|\leq c,\ \text{a.e. } t\in J\right\},$$

- ♦ $\mathcal{C}$ is the space defined by

$$\mathcal{C} = \{y : (-\infty, T] \to E \text{ such that } y|_J \in C(J, E) \text{ and } y_0 \in \mathcal{B}\},$$

we denote by $y|_J$ the restriction of y to J.

- ♦ $B(E)$ is the Banach space of bounded linear operators from E into E, with the usual supreme norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

Definition 1.1.1. *A function $f : \mathbb{R} \to E$ is called strongly measurable if there exists a sequence of simple functions $(f_n)_n$ such that*

$$\lim_{n\to\infty} |f_n(t) - f(t)| = 0.$$

Definition 1.1.2. *A function $f : \mathbb{R} \to E$ is Bochner integrable on J if it is strongly measurable and*

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(t) - f(t)|dt = 0$$

for any sequence of simple functions $(f_n)_n$.

Theorem 1.1.1. *A strongly measurable function $f : \mathbb{R} \to E$ is Bochner integrable if and only if $|f|$ is Lebesgue integrable.*

We refer to [97,118] for more details.

Definition 1.1.3. *A map $f : J \times E \to E$ is Carathéodory if*

(i) $t \longmapsto f(t, y)$ *is measurable for all* $y \in E$*, and*

(ii) $y \longmapsto f(t, y)$ *is continuous for almost each* $t \in J$*.*

If, in addition,

(iii) *for each* $r > 0$*, there exists* $g_r \in L^1(J, \mathbb{R}_+)$ *such that*

$$|f(t, y) \le g_r(t) \text{ for all } |y| \le r \text{ and almost each } t \in J,$$

then we say that the map is L^1-Carathéodory.

Lemma 1.1.1. *(Cauchy formula). If $f : \mathbb{R}_+ \to \mathbb{R}_+$ continuous function then*

$$\int_a^t \int_a^{s_1} \int_a^{s_2} \cdots \int_a^{s_n} f(s_{n+1}) ds_{n+1} ds_n \cdots ds_1 = \frac{1}{n!} \int_a^t f(s)(t-s)^n ds$$

for each $t \ge a$.

1.2 Some properties of evolution system

In this thesis, we will use the concept of evolution systems $\mathcal{U}(t,s)$ associated with problem

$$y'' = A(t)y,\ t \in J, \tag{1.1}$$

$$y(0) = y_0, y'(0) = y_1, \tag{1.2}$$

introduced by Kozak in [89]. With this purpose, we assume that $\{A(t),\ t \in J\}$ is a family of closed densely defined linear unbounded operators on the Banach space E and with domain $D(A(t))$ independent of t.

Definition 1.2.1. *A family $\mathcal{U}$ of bounded operators $\mathcal{U}(t,s) : E \to E$, $(t,s) \in \Delta := \{(t,s) \in J \times J : s \leq t\}$, is said to be an evolution operator generated by the family $\{A(t),\ t \geq 0\}$ if the following conditions are fulfilled:*

(Π_1) *For any $x \in E$ the map $(t,s) \longmapsto \mathcal{U}(t,s)x$ is of continuously differentiable and*

- **(a)** *for any $t \in J, \mathcal{U}(t,t) = 0$.*
- **(b)** *for all $(t,s) \in \Delta$ and for any $x \in E$,* $\frac{\partial}{\partial t}\mathcal{U}(t,s)x\big|_{t=s} = x$ *and* $\frac{\partial}{\partial s}\mathcal{U}(t,s)x\big|_{t=s} = -x$.

(Π_2) *For all $(t,s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, the map $(t,s) \longmapsto \mathcal{U}(t,s)x$ is of class C^2 and*

- **(a)** $\frac{\partial^2}{\partial t^2}\mathcal{U}(t,s)x = A(t)\mathcal{U}(t,s)x$.
- **(b)** $\frac{\partial^2}{\partial s^2}\mathcal{U}(t,s)x = \mathcal{U}(t,s)A(s)x$
- **(c)** $\frac{\partial^2}{\partial s \partial t}\mathcal{U}(t,s)x\big|_{t=s} = 0$.

(Π_3) *For all $(t,s) \in \Delta$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, there exist $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x$ and*

- **(a)** $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x = A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t,s)x$.
  Moreover, the map $(t,s) \longmapsto A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t,s)x$ is continuous.
- **(b)** $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x = \frac{\partial}{\partial t}\mathcal{U}(t,s)A(s)x$.

Kozak [89] has proved that the problem (1.1)-(1.2) has a unique solution

$$y(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 \text{ for } t \in J$$

More details on evolution systems and their properties could be found on Kozak [89].

1.3 Some properties of measure of noncompactness

The theory of measures of non–compactness has many applications in Topology, Functional analysis and Operator theory. For more information the reader can see [28]. Throughout thesis, we will also accept the following definition of the concept of measure of noncompactness [28].

Definition 1.3.1. *Let E be a Banach space and Ω_D the family of all nonempty and bounded subsets D of E. The measure of noncompactness is a map*

$$\mu : \Omega_D \to [0; +\infty)$$

satisfying the following properties:

(i_1) $\mu(D) = 0$ if only if D is relatively compact,

(i_2) $\mu(\overline{D}) = \mu(D)$; $\overline{D}$ the closure of D,

(i_3) $\mu(C) \leq \mu(D)$ when $C \subset D$;

(i_4) $\mu(C + D) \leq \mu(C) + \mu(D)$ where $C + D = \{x \mid x = y + z; y \in C; z \in D\}$,

(i_5) $\mu(aD) = |a|\mu(D)$ for any $a \in \mathbb{R}$,

(i_6) $\mu(ConvD) = \mu(D)$; $ConvD$ the convex hull of D,

(i_7) $\mu(C \cup D) = \max(\mu(C), \mu(D))$,

(i_8) $\mu(C \cup \{x\}) = \mu(D)$ for any $x \in E$.

We will use a measure of noncompactness μ in the space $C(J, E)$ which was considered in [28]. Denote by $\omega^T(y, \varepsilon)$ the modulus of continuity of y on the interval J, i.e.

$$\omega^T(y, \varepsilon) = \sup \{|y(t) - y(s)| : t, s \in J, |t - s| \leq \varepsilon\} .$$

Moreover, let us put

$$\omega^T(D, \varepsilon) = \sup \left\{\omega^T(y, \varepsilon) : y \in D\right\}$$

$$\omega_0^T(D) = \lim_{\varepsilon \to 0} \sup \omega^T(D, \varepsilon).$$

Lemma 1.3.1. *[28] If $\{D\}_{n=0}^{+\infty}$ is sequence of nonempty, bounded and closed subsets of E such that $D_{n+1} \subset D_n (n = 1, 2, 3...)$ and if $\lim\limits_{n\to\infty} \mu(D_n) = 0$, then the intersection*

$$D_\infty = \cap_{n=0}^{+\infty} D_n$$

is nonempty and compact.

Lemma 1.3.2. *([28]). If μ is a regular measure of noncompactness, then*

$$|\mu(D_1) - \mu(D_2)| \leq \mu(B(0,1))d_H(D_1, D_2)$$

for any bounded subset $D_1, D_2 \in E$, where d_H is Hausdorff distance between D_1 and D_2.

Lemma 1.3.3. *[102] If $\mathcal{D} \subset C(J; E)$ is bounded and equicontinuous, then the map $t \to \mu(D(t))$ is continuous on J and*

$$\mu(\mathcal{D}) = \sup_{t \in J} \mu(\mathcal{D}(t)),$$

$$\mu\left(\int_0^t \mathcal{D}(s)ds\right) \leq \int_0^t \mu(\mathcal{D}(s))ds \text{ for each } t \in J.$$

where

$$\mathcal{D}(t) = \{y(t) : y \in \mathcal{D}\}$$

and

$$\int_0^t \mathcal{D}(s)ds = \left\{\int_0^t y(s)ds : y \in \mathcal{D}\right\}.$$

Lemma 1.3.4. *[43] Assume that a set $\mathcal{D} \subset C(J, E)$ is bounded, then*

$$\sup_{t \in J} \mu(\mathcal{D}(t)) \leq \mu(\mathcal{D}(J)) \leq \omega_0^T(\mathcal{D}) + \sup_{t \in J} \mu(\mathcal{D}(t)),$$

$$\sup_{t \in J} \mu(\mathcal{D}(t)) \leq \eta(\mathcal{D}) \leq \omega_0^T(\mathcal{D}) + \sup_{t \in J} \mu(\mathcal{D}(t)),$$

where η is a measure of noncompactness in $C(J, E)$.

Lemma 1.3.5. *[77] If E separable Banach space and C is nonempty, bounded subset of $BC(\bar{J}; E)$ then the function $t \to \mu(C(t))$ is measurable and*

$$\mu\left(\int_0^t C(s)ds\right) \leq \int_0^t \mu(C(s))ds \text{ for each } t \geq 0,$$

Lemma 1.3.6. *[43] If B is bounded subset of Banach space, then for each $\varepsilon > 0$ there is a sequence function $\{b_n\}_{n=0}^{\infty} \subset B$ such that*

$$\mu(B) \leq 2\mu(\{b_n\}_{n=0}^{\infty}) + \varepsilon.$$

We recall that a subset $\tilde{B} \subset L^1(J; E)$ is uniformly integrable if there exists $\xi \in L^1(J; \mathbb{R}^+)$ such that

$$|x(s)| \leq \xi(s) \text{ for } x \in \tilde{B} \text{ and a.e. } s \in J.$$

Lemma 1.3.7. *[99] If $\{B_n\}_{n=0}^{\infty} \subset L^1(J, E)$ is uniformly integrable, then the function $t \to \mu(\{B_n(t)\}_{n=0}^{\infty})$ is mesurable and*

$$\mu\left\{\int_0^t B_n(s))ds\right\}_{n=0}^{\infty} \leq 2\int_0^t \mu(\{B_n(s)\}_{n=0}^{\infty}).$$

Definition 1.3.2. *[53] Let X be a Banach space, and let S be a nonempty subset of X. A continuous mapping $\Psi : S \to X$ is called to be $\mu-$contraction if there exists a constant $k \in [0;1)$ such that, for every bounded set $\nu \subset S$,*

$$\mu(\Psi(\nu) \leq k\mu(\nu).$$

Lemma 1.3.8. *[27] If the map $\Psi : \mathcal{D}(\Psi) \subset X \to Y$ is Lipschitz continuous with constant k, then $\mu(\Psi(V)) \leq k\mu(V)$ for any bounded subset $V \subset \mathcal{D}(\Psi)$, where Y is another Banach space.*

1.4 Some properties of phase spaces

For any continuous function y and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t+\theta) \text{ for } \theta \in (-\infty, 0].$$

Here $y_t(\cdot)$ represents the history of the state up to the present time t. We assume that the histories y_t belong to $\mathcal{B}$.
In this thesis, we will employ an axiomatic definition of the *phase space* $\mathcal{B}$ introduced by Hale & Kato [74] and follow the terminology used in [85]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(Λ_1) If $y : (-\infty, T) \to E, T > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in [0, T)$ the following conditions hold :

- (i) $y_t \in \mathcal{B}$;
- (ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
- (iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{\, |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(Λ_2) For the function y in (Λ_1), y_t is a $\mathcal{B}-$valued continuous function on J.

(Λ_3) The space $\mathcal{B}$ is complete.

Remark 1.4.1. *In the sequel, we get*

$$\gamma := \max\left\{\sup_{t\in J}\{K(t)\}, \sup_{t\in J}\{M(t)\}\right\}.$$

1.4.1 Examples of phase spaces

Example 1.4.1. *Let:*
BC *the space of bounded continuous functions defined from* $(-\infty, 0]$ *to* E*,*
BUC *the space of bounded uniformly continuous functions defined from* $(-\infty, 0]$ *to* E*,*
$C^\infty := \{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \; exists \; in \; E\}$
$C^0 := \{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0\}$*, endowed with the uniform norm*

$$\|\phi\| = \sup\{|\phi(\theta)| : \theta \leq 0\}.$$

We have that the spaces BUC*,* C^∞ *and* C^0 *satisfy conditions* $(\Lambda_1) - (\Lambda_3)$*. However,* BC *satisfies* $(\Lambda_1), (\Lambda_3)$*, but* (Λ_2) *is not satisfied.*

Example 1.4.2. *Let* g *be a positive continuous function on* $(-\infty, 0]$*. We define:*

$$C_g := \left\{\phi \in C((-\infty, 0]), E) : \frac{\phi(\theta)}{g(\theta)} \; is \; bounded \; on \; (-\infty, 0]\right\}$$

and

$$C_g^0 := \left\{\phi \in C_g : \lim_{\theta \to -\infty} \frac{\phi(\theta)}{g(\theta)} = 0\right\},$$

endowed with the uniform norm

$$\|\phi\| = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)} : \theta \leq 0\right\}.$$

Then we have that the spaces C_g *and* C_g^0 *satisfy condition* (Λ_3)*. We consider the following condition on the function* g*.*
(g_1) *For all* $a > 0$*,*

$$\sup_{0 \leq t \leq a} \sup\left\{\frac{\phi(t+\theta)}{g(\theta)} : \; -\infty < t \leq -t\right\} \leq \infty.$$

Then C_g *and* C_g^0 *satisfy conditions* (Λ_1) *and* (Λ_2) *if* (g_1) *holds.*

Example 1.4.3. *The space* C_γ *For any real positive constant* γ*, we define the functional space* C_γ *by*

$$C_\gamma := \{\phi \in C((-\infty, 0]), E) : \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \; exist \; in \; E\}$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \theta \leq 0\}.$$

Then in the space C_γ *the axioms* $(\Lambda_1) - (\Lambda_3)$ *are satisfied.*

For other details we refer, for instance to the book by Hino *et al.* [85].

1.5 Some Properties of Set-Valued Maps

Let (X, d) be a metric space and A be a subset of X. We denote:

$$P(X) = \{A \subset X : A \neq \emptyset\}$$

and

$$P_b(X) = \{A \subset X : A \text{ bounded } \}, \qquad P_{cl}(X) = \{A \subset X : A \text{ closed } \}.$$

$$P_{cp}(X) = \{A \subset X : A \text{ compact } \}, \qquad P_{cv}(X) = \{A \subset X : A \text{ convexe } \}.$$

$$P_{cv,cp}(X) = P_{cv}(X) \cap P_{cp}(X).$$

Let $A_1, A_2 \in P(X)$, consider $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$ the Hausdorff distance between A_1 and A_2 given by:

$$H_d(A_1, A_2) = \max \left\{ \sup_{a_1 \in A_1} d(a_1, A_2) \ , \ \sup_{a_2 \in A_2} d(A_1, a_2) \right\},$$

where $d(a_1, A_2) = \inf\{d(a_1, a_2) : a_2 \in A_2\}$ and $d(A_1, a_2) = \inf\{d(a_1, a_2) : a_1 \in A_1\}$.
As usual, $d(x, \emptyset) = +\infty$.
Then, $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized (complete) metric space.

Definition 1.5.1. *A multivalued map $F : J \to P_{cl}(X)$ is said to be measurable if, for each $x \in X$, the function $g : J \to X$ defined by*

$$g(t) = d(x, F(t)) = \inf\{d(x, z) : z \in F(t)\},$$

is measurable.

Definition 1.5.2. *Let X and Y be metric spaces. A set-valued map F from X to Y is characterized by its graph $Gr(F)$, the subset of the product space $X \times Y$ defined by*

$$Gr(F) := \{(x, y) \in X \times Y : y \in F(x)\}$$

Definition 1.5.3.

1. *A measurable multivalued function $F : J \to P_{b,cl}(X)$ is said to be integrably bounded if there exists a function $g \in L^1(\mathbb{R}_+)$ such that $|f| \leq g(t)$ for almost $t \in J$ for all $f \in F(t)$.*

2. *F is bounded on bounded sets if $F(\mathcal{W}) = \bigcup_{x \in B} F(x)$ is bounded in X for all $\mathcal{W} \in P_b(X)$, i.e. $\sup_{x \in \mathcal{W}}\{\sup\{|y| : y \in F(x)\}\} < \infty$.*

3. *A set-valued map F is called upper semi-continuous (u.s.c. for short) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X and for each open set U of X containing $F(x_0)$, there exists an open neighborhood V of x_0 such that $F(V) \subset U$. A set-valued map F is said to be upper semi-continuous if it is so at every point $x_0 \in X$.*

4. *A set-valued map F is called lower semi-continuous (l.s.c) at $x_0 \in X$ if for any $y_0 \in F(x_0)$ and any neighborhood V of y_0 there exists a neighborhood U of x_0 such that*
$$F(x_0) \cap V \neq \quad \text{for all } \ x_0 \in U.$$
A set-valued map F is said to be lower semi-continuous if it is so at every point $x_0 \in X$.

