Ministère De L'Enseignement Supérieur Et De La Recherche Scientifique

Université Djillali Liabès De Sidi Bel Abbès

THÈSE

Préparée au Département de Mathématiques

de la Faculté Des Sciences Exactes

Présenté par

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Spécialité : Mathématique

Option : Étude de quelques classes d'équations et d'inclusions différentielles

Intitulé

Une Contribution Dans L'étude Des Equations Différentielles Fractionnaires Implicites

Pour obtenir

Le grade de **DOCTEUR**

Thèse présentée et soutenue publiquement le 17/12/2015

devant le Jury composé de

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Dédicace

Je dédie ce travail à :
L'âme de mon cher père décédé ces derniers mois.
Ma chère mère.
Ma femme et mes enfants Marwa, Noureddine et Ismail
Mes frères et soeurs.

Remerciement

• **J**e souhaiterais avant tout remercier toutes les personnes qui, de près ou de loin, ont permis à l'aboutissement de ce travail. Bien sur, cette liste ne saurait être exhaustive, et je tiens par avance à m'excuser auprès de ceux que j'aurais pu oublier.

• **J**e pense donc en particulier à mon directeur de thèse *Prof Mouffak Benchohr*a, qui m'a proposé ce projet et m'a fait bénéficié de ses compétences scientifiques et ses conseils précieux, qui a bien voulu porter une attention toute particulière à mon travail et les encouragements qu'il m'a constamment apporté, qui a partagé avec moi son expérience et ses idées, et m'avoir initié à la recherche mathématique. Qu'il trouve ici l'expression de mon profond respect.

• **T**rès vivement, Monsieur *Prof Abdelghani Ouahab*, pour m'avoir fait l'honneur de présider le jury. Et qui n'a pas épargné son temps pour m'aider avec ses idées et orientations.

• **J**e suis très heureux que, Messieurs le *Pr : Belmekki Mohammed , Dr : Saïd Abbass, Dr : Boualam Attou Slimani, Dr : Farida Barhoun*, aient participés et acceptés de faire partie de mon Jury, et je les remercie également pour cela.

• **D**e manière plus personnelle, mes pensées les plus profondes et intimes vont vers ma famille : Ma femme toujours à mon écoute et parfois pour son plus grand désarroi, ennui et malheur, car c'est bien la seule personne à qui malheureusement j'ose confier mes inquiétudes et sur qui je reporte trop souvent mon stress, omniprésents durant ces années ; Ma mère (la voix de la sagesse et de l'amour) a qui je dois beaucoup, malgré, je sais que je pourrais jamais te rendre ce que tu m'as donnée. Mais, comme, tu le dis " notre réussite c'est ton bonheur".

• **E**nfin, je saisie l'occasion pour remercier mes chers amis et collègues : Benguassoum Aissa, Kada Maazouz, Abdessalem Baliki, Lazreg Jamel Eddine, Larbi Cheikh et Bekkouche Mohamed Ali.

Publications

1. M. Benchohra and **M. S. Souid**, Integrable Solutions for Implicit Fractional Order Differential Equations. *Transylvanian Journal of Mathematics and Mechanics* **6** (2014), No. 2, 101-107.

2. M. Benchohra and **M. S. Souid**, L^1-Solutions for Implicit Fractional Order Differential Equations with Nonlocal Condition, (to appear).

3. M. Benchohra and **M. S. Souid**, Integrable Solutions For Implicit Fractional Order Functional Differential Equations with Infinite Delay, *Archivum Mathematicum (BRNO) Tomus* **51** (2015), 67-76 .

4. M. Benchohra and **M. S. Souid**, A New Result of Integrable Solutions for Implicit Fractional Order Differential Equations, (submitted).

5. M. Benchohra and **M. S. Souid**, Integrable Solutions for Implicit Fractional Order Differential Inclusions, (submitted).

6. M. Benchohra and **M. S. Souid**, L^1-Solutions of Boundary Value Problems for Implicit Fractional Order Differential Equations, (to appear).

Abstract

In this thesis, we discuss the existence and uniqueness of integral solutions for a class of initial value problem and of boundary value problem for nonlinear ***implicit*** fractional differential equations and inclusions (NIFDE for short) with ***Caputo*** fractional derivative. Our results will be obtained by means of fixed points theorems and by the technique of measures of noncompactness.

Key words and phrases : Initial value problem, boundary value problems, Caputo's fractional derivative, Implicit fractional-order differential equation, infinite delay, Banach space, fixed point, integrable solution, inclusion, measure of noncompactness, local and nonlocal conditions.
AMS (MOS) Subject Classifcations : 26A33, 34A08, 34A37, 34A60.

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INTRODUCTION

Fractional calculus is a generalization of differentiation and integration to arbitrary order (non-integer) fundamental operator D_{a+}^{α} where $a, \alpha \in \mathbb{R}$. Several approaches to fractional derivatives exist : Riemann-Liouville (RL), Hadamard, Grunwald-Letnikov (GL), Weyl and Caputo etc. The Caputo fractional derivative is well suitable to the physical interpretation of initial conditions and boundary conditions. We refer readers, for example, to the books [1, 18, 29, 54, 80, 89, 95, 101, 102, 104, 105] and the references therein. ***In this thesis, we always use the Caputo's derivative***.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [29, 70, 91, 101, 102, 110]). There has been a significant development in ordinary and partial fractional differential equations in recent years ; see the monographs of Abbas *et al.* [1], Kilbas *et al.* [80], Lakshmikantham *et al.* [89], and the papers by Agarwal *et al* [8, 9], Belarbi *et al.* [28], Benchohra *et al.* [32], and the references therein.

Fractional differential equations with nonlocal conditions have been discussed in ([7, 11, 59, 67, 90, 99, 100]) and references therein. Nonlocal conditions were initiated by Byszewski [45] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems (C.P. for short). As remarked by Byszewski ([43, 44]), the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. It is worthwhile mentioning that several important problems of the theory of ordinary and delay differential equations lead to investigations of functional differential equations of various types (see the books by Hale and Verduyn Lunel

[69], Wu [117], and the references therein).

Differential delay equations, or functional differential equations, have been used in modelling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case is called distributed delay ; see for instance the books ([69, 84, 117]), and the papers ([47, 68]).

In the literature devoted to equations with infinite delay, the state space is usually the space of all continuous function on $[-r, 0]$, $r > 0$ and $\alpha = 1$ endowed with the uniform norm topology, see the book of Hale and Lunel [69]. When the delay is infinite, the notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato in [68], see also Corduneanu and Lakshmikantham [47], Kappel and Schappacher [76] and Schumacher [106]. For detailed discussion and applications on this topic, we refer the reader to the book by Hale and Verduyn Lunel [69], Hino et al. [71] and Wu [117].

Differential inclusions are generalization of differential equations, therefore all problems considered for differential equations, that is, existence of solutions,continuation of solutions, dependence on initial conditions and parameters, are present in the theory of differential inclusions. Since a differential inclusion usually has many solutions starting at a given point, new issues appear, such as investigation of topological properties of the set of solutions, and selection of solutions with given properties. As a consequence, differential inclusions have been the subject of an intensive study of many researchers in the recent decades ; see, for example, the monographs [19, 20, 41, 64, 73, 78, 108, 111] and the papers of Bressan and Colombo [39, 40].

Implicit differential equations involving the regularized fractional derivative were analyzed by many authors, in the last year ; see for instance [3, 4, 5, 6, 114, 114, 115] and the references therein.

There are two measures which are the most important ones. The Kuratowski measure of noncompactness $\alpha(B)$ of a bounded set B in a metric space is defined as infimum of numbers $r > 0$ such that B can be covered with a finite number of sets of diameter smaller than r. The Hausdorf measure of noncompactness $\chi(B)$ defined as infimum of numbers $r > 0$ such that B can be covered with a finite number of balls of radii smaller than r. Several authors have studied the measures of noncompactness in Banach spaces. See, for example, the books such as [15, 23, 112] and the articles [17, 24, 27, 31, 35, 37, 72, 94],and the references cited therein. Recently, considerable attention has been given to the existence of solutions of initial value problem and boundary conditions for implicit fractional differential equations and integral equations with Caputo fractional derivative. See for example [10, 12, 13, 14, 21, 38, 35, 74, 85, 86, 87, 88, 109, 119], and the references therein.

The problem of the existence of solutions of Cauchy-type problems for ordinary differential equations of fractional order and without delay in spaces of integrable functions was studied in some works [79, 107]. The similar problem in spaces of continuous functions was studied in [116].

To our knowledge, the literature on integral solutions for fractional differential equations is very limited. El-Sayed and Hashem [56] studies the existence of integral and continuous solutions for quadratic integral equations. El-Sayed and Abd El Salam considered L^p-solutions for a weighted Cauchy problem for differential equations involving the Riemann-Liouville fractional derivative.

Motivated by the above works, this thesis is devoted to the existence of integral solutions for initial value problem (IVP for short), and boundary value problem (BVP for short)for fractional order implicit differential equation.

In the following we give an outline of our thesis organization, Consisting of **7 chapters**.

The **first chapter** gives some notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

In Chapter 2, we study the existence of solutions for initial value problem (IVP for short), for fractional order implicit differential equation

$$^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)), \ t \in J = [0, T], \ 0 < \alpha \leq 1,$$

$$y(0) = y_0,$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_0 \in \mathbb{R}$, and $^cD^\alpha$ is the Caputo fractional derivative.

In Chapter 3, we deal with the existence of solutions of the nonlocal problem, for fractional order implicit differential equation

$$^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)), \ a.e, \ t \in J =: (0, T], \ 0 < \alpha \leq 1,$$

$$\sum_{k=1}^{m} a_k y(t_k) = y_0,$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_0 \in \mathbb{R}$, $a_k \in \mathbb{R}$, $^cD^\alpha$ is the Caputo fractional derivative, and $0 < t_1 < t_2 < ..., t_m < T, \ k = 1, 2, ..., m.$

In Chapter 4, we shall be concerned with the existence of solutions for initial value problem (IVP for short), for implicit fractional order functional differential equations with infinite delay

$$^cD^\alpha y(t) = f(t, y_t, {}^c D^\alpha y_t), \ t \in J := [0, b], \ 0 < \alpha \leq 1,$$

$$y(t) = \phi(t), \ t \in (-\infty, 0],$$

where $^cD^\alpha$ is the Caputo fractional derivative, and $f : J \times \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ is a given function satisfying some assumptions that will be specified later, and $\mathcal{B}$ is called a phase space that will be defined later (see Section **1.5**). For any function y defined on $(-\infty, b]$ and any $t \in J$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta), \ \theta \in (-\infty, 0]$. Here $y_t(.)$ represents the history of the state from time $-\infty$ up to the present time t.

In Chapter 5, we study the existence of integrable solutions for the Initial Value Problem (IVP for short), for implicit fractional order differential equation

$$^cD^\alpha y(t) = f(t, y(t), {}^c D^\alpha y(t)), \ t \in J = [0, T], \ 0 < \alpha \leq 1,$$

$$y(0) = y_0,$$

where $^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function satisfying some assumptions that will be specified later. We will use the technique of measures of noncompactness which is often used in several branches of nonlinear analysis.

In Chapter 6, deals with the existence existence of solutions for initial value problem (IVP for short), for fractional order implicit differential inclusions

$$^cD^\alpha y(t) \in F(t, y(t), {}^c D^\alpha y(t)), \ a.e. \ t \in J := [0, T], \ 0 < \alpha \leq 1,$$

$$y(0) = y_0,$$

where $^cD^\alpha$ is the Caputo fractional derivative, $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact values ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), $y_0 \in \mathbb{R}$.

In Chapter 7, In section **7.1** we study the the existence of solutions for boundary value problem (BVP for short), for fractional order implicit differential equation

$$^cD^\alpha y(t) = f(t, y, {}^c D^\alpha y(t)), \ t \in J := [0, T], \ 1 < \alpha \leq 2,$$

$$y(0) = y_0, \ y(T) = y_T$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_0, \ y_T \in \mathbb{R}$, and $^cD^\alpha$ is the Caputo fractional derivative.

In section **7.3** is devoted to some existence and uniqueness results for the following class of nonlocal problems

$$^cD^\alpha y(t) = f(t, y, {}^c D^\alpha y(t)), \ t \in J := [0, T], \ 1 < \alpha \leq 2,$$

$$y(0) = g(y),\ y(T) = y_T$$

where $g : L^1(J, \mathbb{R}) \to \mathbb{R}$ a continuous function. The nonlocal condition can be applied in physics with better effect than the classical initial condition $y(0) = y_0$. For example, $g(y)$ may be given by

$$g(y) = \sum_{i=1}^{p} c_i y(t_i).$$

where $c_i, i = 1, 2, ..., p$ are given constants and $0 < ... < t_p < T$.

Chapitre 1

Preliminaires

We introduce in this Chapter notations, definitions, fixed point theorems and preliminary facts from multi-valued analysis which are used throughout this thesis.

1.1 Notations and definitions

Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from $J := [0,T]$ into $\mathbb{R}$ with the usual norm

$$\|y\| = sup\{|y(t)| : 0 < t < T\}.$$

$L^1(J,\mathbb{R})$ denote the Banach space of functions : $J \to \mathbb{R}$ that are Lebesgue integrable with the norm

$$\|y\|_{L_1} = \int_0^T |y(t)|dt.$$

Definition 1.1 *[49]. A map $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to be L^1-Carathéodory if*

(i) *the map $t \longmapsto f(t,x,y)$ is measurable for each $(x,y) \in \mathbb{R} \times \mathbb{R}$,*

(ii) *the map $(x,y) \longmapsto f(t,x,y)$ is continuous for almost all $t \in J$,*

(iii) *For each $q > 0$, there exists $\varphi_q \in L^1(J,\mathbb{R})$ such that*

$$|f(t,x,y)| \leq \varphi_q(t)$$

for all $|x| \leq q,\ |y| \leq q$ and for a.e. $t \in J$.

The map f is said of Carathéodory if it satisfies (i) and (ii).

Definition 1.2 *An operator $T : E \longrightarrow E$ is called compact if the image of each bounded set $B \in E$ is relatively compact i.e ($\overline{T(B)}$ is compact). T is called completely continuous operator if it is continuous and compact.*

Theorem 1.1 *(Kolmogorov compactness criterion [50]). Let $\Omega \subseteq L^p(J,\mathbb{R}),\ 1 \leq p \leq \infty$. If*

(i) *Ω is bounded in $L^p(J,\mathbb{R})$, and*

(ii) *$u_h \longrightarrow u$ as $h \longrightarrow 0$ uniformly with respect to $u \in \Omega$,*

then Ω is relatively compact in $L^p(J,\mathbb{R})$,

where

$$u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)ds.$$

1.2 Fractional Calculus.

Definition 1.3 *([80, 104]). The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)ds,$$

where $\Gamma(.)$ is the gamma function. When $a=0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t>0$, and $\varphi_\alpha(t)=0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 1.4 *. ([80, 104]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of function $h \in L^1([a,b],\mathbb{R}_+)$, is given by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)}\Big(\frac{d}{dt}\Big)^n \int_a^t (t-s)^{n-\alpha-1}h(s)ds,$$

Here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α. If $\alpha \in (0,1]$, then

$$(D_{a+}^\alpha h)(t) = \frac{d}{dt}I_{a+}^{1-\alpha}h(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{ds}\int_a^t (t-s)^{-\alpha}h(s)ds.$$

Definition 1.5 *([80]). The Caputo fractional derivative of order $\alpha \in (0,1]$, of function $h \in L^1([a,b],\mathbb{R}_+)$ is given by*

$$({}^cD_{a+}^\alpha h)(t) = I_{a+}^{1-\alpha}\frac{d}{dt}h(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\frac{d}{ds}h(s)ds,$$

where $n = [\alpha]+1$.

The following properties are some of the main ones of the fractional derivatives and integrals.

Lemma 1.1 *([80]). Let $\alpha > 0$, then the differential equation*

$${}^cD^\alpha h(t) = 0$$

has solution

$$h(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},\ c_i \in \mathbb{R},\ i = 0,1,2,...,n-1,\ n = [\alpha]+1.$$

Lemma 1.2 *([80]). Let $\alpha > 0$, then*

$$I^{\alpha c} D^{\alpha} h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for arbitrary $c_i \in \mathbb{R}, \ i = 0, 1, 2, ..., n-1, \ n = [\alpha] + 1$.

