N° d'ordre :

REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBÈS

THESE DE DOCTORAT

Présentée par

Abderrahmane BENIANI

Spécialité : MATHEMATIQUES

Option : EQUATIONS AUX DERIVEES PARTIELLES

Intitulée

Existence globale et stabilisation de certains problèmes d'évolution linéaires et non linéaires dans des domaines non bornés

Soutenue le 17/05/2017.

Devant le jury composé de :

Président *Ali HAKEM* Professeur à l'Université de Sidi Bel Abbes.

Examinateurs *Kacem BELGHABA* Professeur à l'Université d'Oran Essenia.

Selma Bendimerad Professeur à l'Université de Sidi Bel Abbes

Khaled Zennir Professeur à l'Universté Al Qassim, KSA (Arabie Saoudite)

Mama ABDELLI MCA à l'Université de Mascara

Directeur de thèse *Abbes BENAISSA* Professeur à l'Université de Sidi Bel Abbes.

Acknowledgements

- I would like to express my deep gratitude to Prof. Abbèes Benaissa , my thesis supervisor, for his patience, motivation and enthusiastic encouragement. His guidance, advice and friendship have been invaluable.
- Huge thanks to D.r Khaled Zennir , for his guidance, encouragement and continuous support through my research. I am very grateful to him.
- My thanks go also to Mr. Ali Hakem for a jury president of this thesis, as well as to my lords Kacem Belghaba, Selma Bendimerad , Khaled Zennir and Mama Abdelli for having accepted to be part of my jury. I thank them for their interest in my work.
- I must thank the members of the Mathematic department of Sidi Bel-Abbès University (Algeria) including the collogues, staffs and students.
- I am especially indebted to the members of Lab of Analysis and Control of PDE of Sidi Bel-Abbès University for their support over the year.
- I owe my loving thanks to my wife and My daughters Nour-Elyakine and Hafsa for being incredibly understanding and supportive.
- Finally, this work is dedicated to my dear parents, for their love and encouragement throughout my studies.

Beniani.

Contents

Publications

The following results were published or submitted:

1. A. Benaissa, A. Beniani and K. Zennir *General decay of solution for coupled system of viscoelastic wave equations of kirchhoff type with density in* $\mathcal{R}^n$, Journal Facta Universitatis. Ser. Math. Inform. Vol. 31, No 5, 1073–1090, 2016.

2. A. Beniani, A. Benaissa and K. Zennir *Polynomial Decay Of Solutions To The Cauchy Problem For A Petrowsky-Petrowsky System In* $\mathcal{R}^n$ Journal Acta Applicandae Mathematicae, Vol.146, Issue 1, pp. 67–79, 2016.

3. A. Beniani, K. Zennir and A. Benaissa *Stability For The Lamé System With A Time Varying Delay Term In A Nonlinear Internal Feedback* Journal Clifford Analysis, Clifford Algebras and their applications. Vol. 5, No.4, PP. 287–298 , 2016.

4. A.Benaissa and A. Beniani *Energy Decay Of Solutions to the Cauchy Problem for Timoshenko Beam Systems With a Linear Dissipative Term,* communications of the korean mathematical society.

5. A. Beniani, A. Benaissa and K. Zennir *Polynomial decay of solutions to the Cauchy problem for a coupled system of wave equations,* Journal Editorial Office Rendiconti del Circolo Matematico di Palermo.

6. A. Benaissa, A. Beniani and K. Zennir *Decay rate of solutions to the Cauchy problem for a coupled system of a viscoelastic wave equations in* $\mathcal{R}^n$, Nonlinear studies the international journal.

7. A. Benaissa, A. Beniani and K. Zennir *Existence and stability to the Cauchy problem for a coupled system of a nonlinear weak viscoelastic wave equations in* $\mathcal{R}^n$*,* Annals polonici mathematici journal.

8. A. Beniani, A. Benaissa and K. Zennir *Decay rate of solutions to the cauchy problem of a viscoelastic plate equations in* $\mathcal{R}^n$ Journal Applied mathematics E-Notes.

9. A. Beniani, A. Benaissa and K. Zennir *General decay for a system of nonlinear weak-viscoelastic wave equations in* $\mathcal{R}^n$ Journal Applied Mathematics and Computation.

10. A. Beniani, A. Benaissa and K. Zennir *Qualitative behavior of solutions to the cauchy problem for the viscoelastic wave equation with structural* δ*-evolutionin* Journal waves, wavelets and fractals-Advanced Analysis.

Abstract

The present thesis is devoted to the study of global existence and asymptotic behaviour in time of solution of Timoshenko system and coupled system .
This work consists of five chapters, will be devoted to the study of the global existence and asymptotic behaviour of some evolution equation with linear, nonlinear dissipative terms and viscoelastic equation. In chapter 1, we recall of some fundamental inequalities. In chapter 2, we consider the Cauchy problem for a coupled system of wave equation, we prove polynomial decay of solution. In chapter 3, we study the Cauchy problem for a coupled system of a viscoelastic wave equation, we prove exponential stability of the solution. In chapter 4, we study the Cauchy problem for a coupled system of a nonlinear weak viscoelastic wave equations, we prove existence and uniqueness of global solution and prove exponential stability of the solution. In this work, the proof an existence and uniqueness for global solution is based on stable set for small data combined with Faedo-Galerkin. The proof an decay estimate is based on multiplier method, Lyapunov functional for some perturbed energy. In chapter 5, we consider a system of viscoelastic wave equations of Petrowsky-Petrowsky type, we use a spaces weighted by density function to establish a very general decay rate of solution.

Key words : Global existence, Coupled system, Exponential decay, Polynomial decay, Multiplier method, Lyapunov method, Galerkin method, Nonlinear dissipation, Viscoelastic equation.

subjclass 2000: 35L05, 58J45, 35L80, 35B40, 35L20, 58G16, 35B40, 35L70

General introduction

The present thesis is devoted of the study of global existence, asymptotic behaviour in time of solution to hyperbolic systems.
The problem of stabilization consists in determining the asymptotic behaviour of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero, they are several type of stabilization:

1. Strong stabilization: $E(t) \to 0$, as $t \to \infty$.
2. Uniform stabilization: $E(t) \leq C \exp(-\delta t)$, $\forall t > 0$, $(c, \delta > 0)$.
3. Polynomial stabilization: $E(t) \leq C t^{-\delta}$, $\forall t > 0$, $(c, \delta > 0)$.
4. Logarithmic stabilization: $E(t) \leq C (\ln(t))^{-\delta}$ $\forall t > 0$, $(c, \delta > 0)$.

For wave equation with dissipation of the form $u'' + \Delta_x u + g(u') = 0$, stabilization problems have been investigated by many authors:
When $g : \mathbb{R} \to \mathbb{R}$ is continuous and increasing function such that $g(0) = 0$, global existence of solutions is known for all initial conditions (u_0, u_1) given in $H_0^1(\Omega) \times L^2(\Omega)$. This result is, for instance, a consequence of the general theory of nonlinear semi-groups of contractions generated by a maximal monotone operator (see Brézis [19]).
Moreover, if we impose on g the condition $\forall \lambda \neq 0, g(\lambda) \neq 0$, then strong asymptotic stability of solutions occurs in $H_0^1(\Omega) \times L^2(\Omega)$.

$$i.e \ (u, u') \to (0, 0) \text{ strongly in } H_0^1(\Omega) \times L^2(\Omega)$$

without speed of convergence. These results follows, for instance, from the invariance principle of Lasalle (see for example A. Haraux [27]). If we add the assumption that g has a polynomial growth near zero, we obtain an explicit decay rate of solutions (see M. Nakao [71]).

This work consists of five chapters:

- In **the chapter 2**: In this chapter, we consider the following initial boundary value problem, that is,

$$\begin{cases} u_{tt} - \phi(x)\Delta u + \alpha v + u_t = 0 & \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+ \\ v_{tt} - \phi(x)\Delta v + \alpha u = 0 & \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+ \\ (u, u_t, v, v_t)(x, 0) = (u_0, u_1, v_0, v_1) & \text{in} \quad \mathbb{R}^n \end{cases} \tag{1}$$

  where the space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ defined in (III.11) and $n \geq 2$, $\phi(x) > 0, \forall x \in \mathbb{R}^n, (\phi(x))^{-1} = \rho(x)$ defined in (***H1***).

  The problem of stabilization of weakly coupled systems has also been studied by several authors. Under certain conditions imposed on the subset where the damping term is effective, Kapitonov [44] proves uniform stabilization of the solutions of a pair of hyperbolic systems coupled in velocities. Alabau and al. [4] studied the indirect internal stabilization of weakly coupled systems where the damping is effective in the whole domain. They prove that the behavior of the first equation is sufficient to stabilize the total system and to have polynomial decay for sufficiently smooth solutions. Alabau [6] proves indirect boundary stabilization (polynomial decay) of weakly coupled equations. She establishes a polynomial decay lemma for non-increasing and nonnegative function which satisfies an integral inequality.

  The purpose of this chapter is to obtain a better decay estimate of solutions to the problem (1). More precisely we show that we can always find initial data in the stable set for which the solution of problem (1) decays polynomials, which is based on the construction of a suitable multiplier method.

- In **the chapter 3**: Let us consider the following problem

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + \alpha u_2 = \phi(x)\Delta_x\left(u_1 + \int_0^t g_1(s)u_1(t-s,x)ds\right), x \in \mathbb{R}^n \times \mathbb{R}^+ \\ \left(|u_2'|^{l-2}u_2'\right)' + \alpha u_1 = \phi(x)\Delta_x\left(u_2 + \int_0^t g_2(s)u_2(t-s,x)ds\right), x \in \mathbb{R}^n \times \mathbb{R}^+ \\ (u_1(0,x), u_2(0,x)) = (u_1^0(x), u_2^0(x)) \in (\mathcal{D}^{1,2}(\mathbb{R}^n))^2, \\ (u_1'(0,x), u_2'(0,x)) = (u_1^1(x), u_2^1(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases} \tag{2}$$

where the space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ defined in (III.11) and $l, n \geq 2$, $\phi(x) > 0, \forall x \in \mathbb{R}^n, (\phi(x))^{-1} = \rho(x)$ defined in (***H2***).

In this chapter we are going to consider the solutions in spaces weighted by the density function $\rho(x)$ in order to compensate for the lack of Poincare's inequality which is useful in the proof.

The same problem treated in [38], was considered in [39], where they consider a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. Conditions used, on the relaxation function g and its derivative g' are different from the usual ones.

The main purpose of this work is to allow a wider class of relaxation functions and improve earlier results in the literature. The basic mechanism behind the decay rates is the relation between the damping and the energy.

- In **the chapter 4**:

Let us consider the following problem

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + f_1(u_1,u_2) = \phi(x)\Delta_x\left(u_1 - \alpha_1(t)\int_0^t g_1(t-s)u_1(s,x)ds\right), x \in \mathbb{R}^n \times \mathbb{R}^+ \\ \left(|u_2'|^{l-2}u_2'\right)' + f_2(u_1,u_2) = \phi(x)\Delta_x\left(u_2 - \alpha_2(t)\int_0^t g_2(t-s)u_2(s,x)ds\right), x \in \mathbb{R}^n \times \mathbb{R}^+ \\ (u_1(0,x), u_2(0,x)) = (u_1^0(x), u_2^0(x)) \in (\mathcal{D}^{1,2}(\mathbb{R}^n))^2, \\ (u_1'(0,x), u_2'(0,x)) = (u_1^1(x), u_2^2(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases} \tag{3}$$

where the space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ defined in (IV.11) and $l \geq 2$, $\phi(x) > 0, \forall x \in \mathbb{R}^n, (\phi(x))^{-1} = \rho(x)$ defined in (***H2***).

In this chapter we consider the solutions in spaces weighted by the density function $\rho(x)$ in order to compensate for the lack of Poincare's inequality.

The main purpose of this work is to prove an existence and uniqueness theorem for global weak solutions in Sobolev spaces using Faedo-Galerkin method and to allow a wider class of relaxation functions and improve earlier results in the literature. The basic mechanism behind the decay rates is the relation between the damping and the energy.

- In **the chapter 5**: In this chapter, derived from [9], we consider the following Petrowsky-

Petrowsky system

$$\begin{cases} u_{tt} + \phi(x)\left(\Delta^2 u - \int_{-\infty}^{t} \mu(t-s)\Delta^2 u(s)ds\right) + \alpha v = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}^+, \\ v_{tt} + \phi(x)\Delta^2 v + \alpha u = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}^+, \\ (u_0, v_0) \in \mathcal{D}^{2,2}(\mathbb{R}^n), \quad (u_1, v_1) \in L^2_\rho(\mathbb{R}^n). \end{cases} \tag{4}$$

where the spaces $\mathcal{D}^{2,2}(\mathbb{R}^n), L^2_\rho(\mathbb{R}^n)$ defined in (V.12) and $\phi(x) > 0, \forall x \in \mathbb{R}^n, \alpha \neq 0, (\phi(x))^{-1} = \rho(x)$, where the function $\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0,1)$ and $\rho \in L^{n/4}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

The purpose of this paper is to extend the above results for any space dimension of Petrowsky-Petrowsky system.

Chapter I

Preliminarie talk and materials needed

All assertions in the first chapter are made without proofs and the scope has been minimized to only material actually needed (See [19], [26], [54], [74]).

1 Function analysis

The universal framework used in nonlinear PDE is based on functional analysis.

1.1 Young, Hölder's inequalities

Notation 1.1 *Let $1 \leq p \leq \infty$, we denote by q the conjugate exponent,*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We define the convolution product of a function $f \in L^1(\mathbb{R}^n)$ with a function $g \in L^p(\mathbb{R}^n)$.

Theorem 1.2 ***(Young)*** *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. Then for a.e.$x \in \mathbb{R}^n$ the function $y \mapsto f(x-y)g(y)$ is integrable on R^n and we define*

$$(f * g)(x) = \int_{R^n} f(x-y)g(y)dy.$$

In addition $(f * g) \in L^p(\mathbb{R}^n)$ and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

The following is an extension of Theorem 1.2.

Theorem 1.3 ***(Young)*** *Assume $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $1 \leq p \leq \infty$, $1 \leq q \leq \infty$*

and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$. *Then* $(f * g) \in L^r(\mathbb{R}^n)$ *and*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Theorem 1.4 ***(Hölder's inequality)*** *Assume that* $f \in L^p$ *and* $g \in L^q$ *with* $1 \leq p \leq \infty$. *Then* $(fg) \in L^1$ *and*

$$\|fg\| \leq \|f\|_p \|g\|_q.$$

Lemma 1.5 ***(Cauchy-Schwarz inequality)*** *Every inner product satisfies the Cauchy-Schwarz inequality*

$$\langle x_1, x_2 \rangle \leq \|x_1\| \|x_2\|.$$

The equality sign holds if and only if x_1 *and* x_2 *are dependent.*

We will give here some integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Lemma 1.6 *Let* $1 \leq p \leq r \leq q$, $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$, *and* $0 \leq \alpha \leq 1$. *Then*

$$\|u\|_{L^r} \leq \|u\|_{L^p}^{\alpha} \|u\|_{L^q}^{1-\alpha}.$$

Since our study based on some known algebraic inequalities, we want to recall few of them here.

Lemma 1.7 *For all* a, $b \in \mathbb{R}^+$, *we have*

$$ab \leq \delta a^2 + \frac{b^2}{4\delta},$$

where δ *is any positive constant.*

Lemma 1.8 *For all* $a, b \geq 0$, *the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where, $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.9 ***(Gronwell lemma in integral form)***

Let $T > 0$, *and let* φ *be a function such that,* $\varphi \in L^1(0,T)$, $\varphi \geq 0$, *almost everywhere and* ϕ *be a function such that,* $\phi \in L^1(0,T)$, $\phi \geq 0$, *almost everywhere and* $\phi\varphi \in L^1[0,T]$, C_1, $C_2 \geq 0$. *Suppose that*

$$\phi(t) \leq C_1 + C_2 \int_0^t \varphi(s)\phi(s)ds, \ \textbf{\textit{for a.e}} \ t \in]0,T[,$$

then,

$$\phi(t) \leq C_1 \exp\left(C_2 \int_0^t \varphi(s)ds \right), \ \textbf{\textit{for a.e}} \ t \in]0,T[.$$

2 Function Spaces

We consider the Euclidean space $\mathbb{R}^n$, $n \geq 1$ endowed with standard Euclidean topology and for Ω a subset of $\mathbb{R}^n$ we will define various spaces of functions $\Omega \to \mathbb{R}^m$. If endowed by a pointwise addition and multiplication the linear space structure of $\mathbb{R}^m$ is inherited by these spaces. Besides, we will endow them by norms, which makes them normed linear (or, mostly even Banach) spaces. Having two such spaces $U \subset V$, we say that the mapping

$$f : U \to V, u \mapsto u.$$

is a continuous embedding (or, that U is embedded continuously to V) if the linear operator f is continuous (hence bounded). This means that

$$\|u\|_V \leq C\|u\|_U$$

for C one can take the norm $\|f\|_{\ell(U,V)}$. If f is compact, we speak about a compact embedding and use the notation $U \subset V$. If U is a dense subset in V , we will speak about a dense embedding; this property obviously depends on the norm of V but not of U. It follows by a general functional-analysis argument that the adjoint mapping

$$f^* : V^* \to U^*.$$

is continuous and injective provided $U \subset V$ continuously and densely, then we can identify V^* as a subset of U^*. Indeed, f^* is injective (because two different linear continuous functionals on V must have also different traces on any dense subset, in particular on U).

2.1 The $L^p(\Omega)$ spaces

Let $1 \leq p \leq \infty$, and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$, by

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \mathbf{u} \ \textit{is measurable and} \ \int_\Omega |u(x)|^p dx < \infty \right\}.$$

Notation 2.1 *For $p \in \mathbb{R}$ and $1 \leq p < \infty$, denote by*

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

If $p = \infty$, we have

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ \middle| \begin{array}{l} \mathbf{u} \ \textit{is measurable and there exists a constant } C \\ \textbf{such that, } |u(x)| \leq C \ \textbf{a.e in } \Omega \end{array} \right\}.$$

with

$$\|u\|_\infty = \inf\{C; |u| \leq C \ \textbf{a.e On } \Omega\}.$$

Theorem 2.2 *It is well known that $L^p(\Omega)$ supplied with the norm $\|.\|_p$ is a Banach space, for all $1 \leq p \leq \infty$.*

Remark 2.3 *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx,$$

is a Hilbert space.

Theorem 2.4 *For $1 < p < \infty$, $L^p(\Omega)$ is reflexive space.*

2.2 The $L^p(0, T, V)$ spaces

Definition 2.5 *Let V be a Banach space, denote by $L^p(0, T, V)$ the space of measurable functions*

$$\begin{aligned} u : \ &]0, T[\to V \\ & t \longmapsto u(t) \end{aligned}$$

such that

$$\left(\int_0^T \|u(t)\|_V^p dt\right)^{\frac{1}{p}} = \|u\|_{L^p(0,T,X)} < \infty, \ \textbf{\textit{for}}\ 1 \leq p < \infty.$$

If $p = \infty$,

$$\|u\|_{L^\infty(0,T,V)} = \sup_{t\in]0,T[} ess \, \|u(t)\|_V \, .$$

Theorem 2.6 *The space* $L^p(0,T,V)$ *is complete.*

We denote by $\mathcal{D}'(0,T,V)$ the space of distributions in $]0,T[$ which take its values in V, and let us define

$$\mathcal{D}'(0,T,V) = (\mathcal{D}\,]0,T[\,,V)\,,$$

where (ϕ,φ) is the space of the linear continuous applications of ϕ to φ. Since $u \in \mathcal{D}'(0,T,V)$, we define the distribution derivation as

$$\frac{\partial u}{\partial t}(\varphi) = -u\left(\frac{d\varphi}{dt}\right), \forall\varphi \in \mathcal{D}\left(]0,T[\right),$$

and since $u \in L^p(0,T,V)$, we have

$$u(\varphi) = \int_0^T u(t)\varphi(t)dt, \forall\varphi \in \mathcal{D}\left(]0,T[\right).$$

We will introduce some basic results on the $L^p(0,T,V)$ space. These results, will be very useful in the other chapters of this thesis.

Lemma 2.7 *Let* $u \in L^p(0,T,V)$ *and* $\frac{\partial u}{\partial t} \in L^p(0,T,V)$, $(1 \leq p \leq \infty)$*, then, the function* u *is continuous from* $[0,T]$ *to* V *.i.e.* $u \in C^1(0,T,V)$.

2.3 Sobolev spaces

Modern theory of differential equations is based on spaces of functions whose derivatives exist in a generalized sense and enjoy a suitable integrability.

Proposition 2.8 *Let* Ω *be an open domain in* $\mathbb{R}^N$, *Then the distribution* $T \in \mathcal{D}'(\Omega)$ *is in*

$L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that

$$\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx, \ \textbf{\textit{for all}}\ \varphi \in \mathcal{D}(\Omega),$$

where $1 \le p \le \infty$, and it's well-known that f is unique.

