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Contents

Introduction

This thesis is devoted to the study of global existence, asymptotic behavior in time of solutions to a Petrovsky system, a system of degenerate Kirchhoff equations and nonlinear vector equation in finite dimensional Hilbert space.
The decreasing of classical energy plays a crucial role in the study of global existence and in stabilization of various distributed systems.
This work consists in three chapter:

- In the chapter one, we prove the global existence and study the decay solutions to a Petrovsky system with a delay term and source term.
- In chapter two, we give decay estimate for a System of degenerate Kirchhoff equation with weakly nonlinear dissipation.
- In chapter three, we establish a general decay result of solution to some nonlinear vector equation in a finite dimensional space.

To prove this different results,we use some classical methods:
The Faedo-Galerkin method to prove the global existence.
The integral inequalities of A. GUESMIA to estimate the decay rate of the energy of some dissipative problems, the multipier method and makes use of some properties of convex funtions.
The purpose of stabilization is to attenuate the vibrations by feedback, it consists to guarantee the decay of the energy of solutions towards 0 in away, more or less fast.
More precisely, we are interested to determine the asymptotic behavior of the energy denoted by $E(t)$ and to give an estimate of the decay rate of the energy.
There are several type of stabilization
1)Strong stabilization : $E(t) \longrightarrow 0$ as $t \longrightarrow \infty$.
2)Logarithmic stabilization : $E(t) \leq c(\log(t))^{-\delta},\ c, \delta > 0$.
3)Polynomial stabilization : $E(t) \leq ct^{-\delta};\ c, \delta > 0$.
4)Uniform stabilization : $E(t) \leq ce^{-\delta t};\ c, \delta > 0$.

Chapter 1:Global existence and energy decay of solutions to a petrovsky system with a delay term and source term.

We consider the Petrovsky equation in bounded domain with a delay term and source term in the internal feedback

(P)

$$\begin{cases} u''(x,t) + \Delta_x^2 u(x,t) + \mu_1 g(u'(x,t)) + \mu_2 g(u'(x,t-\tau)) = bu|u|^{p-2} & \text{in } \Omega\times]0,+\infty[, \\ u = \partial_\nu u = 0 & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) & \text{in } \Omega, \\ u'(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau[, \end{cases}$$

We prove the global existence of its solutions in Sobolev spaces by means of the energy method combined with the Faedo-Galerkin procedure under a condition between the weight of the delay term in the feedback and the weight of the term without delay. Furthermore, we study the asymptotic behavior of solutions using multiplier method and general weighted integral inequalities.

Chapter 2: Energy decay for a system of degenerate Kirchhoff equations with weakly nonlinear dissipation

In this chapter, we consider the initial boundary value problem for the nonlinear Kirchhoff equation

$$(1) \quad (|u_t|^{l-2}u_t)' - \Big(\int_\Omega |\nabla_x u|^2\,dx + \int_\Omega |\nabla_x v|^2\,dx\Big)^\gamma \Delta_x u + \alpha(t) g_1(u_t) = 0, \ \text{ in } \Omega\times(0,\infty),$$

$$(2) \quad (|v_t|^{l'-2}v_t)' - \Big(\int_\Omega |\nabla_x u|^2\,dx + \int_\Omega |\nabla_x v|^2\,dx\Big)^\gamma \Delta_x v + \alpha(t) g_2(v_t) = 0, \ \text{ in } \Omega\times(0,\infty),$$

$$(3) \qquad u = 0, v = 0 \quad \text{on} \quad \partial\Omega\times(0,\infty)$$

$$(4) \qquad u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \ \ x\in\Omega.$$

$$(5) \qquad v(x,0) = v^0(x), \quad v_t(x,0) = v^1(x), \ \ x\in\Omega.$$

In this chapter we use some technique from [**48**], we establish an explicit and general decay result, depending on g and α. The proof is based on the multiplier method and makes use of some properties of convex functions including the use of the general Young's inequality and Jensen's inequality. These convexity arguments were introduced and developed by Lasieka and co-workers ([**33**], [**36**], [**37**]) and used by Liu and Zuazua [**41**] and Alabau-Boussouira [**4**].

Chapter 3: General decay of solution to some nonlinear vector equation in a finite dimensional Hilbert space

Let H be a finite dimensional real Hilbert space, with norm denoted by $\|.\|$. We consider first the following nonlinear equation

$$u'' + \phi(\|A^{\frac{1}{2}}u\|^2)Au + g(u') = 0,$$

we use some technique from to establish an explicit and general decay result, depending on g and ϕ. The proof is based on the multiplier method and makes use of some properties of convex functions, the general Young inequality and Jensen's inequality.

Preliminaries

0.1 Sobolev spaces

We denote by Ω an open domain in $\mathbb{R}^n, n \geq 1$, with a smooth boundary $\Gamma = \partial\Omega$. In general, some regularity of Ω will be assumed. We will suppose that either

$$\Omega \text{ is Lipschitz,}$$

i.e., the boundary Γ is locally the graph of a Lipschitz function, or

$$\Omega \text{ is of class } \mathcal{C}^r, r \geq 1,$$

i.e., the boundary Γ is a manifold of dimension $n \geq 1$ of class $\mathcal{C}^r$. In both cases we assume that Ω is totally on one side of Γ. These definitions mean that locally the domain Ω is below the graph of some function ψ, the boundary Γ is represented by the graph of ψ and its regularity is determined by that of the function ψ. Moreover, it is necessary to note that a domain with a continuous boundary is never on both sides of its boundary at any point of this boundary and that a Lipschitz boundary has almost everywhere a unit normal vector ν.

We will also use the following multi-index notation for partial differential derivatives of a function:

$$\partial_i^k u = \frac{\partial^k u}{\partial x_i^k} \text{ for all } k \in \mathbb{N} \text{ and } i = 1, ..., n,$$
$$D^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} u = \frac{\partial^{\alpha_1 + \ldots + \alpha_n} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}},$$
$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + \ldots + \alpha_n.$$

We denote by $\mathcal{C}(D)$ (respectively $\mathcal{C}^k(D), k \in \mathbb{N}$ or $k = +\infty$) the space of real continuous functions on D (respectively the space of k times continuously differentiable functions on D), where D plays the role of Ω or its closure $\overline{\Omega}$. The space of real $\mathcal{C}^\infty$ functions on Ω with a compact support in Ω is denoted by $\mathcal{C}_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$ as in the distributions theory of Schwartz.The distributions space on Ω is denoted by $\mathcal{D}'(\Omega)$, i.e., the space of continuous linear form over $\mathcal{D}(\Omega)$.

For $1 \leq p \leq \infty$, we call $L^p(\Omega)$ the space of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

$$\|f\|_{L^\infty(\Omega)} = \sup_\Omega |f(x)| < +\infty \quad \text{for} \quad p = +\infty$$

The space $L^p(\Omega)$ equipped with the norm $f \longrightarrow \|f\|_{L^p}$ is a Banach space: it is reflexive and separable for $1 < p < \infty$ (its dual is $L^{\frac{p}{p-1}}(\Omega)$), separable but not reflexive for $p = 1$ (its dual is $L^\infty(\Omega)$), and not separable, not reflexive for $p = \infty$ (its dual contains strictly $L^1(\Omega)$). In particular the space $L^2(\Omega)$ is a Hilbert space equipped with the scalar product defined by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

We denote by $L^p_{loc}(\Omega)$ the space of functions which are L^p on any bounded sub-domain of Ω.

Similar space can be defined on any open set other than Ω, in particular, on the cylinder set $\Omega \times]a, b[$ or on the set $\Gamma \times]a, b[$, where $a, b \in \mathbb{R}$ and $a < b$.

Let U be a Banach space, $1 < p < +\infty$ and $-\infty \leq a < b \leq +\infty$, then $L^p(a, b; U)$ is the space of L^p functions f from (a, b) into U which is a Banach space for the norm

$$\|f\|_{L^p(a,b;U)} = \left(\int_a^b \|f(x)\|_U^p \, dt\right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

and for the norm

$$\|f\|_{L^\infty(a,b;U)} = \sup_{t\in(a,b)} \|f(x)\|_U < +\infty \quad \text{for} \quad p = +\infty$$

Similarly, for a Banach space $U, k \in \mathbb{N}$ and $-\infty < a < b < +\infty$, we denote by $C([a, b]; U)$ (respectively $C^k([a, b]; U)$) the space of continuous functions (respectively the space of k times continuously differentiable functions) f from $[a, b]$ into U, which are Banach spaces, respectively, for the norms

$$\|f\|_{\mathcal{C}(a,b;U)} = \sup_{t\in(a,b)} \|f(x)\|_U, \quad \|f\|_{\mathcal{C}^k(a,b;U)} = \sum_{i=0}^{k} \left\|\frac{\partial^i f}{\partial t^i}\right\|_{\mathcal{C}(a,b;U)}$$

0.1.1 Definition of Sobolev Spaces

Now, we will introduce the Sobolev spaces: The Sobolev space $W^{k,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order k have a finite L^p norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega). \ \forall\alpha; |\alpha| \leq k\} \ ,$$

With this definition, the Sobolev spaces admit a natural norm,

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^p(\Omega)}^p\right)^{1/p}, \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^\infty(\Omega)} \ , \text{ for } p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\| \,.\, \|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f,g)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)} \ .$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H_0^m(\Omega)$ (respectively $W_0^{k,p}(\Omega)$).

Now, we introduce a space of functions with values in a space X (a separable Hilbert space).

The space $L^2(a,b;X)$ is a Hilbert space for the inner product

$$(f,g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X \, dt$$

We note that $L^\infty(a,b;X) = (L^1(a,b;X))'$.

Now, we define the Sobolev spaces with values in a Hilbert space X

For $k \in \mathbb{N}$, $p \in [1,\infty]$, we set:

$$W^{k,p}(a,b;X) = \left\{ v \in L^p(a,b;X); \frac{\partial v}{\partial x_i} \in L^p(a,b;X). \ \ \forall i \leq k \right\} \ ,$$

The Sobolev space $W^{k,p}(a,b;X)$ is a Banach space with the norm

$$\begin{aligned} \|f\|_{W^{k,p}(a,b;X)} &= \left(\sum_{i=0}^{k} \left\| \frac{\partial f}{\partial x_i} \right\|_{L^p(a,b;X)}^p \right)^{1/p} , \text{ for } p < +\infty \\ \|f\|_{W^{k,\infty}(a,b;X)} &= \sum_{i=0}^{k} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^\infty(a,b;X)} , \qquad \text{for } p = +\infty \end{aligned}$$

The spaces $W^{k,2}(a,b;X)$ form a Hilbert space and it is noted $H^k(0,T;X)$. The $H^k(0,T;X)$ inner product is defined by:

$$(u,v)_{H^k(a,b;X)} = \sum_{i=0}^{k} \int_a^b \left(\frac{\partial u}{\partial x^i}, \frac{\partial v}{\partial x^i} \right)_X dt \ .$$

Theorem 0.1.1 *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

where p^* is given by $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$ (where $p^* = \infty$ if $p = n$). Moreover there exists a constant $C = C(p,n)$ such that

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)} \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corollary 0.1.1 *Let $1 \leq p < n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [p, p^*]$$

with continuous imbedding.

For the case $p \geq \dfrac{n}{n+1}$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [n, +\infty[$$

Theorem 0.1.2 *Let $p > n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

with continuous imbedding.

Corollary 0.1.2 *Let Ω be a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & 1 \leq p < \infty, \text{ then } W^{1,p}(\Omega) \subset L^{p^*}(\Omega) \text{ where } \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if & p = n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n, \text{ then } W^{1,p}(\Omega) \subset L^\infty(\Omega) \end{array}$$

with continuous imbedding.
Moreover, if $p > n$, we have: $\forall u \in W^{1,p}(\Omega)$,

$$|u(x) - u(y)| \leq C|x - y|^\alpha \|u\|_{W^{1,p}(\Omega)} \; a.e \; x, y \in \Omega$$

with $\alpha = 1 - \dfrac{n}{p} > 0$ and C is a constant which depend on p, n and Ω. In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Corollary 0.1.3 *Let Ω be a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & p < n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega) \forall q \in [1, p^*[\text{ where } \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if & p = n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n, \text{ then } W^{1,p}(\Omega) \subset C(\overline{\Omega}) \end{array}$$

with compact imbedding.

Remark 0.1.1 *We remark in particular that*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

with compact imbedding for $1 \leq p \leq \infty$ and for $p \leq q < p^*$.

Corollary 0.1.4

$$
\begin{aligned}
&if \quad \frac{1}{p}-\frac{m}{n}>0,\ then\ W^{m,p}(\mathbb{R}^n)\subset L^q(\mathbb{R}^n)\ where\ \frac{1}{q}=\frac{1}{p}-\frac{m}{n}.\\
&if \quad \frac{1}{p}-\frac{m}{n}=0,\ then\ W^{m,p}(\mathbb{R}^n)\subset L^q(\mathbb{R}^n),\forall q\in[p,+\infty[.\\
&if \quad \frac{1}{p}-\frac{m}{n}<0,\ then\ W^{m,p}(\mathbb{R}^n)\subset L^\infty(\mathbb{R}^n)
\end{aligned}
$$

with continuous imbedding.

0.2 Weak convergence

Let $(E; \|.\|_E)$ a Banach space and E' its dual space, i.e., the Banach space of all continuous linear forms on E endowed with the norm $\|.\|'_E$ defined by

$$\|f\|_{E'} =: \sup_{x\neq 0} \frac{|\langle f, x\rangle|}{\|x\|}$$

; where $\langle f, x\rangle$; denotes the action of f on$x$, i.e.$\langle f, x\rangle := f(x)$. In the same way, we can define the dual space of E' that we denote by E''. (The Banach space E'' is also called the bi-dual space of E.) An element x of E can be seen as a continuous linear form on E' by setting $x(f) := \langle x, f\rangle$, which means that $E \subset E''$:

Definition 0.2.1 *The Banach space E is said to be reflexive if $E = E''$.*

Definition 0.2.2 *The Banach space E is said to be separable if there exists a countable subset D of E which is dense in E, i.e. $\overline{D} = E$.*

Theorem 0.2.1 *(Riesz). If $(H; \langle ., .\rangle)$ is a Hilbert space, $\langle ., .\rangle$ being a scalar product on H, then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $x \in H$ such that $f = \langle x, .\rangle$ and $\|f\|'_H = \|x\|_H$*

Remark : From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.

Proposition 0.2.1 *If E is reflexive and if F is a closed vector subspace of E, then F is reflexive.*

Corollary 0.2.1 *The following two assertions are equivalent: (i) E is reflexive; (ii) E' is reflexive.*

0.2.1 Weak, weak star and strong convergence

Definition 0.2.3 *(Weak convergence in E). Let $x \in E$ and let $\{x_n\} \subset E$. We say that $\{x_n\}$ weakly converges to x in E, and we write $x_n \rightharpoonup x$ in E, if*

$$\langle f, x_n\rangle \to \langle f, x\rangle$$

for all $f \in E'$.

Definition 0.2.4 *(weak convergence in E'). Let $f \in E'$ and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly converges to f in E', and we write $f_n \rightharpoonup f$ in E', if*

$$\langle f_n, x\rangle \to \langle f, x\rangle$$

for all $x \in E''$.

Definition 0.2.5 *(weak star convergence). Let $f \in E'$ and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly star converges to f in E', and we write $f_n \rightharpoonup *f$ in E' if;*

$$\langle f_n, x\rangle \to \langle f, x\rangle$$

for all $x \in E$.

Remark As $E \subset E''$ we have $f_n \rightharpoonup f$ in E' imply $f_n \rightharpoonup *f$ in E'. When E is reflexive, the last definitions are the same, i.e, weak convergence in E' and weak star convergence coincide.

Definition 0.2.6 *(strong convergence). Let $x \in E$(resp. $f \in E'$) and let $\{x_n\} \subset E$ (resp $\{f_n\} \subset E'$). We say that $\{x_n\}$ (resp. $\{f_n\}$) strongly converges to x (resp. f), and we write $x_n \to x$ in E (resp. $f_n \to f$ in E'), if*

$$\lim_n \|x_n - x\|_E = 0; (resp.\ \lim_n \|f_n - f\|_{E'} = 0)$$

Proposition 0.2.2 *Let $x \in E$, let $\{x_n\} \subset E$, let $f \in E'$ and let $\{f_n\} \subset E'$.*

i. If $x_n \to x$ in E then $x_n \rightharpoonup x$ in E.

ii. If $x_n \rightharpoonup x$ in E then $\{x_n\}$ is bounded.

iii. If $x_n \rightharpoonup x$ in E then $\liminf_{n\to\infty} \|x_n\|_E \geq \|x\|_E$

iv. If $f_n \to f$ in E' then $f_n \rightharpoonup f$ inE' (and so $f_n \overset{*}{\rightharpoonup} f$ in E').

v. If $f_n \rightharpoonup f$ in E' then $\{f_n\}$ is bounded.

vi. If $f_n \rightharpoonup f$ in E' then $\liminf_{n\to\infty} \|f_n\|'_E \geq \|f\|'_E$

Proposition 0.2.3 *(finite dimension). If $\dim E < \infty$ then strong, weak and weak star convergence are equivalent.*

0.2.2 Weak and weak star compactness

In finite dimension, i.e, $\dim E < \infty$, we have Bolzano-Weierstrass's theorem (which is a strong compactness theorem).

Theorem 0.2.2 *(Bolzano-Weierstrass). If $\dim E < \infty$ and if $\{x_n\} \subset E$) is bounded, then there exist $x \in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ strongly converges to x.*

The following two theorems are generalizations, in infinite dimension, of Bolzano- Weierstrass's theorem.

Theorem 0.2.3 *(weak star compactness, Banach-Alaoglu-Bourbaki). Assume that E is separable and consider $\{f_n\} \subset E'$) . If $\{x_n\}$ is bounded, then there exist $f \in E'$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ weakly star converges to f in E'.*

Theorem 0.2.4 *(weak compactness, Kakutani-Eberlein). Assume that E is reflexive and consider $\{x_n\} \subset E$). If $\{x_n\}$ is bounded, then there exist $x \in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ weakly converges to x in E.*

Weak, weak star convergence and compactness in $L^p(\Omega)$.