5. *F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_b(X)$. If the multivalued map f is completely continuous with nonempty compact values, then f is upper semi-continuous if and only if f has closed graph.*

Proposition 1.5.1. *Let $F : X \to Y$ be an u.s.c map with closed values. Then $Gr(F)$ is closed.*

Definition 1.5.4. *Let E be a Banach space. A multivalued map $F : J \times E \to E$ is said to be L^1-Carathéodory if*

(i) *$t \longmapsto F(t, y)$ is measurable for all $y \in E$,*

(ii) *$y \longmapsto F(t, y)$ is upper semicontinuous for almost each $t \in J$,*

(iii) *for each $\rho > 0$, there exists $\psi_\rho \in L^1(J, \mathbb{R}_+)$ such that*
$$\|F(t, y)\|_P \leq \psi_\rho(t), \quad \text{for all } |y| \leq \rho \text{ and a. e. } \ t \in J,$$

such that $\|F(t, y)\|_P = \sup\{|f| : f \in F(t, y)\}$.

Definition 1.5.5. *Let X, Y be nonempty sets and $F : X \to P(Y)$. The single-valued operator $f : X \to Y$ is called a selection of F if and only if $f(x) \in F(x)$, for each $x \in X$. The set of all selection functions for F is denoted by S_F.*

The following lemmas is very important to prouve our result.

Lemma 1.5.1 (Lasota and Opial [92])**.** *Let E be a Banach space and I bounded closed interval, and F be an L^1-Caratheodory multivalued map with compact convex values, and let $\mathcal{L} : L^1(I, E) \to C(J, E)$ be a linear continuous mapping. Then the operator*
$$\mathcal{L} \circ S_F : C(I, E) \to P_{cp,cv}(C(I, E)),$$
is a closed graph operator in $C(I, E) \times C(I, E)$.

Lemma 1.5.2. *[68] Let X be a separable metric space. Then every measurable multi-valued map $F : X \to P_{cl}(X)$ has a measurable selection.*

Definition 1.5.6. *Let $\mathcal{T} : X \to P(X)$ be a multi-valued map. An element $x \in X$ is said to be a fixed point of $\mathcal{T}$ if $x \in \mathcal{T}(x)$.*

For more details on multivalued maps and the proof of the known results cited in this section we refer interested reader to the books of Deimling [54], Gorniewicz [68], Hu and Papageorgiou [86], Smirnov [111] and Tolstonogov [112].

1.6 Fixed Point Theorems

Fixed point theory plays an important role in our existence results, therefore we state the following fixed point theorems.

Theorem 1.6.1. *(Schauder [60])*
Let $\mathcal{X}$ be a Banach space, $\mathcal{K}$ compact convex subset of $\mathcal{X}$ and $N : \mathcal{K} \to \mathcal{K}$ continuous map.Then N has at least one fixed point in $\mathcal{K}$.

Theorem 1.6.2. *(Tykhonoff [60])*
Let $\mathcal{X}$ be a locally convex space, $\mathcal{K}$ compact convex subset of $\mathcal{X}$ and $N : \mathcal{K} \to \mathcal{K}$ is a continuous map. Then N has at least one fixed point in $\mathcal{K}$.

Theorem 1.6.3. *(Darbo-Sadovskii [28])*
Let $\mathcal{X}$ be a Banach space, $\mathcal{K}$ bounded, closed and convex subset of X and $N : \mathcal{K} \to \mathcal{K}$ continuous map and $\mu-$contraction. Then N has at least one fixed point in $\mathcal{K}$.

Theorem 1.6.4. *(Bohnenblust-Karlin [42])*
Let $\mathcal{K} \in \mathcal{P}_{cp,cv}(\mathcal{X})$, and $N : \mathcal{K} \to \mathcal{P}_{cl,cv}(\mathcal{K})$ be an upper semicontinuous operator. Then N has at least one fixed point in $\mathcal{K}$.

CHAPTER 2

SEMILINEAR DIFFERENTIAL EQUATIONS ON BOUNDED INTERVALS [(1)]

2.1 Introduction

The theory of abstract nonlinear second order functional differential and integrodifferential equations has received considerable attention in recent years. Several papers have also appeared for the existence of solutions of the nonlinear second-order diffrential equations in Banach spaces [19, 21]. Non–autonomous second order problems have received much attention in recent years due to their applications in different fields. Specially, many authors [41, 78, 70, 110, 116] have studied the Cauchy problem for second order evolution.
The aim of this chapter is to study the existence of mild solutions for second order semi-linear functional evolution equations. Our analysis is based on the technique of measure of noncompactness and Schauder's fixed point theorem.
Consider the following problem

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in J = [0; T], \tag{2.1}$$

$$y(0) = y_0,\ y'(0) = y_1, \tag{2.2}$$

where $\{A(t)\}_{0 \leq t < T}$ is a family of linear closed operators from E into E that generate an evolution system of linear bounded operators $\{\mathcal{U}(t, s)\}$ for $0 \leq s \leq t < T$,
$f : J \times E \to E$ be a Carathéodory function and $(E, |\cdot|)$ a real Banach space.

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(1) [40] M. Benchohra; N. Rezzoug, Measure of noncompactness and second order evolution equations. Gulf J. Math. **4** (2016), no. 2, 71–79.

2.2 Existence of solutions

Let us start by defining what we mean by a mild solution of the problem (2.1) -(2.2).

Definition 2.2.1. *A function $y \in C(J, E)$ is called a mild solution to the problem (2.1) -(2.2) if y satisfies the integral equation*

$$y(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f(s,y(s))ds. \tag{2.3}$$

To prove our results we introduce the following conditions:

$(\boldsymbol{H_1})$ There exists a constant $M \geq 1$ such that

$$\|\mathcal{U}(t,s)\|_{B(E)} \leq M.$$

$(\boldsymbol{H_2})$ There exists a constant $\tilde{M} \geq 0$ such that

$$\left\|\frac{\partial}{\partial s}\mathcal{U}(t,s)\right\|_{B(E)} \leq \tilde{M}.$$

$(\boldsymbol{H_3})$ There exist integrable function $p : J \to \mathbb{R}_+$ and a continuous nondecreasing function $\psi : [0,\infty) \to (0;\infty)$ such that:

$$|f(t,u)| \leq p(t)\psi(|u|) \text{ for a.e } t \in J \text{ and each } u \in E.$$

$(\boldsymbol{H_4})$ There exists a constant $R > 0$ such that

$$\tilde{M}|y_0| + M(|y_1| + \psi(R)\,\|p\|_{L^1} \leq R$$

$(\boldsymbol{H_5})$ There exists a function $\gamma \in L^\infty(J, \mathbb{R}_+)$ such that for any nonempty set $D \subset E$ we have :

$$\mu(f(J \times D)) \leq \gamma(t)\mu(D).$$

Theorem 2.2.1. *If assumptions $(\mathbf{H_1})-(\mathbf{H_5})$ are satisfied, then the problem (2.1)-(2.2) admits at least one mild solution.*

Proof. Consider the operator $N : C(J,E) \to C(J,E)$ defined by

$$(Ny)(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f(s,y(s))ds.$$

Let $s, t \in J$ with $t > s$. Then we have

$$\begin{aligned}
|(Ny)(t)-(Ny)(s)| &\leq \left|\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0-\frac{\partial}{\partial s}\mathcal{U}(s,0)y_0\right|+|\mathcal{U}(t,0)y_0-\mathcal{U}(s,0)y_0|\\
&+ \int_0^s \|\mathcal{U}(t,\tau)-\mathcal{U}(s,\tau)\|_{B(E)}\,|f(\tau,y(\tau)|\,d\tau\\
&+ \int_s^t \|\mathcal{U}(t,\tau)\|_{B(E)}\,|f(\tau,y(\tau)|\,d\tau\\
&\leq \tilde{\omega}^T(\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0,\varepsilon)+\omega^T(U(t,0)y_0,\varepsilon)\\
&+ \mathcal{A}^T(\mathcal{U},\varepsilon)\int_0^s p(\tau)\psi(|y(\tau)|)d\tau\\
&+ M\sup\left\{\int_s^t p(\tau)\psi(|y(\tau)|)d\tau; t\leq s\leq T, |t-s|\leq\varepsilon\right\},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\omega}^T(\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0,\varepsilon) &= \sup\left\{\left|\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0-\frac{\partial}{\partial s}\mathcal{U}(s,0)y_0\right|; t\leq s\leq T, |t-s|\leq\varepsilon\right\},\\
\omega^T(\mathcal{U}(t,0)y_0,\varepsilon) &= \sup\left\{|\mathcal{U}(t,0)y_0-\mathcal{U}(s,0)y_0|; t\leq s\leq T, |t-s|\leq\varepsilon\right\},\\
\mathcal{A}^T(\mathcal{U},\varepsilon) &= \sup\left\{\|\mathcal{U}(t,\tau)-\mathcal{U}(s,\tau)\|_{B(E)}; \tau\leq t\leq s\leq T, |t-s|\leq\varepsilon\right\}.
\end{aligned}$$

Puting

$$\begin{aligned}
\Omega(T,\varepsilon) &= \tilde{\omega}^T(\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0,\varepsilon)+\omega^T(\mathcal{U}(t,0)y_0,\varepsilon)\\
&+ \mathcal{A}^T(\mathcal{U},\varepsilon)\int_0^s p(\tau)\psi(|y(\tau)|)d\tau\\
&+ M\sup\left\{\int_s^t p(\tau)\psi(|y(\tau)|)d\tau; t\leq s\leq T, |t-s|\leq\varepsilon\right\}.
\end{aligned}$$

We have

$$|(Ny)(t)-(Ny)(s)|\leq\Omega(T,\varepsilon)$$

i.e

$$\omega^T(Ny,\varepsilon)\leq\Omega(T,\varepsilon). \tag{2.4}$$

The assumptions $(\mathbf{H_1})-(\mathbf{H_4})$ yield that

$$\lim_{\varepsilon\to 0}\Omega(T,\varepsilon)=0.$$

We define

$$D = \left\{ y \in C(J, E) : \|y\| \le R \text{ and } \lim_{\varepsilon \to 0} \Omega(T, \varepsilon) = 0 \right\}.$$

The set D is nonempty convex and closed.
Now, we have

$$\begin{aligned} |Ny(t)| &\le \left\| \frac{\partial}{\partial s}\mathcal{U}(t,0) \right\|_{B(E)} |y_0| \\ &+ \|\mathcal{U}(t,s)\|_{B(E)} |y_1| + \|\mathcal{U}(t,s)\|_{B(E)} \int_0^t p(s)\psi(|y(s)|)ds \\ &\le \tilde{M} + M \int_0^t p(s)\psi(R)ds \\ &\le \tilde{M} + M\psi(R) \|P\|_{L^1} \\ &\le R. \end{aligned} \tag{2.5}$$

The conditions (2.4) and (2.5) ensure that the operator N transforms the set D into itself.

Step 1. N continuous.

Let $(y_n)_{n \in N}$ be a sequence in D such that $y_n \to y$ in D. We have

$$|Ny_n(t) - Ny(t)| \le M \int_0^t |f(s, y_n(s)) - f(s, y(s))| \, ds.$$

Since f is Caratheory we obtain by the Lebesgue dominated convergence theorem that

$$\lim_{t \to +\infty} \|N(y_n) - N(y)\| = 0.$$

We deduce N is continuous.

Consider the mesure of noncompacteness $\mu^*(D)$ defined on the family of bounded subsets of the space $C(J, E)$ by

$$\mu^*(D) = \omega_0^T(D) + \sup_{t \in J} \mu(D(t))$$

Step 2. $D_\infty = \cap_{n=0}^{+\infty} D_n$ is compact.

In the sequel, we consider the sequence of sets $\{D_n\}_{n=0}^{+\infty}$ defined by induction as follows :

$$D_0 = D, \; D_{n+1} = Conv N(D_n) \text{ for } n = 0, 1, 2, \cdots \text{ and } D_\infty = \cap_{n=0}^{+\infty} D_n$$

this sequence is nondecreasing, i.e. $D_{n+1} \subset D_n$ for each n.

Claim 1. $\lim_{n\to+\infty} \omega_0^T(D_n) = 0.$

This is a consequence from the equicontinuity of the set D on compact intervals.

Claim 2. $\lim_{n\to+\infty} \sup_{t\in J} \mu(D_n(t)) = 0.$

Set

$$\alpha_n(t) = \mu(D_n(t)).$$

In view of Lemma 1.3.8 and (2.4) we have

$$|\alpha_n(t) - \alpha_n(s)| \le \mu(B(0,1))\Omega(T, |t-s|)$$

which together with proves the continuity $\alpha_n(t)$ on J.
Using the properties of μ, Lemma 1.3.2 and $(\mathbf{H_5}))$ we get

$$\begin{aligned}
\alpha_{n+1}(t) &= \mu(CovN(D_n)) = \mu(N(D_n)) = \mu\left(\int_0^t \mathcal{U}(t,s)f(s,y(s))ds\right) \\
&\le \|\mathcal{U}(t,s)\|_{B(E)} \int_0^t \mu(f(s,y(s)))ds \\
&\le M\int_0^t \gamma(s)\mu(D(s))ds \\
&\le M\int_0^t \gamma(s)\alpha_n(s)ds.
\end{aligned}$$

Using Lemma 1.1.1 we derive

$$\begin{aligned}
\alpha_{n+1}(t) &\le M^{n+1}\int_0^t \gamma(s_1)\int_0^{s_1}\gamma(s_2)\int_0^{s_2}\cdots\int_0^{s_n}\gamma(s_{n+1})\alpha_0(s_{n+1})ds_1ds_2\cdots ds_n ds_{n+1} \\
&\le M^{n+1}\tilde{\gamma}^{n+1}\int_0^t\int_0^{s_1}\int_0^{s_2}\cdots\int_0^{s_n}\alpha_0(s_{n+1})ds_{n+1}ds_n\cdots ds_2 ds_1 \\
&\le \frac{M^{n+1}\tilde{\gamma}^{n+1}(t)T^n}{n!}\int_0^t \alpha_0(s)ds,
\end{aligned}$$

where $\tilde{\gamma}(t) = ess\sup\{\gamma(s) : s \le t\}$.
We get

$$\lim_{n\to+\infty} \alpha_n(t) = 0.$$

Then

$$\lim_{n\to+\infty} \mu(D_n(t)) = 0.$$

Form **Claim** 1 and **Claim** 2 we conclude

$$\lim_{n\to+\infty}\omega_0^T(D_n)+\lim_{n\to+\infty}\sup_{t\in J}\mu(D_n(t))=0.$$

Taking into account Lemma 1.3.8 we infer that $D_\infty = \cap_{n=0}^{+\infty} D_n$ is nonempty, convex and compact. Thus, by Schauder's fixed point theorem the operator $N : D_\infty \to D_\infty$ has at least one fixed point which is a mild solution of problem (2.1)-(2.2).

2.3 An example

Consider the second order Cauchy problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}z(t,\tau) & = \dfrac{\partial^2}{\partial \tau^2}z(t,\tau)+a(t)\dfrac{\partial}{\partial t}z(t,\tau) \\ & +e^{-t}\displaystyle\int_0^t \frac{z(t,\tau)}{1+z^2(t,\tau)}d\tau, \qquad t\in J,\ \tau\in[0,\pi], \\ z(t,0)= & z(t,\pi)=0 \qquad t\in J, \\ \dfrac{\partial}{\partial t}z(0,\tau) & =\psi(\tau) \qquad \tau\in[0,\pi], \end{cases} \tag{2.6}$$

where we assume that $a : J \to \mathbb{R}$ is a Hölder continuous function.