Definition 2.9 *Let $m \in \mathbb{N}$ and $p \in [0,\infty]$. The $W^{m,p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as*

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega),\ \textbf{\textit{such that}}\ \partial^\alpha f \in L^p(\Omega)\ \textbf{\textit{for all}}\ \alpha \in \mathbb{N}^m\ \textbf{\textit{such that}}$$
$$|\alpha| = \sum_{j=1}^{n} \alpha_j \le m,\ \textbf{\textit{where,}}\ \partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}..\partial_n^{\alpha_n}\}.$$

Theorem 2.10 *$W^{m,p}(\Omega)$ is a Banach space with their usual norm*

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{L^p}, 1 \le p < \infty\textbf{\textit{, for all}}\ f \in W^{m,p}(\Omega).$$

Definition 2.11 *Denote by $W_0^{m,p}(\Omega)$ the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$.*

Definition 2.12 *When $p = 2$, we prefer to denote by $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ supplied with the norm*

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\alpha|\le m} (\|\partial^\alpha f\|_{L^2})^2\right)^{\frac{1}{2}},$$

which do at $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\alpha|\le m} \int_\Omega \partial^\alpha u \partial^\alpha v dx$$

Theorem 2.13 ***1)*** *$H^m(\Omega)$ supplied with inner product $\langle .,. \rangle_{H^m(\Omega)}$ is a Hilbert space.*
2) *If $m \ge m'$, $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$, with continuous imbedding .*

Lemma 2.14 *Since $\mathcal{D}(\Omega)$ is dense in $H_0^m(\Omega)$, we identify a dual $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ in a weak subspace on Ω, and we have*

$$\mathcal{D}(\Omega) \hookrightarrow H_0^m(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-m}(\Omega) \hookrightarrow \mathcal{D}'(\Omega),$$

The next results are fundamental in the study of partial differential equations

Theorem 2.15 *Assume that* Ω *is an open domain in* $\mathbb{R}^N$ $(N \geq 1)$, *with smooth boundary* $\partial\Omega$. *Then,*

(**i**) *if* $1 \leq p \leq n$, *we have* $W^{1,p} \subset L^q(\Omega)$, *for every* $q \in [p, p^*]$, *where* $p^* = \frac{np}{n-p}$.

(**ii**) *if* $p = n$ *we have* $W^{1,p} \subset L^q(\Omega)$, *for every* $q \in [p, \infty)$.

(**iii**) *if* $p > n$ *we have* $W^{1,p} \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$, *where* $\alpha = \frac{p-n}{p}$.

Theorem 2.16 *If* Ω *is a bounded, the embedding* (*ii*) *and* (*iii*) *of Theorem 2.15 are compacts. The embedding* (*i*) *is compact for all* $q \in [p, p^*)$.

Remark 2.17 *For all* $\varphi \in H^2(\Omega)$, $\Delta\varphi \in L^2(\Omega)$ *and for* $\partial\Omega$ *sufficiently smooth, we have*

$$\|\varphi(t)\|_{H^2(\Omega)} \leq C \|\Delta\varphi(t)\|_{L^2(\Omega)}.$$

2.4 Weak convergence

Let $(E; \|.\|_E)$ a Banach space and E' its dual space, i.e., the Banach space of all continuous linear forms on E endowed with the norm $\|.\|_{E'}$ defined by

$$\|f\|_{E'} =: \sup_{x \neq 0} \frac{| < f, x > |}{\|x\|}$$

where $< f, x >$ denotes the action of f on x, i.e $< f, x >= f(x)$. In the same way, we can define the dual space of E' that we denote by E''. (The Banach space E'' is also called the bi-dual space of E). An element x of E can be seen as a continuous linear form on E'' by setting $x(f) =:< x, f >$, which means that $E \subset E''$.

2.4.1 Weak, weak star and strong convergence

Definition 2.18 *(**Weak convergence in** E). Let* $x \in E$ *and let* $\{x_n\} \subset E$. *We say that* $\{x_n\}$ *weakly converges to* x *in* E, *and we write* $x_n \rightharpoonup x$ *in* E, *if*

$$< f, x_n > \to < f, x >$$

for all $x \in E'$.

Definition 2.19 *(**weak convergence in E'**). Let* $f \in E'$ *and let* $\{f_n\} \subset E'$ *We say that* $\{f_n\}$ *weakly converges to* f *in* E', *and we write* $f_n \rightharpoonup f$ *in* E', *if*

$$< f_n, x > \to < f, x >$$

for all $x \in E''$.

Definition 2.20 *(**weak star convergence**). Let* $f \in E'$ *and let* $\{f_n\} \subset E'$ *We say that* $\{f_n\}$ *weakly star converges to* f *in* E'*, and we write* $f_n \overset{*}{\rightharpoonup} f$ *in* E' *if*

$$< f_n, x > \to < f, x >$$

for all $x \in E$.

Remark 2.21 *As* $E \subset E''$ *we have* $f_n \rightharpoonup f$ *in* E' *imply* $f_n \overset{*}{\rightharpoonup} f$ *in* E'*. When* E *is reflexive, the last definitions are the same, i.e, weak convergence in* E' *and weak star convergence coincide.*

Definition 2.22 *(**strong convergence**). Let* $x \in E$ *(resp.*$f \in E'$*) and let* $\{x_n\} \subset E$ *(resp.*$\{f_n\} \subset E'$*). We say that* $\{x_n\}$ *(resp.*$\{f_n\}$*) strongly converges to* x *(resp.*f *), and we write* $x_n \to x$ *in* E *(resp.*$f_n \to f$ *in* E'*), if*

$$\lim_{n\to\infty} \|x_n - x\|_E = 0, (resp. \lim_{n\to\infty} \|f_n - f\|_{E'} = 0).$$

Definition 2.23 *(**weak convergence in*** $L^p(\Omega)$ ***with*** $1 \le p < \infty$ *). Let* Ω *an open subset of* $\mathbb{R}^n$*. We say that the sequence* $\{f_n\}$ *of* $L^p(\Omega)$ *weakly converges to* $f \in L^p(\Omega)$*, if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^q,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Definition 2.24 *(**weak convergence in*** $W^{1,p}(\Omega)$ ***with*** $1 < p < \infty$ *)We say the* $\{f_n\} \subset W^{1,p}(\Omega)$ *weakly converges to* $f \in W^{1,p}(\Omega)$*, and we write* $f_n \rightharpoonup f$ *in* $W^{1,p}(\Omega)$*, if*

$$f_n \rightharpoonup f \text{ in } L^p(\Omega) \text{ and } \nabla f_n \rightharpoonup \nabla f \text{ in } L^p(\Omega; \mathbb{R}^n)$$

2.4.2 Weak and weak star compactness

In finite dimension, i.e, $\dim E < \infty$, we have Bolzano-Weierstrass's theorem (which is a strong compactness theorem).

Theorem 2.25 *(**Bolzano-Weierstrass**). If* $\dim E < \infty$ *and if* $\{x_n\} \subset E$ *is bounded, then there exist* $x \in E$ *and a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\{x_{n_k}\}$ *strongly converges to* x.

The following two theorems are generalizations, in infinite dimension, of Bolzano- Weierstrass's theorem.

Theorem 2.26 ***(weak star compactness, Banach-Alaoglu-Bourbaki)***. *Assume that E is separable and consider $\{f_n\} \subset E'$. If $\{x_n\}$ is bounded, then there exist $f \in E'$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ weakly star converges to f in E'.*

Theorem 2.27 ***(weak compactness, Kakutani-Eberlein)***. *Assume that E is reflexive and consider $\{x_n\} \subset E$. If $\{x_n\}$ is bounded, then there exist $x \in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ weakly converges to x in E.*

Theorem 2.28 *(weak compactness in $L^p(\Omega)$) with $1 < p < \infty$. Given $\{f_n\} \subset L^p(\Omega)$, if $\{f_n\}$ is bounded, then there exist $f \in L^p(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \rightharpoonup f$ in $L^p(\Omega)$.*

Theorem 2.29 ***(weak star compactness in*** $L^\infty(\Omega)$***)***
Given $\{f_n\} \subset L^\infty(\Omega)$, if $\{f_n\}$ is bounded, then there exist $f \in L^\infty(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$.

2.5 Aubin -Lions Lemma

The Aubin-Lions Lemma is a result in the theory of Sobolev spaces of Banach space-valued functions. More precisely, it is a compactness criterion that is very useful in the study of nonlinear evolutionary partial differential equations. The result is named after the French mathematicians Thierry Aubin and Jacques-Louis Lions.

Lemma 2.30 *Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Assume that X_0 is compactly embedded in X and that X is continuously embedded in X_1; assume also that X_0 and X_1 are reflexive spaces. For $1 < p, q < +\infty$, let*

$$W = \{u \in L^p([0,T]; X_0) / \ \ \dot{u} \in L^q([0,T]; X_1)\}$$

Then the embedding of W into $L^p([0,T]; X)$ is also compact.

Chapter II

Polynomial decay of solutions to the Cauchy problem for a coupled system of wave equations

1 Introduction

In this chapter, we consider the following initial boundary value problem, that is,

$$\begin{cases} u_{tt} - \phi(x)\Delta u + \alpha v + u_t = 0, & \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+, \\ v_{tt} - \phi(x)\Delta v + \alpha u = 0, & \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+, \\ (u, u_t, v, v_t)(x, 0) = (u_0, u_1, v_0, v_1), & \text{in} \quad \mathbb{R}^n. \end{cases} \tag{II.1}$$

with initial conditions (u_0, u_1), (v_0, v_1) in appropriate function spaces. Throughout the chapter we assume that the functions $\phi(x)$ and ρ satisfy the following hypotheses:

(H1) The function $\phi, \ \rho : \mathbb{R}^n \to \mathbb{R}_+^*, (\phi(x))^{-1} =: \rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0, 1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n - qn + 2q}$.

The purpose of this chapter is to obtain a better decay estimate of solutions to the problem (II.1). More precisely we show that we can always find initial data in the stable set for which the solution of problem (II.1) decays polynomials, which is based on the construction of a suitable multiplier method.

2 Preliminaries

In this section, we briefly mention some facts, notation and results from paper [43]. The space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is defined as the closure $C_0^\infty(\mathbb{R}^n)$ functions with respect to the energy norm

$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} =: \int_{\mathbb{R}^n} |\nabla u|^2 dx$. It is known that

$$\mathcal{D}^{1,2}(\mathbb{R}^n) = \left\{ u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) : \nabla u \in (L^2(\mathbb{R}^n))^n \right\},$$

and $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is embedded continuously in $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$.i.e., there exists $k > 0$ such that

$$\|u\|_{\frac{2n}{n-2}} \leq k \|u\|_{\mathcal{D}^{1,2}}. \tag{II.2}$$

We shall frequently use the following version of the generalized Poincaré's inequality

Lemma 2.1 *Suppose $\rho \in L^{\frac{n}{2}}(\mathbb{R}^n)$. Then there exists $\gamma > 0$ such that*

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \gamma \int_{\mathbb{R}^n} \rho u^2 dx, \tag{II.3}$$

for all $u \in C_0^\infty$

Proof of lemma 2.1 Since $\mathcal{D}^{1,2}$ can be embedded continuously in $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$, there exists $k > 0$ such that, for all $u \in C_0^\infty$,

$$\|u\|_{\frac{2n}{n-2}} \leq k \|u\|_{\mathcal{D}^{1,2}}.$$

Thus, if $u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\rho| u^2 dx &\leq \left(\int_{\mathbb{R}^n} |\rho|^{\frac{n}{2}} dx \right)^{\frac{2}{n}} . \left(\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \\ &= \|\rho\|_{\frac{n}{2}} . \|u\|^2_{\frac{2n}{n-2}}, \end{aligned} \tag{II.4}$$

and so

$$\int_{\mathbb{R}^n} |\rho| u^2 dx \leq k^2 \|\rho\|_{\frac{n}{2}} . \|u\|^2_{\mathcal{D}^{1,2}}.$$

which completes the proof.

It is shown that $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is a separable Hilbert space. The space $L^2_\rho(\mathbb{R}^n)$ is defined to be the closure of C_0^∞ functions with respect to the inner product

$$(u, v)_{L^2_\rho(\mathbb{R}^n)} =: \int_{\mathbb{R}^n} \rho u v dx. \tag{II.5}$$

It is clear that $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space. Moreover, we have the following compact embedding.

Lemma 2.2 *Let $\rho \in L^{\frac{n}{2}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then the embedding $\mathcal{D}^{1,2} \subset L^2_\rho$ is compact.*

So we have established the evolution triple

$$\mathcal{D}^{1,2}(\mathbb{R}^n) \subset L^2_\rho(\mathbb{R}^n) \subset \mathcal{D}^{-1,2}(\mathbb{R}^n),$$

where all the embedding are dense and compact.

In order to deal with (II.1), we need information concerning the properties of the operator $-\phi\Delta$. We consider the equation

$$-\phi(x)\Delta u(x) = \eta(x), \quad x \in \mathbb{R}^n, \tag{II.6}$$

without boundary conditions. Since for every u, v in $C_0^\infty(\mathbb{R}^n)$

$$(-\phi\Delta u, v)_{L^2_\rho} = \int_{\mathbb{R}^n} \nabla u \nabla v dx, \tag{II.7}$$

and $L^2_\rho(\mathbb{R}^n)$ is defined as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the inner product (IV.12), we may consider equation (II.6) as an operator equation:

$$A_0 u = \eta, A_0 : D(A_0) \subseteq L^2_\rho(\mathbb{R}^n) \to L^2_\rho(\mathbb{R}^n), \text{for any } \eta \in L^2_\rho(\mathbb{R}^n). \tag{II.8}$$

Relation (II.7) implies that the operator $A_0 = -\phi\Delta$ with domain of definition $D(A_0) = C_0^\infty(\mathbb{R}^n)$ being symmetric. Let us note that the operator A_0 is not symmetric in the standard Lebesgue space $L^2(\mathbb{R}^n)$, because of the appearance of $\phi(x)$. For comments of the same nature on a similar model in the case of a bounded weight we refer to [[74], pages 185-187]. From (II.22) and equation (II.7) we have

$$(A_0 u, u)_{L^2_\rho} \geq c\|u\|_{L^2_\rho}, \text{for all } u \in D(A_0). \tag{II.9}$$

From (II.7) and (II.9) we conclude that A_0 is a symmetric, strongly monotone operator on $L^2_\rho(\mathbb{R}^n)$. Hence, the Friedrichs extension theorem (see [86]) is applicable. The energy scalar product given by (II.7) is

$$(u, v)_E = \int_{\mathbb{R}^n} \nabla u \nabla v dx, \tag{II.10}$$

and the energy space is the completion of $D(A_0)$ with respect to $(u, v)_E$. It is obvious that the energy space X_E is the homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^n)$. The energy extension

$A_E = -\phi\Delta$ of A_0, namely

$$-\phi\Delta : \mathcal{D}^{1,2}(\mathbb{R}^n) \to \mathcal{D}^{-1,2}(\mathbb{R}^n), \tag{II.11}$$

is defined to be the duality mapping of $\mathcal{D}^{1,2}(\mathbb{R}^n)$. For every $\eta \in \mathcal{D}^{-1,2}(\mathbb{R}^n)$ the equation (II.6) has a unique solution. Define $D(A)$ to be the set of all solutions of the equation (II.6), for arbitrary $\eta \in L^2_\rho(\mathbb{R}^n)$. The *Friedrichs extension* A of A_0 is the restriction of the energy extension A_E to the set $D(A)$. The operator A is self-adjoint and therefore graph-closed. Its domain $D(A)$, is a Hilbert space with respect to the graph scalar product

$$(u,v)_{D(A)} = (u,v)_{L^2_\rho} + (Au, Av)_{L^2_\rho}, \text{for all} u,v \in D(A).$$

The norm induced by the scalar product $(u,v)_{D(A)}$ is

$$\|u\|_{D(A)} = \left\{\int_{\mathbb{R}^n} \rho|u|^2 dx + \int_{\mathbb{R}^n} \phi|\Delta u|^2 dx\right\}^{\frac{1}{2}},$$

which is equivalent to the norm $\|Au\|_{L^2_\rho} = \left\{\int_{\mathbb{R}^n} \phi|\Delta u|^2 dx\right\}^{\frac{1}{2}}$. A consequence of the compactness of the embedding in (IV.17) is that for the eigenvalue problem

$$-\phi(x)\Delta u = \mu u, x \in \mathbb{R}^n, \tag{II.12}$$

there exists a complete system of eigensolutions $\{w_n, \mu_n\}$ with the following properties:

$$\begin{cases} -\phi\Delta w_j = \mu w_j, & j = 1,2\cdots, \quad w_j \in \mathcal{D}^{1,2}(\mathbb{R}^n), \\ 0 < \mu_1 \le \mu_2 \le \cdots, & \mu_j \to \infty, \quad as j \to \infty. \end{cases} \tag{II.13}$$

It can be shown, as in [18], that every solution of (II.12) is such that

$$u(x) \longrightarrow 0, as\ |x| \longrightarrow \infty, \tag{II.14}$$

uniformly with respect to x. Finally, we give the definition of *weak solutions* for the problem (II.1).

we give the definition of the weak solution for this problem.

Definition 2.3 *A weak solution of the problem (II.1) is a function $(u,v)(t,x)$ such that*

(i) $u, v \in L^2[0,T;\mathcal{D}^{1,2}(\mathbb{R}^n)]$, $u_t, v_t \in L^2[0,T;L^2_\rho(\mathbb{R}^n)]$, $u_{tt}, v_{tt} \in L^2[0,T;\mathcal{D}^{-1,2}(\mathbb{R}^n)]$,

(ii) for all $s \in C_0^\infty([0,T]\times\mathbb{R}^n)$*,* (u,v) *satisfies the generalized formula*

$$\begin{aligned}&\int_0^T (u_{tt}(\tau), s(\tau))_{L^2_\rho} d\tau + \alpha \int_0^T (v(\tau), s(\tau))_{L^2_\rho} d\tau + \int_0^T (u_t(\tau), s(\tau))_{L^2_\rho} d\tau \\ &+ \int_0^T \int_{\mathbb{R}^n} \nabla u(\tau) \nabla s(\tau) dx d\tau = 0,\end{aligned} \tag{II.15}$$

and

$$\int_0^T (v_{tt}(\tau), s(\tau))_{L^2_\rho} d\tau + \alpha \int_0^T (u(\tau), s(\tau))_{L^2_\rho} d\tau + \int_0^T \int_{\mathbb{R}^n} \nabla v(\tau) \nabla s(\tau) dx d\tau = 0, \tag{II.16}$$

(iii) (u,v) *satisfies the initial conditions*

$$(u_0(x), v_0(x)) \in (\mathcal{D}^{1,2}(\mathbb{R}^n))^2, \qquad (u_1(x), v_1(x)) \in (L^2_\rho(\mathbb{R}^n))^2.$$

The energy of the problem is defined as

$$E(t) := \frac{1}{2}\left(\|u_t\|^2_{L^2_\rho} + \|v_t\|^2_{L^2_\rho} + \|\nabla u\|^2_{L^2} + \|\nabla v\|^2_{L^2} + 2\alpha \int_{\mathbb{R}^n} \rho uv dx \right). \tag{II.17}$$

For α small enough we deduce that:

$$E(t) \geq \frac{1}{2}(1 - |\alpha|\gamma) \left(\|u_t\|^2_{L^2_\rho} + \|v_t\|^2_{L^2_\rho} + \|\nabla u\|^2_{L^2} + \|\nabla v\|^2_{L^2} \right).$$

From problem (II.1), we can have

$$E'(t) = -\int_0^t \|u_t(\tau)\|^2_{L^2_\rho} d\tau \leq 0. \tag{II.18}$$

We denote by $\mathcal{A}$ the unbounded operator in the energy space:
$\mathcal{H}$=$H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ defined by:

$$\mathcal{D}(\mathcal{A}) = (H^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n))^2 \times (H^1(\mathbb{R}^n))^2$$

and

$$\mathcal{A}\mathcal{U} = (\widetilde{u}, \phi(x)\Delta u - \alpha v - \widetilde{u}, \widetilde{v}, \phi(x)\Delta v - \alpha u)^T \text{ and } \mathcal{U} = (u, \widetilde{u}, v, \widetilde{v})^T.$$

The problem (II.1) can then be reformulated under the abstract from

$$\mathcal{U}' + \mathcal{A}\mathcal{U} = 0.$$

3 Linear polynomial stability

Our main result reads as follows.