Definition 0.2.7 *(weak convergence in $L^p(\Omega)$ with $1 \leq p < \infty$). Let Ω an open subset of $\mathbb{R}^n$ .We say that the sequence $\{f_n\}$ of $L^p(\Omega)$ weakly converges to $f \in L^p(\Omega)$, if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^q;\ (\frac{1}{p} + \frac{1}{q} = 1)$$

Definition 0.2.8 *(weak star convergence in $L^\infty(\Omega)$). We say that the sequence $\{f_n\} \subset L^\infty(\Omega)$ weakly star converges to $f \in L^\infty(\Omega)$, if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^1(\Omega)$$

Theorem 0.2.5 *(weak compactness in $L^p(\Omega)$) with $1 < p < \infty$. Given $\{f_n\} \subset L^p(\Omega)$, if $\{f_n\}$ is bounded, then there exist $f \in L^p(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \rightharpoonup f$ in $L^p(\Omega)$.*

Theorem 0.2.6 *(weak star compactness in $L^\infty(\Omega)$.
Given $\{f_n\} \subset L^\infty(\Omega)$, if $\{f_n\}$ is bounded, then there exist $f \in L^\infty(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$.*

Generalities. In what follows, Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and $1 \leq p \leq \infty$.

Weak and weak star convergence in Sobolev spaces

For $1 \leq p \leq \infty$, $W^{1;p}(\Omega)$ is a Banach space. Denote the space of all restrictions to Ω of C^1-differentiable functions from $\mathbb{R}^N$ to $\mathbb{R}$ with compact support in R^N by $C^1(\overline{\Omega})$.

Theorem 0.2.7 *for every $1 \leq p \leq \infty$ $C^1(\overline{\Omega}) \subset W^{1;p}(\Omega) \subset L^p(\Omega)$, and, for $1 < p < \infty$, $C^1(\overline{\Omega})$ is dense in $W^{1;p}(\Omega)$.*

Definition 0.2.9 *(weak convergence in $W^{1;p}(\Omega)$ with $1 \leq p < \infty$).)
We say the $\{f_n\} \subset W^{1;p}(\Omega)$ weakly converges to $f \in W^{1;p}(\Omega)$, and we write $f_n \rightharpoonup f$ in $W^{1;p}(\Omega)$, if $f_n \rightharpoonup f$ in $L^p(\Omega)$ and $\nabla f_n \rightharpoonup \nabla f$ in $L^p(\Omega;\ \mathbb{R}^N)$*

Definition 0.2.10 *(weak convergence in $W^{1;\infty}(\Omega)$
. We say the $\{f_n\} \subset W^{1;\infty}(\Omega)$ weakly star converges to $f \in W^{1;\infty}(\Omega)$, and we write $f_n \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$, if $f_n \overset{*}{\rightharpoonup} f$ in $L^p(\Omega)$ and $\nabla f_n \overset{*}{\rightharpoonup} \nabla f$ in $L^\infty(\Omega;\ \mathbb{R}^N)$*

Theorem 0.2.8 *(Rellich). Let $1 \leq p \leq \infty$, $\{f_n\} \subset W^{1;p}(\Omega)$ and $f \in W^{1;p}(\Omega)$; if $f_n \rightharpoonup f$ in $W^{1;p}(\Omega)$ when $1 \leq p < \infty$ (resp.$f_n \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$) when $p = \infty$) then $f_n \to f$ in $L^p(\Omega)$), which means that for every $1 \leq p \leq \infty$, the weak convergence in $W^{1;p}(\Omega)$ imply the strong convergence in $L^p(\Omega)$.*

Theorem 0.2.9 *Let $1 < p \leq \infty$ and let $\{f_n\} \subset W^{1;p}(\Omega)$. If $\{f_n\}$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \rightharpoonup f$ in $W^{1;p}(\Omega)$ when $1 < p < \infty$ (resp. $f_{n_k} \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$)*

As a consequence of this theorem we have

Corollary 0.2.2 *Let $1 < p \leq \infty$and let $\{f_n\} \subset W^{1;p}(\Omega)$. If $\{f_n\}$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ in $L^p(\Omega)$ and $\nabla f_{n_k} \rightharpoonup \nabla f$ in $L^p(\Omega)$ when $1 < p < \infty$ (resp.$\nabla f_{n_k} \overset{*}{\rightharpoonup} \nabla f$ in $L^\infty(\Omega)$)*

Theorem 0.2.10 . *If$N < p \leq \infty$ and if $\{f_n\} \subset W^{1;p}(\Omega)$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ converges uniformly to f, and $\nabla f_{n_k} \rightharpoonup \nabla f$ in $W^{1;p}(\Omega)$ when $N < p < \infty$ (resp. $\nabla f_{n_k} \overset{*}{\rightharpoonup} \nabla f$ in $W^{1;\infty}$)*

0.3 Fadeo-Galerkin method

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$.

$$(P) \qquad \begin{cases} u''(t) + A(t)u(t) = f(t), & t \in [0,T] \\ u(x,0) = u_0(x), \ u'(x,0) = u_1(x); \end{cases}$$

where u and f are unknown and given function, respectively, mapping the closed interval $[0,T] \subset \mathbb{R}$ into a real separable Hilbert space H ,$A(t)$ $(0 \leq t \leq T\ )$ are linear bounded operators in H acting in the energy space $V \subset H$.
Assume that $\langle A(t)u(t), v(t) \rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t; .,.)$ is a bilinear continuous in V.
The problem (P) can be formulated as: Found the solution $u(t)$ such that

$$(\widetilde{P}) \qquad \begin{cases} u \in C([0,T]; V), u' \in C([0,T]; H) \\ \langle u''(t), v \rangle + a(t; u(t), v) = \langle f, v \rangle \text{ in } D'(]0,T[) \\ u_0 \in V\ ,\ u_1 \in H; \end{cases}$$

This problem can be resolved with the approximation process of Fadeo-Galerkin.

0.3.1 General method

Let V_m a sub-space of V with the finite dimension d_m, and let $\{w_{jm}\}$ one basis of V_m.
we define the solution u_m of the approximate problem

$$(P_m) \qquad \begin{cases} u_m(t) = \sum_{j=1}^{d_m} g_j(t) w_{jm} \\ u_m \in C([0,T]; V_m), u'_m \in C([0,T]; V_m) \ \ , u_m \in L^2(0,T; V_m) \\ \langle u''_m(t), w_{jm} \rangle + a(t; u_m(t), w_{jm}) = \langle f, w_{jm} \rangle, \quad 1 \leq j \leq d_m \\ u_m(0) = \sum_{j=1}^{d_m} \xi_j(t) w_{jm}\ ,\ u'_m(0) = \sum_{j=1}^{d_m} \eta_j(t) w_{jm} \end{cases}$$

where

$$\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \longrightarrow u_0 \text{ in V as } m \longrightarrow \infty$$

$$\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \longrightarrow u_1 \text{ in V as } m \longrightarrow \infty$$

By virtue of the theory of ordinary differential equations,the system (P_m) has unique local solution which is extended to a maximal interval $[0, t_m[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$, to obtain one solution defined for all $t > 0$.

0.3.2 A priori estimation and convergence

Using the following estimation

$$\|u_m\|^2 + \|u'_m\|^2 \leq C(\|u_m(0)\|^2 + \|u'_m(0)\|^2 + \int_0^T |f(s)|^2 ds) \; ; \quad 0 \leq t \leq T$$

and the Gronwall lemma we deduce that the solution u_m of the approximate problem (P_m) converges to the solution u of the initial problem (P).The uniqueness proves that u is the solution.

0.3.3 Gronwall lemma

Lemma 0.3.1 *Let $T > 0$, $g \in L^1(0, T)$, $g \geq 0$ a.e and c_1, c_2 are positives constants.Let $\varphi \in L^1(0, T)$ $\varphi \geq 0$ a.e such that $g\varphi \in L^1(0, T)$ and*

$$\varphi(t) \leq c_1 + c_2 \int_0^t g(s)\varphi(s) ds \quad a.e \ in \ (0, T).$$

then, we have

$$\varphi(t) \leq c_1 exp \ (c_2 \int_0^t g(s) ds) \quad a.e \ in \ (0, T).$$

0.4 Convex analysis

0.4.1 Fenchel conjugate functions

Let V be a topological vector space and let V' be its dual space with bilinear duality form $\langle ., . \rangle_{V,V'}$.

Definition 0.4.1 *(Conjugate function)*
Let $F : V \longrightarrow \overline{\rm I\!R}$ be an extend real valued function. The function $F^* : V' \longrightarrow \overline{\rm I\!R}$ defined by

$$F^*(f) = \sup_{u \in V}(\langle f, u\rangle_{V,V'} - F(u)), \quad \forall f \in V'$$

is said to by Fenchel (convex) conjugate or conjugate function of F.
The mapping $F \longrightarrow F^*$ is called the Legendre -Fenchel transformation.

Proposition 0.4.1 *Let $F : V \longrightarrow \overline{\rm I\!R}$ be a given extend real valued function, the following statements are true*

- *i. $F^*(f) + F(u) \geq \langle f, u\rangle_{V,V'}, \quad \forall f \in V', \ \forall u \in V$*
- *ii. Let f be in the dual $V'$of V and $\lambda \in \ {\rm I\!R}$,the conjugate of affine function $u \longrightarrow (\langle f, u\rangle_{V,V'} - \lambda$ is less than F if and only if*

  $$F^*(f) \leq \lambda$$

- *iii.If F is identically equal to $+\infty$ then F^* is identically equal to $-\infty$ .Moreover ,if F is proper ,then the relation:$F^*(f) = \sup_{u \in V}(\langle f, u\rangle_{V,V'} - F(u))$ may be restricted to the points u in the effective domain of F $(dom(F))$.*
- *iv.The function F^* is always in $\Gamma(V')$ (since F^* is the point-wise supremum of a family of affine continuous functions of v').Therefore, F^* is always o lower semi-continuous convex function on V'. Moreover, if F^*takes the value $-\infty$ then F^* is identically equal to $-\infty$.*

Proposition 0.4.2 *(i)Let F and G be tow given extend real valued functions of V into $\overline{\rm I\!R}$, the following properties hold:*

1. *$F^*(0) = -(\inf_{u \in V})F(u)$.*
2. *If F is less than G then G^* is less than F^*.*
3. *If $G(u) = F(\alpha u), \ \forall u \in V, \ $ with $\ \alpha \neq 0$ then $G^*(f) = F^*(f/\alpha)$,$\forall f \in V'$.*
4. *$(\alpha F)^*(f) = \alpha F^*(f/\alpha), \ \forall \in V', \quad \forall \alpha > 0$.*
5. *$(F + \beta)^* = F^* - \beta, \quad \forall \beta \in \ {\rm I\!R}$.*

(ii)Given a family $(F_i)_{i \in J}$ of functions from V into $\overline{\rm I\!R}$, we have

$$(\inf_{i \in J} F_i)^* = \sup_{i \in J} F_i^*$$

$$\sup_{i \in J} F_i^* \leq \inf_{i \in J}(F_i)^*$$

(iii)For every $a \in V$ we denote by F_a the translated function (i.e,$F_a(u) = F(u - a), \forall u \in V$).Then ${F_a}^*(f) = F^*(f) + \langle f, a\rangle_{V,V'}, \quad \forall f \in V'$.

Theorem 0.4.1 *(Fenchel duality)Let V be a locally convex Hausdorff topological vector space over $\mathbb{R}$ with its dual V'. Let F and G be two power convex functions of V into $\overline{\mathbb{R}}$ .Assume that there exists $u_0 \in dom(F) \cap dom(G)$ such that F is continuous in u_0 .Then*

$$\inf_{u \in V}(F(u) + G(u)) = \sup_{f \in V'}(-F^*(-f) - G^*(f)).$$

Proof: From Fenchel inequality, we have for any function H

$$H^*(f) + H(u) \geq (\langle f, u\rangle_{V,V'}, \quad \forall u \in V \ , \ \forall f \in V'$$

consequently, we have that

$$\inf_{u \in V}(F(u) + G(u)) \geq \sup_{f \in V'}(-F^*(-f) - G^*(f)).$$

(this fact is usually referred to as weak duality).
Denote $p := \inf_{u \in V}(F(u) + G(u))$, $q := \sup_{f \in V'}(-F^*(-f) - G^*(f))$ and $C := epiF$. To complete the proof, we show that $p \leq q$.
If $p = -\infty$ there is nothing to prove.Suppose now that $p \neq -\infty$.
It is clear that the interior of C : $intC$ is not empty (because F is continuous in u_0).
We introduce now the following sets:

$$A := intC,$$

$$B := \{(\lambda, u) \in V \times \mathbb{R} : \lambda \leq p - G(u)\}$$

The set A and B are convex (since F and G are convex) and disjoint (according to the definition of p), therefore, (because of Hahn-Banach's first geometric form) there exist a non zero continuous linear function $f \in V'$ and $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$H = \{(\lambda, u) \in V \times \mathbb{R} : (\langle f, u\rangle_{V,V'} + \alpha\lambda = \beta\}$$

and

$$(6) \qquad \begin{array}{l} (\langle f, u\rangle_{V,V'} + \alpha\lambda \geq \beta, \forall (u, \lambda) \in C, \\ (\langle f, u\rangle_{V,V'} + \alpha\lambda \leq \beta, \forall (u, \lambda) \in B, \end{array}$$

By taking $u = u_0$ in the first part of the last inequality and by passing to the limit on $(\lambda \longrightarrow +\infty)$ we can deduce that $\alpha \geq 0$.
Prove now that $\alpha \neq 0$; for this we proceed by contradiction. Assume that $\alpha = 0$, then according to the last inequalities, we arrive at

$$\langle f, u\rangle_{V,V'} \geq \beta, \forall u \in dom(F), \text{ and }, \langle f, u\rangle_{V,V'} \leq \beta, \forall u \in dom(G).$$

In particular $\langle f, u_0\rangle_{V,V'} = \beta$ (since $u_0 \in dom(F) \cap dom(G)$) and then $\langle f, u - u_0\rangle_{V,V'} \geq 0$ for all u in $dom(F)$ .Consequently, $f = 0$ since $dom(F)$ is neighborhood of u_0. We thus have $\alpha > 0$.
According to

$$(7) \qquad \begin{array}{l} \langle f, u\rangle_{V,V'} + \alpha\lambda \geq \beta, \forall (u, \lambda) \in C, \\ \langle f, u\rangle_{V,V'} + \alpha\lambda \leq \beta, \forall (u, \lambda) \in B, \end{array}$$

and dividing by $\alpha > 0$, we obtain easily that

$$F^*(-f_\alpha) \leq -\beta_\alpha,$$
$$G^*(f_\alpha) \leq \beta_\alpha - p$$

and then $f_\alpha = f/\alpha$ and $\beta_\alpha = \beta/\alpha$.
Therefore ,$p \leq q$.This complete the proof.
Examples
1.Let C be a non-empty subset of topological vector space V and χ_C be its indicator function. Then the conjugate function ${\chi_C}^*$ is defined by

$${\chi_C}^*(f) = \sup_{u\in C}\langle f, u\rangle_{V,V'}$$

and is called the support function of C. Moreover, if C is a closed and convex set, χ_C is closed and convex, and by the conjugacy theorem the conjugate of its support function is its indicator function.
2.Let $(V, \|.\|)$ be a Banach space, $(V', \|.\|_*)$ its dual, $\Psi_\alpha : t \in \mathbb{R} \longrightarrow |t|^\alpha/\alpha$ and $F_\alpha : V \longrightarrow \mathbb{R}$ such that $F_\alpha(u) = \Psi_\alpha(\|u\|)$, where $1 < \alpha < \infty$. Then

$$\begin{aligned} F_\alpha^*(f) &= \sup_{u\in V}(\langle f, u\rangle_{V,V'} - F_\alpha(u)) \\ &= \sup_{\lambda\geq 0}\left(\|f\|_*\lambda - \frac{\lambda^\alpha}{\alpha}\right) \end{aligned}$$

Hence (by analyzing the function $r(\lambda) := \theta\lambda - \lambda^\alpha/\alpha$ where $\theta := \|f\|_*$ and $\lambda \in [0, +\infty[$, $F_\alpha^*(f) = \|f\|_*^{\alpha^*}/\alpha^*$ where $1/\alpha + 1/\alpha^* = 1$.Consequently

$$F_\alpha^*(f) = \Psi_{\alpha^*}(\|f\|_*)$$

3.We finish with an interesting example for the boundary valued problems in a lemma form.

Lemma 0.4.1 *Let $(V, \|.\|)$ be a Banach space, $(V', \|.\|_*)$ its dual and C be a non-empty closed and convex subset or V. Consider the convex and lower semi-continuous real-valued function F on V given by*

$$F(v) := \langle f, v\rangle_{V,V'} + \chi_C(v - u) \quad \forall v \in V$$

where $u \in V$ and $f \in V'$ are given elements.
then the conjugate of F is

$$F^*(g) = \langle g - f, u\rangle_{V,V'} + \chi_{C^*}(g - f) \quad \forall g \in V'$$

where $C^* = \{g \in V' : \langle g, v\rangle_{V,V'} = 0 \ \ \forall v \in C\}$ (which is said to be the polar set of C)

Proof.Let $g \in V'$, we have

$$\begin{aligned} F^*(g) &= \sup_{v\in V}(\langle g, v\rangle_{V,V'} - \langle f, v\rangle_{V,V'} - \chi_C(v - u)) \\ &= \sup_{w\in C}\langle g - f, w + u\rangle_{V,V'} \\ &= \langle g - f, u\rangle_{V,V'} + \sup_{w\in C}\langle g - f, w\rangle_{V,V'} \end{aligned}$$

This completes the proof (since $\sup_{w\in C}\langle g - f, w\rangle_{V,V'} = \chi_C^*(g - f) = \chi_{C^*}(g - f)$).

0.4.2 Legendre transformation

In mathematics, the Legendre transformation or Legendre transform, named after Adrien-Marie Legendre, is an operation that transforms one real-valued function of a real variable into another. Specifically, the Legendre transform of a convex function F is the function F^* defined by

$$F^*(p) = \sup(px - F(x))$$

where "sup" represents the supremum. If F is differentiable, then $F^*(p)$ can be interpreted as the negative of the y-intercept of the tangent line to the graph of F that has slope p. In particular, the value of x that attains the maximum has the property :$F'(x) = p$

That is, the derivative of the function F becomes the argument to the function F^*. In particular, if F is convex (or concave up), then F^* satisfies the functional equation

$$F^*(F'(x)) = xF'(x) - F(x)$$

The Legendre transform is its own inverse. Like the familiar Fourier transform, the Legendre transform takes a function $F(x)$ and produces a function of a different variable p. However, while the Fourier transform consists of an integration with a kernel, the Legendre transform uses maximization as the transformation procedure. The transform is especially well behaved if $F(x)$ is a convex function. The Legendre transformation is an application of the duality relationship between points and lines. The functional relationship specified by $F(x)$ can be represented equally well as a set of (x, y) points, or as a set of tangent lines specified by their slope and intercept values. The Legendre transformation can be generalized to the Legendre-Fenchel transformation. It is commonly used in thermodynamics and in the Hamiltonian formulation of classical mechanics.