Let $E = L^2([0,\pi],\mathbb{R})$ the space of 2-integrable functions from $[0,\pi]$ into $\mathbb{R}$, and $H^2([0,\pi],\mathbb{R})$ denotes the Sobolev space of functions $x : [0,\pi] \to \mathbb{R}$ such that $x'' \in L^2([0,\pi],\mathbb{R})$. We consider the operator $A_1y(\tau) = y''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R},\mathbb{C})$, infinitesimal generator of strongly continuous cosine function $C(t)$ on E. Moreover, we take $A_2(t)y(s) = a(t)y'(s)$ defined on $H^1([0,\pi],\mathbb{R})$, and consider the closed linear operator $A(t) = A_1 + A_2(t)$ which generates an evolution operator $\mathcal{U}$ defined by

$$\mathcal{U}(t,s)=\sum_{n\in\mathbb{Z}} z_n(t,s)\langle x,w_n\rangle w_n,$$

where z_n is a solution to the following scalar initial value problem

$$\begin{cases} z''(t)=-n^2z(t)+ina(t)z(t) \\ z(s)=0, \qquad z'(s)=z_1. \end{cases} \tag{2.7}$$

Set

$$y(t)(\tau)=w(t)(\tau),\ t\ge 0,\ \tau\in[0,\pi],$$

$$f(t,y)(\tau)=e^{-t}\int_0^t \frac{y(t,\tau)}{1+y^2(t,\tau)}d\tau,$$

and

$$\frac{\partial}{\partial t} y(0)(\tau) = \frac{d}{dt} w(0)(\tau),\ \tau \in [0, \pi].$$

Consequently, (2.6) can be written in the abstract form (2.1)-(2.2) with $A(t)$ and f defined above. The existence of a mild solution can be deduced from an application of Theorem 2.2.1.

CHAPTER 3

SEMILINEAR DIFFERENTIAL EQUATIONS ON UNBOUNDED INTERVALS IN BANACH SPACE (2)

3.1 Introduction

Differential equations on infinite intervals frequently occur in mathematical modelling of various applied problems see [3, 104]. For example, in the study of unsteady flow of a gas through a semi-infinite porous medium Agarwal & O'Regan [5], Kidder [88], analysis of the mass transfer on a rotating disk in a non-Newtonian fluid Agarwal & O'Regan [6], heat transfer in the radial flow between parallel circular disks Na [100], investigation of the temperature distribution in the problem of phase change of solids with temperature dependent thermal conductivity Na [100], as well as numerous problems arising in the study of circular membranes Agarwal & O'Regan [4], Dickey [57, 58], plasma physics Agarwal & O'Regan [6], nonlinear mechanics, and non-Newtonian fluid flows Agarwal & O'Regan [4].
The aim of this chapter is to study the existence of mild solutions for second order semi-linear functional evolution equations on unbounded intervals in Banch space.
Consider the following problem

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in \bar{J} = [0, \infty), \tag{3.1}$$

$$y(0) = y_0,\ y'(0) = y_1, \tag{3.2}$$

where $\{A(t)\}_{0\leq t<+\infty}$ is a family of linear closed operators from E into E that generate an evolution system of linear bounded operators $\{\mathcal{U}(t,s)\}_{(t,s)\bar{J}\times\bar{J}}$ for $0 \leq s \leq t < +\infty$, $f : \bar{J} \times E \to E$ be a Carathéodory function, $y_0, y_1 \in E$ and $(E, |\cdot|)$ a real separable Banach space.

(2) [36] M. Benchohra, J. Henderson and N. Rezoug, Global existence results for second order evolution equations, *Communications on Applied Nonlinear Analysis.* Vol 23, (2016) no. 3, 57 - 67.

3.2 Existence of solutions

A mild solution of (3.1)- (3.2) is defined as follows.

Definition 3.2.1. *A function $y \in C(\bar{J}, E)$ is called a mild solution to the problem (3.1) -(3.2) if y satisfies the integral equation*

$$y(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f(s,y(s))ds. \tag{3.3}$$

Now, we give the basic assumptions to prove our results

($\boldsymbol{H_1}$) There exists a constant $M \geq 1$ and such that

$$\|\mathcal{U}(t,s)\|_{B(E)} \leq M, (t,s) \in \Delta.$$

($\boldsymbol{H_2}$) There exists a constant $\tilde{M} \geq 0$ such that

$$\left\| \frac{\partial}{\partial s}\mathcal{U}(t,s) \right\|_{B(E)} \leq \tilde{M}, (t,s) \in \Delta.$$

($\boldsymbol{H_3}$) There exist an integrable function $p : \bar{J} \to \mathbb{R}_+$ and a continuous nondecreasing function $\psi : [0,\infty) \to (0;\infty)$ such that:

$$|f(t,u)| \leq p(t)\psi(|u|) \text{ for a.e } t \in \bar{J} \text{ and each } u \in E.$$

($\boldsymbol{H_4}$) There exists a constant $R > 0$ such that

$$\tilde{M}|y_0| + M(|y_1| + \psi(R)\,\|p\|_{L^1} \leq R$$

($\boldsymbol{H_5}$) There exists a locally integrable function $\sigma : \bar{J} \to \mathbb{R}_+$ such that for any nonempty set $D \subset E$ we have :

$$\mu(f(t,D)) \leq \sigma(t)\mu(D) \text{ for a.e } t \in \bar{J} \text{ and each } D \subset E.$$

To establish our main theorem, we need the following lemma.

Lemma 3.2.1. *[105] Assume that the hypotheses $(\mathbf{H_1}) - (\mathbf{H_4})$ hold and a set $D \subset E$ is bouded. Then*

$$\omega_0^T(F(D)) \leq 2TM\mu(f([0,T],D).$$

Where $F(y(t)) = \displaystyle\int_0^t \mathcal{U}(t,s)f(s,y(t)ds$ for a.e $t \in J$ and $y \in D$.

Theorem 3.2.1. *Assume that the hypotheses* $(\mathbf{H_1}) - (\mathbf{H_4})$ *are satisfied, and if*

$$\lim_{t\to+\infty}\int_0^t \sigma(s)ds = +\infty.$$

Then the problem (3.1)-(3.2) admits at least one mild solution.

Proof. Consider the operator $N : BC(\bar{J}, E) \to BC(\bar{J}, E)$ defined by

$$(Ny)(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f(s,y(s))ds.$$

We define

$$D = \left\{y \in BC(\bar{J}, E) : \|y\| \le R\right\}.$$

The set D is nonempty convex and closed.
Now, we have

$$\begin{aligned}
|Ny(t)| &\le \left\|\frac{\partial}{\partial s}\mathcal{U}(t,0)\right\|_{B(E)} |y_0| \\
&+ \|\mathcal{U}(t,s)\|_{B(E)} |y_1| + \|\mathcal{U}(t,s)\|_{B(E)} \int_0^t p(s)\psi(|y(s)|)ds \\
&\le \tilde{M}|y_1| + M|y_0| + M\int_0^t p(s)\psi(R)ds \\
&\le \tilde{M}|y_0| + M(|y_1| + \psi(R)\|p\|_{L^1}) \\
&\le R. \qquad (3.4)
\end{aligned}$$

The condition (3.4) ensure that the operator N transforms the set D into itself.

Step 1. N continuous.

Let $(y_n)_{n\in N}$ be a sequence in D such that $y_n \to y$ in D. We have

$$|Ny_n(t) - Ny(t)| \le M\int_0^t |f(s,y_n(s)) - f(s,y(s))|\, ds.$$

Hence, from the Carathéodory of the function f and the Lebesgue dominated convergence theorem we obtain

$$\|Ny_n - Ny\| \to 0 \quad \text{as } n \to +\infty.$$

So N is continuous.
Consider the mesure of noncompacteness $\mu^*(D)$ defined on the family of bounded subsets of the space $BC(\bar{J}, E)$ by

$$\mu^*(D) = \tilde{\omega}_0^T(D) + \lim_{T\to+\infty}\sup\{e^{-\tau\tilde{\sigma}(t)}\,|y(t)| : t > T\} + \sup_{t\in J}\bar{\mu}(D(t)),$$

where $\tilde{\sigma}(t) = M\displaystyle\int_0^t \sigma(s)ds, \qquad \tau > 1, \qquad \tilde{\omega}_0^T(D) = e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D)$ and

$$\bar{\mu}(D_n) = e^{-\tau\tilde{\sigma}(t)}\mu(D).$$

Step 2. $D_\infty = \cap_{n=0}^{+\infty} D_n$ is compact.

In the sequel, we consider the sequence of sets $\{D_n\}_{n=0}^{+\infty}$ defined by induction as follows :

$$D_0 = D,\ D_{n+1} = Conv(N(D_n)) \text{ for } n = 0, 1, 2, \cdots \text{and } D_\infty = \cap_{n=0}^{+\infty} D_n$$

this sequence is nondecreasing, i.e. $D_{n+1} \subset D_n$ for each n.

Claim 1. $\lim\limits_{n\to+\infty} \overline{\mu}(D_n(t)) = 0.$

Using the properties of μ, Lemma 1.3.5 and $((\mathbf{H_4})$ we get

$$\begin{aligned}
\mu(D_{n+1}(t)) &= \mu(ConvN(D_n)) = \mu(N(D_n)) = \mu\left(\int_0^t \mathcal{U}(t,s)f(s, D_n(s))ds\right) \\
&\leq \|\mathcal{U}(t,s)\|_{B(E)} \int_0^t \mu(f(s, D_n(s))ds \\
&\leq M\int_0^t \sigma(s)\mu(D_n(s))ds. \\
&\leq M\int_0^t \sigma(s)e^{\tau\tilde{\sigma}(s)}e^{-\tau\tilde{\sigma}(s)}\mu(D_n(s))ds \\
&\leq \frac{1}{\tau}e^{\tau\tilde{\sigma}(t)}\overline{\mu}(D_n(t)),
\end{aligned}$$

we obtain

$$e^{-\tau\tilde{\sigma}(t)}\mu(D_{n+1}(t)) \leq \frac{1}{\tau}\overline{\mu}(D_n(t))$$

$$\overline{\mu}(D_{n+1}(t)) \leq \frac{1}{\tau}\overline{\mu}(D_n(t)).$$

By method of mathematical induction, we can prove

$$\overline{\mu}(D_{n+1}(t)) \ \ \leq \ \ \left(\frac{1}{\tau}\right)^{n+1}\overline{\mu}(D_0(t)).$$

We get

$$\lim_{n\to+\infty} \overline{\mu}(D_n(t)) = 0.$$

Claim 2. $\lim\limits_{n\to+\infty} \tilde{\omega}_0^T(D_n) = 0.$

It is eough to prove that
$\forall\delta > 0 \qquad \exists n \geq n_0 \qquad \forall T \geq 0 \qquad \omega_0^T(D_n) \leq \delta.$
Observe thas, for arbitrary $T \geq T_0$, $t_2, t_1 \in [T_0, T]$ and $y \in D_n$ we have

$$\begin{aligned} |(Ny)(t_2) - (Ny)(t_1)| &\leq |(Ny)(t_2)| + |(Ny)(t_1)| \\ &\leq 2R, \end{aligned}$$

we get

$$e^{-\tau\tilde{\sigma}(t)}|(Ny)(t_2) - (Ny)(t_1)| \leq 2Re^{-\tau\tilde{\sigma}(t)}.$$

Hence, there exists T_0 such that

$$e^{-\tau\tilde{\sigma}(t)}|(Ny)(t) - (Ny)(t)| \leq \delta. \tag{3.5}$$

For $T > T_0, t_2, t_1 \in [T_0, T]\,, y_n \in D_n$
Let us fix any $T_1 > T_0$ and $\varepsilon \in (0, T_1 - T_0)$.
For arbitrary $T > T_0$ and for arbitrary $t_2, t_1 \in [T_0, T]$ with $|t_2 - t_1| \leq \varepsilon$, at least one of the following two cases is fulfilled:
$a)$ $t_2, t_1 \in [T_0, T]$
or
$b)$ $t_2, t_1 \in [0, T_1]$

case $a)$ A vertu (3.5), we have

$$e^{-\tau\tilde{\sigma}(t)}|(Ny)(t) - (Ny)(t)| \leq \delta.$$

case $b)$ Implies that

$$e^{-\tau\tilde{\sigma}(T)}|(Ny)(t_2) - (Ny)(t_1)| \leq e^{-\tau\tilde{\sigma}(T_1)}|(Ny)(t_2) - (Ny)(t_1)| \leq e^{-\tau\tilde{\sigma}(T_1)}\omega^{T_1}(D_n, \varepsilon).$$

Joining these two facts we get

$$e^{-\tau\tilde{\sigma}(T)}\omega^T(D_n, \varepsilon) \leq \max\{\delta, e^{-\tau\tilde{\sigma}(T_1)}\omega^{T_1}(D_n, \varepsilon)\} \quad \text{for } \ \varepsilon \in (0, T_1 - T_0), T > T_0.$$

Letting $\varepsilon \to 0$ and keeping in mind that the function $\omega_0^{T_1}(.)$ nondecreasing with rspect to T, we obtain

$$e^{-\tau\tilde{\sigma}(T)}\omega^T(D_n, \varepsilon) \leq \max\{\delta, e^{-\tau\tilde{\sigma}(T_1)}\omega^{T_1}(D_n, \varepsilon)\} \quad \text{for } \ T \geq 0. \tag{3.6}$$

Now, applying Lemma (6.2.1) we derive

$$\begin{aligned}\omega^{T_1}(D_{n+1}) &= \omega^{T_1}(Conv(N(D_n)) = \omega^{T_1}(F(D_n)) \\ &\leq 2T_1 M\mu(\{f(s,y(s)) : s \leq T_1, y \in D_n\}) \\ &\leq M\sigma(T_1)\mu(D_n)\end{aligned}$$

we have

$$e^{-\tau\tilde{\sigma}(T_1)}\omega^{T_1}(D_{n+1}) \leq 2MT\sigma(T_1)e^{-\tau\tilde{\sigma}(T_1)}\mu(D_n),$$

than

$$\tilde{\omega}^{T_1}(D_{n+1}) \leq 2MT\sigma(T_1)\bar{\mu}(D_n).$$

From **Claim 3** and (3.6) we obtain

$$\lim_{n\to+\infty}\tilde{\omega}_0^T(D_n) = 0.$$

Claim 3. $\lim\limits_{T\to+\infty}\sup\{e^{-\tau\tilde{\sigma}(t)}|(Ny)(t)| : t \geq T\}$.
We have

$$e^{-\tau\tilde{\sigma}(t)}|(Ny)(t)| \leq Re^{-\tau\tilde{\sigma}(t)}.$$

Then

$$\lim_{t\to+\infty}\sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}|(Ny)(t)| = 0.$$

Form **Claim 1, Claim 2,** and **Claim 3** we conclude

$$\lim_{n\to+\infty}\mu^*(D_n) = 0.$$

Taking into account Lemma 1.3.8 we infer that $D_\infty = \cap_{n=0}^{+\infty}D_n$ is nonempty, convex and compact. Thus, by Schauder's fixed point theorem the operator $N : D_\infty \to D_\infty$ has at least one fixed point which is a mild solution of problem (3.1)-(3.2).

Theorem 3.2.2. *Assume that the hypotheses* $(\mathbf{H_1}) - (\mathbf{H_4})$ *are satisfied, and if*

$$\lim_{t\to+\infty}\int_0^t \sigma(s)ds = c < \infty.$$

Then the problem (3.1)-(3.2) admits at least one mild solution.