Theorem 3.1 *Let* $(u_0, u_1), (v_0, v_1) \in \mathcal{D}^{1,2}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)$*and suppose that* (***H1***) *holds. Then the solution of the problem (II.1) satisfies the following energy decay rate estimate:*

$$E(t) \leq \frac{C(E(0) + E_1(0))}{t}, \qquad \forall t \geq 0. \tag{II.19}$$

Where C *is a positive constants.*

Proof of theorem 3.1 We multiply the first equation in (II.1) by ρu and the second equation by ρv and integrating over $(0, T) \times \mathbb{R}^n$, we get respectively

$$\int_{\mathbb{R}^n} \int_0^T \frac{d}{dt} \left[(\rho u u_t) - \rho u_t^2\right] dtdx + \int_{\mathbb{R}^n} \int_0^T |\nabla u|^2 dtdx + \alpha \int_{\mathbb{R}^n} \int_0^T \rho uv dtdx + \int_{\mathbb{R}^n} \int_0^T \rho u u_t dtdx = 0, \tag{II.20}$$

and

$$\int_{\mathbb{R}^n} \int_0^T \frac{d}{dt} \left[(\rho v v_t) - \rho v_t^2\right] dtdx + \int_{\mathbb{R}^n} \int_0^T |\nabla v|^2 dtdx + \alpha \int_{\mathbb{R}^n} \int_0^T \rho uv dtdx = 0, \tag{II.21}$$

Summing up (II.20) and (II.21), we obtain

$$\begin{aligned} &\int_0^T \left(\|\nabla u\|^2_{L^2} + \|\nabla v\|^2_{L^2} + \|u_t\|^2_{L^2_\rho} + \|v_t\|^2_{L^2_\rho}\right) dt + 2\alpha \int_{\mathbb{R}^n} \int_0^T \rho uv dtdx \\ &= -\left[\int_{\mathbb{R}^n} (\rho u u_t + \rho v v_t) dx\right]_0^T + 2 \int_{\mathbb{R}^n} \int_0^T (\rho u_t^2 + \rho v_t^2) dtdx - \int_{\mathbb{R}^n} \int_0^T \rho u u_t dtdx. \end{aligned}$$

Then we have

$$\int_0^T E(t) dt = -\left[\int_{\mathbb{R}^n} (\rho u u_t + \rho v v_t) dx\right]_0^T + 2 \int_{\mathbb{R}^n} \int_0^T (\rho u_t^2 + \rho v_t^2) dtdx - \int_0^T \int_{\mathbb{R}^n} \rho u u_t dxdt, \tag{II.22}$$

Let us estimate the right-hand side terms:
By using Cauchy-Schwartz inequality and Poincare inequality, we obtain, for any $\varepsilon > 0$,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \rho u u_t dxdt &\leq \tfrac{\varepsilon}{2} \int_0^T \int_{\mathbb{R}^n} \rho u^2 dxdt + \frac{1}{2\varepsilon} \int_0^T \int_{\mathbb{R}^n} \rho u_t^2 dxdt, \\ &\leq \tfrac{\varepsilon\gamma}{2} \int_0^T \int_{\mathbb{R}^n} |\nabla u|^2 dxdt + \frac{1}{2\varepsilon} \int_0^T (-E'(t))\, dt, \\ &\leq \tfrac{\varepsilon\gamma}{2} \int_0^T E(t) dt + \frac{1}{2\varepsilon} E(0). \end{aligned} \tag{II.23}$$

and

$$\left[\int_{\mathbb{R}^n} (\rho u u_t + \rho v v_t) dx\right]_0^T \leq CE(0). \tag{II.24}$$

here γ is the Poincare constant.

Thus,

$$\left(1 - \frac{\varepsilon\gamma}{2}\right) \int_0^T E(t) dt \leq CE(0) + 2 \int_{\mathbb{R}^n} \int_0^T \rho v_t^2. \tag{II.25}$$

To estimate the term $\int_0^T \int_{\mathbb{R}^n} \rho v^2 dx dt$, we multiply the first equation in (II.1) by ρv and the second equation by ρu and integrating over $(0, T) \times \mathbb{R}^n$, we get respectively

$$\int_0^T \int_{\mathbb{R}^n} \rho v u_{tt} dx dt - \int_0^T \int_{\mathbb{R}^n} v \Delta u dx dt + \alpha \int_0^T \int_{\mathbb{R}^n} \rho v^2 dx dt + \int_0^T \int_{\mathbb{R}^n} \rho v u_t dx dt = 0, \tag{II.26}$$

and

$$\int_0^T \int_{\mathbb{R}^n} \rho u v_{tt} dx dt - \int_0^T \int_{\mathbb{R}^n} u \Delta v dx dt + \alpha \int_0^T \int_{\mathbb{R}^n} \rho u^2 dx dt = 0, \tag{II.27}$$

The difference (II.26) and (II.27), we obtain

$$\int_0^T \int_{\mathbb{R}^n} \rho[v u_{tt} - u v_{tt}] dx dt + \alpha \int_0^T \int_{\mathbb{R}^n} \rho v^2 dx dt - \alpha \int_0^T \int_{\mathbb{R}^n} \rho u^2 dx dt + \int_0^T \int_{\mathbb{R}^n} \rho v u_t dx dt = 0, \tag{II.28}$$

Thus, we get

$$\alpha \int_0^T \int_{\mathbb{R}^n} \rho v^2 dx dt = \alpha \int_0^T \int_{\mathbb{R}^n} \rho u^2 dx dt - \int_0^T \int_{\mathbb{R}^n} \rho v u_t dx dt + \left[\int_{\mathbb{R}^n} \rho (u v_t - v u_t)\right]_0^T, \tag{II.29}$$

We estimate the second term as follows:

$$\int_0^T \int_{\mathbb{R}^n} \rho v u_t dx dt \leq \frac{\varepsilon}{2} \int_0^T \int_{\mathbb{R}^n} \rho v^2 dx dt + \frac{1}{2\varepsilon} E(0). \tag{II.30}$$

Then it follows from (II.29) and (II.30) that

$$\left(\alpha - \frac{\varepsilon}{2}\right) \int_0^T \int_{\mathbb{R}^n} \rho v^2 dx dt \leq CE(0) + \alpha \int_0^T \int_{\mathbb{R}^n} \rho u^2 dx dt. \tag{II.31}$$

From (II.31) yield

$$\int_0^T \int_{\mathbb{R}^n} \rho v^2 dx dt \leq CE(0) + C\alpha \int_0^T \int_{\mathbb{R}^n} \rho u^2 dx dt. \tag{II.32}$$

The only term whose estimation is not easy is the last term $\int_0^T \int_{\mathbb{R}^n} \rho v_t^2 dxdt$:
Using inequality (II.31) with the derivatives, we obtain:

$$\int_0^T \int_{\mathbb{R}^n} \rho v_t^2 dxdt \leq CE(U'(0)) + C \int_0^T \int_{\mathbb{R}^n} \rho u_t^2 dxdt \leq C(E(0) + E(U'(0))). \tag{II.33}$$

Combining the last inequalities and (II.25), we obtain

$$\int_0^T E(t)dt \leq C(E(U(0)) + E(U'(0))) \tag{II.34}$$

Using then the following result of Alabau [6], we prove the polynomial energy decay of the solution of the system (II.1):
If E is a non-increasing function which verifies (II.34) for all $U \in \mathcal{D}(\mathcal{A})$, then the full energy of the solution U of system (II.1) decays polynomially, i.e.

$$\forall t > 0, \quad E(U(t)) \leq \frac{C}{t}(E(U(0) + E(U'(0))).$$

Moreover if the initial data are more regular, i.e. $U_0 = (u_0, v_0, u_1, v_1) \in \mathcal{D}(\mathcal{A}^{\mathrm{n}})$, for a certain positive integer n then the following inequality holds:

$$\forall t > 0, \quad E(U(t)) \leq \frac{C}{t^n} \sum_{p=0}^{p=n} (E(U^p(0)).$$

This completes the proof of Theorem 3.1

Chapter III

Decay rate of solutions to the Cauchy problem for a coupled system of a viscoelastic wave equations in $\mathbb{R}^n$

1 Introduction and statement

Let us consider the following problem

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + \alpha u_2 = \phi(x)\Delta_x \left(u_1 + \displaystyle\int_0^t g_1(s)u_1(t-s,x)ds\right), x \in \mathbb{R}^n \times \mathbb{R}^+ \\ \left(|u_2'|^{l-2}u_2'\right)' + \alpha u_1 = \phi(x)\Delta_x \left(u_2 + \displaystyle\int_0^t g_2(s)u_2(t-s,x)ds\right), x \in \mathbb{R}^n \times \mathbb{R}^+ \\ (u_1(0,x), u_2(0,x)) = (u_1^0(x), u_2^0(x)) \in (\mathcal{D}^{1,2}(\mathbb{R}^n))^2, \\ (u_1'(0,x), u_2'(0,x)) = (u_1^1(x), u_2^1(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases} \tag{III.1}$$

where the space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ defined in (III.11) and $l, n \geq 2$, $\phi(x) > 0, \forall x \in \mathbb{R}^n, (\phi(x))^{-1} = \rho(x)$ defined in (***H2***).

In this chapter we are going to consider the solutions in spaces weighted by the density function $\rho(x)$ in order to compensate for the lack of Poincare's inequality which is useful in the proof.

In this framework, (see [38], [43]), it is well known that, for any initial data $(u_1^0, u_2^0) \in (\mathcal{D}^{1,2}(\mathbb{R}^n))^2, (u_1^1, u_2^1) \in (L^l_\rho(\mathbb{R}^n))^2$, then problem (III.1) has a global solution $(u_1, u_2) \in (C([0,T), \mathcal{D}^{1,2}(\mathbb{R}^n)))^2, (u_1', u_2') \in (C([0,T), L^l_\rho(\mathbb{R}^n))^2$ for T small enough, under hypothesis (***H1***-***H2***).

The energy of (u_1, u_2) at time t is defined by

$$E(t) = \frac{(l-1)}{l}\sum_{i=1}^{2}\|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l + \frac{1}{2}\sum_{i=1}^{2}(1-\int_0^t g_i(s)ds)\|\nabla_x u_i\|_2^2 + \frac{1}{2}\sum_{i=1}^{2}(g_i \circ \nabla_x u_i)$$
$$+\alpha\int_{\mathbb{R}^n}\rho u_1 u_2 dx. \tag{III.2}$$

For α small enough we deduce that:

$$E(t) \geq \frac{1}{2}(1-|\alpha|\|\rho\|_{L^s}^{-1})\left[\frac{2(l-1)}{l}\sum_{i=1}^{2}\|u_i'\|_{L^l_\rho}^l + \sum_{i=1}^{2}\left(1-\int_0^t g_i(s)ds\right)\|\nabla_x u_i\|_2^2 + \sum_{i=1}^{2}(g_i \circ \nabla_x u_i)\right],$$

and the following energy functional law holds, which means that, our energy is uniformly bounded and decreasing along the trajectories.

$$E'(t) = \frac{1}{2}\sum_{i=1}^{2}(g_i' \circ \nabla_x u_i)(t) - \frac{1}{2}\sum_{i=1}^{2}g_i(t)\|\nabla_x u_i(t)\|_2^2, \quad \forall t \geq 0. \tag{III.3}$$

The following notation will be used throughout this chapter

$$(\Phi^s \circ \Psi)(t) = \int_0^t \Phi^s(t-\tau)\,\|\Psi(t)-\Psi(\tau)\|_2^2\,d\tau. \tag{III.4}$$

For the literature, in $\mathbb{R}^n$ we quote essentially the results of [1], [38], [41], [39], [43], [57]. In [41], authors showed for one equation that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem (III.1) with $l = 2,\ \rho(x) = 1$ is polynomial. The finite-speed propagation is used to compensate for the lack of Poincare's inequality. In the case $l = 2$, in [38], author looked into a linear Cauchy viscoelastic equation with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincare's inequality. The same problem treated in [38], was considered in [39], where they consider a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. Conditions used, on the relaxation function g and its derivative g' are different from the usual ones.

The problem (III.1) for the case $l = 2,\ \rho(x) = 1$, in a bounded domain $\Omega \subset \mathbb{R}^n, (n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive nonincreasing function was considered as equation in [57], where they established an explicit and general decay rate result for relaxation functions satisfying:

$$g'(t) \leq -H(g(t)), t \geq 0, H(0) = 0, \tag{III.5}$$

for a positive function $H \in C^1(\mathbb{R}^+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$. Which improve the conditions considered recently by Alabau-Boussouira and Cannarsa [1] on the relaxation functions

$$g'(t) \leq -\chi(g(t)), \chi(0) = \chi'(0) = 0, \tag{III.6}$$

where χ is a non-negative function, strictly increasing and strictly convex on $(0, k_0], k_0 > 0$. They required that

$$\int_0^{k_0} \frac{dx}{\chi(x)} = +\infty, \int_0^{k_0} \frac{xdx}{\chi(x)} < 1, \lim \inf_{s \to 0^+} \frac{\chi(s)/s}{\chi'(s)} > \frac{1}{2}, \tag{III.7}$$

and proved a decay result for the energy of equation (III.1) with $\alpha = 0, l = 2, \rho(x) = 1$ in a bounded domain. In addition to these assumptions, if

$$\lim \sup_{s \to 0^+} \frac{\chi(s)/s}{\chi'(s)} < 1, \tag{III.8}$$

then, in this case, an explicit rate of decay is given.

We omit the space variable x of $u(x, t), u'(x, t)$ and for simplicity reason denote $u(x, t) = u$ and $u'(x, t) = u'$, when no confusion arises. We denote by $|\nabla_x u|^2 = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2, \Delta_x u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$. The constants c used throughout this paper are positive generic constants which may be different in various occurrences also the functions considered are all real valued, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$.

The main purpose of this work is to allow a wider class of relaxation functions and improve earlier results in the literature. The basic mechanism behind the decay rates is the relation between the damping and the energy.

First we recall and make use the following assumptions on the functions ρ and g for $i = 1, 2$ as:

H1: To guarantee the hyperbolicity of the system, we assume that the function g_i :

$\mathbb{R}^+ \longrightarrow \mathbb{R}^+$(for $i = 1, 2$) is of class C^1 satisfying:

$$1 - \int_0^\infty g_i(t)dt \geq k_i > 0, g_i(0) = g_{i0} > 0, \tag{III.9}$$

and there exist nonincreasing continuous functions ξ_1,ξ_2: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$g_i'(t) \leq -\xi_i g_i(t). \tag{III.10}$$

H2: The function $\rho : \mathbb{R}^n \to \mathbb{R}_+^*, \rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0, 1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n-qn+2q}$.

Definition 1.1 ([38], [73]) *We define the function spaces of our problem and its norm as follows:*

$$\mathcal{D}^{1,2}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in L^2(\mathbb{R}^n) \right\}, \tag{III.11}$$

and the spaces $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f, h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx.$$

For $1 < p < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{1/q}. \tag{III.12}$$

Corrolary 1.2 *The separable Hilbert space $L^2_\rho(\mathbb{R}^n)$ with*

$$(f, f)_{L^2_\rho(\mathbb{R}^n)} = \|f\|^2_{L^2_\rho(\mathbb{R}^n)}.$$

consist of all f for which $\|f\|_{L^q_\rho(\mathbb{R}^n)} < \infty, 1 < q < +\infty$.

The following technical lemma will play an important role in the sequel.

Lemma 1.3 *[22] For any two functions g, $v \in C^1(\mathbb{R})$ and $\theta \in [0, 1]$ we have*

$$\begin{aligned} \int_{\mathbb{R}^n} v'(t) \int_0^t g(t-s)v(s)dsdx &= -\frac{1}{2}\frac{d}{dt}(g \circ v)(t) + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)ds \right) \|v(t)\|_2^2 \\ &\quad + \frac{1}{2}(g' \circ v)(t) - \frac{1}{2}g(t)\|v(t)\|_2^2. \end{aligned} \tag{III.13}$$

and

$$\int\limits_{\mathbb{R}^n} \left(\int_0^t g(t-s)|v(s)-v(t)|ds\right)^2 dx \leq \left(\int_0^t g^{2(1-\theta)}(s)ds\right)(g^{2\theta}\circ v). \tag{III.14}$$

Lemma 1.4 *[42] Let ρ satisfies (**H2**), then for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$*

$$\|u\|_{L^q_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^s(\mathbb{R}^n)}\|\nabla_x u\|_{L^2(\mathbb{R}^n)}, \quad \textit{with } s = \frac{2n}{2n-qn+2q}, 2 \leq q \leq \frac{2n}{n-2}.$$

Proof of lemma 1.4 The Lemma is consequence of Hölder's inequality. In fact,

$$\begin{aligned} \int_{\mathbb{R}^n} \rho u^q dx &\leq \left(\int_{\mathbb{R}^n} \rho^s dx\right)^{\frac{1}{s}} \left(\int_{\mathbb{R}^n} |u|^{pb} dx\right)^{\frac{1}{b}}, \\ &\leq \left(\int_{\mathbb{R}^n} \rho^s dx\right)^{\frac{1}{s}} \left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right)^{\frac{p}{2}}, \end{aligned} \tag{III.15}$$

where $s = \dfrac{2n}{(2n-pn+2p)}$ and $b = \dfrac{2n}{(n-2)p}$.

Corrolary 1.5 *If $q=2$, then Lemma 1.4. yields*

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^{n/2}(\mathbb{R}^n)}\|\nabla_x u\|_{L^2(\mathbb{R}^n)},$$

where we can assume $\|\rho\|_{L^{n/2}(\mathbb{R}^n)} = C_0 > 0$ to get

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \leq C_0\|\nabla_x u\|_{L^2(\mathbb{R}^n)}. \tag{III.16}$$

Lemma 1.6 *Let ρ satisfy condition (**H2**). If $1 \leq q < p < p^* = \frac{2n}{n-2}$, then there exists $C_0 > 0$ such that the weighted inequality*

$$\|u\|_{L^p_\rho} \leq C_0\|u\|^{1-\theta}_{L^q_\rho}\|u\|^{\theta}_{\mathcal{D}^{1,2}(\mathbb{R}^n)}, \tag{III.17}$$

is valid for all $(0,1)$ which satisfy the relation $\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{p^*}$.

Proof of lemma 1.6 We get relation (III.17) by using the weighted interpolation inequality

$$\|u\|_{L^p_\rho} \leq C_0\|u\|^{1-\theta}_{L^p_\rho}\|u\|^{\theta}_{L^{p^*}_\rho}. \tag{III.18}$$

(see [75]) and inequality (II.33). Here $C_0 = k^\theta$.

Using Cauchy-Schwarz and Poincare's inequalities, the proof of the following Lemma is immediate.

Lemma 1.7 *There exist constants $c, c' > 0$ such that*

$$\int_{\mathbb{R}^n} \left(\int_0^t g_i(t-s)(u_i(t) - u_i(s))ds \right)^2 dx \leq c(g_i \circ u_i)(t) \leq c'(g_i' \circ \nabla u_i)(t), \tag{III.19}$$

for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$.

We are now ready to state and prove our main results.

2 Exponential decay

For the purpose of constructing a Lyapunov functional L equivalent to E, we introduce the next functionals

$$\psi_1(t) = \sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) u_i |u_i'|^{l-2} u_i' dx, \tag{III.20}$$

$$\psi_2(t) = -\sum_{i=1}^{2} \int_{\mathbb{R}^n} \rho(x) |u_i'|^{l-2} u_i' \int_0^t g_i(t-s)(u_i(t) - u_i(s)) ds dx, \tag{III.21}$$

Lemma 2.1 *Under the assumptions (**H1-H2**), the functional ψ_1 satisfies, along the solution of (III.1), for any $\delta \in (0,1)$*

$$\psi_1'(t) \leq \sum_{i=1}^{2} \|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - (k + |\alpha| C_0 - \delta - 1) \sum_{i=1}^{2} \|\nabla_x u_i\|_2^2 + \frac{(1-k)}{4\delta} \sum_{i=1}^{2} (g_i \circ \nabla_x u_i), \tag{III.22}$$

Proof of lemma 2.1 From (III.20), integrate by parts over $\mathbb{R}^n$, we have

$$\begin{aligned}
\psi_1'(t) &= \int_{\mathbb{R}^n} \rho(x) u_1'^l dx + \int_{\mathbb{R}^n} \rho(x) u_1 \left(|u_1'|^{l-2} u_1' \right)' dx \\
&+ \int_{\mathbb{R}^n} \rho(x) u_2'^l dx + \int_{\mathbb{R}^n} \rho(x) u_2 \left(|u_2'|^{l-2} u_2' \right)' dx, \\
&= \int_{\mathbb{R}^n} \left(\rho(x) u_1'^l + u_1 \Delta_x u_1 - \alpha \rho(x) u_1 u_2 - u_1 \int_0^t g_1(t-s) \Delta_x u_1(s,x) ds \right) dx \\
&+ \int_{\mathbb{R}^n} \left(\rho(x) u_2'^l + u_2 \Delta_x u_2 - \alpha \rho(x) u_1 u_2 - u_2 \int_0^t g_2(t-s) \Delta_x u_2(s,x) ds \right) dx, \\
&\leq \sum_{i=1}^{2} \|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \sum_{i=1}^{2} k_i \|\nabla_x u_i\|_2^2 - 2\alpha \int_{\mathbb{R}^n} \rho(x) u_1 u_2 dx \\
&+ \sum_{i=1}^{2} \int_{\mathbb{R}^n} \nabla_x u_i \int_0^t g_i(t-s)(\nabla_x u_i(s) - \nabla_x u_i(t)) ds dx.
\end{aligned}$$

Using Young's, Poincare's inequalities, Lemma 1.3 and Lemma 1.4, we obtain

$$\begin{aligned}
\psi_1'(t) &\leq \sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \sum_{i=1}^{2} k_i\|\nabla_x u_i\|_2^2 + (1-|\alpha|\|\rho\|^{-1}_{L^s(\mathbb{R}^n)})\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2\\
&+ \delta\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2 + \frac{1}{4\delta}\sum_{i=1}^{2}\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)|\nabla_x u_i(s)-\nabla_x u_i(t)|ds\right)^2 dx,\\
&\leq \sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - (k+|\alpha|C_0-\delta-1)\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2 + \frac{(1-k)}{4\delta}\sum_{i=1}^{2}(g_i\circ\nabla_x u_i).
\end{aligned}$$

For α small enough and $k=\max\{k_1,k_2\}$.