0.4.3 Jensen inequality

Let (Ω, A, μ) be a measure space, such that $\mu(\Omega) = 1$. If g is a real-valued function that is μ-integrable, and if φ is a convex function on the real line, then:

$$\varphi\left(\int_\Omega g\ d\mu\right) \leq \int_\Omega \varphi \circ g\ d\mu$$

In real analysis, we may require an estimate on $\varphi\left(\int_a^b g(x)\ dx\right)$ where a, b are real numbers, and g is a non-negative real-valued function that is Lebesgue-integrable. In this case, the Lebesgue measure of $[a, b]$ don't need to be unity. However, by integration by substitution, the interval can be rescaled so that it has measure unity. Then Jensen's inequality can be applied to get

$$\varphi\left(\int_a^b g(x)\ dx\right) \leq \frac{1}{b-a}\int_a^b \varphi((b-a)g(x))\ dx$$

0.5 Aubin -Lions lemma

The Aubin Lions lemma is a result in the theory of Sobolev spaces of Banach space-valued functions. More precisely, it is a compactness criterion that is very useful in the study of nonlinear evolutionary partial differential equations. The result is named after the French mathematicians Thierry Aubin and Jacques-Louis Lions. We complete the preliminaries by the useful inequalities of Gagliardo-Nirenberg and Sobolev-Poincaré.

Lemma 0.5.1 *LetX_0,X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Assume that$X_0$ is compactly embedded in X and that X is continuously embedded in X_1; assume also that X_0 and X_1 are reflexive spaces. For $1 < p, q < +\infty$, let*

$$W = \{u \in L^p([0,T]; X_0)/ \quad \dot{u} \in L^q([0,T]; X_1)\}$$

Then the embedding of W into $L^p([0,T]; X)$ is also compact.

Lemma 0.5.2 (Gagliardo-Nirenberg) *Let $1 \leq r < q \leq +\infty$ and $p \leq q$. Then, the inequality*

$$\|u\|_{W^{m,q}} \leq C\|u\|^{\theta}_{W^{m,p}}\|u\|^{1-\theta}_{r} \quad \textit{for} \quad u \in W^{m,p} \bigcap L^r$$

holds with some $C > 0$ and

$$\theta = \left(\frac{k}{n} + \frac{1}{r} - \frac{1}{q}\right)\left(\frac{m}{n} + \frac{1}{r} - \frac{1}{p}\right)^{-1}$$

provided that $0 < \theta \leq 1$ (we assume $0 < \theta < 1$ if $q = +\infty$).

Lemma 0.5.3 (Sobolev-Poincaré inequality) *Let q be a number with $2 \leq q < +\infty$ ($n = 1, 2$) or $2 \leq q \leq 2n/(n-2)$ ($n \geq 3$), then there is a constant $c_* = c(\Omega, q)$ such that*

$$\|u\|_q \leq c_*\|\nabla u\|_2 \quad \textit{for} \quad u \in H^1_0(\Omega).$$

Chapter 1

GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS TO A PETROVSKY SYSTEM WITH A DELAY TERM AND SOURCE TERM

1.1 Introduction

In this chapter, we investigate the existence and decay properties of solutions for the initial boundary value problem of system of petrovsky of the type
(P)

$$\begin{cases} u''(x,t) + \Delta_x^2 u(x,t) + \mu_1 g(u'(x,t)) + \mu_2 g(u'(x,t-\tau)) = bu|u|^{p-2} & \text{in } \Omega\times]0,+\infty[, \\ u = \partial_\nu u = 0 & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) & \text{in } \Omega, \\ u'(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau[, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$, $\tau > 0$ is a time delay, μ_1 and μ_2 are positive real numbers, and the initial data (u_0, u_1, f_0) belong to a suitable space.

Time delays so often arise in many physical, chemical, biological phenomena. In recent years, the control of PDEs with time delay effects has become an active area of research. The presence of delay may be a source of instability.

When $\mu_1 \neq 0$,$\mu_2 = 0$ and $g(s) = \delta|s|^{m-2}s$ $(m \geq 1)$ S. A. Messaoudi [**44**] determined suitable relations between m and p, for which there is global existence or alternatively finite time blow up. More precisely: he showed that solutions with any initial data continue to exist globally in time if $m \geq p$ and blow up in finite time if $m < p$ and the initial energy is negative. To prove global existence he used a new method introduced by Georgiev and Todorova [**20**] based on a fixed point theorem.

For the wave equation ($\Delta_x u$ instead of $\Delta_x^2 u$ in (P)),when g is linear and $f(u) = 0$, it is well known that if $\mu_2 = 0$, that is, in absence of delay, the energy of problem (P) is exponentially decaying to zero (see for instance **[32]**, **[14]**, **[15]** and **[49]**). On the contrary, if $\mu_1 = 0$, that is, there exits only the delay part in the internal, the system (P) becomes unstable (see, for instance **[17]**). In recent years, the PDEs with time delay effects have become an active area of research and arise in many pratical problems . In **[17]**, the authors showed that a small delay in a boundary control could turn such well-behave hyperbolic system into a wild one and therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see **[50]**, **[61]**, **[51]**). In **[50]** the authors examined the problem (P) and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they found a sequence of delays for which the corresponding solution of (P) will be instable if $\mu_2 \geq \mu_1$. The main approach used in **[50]**, is an observability inequality obtained with a Carleman estimate. The same results were showed if both the damping and the delay acting in the boundary domain. We also recall the result by Xu, Yung and Li **[61]**, where the authors proved the same result as in **[50]** for the one space dimension by adopting the spectral analysis approach.

When g is nonlinear and in the case $\mu_2 = 0$, the problem of existence and energy decay have been previously studied by several authors (see **[27]**, **[32]**,**[9]**) and many energy estimates have been derived for arbitrary growing feedbacks (polynomial, exponential or logarithmic decay).

In this article,we give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (P) for a nonlinear damping and a delay term. To obtain global solutions to the problem (P), we use the argument combining the Galerkin approximation scheme (see **[40]**) with the energy estimate method. This work extends the result obtained by A.Benaissa and Naima Louhibi for a wave equation with a delay. The technic based on the theory of nonlinear semigroups used in **[50]** does not seem to be applicable in the nonlinear case.

To prove decay estimates, we use a multiplier method and some properties of convex functions. These arguments of convexity were introduced and developed by Lasiecka et al. **[12]**, **[16]**, **[35]**, **[38]**and **[36]**, and used by Liu and Zuazua **[41]**, Eller et al **[19]** and Alabau-Boussouira **[4]**.

1.2 Preliminaries and main results

In this section, we present some materials needed in the proof of our results.

(H1) $g : \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of the class $C^1(\mathbb{R})$ such that there exist ϵ_1, c_1, $c_2, c_3, \alpha_1, \alpha_2 > 0$ and a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of the class $C^1(\mathbb{R}_+) \cap C^2(]0, \infty[)$ satisfying $H(0) = 0$, and H linear on $[0, \epsilon_1]$ or ($H'(0) = 0$ and $H'' > 0$ on $]0, \epsilon_1]$), such that

$$c_1|s| \leq |g(s)| \quad \text{if } |s| \geq \epsilon_1, \tag{1.1}$$

$$s^2 + g^2(s) \leq H^{-1}(sg(s)) \quad \text{if } |s| \leq \epsilon_1. \tag{1.2}$$

(1.3) $$|g'(s)| \le c_3$$

(1.4) $$\alpha_1\ sg(s) \le G(s) \le \alpha_2\ sg(s).$$

where

$$G(s) = \int_0^s g(r)\,dr$$

(H2)

(1.5) $$\alpha_2\mu_2 < \alpha_1\mu_1.$$

(H3)

(1.6) $$2 \le p \le \frac{2n-2}{(n-4)^+}.$$

1.3 Technical Lemmas

Lemma 1.3.1 (Sobolev-Poincaré's inequality) *Let q be a number with $2 \le q < +\infty$ ($n = 1, 2, 3, 4$) or $2 \le q \le 2n/(n-4)$ ($n \ge 5$). Then there is a constant $c_* = c_*(\Omega, q)$ such that*

(1.7) $$\|u\|_q \le c_*\|\Delta u\|_2 \quad \textit{for} \quad u \in H_0^2(\Omega).$$

Lemma 1.3.2 ([19], [21]) *Let $E: \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing differentiable function and $\Psi: \mathbb{R}_+ \to \mathbb{R}_+$ a convex and increasing function such that $\Psi(0) = 0$. Assume that*

$$\int_s^T \Psi(E(t))\,dt \le E(s), \quad \forall 0 \le s \le T$$

Then E satisfies the following estimate:

(1.8) $$E(t) \le \psi^{-1}\Big(h(t) + \psi(E(0))\Big)\Big), \ \forall t \ge 0,$$

where $\psi(t) = \int_t^1 \frac{1}{\Psi(s)}\,ds$ for $t > 0$, $h(t) = 0$ for $0 \le t \le \frac{E(0)}{\Psi(E(0))}$, and

$$h^{-1}(t) = t + \frac{\psi^{-1}\Big(t + \psi(E(0))\Big)}{\Psi\Big(\psi^{-1}\Big(t + \psi(E(0))\Big)\Big)}, \ \forall t \ge \frac{E(0)}{\Psi(E(0))}.$$

We introduce as in **[50]** the new variable

(1.9) $$z(x, \rho, t) = u_t(x, t - \tau\rho), x \in \Omega,\ \rho \in (0, 1), \quad t > 0.$$

Then, we have

(1.10) $$\tau z'(x, \rho, t) + z_\rho(x, \rho, t) = 0, \quad \text{in } \Omega \times (0, 1) \times (0, +\infty).$$

Therefore, problem (P) is equivalent to:

$$
(1.11)\qquad \begin{cases} u''(x,t)+\Delta_x^2 u(x,t)+\mu_1 g(u'(x,t)) & \\ \qquad +\mu_2 g(z(x,1,t)) = bu|u|^{p-2}, & \text{in } \Omega\times]0,+\infty[, \\ \tau z'(x,\rho,t)+z_\rho(x,\rho,t)=0, & \text{in } \Omega\times]0,1[\times]0,+\infty[\\ u(x,t)=0, & \text{on } \partial\Omega\times[0,+\infty[\\ z(x,0,t)=u'(x,t) & \text{on } \Omega\times[0,+\infty[\\ u(x,0)=u_0(x),\ u'(x,0)=u_1(x) & \text{in } \Omega \\ z(x,\rho,0)=f_0(x,-\rho\tau) & \text{in } \Omega\times]0,1[\end{cases}
$$

Let ξ be a positive constant such that

$$
(1.12)\qquad \tau\frac{\mu_2(1-\alpha_1)}{\alpha_1}<\xi<\tau\frac{\mu_1-\alpha_2\mu_2}{\alpha_2}.
$$

In order to state and prove our main result we fisrt introduce the following

$$
(1.13)\qquad \begin{aligned} I(t) &= I(u(t)) = \|\Delta_x u(t)\|_2^2 - b\|u(t)\|_p^p \\ J(t) &= J(u(t)) = \frac{1}{2}\|\Delta_x u(t)\|_2^2-\frac{b}{p}\|u(t)\|_p^p \end{aligned}
$$

We define the energy associated to the solution of the problem (1.11) by the following formula:

$$
(1.14)\qquad E(t)=J(t)+\frac{1}{2}\|u'(t)\|_2^2+\xi\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx.
$$

Then we can define the stable set as

$$
H=\Big\{u\in H_0^2(\Omega)\ \ /\ \ I(u)>0\Big\}\cup\{0\}.
$$

We have the following theorem.

Theorem 1.3.1 *Let $(u_0,u_1,f_0)\in H^4\cap H_0^2\times H_0^2\cap L^2\times H_0^1(\Omega;H^1(0,1))$ satisfy the compatibility condition*

$$
f_0(.,0)=u_1.
$$

Assume that the hypotheses $(H1)-(H2)$ hold. Then the problem (P) admits a unique solution

$$
u\in L^\infty((-\tau,\infty);H^4\cap H_0^2(\Omega)), u'\in L^\infty((-\tau,\infty);H_0^2\cap L^2(\Omega)), u''\in L^\infty((-\tau,\infty);L^2(\Omega))
$$

and, for some constants ω,ϵ_0 we obtain the following decay property:

$$
(1.15)\qquad E(t)\le\psi^{-1}\Big(h(t)+\psi(E(0))\Big),\quad \forall t>0,
$$

where $\psi(t)=\int_t^1\frac{1}{\omega\varphi(\tau)}d\tau$ for $t>0$, $h(t)=0$ for $0\le t\le\frac{E(0)}{\omega\varphi(E(0))}$,

$$
h^{-1}(t)=t+\frac{\psi^{-1}\Big(t+\psi(E(0))\Big)}{\omega\varphi\Big(\psi^{-1}\Big(t+\psi(E(0))\Big)\Big)},\quad \forall t>0,
$$

$$\varphi(s) = \begin{cases} s & \text{if } H \text{ is linear on } [0, \epsilon_1], \\ sH'(\epsilon_0 s) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \epsilon_1]. \end{cases}$$

Example. Let $g(s) = s^p(-\ln s)^q$, where $p \geq 1$, $q \in \mathbb{R}$. Then $H(s) = cs^{\frac{p+1}{2}}(-\ln \sqrt{s})^q$. We have

$$H'(s) = cs^{\frac{p-1}{2}}\left(-\ln \sqrt{s}\right)^{q-1}\left(\frac{p+1}{2}\left(-\ln \sqrt{s}\right) - \frac{q}{2}\right).$$

Thus

$$\varphi(s) = cs^{\frac{p+1}{2}}\left(-\ln \sqrt{s}\right)^{q-1}\left(\frac{p+1}{2}\left(-\ln \sqrt{s}\right) - \frac{q}{2}\right)$$

and

$$\begin{aligned} \psi(t) &= c\int_t^1 \frac{1}{s^{\frac{p+1}{2}}\left(-\ln \sqrt{s}\right)^{q-1}\left(\frac{p+1}{2}\left(-\ln \sqrt{s}\right) - \frac{q}{2}\right)}\,ds \\ &= c\int_1^{\frac{1}{\sqrt{t}}} \frac{z^{p-2}}{(\ln z)^{q-1}\left(\frac{p+1}{2}\ln z - \frac{q}{2}\right)}\,dz. \end{aligned}$$

We obtain

$$\psi(t) \equiv \begin{cases} c\dfrac{1}{t^{\frac{p-1}{2}}\left(-\ln t\right)^q} & \text{if } p > 1, \\ c\left(-\ln t\right)^{1-q} & \text{if } p = 1,\ q < 1, \\ c\left(\ln(-\ln t)\right) & \text{if } p = 1,\ q = 1. \end{cases}$$

and then

$$\psi^{-1}(t) \equiv \begin{cases} ct^{-\frac{2}{p-1}}(\ln t)^{-\frac{2q}{p-1}} & \text{if } p > 1, \\ ce^{-t^{\frac{1}{1-q}}} & \text{if } p = 1,\ q < 1, \\ ce^{-e^t} & \text{if } p = 1,\ q = 1. \end{cases}$$

Using the fact that $h(t) = t$ as t goes to infinity, then

$$E(t) \leq \begin{cases} ct^{-\frac{2}{p-1}}(\ln t)^{-\frac{2q}{p-1}} & \text{if } p > 1, \\ ce^{-t^{\frac{1}{1-q}}} & \text{if } p = 1,\ q < 1, \\ ce^{-e^t} & \text{if } p = 1,\ q = 1. \end{cases}$$

Proof of Theorem 1.3.1. We finish this section by giving an explicit upper bound for the derivative of the energy.

Lemma 1.3.3 *Let (u, z) be a solution of the problem (1.9). Then, the energy functional defined by (1.14) satisfies*

$$\begin{aligned} E'(t) &\leq -\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_\Omega u'g(u')\,dx \\ &\qquad - \left(\frac{\xi}{\tau}\alpha_1 - \mu_2(1-\alpha_1)\right)\int_\Omega z(x,1,t)g(z(x,1,t))\,dx \\ &\leq 0 \end{aligned} \tag{1.16}$$

Proof. Multiplying the first equation in (1.11) by u', integrating over Ω and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}(\|u'\|_2^2+\|\Delta_x u\|_2^2)-\frac{b}{p}\frac{d}{dt}\|u(t)\|_p^p+\mu_1\int_\Omega u'g(u')\,dx+\mu_2\int_\Omega u'(x,t)g(z(x,1,t))\,dx=0. \tag{1.17}$$

We multiply the second equation in (1.11) by $\xi g(z)$ and integrate the result over $\Omega\times(0,1)$, to obtain:

$$\begin{aligned}\xi\int_\Omega\int_0^1 z'g(z(x,\rho,t))\,d\rho\,dx &= -\frac{\xi}{\tau}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}G(z(x,\rho,t))\,d\rho\,dx\\ &= -\frac{\xi}{\tau}\int_\Omega(G(z(x,1,t))-G(z(x,0,t)))\,dx.\end{aligned} \tag{1.18}$$

Then

$$\xi\frac{d}{dt}\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx=-\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx+\frac{\xi}{\tau}\int_\Omega G(u')\,dx. \tag{1.19}$$

From (1.17), (1.19) and using Young inequality we get

$$E'(t)=-\left(\mu_1-\frac{\xi\alpha_2}{\tau}\right)\int_\Omega u'g(u')\,dx-\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx-\mu_2\int_\Omega u'(t)g(z(x,1,t))\,dx \tag{1.20}$$

Let us denote G^* to be the conjugate function of the convex function G, i.e., $G^*(s)=\sup_{t\in\mathbb{R}^+}(st-G(t))$. Then G^* is the Legendre transform of G, which is given by (see Arnold [**8**], p. 61-62, and Lasiecka [**12**], [**16**], [**35**]-[**36**])

$$G^*(s)=s(G')^{-1}(s)-G[(G')^{-1}(s)],\quad\forall s\geq 0 \tag{1.21}$$

and satisfies the following inequality

$$st\leq G^*(s)+G(t),\quad\forall s,t\geq 0. \tag{1.22}$$

Then, from the definition of G, we get

$$G^*(s)=sg^{-1}(s)-G(g^{-1}(s))$$

Hence

$$\begin{aligned}G^*(g(z(x,1,t))) &= z(x,1,t)g(z(x,1,t))-G(z(x,1,t))\\ &\leq (1-\alpha_1)z(x,1,t)g(z(x,1,t))\end{aligned} \tag{1.23}$$

Making use of (1.20), (1.22) and (1.23), we have

$$\begin{aligned}E'(t) &\leq -\left(\mu_1-\frac{\xi\alpha_2}{\tau}\right)\int_\Omega u'g(u')\,dx-\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx+\mu_2\int_\Omega(G(u')+G^*(g(z(x,1,t))))\,dx\\ &\leq -\left(\mu_1-\frac{\xi\alpha_2}{\tau}-\mu_2\alpha_2\right)\int_\Omega u'g(u')\,dx-\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx+\mu_2\int_\Omega G^*(g(z(x,1,t)))\,dx.\end{aligned} \tag{1.24}$$

Using (1.4) and (1.12), we obtain

$$\begin{aligned}E'(t) &\leq -\left(\mu_1-\frac{\xi\alpha_2}{\tau}-\mu_2\alpha_2\right)\int_\Omega u'g(u')\,dx\\ &\qquad -\left(\frac{\xi}{\tau}\alpha_1-\mu_2(1-\alpha_1)\right)\int_\Omega z(x,1,t)g(z(x,1,t))\,dx\\ &\leq 0\end{aligned}$$

1.4 Global Existence

We are now ready to start proving Theorem 1.3.1 in the next two sections.