Proposition 1.1 *[80]. Let $\alpha, \ \beta > 0$. Then we have*

(1) $I^{\alpha}: \ L^1(J, \mathbb{R}) \to L^1(J, \mathbb{R})$, *and if $f \in L^1(J, \mathbb{R})$, then*

$$I^{\alpha} I^{\beta} f(t) = I^{\beta} I^{\alpha} f(t) = I^{\alpha+\beta} f(t).$$

(2) *If $f \in L^p(J, \mathbb{R}), \ \ 1 \leq p \leq +\infty$, then $\|I^{\alpha} f\|_{L^p} \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|f\|_{L^p}$.*

(3) *The fractional integration operator I^{α} is linear.*

(4) *The fractional order integral operator I^{α} maps $L^1(J, \mathbb{R})$ into itself.*

(5) *When $\alpha = n \in \mathbb{N}, I_0^{\alpha}$ is the n-fold integration.*

(6) *The Caputo and Riemann-Liouville fractional derivative are linear*

(7) *The Caputo derivative of a constant is equal to zero.*

1.3 Multi-valud analysis

Let $(X, \|.\|)$ be a Banach space and K be a subset of X. We denote by :

$$\mathcal{P}(X) = \{K \subset X : K \neq \emptyset\},$$

$$\mathcal{P}_{cl}(X) = \{K \subset \mathcal{P}(X) : K \textit{ is closed}\},$$

$$\mathcal{P}_{b}(X) = \{K \subset \mathcal{P}(X) : K \textit{ is bounded}\},$$

$$\mathcal{P}_{cv}(X) = \{K \subset \mathcal{P}(X) : K \textit{ is convex}\},$$

$$\mathcal{P}_{cp}(X) = \{K \subset \mathcal{P}(X) : K \textit{ is compact}\},$$

$$\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X).$$

Let $A, B \in \mathcal{P}(X)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ the Hausdorff distance between A and B given by :

$$H_d(A, B) = max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where $d(A, b) = \inf_{a \in A} d(a, b)$ and $d(a, B) = \inf_{b \in B} d(a, b)$. As usual, $d(x, \emptyset) = +\infty$.
Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized (complete) metric space (see [83]).

Definition 1.6 *A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called :*
(a) *γ-Lipschitz if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \text{ for all } x,\ y \in X;$$

(b) *a contraction if it is γ-Lipschitz with $\gamma < 1$.*

Definition 1.7 *A multivalued map $F : J \to \mathcal{P}_{cl}(X)$ is said to be measurable if, for each $y \in X$, the function*

$$t \longmapsto d(y, F(t)) = inf\{d(x, z) : z \in F(t)\}$$

is measurable.

Definition 1.8 *Let X and Y be metric spaces. A set-valued map F from X to Y is characterized by its graph $Gr(F)$, the subset of the product space $X \times Y$ defined by*

$$Gr(F) := \{f(x, y) \in X \times Y : y \in F(x)\}$$

Definition 1.9

1. *A multi-valued map $F : X \to \mathcal{P}(X)$ is convex (closed) if $F(x)$ is convex (closed) for all $x \in X$.*
2. *F is bounded on bounded sets if $F(\mathcal{B}) = \cup_{x\in\mathcal{B}} F(x)$ is bounded in X for all $\mathcal{B} \in \mathcal{P}_b(X)$, i.e. $sup_{x\in\mathcal{B}}\{sup\{|y| : y \in F(x)\}\} < \infty$.*
3. *A multi-valued map F is called upper semi-continuous (u.s.c. for short) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X and for each open set U of X containing $F(x_0)$, there exists an open neighborhood V of x_0 such that $F(V) \subset U$.*
4. *F is said to be completely continuous if $F(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in \mathcal{P}_b(X)$. If the multi-valued map F is completely continuous with nonempty compact values, then F is upper semi-continuous if and only if F has closed graph (i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in F(x_*)$).*
5. *F has a fixed point if there exists $x \in X$ such that $x \in Fx$. The set of fixed points of the multi-valued operator G will be denoted by $FixF$.*
6. *A measurable multi-valued function $F : J \to \mathcal{P}_{b,cl}(X)$ is said to be integrably bounded if there exists a function $g \in L^1(\mathbb{R}_+)$ such that $|f| \leq g(t)$ for almost $t \in J$ for all $f \in F(t)$.*

Proposition 1.2 *[75] Let $F : X \to Y$ be an u.s.c map with closed values. Then $Gr(F)$ is closed.*

Definition 1.10 *A multi-valued map $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if*

(i) *$t \to F(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$;*

(ii) $x \to F(t,x,y)$ *is upper semicontinuous for almost all* $t \in J$ *;*

(iii) *For each* $q > 0$*, there exists* $\varphi_q \in L^1(J, \mathbb{R}_+)$ *such that*

$$\|F(t,x,y)\|_{\mathcal{P}} = sup\{|f| : f \in F(t,x,y)\} \leq \varphi_q(t)$$

for all $|x| \leq q,\ |y| \leq q$ *and for a.e.* $t \in J$*.*

The multi-valued map F is said of Carathéodory if it satisfies (i) and (ii).

Definition 1.11 *. Let* X, Y *be nonempty sets and* $F : X \to \mathcal{P}(Y)$*. The single-valued operator* $f : X \to Y$ *is called a selection of* F *if and only if* $f(x) \in F(x)$*, for each* $x \in X$*. The set of all selection functions for* F *is denoted by* S_F*.*

Lemma 1.3 *([64]) Let* X *be a separable metric space. Then every measurable multi-valued map* $F : X \to \mathcal{P}_{cl}(X)$ *has a measurable selection.*

For more details on multivalued maps and the proof of the known results cited in this section we refer interested reader to the books of Aubin and Cellina [19], Deimling [51], Gorniewicz [64], Hu and Papageorgiou [75], Smirnov [108], Tolstonogov [113], Djebali and al [52] and Graef and al [62] .

1.4 Measure of noncompactness

We define in this Section the Kuratowski (1896-1980) and Hausdorf (1868-1942) measures of noncompactness ($\mathcal{MNC}$ for short) and give their basic properties, and introduce the notion of measure of noncompactness in $L^1(J)$.
Now let us recall some fundamental facts of the notion of measure of noncompactness in Banach space.

Let (X, d) be a complete metric space and $\mathcal{P}_{bd}(X)$ be the family of all bounded subsets of X. Analogously denote by $\mathcal{P}_{rcp}(X)$ the family of all relatively compact and nonempty subsets of X. Let $B \subset X$ recall that

$$diam(B) := \begin{cases} \sup\limits_{(x,y)\in B^2} d(x,y) & \text{si} \quad B \neq \phi \\ \quad 0 & \text{si} \quad B = \phi \end{cases}$$

is recalled the diameter of B.

Definition 1.12 *([23]) Let* X *be a Banach space and* $\mathcal{P}_{bd}(X)$ *the family of bounded subsets of* X*. For every* $B \in \mathcal{P}_{bd}(X)$ *the Kuratowski measure of noncompactness is the map* $\alpha : \mathcal{P}_{bd}(X) \to [0, +\infty]$ *defined by*

$$\alpha(B) = inf\{r > 0 : \ B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) < r\}.$$

Remark 1.1 *It is clear that* $0 \leq \alpha(B) \leq diam(B) < +\infty$ *for each nonempty bounded subset* B *of* X *and that* $diam(B) = 0$ *if and only if* B *is an empty set or consists of exactly one point.*

The Kuratowski measure of noncompactness satisfies the following properties :

Proposition 1.3 *([15, 23, 24, 81]). Let* X *be a Banach space. Then for all bounded subsets* $A,\ B$ *of* X *the following assertions hold*

1. $\alpha(B) = 0 \Longleftrightarrow \overline{B}$ *is compact (*B *is relatively compact).*
2. $\alpha(\phi) = 0$.
3. $\alpha(B) = \alpha(\overline{B}) = \alpha(convB)$, *where* $convB$ *is the convex hull of* B.
4. *monotonicity :* $(A \subset B) \Longrightarrow \alpha(A) \leq \alpha(B)$.
5. *algebraic semi-additivity :* $\alpha(A+B) \leq \alpha(A) + \alpha(B)$, *where* $A+B = \{x+y : x \in A; y \in B\}$.
6. *semi-homogencity :* $\alpha(\lambda B) = |\lambda|\alpha(B), \lambda \in \mathbb{R}$, *where* $\lambda(B) = \{\lambda x : x \in B\}$.
7. *semi-additivity :* $\alpha(A \cup B) = max\{\alpha(A), \alpha(B)\}$.
8. *semi-additivity :* $\alpha(A \cap B) = min\{\alpha(A), \alpha(B)\}$.
9. *invariance under translations :*$\alpha(B + x_0) = \alpha(B)$ *for any* $x_0 \in X$.

Remark 1.2 *The* α*-measure of noncompactness was introduced by Kuratowski in order to generalize the Cantor intersection theorem.*

Theorem 1.2 *([23]) Let (*X*; d) be a complete metric space and* $\{B_n\}$ *be a decreasing sequence of nonempty, closed and bounded subsets of* X*. If* $\lim_{n\to+\infty} \alpha(B_n) = 0$ *then* $A_\infty = \bigcap_{n=1}^{\infty} B_n$ *is a nonempty compact set.*

In the definition of the Kuratowski measure we can consider balls instead of arbitrary sets. In this way we get the defnition of the Hausdorf measure :

Definition 1.13 *([23])Let* X *be a Banach space and* $\mathcal{P}_{bd}(X)$ *the family of bounded subsets of* X*. For every* $B \in \mathcal{P}_{bd}(X)$ *The Hausdorf measure of noncompactness is the map* $\chi : \mathcal{P}_{bd}(X) \to [0, +\infty]$ *defined by*

$$\chi(B) = inf\{r > 0 : B \ admits\ a\ finite\ covering\ by\ balls\ of\ radius \leq r\}.$$

Theorem 1.3 *([23]) The Kuratowski and Hausdorf (*$\mathcal{MNC}$*) are related by the inequalities*

$$\chi(B) \leq \alpha(B) \leq 2\chi(B)$$

In the class of all infinite dimensional Banach spaces these inequalities are the best possible.

•Measure of noncompactness in $L^1(J)$
To introduce the notion of measure of noncompactness in $L^1(J)$ we let $\mathcal{M}_{bd}$ be the family of all bounded subsets of $L^1(J)$. Analogously denote by $\mathcal{N}_{rcp}$ the family of all relatively compact and nonempty subsets of $L^1(J)$.
We will adopt the following definition of measure of noncompactness [23].

Definition 1.14 *A function $\mu : \mathcal{M}_{bd} \longrightarrow \mathbb{R}_+$ will be called a measure of noncompactnes if it satisfies to the following conditions :*

1. $Ker\mu(A) = \{A \in \mathcal{M}_{bd} : \mu(A) = 0\}$ *is nonempty and* $ker\mu(A) \subset \mathcal{N}_{rcp}$.
2. $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.
3. $\mu(\overline{A}) = \mu(A)$.
4. $\mu(convA) = \mu(A)$.
5. $\mu(\lambda A + (1-\lambda)B) \leq \lambda\mu(A) + (1-\lambda)\mu(B)$ *pour* $\lambda \in [0,1]$.
6. *If* $(A_n)_{n\geq 1}$ *Is a sequence of closed sets from* $\mathcal{M}_{bd}$ *such that*

$$A_{n+1} \subset A_n \ (n = 1, 2, \ldots)$$

and

$$\lim_{n\to+\infty} \mu(A_n) = 0.$$

Then the intersection set $A_\infty = \bigcap_{n=1}^{\infty} A_n$ *is nonempty.*

In particular*, the measure of noncompactness in $L^1(J)$ is defined as follows. Let X be a fixed nonempty and bounded subset of $L^1(J)$. For $x \in X$, denote by*

$$\mu(X) = \lim_{\delta\to 0}\left\{\sup\left\{\sup\left(\int_0^T |x(t+h) - x(t)|dt\right),\ |h| \leq \delta\right\},\ x \in X\right\}. \qquad (1.1)$$

It can be easily shown, that μ is measure of noncompactness in $L^1(J)$ (see [23]).

For more details on measure of noncompactness and the proof of the known results cited in this section we refer the reader to Akhmerov *et al.* [15], Alvarez [17], Banas *et al.* [22, 23, 24, 25, 26], Guo *et al.* [66].

1.5 Phase spaces

In this paper, we assume that the state space $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and satisfying the following fundamental axioms which were introduced by Hale and Kato in [68].

(A_1) If $y : (-\infty, b] \to \mathbb{R}$, and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold :
(i) $y_t \in \mathcal{B}$
(ii) $\|y_t\|_{\mathcal{B}} \leq K(t) \int_0^t |y(s)|ds + M(t)\|y_0\|_{\mathcal{B}}$,
(iii) $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$, where $H \geq 0$ is a constant, $K : J \to [0, \infty)$ is continuous, $M : [0, \infty) \to [0, \infty)$ is locally bounded and H, K, M, are independent of $y(.)$.

(A_2) For the function $y(.)$ in (A_1), y_t is a $\mathcal{B}$-valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Denote $K_b = supK(t) : t \in J$ and $M_b = supM(t) : t \in J$

Remark 1.3 *1. (A_1)(ii) is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.*
2. Since $\|.\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$
3.From the equivalence in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$: We necessarily have that $\phi(0) = \psi(0)$.

Now We indicate some examples of phase spaces. For other details we refer, for instance to the book by Hino et al [71]

1.5.1 Examples of phase spaces

Example 1.1 *Let :*
BC the space of bounded continuous functions defined from $(-\infty, 0] \to E$,
BUC the space of bounded uniformly continuous functions defined from $(-\infty, 0] \to E$,
$C^\infty := \{\phi \in BC : \lim\limits_{\theta \to -\infty} \phi(\theta)$ *exist in* $E\}$
$C^0 := \{\phi \in BC : \lim\limits_{\theta \to -\infty} \phi(\theta) = 0\}$,*endowed with the uniform norm*

$$\|\phi\| = \sup\{|\phi(\theta)| : \theta \leq 0\}.$$

We have that the spaces BUC, C^∞ and C^0 satisfy conditions (A1)-(A3). However, BC satisfies $(A1), (A3)$ but (A2) is not satisfied.

Example 1.2 *Let g be a positive continuous function on $(-\infty, 0]$. We define :*
$C_g := \left\{\phi \in C((-\infty, 0]), E) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on}(-\infty, 0]\right\}$,
$C_g^0 := \left\{\phi \in C_g : \lim\limits_{\theta \to -\infty} \dfrac{\phi(\theta)}{g(\theta)} = 0\right\}$, *endowed with the uniform norm*

$$\|\phi\| = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)} : \theta \leq 0\right\}.$$

Then we have that the spaces C_g and C_g^0 satisfy condition (A3). We consider the following condition on the function g.
(g_1) For all $a > 0$, $\sup_{0 \leq t \leq a} \sup\{\frac{\phi(t+\theta)}{g(\theta)} : -\infty < t \leq -t\}$ Then C_g and C_g^0 satisfy conditions (A1) and (A2) if (g_1) holds.

Example 1.3 *The space C_γ For any real positive constant γ, we define the functional space C_γ by*

$$C_\gamma := \{\phi \in C((-\infty, 0]), E) : \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exist in } E$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \theta \leq 0\}.$$

Then in the space C_γ the axioms (A1)-(A3) are satisfied.

1.6 Some fixed point theorems

Definition 1.15 *([16]) Let (M, d) be a metric space. The map $T : M \longrightarrow M$ is said to be Lipschitzian if there exists a constant $k > 0$ (called Lipschitz constant) such that*

$$d(T(x), T(y)) \leq kd(x, y), \ forall \ x, \ y \in M$$

A Lipschitzian mapping with a Lipschitz constant $k < 1$ is called contraction.

Theorem 1.4 *(Banach's fixed point theorem [61]). Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 1.5 *(Schauder fixed point theorem ([50]) Let E a Banach space and Q be a convex subset of E and $T : Q \longrightarrow Q$ is compact, and continuous map. Then T has at least one fixed point in Q.*

In the next definition we will consider a special class of continuous and bounded operators.