Proof. The prouf of this theorem is similar to the prouf of theorem 3.2.1 and we used the mesure of noncompacteness $\mu^*(D)$ defined on the family of bounded subsets of the space $BC(J,E)$ by

$$\mu^*(D) = e^{-\tau(\tilde{\sigma}(t)+t)}\omega_0^T(D) + \lim_{T\to+\infty} \sup\{e^{-\tau(\tilde{\sigma}(t)+t)}\,|y(t)| : t \geq T\} + \sup_{t\in J} e^{-\tau(\tilde{\sigma}(t)+t)}\mu(D(t)).$$

and

$$\begin{aligned}
\mu(D_{n+1}) &= \mu(ConvN(D_n)) = \mu(N(D_n)) = \mu\left(\int_0^t \mathcal{U}(t,s)f(s,D_n(s))ds\right) \\
&\leq \|\mathcal{U}(t,s)\|_{B(E)} \int_0^t \mu(f(s,D_n(s))ds \\
&\leq M\int_0^t \sigma(s)\mu_n(D_n(s))ds. \\
&\leq M\int_0^t (\sigma(s)+1)e^{\tau(\tilde{\sigma}(s)+t)}e^{-\tau(\tilde{\sigma}(s)+t)}\mu(D_n(s))ds \\
&\leq \frac{1}{\tau}e^{\tau(\tilde{\sigma}(t)+t)}\overline{\mu}(D_n(s)).
\end{aligned}$$

3.3 An example

Consider the second order Cauchy problem

$$\begin{cases}
\dfrac{\partial^2}{\partial t^2}z(t,\tau) &= \dfrac{\partial^2}{\partial \tau^2}z(t,\tau) + a(t)\dfrac{\partial}{\partial t}z(t,\tau) \\
&\quad + \displaystyle\int_0^t \frac{\sigma(t-\tau)\ln(1+|z(t,\tau)|)}{1+z^2(t,\tau)}d\tau, \quad t\in\bar{J},\ \tau\in[0,\pi], \\
z(t,0) = & z(t,\pi) = 0 \qquad t\in\bar{J}, \\
\dfrac{\partial}{\partial t}z(0,\tau) &= \psi(\tau) \qquad \tau\in[0,\pi]
\end{cases} \tag{3.7}$$

where we assume that $a : \bar{J} \to \mathbb{R}$ is a Hölder continuous function and $\sigma : J \to \mathbb{R}$ essentially bounded measurable function. Let $E = L^2([0,\pi],\mathbb{R})$ the space of 2-integrable functions from $[0,\pi]$ into $\mathbb{R}$, and $H^2([0,\pi],\mathbb{R})$ denotes the Sobolev space of functions $x : [0,\pi] \to \mathbb{R}$ such that $x'' \in L^2([0,\pi],\mathbb{R})$. We consider the operator $A_1y(\tau) = y''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R},\mathbb{C})$, infinitesimal generator of strongly continuous cosine function $C(t)$ on E. Moreover, we take $A_2(t)y(s) = a(t)y'(s)$ defined on $H^1([0,\pi],\mathbb{R})$, and consider the closed linear operator $A(t) = A_1 + A_2(t)$ which generates an evolution operator $\mathcal{U}$ defined by

$$\mathcal{U}(t,s) = \sum_{n\in\mathbb{Z}} z_n(t,s)\langle x, w_n\rangle w_n,$$

where z_n is a solution to the following scalar initial value problem

$$\begin{cases} z''(t) = -n^2z(t) + ina(t)z(t) \\ z(s) = 0, \qquad z'(s) = z_1. \end{cases} \tag{3.8}$$

Set

$$z(t)(\tau) = w(t)(\tau),\ t \geq 0,\ \tau \in [0, \pi],$$

$$f(t,z)(\tau) = \int_0^t \frac{\sigma(t-\tau)\ln(1+|z(t,\tau)|)}{1+z^2(t,\tau)} d\tau,$$

and

$$\frac{\partial}{\partial t} z(0)(\tau) = \frac{d}{dt} w(0)(\tau),\ \tau \in [0, \pi].$$

Consequently, (3.7) can be written in the abstract form (3.1)-(3.2) with $A(t)$ and f defined above. The existence of a mild solution can be deduced from an application of Theorem 3.2.1.

CHAPTER 4

SEMILINEAR DIFFERENTIAL EQUATIONS IN FRÉCHET SPACE WITH NON–LOCAL CONDITIONS [(3)]

4.1 Introduction

Evolution equations with non–local initial conditions generalize evolution equations with classical initial conditions. This notion is more complete in describing nature phenomena than the classical one because additional information is taken into account. For the importance of nonlocal conditions in different fields of applied sciences see [9,22,37,38,120,121] and the references therein. The earliest works in this areaweremade by Byszewski in [44,45,46,47].

This chapter is devoted to study the existence of mild solutions of non–local initial value problem described as a second order non–autonomous abstract differential problem.

Consider the following problem

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in \bar{J} = [0, \infty), \tag{4.1}$$

$$y(0) = g(y),\ y'(0) = h(y), \tag{4.2}$$

where $\{A(t)\}_{0 \le t < +\infty}$ is a family of linear closed operators from E into E that generate an evolution system of linear bounded operators $\{\mathcal{U}(t,s)\}_{(t,s)\in \bar{J}\times\bar{J}}$ for $0 \le s \le t < +\infty$, $f : \bar{J} \times E \to E$ be a Carathéodory function, $g, h : C(\bar{J}; E) \to E$ are given functions and $(E, |\cdot|)$ a real Banach space.

(3) M. Benchohra, J. Nieto and N. Rezoug, Second order evolution equations nonlocal conditions, (submitted)

4.2 Existence of solutions

A mild solution of (4.1)- (4.2) is defined as follows.

Definition 4.2.1. *A function $y \in C(\bar{J}, E)$ is called a mild solution to the problem (4.1)-(4.2) if y satisfies the integral equation*

$$y(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)g(y) + \mathcal{U}(t,0)h(y) + \int_0^t \mathcal{U}(t,s)f(s,y(s))ds. \tag{4.3}$$

To prove our results we introduce the following conditions:

($\boldsymbol{H_1}$) There exists a constant $M \geq 1$ and such that

$$\|\mathcal{U}(t,s)\|_{B(E)} \leq M, (t,s) \in \Delta.$$

($\boldsymbol{H_2}$) There exists a constant $\tilde{M} \geq 0$ such that

$$\left\|\frac{\partial}{\partial s}\mathcal{U}(t,s)\right\|_{B(E)} \leq \tilde{M}, (t,s) \in \Delta.$$

($\boldsymbol{H_3}$) There exist an integrable function $p : \bar{J} \to \mathbb{R}_+$ and a continuous nondecreasing function $\psi : [0,\infty) \to (0;\infty)$ such that:

$$|f(t,u)| \leq p(t)\psi(|u|) \text{ for a.e } t \in \bar{J} \text{ and each } u \in E.$$

($\boldsymbol{H_4}$) There exists a locally integrable function $\sigma : \bar{J} \to \mathbb{R}_+$ such that for any nonempty bounded set $D \subset E$ we have :

$$\mu(f(t,D)) \leq \sigma(t)\mu(D) \text{ for a.e } t \in \bar{J}.$$

($\boldsymbol{H_5}$) $g, h : C(\bar{J}, E) \to E$ is a continuous mapping and

$$\sup_{y \in D} |g(y)| < \infty, \quad \sup_{y \in D} |h(y)| < \infty$$

for any nonempty bounded set $D \subset C(\bar{J}, E)$.

($\boldsymbol{H_6}$) There exist $L_i > 0$ $(i = 1, 2)$ such that

$$\mu(g(D)) \leq L_1 \eta(D)$$

and

$$\mu(h(D)) \leq L_2 \eta(D).$$

for any nonempty bounded set $D \subset C(\bar{J}, E)$.

($\boldsymbol{H_7}$) There exists a constant $R > 0$ such that

$$\tilde{M} \sup_{y \in B_R} |g(y)| + M \sup_{y \in B_R} |h(y)| + M\psi(R) \|p\|_{L^1} \leq R,$$

where B_R is the closed ball in $C(\bar{J}; E)$ centered at zero O and with radius R.

Consider the operators $N_i : C(\bar{J}, E) \to C(\bar{J}, E)(i = 1, 2, 3)$ defined by

$$(N_1 y)(t) = -\frac{\partial}{\partial s}\mathcal{U}(t, 0)g(y),$$

$$(N_2 y)(t) = \mathcal{U}(t, 0)h(y),$$

$$(N_3 y)(t)) = \int_0^t \mathcal{U}(t, s)f(s, y(s))ds.$$

Lemma 4.2.1. *[105] Assume that the hypotheses* $(\mathbf{H_1}) - (\mathbf{H_7})$ *hold and a set* $D \subset E$ *is bounded. Then*

$$\omega_0^T(N_1(D)) \leq 2M\mu(g(D)),$$

$$\omega_0^T(N_2(D)) \leq 2M\mu(h(D)),$$

$$\omega_0^T(N_3(D)) \leq 2M \int_0^T \mu(f(s, D(s))ds.$$

Theorem 4.2.1. *Assume that the hypotheses* $(\mathbf{H_1}) - (\mathbf{H_7})$ *are satisfied. If*

$$3\tilde{M}L_1 + 3ML_2 + \frac{6}{\tau} < 1, \ \tau > 6,$$

then the problem (4.1)-(4.2) admits at least one mild solution.

Proof. Consider the operator $N : C(\bar{J}, E) \to C(\bar{J}, E)$ defined by

$$(Ny)(t) = -\frac{\partial}{\partial s}\mathcal{U}(t, 0)g(y) + \mathcal{U}(t, 0)h(y) + \int_0^t \mathcal{U}(t, s)f(s, y(s))ds.$$

We define

$$D = B_R = \left\{y \in C(\bar{J}, E) : \|y\|_T \leq R\right\}.$$

The set B_R is nonempty convex and closed.
Now, for $t \in J, T > 0$ we have

$$\begin{aligned}
|(Ny)(t)| &\leq \left\| \frac{\partial}{\partial s}\mathcal{U}(t,0) \right\|_{B(E)} |g(y)| \\
&+ \|\mathcal{U}(t,s)\|_{B(E)} |h(y)| + \|\mathcal{U}(t,s)\|_{B(E)} \int_0^t p(s)\psi(|y(s)|)ds \\
&\leq \tilde{M}|g(y)| + M|h(y)| + M\int_0^t p(s)\psi(R)ds \\
&\leq \tilde{M}\sup_{y\in B_R}|g(y)| + M\sup_{y\in B_R}|h(y)| + M\psi(R)\,\|p\|_{L^1} \\
&\leq R. \qquad (4.4)
\end{aligned}$$

(4.4) ensures that the operator N transforms the set B_R into itself.

Step 1. N is continuous.

Let $(y_n)_{n\in N}$ be a sequence in B_R such that $y_n \to y$ in B_R.
For $t \in J, T \geq 0$ we have

$$\begin{aligned}
|(Ny_n)(t) - (Ny)(t)| &\leq |g(y_n) - g(y)| + |h(y_n) - h(y)| \\
&+ M\int_0^t |f(s, y_n(s)) - f(s, y(s))|\, ds.
\end{aligned}$$

Hence, since the functions g, h are continuous and f is Carathéodory, the Lebesgue dominated convergence theorem implies that

$$\|Ny_n - Ny\|_T \to 0 \quad \text{as } n \to +\infty.$$

So N is continuous.
Consider the measure of noncompacteness $\mu^*(D)$ defined on the family of bounded subsets of the space $C(\bar{J}, E)$ by

$$\mu^*(D) = \sup\left\{ e^{-\tau\tilde{\sigma}(T)}(\omega_0^T(D) + \sup_{t\in[0,T]}\mu(D_n(t)) : T \geq 0 \right\},$$

where

$$\tilde{\sigma}(t) = M\int_0^t \sigma(s)ds, \qquad \tau > 6.$$

Step 2. $D_\infty = \cap_{n=0}^{+\infty} D_n$ is compact.

In the sequel, we consider the sequence of sets $\{D_n\}_{n=0}^{+\infty}$ defined by induction as follows :

$$D_0 = B_R,\ D_{n+1} = Conv(N(D_n)) \text{ for } n = 0, 1, 2, \cdots \text{and } D_\infty = \cap_{n=0}^{+\infty} D_n$$

this sequence is nondecreasing, i.e. $D_{n+1} \subset D_n$ for each n.

Claim 1. $\lim_{n\to+\infty} \mu^*(D_n) = 0.$
We know from Lemma 1.3.6 that for each $\varepsilon > 0$ there is a sequence function $\{W_k\}_{k=0}^{\infty} \subset (N_3D_n)(s)$ such that

$$\mu(N_3D_n)(s) \leq 2\mu(\{W_k\}_0^{\infty}) + \varepsilon.$$

This implies that there is a sequence $\{Q_k\}_{k=0}^{\infty} \subset W_k(s)$ such that

$$W_k = (N_3Q_k)(s) \text{ for } k = 1, 2, \dots$$

Using the properties of μ, Lemma 1.3.6, Lemma 1.3.7 and assumptions $(\mathbf{H_4}), (\mathbf{H_5})$), we get

$$\begin{aligned}
\mu(D_{n+1}(t)) &= \mu(ConvN(D_n)(t)) \\
&= \mu((N_1D_n)(t)) + \mu((N_2D_n)(t)) + \mu((N_3D_n)(t)) \\
&= \mu(g(D_n)) + \mu(h(D_n)) + 2\mu(\{W_k\}_{k=0}^{\infty}) + \varepsilon \\
&= \mu(g(D_n) + \mu(h(D_n) + 2\mu(\{(N_3Q_k)(s)\}_{k=0}^{\infty}) + \varepsilon \\
&= \mu(g(D_n)) + \mu(h(D_n)) \\
&+ 4\mu\left(\left\{\int_0^t \mathcal{U}(t,s)f(s,(N_3Q_k)(s))ds\right\}_{k=0}^{\infty}\right) + \varepsilon \\
&\leq \tilde{M}L_1\eta(D_n) + MK_2\eta(D_n) \\
&+ 4M\int_0^t \sigma(s)\mu(\{Q_k(s)\}_{k=0}^{\infty})ds + \varepsilon. \\
&\leq \tilde{M}L_1\eta(D_n) + MK_2\eta(D_n) + 4M\int_0^t \sigma(s)\mu(D_n(s))ds + \varepsilon.
\end{aligned}$$

Since ε is arbitrary and using Lemma 1.3.4 we obtain

$$\begin{aligned}
\mu(D_{n+1}(t)) &\leq \tilde{M}L_1(\omega_0^T(D_n) + \sup_{t\in[0,T]} \mu(D_n(t))) \\
&+ ML_2(\omega_0^T(D_n) + \sup_{t\in[0,T]} \mu(D_n(t))) \\
&+ 4M\int_0^t \sigma(s)(\omega_0^T(D_n) + \sup_{s\in[0,T]} \mu(D_n(s)))ds.
\end{aligned} \tag{4.5}$$

Now, applying Lemma 6.2.1 and using assumptions $(\mathbf{H_4}), (\mathbf{H_5})$, we derive

$$\begin{aligned}
\omega_0^T(D_{n+1}) &= \omega_0^T(Conv(ND_n)) \\
&= \omega_0^T(N_1D_n) + \omega_0^T(N_2D_n) + \omega_0^T(N_3D_n)) \\
&\leq \tilde{M}L_1\eta(D_n) + ML_2\eta(D_n) + 2M\int_0^t \sigma(s)\mu(D_n(s))ds \\
&\leq 2\tilde{M}L_1(\omega_0^T(D_n) + \sup_{t\in[0,T]}\mu(D_n(t)) \\
&+ 2ML_2(\omega_0^T(D_n) + \sup_{t\in[0,T]}\mu(D_n(t))) \\
&+ 2M\int_0^t \sigma(t)(\omega_0^T(D_n) + \sup_{s\in[0,T]}\mu(D_n(s))ds.
\end{aligned} \tag{4.6}$$

From (6.9) and (4.6), we have

$$\begin{aligned}
&\omega_0^T(D_{n+1}) + \sup_{t\in[0,T]}\mu(D_{n+1}(t)) \\
&\leq (3\tilde{M}L_1 + 3ML_2)(\omega_0^T(D_n) + \sup_{t\in[0,T]}\mu(D_n(t))) \\
&+ \quad 6M\int_0^t \sigma(s)(\omega_0^T(D_n) + \sup_{s\in[0,T]}\mu(D_n(s)))ds.
\end{aligned}$$

Then

$$\begin{aligned}
&\omega_0^T(D_{n+1}) + \sup_{t\in[0,T]}\mu(D_{n+1}(t)) \\
&\leq (3\tilde{M}K_1 + 3MK_2)(\omega_0^T(D_n) + \sup_{t\in[0,T]}\mu(D_n(t))) \\
&+ \quad 6M\int_0^t \sigma(s)e^{-\tau\tilde{\sigma}(t)}e^{\tau\tilde{\sigma}(t)}(\omega_0^T(D_n) + \sup_{s\in[0,T]}\mu(D_n(s)))ds.
\end{aligned}$$

We obtain

$$\begin{aligned}
&e^{-\tau\tilde{\sigma}(T)}(\omega_0^T(D_{n+1}) + \sup_{t\in[0,T]}\mu(D_n(t))) \\
&\leq \left(3\tilde{M}K_1 + 3MK_2 + \frac{6}{\tau}\right)\sup\{e^{-\tau\tilde{\sigma}(T)}(\omega_0^T(D_n) + \sup_{t\in[0,T]}\mu(D_n(t))) : T \geq 0\}.
\end{aligned}$$

Hence, we get

$$\mu^*(D_{n+1}) \leq \left(3\tilde{M}K_1 + 3MK_2 + \frac{6}{\tau}\right)\mu^*(D_n).$$

By method of mathematical induction, we can prove

$$\mu^*(D_{n+1}) \leq \left(3\tilde{M}K_1 + 3MK_2 + \frac{6}{\tau}\right)^{n+1}\mu^*(D_0).$$

Hence, in view of the assumption (H_6), we get

$$\lim_{n\to+\infty} \mu^*(D_n) = 0.$$

Taking into account Lemma 1.3.8 we infer that $D_\infty = \cap_{n=0}^{+\infty} D_n$ is nonempty, convex and compact. Thus, by Tykhonoff's fixed point theorem the operator $N : D_\infty \to D_\infty$ has at least one fixed point which is a mild solution of problem (4.1)-(4.2).