Throughout this section we assume $u_0 \in H^4(\Omega) \cap H$ and $u_1 \in H_0^2 \cap L^2(\Omega), f_0 \in H_0^1(\Omega; H^1(0,1))$.

We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$ where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^4 \cap H_0^2$.

Now, we define for $1 \leq j \leq k$ the sequence $\phi_j(x, \rho)$ as follows:

$$\phi_j(x, 0) = w_j.$$

Then, we may extend $\phi_j(x, 0)$ by $\phi_j(x, \rho)$ over $L^2(\Omega \times [0, 1])$ such that $(\phi_j)_j$ form a basis of $L^2(\Omega; H^1(0, 1))$ and denote Z_k the space generated by $\{\phi_1, \phi_2, \ldots, \phi_k\}$.

We construct approximate solutions $(u_k, z_k)(k = 1, 2, 3, \ldots,)$ in the form

$$u_k(t) = \sum_{j=1}^{k} g_{jk}(t) w_j$$

$$z_k(t) = \sum_{j=1}^{k} h_{jk}(t) \phi_j$$

where g_{ik} and h_{ik} $(j = 1, 2, \ldots, k)$ are determined by the following ordinary differential equations:

$$\begin{cases} (u_k''(t), w_j) + (\Delta_x u_k(t), \Delta_x w_j) + \mu_1(g(u_k'), w_j) + \mu_2(g(z_k(., 1)), w_j) = (b|u_k(t)|^{p-2} u_k(t), w_j) \\ 1 \leq j \leq k, \\ z_k(x, 0, t) = u_k'(x, t) \end{cases}$$
(1.25)

$$u_k(0) = u_{0k} = \sum_{j=1}^{k} (u_0, w_j) w_j \to u_0 \text{ in } H^4 \cap H_0^2 \text{ as } k \to +\infty, \tag{1.26}$$

$$u_k'(0) = u_{1k} = \sum_{j=1}^{k} (u_1, w_j) w_j \to u_1 \text{ in } H_0^2 \cap L^2 \text{ as } k \to +\infty. \tag{1.27}$$

and

$$\begin{cases} (\tau z_{kt} + z_{k\rho}, \phi_j) = 0 \\ 1 \leq j \leq k, \end{cases} \tag{1.28}$$

$$z_k(\rho, 0) = z_{0k} = \sum_{j=1}^{k} (f_0, \phi_j) \phi_j \to f_0 \text{ in } H_0^1(\Omega; H^1(0, 1)) \text{ as } k \to +\infty, \tag{1.29}$$

By virtue of the theory of ordinary differential equations, the system (1.25)-(1.29) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \leq +\infty$) by Zorn lemma since the nonlinear terms in (1.25) are locally Lipschitz continuous. Note that $u_k(t)$ is C^2-class.

In the next step, we obtain a priori estimates for the solution, so that it can be extended outside $[0, T_k[$ to obtain one solution defined for all $t > 0$.

We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for (u_k, z_k).

Lemma 1.4.1 *Assume that* $(H3)$ *holds. Let* $u(t)$ *be a solution with the initial data* $\{u_0, u_1, f_0\}$ *satisfying* $u_0 \in H, u_1 \in L^2(\Omega), f_0 \in H_0^1(\Omega; H^1(0,1))$.
If $\{u_0, u_1, f_0\}$ *satisfies*

$$\eta = 1 - b\ C_*^p \left(\frac{2p}{(p-2)} E(u_0, u_1)\right)^{(p-2)/2} > 0, \tag{1.30}$$

then $u(t) \in H$ *for all* $t \in [0, +\infty[$.

Proof. Since $I(u_0) > 0$, it follows from the continuity of $u(t)$ that

$$I(u(t)) \geq 0 \tag{1.31}$$

for some interval near $t = 0$. Let t_{max} be a maximal time ($t_{max} = T_{max}$), when (1.31) holds on $[0, t_{max})$. On the other hand,

$$\begin{aligned} J(t) &= \frac{1}{2}\|\Delta_x u(t)\|_2^2 - \frac{b}{p}\|u(t)\|_p^p \\ &= \frac{p-2}{2p}\|\Delta_x u(t)\|_2^2 + \frac{1}{p} I(u(t)) \\ &\geq \frac{p-2}{2p}\|\Delta_x u(t)\|_2^2 \qquad \forall t \in [0, t_{\max}); \end{aligned} \tag{1.32}$$

hence,

$$\begin{aligned} \|\Delta_x u(t)\|_2^2 &\leq \frac{2p}{p-2} J(t) \leq \frac{2p}{p-2} E(t) \\ &\leq \frac{2p}{p-2} E(u_0, u_1, f_0), \qquad \forall t \in [0, t_{\max}) \end{aligned} \tag{1.33}$$

Using (1.3.1),(1.30) and (1.33), we deduce that

$$\begin{aligned} b\|u(t)\|_p^p &\leq b\ C_*^p \|\Delta_x u(t)\|_2^p \leq b\ C_*^p \left(\frac{2p}{p-2} E(u_0, u_1, f_0)\right)^{(p-2)/2} \|\Delta_x u(t)\|_2^2 \\ &< \|\Delta_x u(t)\|_2^2, \qquad \forall t \in [0, t_{\max}); \end{aligned} \tag{1.34}$$

Therefore we get

$$\|\Delta_x u(t)\|_2^2 - b\|u(t)\|_p^p > 0 \text{ on } [0, t_{\max}).$$

This implies that we can take $t_{\max} = T_m$. Furthermore, by the fact that the energy is non-increasing we have

$$\begin{aligned} E(u_0, u_1, f_0) &\geq E(t) = \frac{1}{2}\|\Delta_x u(t)\|_2^2 - \frac{b}{p}\|u(t)\|_p^p + \frac{1}{2}\|u'(t)\|_2^2 + \xi \int_\Omega \int_0^1 G(z(x,\rho,t))\, d\rho\, dx \\ &= \frac{p-2}{2p}\|\Delta_x u(t)\|_2^2 + \frac{1}{p}I(u(t)) + \frac{1}{2}\|u'(t)\|_2^2 + \xi \int_\Omega \int_0^1 G(z(x,\rho,t))\, d\rho\, dx \quad , \\ &\geq \frac{p-2}{2p}\|\Delta_x u(t)\|_2^2 + \frac{1}{2}\|u'(t)\|_2^2 \text{ on } [0, t_{\max}) \end{aligned} \tag{1.35}$$

These estimates imply that the (approximated) solution $u(t)$ exists globally in $[0, +\infty)$. This ends the proof of lemma 1.4.1.

The first estimate.

Since the sequences u_{0k}, u_{1k} and z_{0k} converge, them from (1.16) we can find a positive constant C independent of k such that

$$E_k(t) + a_1 \int_0^t \int_\Omega u_k' g(u_k')\, dxds + a_2 \int_0^t \int_\Omega z_k(x,1,t) g(z_k(x,1,t))\, dxds \leq E_k(0) \leq C. \tag{1.36}$$

where

$$E_k(t) = \frac{1}{2}\|u_k'(t)\|_2^2 + \frac{1}{2}\|\Delta_x u_k(t)\|_2^2 - \frac{b}{p}\|u_k(t)\|_p^p + \xi \int_\Omega \int_0^1 G(z_k(x,\rho,t))\, d\rho\, dx, \tag{1.37}$$

$$a_1 = \mu_1 - \frac{\xi \alpha_2}{\tau} - \mu_2 \alpha_2 \text{ and } a_2 = \frac{\xi}{\tau}\alpha_1 - \mu_2(1-\alpha_1)$$

These estimates imply that the solution (u_k, z_k) exists globally in $[0, +\infty[$. Estimate (1.36) yields

$$u_k \text{ is bounded in } L^\infty_{loc}(0,\infty; H_0^2(\Omega)) \tag{1.38}$$

$$u_k' \text{ is bounded in } L^\infty_{loc}(0,\infty; H_0^2(\Omega)) \tag{1.39}$$

$$u_k'(t) g(u_k'(t)) \text{ is bounded in } L^1(\Omega \times (0,T)) \tag{1.40}$$

$$G(z_k(x,\rho,t)) \text{ is bounded in } L^\infty_{loc}(0,\infty; L^1(\Omega \times (0,1))) \tag{1.41}$$

$$z_k(x,1,t) g(z_k(x,1,t)) \text{ is bounded in } L^1(\Omega \times (0,T)) \tag{1.42}$$

The second estimate.

first of all, we are going to estimate $u_k''(0)$. Testing (1.25) by $g_{jk}''(t)$ and choosing $t=0$, we obtain:

$$\begin{aligned} \|u_k''(0)\|_2 &\leq \|\Delta_x^2 u_{0k}\|_2 + \mu_1 \|g(u_{1k})\|_2 + \mu_2 \|g(z_{0k})\|_2 + \|f(u_{0k})\|_2 \\ &\leq \|\Delta_x^2 u_{0k}\|_2 + \mu_1 \|g(u_{1k})\|_2 + \mu_2 \|g(z_{0k})\|_2 + c\|\Delta_x u_{0k}\|_2^{p-1}, \end{aligned}$$

where we set $f(u) = bu|u|^{p-2}$. Using the Gagliardo-Nirenberg inequality, we have

$$\|f(u_{0k})\|_2 < c\|\Delta_x^2 u_{0k}\|_2^{p-1}$$

Since $g(u_{1k}), g(z_{0k})$ is bounded in $L^2(\Omega)$ by (**H1**) hence from (1.26), (1.27) and (1.29):

$$\|u''_k(0)\|_2 \leq C.$$

Differentiating (1.25) with respect to t, we get

$$(u'''_k(t) + \Delta_x^2 u'_k(t) + \mu_1 u''_k(t) g'(u'_k) + \mu_2 z'_k g'(z_k) - u'_k f'(u_k), w_j) = 0.$$

Multiplying by $g''_{jk}(t)$, summing over j from 1 to k, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|u''_k(t)\|_2^2 + \|\Delta_x u'_k(t)\|_2^2\right) + \mu_1 \int_\Omega u''^2_k(t) g'(u'_k(t))\,dx \\ +\mu_2 \int_\Omega u''_k(t) z'_k(x,1,t) g'(z_k(x,1,t))\,dx \leq b(p-1)\int_\Omega |u''_k(t)||u'_k(t)||u_k(t)|^{p-2}\,dx,\end{aligned} \tag{1.43}$$

Next, we are going to analyze the term on the right-hand side of (1.43).
Making use of the generalized Hölder inequality, observing that $\frac{p-2}{2(p-1)} + \frac{1}{2(p-1)} + \frac{1}{2} = 1$, using Lemmas 1.3.1 and 1.4.1 we conclude that

$$\begin{aligned}\left|\int_\Omega u''_k(t) u'_k(t) f'(u_k(t))\,dx\right| &\leq b(p-1)\|u_k(t)\|_{2(p-1)}^{p-2}\|u'_k(t)\|_{2(p-1)}\|u''_k(t)\|_2 \\ &\leq C_1 \|\Delta_x u_k(t)\|_2^{p-2}\|\Delta_x u'_k(t)\|_2\|u''_k(t)\|_2 \\ &\leq C_2\left(\|\Delta_x u'_k(t)\|_2^2 + \|u''_k(t)\|_2^2\right)\end{aligned} \tag{1.44}$$

Differentiating (1.28) with respect to t, we get

$$(\tau z''_k(t) + \frac{\partial}{\partial \rho} z'_k, \phi_j) = 0.$$

Multiplying by $h'_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\tau\frac{d}{dt}\|z'_k(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z'_k(t)\|_2^2 = 0 \tag{1.45}$$

Taking the sum of (1.43) and (1.45), we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u''_k(t)\|_2^2 + \|\Delta_x u'_k(t)\|_2^2 + \tau\|z'_k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\right) + \mu_1\int_\Omega u''^2_k(t) g'(u'_k(t))\,dx \\ &-\int_\Omega |u''_k(t)||u'_k(t)||f'(u_k(t))|\,dx + \frac{1}{2}\int_\Omega |z'_k(x,1,t)|^2\,dx = \\ &-\mu_2\int_\Omega u''_k(t) z'_k(x,1,t) g'(z_k(x,1,t))\,dx + \frac{1}{2}\|u''_k(t)\|_2^2\end{aligned}$$

Using (1.3),(1.44), Cauchy-Schwartz and Young inequalities, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u''_k(t)\|_2^2 + \|\Delta_x u'_k(t)\|_2^2 + \tau\|z'_k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\right) + \mu_1\int_\Omega u''^2_k(t) g'(u'_k(t))\,dx \\ &+c\int_\Omega |z'_k(x,1,t)|^2\,dx \leq c'(\|\Delta_x u'_k(t)\| + \|u''_k(t)\|_2^2).\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall Lemma, we obtain

$$\|u_k''(t)\|_2^2 + \|\Delta_x u_k'(t)\|_2^2 + \|z_k'(x,\rho,t)\|_{L^2(\Omega\times(0,1))}^2 \leq e^{cT}\left(\|u_k''(0)\|_2^2 + \|\Delta_x u_k'(0)\|_2^2 + \|z_k'(x,\rho,0)\|_{L^2(\Omega\times(0,1))}^2\right)$$

for all $t \in \mathbb{R}_+$, therefore, we conclude that

$$u_k'' \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2) \tag{1.46}$$

$$u_k' \text{ is bounded in } L^\infty_{loc}(0,\infty;H_0^2) \tag{1.47}$$

$$z_k' \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2(\Omega\times(0,1))) \tag{1.48}$$

THE THIRD ESTIMATE.
Replacing w_j by $-\Delta_x w_j$ in (1.25), multiplying by $g_{jm}'(t)$, summing over j from 1 to k, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\nabla_x u_k'(t)\|_2^2 + \|\nabla_x \Delta u_k(t)\|_2^2 - \frac{b}{p}\|\nabla_x u_k(t)\|_p^p\right) + \mu_1 \int_\Omega |\nabla_x u_k'(t)|^2 g'(u_k'(t))\,dx \\ +\mu_2 \int_\Omega \nabla_x u'_k(t)\nabla_x z_k'(x,1,t) g'(z_k(x,1,t))\,dx = 0\end{aligned} \tag{1.49}$$

Replacing ϕ_j by $-\Delta_x\phi_j$ in (1.28), multiplying by $h_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\tau\frac{d}{dt}\|\nabla_x z_k(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|\nabla_x z_k(t)\|_2^2 = 0 \tag{1.50}$$

From (1.49) and (1.50), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla_x u_k'(t)\|_2^2 + \|\nabla \Delta u_k(t)\|_2^2 - \frac{b}{p}\|\nabla_x u_k(t)\|_p^p + \tau\|\nabla_x z_k(x,\rho,t)\|_{L^2(\Omega\times(0,1))}^2\right) + \mu_1\int_\Omega |\nabla_x u_k'(t)|^2 g'(u_k'(t))\,dx \\ &+\frac{1}{2}\int_\Omega |\nabla_x z_k(x,1,t)|^2\,dx = \\ &-\mu_2\int_\Omega \nabla_x u'_k(t)\nabla_x z_k'(x,1,t)g'(z_k(x,1,t))\,dx + \frac{1}{2}\|\nabla_x u_k'(t)\|_2^2\end{aligned}$$

Using (1.3), Cauchy-Schwartz and Young inequalities, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla_x u_k'(t)\|_2^2 + \|\nabla \Delta u_k(t)\|_2^2 - \frac{b}{p}\|\nabla_x u_k(t)\|_p^p + \tau\|\nabla_x z_k(x,\rho,t)\|_{L^2(\Omega\times(0,1))}^2\right) + \mu_1\int_\Omega |\nabla_x u_k'(t)|^2 g'(u_k'(t))\,dx \\ &+c\int_\Omega |\nabla_x z_k'(x,1,t)|^2\,dx \leq c'\|\nabla_x u_k'(t)\|_2^2.\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall Lemma, we obtain

$$\begin{aligned}&\|\nabla_x u_k'(t)\|_2^2 + \|\nabla\Delta u_k(t)\|_2^2 - \|\nabla_x u_k(t)\|_p^p + \|\nabla_x z_k(x,\rho,t)\|_{L^2(\Omega\times(0,1))}^2 \leq \\ &\quad e^{cT}\left(\|\nabla_x u_k'(0)\|_2^2 + \|\nabla_x\Delta u_k(0)\|_2^2 - \|\Delta_x u_k(0)\|_p^p + \|\nabla_x z_k(x,\rho,0)\|_{L^2(\Omega\times(0,1))}^2\right)\end{aligned}$$

for all $t \in \mathbb{R}_+$, therefore, we conclude that

$$u_k \text{ is bounded in } L^\infty_{loc}(0,\infty;H^4\cap H_0^2) \tag{1.51}$$

$$z_k \text{ is bounded in } L^\infty_{loc}(0,\infty; H^1_0(\Omega; L^2(0,1))) \tag{1.52}$$

Applying Dunford-Pettis theorem we conclude from (1.38),(1.39), (1.40), (1.41), (1.46), (1.47), (1.48) (1.51) and (1.51) replacing the sequence u_k with a subseqence if needed, that

$$u_k \to u \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty; H^4(\Omega)\cap H^2_0(\Omega)) \tag{1.53}$$

$$\begin{aligned} u'_k &\to u' \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty; H^2_0(\Omega)) \\ u''_k &\to u'' \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty; H^1_0) \\ g(u'_k) &\to \chi \ \text{ weak-star in } \ L^2(\Omega\times(0,T)) \end{aligned} \tag{1.54}$$

$$\begin{aligned} z_k &\to z \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty; H^1_0(\Omega; L^2(0,1)) \\ z'_k &\to z' \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty; L^2(\Omega\times(0,1))) \\ g(z_k(x,1,t)) &\to \psi \ \text{ weak-star in } \ L^2(\Omega\times(0,T)) \end{aligned} \tag{1.55}$$

for suitable functions $u \in L^\infty(0,T;H^2_0)$, $z \in L^\infty(0,T;L^2(\Omega\times(0,1)))$, $\chi \in L^2(\Omega\times(0,T))$, $\psi \in L^2(\Omega\times(0,T))$ for all $T \geq 0$. We have to show that (u,z) is a solution of (1.11).