Definition 1.16 *Let $T : M \subset E \longrightarrow E$ be a bounded operator from a Banach space E into itself. The operator T is called a k-set contraction if there is a number $k \geq 0$ such that*

$$\mu(T(A)) \leq k\mu(A)$$

for all bounded sets A in M. The bounded operator T is called condensing if $\mu(T(A)) < \mu(A)$ for all bounded sets A in M with $\mu(M) > 0$.

Obviously, every k-set contraction for $0 \leq k < 1$ is condensing. Every compact map T is a k-set contraction with $k = 0$.

Theorem 1.6 *(Darbo's fixed point theorem [23]) Let M be nonempty, bounded, convex and closed subset of a Banach space E and $T : M \longrightarrow M$ is a continuous operator satisfying $\mu(TA) \leq k\mu(A)$ for any nonempty subset A of M and for some constant $k \in [0, 1)$. Then T has at least one fixed point in M.*

Next we state two multi-valued fixed point theorems

Lemma 1.4 *(Bohnenblust-Karlin **1950**)([42]). Let X be a Banach space and $K \in \mathcal{P}_{cl,cv}(X)$ and suppose that the operator $G : K \to \mathcal{P}_{cl,cv}(K)$ is upper semicontinuous and the set $G(K)$ is relatively compact in X. Then G has a fixed point in K.*

Lemma 1.5 *(Covitz-Nadler [48]). Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \phi$.*

Chapitre 2

Integrable Solutions for Implicit Fractional Order Differential Equations [(1)]

2.1 Introduction and Motivations

In this chapter we deal with the existence of integrable solutions for initial value problem (IVP for short), for fractional order implicit differential equation

$$^cD^\alpha y(t) = f(t, y(t), ^cD^\alpha y(t)),\ t \in J = [0, T],\ 0 < \alpha \leq 1, \tag{2.1}$$

$$y(0) = y_0, \tag{2.2}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_0 \in \mathbb{R}$, and $^cD^\alpha$ is the Caputo fractional derivative.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [29, 70, 91, 102, 110]). There has been a significant development in ordinary and partial fractional differential equations in recent years ; see the monographs of Abbas *et al.* [1, 2], Kilbas *et al.* [80], Lakshmikantham *et al.* [89], and the papers by Agarwal *et al.* [8, 9], Belarbi *et al.* [28], Benchohra *et al.* [32], and the references therein.

[(1)] M. Benchohra and **M. S. Souid**, Integrable Solutions for Implicit Fractional Order Differential Equations. *Transylvanian Journal of Mathematics and Mechanics* **6** (2014), No. 2, 101-107.

More recently, some authors have considered initial value problems for fractional differential equations depending on the fractional derivative.
In [30], Benchohra et al. studied the problem involving Caputo's derivative :

$$^{c}D^{\alpha}u(t) = f(t, u(t), ^{c}D^{\alpha-1}u(t)),\ t \in J = [0, +\infty),\ 1 < \alpha \leq 2$$

$$u(0) = u_0,\ u \textit{ is bounded on } J :$$

In [36], Benchohra and Lazreg, studied the existence of continuous solutions for the problem (2.1)-(2.2)

This chapter is organized as follows. In Section **2.2**, we give two results, the first one is based on Schauder's fixed point theorem (Theorem 2.1) and the second one on the Banach contraction principle (Theorem 2.2). An example is given in Section **2.3** to demonstrate the application of our main results. These results can be considered as a contribution to this emerging field.

2.2 Existence of solutions

Let us start by defining what we mean by an integrable solution of the problem (2.1) – (2.2).

Definition 2.1 . *A function $y \in L^1(J, \mathbb{R})$ is said to be a solution of IVP* (2.1) – (2.2) *if y satisfies (2.1) and (2.2).*

For the existence of solutions for the problem (2.1) – (2.2), we need the following auxiliary lemma.

Lemma 2.1 *The solution of the IVP* (2.1) – (2.2) *can be expressed by the integral equation*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \tag{2.3}$$

where $x \in L^1(J, \mathbb{R})$ is the solution of the functional integral equation

$$x(t) = f\left(t, y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, x(t)\right). \tag{2.4}$$

Proof. Let $^{c}D^{\alpha}y(t) = x(t))$ in equation (2.1), then

$$x(t) = f(t, y(t), x(t)) \tag{2.5}$$

and

$$\begin{aligned} y(t) &= y(0) + I^{\alpha}x(t)) \\ &= y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds. \end{aligned} \tag{2.6}$$

Leu us introduce the following assumptions :

(H1) $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in J$, for any $(u_1, u_2) \in \mathbb{R}^2$ and continuous in $(u_1, u_2) \in \mathbb{R}^2$, for almost all $t \in J$.

(H2) There exist a positive function $a \in L^1(J, \mathbb{R})$ and constants, $b_i > 0; i = 1, 2$ such that :

$$|f(t, u_1, u_2)| \leq a(t) + b_1|u_1| + b_2|u_2|, \forall (t, u_1, u_2) \in J \times \mathbb{R}^2.$$

Our first result is based on Schauder fixed point theorem.

Theorem 2.1 *Assume that the assumptions* $(H1) - (H2)$ *are satisfied. If*

$$\frac{b_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{b_2 T^{\alpha}}{\Gamma(\alpha+1)} < 1, \tag{2.7}$$

then the IVP $(2.1) - (2.2)$ *has at least one solution* $y \in L^1(J, \mathbb{R})$.

Proof. Transform the problem $(2.1) - (2.2)$ into a fixed point problem. Consider the operator

$$H : L^1(J, \mathbb{R}) \longrightarrow L^1(J, \mathbb{R})$$

defined by :

$$(Hx)(t) = y_0 + I^{\alpha} x(t), \tag{2.8}$$

where

$$x(t) = f(t, y_0 + I^{\alpha} x(t), x(t)).$$

The operator H is well defined, indeed, for each $x \in L^1(J, \mathbb{R})$, from assumptions **(H1)** and **(H2)**, we obtain

$$\begin{aligned}
\|Hx\|_{L_1} &= \int_0^T |Hx(t)| dt \\
&= \int_0^T |y_0 + I^{\alpha} x(t))| dt \\
&\leq T|y_0| + \int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s))| ds \Big) dt \\
&\leq T|y_0| + \int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y_0 + I^{\alpha} x(s), x(s))| ds \Big) dt \\
&\leq T|y_0| + \int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |a(s) + b_1(y_0 + I^{\alpha} x(s)) + b_2(x(s)| ds \Big) dt \\
&\leq T|y_0| + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|a\|_{L_1} + \frac{b_1 |y_0| T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{b_2 T^{\alpha}}{\Gamma(\alpha+1)} \|x\|_{L_1} \\
&\quad + b_1 \int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} I^{\alpha} |x(s)| ds \Big) dt \\
&\leq T|y_0| + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|a\|_{L_1} + \frac{b_1 |y_0| T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{b_2 T^{\alpha}}{\Gamma(\alpha+1)} \|x\|_{L_1} \\
&\quad + \frac{b_1 T^{2\alpha}}{\Gamma(2\alpha+1)} \|x\|_{L_1} < +\infty.
\end{aligned} \tag{2.9}$$

Let

$$r = \frac{T|y_0| + \left(\frac{T^\alpha \|a\|_{L_1} + b_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)}\right)}{1 - \left(\frac{b_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\right)},$$

and consider the set

$$B_r = \{x \in L^1(J, \mathbb{R}) : \|x\|_{L_1} \leq r.\}.$$

Clearly B_r is nonempty, bounded, convex and closed.

Now, we will show that $HB_r \subset B_r$, indeed, for each $x \in B_r$, from (2.7) and (2.9) we get

$$\begin{aligned} \|Hx\|_{L_1} &\leq T|y_0| + \left(\frac{T^\alpha \|a\|_{L_1} + b_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)}\right) \\ &+ \left(\frac{b_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\right) \|x\|_{L_1} \\ &\leq r. \end{aligned}$$

Then $HB_r \subset B_r$. Assumption **(H1)** implies that H is continuous. Now, we will show that H is compact, this is HB_r is relatively compact. Clearly HB_r is bounded in $L^1(J, \mathbb{R})$, i.e condition **(i)** of Kolmogorov compactness criterion is satisfied. It remains to show $(Hx)_h \longrightarrow (Hx)$ in $L^1(J, \mathbb{R})$ for each $x \in B_r$.
Let $x \in B_r$, then we have

$$\begin{aligned} & \|(Hx)_h - (Hx)\|_{L^1} \\ = & \int_0^T |(Hx)_h(t) - (Hx)(t)| dt \\ = & \int_0^T \left| \frac{1}{h} \int_t^{t+h} (Hx)(s) ds - (Hx)(t) \right| dt \\ \leq & \int_0^T \left(\frac{1}{h} \int_t^{t+h} |(Hx)(s) - (Hx)(t)| ds \right) dt \\ \leq & \int_0^T \left(\frac{1}{h} \int_t^{t+h} |I^\alpha x(s) - I^\alpha x(t)| ds \right) dt \\ \leq & \int_0^T \frac{1}{h} \int_t^{t+h} |I^\alpha f(s, y_0 + I^\alpha x(s), x(s)) - I^\alpha f(t, y_0 + I^\alpha x(t), x(t))| ds dt. \end{aligned}$$

Since $x \in B_r \subset L^1(J, \mathbb{R})$ and assumption **(H2)** that implies $f \in L^1(J, \mathbb{R})$ and by Proposition 1.1 **(4)**, it follows that $I^\alpha f \in L^1(J, \mathbb{R})$, then we have

$$\frac{1}{h} \int_t^{t+h} |I^\alpha f(s, y_0 + I^\alpha x(s), x(s)) - I^\alpha f(t, y_0 + I^\alpha x(t), x(t)| ds \longrightarrow 0 \ as \ h \longrightarrow 0, \ t \in J.$$

Hence

$$(Hx)_h \longrightarrow (Hx) \ \text{ uniformly as } \ h \longrightarrow 0.$$

Then by Kolmogorov compactness criterion, $H(B_r)$ is relatively compact. As a consequence of Schauder's fixed point theorem the IVP $(2.1)-(2.2)$ has at least one solution in B_r. □

The following result is based on the Banach contraction principle.

Theorem 2.2 *Assume that **(H1)** and the following condition hold.*

(H3) *There exist constants $k_1, k_2 > 0$ such that*

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq k_1|x_1 - x_2| + k_2|y_1 - y_2|, \ t \in J, \ x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

If

$$\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 T^{\alpha}}{\Gamma(\alpha+1)} < 1, \tag{2.10}$$

then the IVP $(2.1)-(2.2)$ has a unique solution $y \in L^1(J,\mathbb{R})$.

Proof. We shall use the Banach contraction principle to prove that H defined by (2.8) has a fixed point. Let $x, y \in L^1(J,\mathbb{R})$, and $t \in J$. Then we have,

$$\begin{aligned}
|(Hx)(t) - (Hy)(t)| &= |I^\alpha [f(t, y_0 + I^\alpha x(t), x(t)) - f(t, y_0 + I^\alpha y(t), y(t))]| \\
&\leq k_1 I^{2\alpha}|x(t) - y(t)| + k_2 I^\alpha |x(t) - y(t)| \\
&\leq \frac{k_1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1}|x(s) - y(s)|ds \\
&\quad + \frac{k_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|x(s) - y(s)|ds.
\end{aligned}$$

Thus

$$\begin{aligned}
\|(Hx) - (Hy)\|_{L_1} &\leq \frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)}\|x - y\|_{L_1} + \frac{k_2 T^\alpha}{\Gamma(\alpha+1)}\|x - y\|_{L_1} \\
&\leq \left(\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 T^\alpha}{\Gamma(\alpha+1)}\right)\|x - y\|_{L_1}.
\end{aligned}$$

Consequently by (2.10) H is a contraction. As a consequence of the Banach contraction principle, we deduce that H has a fixed point which is a solution of the problem $(2.1)-(2.2)$. □

2.3 Example

Let us consider the following fractional initial value problem,

$${}^cD^\alpha y(t) = \frac{e^{-t}}{(e^t+8)(1+|y(t)|+|{}^cD^\alpha y(t)|)}, \ t \in J := [0,1], \ \alpha \in (0,1], \tag{2.11}$$

$$y(0) = 1. \tag{2.12}$$

Set

$$f(t,y,z)=\frac{e^{-t}}{(e^t+8)(1+y+z)},\ (t,y,z)\in J\times[0,+\infty)\times[0,+\infty).$$

Let $y_1,y_2,z_1,z_2\in[0,+\infty)$ and $t\in J$. Then we have

$$\begin{aligned}|f(t,y_1,z_1)-f(t,y_2,z_2)| &= \left|\frac{e^{-t}}{e^t+8}\left(\frac{1}{1+y_1+z_1}-\frac{1}{1+y_2+z_2}\right)\right|\\ &\leq \frac{e^{-t}(|y_1-y_2|+|z_1-z_2|)}{(e^t+8)(1+y_1+z_1)(1+y_2+z_2)}\\ &\leq \frac{e^{-t}}{(e^t+8)}(|y_1-y_2|+|z_1-z_2|)\\ &\leq \frac{1}{9}|y_1-y_2|+\frac{1}{9}|z_1-z_2|.\end{aligned}$$

Hence the condition **(H3)** holds with $k_1=k_2=\frac{1}{9}$. We shall check that condition (2.10) is satisfied with $T=1$. Indeed

$$\frac{k_1T^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{k_2T^{\alpha}}{\Gamma(\alpha+1)}=\frac{1}{9\Gamma(2\alpha+1)}+\frac{1}{9\Gamma(\alpha+1)}<1. \tag{2.13}$$

Then by Theorem 2.2, the problem (2.11) – (2.12) has a unique integrable solution on $[0,1]$.

Chapitre 3

L^1-Solutions for Implicit Fractional Order Differential Equations with Nonlocal Condition[(2)]

3.1 Introduction

In this chapter we deal with the existence of solutions of the nonlocal problem, for fractional order implicit differential equation

$$ {}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)), \ a.e, \ t \in J =: (0, T], \ 0 < \alpha \leq 1, \tag{3.1}$$

$$\sum_{k=1}^{m} a_k y(t_k) = y_0, \tag{3.2}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_0 \in \mathbb{R}$, $a_k \in \mathbb{R}$, ${}^cD^\alpha$ is the Caputo fractional derivative, and $0 < t_1 < t_2 < ..., t_m < T, \ k = 1, 2, ..., m$.

Fractional differential equations with nonlocal conditions have been discussed in ([7, 11, 59, 67, 90, 99, 100]) and references therein. Nonlocal conditions were initiated by Byszewski [45] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems (C.P. for short). As remarked by Byszewski ([43, 44]), the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

This chapter is organized as follows. In Section **3.2**, we give two results, the first one is based on Schauder's fixed point theorem (Theorem 3.1) and the second one on

(2) M. Benchohra and **M. S. Souid**, L^1-Solutions for Implicit Fractional Order Differential Equations with Nonlocal Condition, (to appear) .

the Banach contraction principle (Theorem 3.2). An example is given in Section **3.3** to demonstrate the application of our main results. These results can be considered as a contribution to this emerging field.

3.2 Existence of solutions

Let us start by defining what we mean by an integrable solution of the nonlocal problem (3.1) – (3.2).