Lemma 2.2 *Under the assumptions (**H1-H2**), the functional ψ_2 satisfies, along the solution of (III.1), for any $\delta\in(0,1)$*

$$\begin{aligned}
\psi_2'(t) &\leq \sum_{i=1}^{2}\left(\delta-\int_0^t g_i(s)ds\right)\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)}\\
&+ \delta\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2 + \frac{c}{\delta}\sum_{i=1}^{2}(g_i\circ\nabla_x u_i) - c_\delta C_0\sum_{i=1}^{2}(g_i'\circ\nabla_x u_i)^{l/2}. \qquad \text{(III.23)}
\end{aligned}$$

Proof of lemma 2.2 Exploiting Eq. in (III.1), to get

$$\begin{aligned}
\psi_2'(t) &= -\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)\left(|u_i'|^{l-2}u_i'\right)'\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx \qquad \text{(III.24)}\\
&- \sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)|u_i'|^{l-2}u_i'\int_0^t g_i'(t-s)(u_i(t)-u_i(s))dsdx - \sum_{i=1}^{2}\int_0^t g_i(s)ds\|u_i'\|^l_{L^l_\rho}.
\end{aligned}$$

To simplify the first term in (III.24), we multiply (III.1) by $\int_0^t g_i(t-s)(u_i(t)-u_i(s))ds$ and integrate by parts over $\mathbb{R}^n$. So we obtain

$$\begin{aligned}
&-\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)\left(|u_i'|^{l-2}u_i'\right)'\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx\\
&= \sum_{i=1}^{2}\int_{\mathbb{R}^n}\Delta u_i(x)\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx\\
&- \sum_{i=1}^{2}\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)(u_i(t)-u_i(s))\int_0^t g_i(t-s)\Delta u_i(s)\right)dx \qquad \text{(III.25)}\\
&- \alpha\int_{\mathbb{R}^n}\left[\rho u_2\int_0^t g_1(t-s)(u_1(t)-u_1(s))ds + \rho u_1\int_0^t g_2(t-s)(u_2(t)-u_2(s))ds\right]dx.
\end{aligned}$$

The first term in the right side of (III.25) is estimated as follows

$$\begin{aligned} & \int_{\mathbb{R}^n} \Delta u_i(x) \int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx \\ \leq & -\int_{\mathbb{R}^n} \nabla_x u_i \int_0^t g_i(t-s)(\nabla_x u_i(t) - \nabla_x u_i(s))dsdx, \\ \leq & \int_{\mathbb{R}^n} \nabla_x u_i \int_0^t g_i(t-s)(\nabla_x u_i(s) - \nabla_x u_i(t))dsdx, \\ \leq & \ \delta \|\nabla_x u_i\|^2 + \frac{1}{4\delta}\left(\int_0^t g_i(s)\right)(g_i \circ \nabla u_i)(t), \\ \leq & \ \delta \|\nabla_x u_i\|^2 + \frac{1-k}{4\delta}(g_i \circ \nabla u_i)(t). \end{aligned}$$

while the second term becomes,

$$\begin{aligned} - & \int_{\mathbb{R}^n} \left(\int_0^t g_i(t-s)(u_i(t)-u_i(s)) \int_0^t g_i(t-s)\Delta u_i(s)\right) dx \\ = & \int_{\mathbb{R}^n} \left(\int_0^t g_i(t-s)(\nabla u_i(t) - \nabla u_i(s)) \cdot \int_0^t g_i(t-s)\nabla u_i(s)\right) dx, \\ \leq & \ \delta \int_{\mathbb{R}^n} \left(\int_0^t g_i(t-s)|\nabla u_i(s) - \nabla u_i(t)) + \nabla u_i(t)|\right)^2 \\ + & \ \frac{1}{4\delta}\int_{\mathbb{R}^n} \left(\int_0^t g_i(t-s)(\nabla u_i(t) - \nabla u_i(s))\right)^2, \\ \leq & \ 2\delta(1-k)^2\|\nabla u_i\|_2^2 + \left(2\delta + \frac{1}{4\delta}\right)(1-k)(g_i \circ \nabla u_i)(t). \end{aligned}$$

Now, using Young's and Poincare's inequalities we estimate

$$\begin{aligned} - & \ \alpha \int_{\mathbb{R}^n} \rho u_2 \int_0^t g_1(t-s)(u_1(t)-u_1(s))dsdx \\ \leq & \ -|\alpha|\delta\|u_2\|^2_{L^2_\rho} - \frac{|\alpha|C_0}{4\delta}(1-k)(g_1 \circ \nabla u_1)(t), \\ \leq & \ -|\alpha|\delta C_0\|\nabla u_2\|^2_{L^2} - \frac{|\alpha|C_0}{4\delta}(1-k)(g_1 \circ \nabla u_1)(t). \end{aligned}$$

By Hölder's and Young's inequalities and Lemma 1.4 we estimate

$$\begin{aligned}
- & \int_{\mathbb{R}^n} \rho(x)|u_i'|^{l-2}u_i' \int_0^t g_i'(t-s)(u_i(t)-u_i(s))dsdx \\
\leq & \left(\int_{\mathbb{R}^n} \rho(x)|u_i'|^l dx\right)^{(l-1)/l} \times \left(\int_{\mathbb{R}^n} \rho(x)|\int_0^t -g_i'(t-s)(u_i(t)-u_i(s))ds|^l\right)^{1/l}, \\
\leq & \delta\|u'\|^l_{L^l_\rho(\mathbb{R}^n)} + \frac{1}{4\delta}\|\rho\|^l_{L^s(\mathbb{R}^n)}\|\int_0^t -g'(t-s)(u(t)-u(s))ds\|^l_{L^l_\rho(\mathbb{R}^n)}, \\
\leq & \delta\|u'\|^l_{L^l_\rho(\mathbb{R}^n)} - \frac{1}{4\delta}C_0(g' \circ \nabla_x u)^{l/2}(t).
\end{aligned}$$

Using Young's and Poincare's inequalities and Lemma 1.3, we obtain

$$\begin{aligned}
\psi_2'(t) \leq & \sum_{i=1}^{2}\left(\delta - \int_0^t g_i(s)ds\right)\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} \\
+ & \delta\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2 + \frac{c}{\delta}\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) - c_\delta C_0\sum_{i=1}^{2}(g_i' \circ \nabla_x u_i)^{l/2}.
\end{aligned}$$

Our main result reads as follows

Theorem 2.3 *Let $(u_1^0, u_1^1), (u_2^0, u_2^1) \in (\mathcal{D}^{1,2}(\mathbb{R}^n) \times L^l_\rho(\mathbb{R}^n))$ and suppose that $(\boldsymbol{H1} - \boldsymbol{H2})$ hold. Then there exist positive constants α_1, ω such that the energy of solution given by (III.1) satisfies,*

$$E(t) \leq \alpha_1 E(t_0)\exp\left(-\omega\int_{t_0}^t \xi(s)ds\right), \forall t \geq t_0 \tag{III.26}$$

where $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}, \quad \forall t \geq 0.$

In order to prove this theorem, let us define

$$L(t) = N_1 E(t) + \psi_1(t) + N_2\psi_2(t), \tag{III.27}$$

for $N_1, N_2 > 1$, We need the next Lemma, which means that there is equivalent between the Lyapunov and energy functions

Lemma 2.4 *For $N_1, N_2 > 1$, we have*

$$\beta_1 L(t) \leq E(t) \leq L(t)\beta_2, \tag{III.28}$$

holds for two positive constants β_1 and β_2.

Proof of lemma 2.4 By applying Young´s inequality to (III.20) and using (III.21) and (III.27), we obtain

$$\begin{aligned}
|L(t) - N_1 E(t)| &\leq |\psi_1(t)| + N_2|\psi_2(t)|, \\
&\leq \sum_{i=1}^{2} \int_{\mathbb{R}^n} \left|\rho(x) u_i |u_i'|^{l-2} u_i'\right| dx \\
&+ N_2 \sum_{i=1}^{2} \int_{\mathbb{R}^n} \left|\rho(x) |u_i'|^{l-2} u_i' \int_0^t g_i(t-s)(u_i(t) - u_i(s)) ds\right| dx.
\end{aligned}$$

Thanks to Hölder and Young's inequalities with exponents $\frac{l}{l-1}$, l, since $\frac{2n}{n+2} \geq l \geq 2$, we have by using Lemma 1.4

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|\rho(x) u_i |u_i'|^{l-2} u_i'\right| dx &\leq \left(\int_{\mathbb{R}^n} \rho(x) |u_i|^l dx\right)^{1/l} \left(\int_{\mathbb{R}^n} \rho(x) |u_i'|^l dx\right)^{(l-1)/l}, \\
&\leq \frac{1}{l}\left(\int_{\mathbb{R}^n} \rho(x) |u_i|^l dx\right) + \frac{l-1}{l}\left(\int_{\mathbb{R}^n} \rho(x) |u_i'|^l dx\right), \\
&\leq c\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + c\|\rho\|^l_{L^s(\mathbb{R}^n)} \|\nabla_x u_i\|_2^l. \qquad \text{(III.29)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|\left(\rho(x)^{\frac{l-1}{l}} |u_i'|^{l-2} u_i'\right)\left(\rho(x)^{\frac{1}{l}} \int_0^t g_i(t-s)(u_i(t) - u_i(s)) ds\right)\right| dx \\
&\leq \left(\int_{\mathbb{R}^n} \rho(x) |u_i'|^l dx\right)^{(l-1)/l} \times \left(\int_{\mathbb{R}^n} \rho(x) |\int_0^t g_i(t-s)(u_i(t) - u_i(s)) ds|^l\right)^{1/l}, \\
&\leq \frac{l-1}{l} \|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \frac{1}{l} \|\int_0^t g_i(t-s)(u_i(t) - u_i(s)) ds\|^l_{L^l_\rho(\mathbb{R}^n)}, \\
&\leq \frac{l-1}{l} \|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \frac{1}{l} \|\rho\|^l_{L^s(\mathbb{R}^n)} (g_i \circ \nabla_x u_i)^{l/2}(t).
\end{aligned}$$

then, since $l \geq 2$, we have

$$\begin{aligned}
|L(t) - N_1 E(t)| &\leq c \sum_{i=1}^{2} \left(\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \|\nabla_x u_i\|_2^l + g_i \circ \nabla_x u_i)^{l/2}(t)\right), \\
&\leq c(E(t) + E^{l/2}(t)), \\
&\leq c(E(t) + E(t) \cdot E^{(l/2)-1}(t)), \\
&\leq c(E(t) + E(t) \cdot E^{(l/2)-1}(0)), \\
&\leq cE(t).
\end{aligned}$$

Consequently, (III.28) follows.

Proof of Theorem2.3 From (III.3), results of Lemmas 2.1 and 2.2, we have

$$\begin{aligned} L'(t) &= N_1E'(t) + \psi_1'(t) + N_2\psi_2'(t), \\ &\leq \left(\frac{1}{2}N_1 - c_\delta C_0 N_2\right)\sum_{i=1}^{2}(g_i' \circ \nabla_x u_i)^{l/2} + \left(\frac{4\xi_2 c + (1-l)}{4\delta}\right)\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) \\ &- M_1\sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - M_2\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2. \end{aligned}$$

At this point, we choose ξ_2 large enough so that

$$M_1 := \left(N_2\left(\int_0^{t_1} g(s)ds - \delta\right) - 1\right) > 0,$$

We choose δ so small that $N_1 > 2c_\delta\|\rho\|^l_{L^s(\mathbb{R}^n)}N_2$. Whence δ is fixed, we can choose ξ_1, ξ_2 large enough so that

$$M_2 := \left(-N_2\sigma + \frac{1}{2}N_1 g(t_1) + (l-\sigma)\right) > 0,$$

and

$$\left(\frac{1}{2}N_1 - c_\delta C_0 N_2\right) > 0.$$

which yields

$$L'(t) \leq M_0\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) - mE(t), \quad \forall t \geq t_1. \tag{III.30}$$

Multiplying (III.30) by $\xi(t)$ gives

$$\xi(t)L'(t) \leq -m\xi(t)E(t) + M_0\xi(t)\sum_{i=1}^{2}(g_i \circ \nabla_x u_i). \tag{III.31}$$

The last term can be estimated, using (**H1**), as follows

$$\begin{aligned} M_0\xi(t)\sum_{i=1}^{2}(g_i\circ\nabla_x u_i) &\leq M_0\sum_{i=1}^{2}\xi_i(t)\int_{\mathbb{R}^n}\int_0^t g_i(t-s)|u_i(t)-u_i(s)|^2, \\ &\leq M_0\sum_{i=1}^{2}\int_{\mathbb{R}^n}\int_0^t \xi_i(t-s)g_i(t-s)|u_i(t)-u_i(s)|^2, \\ &\leq -M_0\sum_{i=1}^{2}\int_{\mathbb{R}^n}\int_0^t g_i'(t-s)|u_i(t)-u_i(s)|^2, \\ &\leq -M_0\sum_{i=1}^{2} g_i'\circ\nabla u_i \leq -M_0E'(t). \end{aligned} \tag{III.32}$$

Thus, (III.30) becomes

$$\xi(t)L'(t)+M_0E'(t) \leq -m\xi(t)E(t), \quad \forall t\geq t_0. \tag{III.33}$$

Using the fact that ξ is a nonincreasing continuous function as ξ_1 and ξ_2 are nonincreasing, and so ξ is differentiable, with $\xi'(t)\leq 0$ for a.e t, then

$$(\xi(t)L(t)+M_0E(t))' \leq \xi(t)L'(t)+M_0E'(t)\leq -m\xi(t)E(t), \quad \forall t\geq t_0. \tag{III.34}$$

Since, using (III.28)

$$F=\xi L+M_0E\sim E, \tag{III.35}$$

we obtain, for some positive constant ω

$$F'(t)\leq -\omega\xi(t)F(t), \quad \forall t\geq t_0. \tag{III.36}$$

Integration over (t_0,t) leads to, for some constant $\omega>0$ such that

$$F(t)\leq \alpha_1 F(t_0)\exp\left(-\omega\int_{t_0}^t \xi(s)ds\right), \forall t\geq t_0. \tag{III.37}$$

Recalling (III.35), estimate (III.37) yields the desired result (III.26). This completes the proof of Theorem 2.3.

Chapter IV

Existence and stability to the Cauchy problem for a coupled system of a nonlinear weak viscoelastic wave equations in $\mathbb{R}^n$

1 Introduction and statement

Let us consider the following problem

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + f_1(u_1,u_2) = \phi(x)\Delta_x\left(u_1 - \alpha_1(t)\int_0^t g_1(t-s)u_1(s,x)ds\right), & \text{in} \quad \mathbb{R}^n\times\mathbb{R}^+, \\ \left(|u_2'|^{l-2}u_2'\right)' + f_2(u_1,u_2) = \phi(x)\Delta_x\left(u_2 - \alpha_2(t)\int_0^t g_2(t-s)u_2(s,x)ds\right), & \text{in} \quad \mathbb{R}^n\times\mathbb{R}^+, \\ (u_1(0,x),u_2(0,x)) = (u_1^0(x),u_2^0(x)) \in (\mathcal{D}^{1,2}(\mathbb{R}^n))^2, \\ (u_1'(0,x),u_2'(0,x)) = (u_1^1(x),u_2^2(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases} \tag{IV.1}$$

where the space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ defined in (IV.11) and $l \geq 2$, $\phi(x) > 0, \forall x \in \mathbb{R}^n, (\phi(x))^{-1} = \rho(x)$ defined in (***H2***).

In this chapter we consider the solutions in spaces weighted by the density function $\rho(x)$ in order to compensate for the lack of Poincare's inequality.

The energy of (u_1,u_2) at time t is defined by

$$E(t) = \frac{(l-1)}{l}\sum_{i=1}^2 \|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \frac{1}{2}\sum_{i=1}^2\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla_x u_i\|_2^2$$
$$+\frac{1}{2}\sum_{i=1}^2 \alpha_i(t)(g_i\circ\nabla_x u_i) + \int_{\mathbb{R}^n}\rho(x)F(u_1,u_2)dx. \tag{IV.2}$$

and the following energy functional law holds, which means that, our energy is uniformly bounded and decreasing along the trajectories.

$$\begin{aligned} E'(t) &= \frac{1}{2}\sum_{i=1}^{2}\alpha_i(t)(g_i'\circ\nabla_x u_i)(t)-\frac{1}{2}\sum_{i=1}^{2}\alpha_i(t)g_i(t)\|\nabla_x u_i(t)\|_2^2 \\ &+ \frac{1}{2}\sum_{i=1}^{2}\alpha_i'(t)(g_i\circ\nabla_x u_i)(t)-\frac{1}{2}\sum_{i=1}^{2}\alpha_i'(t)\int_0^t g_i(s)ds\|\nabla_x u_i(t)\|_2^2, \forall t\geq 0. \\ &\leq \frac{1}{2}\sum_{i=1}^{2}\alpha_i(t)(g_i'\circ\nabla_x u_i)(t)-\frac{1}{2}\sum_{i=1}^{2}\alpha_i'(t)\int_0^t g_i(s)ds\|\nabla_x u_i(t)\|_2^2, \forall t\geq 0. \end{aligned} \tag{IV.3}$$

The following notation will be used throughout this chapter

$$(g^s\circ\Psi)(t)=\int_0^t g^s(t-\tau)\left\|\Psi(t)-\Psi(\tau)\right\|_2^2 d\tau, \tag{IV.4}$$

for any $\psi\in L^\infty(0,T;L^2(\mathbb{R}^n))$.

For the literature, in $\mathbb{R}^n$ we quote essentially the results of [1], [38], [41], [39], [43], [57]. In [41], authors showed for one equation that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem (IV.1) with $l=2,\alpha_i=\rho(x)=1$ is polynomial. The finite-speed propagation is used to compensate for the lack of Poincare's inequality. In the case $l=2,\alpha_i=1$, in [38], author looked into a linear Cauchy viscoelastic equation with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincare's inequality. The same problem treated in [38], was considered in [39], where they consider a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. Conditions used, on the relaxation function g and its derivative g' are different from the usual ones.

The problem of stabilization of weakly coupled systems has also been studied by several authors. Under certain conditions imposed on the subset where the damping term is effective, Kapitonov [44] proves uniform stabilization of the solutions of a pair of hyperbolic systems coupled in velocities. Alabau and al. [4] studied the indirect internal stabilization of weakly coupled systems where the damping is effective in the whole domain. They prove that the behavior of the first equation is sufficient to stabilize the total system and to have polynomial decay for sufficiently smooth solutions. Alabau [6] proves indirect boundary stabilization (polynomial decay) of weakly coupled equations. She establishes a polynomial decay lemma for non-increasing and nonnegative function which satisfies an integral inequality.

The main purpose of this work is to prove an existence and uniqueness theorem for global weak solutions in Sobolev spaces using Faedo-Galerkin method and to allow a wider class of relaxation functions and improve earlier results in the literature. The basic mechanism behind the decay rates is the relation between the damping and the energy.

First we recall and make use the following assumptions on the functions ρ and g for $i = 1, 2$ as:

H1: To guarantee the hyperbolicity of the system, we assume that the function g_i : $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$(for $i = 1, 2$) is of class C^1 satisfying:

$$1 - \alpha_i(t) \int_0^t g_i(t)dt \geq k_i > 0, \int_0^\infty g_i(t)dt < +\infty, \ \alpha_i(t) > 0, \tag{IV.5}$$

$$\lim_{t\to+\infty} \frac{-\alpha'(t)}{\alpha(t)\xi(t)} = 0, \tag{IV.6}$$

and there exist nonincreasing continuous functions ξ_1,ξ_2: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$g_i'(t) \leq -\xi_i(t)g_i(t), \forall t \geq 0. \tag{IV.7}$$

H2: The function $\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0, 1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n-qn+2q}$.

H3: The function $f_i : \mathbb{R}^2 \to \mathbb{R}$ (for i=1,2) is of class C^1 and there exists a function F such that

$$f_1(x, y) = \frac{\partial F}{\partial x}, \qquad f_2(x, y) = \frac{\partial F}{\partial y},$$
$$F \geq 0, \ xf_1(x, y) + yf_2(x, y) - F(x, y) \geq 0.$$

and

$$\left|\frac{\partial f_i}{\partial x}(x, y)\right| + \left|\frac{\partial f_i}{\partial y}(x, y)\right| \leq d(1 + |x|^{\beta_{i1}-1} + |y|^{\beta_{i2}-1}), \quad \forall (x, y) \in \mathbb{R}^2, \tag{IV.8}$$

for some constant $d > 0$ and $1 \leq \beta_{ij} \leq \frac{n}{n-2}$ for $i, j = 1, 2$.

H4: There exists a positive constant k such that

$$|f_i(x, y)| \leq k(|x| + |y| + |x|^{\beta_{i1}} + |y|^{\beta_{i2}}), \tag{IV.9}$$

and

$$|f_i(x, y) - f_i(r, s)| \leq k(1 + |x|^{\beta_{i1}-1} + |y|^{\beta_{i2}-1} + |r|^{\beta_{i1}-1} + |s|^{\beta_{i2}-1})(|x - r| + |y - s|), \tag{IV.10}$$

for all $(x,y),(r,s) \in \mathbb{R}^2$ and $i = 1,2$.

Definition 1.1 *[73] We define the function spaces related with our problem and its norm as follows:*

$$\mathcal{D}^{1,2}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in (L^2(\mathbb{R}^n))^n \right\}, \tag{IV.11}$$

and the separable Hilbert space $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f,h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx. \tag{IV.12}$$

For $1 < p < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{1/q}. \tag{IV.13}$$

The next Lemma can be shown (see [42], Lemma 2.1).