4.3 An example

Consider the following partial differential equation with nonlocal conditions;

$$\begin{cases} \dfrac{\partial^2 z(t,\tau)}{\partial t^2} & = \dfrac{\partial^2 z(t,\tau)}{\partial \tau^2} + a(t)\dfrac{\partial z(t,\tau)}{\partial t} \\ & +f_1(t, z(t,\tau)), & t \in \bar{J},\ \tau \in [0,\pi], \\ z(t,0) = & z(t,\pi) & t \in \bar{J}, \\ \dfrac{\partial}{\partial t} z(t,0) = & \dfrac{\partial}{\partial t} z(t,0) & t \in \bar{J}, \\ z(0,\tau) = & \displaystyle\int_0^{+\infty} g_1(t, z(t,\tau))dt, & \tau \in [0,\pi], \\ \dfrac{\partial}{\partial t} z(0,\tau) & = \displaystyle\int_0^{+\infty} h_1(t, z(t,\tau))dt, & \tau \in [0,\pi] \end{cases} \tag{4.7}$$

where we assume that $a : \bar{J} \to \mathbb{R}$ is a Hölder continuous function and $h_1, g_1 : \bar{J} \times \mathbb{R} \to \mathbb{R}$ are given functions.
Let $E = L^2([0,\pi], \mathbb{C})$ the space of 2-integrable functions from $[0,\pi]$ into $\mathbb{R}$, and $H^2([0,\pi], \mathbb{C})$ denotes the Sobolev space of functions $x : [0,\pi] \to \mathbb{R}$ such that $x'' \in L^2([0,\pi], \mathbb{C})$. We consider the operator $A_1 y(\tau) = y''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R}, \mathbb{C})$, infinitesimal generator of strongly continuous cosine function $C(t)$ on E. Moreover, we take $A_2(t)y(s) = a(t)y'(s)$ defined on $H^1([0,\pi], \mathbb{C})$, and consider the closed linear operator $A(t) = A_1 + A_2(t)$ which generates an evolution operator $\mathcal{U}$ defined by

$$\mathcal{U}(t,s) = \sum_{n\in\mathbb{Z}} z_n(t,s)\langle x, w_n\rangle w_n,$$

where z_n is a solution to the following scalar initial value problem

$$\begin{cases} z''(t) = -n^2 z(t) + ina(t)z(t) \\ z(0) = 0, \qquad z'(0) = 1. \end{cases}$$

Set

$$w(t)(\tau) = z(t,\tau),\ t \geq 0,\ \tau \in [0,\pi],$$

$$f(t, z(t,\tau)) = f_1(t, z(t,\tau))$$

$$g(z)(\tau) = \int_0^{+\infty} g_1(t, z(t,\tau))dt, \tau \in [0,\pi],$$

$$h(z)(\tau) = \int_0^{+\infty} h_1(t, z(t, \tau))ds, \tau \in [0, \pi].$$

We now assume that:

(1) The maps f is Carathéodory and satisfy conditions $(\mathbf{H_3}), (\mathbf{H_4})$.

(3) The maps g and h satisfy Carathéodory conditions and there exist functions $\varrho_i \in L^2(J)$ $(i = 1, 2)$ such that

$$|g_1(t; s)| \leq \varrho_1(t)s \text{ for a.e. } t, s \in \mathbb{R};$$

and

$$|h_1(t; s)| \leq \varrho_2(t)s \text{ for a.e. } t, s \in \mathbb{R};$$

Hence, reasoning similarly as in the proof of Claim 1 and using Lemma 1.3.4 we infer that for any $D \subset C(\bar{J}; E)$

$$\begin{aligned} \mu(g(D)) &\leq 4\left(\int_0^T \varrho_1^2(t)dt\right)^{\frac{1}{2}} \sup_{t\in J} \mu(D(t)) \\ &\leq 4\left(\int_0^T \varrho_1^2(t)dt\right)^{\frac{1}{2}} \eta(D). \end{aligned}$$

and

$$\begin{aligned} \mu(h(D)) &\leq 4\left(\int_0^T \varrho_2^2(t)dt\right)^{\frac{1}{2}} \sup_{t\in J} \mu(D(t)) \\ &\leq 4\left(\int_0^T \varrho_2^2(t)dt\right)^{\frac{1}{2}} \eta(D). \end{aligned}$$

These show that the maps g and h satisfy conditions $(\mathbf{H_5})$ and $(\mathbf{H_6})$ with the constants

$$L_i = \left(\int_0^T \varrho_i^2(t)dt\right)^{\frac{1}{2}}, \ i = 1, 2.$$

Consequently, problem (4.7) can be written in the abstract form (4.1)-(4.2) with $A(t)$ and f defined above. The existence of a mild solution can be deduced from an application of Theorem 4.2.1.

CHAPTER 5

SEMILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY [(4)]

5.1 Introduction

In recent years we see an increasing interest in infinite delay equations. The main reason is that equations of this type become more and more important for different applications. When the delay is infinite, the notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato in [74], see also the books by Ahmed [8], Corduneanu and Lakshmikantham [50], Kappel and Schappacher [87]. For detailed discussion and applications on this topic, we refer the reader to the book by Hale and Verduyn Lunel [76], Hino et al. [85], Wu [117], Baghli and Benchohra [16,17,18] and Baliki and Benchohra [24,25] and the references therein.
This Chapter is dedicated to study the existence and uniqueness of mild solution the following second order evolution equation.
Consider the following problem

$$
\begin{aligned}
y''(t) - A(t)y(t) &= f(t, y_t), \ a.e. \ t \in J = [0; T], && (5.1)\\
y(t) &= \phi(t), t \in (-\infty, 0], && (5.2)\\
y'(0) &= \tilde{y}, && (5.3)
\end{aligned}
$$

where $\{A(t)\}_{0 \le t < +\infty}$ is a family of linear closed operators from E into E that generate an evolution system of operators $\{\mathcal{U}(t,s)\}_{(t,s) \in J \times J}$ for $0 \le s \le t \le T$, $f : J \times \mathcal{B} \to E$ be a Carathéodory function and $\mathcal{B}$ is an abstract phase space to be specified later, $\tilde{y} \in E$, $\phi \in \mathcal{B}$ and $(E, |\cdot|)$ a real Banach space.

[(4)] M. Benchohra and N. Rezoug, Second order functional evolution equations with infinite delay, (submitted).

5.2 Existence of solutions

Definition 5.2.1. *A function $y \in \mathcal{C}$ is called a mild solution to the problem (5.1) -(5.3), if y is continuous and*

$$y(t) = \begin{cases} \phi(t), & \text{if } t \leq 0 \\ -\dfrac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0) + \mathcal{U}(t,0)\tilde{y} + \displaystyle\int_0^t \mathcal{U}(t,s)f(s,y_s)ds, & \text{if } t \in J. \end{cases} \tag{5.4}$$

To prove our results we introduce the following conditions:

$(\boldsymbol{H_1})$ There exists a constant $M \geq 1$ such that

$$\|\mathcal{U}(t,s)\|_{B(E)} \leq M \quad \text{for any } (t,s) \in \Delta.$$

$(\boldsymbol{H_2})$ There exists a constant $\tilde{M} \geq 0$ such that

$$\left\|\frac{\partial}{\partial s}\mathcal{U}(t,s)\right\|_{B(E)} \leq \tilde{M}.$$

$(\boldsymbol{H_3})$ There exist a function $p \in L^1(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : [0,+\infty) \to (0,+\infty)$ such that:

$$|f(t,u)| \leq p(t)\psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t \in J \text{ and any } u \in \mathcal{B}.$$

$(\boldsymbol{H_4})$ There exists a constant $R > 0$ such that

$$M\psi\Big(\gamma R + \gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1) + \gamma M|\tilde{y}|\Big)\|p\|_{L^1} \leq R$$

.

$(\boldsymbol{H_5})$ There exists a function $\sigma \in L^1(J, \mathbb{R}_+)$ such that for any nonempty bounded set $D \subset \mathcal{B}$ we have :

$$\mu(f(t,D)) \leq \sigma(t) \sup_{\theta \in (-\infty,0]} \mu(D(\theta)) \text{ for a.e } t \in J.$$

To establish our main theorem, we need the following lemma.

Lemma 5.2.1. *[104] Assume that the hypotheses $(\mathbf{H_1})$, $(\mathbf{H_3})$ and $(\mathbf{H_5})$ hold and a set $D \subset \mathcal{C}$ is bounded. Then*

$$\omega_0^T(F(D)) \leq 2M \int_0^T \mu(f(s,D_s))ds,$$

where

$$F(D) = \{Fy : y \in D\},$$

and

$$(Fy)(t) = \int_0^T \mathcal{U}(t,s)f(s,y_s)ds.$$

Theorem 5.2.1. *Assume that the hypotheses* $(\mathbf{H_1}) - (\mathbf{H_5})$ *are satisfied. Then the problem (5.1) -(5.3) admits at least one mild solution.*

Proof. It is clear that the fixed point of the operator $N : \mathcal{C} \to \mathcal{C}$ defined by $(Ny)(t) = \phi(t)$ if $t \leq 0$ and

$$(Ny)(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0) + \mathcal{U}(t,0)\tilde{y} + \int_0^t \mathcal{U}(t,s)f(s,y_s)ds, \text{ if } t \in J, \tag{5.5}$$

are mild solutions of problem (5.1) -(5.3).
For $\phi \in \mathcal{B}$, Let $x : (-\infty, T] \to E$ be the function defined by

$$x(t) = \begin{cases} -\frac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0) + \mathcal{U}(t,0)\tilde{y}, & \text{if } t \in J \\ \\ \phi(t) & \text{if } t \in (-\infty, 0]. \end{cases}$$

Then $x_0 = \phi$. For any function $z \in \mathcal{C}$, we denote

$$y(t) = x(t) + z(t).$$

It is obvious that y satisfies (5.4) if and only if z satisfies $z_0 = 0$ and for all $t \in J$

$$z(t) = \int_0^t \mathcal{U}(t,s)f(t, x_s + z_s)ds. \tag{5.6}$$

In the sequel, we always denote $\mathcal{C}_0$ as a Banach space defined by

$$\mathcal{C}_0 = \{z \in \mathcal{C} : \ z_0 = 0\},$$

endowed with the family of seminorms

$$\begin{aligned} \|z\|_{\mathcal{C}_0} &= \sup\{|z(t)| : t \in J\} + \|z_0\|_{\mathcal{B}} \\ &= \sup\{|z(t)| : t \in J\} \end{aligned}$$

Now, we can consider the operator $L : \mathcal{C}_0 \to \mathcal{C}_0$ given by

$$(Lz)(t) = \int_0^t \mathcal{U}(t,s)f(s, z_s + x_s)ds, \text{ for } t \in J.$$

The problem (5.1) -(5.3) has a solution is equivalent to L has a fixed point. To prove this end, we start with the following estimation.
For any $z \in \mathcal{C}_0$ and $t \in J$, we have

$$\begin{aligned} \|z_t + x_t\|_{\mathcal{B}} &\leq \|z_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}} \\ &\leq K(t)|z(t)| + K(t)\|\frac{\partial}{\partial s}\mathcal{U}(t,0)\|_{B(E)}\|\phi\|_{\mathcal{B}} \\ &+ K(t)\|\mathcal{U}(t,0)\|_{B(E)}|\tilde{y}| + M(t)\|\phi\|_{\mathcal{B}} \\ &\leq \gamma\|z\|_{\mathcal{C}_0} + \gamma\tilde{M}\|\phi\|_{\mathcal{B}} + \gamma M|\tilde{y}| + \gamma\|\phi\|_{\mathcal{B}} \\ &\leq \gamma\|z\|_{\mathcal{C}_0} + \gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1) + \gamma M|\tilde{y}|. \end{aligned} \tag{5.7}$$

Now, we will show that the operator L satisfied the conditions of Schauder's fixed point theorem.
We define

$$B_R = \{z \in \mathcal{C}_0 : \|z\|_{\mathcal{C}_0} \leq R\}.$$

The set B_R is nonempty convex and closed. Let $z \in B_R$

$$\begin{aligned} |L(z)(t)| &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)} |f(s, x_s + z_s)| \, ds \\ &\leq M \int_0^t p(s)\psi(\|z_s + x_s\|_{\mathcal{B}}) ds. \\ &\leq M\psi\Big(\gamma\|z\|_{\mathcal{C}_0} + \gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1) + \gamma M|\widetilde{y}|\Big) \int_0^t p(s) ds \\ &\leq M\psi\big(\gamma R + \gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1) + \gamma M|\widetilde{y}|\big)\|p\|_{L^1} \leq R. \end{aligned}$$

Thus the operator L maps B_R into itself.

Step 1. L is continuous.
Let $(z^n)_{n\in\mathbb{N}}$ be a sequence in B_R such that $z^n \to z$ in B_R, then for any $t \in J$ we obtain

$$\begin{aligned} |(Lz^n)(t) - (Lz)(t)| &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)} |f(t, x_s + z_s^n) - f(t, x_s + z_s)| \, ds \\ &\leq M \int_0^t |f(s, z_s^n + x_s) - f(s, z_s + x_s)| \, ds. \end{aligned}$$

Hence, from the continuity of the function f and the Lebesgue dominated convergence theorem we obtain

$$\|Lz_k - Lz\|_{\mathcal{C}_0} \to 0 \quad \text{as } n \to +\infty.$$

So L is continuous.
Consider the mesure of noncompacteness μ^* defined on the family of bounded subsets of the space $\mathcal{C}_0$ by

$$\mu^*(D) = e^{-\tau\tilde{\sigma}(T)}\omega_0^T(D) + \sup_{t\in J} e^{-\tau\tilde{\sigma}(T)}\mu(D(t)),$$

where

$$\tilde{\sigma}(t) = M\int_0^t \sigma(s) ds, \qquad \tau \geq 6.$$

Step 2. $D_\infty = \cap_{n=0}^{+\infty} D_n$ is compact.

In the sequel, we consider the sequence of sets $\{D_n\}_{n=0}^{+\infty}$ defined by induction as follows :

$$D_0 = B_R,\ D_{n+1} = Conv(N(D)_n)) \text{ for } n = 0, 1, 2, \cdots \text{and } D_\infty = \cap_{n=0}^{+\infty} D_n$$

this sequence is nondecreasing, i.e. $D_{n+1} \subset D_n$ for each n.