Definition 3.1 . *A function $y \in L^1([0,T],\mathbb{R})$ is said to be a solution of IVP* (3.1) – (3.2) *if y satisfies (3.1) and (3.2).*

For the existence of solutions for the nonlocal problem (3.1) – (3.2), we need the following auxiliary lemma.
Set

$$a = \frac{1}{\sum_{k=1}^{m} a_k}.$$

Lemma 3.1 *Assume that $\sum_{k=1}^{m} a_k \neq 0$, the nonlocal problem* (3.1) – (3.2) *is equivalent to the integral equation*

$$y(t) = ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, \tag{3.3}$$

where x is the solution of the functional integral equation

$$x(t) = f\left(t, ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds) + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, x(t)\right). \tag{3.4}$$

Proof. Let ${}^cD^\alpha y(t) = x(t))$ in equation (3.1), then

$$x(t) = f(t, y(t), x(t)) \tag{3.5}$$

and

$$\begin{aligned} y(t) &= y(0) + I^\alpha x(t)) \\ &= y(0) + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds. \end{aligned} \tag{3.6}$$

Let $t = t_k$ in (3.6), we obtain

$$y(t_k) = y(0) + \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds,$$

and

$$\sum_{k=1}^{m} a_k y(t_k) = \sum_{k=1}^{m} a_k y(0) + \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds. \tag{3.7}$$

Substitute from (3.2) into (3.7), we get

$$y_0 = \sum_{k=1}^{m} a_k y(0) + \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds,$$

and

$$y(0) = a\left(y_0 - \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds\right). \tag{3.8}$$

Substitute from (3.8) into (3.6) and (3.5), we obtain (3.3) and (3.4).
For complete the proof, we prove that equation (3.3) satisfies the nonlocal problem (3.1) – (3.2). Differentiating (3.3), we get

$$^cD^\alpha y(t) = x(t) = f(t, y(t), ^c D^\alpha y(t)).$$

Let $t = t_k$ in (3.3), we obtain

$$\begin{aligned} y(t_k) &= ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds) + \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \\ &= ay_0 + \left(1 - a\sum_{k=1}^{m} a_k\right) \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds. \end{aligned}$$

Then

$$\sum_{k=1}^{m} a_k y(t_k) = \sum_{k=1}^{m} a_k a y_0 + \sum_{k=1}^{m} a_k \left(1 - a\sum_{k=1}^{m} a_k\right) \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds = y_0.$$

This complete the proof of the equivalent between the nonlocal problem (3.1)-(3.2) and the integral equation (3.3).
Leu us introduce the following assumptions :

- **(H1)** $f : [0, T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in [0, T]$, for any $(u_1, u_2) \in \mathbb{R}^2$ and continuous in $(u_1, u_2) \in \mathbb{R}^2$, for almost all $t \in [0, T]$.
- **(H2)** There exist a positive function $a \in L^1[0, T]$ and constants, $b_i > 0; i = 1, 2$ such that :

$$|f(t, u_1, u_2)| \leq a(t) + b_1|u_1| + b_2|u_2|, \forall (t, u_1, u_2) \in [0, T] \times \mathbb{R}^2.$$

Our first result is based on Schauder's fixed point theorem.

Theorem 3.1 *Assume that the assumptions* $(H1)-(H2)$ *are satisfied. If*

$$\frac{2b_1T^\alpha}{\Gamma(\alpha+1)}+b_2<1, \tag{3.9}$$

then the IVP $(3.1)-(3.2)$ *has at least one solution* $y\in L^1([0,T],\mathbb{R})$.

Proof. Transform the nonlocal problem $(3.1)-(3.2)$ into a fixed point problem. Consider the operator

$$H:L^1([0,T],\mathbb{R})\longrightarrow L^1([0,T],\mathbb{R})$$

defined by :

$$(Hx)(t)=f\left(t,ay_0-a\sum_{k=1}^m a_k\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds,x(t)\right), \tag{3.10}$$

Let

$$r=\frac{Tab_1|y_0|+\|a\|_{L_1}}{1-\left(\frac{2b_1T^\alpha}{\Gamma(\alpha+1)}+b_2\right)},$$

and consider the set

$$B_r=\{x\in L^1([0,T],\mathbb{R}):\|x\|_{L_1}\leq r.\}$$

Clearly B_r is nonempty, bounded, convex and closed.

Now, we will show that $HB_r\subset B_r$, indeed, for each $x\in B_r$, from (3.9) and (3.10) we get

$$\begin{aligned}\|Hx\|_{L_1} &= \int_0^T|Hx(t)|dt\\ &= \int_0^T\left|f\left(t,ay_0-a\sum_{k=1}^m a_k\int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds,x(t)\right)\right|dt\\ &\leq \int_0^T\left[|a(t)|+b_1|ay_0-a\sum_{k=1}^m a_kI^\alpha x(t)|_{t=t_k}+I^\alpha x(t)|+b_2|x(t)|\right]dt\\ &\leq Tab_1|y_0|+\|a\|_{L_1}+\frac{b_1a\sum_{k=1}^m a_kt_k^\alpha}{\Gamma(\alpha+1)}\|x\|_{L_1}+\frac{b_1T^\alpha}{\Gamma(\alpha+1)}\|x\|_{L_1}+b_2\|x\|_{L_1}\\ &\leq Tab_1|y_0|+\|a\|_{L_1}+\left(\frac{2b_1T^\alpha}{\Gamma(\alpha+1)}+b_2\right)\|x\|_{L_1}\\ &\leq r.\end{aligned}$$

Then $HB_r\subset B_r$. Assumption **(H1)** implies that H is continuous. Now, we will show that H is compact, this is HB_r is relatively compact. Clearly HB_r is bounded in

$L^1([0,T],\mathbb{R})$, i.e condition **(i)** of Kolmogorov compactness criterion is satisfied. It remains to show $(Hx)_h \longrightarrow (Hx)$ in $L^1([0,T],\mathbb{R})$ for each $x \in B_r$.
Let $x \in B_r$, then we have

$$\begin{aligned}
& \|(Hx)_h - (Hx)\|_{L^1} \\
= & \int_0^T |(Hx)_h(t) - (Hx)(t)|dt \\
= & \int_0^T \left| \frac{1}{h}\int_t^{t+h} (Hx)(s)ds - (Hx)(t) \right| dt \\
\leq & \int_0^T \left(\frac{1}{h}\int_t^{t+h} |(Hx)(s) - (Hx)(t)|ds \right) dt \\
\leq & \int_0^T \frac{1}{h}\int_t^{t+h} |f\left(t, ay_0 - a\sum_{k=1}^m a_k \int_0^{s_k} \frac{(s_k-\tau)^{\alpha-1}}{\Gamma(\alpha)}x(\tau)d\tau + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}x(\tau)d\tau, x(s)\right) \\
& -f\left(t, ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds, x(t)\right) |dsdt.
\end{aligned}$$

Since $x \in B_r \subset L^1([0,T],\mathbb{R})$ and assumption (**H2**) that implies $f \in L^1([0,T],\mathbb{R})$, it follows that
$\frac{1}{h}\int_t^{t+h} \left| f\left(t, ay_0 - a\sum_{k=1}^m a_k \int_0^{s_k} \frac{(s_k-\tau)^{\alpha-1}}{\Gamma(\alpha)}x(\tau)d\tau + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}x(\tau)d\tau, x(s)\right)\right.$
$\left. - f\left(t, ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds, x(t)\right) \right| ds \to 0 \ as \ h \to 0.$
Hence

$$(Hx)_h \to (Hx) \ \text{ uniformly as } \ h \to 0.$$

Then by Kolmogorov compactness compactness criterion, $H(B_r)$ is relatively compact. As a consequence of Schauder's fixed point theorem the nonlocal problem (3.1) – (3.2) has at least one solution in B_r. □

The following result is based on the Banach contraction principle.

Theorem 3.2 *Assume that* ***(H1)*** *and the following condition hold.*

(H3) *There exist constants* $k_1, k_2 > 0$ *such that*

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq k_1|x_1 - x_2| + k_2|y_1 - y_2|, \ t \in [0,T], \ x_1,x_2,y_1,y_2 \in \mathbb{R}.$$

If

$$\frac{2k_1 T^\alpha}{\Gamma(\alpha+1)} + k_2 < 1, \tag{3.11}$$

then the IVP (3.1) – (3.2) *has a unique solution* $y \in L^1([0,T],\mathbb{R})$.

Proof. We shall use the Banach contraction principle to prove that H defined by (3.10) has a fixed point. Let $x, y \in L^1([0,T],\mathbb{R})$, and $t \in [0,T]$. Then we have,

$$\begin{aligned}
&|(Hx)(t)-(Hy)(t)|\\
= &\Big|f(t, ay_0 - a\sum_{k=1}^{m} a_k I^\alpha x(t)|_{t=t_k} + I^\alpha x(t), x(t))\\
&-f(t, ay_0 - a\sum_{k=1}^{m} a_k I^\alpha y(t)|_{t=t_k} + I^\alpha y(t), y(t))\Big|\\
\leq &k_1 a \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)-y(s)|ds\\
&+k_1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)-y(s)|ds + k_2|x-y|.
\end{aligned}$$

Thus

$$\begin{aligned}
\|(Hx)-(Hy)\|_{L_1} \leq &\frac{k_1 t_k^\alpha a\sum_{k=1}^m a_k}{\Gamma(\alpha+1)}\int_0^T |x(t)-y(t)|dt + \frac{k_1 T^\alpha}{\Gamma(\alpha+1)}\int_0^T |x(t)-y(t)|dt\\
&+k_2\int_0^T |x(t)-y(t)|dt\\
\leq &\frac{2k_1T^\alpha}{\Gamma(\alpha+1)}\|x-y\|_{L_1} + k_2\|x-y\|_{L_1}\\
\leq &\left(\frac{2k_1T^\alpha}{\Gamma(\alpha+1)} + k_2\right)\|x-y\|_{L_1}.
\end{aligned}$$

Consequently by (3.11) H is a contraction. As a consequence of the Banach contraction principle, we deduce that H has a fixed point which is a solution of the nonlocal problem (3.1) – (3.2). □

3.3 Example

Let us consider the following fractional nonlocal problem,

$$^cD^\alpha y(t) = \frac{1}{(e^t+5)(1+|y(t)|+|^cD^\alpha y(t)|)},\ t \in J := [0,1],\ \alpha \in (0,1], \tag{3.12}$$

$$\sum_{k=1}^{m} a_k y(t_k) = 1, \tag{3.13}$$

where $a_k \in \mathbb{R},\ 0 < t_1 < t_2 < ... < 1$.
Set

$$f(t,y,z) = \frac{1}{(e^t+5)(1+y+z)},\ (t,y,z) \in J \times [0,+\infty) \times [0,+\infty).$$

Let $y_1, y_2, z_1, z_2 \in [0, +\infty)$ and $t \in J$. Then we have

$$\begin{aligned}
|f(t, y_1, z_1) - f(t, y_2, z_2)| &= \left| \frac{1}{e^t + 5} \left(\frac{1}{1 + y_1 + z_1} - \frac{1}{1 + y_2 + z_2} \right) \right| \\
&\leq \frac{|y_1 - y_2| + |z_1 - z_2|}{(e^t + 5)(1 + y_1 + z_1)(1 + y_2 + z_2)} \\
&\leq \frac{1}{(e^t + 5)} (|y_1 - y_2| + |z_1 - z_2|) \\
&\leq \frac{1}{6}|y_1 - y_2| + \frac{1}{6}|z_1 - z_2|.
\end{aligned}$$

Hence the condition **(H3)** holds with $k_1 = k_2 = \frac{1}{6}$. We shall check that condition (3.11) is satisfied. Indeed

$$\frac{2k_1}{\Gamma(\alpha + 1)} + k_2 = \frac{1}{3\Gamma(\alpha + 1)} + \frac{1}{6} < 1. \tag{3.14}$$

Then by Theorem 3.2, the nonlocal problem (3.12) − (3.13) has a unique integrable solution on $[0, 1]$.

Chapitre 4

Integrable Solutions For Implicit Fractional Order Functional Differential Equations with Infinite Delay [3]

4.1 Introduction

In this chapter we deal with the existence of solutions for initial value problem (IVP for short), for implicit fractional order functional differential equations with infinite delay

$$^{c}D^{\alpha}y(t) = f(t, y_t, {}^{c}D^{\alpha}y_t),\ t \in J := [0, b],\ 0 < \alpha \leq 1, \tag{4.1}$$

$$y(t) = \phi(t),\ t \in (-\infty, 0], \tag{4.2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, and $f : J \times \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ is a given function satisfying some assumptions that will be specified later, and $\mathcal{B}$ is called a phase space. For any function y defined on $(-\infty, b]$ and any $t \in J$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t+\theta),\ \theta \in (-\infty, 0]$. Here $y_t(.)$ represents the history of the state from time$-\infty$ up to the present time t.

In the literature devoted to equations with finite delay, the state space is usually the space of all continuous function on $[-r, 0],\ r > 0$ and $\alpha = 1$ endowed with the uniform norm topology ; see the book of Hale and Lunel [69]. When the delay is infinite, the selection of the state $\mathcal{B}$ (i.e. phase space) plays an important role in the study of both qualitative and quantitative theory for functional differential equations.

[3] M. Benchohra and **M. S. Souid**, Integrable Solutions For Implicit Fractional Order Functional Differential Equations with Infinite Delay, *Archivum Mathematicum (BRNO)Tomus* **51** (2015), 13-22.

A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato [68] (see also Kappel and Schappacher [77] and Schumacher [106]). For a detailed discussion on this topic we refer the reader to the book by Hino *et al.* [71].

This chapter is organized as follows. In Section **4.2**, we give two results, the first one is based on the Banach contraction principle (Theorem 4.1) and the second one on Schauder type fixed point theorem (Theorem 4.2). An example is given in Section **4.3** to demonstrate the application of our main results. These results can be considered as a contribution to this emerging field.

4.2 Existence of solutions

Let us start by defining what we mean by an integrable solution of the problem (4.1) – (4.2).
Let the space

$$\Omega = \{y : (-\infty, b] \to \mathbb{R} : y|_{(-\infty,0]} \in \mathcal{B} \text{ and } y|_J \in L^1(J)\}.$$

Definition 4.1 . *A function $y \in \Omega$ is said to be a solution of IVP* (4.1) – (4.2) *if y satisfies (4.1) and (4.2).*

For the existence of solutions for the problem (4.1) – (4.2), we need the following auxiliary lemma.

Lemma 4.1 *The solution of the IVP* (4.1) – (4.2) *can be expressed by the integral equation*

$$y(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \ t \in J, \tag{4.3}$$

$$y(t) = \phi(t), \ t \in (-\infty, 0], \tag{4.4}$$

where x is the solution of the functional integral equation

$$x(t) = f\left(t, \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x_s ds, x_t\right). \tag{4.5}$$

Proof. Let y be solution of (4.3) – (4.4), then for $t \in J$ and $t \in (-\infty, 0]$, we have (4.1) and (4.2), respectively. □

To present the main result, let us introduce the following assumptions :

(H1) $f : J \times \mathcal{B}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in J$, for any $(u_1, u_2) \in \mathcal{B}^2$ and continuous in $(u_1, u_2) \in \mathcal{B}^2$, for almost all $t \in J$.

(H2) There exist constants $k_1, k_2 > 0$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq k_1 \|x_1 - x_2\|_{\mathcal{B}} + k_2 \|y_1 - y_2\|_{\mathcal{B}},$$

for $t \in J$, and every $x_1, x_2, y_1, y_2 \in \mathcal{B}$.

Our first existence result for the IVP (4.1) – (4.2) is based on the Banach contraction principle.
Set

$$K_b = sup\{|K(t)| : t \in J\}.$$

Theorem 4.1 *Assume that the assumptions* $(H1) - (H2)$ *are satisfied. If*

$$\frac{k_1 K_b b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 K_b b^{\alpha}}{\Gamma(\alpha+1)} < 1, \tag{4.6}$$

then the IVP (4.1) – (4.2) *has a unique solution on the interval* $(-\infty, b]$.