Lemma 1.4.2 *For each $T>0$, $g(u'), g(z(x,1,t)) \in L^1(Q)$ and $\|g(u')\|_{L^1(Q)}, \|g(z(x,1,t))\|_{L^1(Q)} \leq K_1$, where K_1 is a contant independent of t.*

Proof: By (**H1**), we have

$$g(u'_k(x,t)) \to g(u'(x,t)) \text{ a.e. in } Q,$$

$$0 \leq g(u'_k(x,t))u'_k(x,t) \to g(u'(x,t))u'(x,t) \text{ a.e. in } Q$$

Hence, by (1.40) and Fatou's lemma we have

$$\int_0^T \int_\Omega u'(x,t)g(u'(x,t))\,dx\,dt \leq K \text{ for } T>0. \tag{1.56}$$

By Cauchy-Schwartz inequality and using (1.56), we have

$$\begin{aligned} \int_0^T \int_\Omega |g(u'(x,t))|\,dx\,dt &\leq c|Q|^{\frac12}\left(\int_0^T\int_\Omega u'g(u')\,dx\,dt\right)^{\frac12} \\ &\leq c|Q|^{\frac12}K^{\frac12} \equiv K_1 \end{aligned}$$

Lemma 1.4.3 *$g(u'_k) \to g(u')$ in $L^1(\Omega\times(0,T))$ and $g(z_k) \to g(z)$ in $L^1(\Omega\times(0,T))$.*

Proof: Let $E \subset \Omega\times[0,T]$ and set

$$E_1 = \left\{(x,t)\in E;\ g(u'_k(x,t)) \leq \frac{1}{\sqrt{|E|}}\right\}, \quad E_2 = E\setminus E_1,$$

where $|E|$ is the measure of E. If $M(r) := \inf\{|s|;\ s \in \mathbb{R} \ \text{ and } \ |g(s)| \geq r\}$,

$$\int_E |g(u_k')|\, dxdt \leq \sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}}\right)\right)^{-1} \int_{E_2} |u_k' g(u_k')|\, dxdt.$$

Applying (1.40) we deduce that $\sup_k \int_E |g(u_k')|\, dxdt \to 0$ as $|E| \to 0$. From Vitali's convergence theorem we deduce that $g(u_k') \to g(u')$ in $L^1(\Omega \times (0,T))$, hence

$$g(u_k') \to g(u') \text{ weak star in } L^2(Q).$$

Similarly, we have

$$g(z_k') \to g(z') \text{ weak star in } L^2(Q),$$

and this implies that

$$\int_0^T \int_\Omega g(u_k')v\, dx\, dt \to \int_0^T \int_\Omega g(u')v\, dx\, dt \text{ for all } v \in L^2(0,T;H_0^2) \tag{1.57}$$

$$\int_0^T \int_\Omega g(z_k)v\, dx\, dt \to \int_0^T \int_\Omega g(z)v\, dx\, dt \text{ for all } v \in L^2(0,T;H_0^2) \tag{1.58}$$

as $k \to +\infty$. Using the compactness of H_0^2 in L^2, we see that

$$\int_0^T \int_\Omega b|u_k|^{p-2}u_k v\, dx\, dt \to \int_0^T \int_\Omega b|u|^{p-2}uv\, dx\, dt \text{ for all } v \in L^2(0,T;H_0^2) \tag{1.59}$$

as $k \to +\infty$. It follows at once from (1.53), (1.54), (1.57), (1.58),(1.59) and (1.55) that for each fixed $v \in L^2(0,T;H_0^2)$ and $w \in L^2(0,T;H_0^1(\Omega \times (0,1)))$

$$\begin{aligned}&\int_0^T \int_\Omega (u_k'' + \Delta_x^2 u_k + \mu_1 g(u_k') + \mu_2 g(z_k) - b|u_k|^{p-2}u_k)v\, dx\, dt\\ &\to \int_0^T \int_\Omega (u'' - \Delta_x u + \mu_1 g(u') + \mu_2 g(z) - b|u|^{p-2}u)v\, dx\, dt\end{aligned}$$

$$\int_0^T \int_0^1 \int_\Omega (\tau z_k' + \frac{\partial}{\partial \rho} z_k) w\, dx\, d\rho\, dt \to \int_0^T \int_0^1 \int_\Omega (\tau z' + \frac{\partial}{\partial \rho} z) w\, dx\, d\rho\, dt$$

as $k \to +\infty$. Hence

$$\int_0^T \int_\Omega (u'' + \Delta_x^2 u + \mu_1 g(u') + \mu_2 g(z) - b|u|^{p-2}u)v\, dx\, dt = 0, \qquad v \in L^2(0,T;H_0^2).$$

$$\int_0^T \int_0^1 \int_\Omega (\tau z' + \frac{\partial}{\partial \rho} z) w\, dx\, d\rho\, dt = 0, \qquad w \in L^2(0,T;H_0^1(\Omega \times (0,1))).$$

Thus the problem (P) admits a global weak solution u.

Uniqueness. Let (u_1, z_1) and (u_2, z_2) be two solutions of problem (1.11). Then $(w, \tilde{w}) = (u_1, z_1) - (u_2, z_2)$ verifies

$$(1.60)\quad \begin{cases} w''(x,t) + \Delta_x^2 w(x,t) + \mu_1 g(u_1'(x,t)) - \mu_1 g(u_2'(x,t)) & \\ \qquad +\mu_2 g(z_1(x,1,t)) - \mu_2 g(z_2(x,1,t)) = b|w|^{p-2}w, & \text{in } \Omega\times]0,+\infty[, \\ \tau \tilde{w}'(x,\rho,t) + \tilde{w}_\rho(x,\rho,t) = 0, & \text{in } \Omega\times]0,1[\times]0,+\infty[\\ w(x,t) = 0, & \text{on } \partial\Omega\times[0,+\infty[\\ \tilde{w}(x,0,t) = u_1'(x,t) - u_2'(x,t) & \text{on } \Omega\times[0,+\infty[\\ w(x,0) = 0,\ w'(x,0) = 0 & \text{in } \Omega \\ \tilde{w}(x,\rho,0) = 0 & \text{in } \Omega\times]0,1[\end{cases}$$

Multiplying the first equation in (1.60) by w, integrating over Ω and using integration by parts, we get

$$(1.61)\quad \begin{array}{l} \frac{1}{2}\frac{d}{dt}(\|w'\|_2^2 + \|\Delta_x w\|_2^2 - \frac{b}{p}\|w\|_p^p) \\ +\mu_1(g(u_1') - g(u_2'), w) + \mu_2(g(z_1'(x,1,t)) - g(u_2'(x,1,t)), w) = 0 \end{array}$$

Multiplying the second equation in (1.60) by $\tilde{w}$, integrating over $\Omega \times (0,1)$ and using integration by parts, we get

$$(1.62)\quad \tau\frac{1}{2}\frac{d}{dt}\int_0^1 \|\tilde{w}'\|_2^2\, d\rho + \frac{1}{2}(\|\tilde{w}(x,1,t)\|_2^2 - \|w'\|_2^2) = 0$$

From (1.61), (1.62), using Cauchy-Schwartz and Sobolev Poincaré inequalities we get

$$\begin{array}{l} \frac{1}{2}\frac{d}{dt}\left(\|w'\|_2^2 + \|\Delta_x w\|_2^2 - \frac{b}{p}\|w\|_p^p + \tau\int_0^1 \|\tilde{w}'\|_2^2\, d\rho\right) + \mu_1(g(u_1') - g(u_2'), w) + \frac{1}{2}\|\tilde{w}(x,1,t)\|_2^2 \\ = -\mu_2(g(z_1'(x,1,t)) - g(z_2'(x,1,t)), w) + \frac{1}{2}\|w'\|_2^2 \\ \leq \frac{1}{2}\|w'\|_2^2 + ||g(z_1'(x,1,t)) - g(z_2'(x,1,t))||_2||\Delta w||_2. \end{array}$$

Using condition (1.3) and Young inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|w'\|_2^2 + \|\Delta_x w\|_2^2 - \frac{b}{p}\|w\|_p^p + \tau\int_0^1 \|\tilde{w}'\|_2^2\, d\rho\right) \leq \frac{1}{2}\|w'\|_2^2 + c||\Delta w||_2^2.$$

where c is a positive constant. Then integrating over $(0,t)$, using Gronwalls lemma, we conclude that

$$\|w'\|_2^2 + \|\Delta_x w\|_2^2 - \frac{b}{p}\|w\|_p^p + \tau\int_0^1 \|\tilde{w}'\|_2^2\, d\rho = 0.$$

1.5 Asymptotic Stability

Before stating and proving the decay result, we start with

Lemma 1.5.1 *suppose that (1.7) holds. If $u_0 \in H$, and $u_1 \in L^2(\Omega)$ satisfying (1.30). Then*

$$b\| \ u(t)\|_p^p \leq (1-\eta)\|\Delta_x u(t)\|_2^2 \tag{1.63}$$

Proof: It suffices to write (1.34) as

$$b\|u(t)\|_p^p \leq \left\{1 - \left[1 - b\ C_*^p \left(\frac{2p}{p-2}E(u_0,u_1)\right)^{(p-2)/2}\right]\right\} \|\Delta_x u(t)\|_2^2.$$

From now on, we denote by c various positive constants which may be different at different occurrences. We multiply the first equation of (1.11) by $\dfrac{\varphi(E)}{E}u$, we obtain that

$$\begin{aligned}
0 &= \int_S^T \frac{\varphi(E)}{E}\int_\Omega u(u'' + \Delta^2 u + \mu_1 g(u'(x,t)) + \mu_2 g(z(x,1,t)) - bu|u|^{p-2})\,dx\,dt \\
&= \left[\frac{\varphi(E)}{E}\int_\Omega uu'\,dx\right]_S^T - \int_S^T (\frac{\varphi(E)}{E})'\int_\Omega uu'\,dxdt - 2\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'^2\,dx\,dt \\
&\quad + \int_S^T \frac{\varphi(E)}{E}\int_\Omega \left(u'^2 + |\Delta u|^2 - \frac{2b}{p}|u|^p\right)dxdt + \int_S^T \frac{\varphi(E)}{E}\int_\Omega b(\frac{2}{p}-1)|u|^p\,dx\,dt \\
&\quad + \mu_1\int_S^T \frac{\varphi(E)}{E}\int_\Omega ug(u')\,dx\,dt + \mu_2\int_S^T \frac{\varphi(E)}{E}\int_\Omega ug(z(x,1,t))\,dxdt.
\end{aligned}$$

Since

$$\begin{aligned}
b\left(1-\frac{2}{p}\right)\int_\Omega |u|^p\,dx &\leq (1-\eta)\frac{p-2}{p}\int_\Omega |\Delta_x u|^2\,dx \\
&\leq (1-\eta)\frac{p-2}{p}\frac{2p}{p-2}E(t) \\
&= 2(1-\eta)E(t),
\end{aligned}$$

Similarly, we multiply the second equation of (1.11) by $\dfrac{\varphi(E)}{E}e^{-2\tau\rho}g(z(x,\rho,t))$, we have

$$\begin{aligned}
0 &= \int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_0^1 e^{-2\tau\rho}g(z)(\tau z' + z_\rho)\,dxd\rho dt \\
&= \left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1 \tau e^{-2\tau\rho}G(z)\,dxd\rho\right]_S^T - \tau\int_S^T (\frac{\varphi(E)}{E})'\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)\,dxd\rho dt \\
&\quad + \int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_0^1 \left(\frac{\partial}{\partial\rho}(e^{-2\tau\rho}G(z)) + 2\tau e^{-2\tau\rho}G(z)\right)dxd\rho dt \\
&= \left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1 \tau e^{-2\tau\rho}G(z)\,dxd\rho\right]_S^T - \tau\int_S^T (\frac{\varphi(E)}{E})'\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)\,dxd\rho dt \\
&\quad + \int_S^T \frac{\varphi(E)}{E}\int_\Omega (e^{-2\tau}G(z(x,1,t)) - G(z(x,0,t)))\,dxdt + 2\tau\int_S^T \frac{\varphi(E)}{E}\int_0^1\int_\Omega e^{-2\tau\rho}G(z)\,dxd\rho dt
\end{aligned}$$

Taking their sum, we obtain that

$$
\begin{aligned}
&A\int_S^T \varphi(E)\,dt \le -\left[\frac{\varphi(E)}{E}\int_\Omega uu'\,dx\right]_S^T + \int_S^T (\frac{\varphi(E)}{E})'\int_\Omega uu'\,dxdt \\
&+2\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'^2\,dxdt - \mu_1\int_S^T \frac{\varphi(E)}{E}\int_\Omega ug(u')\,dx\,dt - \mu_2\int_S^T \frac{\varphi(E)}{E}\int_\Omega ug(z(x,1,t))\,dxdt \\
&-\left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1 \tau e^{-2\tau\rho}G(z)\,dxd\rho\right]_S^T + \tau\int_S^T(\frac{\varphi(E)}{E})'\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)\,dxd\rho dt \\
&-\int_S^T \frac{\varphi(E)}{E}\int_\Omega (e^{-2\tau}G(z(x,1,t)) - G(z(x,0,t)))\,dxdt,
\end{aligned}
\tag{1.64}
$$

where $A = 2\min\{\eta, \tau e^{-2\tau}/2\xi\}$. Since E is non-increasing, we find that

$$
\begin{aligned}
-\left[\frac{\varphi(E)}{E}\int_\Omega uu'\,dx\right]_S^T &= \frac{\varphi(E(S))}{E(S)}\int_\Omega u(S)u'(S)\,dx - \frac{\varphi(E(T))}{E(T)}\int_\Omega u(T)u'(T)\,dx \\
&\le C\varphi(E(S))
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_S^T (\frac{\varphi(E)}{E})'\int_\Omega uu'\,dxdt\right| &\le c\int_S^T |(\frac{\varphi(E)}{E})'|E\,dt \\
&\le c\varphi(E(S))
\end{aligned}
$$

$$
\begin{aligned}
&-\left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)\,dxd\rho\right]_S^T \\
&= \frac{\varphi(E(S))}{E(S)}\int_\Omega\int_0^1 e^{-2\tau\rho}G(z(x,\rho,S))\,dxd\rho - \frac{\varphi(E(T))}{E(T)}\int_\Omega\int_0^1 e^{-2\tau\rho}G(z(x,\rho,T))\,dxd\rho \\
&\le C\varphi(E(S))
\end{aligned}
$$

$$
\begin{aligned}
\int_S^T ((\frac{\varphi(E)}{E})')\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)\,dxd\rho dt &\le c\int_S^T (-(\frac{\varphi(E)}{E})')E\,dt \\
&\le c\varphi(E(S))
\end{aligned}
$$

$$
\begin{aligned}
\int_S^T \frac{\varphi(E)}{E}\int_\Omega e^{-2\tau}G((x,1,t))\,dxdt &\le c\int_S^T \frac{\varphi(E)}{E}(-E')\,dt \\
&\le c\varphi(E(S))
\end{aligned}
$$

$$
\begin{aligned}
\int_S^T \frac{\varphi(E)}{E}\int_\Omega G(z(x,0,t))\,dxdt &= \int_S^T \frac{\varphi(E)}{E}\int_\Omega G(u'(x,t))\,dxdt \\
&\le c\int_S^T \frac{\varphi(E)}{E}(-E')\,dt \\
&\le c\varphi(E(S))
\end{aligned}
$$

where we have also used the Cauchy-Schwartz inequality. Using these estimates we conclude from (1.64) that

$$
\begin{aligned}
A\int_S^T \varphi(E)\,dt \le c\varphi(E(S)) + \mu_1\int_S^T \frac{\varphi(E)}{E}\int_\Omega |u||g(u')|\,dx\,dt + 2\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'^2\,dxdt \\
+\mu_2\int_S^T \frac{\varphi(E)}{E}\int_\Omega |u||g(z(x,1,t))|\,dxdt
\end{aligned}
\tag{1.65}
$$

Now, we estimate the terms of the right-hand side of (1.65) in order to apply the results of Lemma 1.3.2.

We distinguish two cases.