Lemma 1.2 *Let ρ satisfies (**H2**), then for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$*

$$\|u\|_{L^q_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^s(\mathbb{R}^n)} \|\nabla_x u\|_{L^2(\mathbb{R}^n)}, \quad \text{with } s = \frac{2n}{2n - qn + 2q}, 2 \leq q \leq \frac{2n}{n-2}.$$

Corrolary 1.3 *If $q = 2$, then Lemma 1.2. yields*

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^{n/2}(\mathbb{R}^n)} \|\nabla_x u\|_{L^2(\mathbb{R}^n)}, \tag{IV.14}$$

where we can assume $\|\rho\|_{L^s(\mathbb{R}^n)} = c > 0$ to get

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \leq c \|\nabla_x u\|_{L^2(\mathbb{R}^n)}. \tag{IV.15}$$

Using Cauchy-Schwarz inequality, the proof of the following Lemma is immediate.

Lemma 1.4 *There exist constants $C > 0$ such that*

$$\int_{\mathbb{R}^n} \left(\int_0^t g_i(t-s)(u(t) - u(s)) ds \right)^2 dx \leq C \left(\int_0^t g_i(s) ds \right) (g_i \circ u)(t), \quad i = 1,2 \tag{IV.16}$$

for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$.

So we are able to construct the necessary *evolution triple* for the space setting of our problem, which is

$$\mathcal{D}^{1,2}(\mathbb{R}^n) \subset L^l_\rho(\mathbb{R}^n) \subset \mathcal{D}^{-1,2}(\mathbb{R}^n) \tag{IV.17}$$

where all the embedding are compact and dense.
The following technical Lemma will play an important role in the sequel.

Lemma 1.5 *For any $v \in C^1\left(0, T, H^1(\mathbb{R}^n)\right)$ we have*

$$\begin{aligned}
& \int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s)\Delta v(s) v'(t) ds dx \\
= \;& \frac{1}{2}\frac{d}{dt}\alpha(t)\left(g \circ \nabla v\right)(t) - \frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(s)\int_{\mathbb{R}^n}\left|\nabla v(t)\right|^2 dxds\right] \\
- \;& \frac{1}{2}\alpha(t)\left(g' \circ \nabla v\right)(t) + \frac{1}{2}\alpha(t) g(t)\int_{\mathbb{R}^n}\left|\nabla v(t)\right|^2 dxds \\
- \;& \frac{1}{2}\alpha'(t)\left(g \circ \nabla v\right)(t) + \frac{1}{2}\alpha'(t)\int_0^t g(s)ds\int_{\mathbb{R}^n}\left|\nabla v(t)\right|^2 dxds.
\end{aligned}$$

Proof of lemma 1.5 It's not hard to see

$$\begin{aligned}
& \int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s)\Delta v(s) v'(t) ds dx \\
= \;& -\alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}\nabla v'(t)\nabla v(s) dxds \\
= \;& -\alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}\nabla v'(t)\left[\nabla v(s) - \nabla v(t)\right] dxds \\
& -\alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}\nabla v'(t)\nabla v(t) dxds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s)\Delta v(s) v'(t) ds dx \\
= \;& \frac{1}{2}\alpha(t)\int_0^t g(t-s)\frac{d}{dt}\int_{\mathbb{R}^n}\left|\nabla v(s) - \nabla v(t)\right|^2 dxds, \\
& -\alpha(t)\int_0^t g(s)\left(\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^n}\left|\nabla v(t)\right|^2 dx\right) ds.
\end{aligned}$$

which implies,

$$\begin{aligned}
& \int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s)\Delta v(s) v'(t) ds dx \\
= & \frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} |\nabla v(s) - \nabla v(t)|^2 dx ds\right] \\
& -\frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(s)\int_{\mathbb{R}^n} |\nabla v(t)|^2 dx ds\right] \\
& -\frac{1}{2}\alpha(t)\int_0^t g'(t-s)\int_{\mathbb{R}^n} |\nabla v(s) - \nabla v(t)|^2 dx ds \\
& +\frac{1}{2}\alpha(t) g(t) \int_{\mathbb{R}^n} |\nabla v(t)|^2 dx ds. \\
& -\frac{1}{2}\alpha'(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} |\nabla v(s) - \nabla v(t)|^2 dx ds \\
& +\frac{1}{2}\alpha'(t)\int_0^s g(s) ds \int_{\mathbb{R}^n} |\nabla v(t)|^2 dx ds.
\end{aligned}$$

This completes the proof.

Definition 1.6 *A weak solution of (IV.1) is a function $u_i(x;t)$ such that*

- $u_i \in L^2[0,T;\mathcal{D}^{1,2}(\mathbb{R}^n)], \quad u_i' \in L^2[0,T;L^l_\rho(\mathbb{R}^n)],$
- *For all $v \in C_0^\infty([0,T]\times\mathbb{R}^n)$, u satisfies the generalized formula*

$$\begin{aligned}
& \int_0^T ((|u_i'|^{l-2}u_i')', v)_{L^l_\rho} + \int_0^T \int_{\mathbb{R}^n} \rho f_i(u_1(s), u_2(s)) v(s) ds \qquad \text{(IV.18)} \\
+ & \int_0^T \int_{\mathbb{R}^n} \nabla u_i(s) \nabla v(s) dx ds - \int_0^T \left(\alpha_i(t)\int_0^s g(s-\tau)\nabla u_i(\tau) d\tau, \nabla v(s)\right) ds = 0.
\end{aligned}$$

- *u satisfies the initial conditions*
  $u_i(x,0) = u_i^0(x) \in \mathcal{D}^{1,2}(\mathbb{R}^n), \quad u_i'(x,0) = u_i^1(x) \in L^l_\rho(\mathbb{R}^n).$

We are now ready to state and prove our main results

2 Well-posedness result

This section is devoted to prove the existence and uniqueness of solutions to system (IV.1)

Theorem 2.1 *Let $(u_1^0, u_1^1), (u_2^0, u_2^1) \in \mathcal{D}^{1,2}(\mathbb{R}^n) \times L^l_\rho(\mathbb{R}^n)$ be given. Assume that $(\boldsymbol{H1} - \boldsymbol{H4})$ are satisfied, then problem (IV.1) has a unique strong solution*

$$u_1, u_2 \in L^\infty\left(\mathbb{R}_+; \mathcal{D}^{1,2}(\mathbb{R}^n)\right) \bigcap W^{1,\infty}\left(\mathbb{R}_+; \mathcal{D}^{1,2}(\mathbb{R}^n)\right) \bigcap W^{2,\infty}\left(\mathbb{R}_+; L^l_\rho(\mathbb{R}^n)\right).$$

Proof of theorem 2.1 The existence is proved using the Galerkin method. In order to do so, we take $\{w_i\}_{i=1}^{\infty}$ be the eigenfunctions of the Laplacian operator subject to Dirichlet boundary conditions. Then $\{w_i\}_{i=1}^{\infty}$ is orthogonal basis of $\mathcal{D}^{1,2}(\mathbb{R}^n)$ which is orthonormal in $L^l_\rho(\mathbb{R}^n)$. Let $V_m = span\{w_1, w_2, \cdots, w_m\}$ and the projection of the initial data on the finite dimensional subspace V_m is given by

$$u_1^{0,m} = \sum_{i=0}^{m} a_j w_j, \quad u_2^{0,m} = \sum_{i=0}^{m} b_j w_j, \quad u_1^{1,m} = \sum_{i=0}^{m} c_j w_j, \quad u_2^{1,m} = \sum_{i=0}^{m} d_j w_j.$$

We search the approximate solutions

$$u_1^m(x,t) := \sum_{i=0}^{m} h_j^m(t) w_j(x), \quad u_2^m(x,t) := \sum_{i=0}^{m} k_j^m(t) w_j(x),$$

of the approximate problem in V_m

$$\begin{cases} \int_{\mathbb{R}^n} \left(\rho(x) \left(|u_1^{m'}|^{l-2} u_1^{m'} \right)' w - \nabla_x u_1^m \nabla w + \alpha_1(t) \int_0^t g_1(t-s) \Delta u_1^m(s,x) w ds + \rho(x) f_1(u_1^m, u_2^m) w \right) dx = 0, \\ \int_{\mathbb{R}^n} \left(\rho(x) \left(|u_2^{m'}|^{l-2} u_2^{m'} \right)' w - \nabla_x u_2^m \nabla w - \alpha_2(t) \int_0^t g_2(t-s) \Delta u_2^m(s,x) w ds + \rho(x) f_2(u_1^m, u_2^m) w \right) dx = 0, \\ u_1^m(0) = u_1^{0,m}, (u_1^m)'(0) = u_1^{1,m}, \ u_2^m(0) = u_2^{0,m}, \ (u_2^m)'(0) = u_2^{1,m}. \end{cases} \tag{IV.19}$$

This system leads to a system of ODEs for unknown functions $h_j^m(t), k_j^m(t)$. Based on standard existence theory for ODE, one can conclude the existence of a solution (u_1^m, u_2^m) of (IV.19) on a maximal time interval $[0, t_m)$, for each $m \in \mathbb{N}$. The a priori estimate that follows implies that in fact $t_m = +\infty$.

• (**A priori estimate 1**): In (IV.19), let $w = (u_1^m)'$ in the first equation and $w = (u_2^m)'$ in the second equation, add the resulting equations, and integrate by parts to obtain

$$\begin{aligned} E^{m'}(t) &= \frac{1}{2}\sum_{i=1}^{2} \alpha_i(t)(g_i' \circ \nabla_x u_i^m)(t) - \frac{1}{2}\sum_{i=1}^{2} \alpha_i(t) g_i(t) \|\nabla_x u_i^m(t)\|_2^2 \\ &+ \frac{1}{2}\sum_{i=1}^{2} \alpha_i'(t)(g_i \circ \nabla_x u_i^m)(t) - \frac{1}{2}\sum_{i=1}^{2} \alpha_i'(t) \int_0^t g_i(s) ds \|\nabla_x u_i^m(t)\|_2^2, \end{aligned} \tag{IV.20}$$

This means, using (**H1**), that, for some positive constant C independent of t and m,

$$E^m(t) \leq E^m(0) \leq C. \tag{IV.21}$$

• (**A priori estimate 2**): In (IV.19), let $w = -\Delta u_1^{m'}$ in the first equation and $w = -\Delta u_2^{m'}$ in the second equation, add the resulting equations, integrate by parts, and use (**H1**) to obtain

$$\begin{aligned}
&\frac{d}{dt}\sum_{i=1}^{2}\left(\frac{l-1}{l}\|\nabla_x u_i^{m'}\|_{L^l_\rho}^l + \frac{1}{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\Delta u_i^m\|_2^2 + \frac{1}{2}\alpha_i(t) g_i \circ \Delta u_i^m\right)\\
=\ & \sum_{i=1}^{2}\left(\frac{1}{2}\alpha_i(t)(g_i' \circ \Delta u_i^m) - \frac{1}{2}\alpha_i(t) g_i(t)\|\Delta u_i^m\|_2^2 + \frac{1}{2}\alpha_i'(t)(g_i \circ \Delta u_i^m)\right)\\
-\ & \frac{1}{2}\sum_{i=1}^{2}\alpha_i'(t)\int_0^t g_i\|\Delta u_i^m\|_2^2 + \sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x) f_i(u_1^m, u_2^m)\Delta u_i^{m'} dx,\\
\leq\ & \sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x) f_i(u_1^m, u_2^m)\Delta u_i^{m'} dx. \qquad \text{(IV.22)}
\end{aligned}$$

Then, integrating over $(0,t)$ yields

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\frac{l-1}{l}\|\nabla_x u_i^{m'}\|_{L^l_\rho}^l + \frac{1}{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\Delta u_i^m\|_2^2 + \frac{1}{2}\alpha_i(t) g_i \circ \Delta u_i^m\right)\\
\leq\ & \sum_{i=1}^{2}\left(\|\nabla_x u_i^{1,m}\|_{L^l_\rho}^l + \|\Delta u_i^{0,m}\|_2^2 + \int_{\mathbb{R}^n}\rho(x) f_i(u_1^m, u_2^m)\Delta u_i^m dx\right)\\
-\ & \sum_{i=1}^{2}\left(\int_{\mathbb{R}^n}\rho(x)\left(f_i(u_1^{0,m}, u_2^{0,m})\Delta u_i^{0,m}\right) + \int_0^t\int_{\mathbb{R}^n}\rho(x)\frac{\partial f_i}{\partial u_i} u_i^{m'}\Delta u_i^m dxds\right)\\
-\ & \int_0^t\int_{\mathbb{R}^n}\rho(x)\left(\frac{\partial f_1}{\partial u_2} u_2^{m'}\Delta u_1^m + \frac{\partial f_2}{\partial u_1} u_1^{m'}\Delta u_2^m\right) dxds. \qquad \text{(IV.23)}
\end{aligned}$$

To estimate the terms on the right hand side of (IV.23), we use (IV.9), Young's inequality, and (IV.15) and take (IV.21) into account to get

$$\begin{aligned}
\int_{\mathbb{R}^n}\rho(x) f_i(u_1^m, u_2^m)\Delta u_i^m \ &\leq\ k\int_{\mathbb{R}^n}\rho(x)\left(|u_1^m| + |u_2^m| + |u_1^m|^{\beta_{i1}} + |u_2^m|^{\beta_{i2}}\right)\Delta u_i^m dx,\\
&\leq\ \delta\|\Delta u_i^m\|_{L^2_\rho}^2 + \frac{c}{\delta}\int_{\mathbb{R}^n}\rho(x)\left(|u_1^m|^2 + |u_2^m|^2 + |u_1^m|^{2\beta_{i1}} + |u_2^m|^{2\beta_{i2}}\right) dx,\\
&\leq\ \delta\|\Delta u_i^m\|_{L^2_\rho}^2 + \frac{c}{\delta}\left(\|u_1^m\|_{L^2_\rho}^2 + \|u_2^m\|_{L^2_\rho}^2 + \|u_1^m\|_{L^2_\rho}^{2\beta_{i1}} + \|u_2^m\|_{L^2_\rho}^{2\beta_{i2}}\right),\\
&\leq\ \delta\|\Delta u_i^m\|_{L^2_\rho}^2 + \frac{c}{\delta}. \qquad \text{(IV.24)}
\end{aligned}$$

Now, we estimate $I := -\int_{\mathbb{R}^n}\rho(x)\frac{\partial f_i}{\partial u_1} u_i^{m'}\Delta u_i^m$ as follows. First, we observe that $\frac{\beta_{1j}-1}{2\beta_{1j}} + \frac{1}{2\beta_{1j}} + \frac{1}{2} = 1$ and use (**H2**) and the generalized Hölder's inequality to infer

$$\begin{aligned}
|I|\ &\leq\ d\int_{\mathbb{R}^n}\rho(x)\left(1 + |u_1^m|^{\beta_{11}-1} + |u_2^m|^{\beta_{12}-1}\right) u_i^{m'}\Delta u_i^m dx,\\
&\leq\ d\left(\|u_i^{m'}\|_{L^2_\rho} + \|u_i^{m'}\|_{L^{2\beta_{11}}_\rho}\|u_1^m\|_{L^{2\beta_{11}}_\rho}^{\beta_{11}-1} + \|u_i^{m'}\|_{L^{2\beta_{12}}_\rho}\|u_2^m\|_{L^{2\beta_{12}}_\rho}^{\beta_{12}-1}\right)\|\Delta u_i^m\|_{L^2_\rho}.
\end{aligned}$$

Then, by (IV.15), (IV.21) and Young's inequality, we arrive at

$$\begin{aligned}|I| &\leq c\left(1+\|\nabla u_1^m\|_2^{\beta_{11}-1}+\|\nabla u_2^m\|_2^{\beta_{12}-1}\right)\|\nabla u_i^{m'}\|_{L^2_\rho}\|\Delta u_i^m\|_{L^2_\rho},\\ &\leq c\left(\|\nabla u_i^{m'}\|_{L^2_\rho}\cdot\|\Delta u_i^m\|_{L^2_\rho}\right)\leq c\|\nabla u_i^{m'}\|^2_{L^2_\rho}+c\|\Delta u_i^m\|^2_{L^2_\rho}. \end{aligned} \tag{IV.25}$$

Since the other terms in (IV.23) can be similarly treated and the norms of the initial data are uniformly bounded, we combine (IV.23)-(IV.25), use (**H1**), and take δ small enough to end up with

$$\sum_{i=1}^2\left(\|\nabla_x u_i^{m'}\|^l_{L^l_\rho}+\|\Delta u_i^m\|_2^2\right) \leq c+c\sum_{i=1}^2\int_0^t\left(\|\nabla_x u_i^{m'}\|^l_{L^l_\rho}+\|\Delta u_i^m\|_2^2\right)ds. \tag{IV.26}$$

Using Gronwall's inequality, this implies that

$$\sum_{i=1}^2\left(\|\nabla_x u_i^{m'}\|^l_{L^l_\rho}+\|\Delta u_i^m\|_2^2\right)\leq C,\quad \forall t\in[0,T] \text{ and } m\in\mathbb{N}. \tag{IV.27}$$

• (**A priori estimate 3**): In (IV.19), let $w=u_1^{m''}$ in the first equation and $w=u_2^{m''}$ in the second equation. Then, by exploiting the previous estimates and using similar arguments, we find

$$\sum_{i=1}^2\|u_i^{m''}\|^l_{L^l_\rho}\leq C,\quad \forall t\in[0,T] \text{ and } m\in\mathbb{N}. \tag{IV.28}$$

From (IV.21), (IV.27), and (IV.28), we conclude that

$$\begin{aligned}&u_i^m \text{ are uniformly bounded in } L^\infty(0,T;\mathcal{D}^{1,2}(\mathbb{R}^n)),\\ &u_i^{m'} \text{ are uniformly bounded in } L^\infty(0,T;\mathcal{D}^{1,2}(\mathbb{R}^n)),\\ &u_i^{m''} \text{ are uniformly bounded in } L^\infty(0,T;L^l_\rho(\mathbb{R}^n)),\end{aligned}$$

which implies that there exist subsequences of $\{u_i^m\}$, which we still denote in the same way, such that

$$u_i^m \overset{*}{\longrightarrow} u_i \text{ in } L^\infty(0,T;\mathcal{D}^{1,2}(\mathbb{R}^n)), \tag{IV.29}$$

$$u_i^{m'} \overset{*}{\longrightarrow} u_i' \text{ in } L^\infty(0,T;\mathcal{D}^{1,2}(\mathbb{R}^n)), \tag{IV.30}$$

$$u_i^{m''} \overset{*}{\longrightarrow} u_i'' \text{ in } L^\infty(0,T;L^l_\rho(\mathbb{R}^n)), \tag{IV.31}$$

In the sequel, we will deal with the nonlinear term. By Aubin's Lemma, we find, up to a subsequence, that

$$u_i^m \to u_i \text{ strongly in } L^2(0,T;L^l_\rho(\mathbb{R}^n)), \tag{IV.32}$$

Then,

$$u_i^m \to u_i \text{ almost everywhere in } (0,T)\times\mathbb{R}^n. \tag{IV.33}$$

and therefore, from (**H4**),

$$f_i(u_1^m, u_2^m) \to f_i(u_1,u_2) \text{ almost every where in } (0,T)\times\mathbb{R}^n, \text{ for } i=1,2. \tag{IV.34}$$

Also, as u_i^m are bounded in $L^\infty(0,T;L^l_\rho(\mathbb{R}^n))$, then the use of (IV.9) and (IV.15) gives that $f_i(u_1^m,u_2^m)$ is bounded in $L^\infty(0,T;L^l_\rho(\mathbb{R}^n))$. From this (IV.34), we can deduce that

$$f_i(u_1^m,u_2^m) \rightharpoonup f_i(u_1,u_2) \text{ in } L^2(0,T;L^l_\rho(\mathbb{R}^n)), \quad \text{for } i=1,2.$$

Combining the results obtained above, we can pass to the limit and conclude that (u_1,u_2) is a strong solution of system (IV.1)

• **For uniqueness**, let us assume that $(u_{11},u_{21}),(u_{12},u_{22})$ are two strong solutions of (IV.1). Then, $(z_1,z_2)=(u_{11}-u_{12},u_{21}-u_{22})$ satisfies, for all $w\in\mathcal{D}^{1,2}(\mathbb{R}^n)$

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n}\left(\rho(x)\left(|z_1'|^{l-2}z_1'\right)' w - \nabla_x z_1\nabla w + \alpha_1(t)\int_0^t g_1(t-s)z_1(s,x)w\,ds + \rho(x)f_1(z_1,z_2)w\right)dx = 0,\\ \displaystyle\int_{\mathbb{R}^n}\left(\rho(x)\left(|z_2'|^{l-2}z_2'\right)' w - \nabla_x z_2\nabla w - \alpha_2(t)\int_0^t g_2(t-s)z_2(s,x)w\,ds + \rho(x)f_2(z_1,z_2)w\right)dx = 0, \end{cases} \tag{IV.35}$$

Substituting $w=z_1'$ in the first equation and $w=z_2'$ in the second equation, adding the resulting equations, integrating by parts, and using (**H1**) yield

$$\begin{aligned} &\frac{d}{dt}\sum_{i=1}^2\left(\frac{l-1}{l}\|z_i'\|^l_{L^l_\rho} + \frac{1}{2}\left(1-\alpha_i(t)\int_0^t g_i(s)ds\right)\|\nabla z_i\|_2^2 + \frac{1}{2}(g_i\circ\nabla z_i)\right)\\ \leq\ & \int_{\mathbb{R}^n}\left([f_1(u_{21},u_{22})+f_1(u_{11},u_{12})]\,z_1' + [f_2(u_{21},u_{22})+f_2(u_{11},u_{12})]\,z_2'\right)dx. \end{aligned} \tag{IV.36}$$

Making use of (IV.10) and following similar arguments that used to obtain (IV.25), we find

$$\begin{aligned} &\int_{\mathbb{R}^n}\left([f_1(u_{21},u_{22})+f_1(u_{11},u_{12})]\,z_1' + [f_2(u_{21},u_{22})+f_2(u_{11},u_{12})]\,z_2'\right)dx\\ \leq\ & k\int_{\mathbb{R}^n}\left(1+|u_{11}|^{\beta_{11}-1}+|u_{12}|^{\beta_{11}-1}+|u_{21}|^{\beta_{12}-1+|u_{22}|^{\beta_{12}-1}}\right)(|z_1|+|z_2|)z_1'dx\\ +\ & k\int_{\mathbb{R}^n}\left(1+|u_{11}|^{\beta_{21}-1}+|u_{12}|^{\beta_{21}-1}+|u_{21}|^{\beta_{22}-1+|u_{22}|^{\beta_{22}-1}}\right)(|z_1|+|z_2|)z_2'dx,\\ \leq\ & c\sum_{i=1}^2\left(\|z_i'\|^l_{L^l_\rho}+\|\nabla z_i\|_2^2\right). \end{aligned} \tag{IV.37}$$

Combining (IV.36) and (IV.37), integrating over $(0,t)$, and using Gronwall's Lemma, then we deduce that

$$\sum_{i=1}^{2}\left(\|z_i'\|_{L^l_\rho}^l + \|z_i\|_2^2\right) = 0, \tag{IV.38}$$

which means that $(u_{11}, u_{21}) = (u_{12}, u_{22})$. This completes the proof.