We know from Lemma 1.3.6 that for each $\varepsilon > 0$ there is a sequence function $\{W_k\}_{k=0}^{\infty} \subset L(D_n(s))$ such that

$$\mu(D_n) \leq 2\mu(\{W_k\}_{k=0}^{\infty}) + \varepsilon.$$

This implies that there is a sequence $\{Q^k\}_{k=0}^{\infty} \subset W_k$ such that

$$W_k = (LQ^k)(s) \text{ for } k = 0, 1, 2, ...$$

Using the properties of μ, Lemma 1.3.6, Lemma 1.3.7 and assumptions $(\mathbf{H_4})$, $(\mathbf{H_5})$, we get

$$\begin{aligned}
\mu(D_{n+1}(t)) &= \mu(ConvL(D_n)(t)) \\
&= 2\mu(\{W_k\}_{k=0}^{\infty}) + \varepsilon \\
&= 2\mu(\{(LQ^k)(s)\}_{k=0}^{\infty}) + \varepsilon \\
&\leq 4\mu\left(\left\{\int_0^t \mathcal{U}(t,s)f(s,Q_s^k)ds\right\}_{k=0}^{\infty}\right) + \varepsilon \\
&\leq 4M\int_0^t \sigma(s) \sup_{\theta\in(-\infty,0]} \mu(\{Q^k(s+\theta))\}_{k=0}^{\infty} + \{x_s\})ds + \varepsilon. \\
&\leq 4M\int_0^t \sigma(s) \sup_{\theta\in(-\infty,0]} \mu(D_n(s+\theta) + \{x_s\}))ds + \varepsilon. \\
&\leq 4M\int_0^t \sigma(s) \sup_{\theta\in(-\infty,0]} \mu(D_n(s+\theta)))ds + \varepsilon. \\
&\leq 4M\int_0^t \sigma(s) \sup_{u\in(-\infty,s]} \mu(D_n(u))ds + \varepsilon. \\
&\leq 4M\int_0^t \sigma(s) \sup_{u\in[0,s]} \mu(D_n(u)))ds + \varepsilon \\
&\leq 2M\int_0^t \sigma(s)\mu(D_n(s))ds + \varepsilon \\
&\leq 4\int_0^t \sigma(s)e^{\tau\tilde{\sigma}(s)}e^{-\tau\tilde{\sigma}(s)}\mu(D_n(s))ds + \varepsilon \\
&\leq 4\int_0^t \sigma(s)e^{\tau\tilde{\sigma}(s)} \sup_{s\in[0,t]} e^{-\tau\tilde{\sigma}(s)}\mu(D_n(s))ds + \varepsilon \\
&\leq 4e^{\tau\tilde{\sigma}(t)} \sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)) + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we get

$$\sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t) \leq \frac{4}{\tau}\left(\sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)\right).$$

Then

$$\sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)) \leq \frac{4}{\tau}\left(e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D_n) + \sup_{t\in J]} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)\right). \tag{5.8}$$

Now, applying Lemma 6.2.1 and using assumptions $(\mathbf{H_4})$, $(\mathbf{H_5})$, we derive

$$\begin{aligned}
\omega_0^T(D_{n+1}) &= \omega_0^T(Conv(LD_n)) \\
&= \omega_0^T(LD_n) \\
&\leq 2M\int_0^T \mu(f(s,(D_n)_s + \{x_s\})ds \\
&\leq 2M\int_0^t \sigma(s) \sup_{\theta\in(-\infty,0]} \mu((D_n)_s) + \{x_s\})ds \\
&\leq \frac{2}{\tau}e^{\tau\tilde{\sigma}(T)}(\sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)).
\end{aligned}$$

Then

$$e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D_n) \leq \frac{4}{\tau}\left(e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D_n) + \sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)\right). \tag{5.9}$$

From (5.8) and (5.9),

$$\begin{aligned}
&e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D_{n+1}) + \sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_{n+1}(t)) \\
&\leq \frac{6}{\tau}\left(e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D_n) + \sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)\right).
\end{aligned}$$

Hence, we get

$$\mu^*(D_{n+1}) \leq \frac{6}{\tau}\mu^*(D_n).$$

By method of mathematical induction, we can prove

$$\mu^*(D_{n+1}) \leq \left(\frac{6}{\tau}\right)^{n+1}\mu^*(D_0).$$

Hence, in view of the assumption $(\mathbf{H_6})$, we get

$$\lim_{n\to+\infty}\mu^*(D_n) = 0.$$

Taking into account Lemma 1.3.8 we infer that $D_\infty = \cap_{n=0}^{+\infty} D_n$ is nonempty, convex and compact. Thus, by Schauder's fixed point theorem the operator $N : D_\infty \to D_\infty$ has at least one fixed point which is a mild solution of problem (5.1)-(5.3).

5.3 An example

Consider the second order Cauchy problem

$$\left\{ \begin{array}{rll} \dfrac{\partial^2}{\partial t^2} y(t,\tau) = & \dfrac{\partial^2}{\partial \tau^2} y(t,\tau) + a(t) \dfrac{\partial}{\partial t} y(t,\tau) & \\ & + \displaystyle\int_{-\infty}^{t} b(t-s) \arctan(y(s,\tau)) ds & t \in J := [0,T],\ \tau \in [0,2\pi], \\ y(t,0) = & y(t,2\pi) = 0 & t \in J, \\ y(\theta,\tau) = & \phi(\theta,\tau), \quad \dfrac{\partial}{\partial t} y(0,\tau) = \psi(\tau) & \theta \in (-\infty,0], \tau \in [0,2\pi] \end{array} \right. \tag{5.10}$$

where we assume that $a,\ b, : J \to \mathbb{R}$ are continuous functions, $\phi(\theta, \cdot) \in \mathcal{B}$.
Let $X = L^2(\mathbb{R}, \mathbb{C})$ the space of 2π-periodic 2-integrable functions from $\mathbb{R}$ into $\mathbb{C}$, and $H^2(\mathbb{R}, \mathbb{C})$ denotes the Sobolev space 2π-periodic functions $x : \mathbb{R} \to \mathbb{C}$ such that $x'' \in L^2(\mathbb{R}, \mathbb{C})$.
We consider the operator $A_1 y(\tau) = y''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R}, \mathbb{C})$, infinitesimal generator of strongly continuous cosine function $C(t)$ on X. Moreover, we take $A_2(t)y(s) = a(t)y'(s)$ defined on $H^1(\mathbb{R}, \mathbb{C})$, and consider the closed linear operator $A(t) = A_1 + A_2(t)$ which generates an evolution operator $\mathcal{U}$ defined by

$$\mathcal{U}(t,s) = \sum_{n \in \mathbb{Z}} z_n(t,s) \langle x, w_n \rangle w_n,$$

where z_n is a solution to the following scalar initial value problem

$$\left\{ \begin{array}{l} z''(t) = -n^2 z(t) + ina(t) z(t) \\ z(s) = 0, \qquad z'(s) = z_1. \end{array} \right. \tag{5.11}$$

Define the operator $f : J \times \mathcal{B} \to X$ by

$$f(t,\varphi)(\tau) = \int_{-\infty}^{t} b(t-s) \varphi(s)(\tau) ds, \ \ \tau \in [0,2\pi],$$

$$w(t)(\tau) = y(t,\tau),\ t \geq 0,\ \tau \in [0,2\pi],$$

$$\phi(s)(\tau) = \arctan(y(s,\tau)),\ -\infty < s \leq 0,\ \tau \in [0,2\pi],$$

and
$$\frac{d}{dt}w(0)(\tau) = \frac{\partial}{\partial t}y(0,\tau),\ \tau \in [0,2\pi].$$

Consequently, (5.10) can be written in the abstract form (5.1)-(5.3) with A and f defined above. Now, the existence of a mild solution can be deduced from an application of Theorem 5.2.1.

CHAPTER 6

NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAYS [5]

6.1 Introduction

Functional differential equations of neutral type play an important role in the theory of functional differential equations because they are used in many fields of science, especially in control, biological, and medical domains. For more details, we refer the reader to [20,72,75,76].

In the literature there are many papers study the problems of neutral differential equations using different methods. Among them, the fixed point method combined by semigroup theory in Fréchet space, see for exemple Baghli and Benchohra [17,18] and Hernandez *et al.* [80,81,82].

In this Chapter, we investigate the existence of mild solutions for the neutral functional differential equation.

$$\frac{d}{dt}[y'(t) - g(t, y_t)] - A(t)y(t) = f(t, y_t), t \in J = [0; T] \tag{6.1}$$

$$y(t) = \phi(t), t \in (-\infty, 0], \tag{6.2}$$

$$y'(0) = \tilde{y}, \tag{6.3}$$

where $\{A(t)\}_{0\leq t\leq T}$ is a family of linear closed operators from E into E that generate an evolution system of operators $\{\mathcal{U}(t,s)\}_{(t,s)\in J\times J}$ for $0 \leq s \leq t < +\infty$, $f : J \times \mathcal{B} \to E$ be a Carathéodory function and $\mathcal{B}$ is an abstract phase space to be specified later,

[5] M. Benchohra and N. Rezoug, Second order neuteral Functional Evolution Equations with Infinite Delay, (submitted)

$\tilde{y} \in E$, $\phi \in \mathcal{B}$ and $(E, |\cdot|)$ a real Banach space.
For any continuous function y and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in (-\infty, 0]$. Here $y_t(\cdot)$ represents the history of the state up to the present time t. We assume that the histories y_t belong to $\mathcal{B}$.

6.2 Existence of solutions

Definition 6.2.1. *A function $y \in \mathcal{C}$ is said to be a mild solution of the problem (6.1)-(6.3), if*

$$y(t) = \begin{cases} \phi(t), & \text{if } t \leq 0 \\ -\dfrac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0) + U(t,0)(\phi(0) - g(0,\phi)) + g(t, y_t) & \\ + \displaystyle\int_0^t U(t,s)A(s)g(s,y_s)ds + \int_0^t U(t,s)f(s,y_s)ds, & \text{if } t \in J. \end{cases} \tag{6.4}$$

To prove our results we introduce the following conditions:

$(\boldsymbol{H_1})$ There exists a constant $M \geq 1$ such that

$$\|\mathcal{U}(t,s)\|_{B(E)} \leq M \quad \text{for any } (t,s) \in \Delta.$$

$(\boldsymbol{H_2 1})$ There exists a constant $\tilde{M} \geq 0$ such that

$$\left\| \frac{\partial}{\partial s}\mathcal{U}(t,s) \right\|_{B(E)} \leq \tilde{M}.$$

$(\boldsymbol{H_3})$ There exist a function $p \in L^1(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : (0, +\infty) \to [0, +\infty)$ such that:

$$|f(t,u)| \leq p(t)\psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t \in J \text{ and any } u \in \mathcal{B}.$$

$(\boldsymbol{H_4})$ There exists a function $\sigma \in L^1(J, \mathbb{R}_+)$ such that for any nonempty bounded set $D \subset \mathcal{B}$ we have :

$$\mu(f(t,D)) \leq \sigma(t) \sup_{\theta \in (-\infty, 0]} \mu(D(\theta)) \text{ for a.e } t \in J.$$

$(\boldsymbol{H_5})$ There exists a constant M^* such that:

$$\|A^{-1}(t)\|_{B(E)} \leq M^* \quad \text{for all} \quad t \in J.$$

($\boldsymbol{H_6}$) There exists a constant $\ell > 0$ such that

$$|A(t)g(t,\phi) - A(s)g(s,\varphi)| \leq \ell(|t-s| + \|\phi - \varphi\|_{\mathcal{B}})$$

for all $t, s \in J$ and $\phi, \varphi \in \mathcal{B}$.

($\boldsymbol{H_7}$) There exists a bounded continuous function $\zeta : J \to \mathbb{R}_+$ such that:

$$|A(t)g(t,\phi)| \leq \xi(t)\|\phi\|_{\mathcal{B}} \quad \text{for all} \quad t \in J,\ \phi \in \mathcal{B}.$$

($\boldsymbol{H_8}$) There exists a constant $R > 0$ such that

$$\begin{array}{l} MM^*\xi^*\|\phi\|_{\mathcal{B}} + M^*\xi^*\gamma(R + \|\phi\|_{\mathcal{B}}(\tilde{M}+1) + M|\tilde{y}|) \\ +M\xi^*\gamma(R + \|\phi\|_{\mathcal{B}}(\tilde{M}+1) + M|\tilde{y}|) \\ +M\gamma\psi(\xi^*\gamma(R + \|\phi\|_{\mathcal{B}}(\tilde{M}+1) + M|\tilde{y}|))\|p\|_{L^1} \\ \leq R, \end{array}$$

where $\xi^* := ess \sup\limits_{t \in J} |\xi(t)|$.

Let $C(J, E)$ the Banach space of all continuous functions y mapping J into E equipped with the norm

$$\|y\| = \sup\{e^{-\tau M \ell \gamma t}|y(t)|, t \in J\}, \qquad \tau > 6.$$

To establish our main theorem, we need the following lemma

Lemma 6.2.1. *[104] Assume that the hypotheses (H_1), (H_3) and (H_5) hold and a set $D \subset E$ is bounded. Then*

$$\omega_0^T(F(D)) \leq 2M \int_0^T \mu(f(s, D_s)ds,$$

where

$$D_s = \{y_s; y \in \mathcal{C}\},$$

and

$$(FD)(t) = \left\{(Fy)(t); (Fy)(t) = \int_0^T \mathcal{U}(t,s)\mu(f(s,y_s)ds\right\}.$$

Theorem 6.2.1. *Assume that $(H_1) - (H_8)$ are fulfilled. if*

$$\ell M^*\gamma + \frac{7}{\tau} \leq 1,$$

then the problem (6.1)-(6.3) admits at least one mild solution.

Proof. It is clear that we will obtain the results if we show that the operator $T : \mathcal{C} \to \mathcal{C}$ defined by:

$$(Ny)(t) = \begin{cases} \phi(t), & \text{if } t \leq 0 \\ \\ -\dfrac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0) + U(t,0)(\phi(0) - g(0,\phi)) + g(t,y_t) \\ + \displaystyle\int_0^t U(t,s)A(s)g(s,y_t)ds + \int_0^t U(t,s)f(s,y_t)ds, & \text{if } t \in J. \end{cases} \tag{6.5}$$

has a fixed point.
For $\phi \in \mathcal{B}$, we can introduce the following function $x : (-\infty, T] \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0] \\ \\ -\dfrac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0) + U(t,0)\phi(0) & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in \mathcal{C}$, set

$$y(t) = x(t) + z(t).$$

It is obvious that y satisfies (6.4) if and only if z satisfies $z_0 = 0$ and for all $t \in J$

$$\begin{aligned} z(t) &= U(t,0)g(0,\phi) + g(t, x_t + z_t) + \int_0^t U(t,s)A(s)g(s, x_s + z_s)ds \\ &+ \int_0^t U(t,s)f(s, x_s + z_s)ds. \end{aligned}$$

Let

$$\mathcal{C}_0 = \{z \in \mathcal{C} : \ z_0 = 0\}.$$

The $\mathcal{C}_0$ is a Banach space with norm

$$\begin{aligned} \|z\|_{\mathcal{C}_0} &= \sup\{e^{-\tau M\ell\gamma t}|z(t)|, t \in J\} + \|z_0\|_{\mathcal{B}} \\ &= \sup\{e^{-\tau M\ell\gamma t}|z(t)|, t \in J\}. \end{aligned}$$

Now, define the operators $N = F + L : \mathcal{C}_0 \to \mathcal{C}_0$ by

$$(Fz)(t) = \int_0^t U(t,s)f(s, z_s + x_s)ds, \text{ for } t \in J,$$

and

$$(Gz)(t) = U(t,0)g(0,\phi) + g(t, x_t + z_t) + \int_0^t U(t,s)A(s)g(s, x_s + z_s)ds.$$

Obviously the problem (6.1)-(6.3) has a solution is equivalent to $F+L$ has a fixed point. To prove this end, we start with the following estimation. For any $z \in \mathcal{C}_0$ and $t \in J$, we have

$$\begin{aligned}
\|z_t + x_t\|_{\mathcal{B}} &\leq \|z_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}} \\
&\leq K(t)|z(t)| + K(t)\|\frac{\partial}{\partial s}\mathcal{U}(t,0)\|_{B(E)}\|\phi\|_{\mathcal{B}} \\
&+ K(t)\|\mathcal{U}(t,0)\|_{B(E)}|\tilde{y}| + M(t)\|\phi\|_{\mathcal{B}} \\
&\leq \gamma\|z\|_{\mathcal{C}_0} + \gamma\tilde{M}\|\phi\|_{\mathcal{B}} + \gamma M|\tilde{y}| + \gamma\|\phi\|_{\mathcal{B}} \\
&\leq \gamma\|z\|_{\mathcal{C}_0} + \gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1) + \gamma M|\tilde{y}|. \qquad (6.6)
\end{aligned}$$