1. H is linear on $[0,\epsilon_1]$:

2.1 If g is nonlinear, we have $C_1|s| \le |g(s)| \le C_2|s|$, for all $s \in \mathbb{R}$, and then, using (1.4) and noting that $s \mapsto \frac{\varphi(E(s))}{E(s)}$ is non-increasing,

$$\int_S^T \frac{\varphi(E)}{E}\int_\Omega |u'|^2 dxdt \le c\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'g(u')dxdt \le c\varphi(E(S)),$$

Using Poincaré's, Young's inequalities and the energy inequality from Lemma 1.3.3, we obtain, for all $\epsilon > 0$,

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_\Omega |ug(u')|dxdt &\le \epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega u^2 dxdt + c_\epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega g^2(u')dxdt\\ &\le \epsilon c\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\Delta u|^2 dxdt + c_\epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'g(u')dxdt\\ &\le \epsilon c\int_S^T \varphi(E)dt + c_\epsilon\varphi(E(S)).\end{aligned}$$

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_\Omega |ug(z(x,1,t))|dxdt &\le \epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega u^2 dxdt + c_\epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega g^2(z(x,1,t))dxdt\\ &\le \epsilon c\int_S^T \frac{\varphi(E)}{E}\int_\Omega |\Delta u|^2 dxdt + c_\epsilon\int_S^T \frac{\varphi(E)}{E}\int_\Omega z(x,1,t)g(z(x,1,t))dxdt\\ &\le \epsilon c\int_S^T \varphi(E)dt + c_\epsilon\varphi(E(S)).\end{aligned}$$

Inserting these two inequalities into (1.65), choosing $\epsilon > 0$ small enough, we deduce that

$$\int_S^T \varphi(E(t))dt \le c\varphi(E(S)).$$

Using Lemma 1.3.2 (Guesmia **[21]**) for E in the particular case where $\varphi(s) = s$, we deduce from (1.8) that

$$E(t) \le ce^{-\omega t}$$

2. $H'(0) = 0$ **and** $H'' > 0$ **on** $]0,\epsilon_1]$**:** for all $t \ge 0$, we denote by

$$\Omega_t^1 = \{x \in \Omega : |u'| \ge \epsilon_1\}, \quad \Omega_t^2 = \{x \in \Omega : |u'| \le \epsilon_1\}$$

Using (1.1), (1.4) and the fact that $s \mapsto \frac{\varphi(s)}{s}$ is non-decreasing, we obtain

$$c\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^1} (|u'|^2 + g^2(u'))dxdt \le c\int_S^T \frac{\varphi(E)}{E}\int_\Omega u'g(u')dxdt \le c\varphi(E(S)).$$

On the other hand, since H is convex and increasing, H^{-1} is concave and increasing. Therefore, (1.2) and the reversed Jensen's inequality for concave function imply that

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^2}(|u'|^2+g^2(u'))\,dxdt &\leq \int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^2} H^{-1}(u'g(u'))\,dxdt \\ &\leq \int_S^T \frac{\varphi(E)}{E}|\Omega|H^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega u'g(u')dx\Big)dt\end{aligned} \tag{1.66}$$

Let us denote H^* to be the conjugate function of the convex function H, i.e., $H^*(s) = \sup_{t\in\mathbb{R}^+}(st-H(t))$. Then H^* is the Legendre transform of H, which is given by (see Arnold [**8**], p. 61-62, and Lasiecka [**12**], [**16**], [**35**]-[**36**])

$$H^*(s) = s(H')^{-1}(s) - H[(H')^{-1}(s)], \quad \forall s \geq 0 \tag{1.67}$$

and satisfies the following inequality

$$st \leq H^*(s) + H(t), \quad \forall s,t \geq 0. \tag{1.68}$$

Due to our choice $\varphi(s) = sH'(\epsilon_0 s)$, we have

$$H^*(\frac{\varphi(s)}{s}) = \epsilon_0 sH'(\epsilon_0 s) - H(\epsilon_0 s) \leq \epsilon_0\varphi(s). \tag{1.69}$$

Making use of (1.66), (1.68) and (1.69), we have

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^2}(|u'|^2+g^2(u'))\,dxdt &\leq c\int_S^T H^*(\frac{\varphi(E)}{E})dt + c\int_S^T\int_\Omega u'g(u')dt \\ &\leq \epsilon_0\int_S^T \varphi(E)dt + cE(S).\end{aligned}$$

Then, choosing $\epsilon_0 > 0$ small enough and using (1.65), we obtain in both cases

$$\begin{aligned}\int_S^{+\infty}\varphi(E(t))dt &\leq c\Big(E(S)+\varphi(E(S))\Big) \\ &\leq c\left(1+\frac{\varphi(E(S))}{E(S)}\right)E(S) \\ &\leq cE(S) \quad \forall S \geq 0.\end{aligned} \tag{1.70}$$

Using Lemma 1.3.2 in the particular case where $\Psi(s) = \omega\varphi(s)$, we deduce from (1.8) our estimate (1.15). The proof of Theorem 1.3.1 is now complete.

Chapter 2

ENERGY DECAY FOR A SYSTEM OF DEGENERATE KIRCHHOFF EQUATION WITH WEAKLY NONLINEAR DISSIPATION

2.1 Introduction

In this chapter we consider the initial boundary value problem for the nonlinear Kirchhoff equation

$$(2.1)\ (|u_t|^{l-2}u_t)' - \Big(\int_\Omega |\nabla_x u|^2\,dx + \int_\Omega |\nabla_x v|^2\,dx\Big)^\gamma \Delta_x u + \alpha(t)g_1(u_t) = 0, \text{ in } \Omega\times(0,\infty),$$

$$(2.2)\ (|v_t|^{l'-2}v_t)' - \Big(\int_\Omega |\nabla_x u|^2\,dx + \int_\Omega |\nabla_x v|^2\,dx\Big)^\gamma \Delta_x v + \alpha(t)g_2(v_t) = 0, \text{ in } \Omega\times(0,\infty),$$

$$(2.3)\qquad u = 0, v = 0 \quad \text{on} \quad \partial\Omega\times(0,\infty)$$

$$(2.4)\qquad u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \ \ x\in\Omega.$$

$$(2.5)\qquad v(x,0) = v^0(x), \quad v_t(x,0) = v^1(x), \ \ x\in\Omega.$$

where $l, l' \geq 2$ and $\gamma \geq 0, \Omega$ is a bounded domain in $\mathbb{R}^n$.
When $l = 2$ and $\gamma = 0$, the problem (2.1)-(2.5) were treated by Mustafa and Massaoudi, they studied the decay property of the energy of (2.1)-(2.5) and they used some properties of convex functions.
Abdelli and Benaissa [**1**] treated system (2.1)-(2.5) for g having a polynomial growth near the origin and established energy decay results depending on α and g and they find the relationship between l and γ.
When $\gamma = 0$ and p-Laplcian type Benaissa and Mimouni [**10**] study the decay rate of solutions used the multiplier technique introduced by Martinez [**42**].
A blow-up result has been proved by Benaissa and Messaoudi [**11**]for system (2.1)-(2.5) of

p-Laplacian type with nonlinear damping and source terms.
In [**9**], Benaissa and Guesmia proved the existence of global solution in Sobolev spaces of ϕ-Laplacien with a general dissipation of the form

$$(|u'|^{l-2}u')' - \triangle_\phi u + \alpha(t)g(u') = 0, \quad \text{in} \quad \Omega \times \mathbb{R}_+,$$

where $\triangle_\phi = \sum_{i=1}^n \partial_{x_i}(\phi(|\partial_{x_i}|^2)\partial_{x_i})$. Then, they proved general stability estimates.
In [**6**] Benaissa and Amroun who constructed exact solution of (2.1)-(2.5) with nonlinear source term without dissipative term for some initial data and showed finite time blowing up results for some other initial data.
In this paper we use some technique from [**48**], we establish an explicit and general decay result, depending on g and α. The proof is based on the multiplier method and makes use of some properties of convex functions including the use of the general Young's inequality and Jensen's inequality. These convexity arguments were introduced and developed by Lasieka and co-workers ([**33**], [**36**], [**37**]) and used by Liu and Zuazua [**41**] and Alabau-Boussouira [**4**].
The paper is organized as follow: in section 2, we give some hypotheses. In section 3, we prove the energy estimates.

2.2 Preliminaries and main results

We use the following hypotheses:
(H1) $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing differentialble function.
(H2) $g_i : \mathbb{R} \to \mathbb{R}$ is non decreasing function of class C^0 such that there exist $\varepsilon, c_1, c_2 > 0$, $l-1 \leq p$ and an convex and increasing function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying $C(0) = 0$ and $G'(0) = 0$ or G is linear on $[0, \varepsilon]$ such that for $i = 1, 2$

$$c_1|s|^{l-1} \leq |g_i(s)| \leq c_2|s|^p \quad \text{if } |s| \geq \varepsilon$$

$$|s|^l + |g_i(s)|^{\frac{l}{l-1}} \leq G^{-1}(sg_i(s)) \quad \text{if } |s| \leq \varepsilon,$$

and p satisfies

$$1 \leq p \leq \frac{n+2}{n-2} \quad \text{if } n > 2$$

$$1 \leq p < \infty \quad \text{if } n \leq 2.$$

Now we define the energy associated to the solution of the system (2.1)-(2.5) by the following formula

$$E(t) = \frac{l-1}{l}\|u_t\|_l^l + \frac{1}{1+\gamma}(\|\nabla_x u\|_2^2 + \|\nabla_x v\|_2^2)^{(\gamma+1)} + \frac{l'-1}{l'}\|v_t\|_{l'}^{l'}. \tag{2.6}$$

Lemma 2.2.1 *Let (u, v) be the solution of (2.1)-(2.5). Then*

$$E'(t) = -\alpha(t)\int_\Omega (u_t g_1(u_t) + v_t g_2(v_t))\, dx \leq 0. \tag{2.7}$$

Lemma 2.2.2 *Assume that (H1), (H2) hold and* $\max(l,l') \geq 2(\gamma+1)$. *Then, for some positive constants M, c and m, the functional F defined by*

$$F(t) = ME(t) + \int_\Omega (u|u_t|^{l-2}u_t + v|v_t|^{l'-2}v_t)\,dx,$$

satisfies, along the solution, the estimate

$$F'(t) \leq -mE(t) + c_1 \int_\Omega (|u_t|^l + |ug_1(u_t)|^{\frac{l}{l-1}} + |v_t|^{l'} + |vg_2(v_t)|^{\frac{l'}{l'-1}})\,dx,$$

and

$$F(t) \sim E(t).$$

Proof. Using the system (2.1)-(2.5), (3.9) and (2.7), we obtain

$$\begin{aligned}
F'(t) &= ME'(t) + \int_\Omega (|u_t|^l + |v_t|^{l'})\,dx + \int_\Omega (u(|u_t|^{l-2}u_t)' + v(|v_t|^{l'-2}v_t)')\,dx \\
&= ME'(t) + \int_\Omega (|u_t|^l + |v_t|^{l'})\,dx + \Big((\|\nabla_x u\|_2^2 + \|\nabla_x v\|_2^2)^\gamma (\int_\Omega \Delta_x u u\,dx + \int_\Omega \Delta_x v v\,dx) \\
&\quad -\alpha(t)\int_\Omega (ug_1(u_t) + vg_2(v_t))\,dx \\
&= ME'(t) + \int_\Omega (|u_t|^l + |v_t|^{l'})\,dx - (\|\nabla_x u\|_2^2 + \|\nabla_x v\|_2^2)^{\gamma+1} - \alpha(t)\int_\Omega (ug_1(u_t) + vg_2(v_t))\,dx, \\
&\leq -mE(t) + c_1 \int_\Omega (|u_t|^l + |ug_1(u_t)| + |v_t|^{l'} + |vg_2(v_t)|)\,dx.
\end{aligned} \tag{2.8}$$

To prove that $F(t) \sim E(t)$, we show that

$$\lambda_1 E(t) \leq F(t) \leq \lambda_2 E(t), \tag{2.9}$$

for some positive constants λ_1 and λ_2. We use (3.9) Poincar's and Young's inequalities with exponents $\frac{l}{l-1}$ and $\frac{1}{l}$ and we assume that $(2 \leq l \leq p+1 \leq \frac{2n}{n+2})$, we get

$$\begin{aligned}
\int_\Omega [u|u_t|^{l-2}u_t + v|v_t|^{l'-2}v_t]\,dx &\leq C_\varepsilon \int_\Omega (|u|^l + |v|^{l'})\,dx + \varepsilon \int_\Omega (|u_t|^l + |v_t|^{l'})\,dx \\
&\leq C_\varepsilon (\|\nabla_x u\|_2^l + \|\nabla_x v\|_2^{l'}) + \varepsilon(\|u_t\|_l^l + \|v_t\|_{l'}^{l'}) \\
&\leq C_\varepsilon (E^{\frac{l}{2(\gamma+1)}}(t) + E^{\frac{l'}{2(\gamma+1)}}(t)) + c\varepsilon E(t) \\
&\leq C_\varepsilon (E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(t) + E^{\frac{l'-2(\gamma+1)}{2(\gamma+1)}}(t))E(t) + c\varepsilon E(t),
\end{aligned} \tag{2.10}$$

we assume that $\max(l,l') \geq 2(\gamma+1)$, we have

$$\int_\Omega [u|u_t|^{l-2}u_t + v|v_t|^{l'-2}v_t]\,dx \leq C_\varepsilon (E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(0) + E^{\frac{l'-2(\gamma+1)}{2(\gamma+1)}}(0))E(t) + c\varepsilon E(t), \tag{2.11}$$

and

$$\begin{aligned}
\int_\Omega [u|u_t|^{l-2}u_t + v|v_t|^{l'-2}v_t]\,dx &\geq -C_\varepsilon \int_\Omega (|u|^l + |v|^{l'})\,dx - \varepsilon \int_\Omega (|u_t|^l + |v_t|^{l'})\,dx \\
&\geq -C_\varepsilon (E^{\frac{l}{2(\gamma+1)}}(t) + E^{\frac{l'}{2(\gamma+1)}}(t)) - c\varepsilon E(t) \\
&\geq -C_\varepsilon (E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(0) + E^{\frac{l'-2(\gamma+1)}{2(\gamma+1)}}(0))E(t) - c\varepsilon E(t)
\end{aligned} \tag{2.12}$$

Then, for M large enough, we obtain (2.9). This completes the proof of Lemma 3.20. Taking $0 < \varepsilon_1 < \varepsilon$ such that

$$sg_i(s) \leq \min\{\varepsilon, G(\varepsilon)\} \quad \text{if } |s| \leq \varepsilon_1, \tag{2.13}$$

and

$$\begin{cases} c_1'|s|^{l-1} \leq |g_i(s)| \leq c_2'|s|^p & \text{if } |s| \geq \varepsilon_1 \\ |s|^l + |g_i(s)|^{\frac{l}{l-1}} \leq G^{-1}(sg_i(s)) & \text{if } |s| \leq \varepsilon_1. \end{cases} \tag{2.14}$$

Considering the following partition of Ω

$$\Omega_1 = \{x \in \Omega : \ |u_t| \leq \varepsilon_1, |v_t| \leq \varepsilon_1'\}, \quad \Omega_2 = \{x \in \Omega : \ |u_t| > \varepsilon_1, |v_t| > \varepsilon_1'\}$$

using the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and Hölder inequality, we get

$$\begin{aligned} \int_{\Omega_2} (|ug_1(u_t)| + |vg_2(v_t)|)\, dx &\leq \Big(\int_{\Omega_2} |u|^{p+1}\, dx\Big)^{\frac{1}{p+1}} \Big(\int_{\Omega_2} |g_1(u_t)|^{1+\frac{1}{p}}\, dx\Big)^{\frac{p}{p+1}} \\ &\quad + \Big(\int_{\Omega_2} |v|^{p+1}\, dx\Big)^{\frac{1}{p+1}} \Big(\int_{\Omega_2} |g_2(v_t)|^{1+\frac{1}{p}}\, dx\Big)^{\frac{p}{p+1}} \\ &\leq c\|u\|_{H_0^1(\Omega)} \Big(\int_{\Omega_2} |g_1(u_t)|^{1+\frac{1}{p}}\, dx\Big)^{\frac{p}{p+1}} + c\|v\|_{H_0^1(\Omega)} \Big(\int_{\Omega_2} |g_2(v_t)|^{1+\frac{1}{p}}\, dx\Big)^{\frac{p}{p+1}}. \end{aligned} \tag{2.15}$$

Using Poincar's inequality and (2.14) yield

$$\begin{aligned} \int_{\Omega_2} [|u_t|^l + |ug_1(u_t)| + |v_t|^{l'} + |vg_2(v_t)|]\, dx &\leq c \int_{\Omega_2} (|u_t|^{l-1}|u_t| + |v_t|^{l'-1}|v_t|)\, dx + \\ &\quad c\Big(\int_{\Omega} |\nabla_x u|^2\, dx\Big)^{\frac{1}{2}} \Big(\int_{\Omega_2} |g_1(u_t)|^{1+\frac{1}{p}}\, dx\Big)^{\frac{p}{p+1}} \\ &\quad + c'\Big(\int_{\Omega} |\nabla_x v|^2\, dx\Big)^{\frac{1}{2}} \Big(\int_{\Omega_2} |g_2(v_t)|^{1+\frac{1}{p}}\, dx\Big)^{\frac{p}{p+1}} \\ &\leq c \int_{\Omega_2} (u_t g_1(u_t) + v_t g_2(v_t))\, dx + \\ &\quad c\Big(\int_{\Omega} |\nabla_x u|^2\, dx\Big)^{\frac{1}{2}} \Big(\int_{\Omega_2} |g_1(u_t)|^{1+\frac{1}{p}}\, dx\Big)^{\frac{p}{p+1}} \\ &\quad + c'\Big(\int_{\Omega} |\nabla_x v|^2\, dx\Big)^{\frac{1}{2}} \Big(\int_{\Omega_2} |g_2(v_t)|^{1+\frac{1}{p}}\, dx\Big)^{\frac{p}{p+1}} \\ &\leq -cE'(t) + cE^{\frac{1}{4(\gamma+1)}} (-E'(t))^{\frac{p}{p+1}}. \end{aligned} \tag{2.16}$$

Then, we use Young's inequality and we assume $p \geq 2\gamma + 1$, we have

$$\begin{aligned} \int_{\Omega_2} [|u_t|^l + |ug_1(u_t)| + |v_t|^{l'} + |vg_2(v_t)|]\, dx &\leq -cE'(t) + c\varepsilon E^{\frac{p+1}{4(\gamma+1)}}(t) + C_\varepsilon(-E'(t)) \\ &\leq c\varepsilon E^{\frac{p+1}{4(\gamma+1)}}(t) - C_\varepsilon E'(t) \\ &\leq c\varepsilon E^{\frac{p-(4\gamma+3)}{4(\gamma+1)}}(0) E(t) - C_\varepsilon E'(t). \end{aligned} \tag{2.17}$$

Similarly, using (3.9) and Young's inequality, we have

$$
\begin{aligned}
\int_{\Omega_1}[|u_t|^l+|ug_1(u_t)|+|v_t|^{l'}+|vg_2(v_t)|]\,dx &\leq \int_{\Omega_1}(|u_t|^l+|v_t|^{l'})\,dx+\varepsilon\int_{\Omega_1}(|u|^l+|v|^{l'})\,dx\\
&\quad +C_\varepsilon\int_{\Omega_1}(|g_1(u_t)|^{\frac{l}{l-1}}+|g_2(v_t)|^{\frac{l'}{l'-1}})\,dx\\
&\leq \int_{\Omega_1}(|u_t|^l+|v_t|^{l'})\,dx+c\varepsilon(E^{\frac{l}{2(\gamma+1)}}(t)+E^{\frac{l'}{2(\gamma+1)}}(t))\\
&\quad +C_\varepsilon\int_{\Omega_1}(|g_1(u_t)|^{\frac{l}{l-1}}+|g_2(v_t)|^{\frac{l'}{l'-1}})\,dx
\end{aligned} \tag{2.18}
$$