3 Decay rate

For the purpose of constructing a Lyapunov functional L equivalent to E, we introduce the next functionals

$$\psi_1(t) = \sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)u_i|u_i'|^{l-2}u_i'dx, \tag{IV.39}$$

$$\psi_2(t) = -\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)|u_i'|^{l-2}u_i'\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx, \tag{IV.40}$$

Lemma 3.1 *Under the assumptions (**H1**- **H4**), the functional ψ_1 satisfies, along the solution of (IV.1), for some $\delta \in (0,1)$*

$$\psi_1'(t) \leq \sum_{i=1}^{2}\|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l - \sum_{i=1}^{2}(k_i-\delta)\|\nabla_x u_i\|_2^2 + \sum_{i=1}^{2}\frac{(1-k_i)}{4\delta}\alpha_i(t)(g_i\circ\nabla_x u_i) - \int_{\mathbb{R}^n}\rho(x)F(u_1,u_2)dx.$$

Proof of lemma 3.1 From (IV.39), integrate by parts over $\mathbb{R}^n$, we have

$$\begin{aligned}
\psi_1'(t) &= \int_{\mathbb{R}^n}\rho(x)u_1'^l dx + \int_{\mathbb{R}^n}\rho(x)u_1\left(|u_1'|^{l-2}u_1'\right)' dx \\
&+ \int_{\mathbb{R}^n}\rho(x)u_2'^l dx + \int_{\mathbb{R}^n}\rho(x)u_2\left(|u_2'|^{l-2}u_2'\right)' dx, \\
&= \int_{\mathbb{R}^n}\left(\rho(x)u_1'^l + u_1\Delta_x u_1 - u_1\alpha_1(t)\int_0^t g_1(t-s)\Delta_x u_1(s,x)ds\right)dx \\
&+ \int_{\mathbb{R}^n}\left(\rho(x)u_2'^l + u_2\Delta_x u_2 - u_2\alpha_2(t)\int_0^t g_2(t-s)\Delta_x u_2(s,x)ds\right)dx \\
&- \int_{\mathbb{R}^n}\rho(x)\left[u_1f_1(u_1,u_2) + u_2f(u_1,u_2)\right], \\
&\leq \sum_{i=1}^{2}\|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l - \sum_{i=1}^{2}k_i\|\nabla_x u_i\|_2^2 - \int_{\mathbb{R}^n}\rho(x)F(u_1,u_2)dx \\
&+ \sum_{i=1}^{2}\int_{\mathbb{R}^n}\nabla_x u_i\alpha_i(t)\int_0^t g_i(t-s)(\nabla_x u_i(s)-\nabla_x u_i(t))dsdx.
\end{aligned}$$

Using Young's inequality, (***H1***), we obtain

$$\begin{aligned}
\psi_1'(t) &\leq \sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \sum_{i=1}^{2}k_i\|\nabla_x u_i\|_2^2 - \int_{\mathbb{R}^n}\rho(x)F(u_1,u_2)dx \\
&+ \delta\sum_{i=1}^{2}\|\nabla_x u_i\|_2^2 + \sum_{i=1}^{2}\frac{\alpha_i^2(t)}{4\delta}\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)|\nabla_x u_i(s)-\nabla_x u_i(t)|ds\right)^2 dx, \\
&\leq \sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \sum_{i=1}^{2}(k_i-\delta)\|\nabla_x u_i\|_2^2 + \sum_{i=1}^{2}\frac{(1-k_i)}{4\delta}\alpha_i(t)(g_i\circ\nabla_x u_i) - \int_{\mathbb{R}^n}\rho(x)F(u_1,u_2)dx.
\end{aligned}$$

Lemma 3.2 *Under the assumptions (**H1-H4**), the functional ψ_2 satisfies, along the solution of (IV.1), for some $\delta\in(0,1)$*

$$\begin{aligned}
\psi_2'(t) &\leq \sum_{i=1}^{2}\left(\delta - \int_0^t g_i(s)ds\right)\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} \\
&+ 2\delta\sum_{i=1}^{2}\left(\frac{1}{2}+(1-k_i)^2+c\right)\|\nabla_x u_i\|_2^2 - \frac{1}{4\delta}c\sum_{i=1}^{2}(g_i'\circ\nabla_x u_i)^{l/2} \\
&+ \sum_{i=1}^{2}\left(\frac{c}{4\delta}+2(\delta\alpha^2(t)+\frac{1}{4\delta})(\int_0^t g_i(s)ds)\right)(g_i\circ\nabla_x u_i).
\end{aligned}$$

Proof of lemma 3.2 Exploiting Eqs. in (IV.1), to get

$$\begin{aligned}
\psi_2'(t) &= -\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)\left(|u_i'|^{l-2}u_i'\right)'\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx \\
&- \sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)|u_i'|^{l-2}u_i'\int_0^t g_i'(t-s)(u_i(t)-u_i(s))dsdx - \sum_{i=1}^{2}\int_0^t g_i(s)ds\|u_i'\|^l_{L^l_\rho}.
\end{aligned} \tag{IV.41}$$

To simplify the first term in (IV.41), we multiply the first Eq. in (IV.1) by $\int_0^t g_1(t-s)(u_1(t)-u_1(s))ds$ and the second by $\int_0^t g_2(t-s)(u_2(t)-u_2(s))ds$, integrate by parts over $\mathbb{R}^n$. So we obtain

$$\begin{aligned}
&-\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)\left(|u_i'|^{l-2}u_i'\right)'\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx \\
&= -\sum_{i=1}^{2}\int_{\mathbb{R}^n}\Delta u_i(x)\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx \\
&+ \sum_{i=1}^{2}\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)(u_i(t)-u_i(s))ds\,\alpha_i(t)\int_0^t g_i(t-s)\Delta u_i(s)ds\right)dx \\
&+ \sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)f_i(u_1,u_2)\int_0^t g_i(t-s)(u_i(t)-u_i(s))dsdx.
\end{aligned} \tag{IV.42}$$

The first term in the right side of (IV.42) is estimated for $i = 1, 2$ as follows

$$\begin{aligned}
- & \int_{\mathbb{R}^n} \Delta_x u_i(x) \int_0^t g_i(t-s)(u_i(t) - u_i(s))dsdx \\
= & \int_{\mathbb{R}^n} \nabla_x u_i \int_0^t g_i(t-s)(\nabla_x u_i(t) - \nabla_x u_i(s))dsdx, \\
\leq & \ \delta\|\nabla_x u_i\|^2 + \frac{1}{4\delta}\left(\int_0^t g_i(s)ds\right)(g_i \circ \nabla_x u_i)(t).
\end{aligned} \tag{IV.43}$$

while the second term becomes,

$$\begin{aligned}
& \int_{\mathbb{R}^n} \left(\int_0^t g_i(t-s)(u_i(t) - u_i(s))ds\alpha_i(t) \int_0^t g_i(t-s)\Delta u_i(s)ds\right)dx \\
= & -\int_{\mathbb{R}^n} \left(\int_0^t g_i(t-s)(\nabla u_i(t) - \nabla u_i(s))ds\alpha_i(t) \int_0^t g_i(t-s)\nabla u_i(s)ds\right)dx, \\
\leq & \ \delta\alpha^2(t) \int_{\mathbb{R}^n} \left(\int_0^t g_i(t-s)(|\nabla u_i(t) - \nabla u_i(s)| + |\nabla u_i(t)|)ds\right)^2 \\
+ & \ \frac{1}{4\delta} \int_{\mathbb{R}^n} \left(\int_0^t g_i(t-s)(\nabla u_i(t) - \nabla u_i(s))ds\right)^2, \\
\leq & \ 2\delta(1-k_i)^2\|\nabla u_i\|_2^2 + \left(2\delta\alpha_i^2(t) + \frac{1}{4\delta}\right)\left(\int_0^t g_i(s)ds\right)(g_i \circ \nabla u_i)(t).
\end{aligned} \tag{IV.44}$$

Now, to estimate the nonlinear terms, we use Young's inequality, (**H4**) and Lemma 1.2 to get

$$\begin{aligned}
& \int_{\mathbb{R}^n} \rho(x) f_i(u_1, u_2) \int_0^t g_i(t-s)(u_i(t) - u_i(s))dsdx \\
\leq & \ \delta c \int_{\mathbb{R}^n} \rho(x)\left(|u_1|^2 + |u_2|^2 + |u_1|^{2\beta_{i1}} + |u_2|^{2\beta_{i2}}\right)dx \\
+ & \ \frac{c}{4\delta} \int_{\mathbb{R}^n} \rho(x)\left(\int_0^t g_i(t-s)(u_i(t) - u_i(s))ds\right)^2 dx, \\
\leq & \ \delta c\left(\|u_1\|^2_{L^2_\rho(\mathbb{R}^n)} + \|u_2\|^2_{L^2_\rho(\mathbb{R}^n)} + \|u_1\|^{2\beta_{i1}}_{L^2_\rho(\mathbb{R}^n)} + \|u_2\|^{2\beta_{i2}}_{L^2_\rho(\mathbb{R}^n)}\right) \\
+ & \ \frac{c}{4\delta} \int_0^t g_i(t-s)\|u_i(t) - u_i(s)\|^2_{L^2_\rho(\mathbb{R}^n)}ds, \\
\leq & \ \|\rho\|_{L^{n/2}(\mathbb{R}^n)}\delta c\left(\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2 + \|\nabla u_1\|_2^{2\beta_{i1}} + \|\nabla u_2\|_2^{2\beta_{i2}}\right) \\
+ & \ \|\rho\|_{L^{n/2}(\mathbb{R}^n)}\frac{c}{4\delta}(g_i \circ \nabla_x u_i).
\end{aligned}$$

Since

$$\frac{1}{2}\sum_{i=1}^{2}\|\nabla u_i\|_2^2 \leq E(t) \leq E(0),$$

and $\beta_{ij} \geq 1, i, j = 1, 2$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho(x) f_i(u_1, u_2) \int_0^t g_i(t-s)(u_i(t) - u_i(s)) ds dx \\
\leq\ & \|\rho\|_{L^{n/2}(\mathbb{R}^n)} \delta c \left(\|\nabla_x u_1\|_2^2 + \|\nabla_x u_2\|_2^2 + \|\nabla_x u_1\|_2^{2(\beta_{i1}-1)} \|\nabla_x u_1\|_2^2 + \|\nabla_x u_2\|_2^{2(\beta_{i2}-1)} \|\nabla_x u_2\|_2^2 \right) \\
+\ & \|\rho\|_{L^{n/2}(\mathbb{R}^n)} \frac{c}{4\delta} (g_i \circ \nabla_x u_i), \\
\leq\ & \|\rho\|_{L^{n/2}(\mathbb{R}^n)} \delta c \left(\|\nabla_x u_1\|_2^2 + \|\nabla_x u_2\|_2^2 + (2E(0))^{2(\beta_{i1}-1)} \|\nabla_x u_1\|_2^2 + (2E(0))^{2(\beta_{i2}-1)} \|\nabla_x u_2\|_2^2 \right) \\
+\ & \|\rho\|_{L^{n/2}(\mathbb{R}^n)} \frac{c}{4\delta} (g_i \circ \nabla_x u_i), \\
\leq\ & c\delta \left(\|\nabla_x u_1\|_2^2 + \|\nabla_x u_2\|_2^2 \right) + \frac{c}{4\delta} (g_i \circ \nabla_x u_i),
\end{aligned} \tag{IV.45}$$

where $\|\rho\|_{L^{n/2}(\mathbb{R}^n)} = c$. By Hölder's and Young's inequalities and Lemma 1.2 we estimate

$$\begin{aligned}
- & \int_{\mathbb{R}^n} \rho(x) |u_i'|^{l-2} u_i' \int_0^t g_i'(t-s)(u_i(t) - u_i(s)) ds dx \\
\leq & \left(\int_{\mathbb{R}^n} \rho(x) |u_i'|^l dx \right)^{(l-1)/l} \times \left(\int_{\mathbb{R}^n} \rho(x) \left| \int_0^t -g_i'(t-s)(u_i(t) - u_i(s)) ds \right|^l \right)^{1/l}, \\
\leq & \delta \|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l + \frac{1}{4\delta} \|\rho\|_{L^s(\mathbb{R}^n)}^l \left\| \int_0^t -g_i'(t-s)(u_i(t) - u_i(s)) ds \right\|_{L^l_\rho(\mathbb{R}^n)}^l \\
\leq & \delta \|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l - \frac{c}{4\delta} (g_i' \circ \nabla_x u_i)^{l/2}(t).
\end{aligned} \tag{IV.46}$$

From (IV.41)-(IV.46), we obtain

$$\begin{aligned}
\psi_2'(t) \leq\ & \sum_{i=1}^2 \left(\delta - \int_0^t g_i(s) ds \right) \|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l \\
+\ & \delta \sum_{i=1}^2 \left(1 + 2(1-k_i)^2 + c \right) \|\nabla_x u_i\|_2^2 - \frac{c}{4\delta} \sum_{i=1}^2 (g_i' \circ \nabla_x u_i)^{l/2} \\
+\ & \sum_{i=1}^2 \left(\frac{c}{4\delta} + 2(\delta \alpha^2(t) + \frac{1}{4\delta}) (\int_0^t g_i(s) ds) \right) (g_i \circ \nabla_x u_i).
\end{aligned}$$

Our next main result reads as follows

Theorem 3.3 *Let (u_1^0, u_1^1), $(u_2^0, u_2^1) \in \mathcal{D}^{1,2}(\mathbb{R}^n) \times L^l_\rho(\mathbb{R}^n)$ and suppose that (**H1** $-$ **H4**) hold. Then there exist positive constants W, ω such that the energy of solution given by (IV.1) satisfies,*

$$E(t) \leq W \exp\left(-\omega \int_0^t \alpha(s) \xi(s) ds \right), \forall t \geq 0 \tag{IV.47}$$

where $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}, \alpha(t) = \min\{\alpha_1(t), \alpha_2(t)\}, \quad \forall t \geq 0.$

In order to prove this theorem, let us define

$$L(t) = N_1 E(t) + \varepsilon\alpha(t)\psi_1(t) + \alpha(t)\psi_2(t), \tag{IV.48}$$

for $N_1, \varepsilon > 0$. We need the next Lemma, which means that there is equivalent between the Lyapunov and energy functions

Lemma 3.4 *For $N_1, \varepsilon > 0$, we have*

$$\beta_1 L(t) \leq E(t) \leq \beta_2 L(t), \tag{IV.49}$$

holds for two positive constants β_1 and β_2.

Proof of lemma 3.4 By applying Young's inequality to (IV.39) and using (IV.40) and (IV.48), we obtain

$$\begin{aligned}
|L(t) - N_1 E(t)| &\leq |\varepsilon\alpha(t)\psi_1(t)| + |\alpha(t)\psi_2(t)|, \\
&\leq \varepsilon\alpha(t)\sum_{i=1}^{2}\int_{\mathbb{R}^n}\left|\rho(x)u_i|u_i'|^{l-2}u_i'\right| dx \\
&+ \alpha(t)\sum_{i=1}^{2}\int_{\mathbb{R}^n}\left|\rho(x)|u_i'|^{l-2}u_i'\int_0^t g_i(t-s)(u_i(t)-u_i(s))ds\right| dx.
\end{aligned}$$

Thanks to Hölder and Young's inequalities with exponents $\frac{l}{l-1}$, l, since $\frac{2n}{n+2} \geq l \geq 2$, we have by using Lemma 1.2

$$\begin{aligned}
\int_{\mathbb{R}^n}\left|\rho(x)u_i|u_i'|^{l-2}u_i'\right| dx &\leq \left(\int_{\mathbb{R}^n}\rho(x)|u_i|^l dx\right)^{1/l}\left(\int_{\mathbb{R}^n}\rho(x)|u_i'|^l dx\right)^{(l-1)/l}, \\
&\leq \frac{1}{l}\left(\int_{\mathbb{R}^n}\rho(x)|u_i|^l dx\right) + \frac{l-1}{l}\left(\int_{\mathbb{R}^n}\rho(x)|u_i'|^l dx\right), \\
&\leq c\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + c\|\rho\|^l_{L^s(\mathbb{R}^n)}\|\nabla_x u_i\|_2^l.
\end{aligned} \tag{IV.50}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left|\left(\rho(x)^{\frac{l-1}{l}}|u_i'|^{l-2}u_i'\right)\left(\rho(x)^{\frac{1}{l}}\int_0^t g_i(t-s)(u_i(t)-u_i(s))ds\right)\right| dx \\
&\leq \left(\int_{\mathbb{R}^n}\rho(x)|u_i'|^l dx\right)^{(l-1)/l} \times \left(\int_{\mathbb{R}^n}\rho(x)|\int_0^t g_i(t-s)(u_i(t)-u_i(s))ds|^l\right)^{1/l}, \\
&\leq \frac{l-1}{l}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \frac{1}{l}\|\int_0^t g_i(t-s)(u_i(t)-u_i(s))ds\|^l_{L^l_\rho(\mathbb{R}^n)}, \\
&\leq \frac{l-1}{l}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \frac{1}{l}\|\rho\|^l_{L^s(\mathbb{R}^n)}(g_i\circ\nabla_x u_i)^{l/2}(t).
\end{aligned} \tag{IV.51}$$

then, since $l \geq 2$, we have

$$\begin{aligned}
|L(t) - N_1 E(t)| &\leq c\sum_{i=1}^{2}\left(\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \alpha_i(t)\|\nabla_x u_i\|_2^l + \alpha_i(t)(g_i \circ \nabla_x u_i)^{l/2}(t)\right), \\
&\leq c(E(t) + E^{l/2}(t)), \\
&\leq c(E(t) + E(t)\cdot E^{(l/2)-1}(t)), \\
&\leq c(E(t) + E(t)\cdot E^{(l/2)-1}(0)), \\
&\leq cE(t).
\end{aligned}$$

Consequently, (IV.49) follows.