Now, we will show that the operator L satisfied the conditions of Schauder's fixed point theorem.
We define

$$B_R = \{z \in \mathcal{C}_0 : \|z\|_{\mathcal{C}_0} \leq R\}.$$

The set B_R is nonempty convex and closed. Let $z \in B_R$

$$\begin{aligned}
|(Nz)(t)| &\leq \|U(t,0)\|_{B(E)}\|A^{-1}(t)\|_{B(E)}|A(t)g(0,\phi)| + \|A^{-1}(t)\|_{B(E)}|A(t)g(t,x_t+z_t)| \\
&+ \int_0^t \|U(t,s)\|_{B(E)}|A(s)g(s,x_s+z_s)|ds + \int_0^t \|U(t,s)\|_{B(E)}|f(s,x_s+z_s)|ds. \\
&\leq MM^*\xi(t)\|\phi\|_{\mathcal{B}} + M^*\xi(t)\|x_t+z_t\|_{\mathcal{B}} \\
&+ M\int_0^t \xi(s)\|x_s+z_s\|_{\mathcal{B}}ds + M\int_0^t p(s)\psi(\|x_s+z_s\|_{\mathcal{B}})ds. \\
&\leq MM^*\xi^*\|\phi\|_{\mathcal{B}} + M^*\xi^*\gamma(\|z\|_{\mathcal{C}_0} + \|\phi\|_{\mathcal{B}}(\tilde{M}+1) + M|\tilde{y}|) \\
&+ M\xi^*\gamma(\|z\|_{\mathcal{C}_0} + \|\phi\|_{\mathcal{B}}(\tilde{M}+1) \\
&+ M|\tilde{y}|) + M\gamma\psi(\xi^*\gamma(\|z\|_{\mathcal{C}_0} + \|\phi\|_{\mathcal{B}}(\tilde{M}+1) + M|\tilde{y}|))\|p\|_{L^1}, \\
&\leq MM^*\xi^*\|\phi\|_{\mathcal{B}} + M^*\xi^*\gamma(R + \|\phi\|_{\mathcal{B}}(\tilde{M}+1) + M|\tilde{y}|) \\
&+ M\xi^*\gamma(R + \|\phi\|_{\mathcal{B}}(\tilde{M}+1) + M|\tilde{y}|) \\
&+ M\gamma\psi(\xi^*\gamma(R + \|\phi\|_{\mathcal{B}}(\tilde{M}+1) + M|\tilde{y}|))\|p\|_{L^1}, \\
&\leq R.
\end{aligned}$$

Thus the operator N maps B_R into itself.

Let $(z^n)_{n\in\mathbb{N}}$ be a sequence in B_R such that $z^n \to z$ in B_R, then for any $t \in J$ we obtain

$$\begin{aligned}
|N(z^n)(t) - N(z)(t)| &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)} |f(t, x_s + z_s^n) - f(t, x_s + z_s)|\, ds \\
&+ |A^{-1}(t)A(t)g(t, z_t^n) - A^{-1}(t)A(t)g(t, z_t)| \\
&+ \int_0^t \|\mathcal{U}(t,s)\|_{B(E)} A(s)g(s, x_s + z_s^n) - A(s)g(s, x_s + z_s) ds \\
&\leq M \int_0^t |f(s, z_s^n + x_s) - f(s, z_s + x_s)|\, ds. \\
&+ M^* |A(t)g(t, z_t^n) - A(t)g(t, z_t)| \\
&+ M \int_0^t A(s)g(s, x_s + z_s^n) - A(s)g(s, x_s + z_s) ds
\end{aligned}$$

Hence, from Carathéodory of the function f, (H_1) and the Lebesgue dominated convergence theorem we obtain

$$\|Nz_k - Nz\|_{\mathcal{C}_0} \to 0 \quad \text{as } n \to +\infty.$$

So N is continuous. Consider the mesure of noncompacteness μ^* defined on the family of bounded subsets of the space $\mathcal{C}_0$ bay

$$\mu^*(D) = e^{-\tau\tilde{\sigma}(T)} \omega_0^T(D) + \sup_{t\in J} e^{-\tau\tilde{\sigma}(t)} \mu(D(t), .$$

where

$$\tilde{\sigma}(t) = M \int_0^t \sigma(s) ds, \qquad \tau > 6.$$

Step 1. F is a μ^*-contraction.
We know from Lemma 1.3.6 that for each $\varepsilon > 0$ there is a sequence function $\{W_k\}_{k=0}^{\infty} \subset L(D_n(s))$ such that

$$\mu(D_n) \leq 2\mu\{W_k\}_{k=0}^{\infty} + \varepsilon.$$

This implies that there is a sequence $\{Q^k\}_{k=0}^{\infty} \subset W_k$ such that

$$W_k = (LQ^k)(s) \text{ for } k = 0, 1, 2, ...$$

Using the properties of μ, Lemma 1.3.6, Lemma 1.3.7 and assumptions (H_4), (H_5), we get

$$\begin{aligned}
\mu(FD(t)) &= 2\mu(\{W_k\}_{k=0}^{\infty}) + \varepsilon \\
&= 2\mu(\{(LQ^k)(s)\}_{k=0}^{\infty}) + \varepsilon \\
&\leq 4\mu\left(\left\{\int_0^t \mathcal{U}(t,s)f(s,Q_s^k)ds\right\}_{k=0}^{\infty}\right) + \varepsilon \\
&\leq 4M\int_0^t \sigma(s) \sup_{\theta\in(-\infty,0]} \mu(\{Q^k(s+\theta))\}_{k=0}^{\infty} + \{x_s\})ds + \varepsilon. \\
&\leq 4M\int_0^t \sigma(s) \sup_{\theta\in(-\infty,0]} \mu(D(s+\theta) + \{x_s\}))ds + \varepsilon. \\
&\leq 4M\int_0^t \sigma(s) \sup_{\theta\in(-\infty,0]} \mu(D(s+\theta)))ds + \varepsilon. \\
&\leq 4M\int_0^t \sigma(s) \sup_{\tau\in(-\infty,s]} \mu(D(\tau))ds + \varepsilon. \\
&\leq 4M\int_0^t \sigma(s) \sup_{\tau\in[0,s]} \mu(D(\tau)))ds + \varepsilon \\
&\leq 4M\int_0^t \sigma(s)\mu(D(s))ds + \varepsilon \\
&\leq 4\int_0^t \sigma(s)e^{\tau\tilde{\sigma}(s)}e^{-\tau\tilde{\sigma}(s)}\mu(D(s))ds + \varepsilon \\
&\leq 4\int_0^t \sigma(s)e^{\tau\tilde{\sigma}(s)} \sup_{s\in[0,t]} e^{-\tau\tilde{\sigma}(s)}\mu(D(s))ds + \varepsilon \\
&\leq 4e^{\tau\tilde{\sigma}(s)} \sup_{t\in J} e^{-\tau'\tilde{\sigma}(t)}\mu(D(t))ds + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we get

$$\sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D(t) \leq \frac{4}{\tau}(\sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D(t)).$$

Then

$$\sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)) \leq \frac{4}{\tau}(e^{-\tau'\tilde{\sigma}(t)}\omega_0^T(D) + \sup_{t\in J]} e^{-\tau\tilde{\sigma}(t)}\mu(D(t)). \tag{6.7}$$

Now, applying Lemma 6.2.1 and using assumptions $(H_4), (H_5)$, we derive

$$\begin{aligned}\omega_0^T(FD) &\leq 2M\int_0^T \mu(f(s,(D_s+\{x_s\})ds \\ &\leq 2M\int_0^t \sigma(s) \sup_{\theta\in(-\infty,0]} \mu((D_s+\{x_s\})ds \\ &\leq \frac{4}{\tau}e^{\tau\tilde{\sigma}(T)}(\sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D(t)).\end{aligned}$$

Then

$$e^{-\tau\tilde{\sigma}(t)}\omega_0^T(FD) \leq \frac{4}{\tau}(e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D) + \sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D(t)). \tag{6.8}$$

From (6.7) and (6.8),

$$\begin{aligned}&e^{-\tau\tilde{\sigma}(t)}\omega_0^T(FD) + \sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(FD(t)) \\ &\leq \frac{6}{\tau}(e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D) + \sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D(t)).\end{aligned}$$

Hence we get

$$\mu^*(FD) \leq \frac{6}{\tau}\mu^*(D).$$

Step 2. G is a μ^*-contraction.

Take $z, \bar{z} \in \mathcal{C}_0$, then for each $t \in J$ and by (H_1), (H_4), (H_5) and (6.6)

$$\begin{aligned}|(Gz)(t)-(G\bar{z})(t)| &\leq |g(t,x_t+z_t)-g(t,x_t+\bar{z}_t)| \\ &+ \int_0^t \|U(t,s)\|_{B(E)}|A(s)g(s,x_s+z_s)-A(s)g(s,x_s+\bar{z}_s)|ds \\ &\leq \|A^{-1}(t)\|_{B(E)}|A(t)g(t,x_t+z_t)-A(t)g(t,x_t+\bar{z}_t)| \\ &+ M\int_0^t |A(s)g(s,x_s+z_s)-A(s)g(s,x_s+\bar{z}_s)|ds \\ &\leq \ell M^*\|z_t-\bar{z}_t\|_{\mathcal{B}} + M\ell\int_0^t \|z_s-\bar{z}_s\|_{\mathcal{B}}ds \\ &\leq \ell M^*K(t)|z-\bar{z}| + M\ell\int_0^t K(s)|z(s)-\bar{z}(s)|ds \\ &\leq \ell M^*\gamma e^{\tau M\ell\gamma t}e^{-\tau M\ell\gamma t}|z-\bar{z}| \\ &+ \int_0^t M\ell\gamma e^{\tau M\ell\gamma s}e^{-\tau M\ell\gamma s}|z(s)-\bar{z}(s)|ds \\ &\leq (\gamma\ell M^*+\frac{1}{\tau})e^{\tau M\ell\gamma t}\|z-\bar{z}\|_{\mathcal{C}_0}.\end{aligned}$$

Therefore

$$\|Gz - G\bar{z}\|_{\mathcal{C}_0} \leq (\gamma \ell M^* + \frac{1}{\tau})\|z - \bar{z}\|_{\mathcal{C}_0}.$$

Thus, G is a Lipschitz operator.
From Lemma 1.3.8 we obtain

$$\mu^*(GD) \leq \left(\gamma \ell M^* + \frac{1}{\tau}\right) \mu^*(D). \tag{6.9}$$

From (4.4) and (6.9), we have

$$\mu^*((F+G)D) \leq \left(\gamma \ell M^* + \frac{1}{\tau} + \frac{6}{\tau}\right) \mu^*(D).$$

Then

$$\mu^*((F+G)D) \leq \left(\gamma \ell M^* + \frac{7}{\tau}\right) \mu^*(D).$$

Thus by Darbo-Sadovskii's fixed point theorem the operator $N : B_R \to B_R$ has at least one fixed point which is a mild solution of problem (6.1)-(6.3)).

6.3 An example

Consider the second order Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\partial}{\partial t} y(t,\tau) - \displaystyle\int_{-\infty}^{t} v(t-s) y(s,\tau) ds\right) = \dfrac{\partial^2}{\partial \tau^2} y(t,\tau) + a(t) \dfrac{\partial}{\partial t} y(t,\tau) \\ + \displaystyle\int_{-\infty}^{t} b(t-s) \arctan(y(s,\tau)) ds \quad t \in J := [0,T],\ \tau \in [0,2\pi], \\ y(t,0) = y(t,2\pi) = 0 \qquad t \in J, \\ y(\theta,\tau) = \phi(\theta,\tau), \quad \dfrac{\partial}{\partial t} y(0,\tau) = \psi(\tau) \quad \theta \in (-\infty, 0], \quad \tau \in [0,2\pi] \end{cases} \tag{6.10}$$

where we assume that $a,\ b, v : J \to \mathbb{R}$ are continuous functions, $\phi(\theta,\cdot) \in \mathcal{B}$ and $\eta : J \times \mathcal{B} \to E$.

Let $X = L^2(\mathbb{R},\mathbb{C})$ the space of 2π-periodic 2-integrable functions from $\mathbb{R}$ into $\mathbb{C}$, and $H^2(\mathbb{R},\mathbb{C})$ denotes the Sobolev space 2π-periodic functions $x : \mathbb{R} \to \mathbb{C}$ such that $x'' \in L^2(\mathbb{R},\mathbb{C})$.

We consider the operator $A_1 y(\tau) = y''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R},\mathbb{C})$, infinitesimal generator of strongly continuous cosine function $C(t)$ on X. Moreover, we take $A_2(t)y(s) = a(t)y'(s)$ defined on $H^1(\mathbb{R},\mathbb{C})$, and consider the closed linear operator $A(t) = A_1 + A_2(t)$ which generates an evolution operator $\mathcal{U}$ defined by

$$\mathcal{U}(t,s) = \sum_{n \in \mathbb{Z}} z_n(t,s) \langle x, w_n \rangle w_n,$$

where z_n is a solution to the following scalar initial value problem

$$\begin{cases} z''(t) = -n^2 z(t) + ina(t)z(t) \\ z(s) = 0, \qquad z'(s) = z_1. \end{cases} \tag{6.11}$$

Define the functions $f : J \times \mathcal{B} \to X$ by

$$f(t, \varphi)(\tau) = \int_{-\infty}^{t} b(t-s) \arctan(\varphi(s, \tau)) ds, \quad \tau \in [0, 2\pi],$$

$$g(t, \varphi)(\tau) = \int_{-\infty}^{t} v(t-s)\varphi(s, \tau) ds$$

Consequently, (6.10) can be written in the abstract form (6.1)-(6.3) with A and f defined above. Now, the existence of a mild solution can be deduced from an application of Theorem 6.2.1.

CHAPTER 7

SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS [(6)]

7.1 Introduction

Differential inclusions are widely studied by many authors by their numerous applications in various fields of science. For this reason, in the literature, there are many papers study the existence, uniqueness, regularity, controllability and behavior of the solution for various classes of semilinear differential inclusion, for instance we refer to [8, 10, 25, 43, 54, 56, 60, 86, 111, 112, 117] This type of equations has received much attention in recent years [1].

In this chapter some sufficient conditions are provided for the existence of mild solutions for the following semilinear evolution differential inclusion.
Consider the following problem

$$y''(t) - A(t)y(t) \in F(t, y(t)),\ t \in J = [0; T], \tag{7.1}$$

$$y(0) = y_0,\ y'(0) = y_1, \tag{7.2}$$

where $\{A(t)\}_{0 \leq t < +\infty}$ is a family of linear closed operators from E into E that generate an evolution system of operators $\{\mathcal{U}(t, s)\}_{(t,s) \in J \times J}$ for $0 \leq s \leq t \leq T$, $f : J \times \mathcal{B} \to E$ be a Carathéodory function and $\mathcal{B}$ is an abstract phase space to be specified later, $\tilde{y} \in E$, $\phi \in \mathcal{B}$ and $(E, |\cdot|)$ a real separable Banach space.

[(5)] N. Rezoug and M. Benchohra, Measure of noncompactness and second-order functional differential inclusions, (submitted).

7.2 Existence of solutions

Definition 7.2.1. *A function $y \in C(J, E)$ is called a mild solution to the problem (7.1) -(7.2) if there exists function $f \in L^1(J, E)$ such that $f(t) \in F(t, y(t))$, a.e. J and y satisfies the integral equation*

$$y(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f(s)ds. \tag{7.3}$$

We will need to introduce the following hypotheses which are assumed thereafter :

$(\boldsymbol{H_1})$ There exists a constant $M \geq 1$ and such that

$$\|\mathcal{U}(t,s)\|_{B(E)} \leq M.$$

$(\boldsymbol{H_2})$ There exists a constant $\tilde{M} \geq 0$ such that

$$\left\|\frac{\partial}{\partial s}\mathcal{U}(t,s)\right\|_{B(E)} \leq \tilde{M}.$$

$(\boldsymbol{H_3})$ The multifunction $F : J \times E \longrightarrow \mathcal{P}(E)$ is Carathéodory with compact and convex values.