Proof. Transform the problem (4.1) – (4.2) into a fixed point problem. Consider the operator $N : \Omega \longrightarrow \Omega$ defined by :

$$(Ny)(t) = \begin{cases} \phi(t), \ t \in (-\infty, 0] \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, I^\alpha y_s, y_s) ds, \ t \in J. \end{cases}$$

We shall use the Banach contraction principle to prove that N has a fixed point.
Let $x(.) : (-\infty, b] \to \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} 0, \ \ if \ t \in J \\ \phi(t), \ \ if \ t \in (-\infty, 0]. \end{cases}$$

Then $x_0 = \phi$. For each $z \in L^1(J, \mathbb{R})$, with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} z(t), \ \ if \ t \in J \\ 0, \ \ if \ t \in (-\infty, 0]. \end{cases}$$

if $y(.)$ satisfies the integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, I^\alpha y_s, y_s) ds,$$

we can decompose $y(.)$ as $y(t) = \overline{z}(t) + x(t), \ 0 \le t \le b$, which implies $y_t = \overline{z}_t + x_t$, for every $0 \le t \le b$, and the function $z(.)$ satisfies

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s) ds.$$

Set

$$L_0 = \{z \in L^1(J, \mathbb{R}) : z_0 = 0\},$$

and let $\|.\|_b$ be the seminorm in L_0 defined by

$$\|z\|_b = \|z_0\|_{\mathcal{B}} + \int_0^b |z(t)| dt = \int_0^b |z(t)| dt, \ z \in L_0.$$

L_0 is a Banach space with norm $\|.\|_b$. Let the operator $P : L_0 \to L_0$ be defined by

$$(Pz)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s) ds, \ t \in J, \tag{4.7}$$

That the operator N has a fixed point is equivalent to P has a fixed point, and so we turn to proving that P has a fixed point. We shall show that $P : L_0 \to L_0$ is a contraction map. Indeed, consider $z, z^* \in L_0$. Then we have for each $t \in J$

$$\begin{aligned}
& |P(z)(t) - P(z^*)(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s) - f(s, I^\alpha(\overline{z}_s^* + x_s), \overline{z}_s^* + x_s)| ds \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [k_1 \|I^\alpha(\overline{z}_s - \overline{z}_s^*)\|_{\mathcal{B}} + k_2 \|\overline{z}_s - \overline{z}_s^*\|_{\mathcal{B}}] ds \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_b \left[k_1 \|I^\alpha(z(s) - z^*(s))\| + k_2 \|z(s) - z^*(s)\|\right] ds \\
\leq\ & \left(\frac{k_1 K_b b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 K_b b^\alpha}{\Gamma(\alpha+1)} \right) \|z - z^*\|_b.
\end{aligned}$$

Therefore

$$\|P(z) - P(z^*)\|_b \ \leq \ \left(\frac{k_1 K_b b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 K_b b^\alpha}{\Gamma(\alpha+1)} \right) \|z - z^*\|_b.$$

Consequently by (4.6) P is a contraction. As a consequence of the Banach contraction principle, we deduce that P has a unique fixed point which is a solution of the problem (4.1) − (4.2). □

The following result is based on Schauder fixed point theorem.

Theorem 4.2 *Assume that* ***(H1)*** *and the following condition hold.*

(H3) *There exist a positive function $a \in L^1(J)$ and constants, $q_i > 0; i = 1, 2$ such that :*

$$|f(t, u_1, u_2)| \leq |a(t)| + q_1 \|u_1\|_{\mathcal{B}} + q_2 \|u_2\|_{\mathcal{B}}, \forall (t, u_1, u_2) \in J \times \mathbb{R}^2.$$

If

$$K_b \left(\frac{q_1 b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 b^\alpha}{\Gamma(\alpha+1)} \right) < 1, \tag{4.8}$$

then the IVP (4.1) − (4.2) *has at least one solution* $y \in L^1(J, \mathbb{R})$.

Proof.

Let $P : L_0 \to L_0$ be defined as in (4.7), and

$$r = \frac{\frac{b^\alpha \|a\|_{L^1}}{\Gamma(\alpha+1)} + M_b \|\phi\|_{\mathcal{B}} (\frac{q_1 b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 b^\alpha}{\Gamma(\alpha+1)})}{1 - K_b (\frac{q_1 b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 b^\alpha}{\Gamma(\alpha+1)})},$$

where $M_b = sup\{|M(t)| : t \in J\}$, and consider the set

$$B_r := \{z \in L_0, \|z\|_b \leq r\}.$$

Clearly B_r is nonempty, bounded, convex and closed. We shall show that the operator P satisfies the assumptions of Schauder fixed point theorem. The proof will be given in three steps.

Step1 : P is continuous.
Let z_n be a sequence such that $z_n \to z$ in L_0. Then

$$\begin{aligned} |(Pz_n)(t) - (Pz)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, I^\alpha(\overline{z}_{n_s} + x_s), \overline{z}_{n_s} + x_s) \\ &- f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s)| ds \end{aligned}$$

Since f is a continuous function, we have

$$\begin{aligned} & \|P(z_n) - P(z)\|_b \\ \leq & \frac{b^\alpha}{\Gamma(\alpha+1)} \|f(., I^\alpha(\overline{z}_{n_{(.)}} + x_{(.)}, \overline{z}_{n_{(.)}}) + x_{(.)}) - f(., I^\alpha(\overline{z}_{(.)} + x_{(.)}), \overline{z}_{(.)} + x_{(.)})\|_{L_1} \to 0 \\ & as\ n \to \infty. \end{aligned}$$

Step2 : P maps B_r into itself.

Let $z \in B_r$. Since f is a continuous functions, we have for each $t \in [0, b]$

$$\begin{aligned} |(Pz)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s)| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [a(t)| + q_1 \|I^\alpha(\overline{z}_s + x_s)\|_{\mathcal{B}} + q_2 \|\overline{z}_s + x_s\|_{\mathcal{B}}] ds \\ &\leq \frac{b^\alpha \|a\|_{L_1}}{\Gamma(\alpha+1)} + \Big(\frac{q_1 b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 b^\alpha}{\Gamma(\alpha+1)}\Big)(K_b r + M_b \|\phi\|_{\mathcal{B}}), \end{aligned}$$

where

$$\|\overline{z}_s + x_s\|_{\mathcal{B}} \leq \|\overline{z}_s\|_{\mathcal{B}} + \|x_s\|_{\mathcal{B}}.$$

Hence $\|(Pz)\|_{L_1} \leq r$. Then $PB_r \subset B_r$.
Step3 : P is compact.

We will show that P is compact, this is PB_r is relatively compact. Clearly PB_r is bounded in L_0, i.e condition **(i)** of Kolmogorov compactness criterion is satisfied. It

remains to show $(Pz)_h \longrightarrow (Pz)$, in L_0 for each $z \in B_r$.
Let $z \in B_r$, then we have

$$\begin{aligned}
&\|(Pz)_h - (Pz)\|_{L^1}\\
=&\int_0^b |(Pz)_h(t) - (Pz)(t)|dt\\
=&\int_0^b \left|\frac{1}{h}\int_t^{t+h}(Pz)(s)ds - (Pz)(t)\right| dt\\
\leq&\int_0^b \left(\frac{1}{h}\int_t^{t+h}|(Pz)(s) - (Pz)(t)|ds\right) dt\\
\leq&\int_0^b \frac{1}{h}\int_t^{t+h}|I^\alpha f(s, \overline{z}_s + x_s), \overline{z}_s + x_s) - I^\alpha f(t, I^\alpha(\overline{z}_t + x_t), \overline{z}_t + x_t)|dsdt.
\end{aligned}$$

Since $z \in B_r \subset L_0$ and assumption **(H3)** that implies $f \in L_0$ and by Proposition 1.1, it follows that $I^\alpha f \in L^1(J, \mathbb{R})$, then we have

$$\frac{1}{h}\int_t^{t+h}|I^\alpha f(\overline{z}_s + x_s), \overline{z}_s + x_s) - I^\alpha f(t, I^\alpha(\overline{z}_t + x_t), \overline{z}_t + x_t)|ds \longrightarrow 0 \; as \; h \longrightarrow 0, \; t \in J.$$

Hence

$$(Pz)_h \longrightarrow (Pz) \text{ uniformly as } h \longrightarrow 0.$$

Then by Kolmogorov compactness criterion, $P(B_r)$ is relatively compact. As a consequence of Schauder's fixed point theorem the IVP $(4.1)-(4.2)$ has at least one solution in B_r. □

4.3 Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional initial value problem,

$${}^cD^\alpha y(t) = \frac{ce^{-\gamma t+t}}{(e^t + e^{-t})(1 + |y_t| + |{}^cD^\alpha y_t|)}, \; t \in J := [0, b], \; \alpha \in (0, 1], \tag{4.9}$$

$$y(t) = \phi(t), \; t \in (-\infty, 0], \tag{4.10}$$

where $c > 1$ is fixed. Let γ be a positive real constant and

$$B_\gamma = \{y \in L^1(-\infty, 0] : \lim_{\theta \to -\infty} e^{\gamma\theta}y(\theta), \; exists \; in \; \mathbb{R}\}.$$

The norm of B_γ is given by

$$\|y\|_\gamma = \int_{-\infty}^0 e^{\gamma\theta}|y(\theta)|d\theta.$$

Let $y : (-\infty, b] \to \mathbb{R}$ be such that $y_0 \in B_\gamma$. Then

$$\begin{aligned}\lim_{\theta\to-\infty} e^{\gamma\theta} y_t(\theta) &= \lim_{\theta\to-\infty} e^{\gamma\theta} y(t+\theta)\\ &= \lim_{\theta\to-\infty} e^{\gamma(\theta-t)} y(\theta)\\ &= e^{-\gamma t} \lim_{\theta\to-\infty} e^{\gamma\theta} y_0(\theta) < \infty.\end{aligned}$$

Hence $y_t \in B_\gamma$. Finally we prove that

$$\|y_t\|_\gamma \le K(t)\int_0^t |y(s)|ds + M(t)\|y_0\|_\gamma,$$

where $K = M = 1$ and $H = 1$. we have

$$|y_t(\theta)| = |y(t+\theta)|.$$

If $\theta + t \le 0$, we get

$$|y_t(\theta)| \le \int_{-\infty}^0 |y(s)|ds.$$

For $t + \theta \ge 0$, then we have

$$|y_t(\theta)| \le \int_0^t |y(s)|ds$$

Thus for all $t + \theta \in J$, we get

$$|y_t(\theta)| \le \int_{-\infty}^0 |y(s)|ds + \int_0^t |y(s)|ds.$$

Then

$$\|y_t\|_\gamma \le \|y_0\|_\gamma + \int_0^t |y(s)|ds.$$

It is clear that $(B_\gamma, \|.\|)$ is a Banach space. We can conclude that B_γ is a phase space. Set

$$f(t,y,z) = \frac{e^{-\gamma t + t}}{c(e^t + e^{-t})(1+y+z)}, \ (t,x,z) \in J \times B_\gamma \times B_\gamma.$$

For $t \in J, y_1, y_2, z_1, z_2 \in B_\gamma$, we have

$$\begin{aligned}|f(t,y_1,z_1) - f(t,y_2,z_2)| &= \frac{e^{-\gamma t+t}}{c(e^t+e^{-t})}\left|\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2}\right|\\ &= \frac{e^{-\gamma t+t}(|y_1-y_2| + |z_1-z_2|)}{c(e^t+e^{-t})(1+y_1+z_1)(1+y_2+z_2)}\\ &\le \frac{e^{-\gamma t}\times e^t(|y_1-y_2| + |z_1-z_2|)}{c(e^t+e^{-t})}\\ &\le \frac{e^{-\gamma t}(\|y_1-y_2\|_\gamma + \|z_1-z_2\|_\gamma)}{c}\\ &\le \frac{1}{c}\|y_1-y_2\|_\gamma + \frac{1}{c}\|z_1-z_2\|_\gamma.\end{aligned}$$

Hence the condition **(H2)** holds. We choose b such that $\frac{K_b b^{2\alpha}}{c\Gamma(2\alpha+1)} + \frac{K_b b^{\alpha}}{c\Gamma(\alpha+1)} < 1$. Since $K_b = 1$, then

$$\frac{b^{2\alpha}}{c\Gamma(2\alpha+1)} + \frac{b^{\alpha}}{c\Gamma(\alpha+1)} < 1.$$

Then by Theorem 4.1, the problem (4.9) – (4.10) has a unique integrable solution on $[-\infty, b]$.

Chapitre 5

A New Result of Integrable Solutions for Implicit Fractional Order Differential Equations (4)

5.1 Introduction

In this chapter deals with the existence of integrable solutions for the Initial Value Problem (IVP for short), for implicit fractional order differential equation

$$^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)),\ t \in J = [0, T],\ 0 < \alpha \leq 1, \tag{5.1}$$

$$y(0) = y_0, \tag{5.2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function satisfying some assumptions that will be specified later. We will use the technique of measures of noncompactness which is often used in several branches of nonlinear analysis. Especially, that technique turns out to be a very useful tool in existence for several types of integral equations; details are found in Akhmerov *et al.* [15], Alvarez [17], Banas *et al.* [22, 23, 24, 25, 26], Guo *et al.* [66].
The principal goal here is to prove the existence of integral solutions for the problem (5.1)-(5.2) using Darbo's fixed point theorem.

Many techniques have been developed for studying the existence and uniqueness of solutions of initial and boundary value problem for fractional differential equations. Several authors tried to develop a technique that depends on the Darbo or the Mönch fixed point theorems with the Hausdorff or Kuratowski measure of noncompactness.

(4) M. Benchohra and **M. S. Souid**, A New Result of Integrable Solutions for Implicit Fractional Order Differential Equations, (submitted).

The notion of the measure of noncompactness was defined in many ways. In 1930, Kuratowski [109] defined the measure of non-compactness, $\alpha(A)$, of a bounded subset A of a metric space $(X; d)$, and in 1955, Darbo [74] introduced a new type of fixed point theorem for noncompactness maps.

This chapter is organized as follows. In Section **5.2**, we give a result (Theorem 5.1). An example is given in Section **5.3** to demonstrate the application of our result. These results can be considered as a contribution to this emerging field.

5.2 Existence of solutions

Let us start by defining what we mean by a solution of the problem (5.1) – (5.2).

Definition 5.1 *A function $y \in L^1(J, \mathbb{R})$ is said to be a solution of IVP* (5.1) – (5.2) *if y satisfies the equation ${}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t))$ on J, and the condition $y(0) = y_0$.*

For the existence of solutions for the problem (5.1) – (5.2), we need the following auxiliary lemma.

Lemma 5.1 *The solution of the IVP* (5.1) – (5.2) *can be expressed by the integral equation*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \tag{5.3}$$

where x is the solution of the functional integral equation

$$x(t) = f\left(t, y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, x(t)\right). \tag{5.4}$$

Leu us introduce the following assumptions :

(H1) $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the Carathéodory conditions.

(H2) There exist a positive function $a \in L^1(J)$ and two constants, $q_1,\ q_2 > 0$ such that :

$$|f(t, u_1, u_2)| \leq a(t) + q_1|u_1| + q_2|u_2|,\ \forall (t, u_1, u_2) \in J \times \mathbb{R} \times \mathbb{R},$$

(H3) We first consider two real numbers $0 < |\rho| < \delta$, there exist a positive valued functions $L_f(.)$ which is continuous in a neighborhood of 0 with $L_f(0) = 0$ and two constants $k_1, k_2 > 0$ such that

$$|f(t+\rho, x_1, y_1) - f(t, x_2, y_2)| \leq L_f(\rho) + k_1|x_1 - x_2| + k_2|y_1 - y_2|,$$

$t \in [0, T],\ x_i, y_i \in \mathbb{R},\ i = 1, 2.$

In this section, we study the existence of a solution of the problem (5.1) – (5.2) by the using the concept of measure of noncompactness in $L^1(J)$.

Theorem 5.1 *Assume that assumptions* $(H1)-(H3)$ *are satisfied. If*

$$\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 T^{\alpha}}{\Gamma(\alpha+1)} < 1, \tag{5.5}$$

then the IVP $(5.1)-(5.2)$ *has at least one solution* $y \in L^1(J, \mathbb{R})$.