Proof of Theorem3.3 From (IV.20), results of Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}
L'(t) &= N_1E'(t) + \varepsilon\alpha'(t)\psi_1(t) + \varepsilon\alpha(t)\psi_1'(t) + \alpha'(t)\psi_2(t) + \alpha(t)\psi_2'(t) \\
&\leq \sum_{i=1}^{2}\left(\alpha_i(t)\frac{1}{2}N_1 - \alpha_i(t)\frac{c}{4\delta}\right)(g_i' \circ \nabla_x u_i)^{l/2} \\
&+ \sum_{i=1}^{2}\left[\left(\frac{c}{4\delta} + 2(\delta\alpha_i^2(t) + \frac{1}{4\delta})(\int_0^t g_i(s)ds)\right) + \varepsilon\frac{(1-k_i)}{4\delta}\alpha_i(t)\right]\alpha(t)(g_i \circ \nabla_x u_i) \\
&- \sum_{i=1}^{2} M_1\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} - \sum_{i=1}^{2} M_2\|\nabla_x u_i\|_2^2 - \int_{\mathbb{R}^n}\rho(x)F(u_1,u_2)dx \\
&+ \varepsilon\alpha'(t)\psi_1(t) + \alpha'(t)\psi_2(t),
\end{aligned}$$

where

$$M_1 := \left(\int_0^t g_i(s)ds - \delta - \varepsilon\right)\alpha(t),$$

and

$$M_2 := \left(-2\delta(\frac{1}{2} + (1-k_i)^2 + c)\alpha(t) + \varepsilon(k_i - \delta)\alpha(t) + \frac{1}{2}N_1(\int_0^t g(s)ds)\alpha'(t)\right),$$

Since $g_i, i = 1, 2$ are positive, we have

$$\int_0^t g_i(s)ds \geq \int_0^{t_0} g_i(s)ds = g_{i0} > 0, \forall t \geq t_0.$$

Then for $\alpha(t)=\min\{\alpha_1(t),\alpha_2(t)\}$, we have

$$\begin{aligned}
L'(t) \leq & \sum_{i=1}^{2}\left(\frac{N_1}{2}-\frac{c}{4\delta}\right)\alpha(t)(g_i'\circ\nabla_x u_i)^{l/2} \\
+ & \sum_{i=1}^{2}\left[\left(\frac{c}{4\delta}+2(\delta\alpha^2(t)+\frac{1}{4\delta})(\int_0^t g_i(s)ds)\right)+\frac{\varepsilon(1-k_i)}{4\delta}\alpha(t)\right]\alpha(t)(g_i\circ\nabla_x u_i) \\
- & \sum_{i=1}^{2}(g_{i0}-\delta-\varepsilon)\,\alpha(t)\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)}-\int_{\mathbb{R}^n}\rho(x)F(u_1,u_2)dx \\
- & \sum_{i=1}^{2}\left(-2\delta(\frac{1}{2}+(1-k_i)^2+c)+\varepsilon(k_i-\delta)+\frac{g_{i0}N_1}{2}\frac{\alpha'(t)}{\alpha(t)}\right)\alpha(t)\|\nabla_x u_i\|_2^2 \\
+ & \varepsilon\alpha'(t)\psi_1(t)+\alpha'(t)\psi_2(t). \qquad \text{(IV.52)}
\end{aligned}$$

By using (IV.50) and (IV.51), we get

$$\begin{aligned}
&\varepsilon\alpha'(t)\psi_1(t)+\alpha'(t)\psi_2(t) \\
&\leq \sum_{i=1}^{2}\varepsilon\alpha'(t)c\|\nabla_x u_i\|_2^l+\sum_{i=1}^{2}\varepsilon c\alpha'(t)\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)}+\sum_{i=1}^{2}c\alpha'(t)(g_i\circ\nabla_x u_i)^{l/2}.
\end{aligned}$$

Hence, (IV.52) becomes

$$\begin{aligned}
L'(t) \leq & \sum_{i=1}^{2}\left(\frac{N_1}{2}-\frac{c}{4\delta}\right)\alpha(t)(g_i'\circ\nabla_x u_i)^{l/2} \\
+ & \sum_{i=1}^{2}\left[\left(\frac{c}{4\delta}+2(\delta\alpha^2(0)+\frac{1}{4\delta})(\int_0^t g_i(s)ds)\right)+\frac{\varepsilon(1-k_i)}{4\delta}\alpha(0)+c\frac{\alpha'(t)}{\alpha(t)}\right]\alpha(t)(g_i\circ\nabla_x u_i) \\
- & \sum_{i=1}^{2}\left(g_{i0}-\delta-\varepsilon-\varepsilon c\frac{\alpha'(t)}{\alpha(t)}\right)\alpha(t)\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)}-\int_{\mathbb{R}^n}\rho(x)F(u_1,u_2)dx \\
- & \sum_{i=1}^{2}\left(-2\delta(\frac{1}{2}+(1-k_i)^2+c)+\varepsilon(k_i-\delta)+(\frac{g_{i0}N_1}{2}-\varepsilon c)\frac{\alpha'(t)}{\alpha(t)}\right)\alpha(t)\|\nabla_x u_i\|_2^2.
\end{aligned}$$

At this point, for $g_0=\min\left\{\int_0^{t_0}g_1(s)ds,\int_0^{t_0}g_2(s)ds\right\}, k=\min\{k_1,k_2\},\ t_0>0$, we choose ε small enough and δ so small such that $g_0-\delta-\varepsilon>0$ and $\varepsilon(k-\delta)>2\delta(\frac{1}{2}+(1-k)^2+c)$. hence ε,δ are fixed, we can choose N_1 large enough so that

$$\left(\frac{N_1}{2}-\frac{c}{4\delta}\right)>0.$$

which yields by assumption on α_i in (**H1**)

$$L'(t) \leq M_0\alpha(t)\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) - m\alpha(t)E(t), \quad \forall t \geq t_0, m, M_0 > 0. \tag{IV.53}$$

Multiplying (IV.53) by $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$ gives

$$\xi(t)L'(t) \leq -m\alpha(t)\xi(t)E(t) + M_0\alpha(t)\xi(t)\sum_{i=1}^{2}(g_i \circ \nabla_x u_i). \tag{IV.54}$$

As in [56], the last term can be estimated, using (**H1**) and (IV.20) as follows

$$\begin{aligned}
\alpha(t)\xi(t)\sum_{i=1}^{2}(g_i \circ \nabla_x u_i) &\leq \alpha(t)\sum_{i=1}^{2}\xi_i(t)\int_{\mathbb{R}^n}\int_0^t g_i(t-s)|\nabla_x u_i(t) - \nabla_x u_i(s)|^2 ds dx,\\
&\leq \alpha(t)\sum_{i=1}^{2}\int_{\mathbb{R}^n}\int_0^t \xi_i(t-s)g_i(t-s)|\nabla_x u_i(t) - \nabla_x u_i(s)|^2 ds dx,\\
&\leq -\alpha(t)\sum_{i=1}^{2}\int_{\mathbb{R}^n}\int_0^t g_i'(t-s)|\nabla_x u_i(t) - \nabla_x u_i(s)|^2 ds dx,\\
&\leq -\alpha(t)\sum_{i=1}^{2}(g_i' \circ \nabla_x u_i),\\
&\leq -2E'(t) - \sum_{i=1}^{2}\alpha'(t)\left(\int_0^t g_i(s)ds\right)\|\nabla_x u_i\|_2^2.
\end{aligned} \tag{IV.55}$$

Thus, (IV.53) becomes

$$\xi(t)L'(t) + 2M_0E'(t) \leq -m\alpha(t)\xi(t)E(t) - M_0\sum_{i=1}^{2}\alpha'(t)\left(\int_0^t g_i(s)ds\right)\|\nabla_x u_i\|_2^2. \tag{IV.56}$$

Using the fact that $\xi'(t) \leq 0$ for a.e t, then

$$\begin{aligned}
(\xi(t)L(t) + 2M_0E(t))' &\leq \xi(t)L'(t) + 2M_0E'(t),\\
&\leq -m\alpha(t)\xi(t)E(t)\\
&- M_0\sum_{i=1}^{2}\alpha'(t)\left(\int_0^t g_i(s)ds\right)\|\nabla_x u_i\|_2^2,\\
&\leq -\alpha(t)\xi(t)\left(m + \frac{2M_0\alpha'(t)}{k\alpha(t)\xi(t)}\sum_{i=1}^{2}\left(\int_0^t g_i(s)ds\right)\right)E(t).
\end{aligned}$$

Using again (**H1**), to get

$$F = \xi(t)L + 2M_0E \sim E, \tag{IV.57}$$

we obtain, for some positive constant w

$$F'(t) \leq -w\alpha(t)\xi(t)F(t), \quad \forall t \geq t_0. \tag{IV.58}$$

Integration over (t_o, t), for some constant $w > 0$ leads to

$$F(t) \leq W \exp\left(-\omega \int_0^t \alpha(t)\xi(s)ds\right), \forall t \geq t_0 \tag{IV.59}$$

The equivalence of F and E completes the proof of Theorem 3.3.

Chapter V

Polynomial decay of solutions to the cauchy problem for a Petrowsky-Petrowsky system in $\mathbb{R}^n$

1 Introduction

The viscoelastic materials show a behavior which is something between that of elastic solids and Newtonian fluids. Indeed, the stresses in these media depend on the entire history of their deformation, not only on their current state of deformation or their current state of motion. This is the reason why they are called materials with memory. The viscoelastic equations with fading memory in a bounded space has been deeply studied by several authors.

1.1 Challenge and statement

In this section, derived from [9], we consider the following Petrowsky-Petrowsky system

$$\begin{cases} u_{tt} + \phi(x)\left(\Delta^2 u - \displaystyle\int_{-\infty}^{t} \mu(t-s)\Delta^2 u(s)ds\right) + \alpha v = 0, & \text{in } \mathbb{R}^n \times \mathbb{R}^+, \\ v_{tt} + \phi(x)\Delta^2 v + \alpha u = 0, & \text{in } \mathbb{R}^n \times \mathbb{R}^+, \\ (u_0, v_0) \in \mathcal{D}^{2,2}(\mathbb{R}^n), \quad (u_1, v_1) \in L^2_\rho(\mathbb{R}^n). \end{cases} \tag{V.1}$$

where the spaces $\mathcal{D}^{2,2}(\mathbb{R}^n), L^2_\rho(\mathbb{R}^n)$ defined in (2.1) and $\phi(x) > 0, \forall x \in \mathbb{R}^n, \alpha \neq 0, (\phi(x))^{-1} = \rho(x)$, where the function $\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0,1)$ and $\rho \in L^{n/4}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Following Dafermos [23], adding a new variable η to the system which corresponds to the relative displacement history. Let us define the auxiliary variable

$$\eta = \eta^t(x,s) = u(x,t) - u(x,t-s), \quad (x,s) \in \mathbb{R}^n \times \mathbb{R}^+,\ t \geq 0. \tag{V.2}$$

By differentiation we have

$$\eta_t^t(x,s) = -\eta_s^t(x,s) + u_t(x,t), \quad (x,s) \in \mathbb{R}^n \times \mathbb{R}^+,\ t \geq 0, \tag{V.3}$$

and we can take as initial condition ($t = 0$)

$$\eta^0(x,s) = u_0(x,0) - u_0(x,-s), \quad (x,s) \in \mathbb{R}^n \times \mathbb{R}^+. \tag{V.4}$$

Thus, the original memory term can be rewritten as

$$\begin{aligned} \int_{-\infty}^t \mu(t-s)\Delta^2 u(s)ds &= \int_0^\infty \mu(s)\Delta^2 u(t-s)ds \\ &= \left(\int_0^\infty \mu(s)ds\right)\Delta^2 u - \int_0^\infty \mu(s)\Delta^2\eta^t(s)ds, \end{aligned}$$

The problem (V.1) is transformed into the system

$$\begin{cases} u_{tt} + \mu_0\phi(x)\Delta^2 u + \int_0^\infty \mu(s)\phi(x)\Delta^2\eta(s)ds + \alpha v = 0, & \text{in } \mathbb{R}^n \times \mathbb{R}^+. \\ v_{tt} + \phi(x)\Delta^2 v + \alpha u = 0, & \text{in } \mathbb{R}^n \times \mathbb{R}^+. \\ \eta_t + \eta_s - u_t = 0, & \text{in } \mathbb{R}^n \times \mathbb{R}^+. \end{cases} \tag{V.5}$$

and initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \eta^0(x,s) = \eta_0(x,s), \tag{V.6}$$

where

$$\eta_0(x,s) = u_0(x,0) - u_0(x,-s) \quad (x,s) \in \mathbb{R}^n \times \mathbb{R}^+.$$

The energy of (u,v) at times t is given by

$$E(t) := \frac{1}{2}\left(\|u_t\|_{L^2_\rho}^2 + \|v_t\|_{L^2_\rho}^2 + \mu_0\|\Delta u\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\Delta\eta^t\|_{L^2_\rho}^2 + 2\alpha\int_{\mathbb{R}^n}\rho uv dx\right) \tag{V.7}$$

For α small enough we deduce that:

$$E(t) \geq \frac{1}{2}(1 - |\alpha|\gamma)\left(\|u_t\|_{L^2_\rho}^2 + \|v_t\|_{L^2_\rho}^2 + \mu_0\|\Delta u\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\Delta\eta^t\|_{L^2_\rho}^2\right).$$

1.2 related results

The systems of nonlinear wave equations go back to Reed [74] who proposed a system in three space dimensions, where this type of system was completely analyzed. Existence and uniqueness of global weak solutions, asymptotic behavior for an analogous hyperbolic-parabolic system of related problems have attracted a great deal of attention in the last decades, and many results have been appeared.

The problem of stabilization of weakly coupled systems has also been studied by several authors. Under certain conditions imposed on the subset where the damping term is effective, Kapitonov [45] showed uniform stabilization of the solutions of a pair of hyperbolic systems coupled in velocities. In [4], the authors developed an approach to prove that, for $\alpha \in \mathbb{R}^+$ with α small enough,

$$\begin{cases} u_{tt} - \Delta u + \alpha v + u_t = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \end{cases} \tag{V.8}$$

is not exponentially stable and the asymptotic behavior of solutions is at least of polynomial type $\frac{1}{t^m}$ with decay rate m depending on the smoothness of initial data. Alabau [6] proved indirect boundary stabilization (polynomial decay) of weakly coupled equations. The author established a polynomial decay lemma for non-increasing and nonnegative function which satisfies an integral inequality.

Recently, Almeida, R.G.C. and Santos, M.L. [9] considered a damping mechanisms of a memory type with past history acting only in one equation. More precisely, the authors looked to the following problem

$$\begin{cases} u_{tt} - \Delta u - \int_0^\infty \mu(s)\Delta u(t-s)ds + \alpha v = 0, \\ v_{tt} - \Delta v + \alpha u = 0, \end{cases} \tag{V.9}$$

in $\Omega \times \mathbb{R}^+$, they showed that the solution of system (V.9) decays polynomially to zero as time goes to infinity. This result was later generalized by M.M. Cavalcanti et al. [21] and A. Guesmia [29]. In these papers, the authors showed that the energy of the system is bounded above by a quantity depending on the growth of the convolution kernel at infinity and the regularity of the initial data.

The purpose of this chapter is to extend the above results for any space dimension of Petrowsky-Petrowsky system unbounded domain.

2 Assumptions

First we recall and make use the following assumptions on the functions μ and ρ as:

H1: We assume that the function $\mu : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying:

$$1 - \int_0^\infty \mu(t)dt = \mu_0 > 0, \quad \forall t \in \mathbb{R}^+, \tag{V.10}$$

and that there exists a constants $k_1 > 0$ such that

$$\mu'(t) + k_1\mu(t) \le 0 \quad \forall t \in \mathbb{R}^+, \tag{V.11}$$

Definition 2.1 *[76] We define the function spaces of our problem and its norm as follows:*

$$\mathcal{D}^{2,2}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-4)}(\mathbb{R}^n) : \Delta_x f \in L^2(\mathbb{R}^n) \right\} \tag{V.12}$$

and the spaces $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f,h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx.$$

For $1 < q < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{1/q}. \tag{V.13}$$

and that $\mathcal{D}^{2,2}(\mathbb{R}^n)$ can be embedded continuously in $L^{2n/(n-4)}(\mathbb{R}^n)$,i.e there exists $k > 0$ such that

$$\|u\|_{L^{2n/(n-4)}} \leq k \|u\|_{\mathcal{D}^{2,2}}. \tag{V.14}$$

We shall frequently use the following version of the generalized Poincare's inequality

$$\int_{\mathbb{R}^n} |\Delta u|^2 dx \geq \gamma \int_{\mathbb{R}^n} \rho u^2 dx \tag{V.15}$$

for all $u \in C_0^\infty$ and $\rho \in L^{\frac{n}{4}}$, where $\gamma =: k^{-2} \|\rho\|^{-1}_{\frac{n}{4}}$ (see[76]). The separable Hilbert space $L^2_\rho(\mathbb{R}^n)$ with

$$(f,f)_{L^2_\rho(\mathbb{R}^n)} = \|f\|^2_{L^2_\rho(\mathbb{R}^n)}.$$

consist of all f for which $\|f\|_{L^q_\rho(\mathbb{R}^n)} < \infty, 1 < q < +\infty$.

In order to consider the relative displacement η as a new variable, we introduce the weighted L^2-spaces

$$L^2_\mu(\mathbb{R}^+; H^2(\mathbb{R}^n)) = \left\{ \xi : \mathbb{R}^+ \to H^2(\mathbb{R}^n) | \int_0^\infty \mu(s) \|\xi(s)\|^2_{H^2} ds < \infty \right\},$$

where $L^2_\mu(\mathbb{R}^+; H^2(\mathbb{R}^n))$ is the Hilbert space of $H^2(\mathbb{R}^n)$-valued functions on $\mathbb{R}^+$ endowed with the inner product

$$(\xi,\xi)_\mu = \int_0^\infty \mu(s)(\xi(s),\xi(s))_V ds, \quad \|\xi\|^2_\mu = \int_0^\infty \mu(s)\|\xi\|^2 ds.$$

It is clear that $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space. Moreover, we have the following compact embedding.

Lemma 2.2 *Let $\rho \in L^{n/4}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then the embedding $\mathcal{D}^{2,2} \subset L^2_\rho$ is compact.*

We give the definition of the weak solution for this problem.

Definition 2.3 *A weak solution of the problem (V.1) is a function (u,v,η) such that* *(i)* $u, v \in L^2[0,T;\mathcal{D}^{2,2}(\mathbb{R}^n)]$, $u_t, v_t \in L^2[0,T;L^2_\rho(\mathbb{R}^n)]$, $u_{tt}, v_{tt} \in L^2[0,T;\mathcal{D}^{-2,2}(\mathbb{R}^n)]$,

(ii) *for all* $\varphi, \phi \in C_0^\infty([0,T]\times\mathbb{R}^n)$, (u,v,η) *satisfies the generalized formula*

$$\int_0^T (u_{tt}(\tau),\varphi(\tau))_{L^2_\rho}d\tau + \alpha\int_0^T (v(\tau),\varphi(\tau))_{L^2_\rho}d\tau + \int_0^T (u_t(\tau),\varphi(\tau))_{L^2_\rho}d\tau$$
$$+(1-\mu_0)\int_0^T\int_{\mathbb{R}^n}\Delta u(\tau)\Delta\varphi(\tau)dxd\tau + \int_0^T\int_{\mathbb{R}^n}\int_0^{+\infty}\mu(s)\Delta\eta^\tau(s)\Delta\varphi(\tau)dsdxd\tau = 0, \tag{V.16}$$

and

$$\int_0^T (v_{tt}(\tau),\phi(\tau))_{L^2_\rho}d\tau + \alpha\int_0^T (u(\tau),\phi(\tau))_{L^2_\rho}d\tau + \int_0^T\int_{\mathbb{R}^n}\Delta v(\tau)\Delta\phi(\tau)dxd\tau = 0, \tag{V.17}$$

(iii) (u,v,η) *satisfies the initial conditions*

$$(u_0(x),v_0(x)) \in (\mathcal{D}^{2,2}(\mathbb{R}^n))^2, \quad (u_1(x),v_1(x)) \in (L^2_\rho(\mathbb{R}^n))^2, \quad \eta_0^t \in L^2_\rho(\mathbb{R}^+,H^1(\mathbb{R}^n)).$$

Lemma 2.4 *For any* $u \in \mathcal{D}^{2,2}(\mathbb{R}^n)$

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \le \|\rho\|_{L^{n/4}(\mathbb{R}^n)}\|\Delta_x u\|_{L^2(\mathbb{R}^n)}. \tag{V.18}$$

3 Main results

We denote by $\mathcal{A}$ the unbounded operator in the energy space:
$\mathcal{H}$=$H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ defined by:
$\mathcal{D}(\mathcal{A})$=$(H^2(\mathbb{R}^n)\cap H^1(\mathbb{R}^n))^2\times(H^1(\mathbb{R}^n))^2$ and
$\mathcal{A}\mathcal{U}$=$\Big(\widetilde{u},(1-\mu_0)\phi(x)\Delta^2 u + \int_0^\infty \mu(s)\phi(x)\Delta^2\eta(s)ds - \alpha v - \widetilde{u}, \widetilde{v}, \phi(x)\Delta^2 v - \alpha u\Big)^T$ and $\mathcal{U}$=$(u,\widetilde{u},v,\widetilde{v},\eta)^T$.
The problem (V.1) can then be reformulated under the abstract from

$$\mathcal{U}' + \mathcal{A}\mathcal{U} = 0 \tag{V.19}$$

Our main result reads as follows.