By Lemma 3.20, (2.17) and (2.18), for ε small enough, that the function $L=F+C_\varepsilon E$ satisfies

$$
\begin{aligned}
L'(t) &\leq \Big(-m+c\varepsilon E^{\frac{p-(4\gamma+3)}{4(\gamma+1)}}(0)+c\varepsilon E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(0)+c\varepsilon E^{\frac{l'-2(\gamma+1)}{2(\gamma+1)}}(0)\Big)E(t)\\
&\quad +\int_{\Omega_1}(|u_t|^l+|v_t|^{l'})\,dx+C_\varepsilon\int_{\Omega_1}(|g_1(u_t)|^{\frac{l}{l-1}}+|g_2(v_t)|^{\frac{l'}{l'-1}}\,dx\\
&\leq -dE(t)+c\int_{\Omega_1}\Big((|u_t|^l+|v_t|^{l'})+(|g_1(u_t)|^{\frac{l}{l-1}}+|g_2(v_t)|^{\frac{l'}{l'-1}})\Big)\,dx
\end{aligned} \tag{2.19}
$$

and

$$L(t)\sim E(t). \tag{2.20}$$

Theorem 2.2.1 *Assume That (H1), (H2) hold* $max(l,l')\geq 2(\gamma+1)$ *and* $p\geq 4\gamma+3$. *The, there exist positive constants* k_1, k_2, k_3 *and* ε_0 *such that the solution of (2.1)-(2.4) satisfies*

$$E(t)\leq k_3G_1^{-1}(k_1\int_0^t\alpha(s)\,ds+k_2)\quad \forall t\geq 0, \tag{2.21}$$

where

$$G_1(t)=\int_t^1\frac{1}{G_2(s)}\,ds \qquad \text{and} \quad G_2(t)=tG'(\varepsilon_0 t).$$

Here, G_1 *is strictly decreasing and convex on* $(0,1]$ *with* $\lim_{t\to 0}G_1(t)=+\infty$.

proof We multiply (2.19) by $\alpha(t)$, we have

$$\alpha(t)L'(t)\leq -d\alpha(t)E(t)+c\alpha(t)\int_{\Omega_1}\Big((|u_t|^l+|v_t|^{l'})+(|g_1(u_t)|^{\frac{l}{l-1}}+|g_2(v_t)|^{\frac{l'}{l'-1}})\Big)\,dx \tag{2.22}$$

- Case 1. G is linear on $[0,\varepsilon]$, then, we deduce that

$$\alpha(t)L'(t)\leq -d\alpha(t)E(t)+c\alpha(t)\int_{\Omega_1}(u_tg_1(u_t)+v_tg_2(v_t))\,dx=-d\alpha(t)E(t)-cE'(t) \tag{2.23}$$

$$(\alpha L+cE)'(t)\leq -d\alpha(t)E(t),$$

using

$$\alpha L+cE\sim E, \tag{2.24}$$

we obtain

$$E(t) \le c' e^{-c'' \int_0^t \alpha(s)\,ds}$$

$$\int_0^t \frac{E'(s)}{E(s)}\,ds \le -d\int_0^t \alpha(s)\,ds$$

$$\log\Big(\frac{E(t)}{E(0)}\Big) \le -d\int_0^t \alpha(s)\,ds$$

$$-\log\Big(\frac{E(t)}{E(0)}\Big) \ge d\int_0^t \alpha(s)\,ds$$

$$G_1\Big(\frac{E(t)}{E(0)}\Big) \ge d\int_0^t \alpha(s)\,ds$$

$$E(t) \le c' G_1^{-1}(c''\int_0^t \alpha(s)\,ds).$$

Then

$$E(t) \le c' e^{-c'' \int_0^t \alpha(s)\,ds} = c' G_1^{-1}(c''\int_0^t \alpha(s)\,ds)$$

- Case 2. G is nonlinear on $[0,\varepsilon]$, we define $I(t)$ by

$$I(t) = \frac{1}{|\Omega_1|}\int_{\Omega_1} (u_t g_1(u_t) + v_t g_2(v_t))\,dx.$$

Jensen's inequality to get

$$G^{-1}(I(t)) \ge c\int_{\Omega_1} G^{-1}(u_t g_1(u_t) + v_t g_2(v_t))\,dx. \tag{2.25}$$

Using (2.14) and (2.25), we get

$$\begin{aligned} \alpha(t)\int_{\Omega_1} [|u_t|^l + |v_t|^{l'}) + (|g_1(u_t)|^{\frac{l}{l-1}} + |g_2(v_t)|^{\frac{l'}{l'-1}})]\,dx &\le \alpha(t)\int_{\Omega_1} G^{-1}(u_t g_1(u_t) + v_t g_2(v_t))\,dx \\ &\le c\alpha(t)G^{-1}(I(t)). \end{aligned} \tag{2.26}$$

Using (2.7) and (2.26), we get

$$H_0'(t) \le -d\alpha(t)E(t) + c\alpha(t)G^{-1}(I(t)) + E'(t) \le -d\alpha(t)E(t) + c\alpha(t)G^{-1}(I(t)) \tag{2.27}$$

where $H_0 = \alpha L + E$, it is clear from (2.20) and (2.24) we have $H_0 \sim E$.
For $\varepsilon_0 < \varepsilon$ and $c_0 > 0$, we define H_1 by

$$H_1(t) = G'(\varepsilon_0 \frac{E(t)}{E(0)})H_0(t) + c_0 E(t).$$

For $a_1,\, a_2 > 0$

$$a_1 H_1(t) \le E(t) \le a_2 H_1(t), \tag{2.28}$$

for $E' \leq 0$, $G' > 0$, $G'' > 0$ on $(0, \varepsilon]$ and applying (3.9) and (2.27) we obtain

$$\begin{aligned} H_1'(t) &= \varepsilon_0 \tfrac{E'(t)}{E(0)} G''(\varepsilon_0 \tfrac{E(t)}{E(0)}) H_0(t) + G'(\varepsilon_0 \tfrac{E(t)}{E(0)}) H_0'(t) + c_0 E'(t) \\ &\leq -d\alpha(t) E(t) G'(\varepsilon_0 \tfrac{E(t)}{E(0)}) + c\alpha(t) G'(\varepsilon_0 \tfrac{E(t)}{E(0)}) G^{-1}(I(t)) + c_0 E'(t). \end{aligned} \tag{2.29}$$

Let G^* be the convex conjugate of G in the sense of Young, then

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)], \quad \text{if } s \in (0, G'(\varepsilon)], \tag{2.30}$$

and G^* satisfies the following Young's inequality

$$AB \leq G^*(A) + G(B), \quad \text{if } A \in (0, G'(\varepsilon)],\ B \in (0, \varepsilon]. \tag{2.31}$$

with $A = G'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big)$ and $B = G^{-1}(I(t))$, using (2.7), (2.13) and (2.29)- (2.31), we get

$$\begin{aligned} H_1'(t) &\leq -d\alpha(t) E(t) G'(\varepsilon_0 \tfrac{E(t)}{E(0)}) + c\alpha(t) G^*\Big(G'(\varepsilon_0 \tfrac{E(t)}{E(0)})\Big) + c\alpha(t) I(t) + c_0 E'(t) \\ &\leq -d\alpha(t) E(t) G'(\varepsilon_0 \tfrac{E(t)}{E(0)}) + c\varepsilon_0 \alpha(t) \tfrac{E(t)}{E(0)} G'(\varepsilon_0 \tfrac{E(t)}{E(0)}) - cE'(t) + c_0 E'(t). \end{aligned} \tag{2.32}$$

Choising $c_0 < c$, we obtain

$$H_1'(t) \leq -k\alpha(t) \frac{E(t)}{E(0)} G'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) = -k\alpha(t) G_2\Big(\frac{E(t)}{E(0)}\Big) \tag{2.33}$$

where $G_2(t) = tG'(\varepsilon_0 t)$
Since

$$G_2'(t) = G'(\varepsilon_0 t) + \varepsilon_0 G''(\varepsilon_0 t).$$

Using the convexity of G on $(, \varepsilon]$, we find that $G_2'(t),\ G_2(t) > 0$ on $(0, 1]$ With $H(t) = \frac{H_1(t)}{E(0)}$ and using (2.28), we have

$$H(t) \sim E(t) \tag{2.34}$$

Applying (2.33), we get

$$H'(t) \leq -k_1 \alpha(t) G_2(H(t))$$

$$G_2(t) = \frac{-1}{G_1'(t)}$$

$$H'(t) \leq k_1 \alpha(t) \frac{1}{G_1'(H(t))}$$

$$H'(t) G_1'(H(t)) \leq k_1 \alpha(t)$$

$$[G_1(H(t))]' \leq k_1 \alpha(t)$$

$$\int_0^t [G_1(H(s))]'\, ds \leq k_1 \int_0^t \alpha(s)\, ds$$

$$G_1(H(s)) - G_1(H(0)) \leq k_1 \int_0^t \alpha(s)\, ds$$

then

$$H(t) \leq G_1^{-1}(k_1 \int_0^t \alpha(s)\, ds + k_2) \tag{2.35}$$

Using (2.34) and (2.35) we obtain (2.21).
The proof of Theorem 2.2.1 is now completed.

Chapter 3

GENERAL DECAY OF SOLUTION TO SOME NONLINEAR VECTOR EQUATION IN A FINITE DIMENSIONAL HILBERT SPACE

3.1 Introduction

Let H be a finite dimensional real Hilbert space, with norm denoted by $\|.\|$. We consider first the following nonlinear equation

$$u'' + \phi(\|A^{\frac{1}{2}}u\|^2)Au + g(u') = 0, \tag{3.1}$$

where A is a positive and symmetric linear operator on H. We denote by $(.,.)$ the inner product in H, A is coercive, which means :

$$\exists \lambda > 0, \ \forall u \in D(A), \ \ (Au, u) \geq \lambda \|u\|^2$$

We also define

$$\forall u \in H, \ \|A^{\frac{1}{2}}u\| := \|u\|_{D(A^{\frac{1}{2}})}$$

a norm equivalent to the norm in H. We assume that g and ϕ are locally Lipschitz continuous.

The consideration of the more complicated problem (3.1) is partially motivated by **[9]** in which a similar but harder (infinite dimensional) problem with general dissipation was studied with application to some PDE in a bounded domain. Under Neumann or Dirichlet boundary conditions, and for nonlinearities asymptotically homogeneous near 0 similar to the ones appearing in (3.1), they proved the existence of a global solution in Sobolev spaces to the initial boundary value problem of the (degenerate or non-degenerate) Kirchhoff equation with weak dissipation and they establish general stability estimates using the multiplier method and general weighted integral inequalities. When $\phi(u) = |u|^\beta u$ and $g(u') = c|u'|^\alpha u'$, Haraux in **[28]** studied the decay rate of the energy of non trivial solutions to the scalar second order ODE with initial data $(u_0, u_1) \in \mathbb{R}^2$. In addition, he showed that if $\alpha > \frac{\beta}{\beta+2}$ all non-trivial solutions are oscillatory and if $\alpha < \frac{\beta}{\beta+2}$ they are non-oscillatory.

We can also consider the equation

$$(\|u'\|^l u')' + \|A^{\frac{1}{2}}u\|^\beta Au + g(u') = 0, \tag{3.2}$$

where g is a locally Lipschitz continuous function. The equation (3.2) has been studied by Abdelli, Anguiano and Haraux **[3]**, they proved the existence and uniqueness of a global solution $u \in C^1(\mathbb{R}^+, H)$ with $\|u'\|^l u' \in C^1(\mathbb{R}^+, H)$ for any initial data $(u_0, u_1) \in H \times H$ they used some techniques from Abdelli and Haraux **[2]**. They used some modified energy function to estimate the rate of decay and they used the method introduced by Haraux **[28]**. Finally, they discuss the optimality of these estimates when $g(s) = c\|s\|^\alpha s$ and $l < \alpha < \frac{\beta(1+l)+l}{\beta+2}$.

In this article, we use some technique from to establish an explicit and general decay result, depending on g and ϕ. The proof is based on the multiplier method and makes use of some properties of convex functions, the general Young inequality and Jensen's inequality.

The plan of this paper is as follows: In Section 2 we establish some basic preliminary inequalities, and in Section 3 we prove the energy estimates.

3.2 Assumptions and preliminary results

In order to state and prove our result, we require the following assumptions:

(A1) $g : H \to H$ and $\phi : H \to H$ are a locally Lipschitz continuous functions.

(A2) $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is of the Class $C^1(\mathbb{R}_+)$ satisfying one of the following tow properties:

Degenerate case: $\phi(s) > 0$ on $]0, +\infty[$ and ϕ is non-decreasing.

Non-degenerate case: there exist $m_0,\ m_1$ such that $\phi(s) \geq m_0$ on $\mathbb{R}_+$ and

$$s\phi(s) \geq m_1 \int_0^s \phi(\tau)\,d\tau \quad \text{on } \mathbb{R}_+. \tag{3.3}$$

(A3) $g : \mathbb{R} \to \mathbb{R}$ is non decreasing function of class C^1 and $G : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying

$$\begin{cases} G(0) = 0 \text{ and } G \text{ is linear on } [0, r_0] \text{ or} \\ G'(0) = 0 \text{ and } G'' > 0 \text{ on }]0, r_0] \text{ such that} \\ c_2\|g(v)\|^2 \leq c_1\|v\|^2 \leq (g(v), v) \text{ if } \|v\| \geq r_0 \\ \|v\|^2 + \|g(v)\|^2 \leq G^{-1}(g(v), v) \text{ if } \|v\| \leq r_0 \end{cases} \tag{3.4}$$

where G^{-1} denotes the inverse function of G and r_0, c_1, c_2 are positive constants.

Remark 3.2.1 *1. In both the degenerate and the non-degenerate cases, we have $\int_0^{+\infty} \phi(\tau)\,d\tau = +\infty$, and then $\widetilde{\phi}(s) = \frac{1}{2}\int_0^s \phi(\tau)\,d\tau$ is a bijection from $\mathbb{R}^+$ to $\mathbb{R}^+$. On the other hand, (3.3) is satisfied in the degenerate case (with $m_1 = 1$) a well.*

2. In the degenerate case, it is enough to suppose that

$$\phi \in C(\mathbb{R}^+) \cap C^1(]0, +\infty[).$$

In this case, one can easily check that $\widetilde{\phi}(s) = \frac{1}{2}\int_0^s \phi(\tau)\,d\tau$ *is a convex function. Indeed, let* $x_1 \neq 0$ *and* $x_2 \neq 0$ *such that* $x_1 < x_2$*. Because* ϕ *is of the class* C^1 *in* $[x_1, x_2]$ *and a non-decreasing function,* $\widetilde{\phi}$ *is a convex function. Now if* $x_1 = 0$*, we have, for all* $0 \leq \lambda \leq 1$*, that*

$$\widetilde{\phi}(\lambda x_2) = \frac{1}{2}\int_0^{\lambda x_2} \phi(s)\,ds = \frac{1}{2}\lambda \int_0^{x_2} \phi(\lambda z)\,dz,$$

where we have made the change of variable $s = \lambda z$ *. As* ϕ *is a non-decreasing function and* $\lambda x_2 \leq x_2$ *for all* $\lambda \in [0, 1]$*, it follows that*

$$\widetilde{\phi}(\lambda x_2) \leq \lambda \widetilde{\phi}(x_2).$$

Proposition 3.2.1 *Let* $(u_0, u_1) \in H \times H$ *and suppose that* g *and* ϕ *satisfies* **(A1)**. *Then the problem (3.1) has a unique global solution*

$$u \in C(\mathbb{R}^+, H), \quad u' \in C(\mathbb{R}^+, H) \quad \textit{and} \quad u(0) = u_0, \quad u'(0) = u_1.$$

We introduce the energy associated to the solution of the problem (3.1) by

$$E(t) = \frac{1}{2}\|u'\|^2 + \frac{1}{2}\widetilde{\phi}(\|A^{\frac{1}{2}}u\|^2), \tag{3.5}$$

where

$$\widetilde{\phi}(s) = \int_0^s \phi(\tau)\,d\tau.$$

By multiplying equation (3.1) by u', we obtain easily

$$\frac{d}{dt}E(t) = -(g(u'), u') \leq 0. \tag{3.6}$$

3.3 Asymptotic behavior

Lemma 3.3.1 *Assume that* **(A2)** *and* **(A3)** *hold, then the functional*

$$F(t) = ME(t) + (u, u'),$$

satisfies the following estimate, for some positive constants $M,\ c,\ m$ *:*

$$F'(t) \leq -mE(t) + c\|u'\|^2 + |(u, g(u'))|, \tag{3.7}$$

and $F(t) \sim E(t)$*.*

Proof 3.3.1 *Using (3.1), (3.5) and (3.6), we obtain*

$$\begin{array}{rcl} F'(t) & = & ME'(t) + \|u'\|^2 + (u, u'') \\ & \leq & \|u'\|^2 - (u, \phi(\|A^{\frac{1}{2}}u\|^2)Au) - (u, g(u')) \ . \\ & \leq & \|u'\|^2 - \phi(\|A^{\frac{1}{2}}u\|^2)\|A^{\frac{1}{2}}u\|^2 - (u, g(u')) \end{array}$$

On the other hand, we have (in both the degenerate and the non-degenerate cases) $s\phi(s) \geq c\widetilde{\phi}(s)$*. Then we deduce that*

$$\begin{array}{rcl} F'(t) & \leq & \|u'\|^2 - c\widetilde{\phi}(\|A^{\frac{1}{2}}u\|^2) + |(u, g(u'))| \\ & \leq & -mE(t) + c\|u'\|^2 + |(u, g(u'))| \end{array}.$$

To prove that $F(t) \sim E(t)$*, we show that for some positive constants* λ_1 *and* λ_2

$$\lambda_1 E(t) \leq F(t) \leq \lambda_2 E(t). \tag{3.8}$$

Using Young's inequality and the definition of E*, we have (note also that* $\widetilde{\phi}$ *is a bijection from* $\mathbb{R}_+$ *to* $\mathbb{R}_+$*)*

$$\begin{array}{rcl} (u, u') & \leq & \frac{1}{2}\|u\|^2 + \frac{1}{2}\|u'\|^2 \\ & \leq & \frac{1}{2}\|A^{\frac{1}{2}}u\|^2 + E(t) \\ & \leq & c\widetilde{\phi}^{-1}(E(t)) + E(t). \end{array}$$