$(\boldsymbol{H_4})$ There exist an integrable function $p : J \to \mathbb{R}_+$ and a continuous nondecreasing function $\psi : [0, \infty) \to (0; \infty)$ such that:

$$\|F(t,u)\|_{P(E)} \leq p(t)\psi(|u|) \text{ for a.e } t \in J \text{ and each } u \in E.$$

$(\boldsymbol{H_5})$ There exists $R > 0$ such that

$$\tilde{M} + M\psi(R)\, \|p\|_{L^1} \leq R.$$

$(\boldsymbol{H_6})$ There exists a function $\gamma \in L^\infty(J, \mathbb{R}_+)$ such that for any nonempty set $D \subset E$ we have :

$$\mu(F(t,D)) \leq \gamma(t)\mu(D) \text{ for a.e } t \in J.$$

Theorem 7.2.1. *Assume that the hypotheses* $(\mathbf{H_4}) - (\mathbf{H_6})$ *hold. Then the problem (7.1) -(7.2) admits at least one mild solution.*

Proof. Consider the multivalued operator $N : C(J, E) \to \mathcal{P}(C(J, E))$ defined by

$$N(y) = \left\{ \begin{array}{ll} h \in C(J,E) : h(t) & = -\dfrac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 \\ & + \displaystyle\int_0^t \mathcal{U}(t,s)f(s)ds, f \in S_{F,y} \quad \text{if } t \in J. \end{array} \right\} \tag{7.4}$$

Let $s, t \in J$ with $t > s$. For any $y \in C(J,E)$ and $h \in N(y)$ we have

$$\begin{array}{rcl} |(hy)(t) - (hy)(s)| & \leq & \left| \dfrac{\partial}{\partial s}\mathcal{U}(t,0)y_0 - \dfrac{\partial}{\partial s}\mathcal{U}(s,0)y_0 \right| + |\mathcal{U}(t,0)y_0 - \mathcal{U}(s,0)y_0| \\ & + & \displaystyle\int_0^s \|\mathcal{U}(t,\tau) - \mathcal{U}(s,\tau)\|_{B(E)} |f(\tau)|\, d\tau \\ & + & \displaystyle\int_s^t \|\mathcal{U}(t,\tau)\|_{B(E)} |f(\tau)|\, d\tau \\ & \leq & \tilde{\omega}^T(\dfrac{\partial}{\partial s}\mathcal{U}(t,0)y_0, \varepsilon) + \omega^T(U(t,0)y_0, \varepsilon) \\ & + & \mathcal{A}^T(\mathcal{U}, \varepsilon) \displaystyle\int_0^s p(\tau)\psi(|y(\tau)|)d\tau \\ & + & M \sup \left\{ \displaystyle\int_s^t p(\tau)\psi(|y(\tau)|)d\tau; t \leq s \leq T, |t-s| \leq \varepsilon \right\}, \end{array}$$

where

$$\begin{array}{rcl} \tilde{\omega}^T(\dfrac{\partial}{\partial s}\mathcal{U}(t,0)y_0, \varepsilon) & = & \sup \left\{ \left| \dfrac{\partial}{\partial s}\mathcal{U}(t,0)y_0 - \dfrac{\partial}{\partial s}\mathcal{U}(s,0)y_0 \right|; t \leq s \leq T, |t-s| \leq \varepsilon \right\}, \\ \omega^T(\mathcal{U}(t,0)y_0, \varepsilon) & = & \sup \left\{ |\mathcal{U}(t,0)y_0 - \mathcal{U}(s,0)y_0|; t \leq s \leq T, |t-s| \leq \varepsilon \right\}, \\ \mathcal{A}^T(\mathcal{U}, \varepsilon) & = & \sup \left\{ \|\mathcal{U}(t,\tau) - \mathcal{U}(s,\tau)\|_{B(E)}; \tau \leq t \leq s \leq T, |t-s| \leq \varepsilon \right\}. \end{array}$$

Puting

$$\begin{array}{rcl} \Omega(T,\varepsilon) & = & \tilde{\omega}^T(\dfrac{\partial}{\partial s}\mathcal{U}(t,0)y_0, \varepsilon) + \omega^T(\mathcal{U}(t,0)y_0, \varepsilon) \\ & + & \mathcal{A}^T(\mathcal{U}, \varepsilon) \displaystyle\int_0^s p(\tau)\psi(|y(\tau)|)d\tau \\ & + & M \sup \left\{ \displaystyle\int_s^t p(\tau)\psi(|y(\tau)|)d\tau; t \leq s \leq T, |t-s| \leq \varepsilon \right\}. \end{array}$$

We have

$$|(hy)(t) - (hy)(s)| \leq \Omega(T,\varepsilon)$$

i.e

$$\omega^T(Ny, \varepsilon) \leq \Omega(T,\varepsilon). \tag{7.5}$$

The assumptions $(\mathbf{H_1})$-$(\mathbf{H_6})$ yield that

$$\lim_{\varepsilon\to 0}\Omega(T,\varepsilon)=0.$$

We define

$$D=\left\{y\in C(J,E):\|y\|\leq R \text{ and } \lim_{\varepsilon\to 0}\Omega(T,\varepsilon)=0\right\}.$$

The set D is nonempty convex and closed.
Now, for any $y\in C(J,E)$ and $h\in N(y)$ we have

$$\begin{aligned}
|(hy)(t)| &\leq \left\|\frac{\partial}{\partial s}\mathcal{U}(t,0)\right\|_{B(E)}|y_0| \\
&+ \|\mathcal{U}(t,s)\|_{B(E)}|y_1|+\|\mathcal{U}(t,s)\|_{B(E)}\int_0^t p(s)\psi(|y(s)|)ds \\
&\leq \tilde{M}+M\int_0^t p(s)\psi(R)ds \\
&\leq \tilde{M}+M\psi(R)\|p\|_{L^1} \\
&\leq R. \qquad (7.6)
\end{aligned}$$

The conditions (7.5) and (7.6) ensure that the operator N transforms the set D into itself.
Consider the mesure of noncompacteness $\mu^*(D)$ defined on the family of bounded subsets of the space $C(J,E)$ by

$$\mu^*(D)=\omega_0^T(D)+\sup_{t\in J}\mu(D(t))$$

In the sequel, we consider the sequence of sets $\{D_n\}_{n=0}^{+\infty}$ defined by induction as follows :

$$D_0=D,\ D_{n+1}=Convh(D_n) \text{ for } n=0,1,2,\cdots \text{and } D_\infty=\cap_{n=0}^{+\infty}D_n$$

this sequence is nondecreasing, i.e. $D_{n+1}\subset D_n$ for each n.

Claim 1. $\lim\limits_{n\to+\infty}\omega_0^T(D_n)=0.$

This is a consequence from the equicontinuity of the set D on compact intervals.

Claim 2. $\lim\limits_{n\to+\infty}\sup\limits_{t\in J}\mu(D_n(t))=0.$
Set

$$\alpha_n(t)=\mu(D_n(t)).$$

In view of Lemma 1.3.8 and (3.1) we have

$$|\alpha_n(t)-\alpha_n(s)| \leq \mu(B(0,1))\Omega(T,|t-s|)$$

which together with proves the continuity $\alpha_n(t)$ on J.
Using the properties of μ, Lemma 1.3.2 and ($\mathbf{H_6}$) we get

$$\begin{aligned}
\alpha_{n+1}(t) &= \mu(Convh(D_n)(t)) = \mu(h(D_n(t))) \\
&= \mu\left(\int_0^t \mathcal{U}(t,s)f(s)ds\right) \\
&\leq \|\mathcal{U}(t,s)\|_{B(E)} \int_0^t \mu(f(s))ds \\
&\leq M\int_0^t \mu(F(s,D(s)))ds \\
&\leq M\int_0^t \gamma(s)\mu(D(s)))ds \\
&\leq M\int_0^t \gamma(s)\alpha_n(s)ds.
\end{aligned}$$

Using Lemma 1.1.1 we derive

$$\begin{aligned}
\alpha_{n+1}(t) &\leq M^{n+1}\int_0^t \gamma(s_1)\int_0^{s_1}\gamma(s_2)\int_0^{s_2}\cdots\int_0^{s_n}\gamma(s_{n+1})\alpha_0(s_{n+1})ds_1ds_2\cdots ds_nds_{n+1} \\
&\leq M^{n+1}\tilde{\gamma}^{n+1}\int_0^t\int_0^{s_1}\int_0^{s_2}\cdots\int_0^{s_n}\alpha_0(s_{n+1})ds_{n+1}ds_n\cdots ds_2ds_1 \\
&\leq \frac{M^{n+1}\tilde{\gamma}^{n+1}(t)T^n}{n!}\int_0^t \alpha_0(s)ds,
\end{aligned}$$

where $\tilde{\gamma}(t) = ess\sup\{\gamma(s) : s \leq t\}$.
We get

$$\lim_{n\to+\infty}\alpha_n(t) = 0.$$

Then

$$\lim_{n\to+\infty}\sup_{t\in J}\mu(D_n(t)) = 0.$$

Form **Claim 1** and **Claim 2** we conclude

$$\lim_{n\to+\infty}\omega_0^T(D_n) + \lim_{n\to+\infty}\sup_{t\in J}\mu(D_n(t)) = 0.$$

Taking into account Lemma 1.3.8 we infer that $D_\infty = \cap_{n=0}^{+\infty}D_n$ is nonempty, convex and compact.

Now we prove that $N : D_\infty \to \mathcal{P}(D_\infty)$ satisfies the assumptions of Bohnenblust-Karlin's fixed theorem. The proof will be given in two steps.

Step 1: We shall show that the operator N is closed and convex. This will be given in two claims.

Claim 1 : $N(y)$ is closed for each $y \in D_\infty$.

Let $(h_n)_{n\geq 0} \in N(y)$ such that $h_n \to \tilde{h}$ in D_∞. Then for $h_n \in D_\infty$ there exists $f_n \in S_{F,y}$ such that:

$$h_n(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f_n(s)ds.$$

Using the fact that F has compact values and from hypotheses $(\mathbf{H_3})$, $(\mathbf{H_4})$ we may pass a subsequence if necessary to get that f_n converges to $f \in L^1(J,E)$ and hence $f \in S_{F,y}$. Then for each $t \in J$,

$$h_n(t) \to \tilde{h}(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f(s)ds.$$

So, $\tilde{h} \in N(y)$.

Claim 2 :$N(y)$ is convex for each $y \in D_\infty$.

Let $h_1, h_2 \in N(y)$, the there exists $f_1, f_2 \in S_{F,y}$ such that, for each $t \in J$ we have :

$$h_i(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f_i(s)ds, i = 1,2.$$

Let $0 \leq \delta \leq 1$. Then, we have for each $t \in J$:

$$\begin{aligned} \delta h_1 + (1-\delta)h_2)(t) &= -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 \\ &+ \int_0^t \mathcal{U}(t,s)[\delta f_1(s) + (1-\delta)f_2(s)]ds. \end{aligned}$$

Since $F(t,y)$ is convex, one has

$$\delta h_1 + (1-\delta)h_2 \in N(y).$$

Step 2: N has closed graph.

Let $\{y_n\}$ be a sequence such that $y_n \to y_*, h_n \in N(y_n)$ and $h_n \to h_*$. We shall show that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $f_n \in S_{F,y_n}$ such that

$$h_n(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f_n(s)ds,\ t \in J.$$

We must prove that there exists f_*

$$h_*(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f_*(s)ds,\ t \in J.$$

Consider the linear and continuous operator $K : L^1(J, E) \to C(J, E)$ defined by

$$K(v)(t) = \int_0^t \mathcal{U}(t, s)v(s)ds.$$

We have

$$|K(f_n)(t) - K(f_*)(t)|$$

$$= |(h_n(t) + \frac{\partial}{\partial s}\mathcal{U}(t, 0)y_0 - \mathcal{U}(t, 0)y_1 - (h_*(t) + \frac{\partial}{\partial s}\mathcal{U}(t, 0)y_0 - \mathcal{U}(t, 0)y_1|$$

$$= |h_n(t) - h_*(t)|$$

$$\leq \|h_n - h_*\|_\infty \to 0, \ as\ n \to \infty.$$

From Lemma 1.5.1 it follows that $K \circ S_F$ is a closed graph operator and from the definition of K has

$$h_n(t) + \frac{\partial}{\partial s}\mathcal{U}(t, 0)y_0 - \mathcal{U}(t, 0)y_1 \in K \circ S_{F,y_n}.$$

As $y_n \to y_*$ and $h_n \to h_*$, there exist $f_* \in S_{F,y_*}$ such that:

$$h_*(t) + \frac{\partial}{\partial s}\mathcal{U}(t, 0)y_0 - \mathcal{U}(t, 0)y_1 = \int_0^t \mathcal{U}(t, s)f_*(s)ds.$$

Hence the multivalued operator N has closed graph, which implies that it is upper semi-continuous.
Thus, by Bohnenblust-Karlin's fixed point theorem the operator $N : D_\infty \to \mathcal{P}(D_\infty)$ has at least one fixed point which is a mild solution of problem (7.1) -(7.2).

7.3 An example

To illustrate the above results we consider the following partial functional differential inclusion:

$$\begin{cases} \frac{\partial}{\partial t}z(t,\tau) & \in a(t,\tau)\frac{\partial^2}{\partial \tau^2}z(t,\tau) + F(t, z(t,\tau)) \quad t \in J,\ \tau \in [0,\pi], \\ z(t,0) & = z(t,\pi) = 0 \qquad\qquad\qquad\qquad t \in J, \\ \frac{\partial}{\partial t}z(0,\tau) & = 0, \qquad\qquad\qquad\qquad\qquad\quad \tau \in [0,\pi], \end{cases} \tag{7.7}$$

where we assume that $a : J \to \mathbb{R}$ is a Hölder continuous function and $\sigma : J \to \mathbb{R}$ essentially bounded measurable function. Let $E = L^2([0, \pi], \mathbb{R})$ the space of 2-integrable functions from $[0, \pi]$ into $\mathbb{R}$, and $H^2([0, \pi], \mathbb{R})$ denotes the Sobolev space of functions $x : [0,\pi] \to \mathbb{R}$ such that $x'' \in L^2([0,\pi],\mathbb{R})$. We consider the operator $A_1 y(\tau) = y''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R},\mathbb{C})$, infinitesimal generator of strongly continuous cosine function $C(t)$ on E. Moreover, we take $A_2(t)y(s) = a(t)y'(s)$ defined on $H^1([0,\pi],\mathbb{R})$, and consider the closed linear operator $A(t) = A_1 + A_2(t)$ which generates an evolution operator $\mathcal{U}$ defined by

$$\mathcal{U}(t,s) = \sum_{n\in\mathbb{Z}} z_n(t,s)\langle x, w_n\rangle w_n,$$

where z_n is a solution to the following scalar initial value problem

$$\begin{cases} z''(t) = -n^2 z(t) + ina(t)z(t) \\ z(s) = 0, \qquad z'(s) = z_1. \end{cases} \tag{7.8}$$

Set

$$w(t)(\tau) = z(t,\tau),\ t \in J,\ \tau \in [0,\pi],$$

$$F(t,z)(\tau) = F(t,z(t,\tau)) = \left\{ v \in E,\ 0 \le v(\tau) \le \frac{e^{-t}z(t,\tau)}{1+z^2(t,\tau)},\ t \in J,\ \tau \in [0,\pi] \right\}$$

and

$$\frac{\partial}{\partial t} z(0)(\tau) = \frac{d}{dt} w(0)(\tau),\ \tau \in [0,\pi].$$

We now assume that Consequently, (7.7) can be written in the abstract form (7.1)-(7.2) with $A(t)$ and f defined above. The existence of a mild solution can be deduced from an application of Theorem 7.2.1.

CONCLUSION AND PERSPECTIVE

In this thesis, we have considered the problem of the existence of solutions of mild solutions for some classes of non–autonomous second order semi-linear functional differential equations and inclusions on bounded or unbounded intervals local and nonlocal conditions. Our approach is based on the technique of measure of noncompactness and the argument of fixed points. Some appropriate fixed point theorems have been used: In particular, we have used Schauder's theorem, Tykhonoff theorem, Darbo-Sadovskii theorem and Bohnenblust-Karlin theorem.
For the perspective, We will study the existence of integrable solutions. Another goal in the future will apply our results in fractional functional differential equations and inclusions. We can also think to apply our results in control theory.

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