Proof. Transform the problem $(5.1)-(5.2)$ into a fixed point problem. Consider the operator $N : L^1(J,\mathbb{R}) \to L^1(J,\mathbb{R})$ defined by :

$$(Nx)(t) = y_0 + I^{\alpha}x(t), \tag{5.6}$$

where $x(t) = f(t, y_0 + I^{\alpha}x(t), x(t))$. Clearly, the fixed point of the operator N are solutions of the problem $(5.1)-(5.2)$. Let

$$r = \frac{T|y_0| + \left(\frac{T^{\alpha}\|a\|_{L^1} + q_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)}\right)}{1 - \left(\frac{q_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 T^{\alpha}}{\Gamma(\alpha+1)}\right)},$$

where

$$\frac{q_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 T^{\alpha}}{\Gamma(\alpha+1)} < 1$$

and consider the set

$$B_r = \{x \in L^1(J,\mathbb{R}) : \|x\|_{L^1} \le r,\ r > 0\}.$$

Clearly, the subset B_r is closed, bounded and convex. We shall show that N satisfies the assumptions of Darbo's fixed point theorem. The proof will be given in three steps.
Step 1. N is continuous.
Let x_n be a sequence such that $x_n \to x$ in B_r. Then for each $t \in J$,

$$\begin{aligned}
& \|N(x_n) - N(x)\|_{L^1} \\
= & \|I^{\alpha}x_n(t) - I^{\alpha}x(t)\|_{L^1} \\
= & \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Big(x_n(s) - x(s)\Big) ds \right\|_{L^1} \\
\le & \int_0^T \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, y_0 + I^{\alpha}x_n(s), x(s)) - f(s, y_0 + I^{\alpha}x(s), x(s))| ds \right) dt.
\end{aligned}$$

Since f is of Carathéodory type, then by the Lebesgue dominated convergence theorem we have

$$\|N(x_n) - N(x)\|_{L^1} \to 0 \ as \ n \to \infty.$$

Step 2. N maps B_r into itself.
Let x an arbitrary element in B_r. Then from assumptions **(H1)-(H2)**, we obtain

$$\begin{aligned}
\|Nx\|_{L^1} &= \int_0^T |Nx(t)|dt \\
&= \int_0^T |y_0 + I^\alpha x(t))|dt \\
&\leq T|y_0| + \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s))|ds \right) dt \\
&\leq T|y_0| + \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y_0 + I^\alpha x(s), x(s))|ds \right) dt \\
&\leq T|y_0| + \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |a(s) + q_1(y_0 + I^\alpha x(s)) + q_2(x(s)|ds \right) dt \\
&\leq T|y_0| + \frac{T^\alpha}{\Gamma(\alpha+1)} \|a\|_{L^1} + \frac{q_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{q_2 T^\alpha}{\Gamma(\alpha+1)} \|x\|_{L^1} \\
&+ q_1 \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} I^\alpha |x(s)|ds \right) dt \\
&\leq T|y_0| + \frac{T^\alpha}{\Gamma(\alpha+1)} \|a\|_{L^1} + \frac{q_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{q_2 T^\alpha}{\Gamma(\alpha+1)} \|x\|_{L^1} + \frac{q_1 T^{2\alpha}}{\Gamma(2\alpha+1)} \|x\|_{L^1} \\
&\leq T|y_0| + \frac{T^\alpha \|a\|_{L^1} + q_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{q_1 T^{2\alpha} r}{\Gamma(2\alpha+1)} + \frac{q_2 T^\alpha r}{\Gamma(\alpha+1)} \leq r.
\end{aligned}$$

Then $\|Nx\|_{L^1} \leq r$, which implies that the operator N maps B_r into itself.

Step 3. N is a contraction, i.e : $\mu(NX) \leq k\mu(X),\ k \in [0, 1)$
Now let us fix a nonempty subset X of B_r. We first consider two real numbers $0 <$

$|\rho| < \delta$ and an arbitrary fixed $x \in X$, by (**H3**) we have

$$\begin{aligned}
& \|Nx(t+\rho) - Nx(t)\|_{L^1} \\
= & \int_0^T |(Nx)(t+\rho) - (Nx)(t)| dt \\
= & \int_0^T |I^\alpha x(t+\rho) - I^\alpha x(t)| dt \\
= & \int_0^T |I^\alpha (x(t+\rho) - x(t))| dt \\
= & \int_0^T \left| I^\alpha \Big(f(t+\rho, y_0 + I^\alpha x(t+\rho), x(t+\rho)) - f(t, y_0 + I^\alpha x(t), x(t)) \Big) \right| dt \\
\leq & \int_0^T \Big(|I^\alpha L_f(\rho)| + k_1 \left| I^{2\alpha}(x(t+\rho) - x(t)) \right| + k_2 \left| I^\alpha (x(t+\rho) - x(t)) \right| \Big) dt \\
\leq & \frac{T^\alpha}{\Gamma(\alpha+1)} \int_0^T |L_f(\rho)|\, dt + \frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} \int_0^T |x(t+\rho) - x(t)|\, dt \\
+ & \frac{k_2 T^\alpha}{\Gamma(\alpha+1)} \int_0^T |x(t+\rho) - x(t)|\, dt.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|Nx(.+\rho) - Nx(.)\|_{L^1} \leq & \frac{T^{\alpha+1}}{\Gamma(\alpha+1)} L_f(\rho) \\
+ & \left(\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 T^\alpha}{\Gamma(\alpha+1)} \right) \|x(.+\rho) - x(.)\|_{L^1}.
\end{aligned}$$

Taking into account that

$$\lim_{\delta \to 0} \sup_{|\rho| \leq \delta} L_f(\rho) = 0,$$

we get

$$\mu(NX) \leq \left(\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 T^\alpha}{\Gamma(\alpha+1)} \right) \mu(X).$$

Here, $\mu(.)$ is the measure of noncompactness in $L^1[0,T]$ given by (1.1). This means that the operator N is a contraction with respect to μ. Finally, since

$$\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 T^\alpha}{\Gamma(\alpha+1)} < 1,$$

then by applying Darbo's fixed point theorem, we conclude that IVP (5.1) – (5.2) has at least one solution belonging to the set $B_r \subset L^1(J, \mathbb{R})$. □

5.3 Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional initial value problem,

$$^cD^\alpha y(t) = \frac{t(1+|y(t)|+|^cD^\alpha y(t)|)}{(t+5)}, \ t \in J := [0,1], \ \alpha \in (0,1], \tag{5.7}$$

$$y(0) = y_0. \tag{5.8}$$

Set

$$f(t,y,z) = \frac{t(1+y+z)}{(t+5)}, \ (t,y,z) \in J \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Clearly, the function f satisfies the Carathéodory conditions. Let $y, z \in \mathbb{R}_+$ and $t \in J$, then we have

$$\begin{aligned} |f(t,y,z)| &= \left|\frac{t}{t+5} + \frac{ty}{t+5} + \frac{tz}{t+5}\right| \\ &\leq \left|\frac{t}{t+5}\right| + \left|\frac{ty}{t+5}\right| + \left|\frac{tz}{t+5}\right| \\ &\leq \left|\frac{t}{t+5}\right| + \frac{1}{6}|y| + \frac{1}{6}|z|. \end{aligned}$$

We first show that $a \in L^1[0,1]$, where $a(t) = \frac{t}{t+5}$, indeed $a(t)$ is a measurable function and

$$\begin{aligned} \int_0^1 a(t)dt &= \int_0^1 \frac{t}{t+5}dt \\ &= [t - 5ln|t+5|)]_0^1 \\ &= 1 - 5ln6 + 5ln5 < \infty. \end{aligned}$$

Then $a \in L^1[0,1]$. Hence the assumption **(H2)** holds with $a(t) = \frac{t}{t+5}$ and $q_1 = q_2 = \frac{1}{6}$. Moreover, for each $t \in [0,1], \ x_i, y_i \in \mathbb{R}_+, \ i = 1,2.$ we have

$$\begin{aligned} & |f(t+\rho, y_1, z_1) - f(t, y_2, z_2)| \\ = & \left|\frac{t+\rho}{t+\rho+5} - \frac{t}{t+5} + \frac{(t+\rho)y_1}{t+\rho+5} - \frac{ty_2}{t+5} + \frac{(t+\rho)z_1}{t+\rho+5} - \frac{tz_2}{t+5}\right| \\ \leq & \frac{\rho}{\rho+5} + \left|\frac{(t+\rho)y_1}{t+\rho+5} - \frac{ty_2}{t+5}\right| + \left|\frac{(t+\rho)z_1}{t+\rho+5} - \frac{tz_2}{t+5}\right| \\ \leq & L_f(\rho) + \left|\frac{(t+\rho)y_1}{t+\rho+5} - \frac{ty_2}{t+5}\right| + \left|\frac{(t+\rho)z_1}{t+\rho+5} - \frac{tz_2}{t+5}\right|, \end{aligned}$$

where $L_f(\rho) = \frac{\rho}{\rho+5}$.
As $\delta \to 0$, then we have

$$|f(t+\rho, y_1, z_1) - f(t, y_2, z_2)| \leq L_f(\rho) + \frac{1}{6}|y_1 - y_2| + \frac{1}{6}|z_1 - z_2|$$

Then the assumption **(H3)** holds with $L_f(\rho) = \frac{\rho}{\rho+5}$ and $k_1 = k_2 = \frac{1}{6}$.
Finally we shall check that condition (5.5) is satisfied for appropriate values of $\alpha \in (0,1]$ with $T = 1$. Indeed

$$\frac{k_1}{\Gamma(2\alpha+1)} + \frac{k_2}{\Gamma(\alpha+1)} < 1 \Leftrightarrow \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(2\alpha+1)} < 6 \tag{5.9}$$

Then by Theorem 5.1 the problem $(5.7) - (5.8)$ has at least one solution on $[0,1]$ for values of α satisfying condition (5.9). For example

- If $\alpha = \frac{1}{2}$ then $\Gamma(\alpha+1) = \Gamma(\frac{3}{2}) \simeq 0,88$ and $\Gamma(2\alpha+1) = \Gamma(2) = 1$ and

$$\frac{k_1}{\Gamma(2\alpha+1)} + \frac{k_2}{\Gamma(\alpha+1)} = \frac{1}{6} + \frac{\frac{1}{6}}{0,88} \simeq 0,35659 < 1.$$

Chapitre 6

Integrable Solutions for Implicit Fractional Order Differential Inclusions [(5)]

6.1 Introduction

In this chapter we deal with the existence of solutions for initial value problem (IVP for short), for fractional order implicit differential inclusions

$$^{c}D^{\alpha}y(t) \in F(t, y(t), {}^{c}D^{\alpha}y(t)), \ a.e. \ t \in J := [0, T], \ 0 < \alpha \leq 1, \tag{6.1}$$

$$y(0) = y_0, \tag{6.2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact values ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), $y_0 \in \mathbb{R}$.

Differential equations and inclusions of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed we can find numerous applications in viscoelasticity, electrochemistry, electromagnetism, and so forth. For details, including some applications and recent results, see the monographs of Kilbas *et al.* [80], Kiryakova [82], Miller and Ross [95], Podlubny [103] and Samko *et al.* [105] , and the papers of Agarwal *et al* [8, 9], Diethelm *et al.* [53], El-Sayed [57, 58], Gaul *et al.* [60], Glockle and Nonnenmacher [63], Lakshmikantham and Devi [85], Mainardi [92], Metzler *et al.* [93], Momani *et al.* [98, 97], Podlubny *et al.*[103], Yu and Gao [118] and the references therein.

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(5) M. Benchohra and **M. S. Souid**, Integrable Solutions for Implicit Fractional Order Differential Inclusions, (submitted).

This chapter is organized as follows. In Section **6.2**, we present an existence result for the problem (6.1) – (6.2) when the right hand side is convex valued by using fixed point theorem of Bohnnenblust-Karlin type. In Section **6.3**, our results are given for nonconvex valued right hand sides, which are based upon a fixed point theorem for contraction multivalued maps due to Covitz and Nadler [48]. An example is given in Section **6.4** to demonstrate the application of our main results. These results can be considered as a contribution to this emerging field.

By $S^1_{F,y}$ we denote the set of all measurable selections of F that belong to the Lebesgue space $L^1(J,\mathbb{R})$, that is,

$$S^1_{F,y} = \{f \in L^1(J,\mathbb{R}) : \ f(t) \in F(t, y(t), {}^cD^\alpha y(t)) \ a.e. \ t \in J\}.$$

Remark 6.1 *Note that for an L^1-Carathéodory multifunction $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ the set $S^1_{F,y}$ is not empty.*

6.2 The Convex Case

In this section, we are concerned with the existence of solutions for the problem (6.1) – (6.2) when the right hand side has convex values. Let us start by defining what we mean by an integrable solution of the problem (6.1) – (6.2).

Definition 6.1 *A function $y \in L^1(J,\mathbb{R})$ such that ${}^cD^\alpha y(t)$ is measurable is said to be a solution of IVP* (6.1) – (6.2) *if y satisfies (6.1) and (6.2).*

For the existence of solutions for the problem (6.1) – (6.2), we need the following auxiliary lemma.

Lemma 6.1 *The solution of the IVP* (6.1) – (6.2) *can be expressed by the integral equation*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \tag{6.3}$$

where x is the solution of the functional integral inclusion

$$x(t) \in F(t, y(t), x(t)). \tag{6.4}$$

Proof. Let ${}^cD^\alpha y(t) = x(t)$ in equation (6.1), then

$$x(t) \in F(t, y(t), x(t))$$

and

$$\begin{aligned} y(t) &= y(0) + I^\alpha x(t) \\ &= y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds. \end{aligned}$$

Let us introduce the following assumptions :

(H1) $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cv,cl}(\mathbb{R})$ is L^1-Carathéodory.

(H2) There exist a positive function $a \in L^1(J)$ and constants, $b_i > 0$; $i = 1, 2$ such that :

$$\|F(t, u_1, u_2)\|_{\mathcal{P}} = sup\{|f| : f \in F(t, u_1, u_2)\} \leq a(t) + b_1|u_1| + b_2|u_2|,$$

for all $u_1,\ u_2 \in \mathbb{R}$, and for a.e. $t \in J$.

(H3) There exist constants $\ell_1,\ \ell_2 > 0$ such that

$$H_d(F(t, x, z)), F(t, \overline{x}, \overline{z})) \leq \ell_1|x - \overline{x}| + \ell_2|z - \overline{z}|\ ,$$

for every $x, \overline{x}, z, \overline{z} \in \mathbb{R}$.

(H4) $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ has the property that $F(., u_1, u_2) : J \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable, and integrably bounded for each $u_1, u_2 \in \mathbb{R}$.

Our first result is based of Bohnenblust-Karlin fixed point theorem.

Theorem 6.1 *Assume that the assumptions* $(H1) - (H3)$ *are satisfied. If*

$$\frac{b_2 T^\alpha}{\Gamma(\alpha + 1)} + \frac{b_1 T^{2\alpha}}{\Gamma(2\alpha + 1)} < 1, \tag{6.5}$$

then the IVP $(6.1) - (6.2)$ *has at least one solution* $y \in L^1(J, \mathbb{R})$.

Remark 6.2 *Note that for an L^1-Carathéodory multifunction $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ the set $S^1_{F,y}$ is not empty.*

Proof. Transform the problem $(6.1) - (6.2)$ into a fixed point problem. Consider the the multivalued operator

$$N : L^1(J, \mathbb{R}) \longrightarrow \mathcal{P}(L^1(J, \mathbb{R}))$$

defined by :

$$(Nx)(t) = \left\{ h \in L^1(J, \mathbb{R}) : h(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) ds \right\} \tag{6.6}$$

where $f \in S^1_{F,x}$. Clearly, from Lemma 1.4, the fixed points of N are solutions to $(6.1) - (6.2)$. We shall show that N satisfies the assumptions of Bohnenblust-Karlin fixed point theorem.

Let

$$r \geq \frac{T|y_0| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|a\|_{L_1} + \frac{b_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)}}{1 - \left(\frac{b_2 T^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 T^{2\alpha}}{\Gamma(2\alpha+1)} \right)},$$

and consider the bounded set

$$B_r := \{x \in L^1(J, \mathbb{R}), \|x\|_{L^1} \leq r\}.$$

The proof will be given in several steps.

Step 1 : $N(x)$ is convex for each $y \in B_r$.
Indeed, if h_1, h_2 belong to $N(x)$, then there exist $f_1, f_2 \in S^1_{F,y}$ such that for each $t \in J$ we have

$$h_i(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_i(s)ds, \ i = 1, 2.$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$, we have

$$(dh_1 + (1-d)h_2)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[df_1(s) + (1-d)f_2(s)]ds.$$

Since $S^1_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(x).$$

Step 2 : $N(B_r)$ is relatively compact.
(a) $N(B_r)$ is bounded.
Let $x \in B_r$ for each $h \in N(x)$ and $t \in J$, we have by **(H2)** and (6.6)

$$\begin{aligned}
\|h\|_{L^1} &= \int_0^T |h(t)|dt \\
&= \int_0^T |y_0 + I^\alpha f(t))|dt \\
&\leq T|y_0| + \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s))|ds \right) dt \\
&\leq T|y_0| + \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [a(s) + b_1|y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds| + b_2|x(s)|]ds \right) dt \\
&\leq T|y_0| + \frac{T^\alpha}{\Gamma(\alpha+1)} \|a\|_{L_1} + \frac{b_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)} \|x\|_{L_1} + \frac{b_1 T^{2\alpha}}{\Gamma(2\alpha+1)} \|x\|_{L_1} \\
&\leq T|y_0| + \frac{T^\alpha}{\Gamma(\alpha+1)} \|a\|_{L_1} + \frac{b_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)} + \left(\frac{b_2 T^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 T^{2\alpha}}{\Gamma(2\alpha+1)} \right) r \leq r.
\end{aligned}$$

Then the above inequalities show that

$$\|N(x)\| = sup\{\|h\|_{L^1} : \ h \in N(x)\} \leq r,$$

which shows that $N(B_r) \subset B_r$ and B_r is bounded, then $N(B_r)$ is bounded.
(b) $(Nx)_\tau \longrightarrow (Nx)$, in $L^1(J, \mathbb{R})$ for each $x \in B_r$.