Using the fact that $s \mapsto \widetilde{\phi}^{-1}(s)$ *is non-decreasing, we obtain*

$$(u, u') \leq c_1 E(t),$$

and

$$\begin{array}{rcl} (u, u') & \geq & -\frac{1}{2}\|u\|^2 - \frac{1}{2}\|u'\|^2 \\ & \geq & -\frac{1}{2}\|A^{\frac{1}{2}}u\|^2 - E(t) \\ & \geq & -c\widetilde{\phi}^{-1}(E(t)) - E(t) \\ & \geq & -c_2 E(t). \end{array}$$

Then, for M *large enough, we obtain (3.8). This completes the proof.*

Theorem 3.3.1 *Assume that* **(A2)** *and* **(A3)** *hold. Let* $\widetilde{\phi}(t) = \int_0^t \phi(\tau)\, d\tau$*. Then there exist* $w,\ k,\ \varepsilon > 0$ *such that the energy* E *satisfies*

A. The degenerate case:

$$E(t) \leq \varphi_1\Big(\psi^{-1}(kt + \psi(E(0)))\Big),\ \forall t \geq 0, \tag{3.9}$$

where $\psi(t) = \int_t^1 \frac{1}{w\varphi(\tau)}\, d\tau$ *for* $t > 0$

$$\left\{ \begin{array}{ll} \varphi_1(s) = \sqrt{s},\ \ \varphi(s) = \widetilde{\phi}(s) & G \ \textit{is linear on }]0, r_0] \\ \varphi_1(s) = s,\ \ \varphi(s) = \frac{s^2}{\widetilde{\phi}^{-1}(s)} G'\Big(\varepsilon \frac{s^2}{\widetilde{\phi}^{-1}(s)}\Big) & \text{if } G'(0) = 0 \ \ \textit{and } G'' > 0 \ \ \textit{on} \ \ ]0, r_0], \end{array} \right. \tag{3.10}$$

B. The non-degenerate case:

$$E(t) \leq \psi^{-1}(kt + \psi(E(0))),\ \forall t \geq 0, \tag{3.11}$$

where $\psi(t) = \int_t^1 \frac{1}{w\varphi(\tau)}\, d\tau$ *for* $t > 0$

$$\begin{cases} \varphi(s) = s & G \ \textit{is linear on }]0, r_0], \\ \varphi(s) = sG'(\varepsilon s) & \text{if } G'(0) = 0 \ \ \textit{and } G'' > 0 \ \ \textit{on} \ \ ]0, r_0]. \end{cases} \tag{3.12}$$

Proof 3.3.2 *We now estimate (3.7).*

The degenerate case: *we distinguish two cases.*

1.G **is linear on** $[0, r_0]$

If $\|u'\| \geq r_0$, *we use Young's inequality and (3.6), for any* $\delta > 0$, *we have*

$$\begin{aligned} |(u, g(u'))| + \|u'\|^2 &\leq \delta\|u\|^2 + C'_\delta\|g(u')\|^2 + c(g(u'), u') \\ &\leq \delta\|A^{\frac{1}{2}}u\|^2 + C_\delta(g(u'), u') \\ &\leq \delta\|A^{\frac{1}{2}}u\|^2 + C_\delta(-E'(t)) \\ &\leq \delta\widetilde{\phi}^{-1}(E(t)) + C_\delta(-E'(t)) \end{aligned} \tag{3.13}$$

If $\|u'\| < r_0$, *we have*

$$\|u'\|^2 + |(u, g(u'))| \leq \delta\widetilde{\phi}^{-1}(E(t)) + C_\delta(-E'(t)) \tag{3.14}$$

We then use (3.13) and (3.14), to deduce from (3.7)

$$F'(t) \leq -\widetilde{\phi}(E(t))\Big(m\frac{E(t)}{\widetilde{\phi}(E(t))} - \delta\frac{\widetilde{\phi}^{-1}(E(t))}{\widetilde{\phi}(E(t))}\Big) + C_\delta(-E'(t)).$$

Using the fact that $\widetilde{\phi}$ *is convex, increasing and choosing* $\delta > 0$ *small enough, we obtain*

$$F'(t) \leq -d\widetilde{\phi}(E(t)) + C_\delta(-E'(t)). \tag{3.15}$$

By Lemma (3.3.1) and (3.15) the function $L(t) = F(t) + C_\delta E(t)$ *satisfies*

$$L'(t) \leq -d\varphi(L(t)), \tag{3.16}$$

where $\varphi(s) = \widetilde{\phi}(s)$, *and*

$$L(t) \sim E(t). \tag{3.17}$$

We choose $\varphi(t) = -\frac{w}{\psi'(t)}$, *where* $\psi(t)$ *is defined in Theorem 3.3.1.*
Using (3.16), we arrive at

$$(\psi(L(t))' = L'(t)\psi'(L(t)) \leq c.$$

A simple integration leads to

$$\psi(L(t)) \leq ct + \psi(L(0)),$$

consequently,

$$L(t) \leq \psi^{-1}(kt + \psi(L(0))).$$

Using (3.20), we obtain (3.9).
2. $G'(0) = 0$ **and** $G'' > 0$ **on** $]0, r_0]$**.**
If $\|u'\| \geq r_0$*. Using Young's inequality, we have, for any* $\delta > 0$,

$$\begin{aligned} |(u, g(u'))| + \|u'\|^2 &\leq \delta\|A^{\frac{1}{2}}u\|^2 + C_\delta\|g(u')\|^2 + \|u'\|^2 \\ &\leq \delta\widetilde{\phi}^{-1}(E(t)) + C_\delta(\|g(u')\|^2 + \|u'\|^2) \\ &\leq \delta\widetilde{\phi}^{-1}(E(t)) + C_\delta(-E'(t)) \end{aligned} \tag{3.18}$$

and if $\|u'\| < r_0$*, we have*

$$\begin{aligned} |(u, g(u'))| + \|u'\|^2 &\leq \delta\|u\|^2 + C_\delta\|g(u')\|^2 + \|u'\|^2 \\ &\leq \delta\widetilde{\phi}^{-1}(E(t)) + C_\delta G^{-1}(g(u'), u') \end{aligned} \tag{3.19}$$

By Lemma (3.3.1), (3.18) and (3.19), for δ *small enough, the function* $L(t) = F(t) + C_\delta E(t)$ *satisfies*

$$L'(t) \leq -\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big(m\frac{\widetilde{\phi}^{-1}(E(t))}{E(t)} - \delta\Big(\frac{\widetilde{\phi}^{-1}(E(t))}{E(t)}\Big)^2\Big) + C_\delta G^{-1}(g(u'), u')$$

and

$$L(t) \sim E(t). \tag{3.20}$$

Using the fact that $s \to \frac{s}{\widetilde{\phi}^{-1}(s)}$ *is non-decreasing and choosing* $\delta > 0$ *small enough, we obtain*

$$L'(t) \quad \leq \quad -d\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))} \ + C_\delta G^{-1}(g(u'), u') \tag{3.21}$$

For $c_0 > 0$*, we define* $\widetilde{E}$ *by*

$$\widetilde{E}(t) = G'\Big(\varepsilon\frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)L(t) + c_0E(t).$$

Then, we see easily that, for $a_1, \ a_2 > 0$

$$a_1\widetilde{E}(t) \leq E(t) \leq a_2\widetilde{E}(t). \tag{3.22}$$

By recalling that $E' \leq 0, \ G' > 0, \ G'' > 0$ *on* $(0, r_0]$ *and using the fact that* $s^2 \mapsto \frac{s}{\widetilde{\phi}^{-1}(s)}$ *is non-decreasing, we obtain making use of (3.5) and (3.21), we obtain*

$$\widetilde{E}'(t) \quad = \quad \varepsilon\Big(\tfrac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)'G''\Big(\varepsilon\tfrac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)L(t) + G'\Big(\varepsilon\tfrac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)L('t) + c_0E'(t) \tag{3.23}$$

making use of (3.5) and (3.21), we obtain from (3.23) that

$$\widetilde{E}'(t) \quad \leq \quad -d\tfrac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}G'\Big(\varepsilon\tfrac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big) + C_\delta G^{-1}(g(u'), u')G'\Big(\varepsilon\tfrac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big) + c_0E'(t). \tag{3.24}$$

On the other hand, let G^* *denote the dual function of the convex function* G *(in the sense of Young, see Arnold* **[8]***, p. 46 , for the definition, and Lasiecka* **[36]***. Because* $G > 0$ *on* $]0, 1]$ *and* $G(0) = 0$*, we can assume, without loss generality, that* G *defines a bijection from*

$\mathbb{R}^+$ to $\mathbb{R}^+$. Then G^* is the Legendre transform of G, which is given by (see Arnold **[8]***, p. 61-62, Lasiecka* **[36]** *, Liu and Zuazua* **[41]***, Alabau-Boussouira* **[4]** *and others).*

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)],$$

and G satises the generalized Young's inequality

$$AB \leq G^*(A) + G(B)$$

with $A = G'\Big(\varepsilon \frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)$ and $B = G^{-1}(g(u'), u')$

$$\begin{aligned} G'\Big(\varepsilon \frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)G^{-1}(g(u'), u') &\leq G^*\Big(G'\Big(\varepsilon \frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big)\Big) + (g(u'), u') \\ &\leq \varepsilon \frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))} G'\Big(\varepsilon \frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big) + (g(u'), u'), \end{aligned} \tag{3.25}$$

Choosing $c_0 > C_\delta$ and ε small enough, we obtain and

$$\widetilde{E}'(t) \leq -k_1 \frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))} G'\Big(\varepsilon \frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big) = -k_1 \varphi\Big(\varepsilon \frac{E^2(t)}{\widetilde{\phi}^{-1}(E(t))}\Big), \tag{3.26}$$

where $\varphi(t) = tG'(\varepsilon t)$. Since

$$\varphi'(t) = G'(\varepsilon t) + t\varepsilon G''(\varepsilon t).$$

and G is convex on $(0, \varepsilon]$, we find that $\varphi'(t) > 0$ and $\varphi(t) > 0$ on $(0, 1]$. By setting $H(t) = \frac{a_1^2 \widetilde{E}^2(t)}{\widetilde{\phi}^{-1}(E(0))}$ (a_1 is given in (3.22)). we easily see that, by (3.22), we have

$$H(t) \sim \widetilde{E}^2(t).$$

using (3.26), we arrive at

$$H'(t) \leq -k_2 \varphi(H(t)),$$

where $\varphi(t) = -\frac{w}{\psi'(t)}$ and $\psi(t) = \int_t^1 \frac{1}{w\varphi(\tau)}\, d\tau$, hence

$$(\psi(H(t))' = H'(t)\psi'(H(t)) \leq k.$$

By integrating over $(0, t)$, we get

$$\psi(H(t)) \leq kt + \psi(H(0)).$$

Consequently,

$$H(t) \leq \psi^{-1}(kt + \psi(H(0))). \tag{3.27}$$

Using (3.22) and (3.27), we obtain (3.9).
The non-degenerate case: *we distinguish two cases.*

1.G **is linear on** $[0, r_0]$

For $\|u'\| \geq r_0$*, we have, thanks to Young's inequality, for any* $\delta > 0$

$$\begin{array}{rcl} |(u, g(u'))| & \leq & \delta\|u\|^2 + C_\delta\|g(u')\|^2 \\ & \leq & \delta\|A^{\frac{1}{2}}u\|^2 + C_\delta(g(u'), u') \\ & \leq & \delta\|A^{\frac{1}{2}}u\|^2 + C_\delta(-E'(t)) \qquad \leq \delta\widetilde{\phi}^{-1}(E(t)) + C_\delta(-E'(t)) \\ & \leq & \delta\frac{\widetilde{\phi}^{-1}(E(t))}{E(t)}E(t) + C_\delta(-E'(t)) \end{array}$$

Using fact that $\widetilde{\phi}^{-1}(s) < cs$ *and choosing* $\delta > 0$ *small enough. we have*

$$|(u, g(u'))| \leq c\delta E(t) + C_\delta(-E'(t)),$$

and

$$\|u'\|^2 \leq c(g(u'), u') \leq c(-E'(t)),$$

then

$$\|u'\|^2 + |(u, g(u'))| \leq c\delta E(t) + C_\delta(-E'(t)), \tag{3.28}$$

and for $\|u'\| < r_0$*, we have*

$$\|u'\|^2 + |(u, g(u'))| \leq c\delta E(t) + C_\delta(-E'(t)) \tag{3.29}$$

By Lemma 3.3.1, (3.28) and (3.29), we obtain

$$\begin{array}{rcl} F'(t) & \leq & -(m - c\delta)E(t) + C_\delta(-E'(t)) \\ & \leq & -dE(t) + C_\delta(-E'(t)), \end{array}$$

we take $L(t) = F(t) + C_\delta E(t)$ *and* $L \sim E$*, we have*

$$E'(t) \leq -dE(t)$$

A simple integration leads to

$$E(t) \leq c'e^{-c''t} = c\psi^{-1}(c''t),$$

where $\varphi(s) = s$*.*

2.G **is non-linear on** $[0, r_0]$

For $\|u'\| \geq r_0$*, we have, thanks to Young's inequality, for any* $\delta > 0$

$$\begin{array}{rcl} |(u, g(u'))| & \leq & \delta\|u\|^2 + C_\delta\|g(u')\|^2 \\ & \leq & \delta\widetilde{\phi}^{-1}(E(t)) + C_\delta(-E'(t)). \end{array}$$

Using fact that $\widetilde{\phi}^{-1}(s) < cs$ *and choosing* $\delta > 0$ *small enough. we have*

$$|(u, g(u'))| \leq c\delta E(t) + C_\delta(-E'(t)),$$

and

$$\|u'\|^2 \leq c(g(u'), u') \leq c(-E'(t)),$$

then

$$\|u'\|^2 + |(u, g(u'))| \leq c\delta E(t) + C_\delta(-E'(t)),$$

and for $\|u'\| < r_0$, *we have*

$$\begin{array}{rcl} \|u'\|^2 + |(u, g(u'))| & \leq & c\delta E(t) + \|u'\|^2 + C(\delta)\|g(u')\|^2 \\ & \leq & c\delta E(t) + c(\|u'\|^2 + \|g(u')\|^2) \\ & \leq & c\delta E(t) + cG^{-1}(g(u'), u') \end{array}$$

$$\begin{array}{rcl} F'(t) & \leq & -(m - c\delta)E(t) + cG^{-1}(g(u'), u') + C_\delta(-E'(t)) \\ & \leq & -dE(t) + cG^{-1}(g(u'), u') + C_\delta(-E'(t)) \end{array}$$

we take $L(t) = F(t) + C_\delta E(t)$ *and* $L \sim E$

$$L'(t) \leq -dE(t) + cG^{-1}(g(u'), u'), \tag{3.30}$$

we define H *by*

$$H(t) = G'\Big(\varepsilon \frac{E(t)}{E(0}\Big)L(t) + c_0 E(t).$$

Then, we see easily that, for $\lambda_1, \lambda_2 > 0$

$$\lambda_1 H(t) \leq E(t) \leq \lambda_2 H(t) \tag{3.31}$$

By recalling that $E' \leq 0, G' > 0,\ G'' > 0$ *on* $(0, r_0]$ *and making use of (3.5) and (3.30), we obtain*

$$\begin{array}{rcl} H'(t) & = & \varepsilon \dfrac{E'(t)}{E(0)} G''\Big(\varepsilon \dfrac{E(t)}{E(0)}\Big)L(t) + G'\Big(\varepsilon \dfrac{E(t)}{E(0)}\Big)L'(t) + c_0 E'(t) \\ & \leq & -dE(t)G'\Big(\varepsilon \dfrac{E(t)}{E(0)}\Big) + cG'\Big(\varepsilon \dfrac{E(t)}{E(0)}\Big)G^{-1}(g(u'), u') + c_0 E'(t). \end{array} \tag{3.32}$$

Let G^* *be the convex conjugate of* G *in the sense of Young (see Arnold* **[8]**, *p. 61-62), then*

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)], \quad \text{if } s \in (0, G'(r_0)], \tag{3.33}$$

and G *satisfies the generalized Young's inequality*

$$AB \leq G^*(A) + G(B) \quad \text{if } A \in (0, G'(r_0)],\ \ B \in (0, r_0], \tag{3.34}$$

with $A = G'(\varepsilon E(t)/E(0))$ *and* $B = G^{-1}(g(u'), u')$, *using (3.6) and (3.32)-(3.34)*

$$\begin{array}{rcl} H'(t) & \leq & -dE(t)G'\Big(\varepsilon \frac{E(t)}{E(0)}\Big) + cG^*\Big(\Big(\varepsilon \frac{E(t)}{E(0)}\Big)\Big) + (g(u'), u') + c_0 E'(t) \\ & \leq & -dE(t)G'\Big(\varepsilon \frac{E(t)}{E(0)}\Big) + c\varepsilon \frac{E(t)}{E(0)} G'\Big(\varepsilon \frac{E(t)}{E(0)}\Big) - cE'(t) + c_0 E'(t). \end{array}$$

Choosing $c_0 > c$ *and* ε *small enough, we obtain*

$$H'(t) \leq -k \frac{E(t)}{E(0)} G'\Big(\varepsilon \frac{E(t)}{E(0)}\Big) = -k\varphi\Big(\frac{E(t)}{E(0)}\Big), \tag{3.35}$$

where $\varphi(s) = sG'(\varepsilon s)$ *and* $\widetilde{E_0}(t) = \frac{\lambda_1 H(t)}{E(0)}$, *(*$\lambda_1$ *is given in (3.31)), we easily see that, by (3.31), we have*

$$\widetilde{E_0}(t) \sim E(t) \tag{3.36}$$

Using (3.35), we arrive at

$$\widetilde{E_0}'(t) \leq -k\varphi(\widetilde{E_0}(t))$$

where $\varphi(t) = -\frac{w}{\psi'(t)}$ *and* $\psi(t) = \int_t^1 \frac{1}{w\varphi(\tau)}\, d\tau$, *hence*

$$(\psi(\widetilde{E_0}(t))' = \widetilde{E_0}'(t)\psi'(t) \leq k.$$

A simple integration leads to

$$\psi(\widetilde{E_0}(t)) \leq kt + \psi(\widetilde{E_0}(0)).$$

Consequently,

$$\widetilde{E_0}(t) \leq \psi^{-1}(kt + \psi(\widetilde{E_0}(0))). \tag{3.37}$$

Using (3.36) and (3.37) we obtain (3.11). This completes the proof of Theorem 3.3.1.

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