Let $x \in B_r$ and $h \in N(x)$ then we have

$$\begin{aligned} & \|h_\tau - h\|_{L^1} \\ = & \int_0^T |h_\tau(t) - h(t)|dt \\ = & \int_0^T \left| \frac{1}{\tau} \int_t^{t+\tau} h(s)ds - h(t) \right| dt \\ \leq & \int_0^T \left(\frac{1}{\tau} \int_t^{t+\tau} |h(s) - h(t)|ds \right) dt \\ \leq & \int_0^T \left(\frac{1}{\tau} \int_t^{t+\tau} |I^\alpha f(s) - I^\alpha f(t)|ds \right) dt \\ \leq & \int_0^T \frac{1}{\tau} \int_t^{t+\tau} |I^\alpha f(s) - I^\alpha f(t)|dsdt. \end{aligned}$$

Since $f \in L^1(J, \mathbb{R})$ and by Proposition 1.1**(4)**, it follows that $I^\alpha f \in L^1(J, \mathbb{R})$, then we have

$$\frac{1}{\tau} \int_t^{t+\tau} |I^\alpha f(s) - I^\alpha f(t)|ds \to 0 \ as \ \tau \to 0, \ t \in J.$$

Hence

$$(Nx)_\tau \to (Nx) \ \text{ uniformly as } \ \tau \to 0.$$

As a consequence of **(a)** and **(b)** together with the Kolmogorov compactness compactness criterion, we can conclude that $N(B_r)$ is relatively compact.

Step 3 : N has a closed graph.
Let $x_n \to x_*$, $h_n \in N(x_n)$, and $h_n \to h_*$. We need to show that $h_* \in N(x_*)$. Now $h_n \in N(x_n)$ implies there exists $f_n \in S^1_{F,x_n}$ such that, for each $t \in J$,

$$h_n(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_n(s)ds.$$

We must show that there exists $f_*(t) \in S^1_{F,y_*}$ such that for each $t \in J$,

$$h_*(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_*(s)ds.$$

Since $F(t, ., .)$ is upper semicontinuous, for every $\epsilon > 0$, there exist $n_0(\epsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$f_n(t) \in F(t, y(t), x(t)) \subset F(t, y_*(t), x_*(t)) + \epsilon B(0, 1) \ a.e. \ t \in J.$$

Since F has compact values, there exists a subsequence $f_{n_m}(.)$ such that

$$f_{n_m}(.) \to f_* \ as \ m \to \infty$$

$$f_*(t) \in F(t, y_*(t), x_*(t)) \ a.e. \ t \in J;$$

For every $\omega(t) \in F(t, y_*(t), x_*(t))$, we have

$$|f_{n_m}(t) - f_*(t)| \leq |f_{n_m}(t) - \omega(t)| + |\omega(t) - f_*(t)|,$$

and so

$$|f_{n_m}(t) - f_*(t)| \leq d(f_{n_m}(t), F(t, y_*(t), x_*(t))).$$

By an analogous relation obtained by interchanging the roles of f_{n_m} and f_* it follows that

$$\begin{aligned} |f_{n_m}(t) - f_*(t)| &\leq H_d(F(t, y_{n_m}(t), x_{n_m}(t))), F(t, y_*(t), x_*(t))) \\ &\leq \ell_1 |y_{n_m} - y_*| + \ell_2 |x_{n_m} - x_*| \\ &\leq \ell_1 |I^\alpha (x_{n_m} - x_*)| + \ell_2 |x_{n_m} - x_*|. \end{aligned}$$

Therefore,

$$\begin{aligned} |h_{n_m}(t) - h_*(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f_{n_m}(s) - f_*(s)| ds \\ \|h_{n_m} - h_*\|_{L_1} &\leq \left(\frac{\ell_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\ell_2 T^\alpha}{\Gamma(\alpha+1)} \right) \|x_{n_m} - x_*\|_{L_1}. \end{aligned}$$

Then

$$\|h_{n_m} - h_*\|_{L_1} \to 0 \ as \ m \ \to \infty.$$

Therefore, we deduce from Bohnenblust-Karlin fixed point theorem that N has a fixed point x in $B_r \subset L^1(J, \mathbb{R})$ which is a solution of IVP (6.1) – (6.2). □

6.3 The Nonconvex Case

This section is devoted to proving the existence of solutions for (6.1) – (6.2) with a nonconvex valued right hand side. Our second result is based on the fixed point theorem for contraction multivalued maps given by Covitz-Nadler [48] ;

Theorem 6.2 *Assume that the assumptions* $(H3) - (H4)$ *are satisfied. If*

$$\frac{\ell_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\ell_2 T^\alpha}{\Gamma(\alpha+1)} < 1, \tag{6.7}$$

then the IVP (6.1) – (6.2) *has at least one solution* $y \in L^1(J, \mathbb{R})$.

For each $y \in L^1(J, \mathbb{R})$, the set $S^1_{F,y}$ is nonempty since, by **(H4)**, F has a measurable selection (see [46], Theorem III.6). **Proof**. We shall show that N given by (6.6) satisfies the assumptions of Covitz and Nadler fixed point theorem. The proof will be given in two steps.

Step 1 : $N(x) \in P_{cl}(L^1(J,\mathbb{R}))$ for all $x \in L^1(J,\mathbb{R})$. Let $(h_n)_{n\geq 0} \in N(x)$ be such that $h_n \to \widetilde{h} \in L^1(J,\mathbb{R})$. Then there exists $f_n \in S^1_{F,y}$ such that, for each $t \in J$,

$$h_n(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_n(s)ds.$$

From **(H3)** and the fact that F has compact values, we may pass to a subsequence if necessary to obtain that f_n converges to v in $L^1(J,\mathbb{R})$ and hence $f \in S^1_{F,y}$. Thus, for each $t \in J$,

$$h_n(t) \to \widetilde{h}(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds,$$

so $\widetilde{h} \in N(x)$.
Step 2 : There exists $\gamma < 1$ such that

$$H_d(N(x), N(\overline{x})) \leq \gamma \|x - \overline{x}\|_{L^1},\ \ for\ all\ x,\ \overline{x} \in L^1(J,\mathbb{R}).$$

Let $x,\ \overline{x} \in L^1(J,\mathbb{R})$ and $h_1 \in N(x)$. Then, there exists $f_1(t) \in F(t,y(t),x(t))$ such that, for each $t \in J$,

$$h_1(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_1(s)ds.$$

From **(H3)** it follows that

$$H_d(F(t,y(t),x(t))), F(t,\overline{y}(t),\overline{x}(t))) \leq \ell_1|y(t)-\overline{y}(t)| + \ell_2|x(t)-\overline{x}(t)|.$$

Hence, there exists $\omega(t) \in F(t,\overline{y}(t),\overline{x}(t))$ such that

$$|f_1(t) - \omega(t)| \leq \ell_1|y(t)-\overline{y}(t)| + \ell_2|x(t)-\overline{x}(t)|,\ t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{\omega \in \mathbb{R} : |f_1(t) - \omega(t)| \leq \ell_1|y(t)-\overline{y}(t)| + \ell_2|x(t)-\overline{x}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t,\overline{y}(t),\overline{x}(t))$ is measurable (see Proposition [46], III.4), there exists a function $f_2(t)$ which is a measurable selection for V. Thus, $f_2(t) \in F(t,\overline{y}(t),\overline{x}(t))$, and for each $t \in J$,

$$\begin{aligned} |f_1(t) - f_2(t)| &\leq \ell_1|y(t)-\overline{y}(t)| + \ell_2|x(t)-\overline{x}(t)| \\ &\leq \ell_1|I^\alpha(x(t)-\overline{x}(t))| + \ell_2|x(t)-\overline{x}(t)| \end{aligned}$$

For each $t \in J$, define

$$h_2(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_2(s)ds.$$

Then, for $t \in J$,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f_1(s) - f_2(s)| ds \\ &\leq \left(\frac{\ell_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\ell_2 T^{\alpha}}{\Gamma(\alpha+1)} \right) |x - \overline{x}|. \end{aligned}$$

Therefore

$$\|h_1 - h_2\|_{L^1} \leq \left(\frac{\ell_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\ell_2 T^{\alpha}}{\Gamma(\alpha+1)} \right) \|x - \overline{x}\|_{L^1}.$$

By an analogous relation, obtained by interchanging the roles of x and $\overline{x}$, it follows that

$$H_d(N(x), N(\overline{x})) \leq \left(\frac{\ell_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\ell_2 T^{\alpha}}{\Gamma(\alpha+1)} \right) \|x - \overline{x}\|_{L_1}.$$

Therefore, by (**??**), N is a contraction, and so by Covitz-Nadler fixed point theorem, N has a fixed point x that is a solution to IVP (6.1) – (6.2). The proof is now complete. □

6.4 Example

As an application of our results we consider the following fractional initial value problem,

$$^cD^{\alpha}y(t) \in F(t, y(t), ^cD^{\alpha}y(t)), \ t \in J := [0,1], \ \alpha \in (0,1], \tag{6.8}$$

$$y(0) = 1, \tag{6.9}$$

where

$$F(t, y(t), ^cD^{\alpha}y(t)) = \{v \in \mathbb{R} : f_1(t, y(t), ^cD^{\alpha}y(t)) \leq v \leq f_2(t, y(t), ^cD^{\alpha}y(t))\},$$

and $f_1, \ f_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are measurable in t. We assume that for each $t \in J, \ f_1(t, ., .)$ is lower semi-continuous (i.e, the set $\{y \in \mathbb{R} : f_1(t, y(t), ^cD^{\alpha}y(t)) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in J, \ f_2(t, ., .)$ is upper semi-continuous (i.e the set $\{y \in \mathbb{R} : f_2(t, y(t), ^cD^{\alpha}y(t)) < \mu\}$ is open for each $\mu \in \mathbb{R}$. Assume that there exists $a \in L^1(J, \mathbb{R}_+)$ such that

$$max(|f_1(t, y(t), x(t))|, |f_2(t, y, x(t))|) \leq \frac{t}{7} + \frac{1}{4}|y(t)| + \frac{1}{4}|x(t)|, \ t \in J.$$

We have $T = 1, \ a(t) = \frac{t}{7}, \ b_1 = b_2 = \frac{1}{4}$. It is easy to see that

$$\frac{b_2 T^{\alpha}}{\Gamma(\alpha+1)} + \frac{b_1 T^{2\alpha}}{\Gamma(2\alpha+1)} = \frac{1}{4\Gamma(\alpha+1)} + \frac{1}{4\Gamma(2\alpha+1)} < 1.$$

Then the condition (6.5) is satisfied for appropriate values of α. It is clear that F is compact and convex valued, and it is upper semi-continuous (see [51]). Since all conditions of Theorem 6.1 are satisfied, IVP (6.8) – (6.9) has at least one solution y on J.

Chapitre 7

L^1-Solutions of Boundary Value Problems for Implicit Fractional Order Differential Equations [(6)]

7.1 Introduction and Motivations

The purpose of this Chapter, is to establish existence and uniqueness of solutions integrable for boundary value problem (BVP for short), for fractional order implicit differential equation

$$^cD^\alpha y(t) = f(t, y(t), ^cD^\alpha y(t)), \ t \in J := [0, T], \ 1 < \alpha \leq 2, \tag{7.1}$$

$$y(0) = y_0, \ y(T) = y_T \tag{7.2}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_0, \ y_T \in \mathbb{R}$, and $^cD^\alpha$ is the Caputo fractional derivative.

More recently, considerable attention has been given to the existence of solutions of boundary value problem and boundary conditions for implicit fractional differential equations and integral equations with Caputo fractional derivative. See for example [10, 12, 13, 14, 21, 38, 35, 74, 85, 86, 87, 88, 109, 119], and the references therein.

In [98], S. Murad and S. Hadid, by means of Schauder fixed-point theorem and the Banach contraction principle, considered the boundary value problem of the fractional differential equation :

$$D^\alpha y(t) = f(t, y(t), D^\beta y(t)), \ t \in J := (0, 1), \ 1 < \alpha \leq 2, \ 0 < \beta < 1, \ 0 < \gamma \leq 1,$$

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(6) M. Benchohra and **M. S. Souid**, L^1-Solutions of Boundary Value Problems for Implicit Fractional Order Differential Equations, (to appear).

$$y(0) = 0, \ y(1) = I_0^{\gamma} y(s)$$

where $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and D^{α} is the Riemann-Liouville fractional derivative.
In [65], A. G-Lakoud and R. Khaldi, studied the following boundary value problem of the fractional integral boundary conditions :

$$^cD^q y(t) = f(t, y(t), ^cD^p y(t)), \ t \in J := (0,1), \ 1 < q \leq 2, \ 0 < p < 1,$$

$$y(0) = 0, \ y'(1) = \alpha I_0^p y(1),$$

where $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and D^{α} is the Caputo fractional derivative. In [109], by means of Schauder fixed-point theorem, Su and Liu studied the existence
of nonlinear fractional boundary value problem involving Caputo's derivative :

$$^cD^{\alpha} u(t) = f(t, u(t), ^cD^{\beta} u(t)), \ t \in J := (0,1), \ 1 < \alpha \leq 2, \ 0 < \beta < 1,$$

$$u(0) = 0 = u'(1) = 0 \ or \ u'(1) = u(1) = 0 \ or \ u(0) = u(1) = 0,$$

where $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.
In [35], Benchohra and Lazreg, studied the existence of continuous solutions for the problem (7.1)-(7.2), and the following implicit fractional-order differential equation :

$$^cD^{\alpha} y(t) = f(t, y(t), ^cD^{\alpha} y(t)), \ t \in J := [0,T], \ 0 < \alpha \leq 1,$$

with boundary condition

$$ay(0) = y_0 + By(T) = c$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $^cD^{\alpha}$ is the Caputo fractional derivative and a, b, c are real constants with $a + b \neq 0$.

This chapter is organized as follows. In Section **7.2**, we give two results, the first one is based on Schauder's fixed point theorem (Theorem 7.1) and the second one on the Banach contraction principle (Theorem 7.2). Some indications to nonlocal problems are given in Section **7.3**. Two examples is given in Section **7.4** to demonstrate the application of our main results. These results can be considered as a contribution to this emerging field..

7.2 Existence of solutions

Let us start by defining what we mean by an integrable solution of the problem $(7.1) - (7.2)$.

Definition 7.1 . *A function $y \in L^1(J, \mathbb{R})$ is said to be a solution of BVP* $(7.1) - (7.2)$ *if y satisfies (7.1) and (7.2).*

For the existence of solutions for the problem (7.1) – (7.2), we need the following auxiliary lemma.

Lemma 7.1 . *Let* $1 < \alpha \leq 2$ *and let* $x \in L^1(J, \mathbb{R})$. *The boundary value problem* (7.1) – (7.2) *is equivalent to the integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, \tag{7.3}$$

where x *is the solution of the functional integral equation*

$$x(t) = f\left(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t)\right). \tag{7.4}$$

and $G(t,s)$ *is the Green's function defined by*

$$G(t,s) := \begin{cases} (t-s)^{\alpha-1} - \frac{t(T-s)^{\alpha-1}}{T}, & 0 \leq s \leq t \leq T, \\ \frac{-t(T-s)^{\alpha-1}}{T}, & 0 \leq t \leq s \leq T, \end{cases} \tag{7.5}$$

Proof.Let ${}^cD^\alpha y(t) = x(t)$ in equation (7.1), then

$$x(t) = f(t, y(t), x(t)) \tag{7.6}$$

and lemma 1.2 implies that

$$y(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds.$$

From (7.2), a simple calculation gives

$$c_0 = y_0$$

and

$$c_1 = -\frac{1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1)} x(s)ds + \frac{(y_T - y_0)}{T}$$

Hence we get equation (7.3).
Inversely, we prove that equation (7.3) satisfies the BVP (7.1) – (7.2).
Differentiating (7.3), we get

$${}^cD^\alpha y(t) = x(t) = f(t, y(t), {}^c D^\alpha y(t)).$$

By (7.3) and (7.5) we have

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}. \tag{7.7}$$

A simple calculation give $y(0) = y_0$ and $y(T) = y_T$. This complete the proof of the equivalent between the BVP (7.1)-(7.2) and the integral equation (7.3).
Leu us introduce the following assumptions :

(H1) $f : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in [0,T]$, for any $(u_1,u_2) \in \mathbb{R}^2$ and continuous in $(u_1,u_2) \in \mathbb{R}^2$, for almost all $t \in [0,T]$.

(H2) There exist a positive function $a \in L^1[0,T]$ and constants, $b_i > 0; i = 1,2$ such that :

$$|f(t,u_1,u_2)| \leq a(t) + b_1|u_1| + b_2|u_2|, \forall (t,u_1,u_2) \in [0,T] \times \mathbb{R}^2.$$

Our first result is based on Schauder fixed point theorem.

Theorem 7.1 *Assume that the assumptions* $(H1)-(H2)$ *are satisfied. If*

$$\frac{b_1 G_0 T}{\Gamma(\alpha)} + b_2 < 1, \tag{7.8}$$

then the BVP $(7.1)-(7.2)$ *has at least one solution* $y \in L^1(J,\mathbb{R})$.

Proof. Transform the problem $(7.1)-(7.2)$ into a fixed point problem. Consider the operator

$$H : L^1(J,\mathbb{R}) \longrightarrow L^1(J,\mathbb{R})$$

defined by :

$$(Hx)(t) = f\left(t, \frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t)\right). \tag{7.9}$$

where G is given by 7.5. let

$$G_0 := \max[|G(t,s)|, (t,s) \in J \times J],$$

and

$$r = \frac{b_1(|y_0| + |y_T|)T + \|a\|_{L_1}}{1 - \left(\frac{b_1 G_0 T}{\Gamma(\alpha)} + b_2\right)}.$$

Consider the set

$$B_r = \{x \in L^1([0,T],\mathbb{R}) : \|x\|_{L_1} \leq r\}.$$

Clearly B_r is nonempty, bounded, convex and closed.

Now, we will show that $HB_r \subset B_r$, indeed, for each $x \in B_r$, from assumption **(H2)** and (7.8) we get

$$\begin{aligned}
\|Hx\|_{L_1} &= \int_0^T |Hx(t)|dt \\
&= \int_0^T \left| f\left(t, \frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t)\right)\right| dt \\
&\leq \int_0^T \left[|a(t)| + b_1\left|\frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)x(s)ds - (\frac{t}{T} - 1)y_0 + \frac{t}{T}y_T\right| + b_2|x(t)|\right] dt \\
&\leq \|a\|_{L_1} + \frac{b_1 G_0 T}{\Gamma(\alpha)}\|x\|_{L_1} + b_1(|y_0| + |y_T|)T + b_2\|x\|_{L_1} \\
&\leq b_1(|y_0| + |y_T|)T + \|a\|_{L_1} + \left(\frac{b_1 G_0 T}{\Gamma(\alpha)} + b_2\right) r \\
&\leq r.
\end{aligned}$$

Then $HB_r \subset B_r$. Assumption **(H1)** implies that H is continuous. Now, we will show that H is compact, this is HB_r is relatively compact. Clearly HB_r is bounded in $L^1(J,\mathbb{R})$, i.e condition **(i)** of Kolmogorov compactness criterion is satisfied. It remains to show $(Hx)_h \longrightarrow (Hx)$ in $L^1(J,\mathbb{R})$ for each $x \in B_r$.
Let $x \in B_r$, then we have

$$\begin{aligned}
&\|(Hx)_h - (Hx)\|_{L^1}\\
= &\int_0^T |(Hx)_h(t) - (Hx)(t)|dt\\
= &\int_0^T \left|\frac{1}{h}\int_t^{t+h}(Hx)(s)ds - (Hx)(t)\right| dt\\
\leq &\int_0^T \left(\frac{1}{h}\int_t^{t+h}|(Hx)(s) - (Hx)(t)|ds\right) dt\\
\leq &\int_0^T \Big(\frac{1}{h}\int_t^{t+h}\Big|f(s,\frac{1}{\Gamma(\alpha)}\int_0^T G(s,\tau)x(\tau)d\tau + y_0 + \frac{(y_T-y_0)s}{T}, x(s))\\
&-f(t,\frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T-y_0)t}{T}, x(t))\Big|ds\Big)dt
\end{aligned}$$

Since $x \in B_r \subset L^1(J,\mathbb{R})$ and assumption (**H2**) that implies $f \in L^1(J,\mathbb{R})$, then we have

$$\begin{aligned}
&\frac{1}{h}\int_t^{t+h}\Big|f(s,\frac{1}{\Gamma(\alpha)}\int_0^T G(s,\tau)x(\tau)d\tau + y_0 + \frac{(y_T-y_0)s}{T}, x(s))\\
-f(t,&\frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T-y_0)t}{T}, x(t))\Big|ds \longrightarrow 0, \ \ as\ h \longrightarrow 0, \ \ t \in J.
\end{aligned}$$

Hence

$$(Hx)_h \longrightarrow (Hx) \ \text{ uniformly as } \ h \longrightarrow 0.$$

Then by Kolmogorov compactness criterion, $H(B_r)$ is relatively compact. As a consequence of Schauder's fixed point theorem the BVP $(7.1)-(7.2)$ has at least one solution in B_r. □

The following result is based on the Banach contraction principle.

Theorem 7.2 *Assume that **(H1)** and the following condition hold.*

(H3) *There exist constants $k_1, k_2 > 0$ such that*

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq k_1|x_1 - x_2| + k_2|y_1 - y_2|, \ \ t \in [0,T], \ \ x_1,x_2,y_1,y_2 \in \mathbb{R}.$$

If

$$\frac{k_1 T G_0}{\Gamma(\alpha)} + k_2 < 1, \tag{7.10}$$

then the BVP $(7.1)-(7.2)$ has a unique solution $y \in L^1([0,T],\mathbb{R})$.

Proof. We shall use the Banach contraction principle to prove that H defined by (7.9) has a fixed point. Let $x, y \in L^1(J, \mathbb{R})$, and $t \in J$. Then we have,

$$\begin{aligned} |(Hx)(t) - (Hy)(t)| &= \Big| f\left(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t)\right) \\ &\quad - f\left(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)y(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, y(t)\right)\Big|. \\ &\leq \frac{k_1}{\Gamma(\alpha)} \int_0^T |G(t,s)(x(s) - y(s))|ds + k_2|x(t) - y(t)| \\ &\leq \frac{k_1 G_0}{\Gamma(\alpha)} \int_0^T |x(s) - y(s)|ds + k_2|x(t) - y(t)| \end{aligned}$$

Thus

$$\begin{aligned} \|(Hx) - (Hy)\|_{L_1} &\leq \frac{k_1 T G_0}{\Gamma(\alpha)} \|x - y\|_{L_1} + k_2 \int_0^T |x(t) - y(t)|dt \\ &\leq \frac{k_1 T G_0}{\Gamma(\alpha)} \|x - y\|_{L_1} + k_2 \|x - y\|_{L_1} \\ &\leq \left(\frac{k_1 T G_0}{\Gamma(\alpha)} + k_2\right) \|x - y\|_{L_1}. \end{aligned}$$

Consequently by (7.10) H is a contraction. As a consequence of the Banach contraction principle, we deduce that H has a fixed point which is a solution of the problem (7.1) – (7.2).

7.3 Nonlocal problems

This section is devoted to some existence and uniqueness results for the following class of nonlocal problems

$$^cD^\alpha y(t) = f(t, y(t), ^c D^\alpha y(t)),\ t \in J := [0, T],\ 1 < \alpha \leq 2, \tag{7.11}$$

$$y(0) = g(y),\ y(T) = y_T \tag{7.12}$$

where $g : L^1(J, \mathbb{R}) \to \mathbb{R}$ a continuous function. The nonlocal condition can be applied in physics with better effect than the classical initial condition $y(0) = y_0$. For example, $g(y)$ may be given by

$$g(y) = \sum_{i=1}^{p} c_i y(t_i).$$

where $c_i, i = 1, 2, ..., p$ are given constants and $0 < ... < t_p < T$. Nonlocal conditions were initiated by Byszewski [43] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [44, 45],

the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.
Let us introduce the following set of conditions on the function g.
(H4) There exists a constant $\tilde{k} > 0$ such that

$$|g(y) - g(\tilde{y}| \leq \tilde{k}|y - \tilde{y}|, \ \textit{for each } y, \tilde{y} \in L^1(J, \mathbb{R}).$$

Theorem 7.3 *Assume that the assumptions **(H1)**,**(H3)**,**(H4)** are satisfied. If*

$$\frac{2k_1 T^\alpha}{\Gamma(\alpha+1)} + k_1\tilde{k} + k_2 < 1, \tag{7.13}$$

then the BVP (7.11) – (7.12) *has a unique solution* $y \in L^1(J, \mathbb{R})$.

Transform the problem (7.11) – (7.12) into a fixed point problem. Consider the operator

$$\tilde{H} : L^1(J, \mathbb{R}) \longrightarrow L^1(J, \mathbb{R})$$

defined by :

$$\begin{aligned} & (\tilde{H}x)(t) \qquad (7.14) \\ = \ & f\left(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}x(s)ds - (\frac{t}{T}-1)g(y) + \frac{t}{T}y_T, x(t)\right). \end{aligned}$$

Proof. We shall use the Banach contraction principle to prove that $\tilde{H}$ defined by (7.14) has a fixed point. Let $x, y \in L^1(J, \mathbb{R})$, and $t \in J$. Then we have,

$$\begin{aligned} & |(\tilde{H}x)(t) - (\tilde{H}y)(t)| \\ = \ & \left| f\left(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}x(s)ds - (\frac{t}{T}-1)g(x) + \frac{t}{T}y_T, x(t)\right)\right. \\ - \ & \left. f\left(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}y(s)ds - (\frac{t}{T}-1)g(y) + \frac{t}{T}y_T, y(t)\right)\right| \\ \leq \ & \frac{k_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|(x(s)-y(s))|ds + \frac{k_1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|(x(s)-y(s))|ds \\ & +k_1|g(x)-g(y)| + k_2|x(t)-y(t)| \end{aligned}$$

Thus

$$\begin{aligned} \|(\tilde{H}x) - (\tilde{H}y)\|_{L_1} & \leq \frac{k_1\|x-y\|_{L_1}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \frac{k_1\|x-y\|_{L_1}}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds \\ & \quad +k_1\tilde{k}\|x-y\|_{L_1} + k_2\|x-y\|_{L_1} \\ & \leq \frac{2k_1T^\alpha}{\Gamma(\alpha+1)}\|x-y\|_{L_1} + k_1\tilde{k}\|x-y\|_{L_1} + k_2\|x-y\|_{L_1} \\ & \leq \left(\frac{2k_1T^\alpha}{\Gamma(\alpha+1)} + k_1\tilde{k} + k_2\right)\|x-y\|_{L_1}. \end{aligned}$$

Consequently by (7.13) $\tilde{H}$ is a contraction. As a consequence of the Banach contraction principle, we deduce that $\tilde{H}$ has a fixed point which is a solution of the problem (7.11) – (7.12). □

7.4 Examples

Example 7.1 *Let us consider the following boundary value problem,*

$$^cD^\alpha y(t) = \frac{e^{-t}}{(e^t+6)(1+|y(t)|+|^cD^\alpha y(t)|)}, \ t \in J := [0,1], \ 1 < \alpha \leq 2, \tag{7.15}$$

$$y(0) = 1, \ y(1) = 2. \tag{7.16}$$

Set

$$f(t,y,z) = \frac{e^{-t}}{(e^t+6)(1+y+z)}, \ (t,y,z) \in J \times [0,+\infty) \times [0,+\infty).$$

Let $y_1, y_2, z_1, z_2 \in [0,+\infty)$ *and* $t \in J$. *Then we have*

$$\begin{aligned} |f(t,y_1,z_1) - f(t,y_2,z_2)| &= \left| \frac{e^{-t}}{e^t+6} \left(\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2} \right) \right| \\ &\leq \frac{e^{-t}(|y_1-y_2|+|z_1-z_2|)}{(e^t+6)(1+y_1+z_1)(1+y_2+z_2)} \\ &\leq \frac{e^{-t}}{(e^t+6)}(|y_1-y_2|+|z_1-z_2|) \\ &\leq \frac{1}{7}|y_1-y_2| + \frac{1}{7}|z_1-z_2|. \end{aligned}$$

*Hence the condition **(H3)** holds with* $k_1 = k_2 = \frac{1}{7}$. *We shall check that condition* (7.10) *is satisfied with* $T = 1$. *Indeed*

$$\frac{k_1 T G_0}{\Gamma(\alpha)} + k_2 = \frac{G_0}{7\Gamma(\alpha)} + \frac{1}{7} < 1. \tag{7.17}$$

Then by Theorem 7.2, the problem (7.15) − (7.16) *has a unique integrable solution on* $[0,1]$ *for values of* α *satisfying condition* (7.17).

Example 7.2 *Let us consider the following nonlocal boundary value problem,*

$$^cD^\alpha y(t) = \frac{e^{-t}}{(e^t+9)(1+|y(t)|+|^cD^\alpha y(t)|)}, \ t \in J := [0,1], \ 1 < \alpha \leq 2, \tag{7.18}$$

$$y(0) = \sum_{i=1}^{n} c_i y(t_i), \ y(1) = 0. \tag{7.19}$$

where $0 < ... < t_n < 1, \ c_i, i = 1,2,...,n$ *are given positive constants with* $\sum_{i=1}^n c_i < \frac{4}{5}$.
Set

$$f(t,y,z) = \frac{e^{-t}}{(e^t+9)(1+y+z)}, \ (t,y,z) \in J \times [0,+\infty) \times [0,+\infty),$$

and

$$g(y) = \sum_{i=1}^{n} c_i y(t_i).$$

Let $y_1, y_2, z_1, z_2 \in [0, +\infty)$ *and* $t \in J$. *Then we have*

$$\begin{aligned}
|f(t,y_1,z_1) - f(t,y_2,z_2)| &= \left| \frac{e^{-t}}{e^t+9}\left(\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2}\right)\right| \\
&\leq \frac{e^{-t}(|y_1-y_2| + |z_1-z_2|)}{(e^t+9)(1+y_1+z_1)(1+y_2+z_2)} \\
&\leq \frac{e^{-t}}{(e^t+9)}(|y_1-y_2| + |z_1-z_2|) \\
&\leq \frac{1}{10}|y_1-y_2| + \frac{1}{10}|z_1-z_2|.
\end{aligned}$$

Hence the condition ***(H5)*** *holds with* $k_1 = k_2 = \frac{1}{10}$. *Also we have*

$$|g(x) - g(y)| \leq \sum_{i=1}^{n} c_i |x-y|.$$

Hence ***(H4)*** *is satisfied with* $\tilde{k} = \sum_{i=1}^{n} c_i$. *We shall check that condition* (7.13) *is satisfied with* $T = 1$. *Indeed*

$$\frac{2k_1T^\alpha}{\Gamma(\alpha+1)} + k_1\tilde{k} + k_2 = \frac{1}{5\Gamma(\alpha+1)} + \frac{1}{10}\sum_{i=1}^{n} c_i + \frac{1}{10} < 1 \Longleftrightarrow \Gamma(\alpha+1) > \frac{10}{41}. \quad (7.20)$$

Then by Theorem 7.3, the problem (7.18) – (7.19) *has a unique integrable solution on* $[0,1]$ *for values of* α *satisfying condition* (7.20).

Conclusion and Perspectives

In this thesis, we have considered the following nonlinear implicit fractional differential (NIFD for short) problem

$$^cD^\alpha y(t) = f(t, y, ^c D^\alpha y(t)), \ t \in J, \ 0 < \alpha \leq 1, \ or \ 1 < \alpha \leq 2$$

with initial value, local and nonlocal conditions, boundary value problems and with infinite delay. Here $^cD^\alpha$ is the Caputo fractional derivative. Also We have discussed and established the existence of integrable solutions for initial value problem for implicit fractional order differential inclusion.
We plan to study the stability problems, for nonlinear implicit fractional differential equations with Caputo fractional derivative. We will study the controlability problem in Frechet space. Also, We will study the existence and uniqueness of integrable solutions for a class of boundary value problem for nonlinear implicit fractional differential equations with Caputo fractional derivative and with integral conditions.

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