Theorem 3.1 *Assume that (**H1**) holds. Let* $u_0, v_0 \in \mathcal{D}^{2,2}(\mathbb{R}^n)$, $u_1, v_1 \in L^2_\rho(\mathbb{R}^n)$ *and* $\eta_0^t \in L^2_\mu(\mathbb{R}^+,H^2(\mathbb{R}^n))$. *Then there exists positive constant* δ_1 *and* δ_2 *such that the solution of* (V.5) *satisfies*

$$E(t) \le \frac{C}{t}(E(0)+E_2(0)+E_3(0)), \qquad \forall t\in\mathbb{R}^+. \tag{V.20}$$

4 Lack of Exponential Stability

Theorem 4.1 *[28] Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim}_{|\beta|\to\infty} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

We use Theorem 4.1 to prove the lack of exponential stability, that is we show that there exists a sequence of values μ_m such that

$$\|(i\mu_m I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \to \infty. \tag{V.21}$$

It is equivalent to prove that there exist a sequence of data $F_m \in \mathcal{H}$ and a sequence of real numbers $\mu_m \in \mathbb{R}$, with $\|F_m\|_{\mathcal{H}} \leq 1$. such that

$$\|(i\mu_m I - \mathcal{A})^{-1} F_m\|_{\mathcal{H}} \to \infty. \tag{V.22}$$

The eigenvalue problem

$$\phi(x)\Delta^2 u = \mu u, \quad x \in \mathbb{R}^n \tag{V.23}$$

has a complete system of eigensolutions w_j, μ_j satisfying the following properties

$$\begin{cases} \phi(x)\Delta^2 w_j = \mu_j w_j, \quad j = 1, \cdots, w_j \in \mathcal{D}^{2,2}(\mathbb{R}^n), \\ 0 < \mu_1 \leq \mu_2 \leq \cdots, \mu_j \to +\infty \text{ as } j \to +\infty. \end{cases} \tag{V.24}$$

Theorem 4.2 *Assume that the kernel is of the form $g(s) = e^{-\mu s}$, $s \in \mathbb{R}^+$, with $\mu > 1$. The semigroup $S(t)$ on $\mathcal{H}$ is not exponentially stable.*

Proof 4.3 *We follow the method in [9]. We will find a sequence of bounded functions $F_m = (f_{1,m}, f_{2,m}, f_{3,m}, f_{4,m}, f_{5,m}) \in \mathcal{H}$ for which the corresponding solutions of the resolvent equations is not bounded. This will prove that the resolvent operator is not uniformly bounded. Let us consider the equation*

$$i\mu U_m - \mathcal{A}U_m = F_m$$

The equation reads

$$\begin{cases} i\mu u - \varphi = f_1, \\ i\mu\varphi + \mu_0\phi(x)\Delta^2 u + \alpha v + \int_0^\infty g(s)\Delta^2\eta(x,s)ds = f_2, \\ i\mu v - \psi = f_3, \\ i\mu\psi + \phi(x)\Delta^2 v + \alpha u = f_4, \\ i\mu\eta - \varphi + \eta_s = f_5. \end{cases} \tag{V.25}$$

Let us consider $f_{1m} = f_{3m} = f_{5m} = 0$ *and* $f_{2m} = f_{4m} = w_m$ *to obtain* $\varphi = i\mu u$ *and* $\psi = i\mu v$. *Then, system* (V.25) *becomes*

$$\begin{cases} -\mu^2 u + \mu_0\phi(x)\Delta^2 u + \alpha v + \int_0^\infty g(s)\Delta^2\eta(x,s)ds = w_m, \\ -\mu^2 v + \phi(x)\Delta^2 v + \alpha u = w_m, \\ i\mu\eta - \varphi + \eta_s = 0. \end{cases} \tag{V.26}$$

We look for solutions of the form

$$u = aw_m, \quad v = bw_m, \quad \varphi = cw_m, \quad \psi = dw_m, \quad \eta(x,s) = \gamma(s)w_m$$

with a, b, c, d *and* $\gamma(s)$ *depend on* μ *and will be determined explicitly in what follows. From* (V.26), *we get* a *and* b *satisfy*

$$\begin{cases} -\mu^2 a + \mu_0 a\mu_m + \alpha b + \int_0^\infty g(s)\gamma(s)ds = 1, \\ -\mu^2 b + \mu_m b + \alpha a = 1, \\ \gamma_s + i\mu\gamma - i\mu a = 0. \end{cases} \tag{V.27}$$

Solving $(\text{V.27})_3$ *we get*

$$\gamma(s) = a - ae^{-i\mu s}. \tag{V.28}$$

Then, from (V.28) *we have*

$$\int_0^\infty g(s)\gamma(s)ds = a(1-\mu_0) - a\int_0^\infty g(s)e^{-i\mu s}ds. \tag{V.29}$$

Now, choosing $\mu = \sqrt{\mu_m}$, *using equations* $(\text{V.27})_1$ *and* $(\text{V.27})_2$ *we obtain*

$$
\begin{aligned}
a &= \frac{1}{\alpha}, \\
b &= \frac{\mu_m(1-\mu_0)}{\alpha^2} - \frac{\mu_m}{\alpha}\int_0^\infty g(s)\gamma(s)ds + \frac{1}{\alpha}, \\
c &= i\tfrac{\sqrt{\mu_m}}{\alpha}, \\
d &= i\sqrt{\mu_m}\left(\frac{\mu_m(1-\mu_0)}{\alpha^2} - \frac{\mu_m}{\alpha}\int_0^\infty g(s)\gamma(s)ds + \frac{1}{\alpha}\right)
\end{aligned}
\tag{V.30}
$$

As

$$\varphi = cw_m = i\frac{\sqrt{\mu_m}}{\alpha}$$

we obtain

$$\|\varphi\|^2_{L^2_\rho} = \frac{\mu_m}{\alpha}$$

Therefore we get

$$\lim_{m\to\infty}\|U_m\|^2_{\mathcal{H}} \geq \lim_{m\to\infty}\|\varphi\|^2_{L^2_\rho} = \lim_{m\to\infty}\frac{\mu_m}{\alpha} = +\infty$$

which completes the proof.

5 Polynomial stability

Let us introduce the second and third–order energy

$$E_2(t) = E(u_t, v_t, \eta_t) \qquad E_3(t) = E(u_{tt}, v_{tt}, \eta_{tt}).$$

Lemma 5.1 *The energies $E(t)$, $E_2(t)$ and $E_3(t)$ associated to solutions of problem* (V.5) *satisfies*

$$\frac{d}{dt}E(t) = -\frac{k_1}{2}\int_0^{+\infty}\mu(s)\|\Delta\eta^t\|_2^2 ds \leq 0, \quad \forall t \geq 0, \tag{V.31}$$

$$\frac{d}{dt}E_2(t) = -\frac{k_1}{2}\int_0^{+\infty}\mu(s)\|\Delta\eta_t^t\|_2^2 ds \leq 0, \quad \forall t \geq 0, \tag{V.32}$$

$$\frac{d}{dt}E_3(t) = -\frac{k_1}{2}\int_0^{+\infty}\mu(s)\|\Delta\eta_{tt}^t\|_2^2 ds \leq 0, \quad \forall t \geq 0, \tag{V.33}$$

Proof 5.2 *Multiplying first Eq. of* (V.5) *by ρu_t and second Eq. of* (V.5) *by ρv_t , respectively, summing the results obtained follows the conclusion of inequality* (V.31)*. The inequalities* (V.32) *and* (V.33) *are obtained using the same procedure as in* (V.31)*.*

Lemma 5.3 *Let (u, v, η) be the solution of* (V.1)*. Let the functional F_1 defined by*

$$F_1(t) := -\int_{\mathbb{R}^n}\rho(x)u_t\left(\int_0^{+\infty}\mu(s)\eta^t(s)ds\right)dx. \tag{V.34}$$

For any $\varepsilon_1 > 0$*, there exists a positive constant* $C_{\varepsilon_1} = \frac{\mu_0 \mu_1}{2\varepsilon_1} + \mu_1 + \frac{\alpha \mu_1}{2\varepsilon_1}$ *such that*

$$\begin{aligned} \frac{d}{dt} F_1(t) \leq & -\frac{\mu_1}{2} \int_{\mathbb{R}^n} \rho(x) u_t^2 dx + \frac{\mu_0 \varepsilon_1}{2} \int_{\mathbb{R}^n} |\Delta u|^2 dx \\ & + \frac{\alpha \varepsilon_1}{2} \int_{\mathbb{R}^n} \rho(x) |v|^2 dx + C_\varepsilon \int_0^{+\infty} \mu(s) \|\Delta \eta^t(s)\|^2 ds, \end{aligned} \tag{V.35}$$

where $\mu_1 = \displaystyle\int_0^\infty \mu(s) ds.$

Proof 5.4 *Using Eqs.* (V.1) *and integrating by parts, we obtain*

$$\begin{aligned} \frac{d}{dt} F_1(t) = & \mu_0 \int_{\mathbb{R}^n} \Delta u \left(\int_0^{+\infty} \mu(s) \Delta \eta^t(s) ds \right) dx + \int_{\mathbb{R}^n} \left| \int_0^{+\infty} \mu(s) \Delta \eta^t(s) ds \right|^2 dx \\ & + \alpha \int_{\mathbb{R}^n} \rho(x) v \left(\int_0^{+\infty} \mu(s) \eta^t(s) ds \right) dx - \mu_1 \int_{\mathbb{R}^n} \rho(x) u_t^2 dx + \int_{\mathbb{R}^n} \rho(x) u_t \left(\int_0^{+\infty} \mu(s) \eta_s^t(s) ds \right) dx \end{aligned}$$

Since

$$\int_{\mathbb{R}^n} \rho(x) u_t \left(\int_0^{+\infty} \mu(s) \eta_s^t(s) ds \right) dx = -\int_{\mathbb{R}^n} \rho(x) u_t \left(\int_0^{+\infty} \mu'(s) \eta^t(s) ds \right) dx, \tag{V.36}$$

and

$$\int_{\mathbb{R}^n} \left| \int_0^{+\infty} \mu(s) \Delta \eta^t(s) ds \right|^2 dx \leq \mu_1 \int_{\mathbb{R}^n} \left(\int_0^{+\infty} \mu(s) |\Delta \eta^t(s)|^2 ds \right) dx. \tag{V.37}$$

Then using Young inequality and hypotheses (V.11) *our conclusion follows.*

Let us consider the following functional

$$F_2(t) = F_1(u_t, v_t, \eta_t^t). \tag{V.38}$$

Then using Lemma 5.3 we obtain

$$\begin{aligned} \frac{d}{dt} F_2(t) \leq & -\frac{\mu_1}{2} \int_{\mathbb{R}^n} \rho(x) u_{tt}^2 dx + \frac{\mu_0 \varepsilon_1}{2} \int_{\mathbb{R}^n} |\Delta u_t|^2 dx \\ & + \frac{\alpha \varepsilon_1}{2} \int_{\mathbb{R}^n} \rho(x) |v_t|^2 dx + C_\varepsilon \int_0^{+\infty} \mu(s) \|\Delta \eta_t^t(s)\|_2^2 ds, \end{aligned} \tag{V.39}$$

Lemma 5.5 *Let* (u, v, η) *be the solution of* (V.1)*. Then the functional* Z_1 *defined*

$$Z_1(t) = \int_{\mathbb{R}^n} \rho(x) u_t . u dx. \tag{V.40}$$

satisfies, along the solution, the estimate

$$\frac{d}{dt}Z_1(t) \leq \int_{\mathbb{R}^n} \rho(x)u_t^2 dx - \frac{\mu_0}{2}\int_{\mathbb{R}^n} |\Delta u|^2 dx - \alpha \int_{\mathbb{R}^n} \rho(x)uv dx + \frac{\mu_1}{2\mu_0}\int_0^{+\infty} \mu(s)\|\Delta\eta^t(s)\|^2 ds, \quad \text{(V.41)}$$

where $\mu_1 = \int_0^{\infty} \mu(s)ds.$

Proof 5.6 *Using first Eq. of* (V.5) *and integrating by parts, we get*

$$\frac{d}{dt}Z_1(t) = -\mu_0 \int_{\mathbb{R}^n} |\Delta u|^2 - \int_{\mathbb{R}^n}\int_0^{+\infty} \mu(s)\Delta\eta^t \Delta u ds dx - \alpha \int_{\mathbb{R}^n} \rho(x)uv dx + \int_{\mathbb{R}^n} \rho(x)u_t^2 dx, \quad \text{(V.42)}$$

then using Young inequality our conclusion follows.

Let us denote by $Z_2(t)$ the following functional

$$Z_2(t) = Z_1(u_t, v_t, \eta_t^t). \quad \text{(V.43)}$$

Then, from (V.41) we have

$$\frac{d}{dt}Z_2(t) \leq \int_{\mathbb{R}^n} \rho(x)u_{tt}^2 dx - \frac{\mu_0}{2}\int_{\mathbb{R}^n} |\Delta u_t|^2 dx - \alpha \int_{\mathbb{R}^n} \rho(x)u_t v_t dx + \frac{\mu_1}{2\mu_0}\int_0^{+\infty} \mu(s)\|\Delta\eta_t^t(s)\|^2 ds, \quad \text{(V.44)}$$

The following Lemma shows the dissipative property of v. For this let us consider the following functional

$$\Phi(t) = \int_{\mathbb{R}^n} u_{tt}v_t dx + \mu_0 \int_{\mathbb{R}^n} \Delta u_t.\Delta v dx + \int_{\mathbb{R}^n} \left(\int_0^{+\infty} \mu(s)\Delta\eta_t^t(s)\Delta v ds\right) dx. \quad \text{(V.45)}$$

Lemma 5.7 *For any* $\varepsilon_2 > 0$ *we have*

$$\begin{aligned}\frac{d}{dt}\Phi(t) \leq& -\alpha\|v_t\|_{L^2_\rho}^2 + \left(\frac{1}{2\varepsilon_2} + \frac{\mu_0}{2\varepsilon_2}\right)\int_{\mathbb{R}^n}\left(\int_0^{\infty} \mu(s)|\Delta\eta_{tt}^t|^2 ds\right) dx \\ &+ \left(\frac{k_2}{2\varepsilon_2} + \frac{\mu_1}{2\varepsilon_2} + \frac{k_2\mu_0}{2\varepsilon_2}\right)\int_{\mathbb{R}^n}\left(\int_0^{\infty} \mu(s)|\Delta\eta^t|^2 ds\right) dx + \frac{\mu(0)}{2\varepsilon_2}\|u_t\|^2 dx \\ &+ \left(\frac{1}{\mu_1} + \frac{\mu(0)+1}{2} + \frac{\mu_1}{\mu_0}\right)\varepsilon_2\|\Delta v\|_2^2 + \frac{\alpha}{2}\|u_{tt}\|_{L^2_\rho}^2 dx + \frac{\alpha C_0}{2}\|\Delta u\|_2^2.\end{aligned}$$

Proof 5.8 *Differentiating* $\Phi(t)$ *with respect to* t *and using first Eq. of* (V.1)*, we obtain*

$$\begin{aligned}\frac{d}{dt}\Phi(t) =& -\alpha \int_{\mathbb{R}^n} \rho(x)|v_t|^2 dx + \int_{\mathbb{R}^n} \Delta u_{tt}.\Delta v dx - \alpha \int_{\mathbb{R}^n} \rho(x)u_{tt}u dx \\ &+ \int_{\mathbb{R}^n}\left(\int_0^{\infty} \Delta\eta_{tt}^t ds\right).\Delta v dx + \mu_0 \int_{\mathbb{R}^n} \Delta u_{tt}\Delta v dx. \quad \text{(V.46)}\end{aligned}$$

On the other hand, from (V.2) *we deduce*

$$\begin{aligned} &\eta_t^t(0) = 0, \quad \eta_s^t(0) = u_t(t), \\ &u_{tt} = \eta_{tt}^t - \eta_{ss}^t. \end{aligned} \tag{V.47}$$

Recalling that $\mu_1 = \int_0^\infty \mu(s)ds$ *and using hypotheses* (V.11) *and* (V.47), *we obtain*

$$\begin{aligned} \mu_1 \int_{\mathbb{R}^n} \Delta u_{tt}.\Delta v dx &= \int_{\mathbb{R}^n} \left(\int_0^\infty \mu(s)\Delta u_{tt} ds\right).\Delta v dx \\ &= \int_{\mathbb{R}^n} \left(\int_0^\infty \mu(s)(\Delta\eta_{tt}^t - \Delta\eta_{ss}^t)ds\right).\Delta v dx \\ &= \int_{\mathbb{R}^n} \left(\int_0^\infty \mu(s)\Delta\eta_{tt}^t\right).\Delta v dx - \int_{\mathbb{R}^n} \left(\int_0^\infty \mu(s)\Delta\eta_{ss}^t)ds\right).\Delta v dx \\ &= \int_{\mathbb{R}^n} \left(\int_0^\infty \mu(s)\Delta\eta_{tt}^t\right).\Delta v dx + \mu(0)\int_{\mathbb{R}^n} \Delta u_t \Delta v dx - \int_{\mathbb{R}^n} \left(\int_0^\infty \mu''(s)\Delta\eta^t\right).\Delta v dx \\ &\leq \int_{\mathbb{R}^n} \left(\int_0^\infty \mu(s)\Delta\eta_{tt}^t\right).\Delta v dx + \mu(0)\int_{\mathbb{R}^n} \Delta u_t \Delta v dx - k_2\int_{\mathbb{R}^n} \left(\int_0^\infty \mu(s)\Delta\eta^t\right).\Delta v dx. \end{aligned} \tag{V.48}$$

Substituting the Eq. (V.48) *into* (V.46) *and using Poincare's and Young inequalities we arrive at*

$$\begin{aligned} \frac{d}{dt}\Phi(t) \leq& -\alpha\|v_t\|_{L^2_\rho}^2 + \frac{1}{2\varepsilon_2\mu_1}\int_{\mathbb{R}^n}\left(\int_0^\infty \mu(s)\Delta\eta_{tt}^t ds\right)^2 dx \\ &+\frac{\varepsilon_2}{2\mu_1}\int_{\mathbb{R}^n}|\Delta v|^2dx + \frac{\alpha}{2}\int_{\mathbb{R}^n}\rho(x)|u_{tt}|^2dx + \frac{\alpha C_0}{2}\int_{\mathbb{R}^n}|\Delta u|^2dx \\ &+\frac{k_2}{2\varepsilon_2\mu_1}\int_{\mathbb{R}^n}\left(\int_0^\infty \mu(s)|\Delta\eta^t|ds\right)^2 dx + \frac{\varepsilon_2}{2\mu_1}\int_{\mathbb{R}^n}|\Delta v|^2dx \\ &+\frac{k_2}{2\varepsilon_2}\int_{\mathbb{R}^n}\left(\int_0^\infty \mu(s)|\Delta\eta^t|ds\right)^2 dx + \frac{\mu(0)+1}{2}\varepsilon_2\int_{\mathbb{R}^n}|\Delta v|^2dx + \frac{\mu(0)}{2\varepsilon_2}\int_{\mathbb{R}^n}|u_t|^2dx \\ &+\frac{\mu_0}{2\varepsilon_2\mu_1}\int_{\mathbb{R}^n}\left(\int_0^\infty \mu(s)\Delta\eta_{tt}^t ds\right)^2 dx + \frac{\varepsilon_2\mu_0}{2\mu_1}\int_{\mathbb{R}^n}|\Delta v|^2dx \\ &+\frac{k_2\mu_0}{2\varepsilon_2\mu_1}\int_{\mathbb{R}^n}\left(\int_0^\infty \mu(s)|\Delta\eta^t|ds\right)^2 dx + \frac{\varepsilon_2\mu_0}{2\mu_1}\int_{\mathbb{R}^n}|\Delta v|^2dx \end{aligned} \tag{V.49}$$

Noting that

$$\int_{\mathbb{R}^n}\left(\int_0^\infty \mu(s)\Delta\eta_{tt}^t\right)^2 dx \leq \mu_1\int_{\mathbb{R}^n}\left(\int_0^\infty \mu(s)|\Delta\eta_{tt}^t|^2 ds\right)dx. \tag{V.50}$$

and

$$\int_{\mathbb{R}^n}\left(\int_0^\infty \mu(s)\Delta\eta_{t}^t\right)^2 dx \leq \mu_1\int_{\mathbb{R}^n}\left(\int_0^\infty \mu(s)|\Delta\eta_{t}^t|^2 ds\right)dx. \tag{V.51}$$

our conclusion follows.

Finally, we define the functional

$$\Psi(t) = \int_{\mathbb{R}^n}\rho(x)v_t.v dx. \tag{V.52}$$

From second Eq. of (V.1)

$$\frac{d}{dt}\Psi(t) = -\int_{\mathbb{R}^n} |\Delta v|^2 dx + \int_{\mathbb{R}^n} \rho(x)|v_t|^2 dx - \alpha \int_{\mathbb{R}^n} \rho(x) u.v dx. \tag{V.53}$$

Proof of Theorem 3.1 We define $\mathcal{L}$ as

$$\mathcal{L}(t) = N_1(E(t) + E_2(t) + E_3(t)) + N_2 \sum_{i=1}^{2} F_i(t) + N_3 \sum_{i=1}^{2} Z_i(t) + N_4\Phi(t) + \Psi(t). \tag{V.54}$$

where N_1, N_2, N_3 and N_4 are positive constants. Then from Lemmas 5.1-5.7 we get

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leq \;& -\Lambda_1 \int_{\mathbb{R}^n} \left(\int_0^\infty \mu(s)|\Delta\eta^t|^2 ds\right) dx - \Lambda_1 \int_{\mathbb{R}^n} \left(\int_0^\infty \mu(s)|\Delta\eta_t^t|^2 ds\right) dx \\ & -\Lambda_1 \int_{\mathbb{R}^n} \left(\int_0^\infty \mu(s)|\Delta\eta_{tt}^t|^2 ds\right) dx - \Lambda_2 \int_{\mathbb{R}^n} |\Delta v|^2 dx \\ & - \left(\frac{\mu_1}{2}N_2 - N_3 - \frac{\alpha}{2}N_4\right) \int_{\mathbb{R}^n} u_t^2 dx - \left(\frac{\mu_1}{2}N_2 - N_3 - \frac{\alpha}{2}N_4\right) \int_{\mathbb{R}^n} u_{tt}^2 dx \\ & - \left(\alpha N_4 - \frac{\alpha\varepsilon_1}{2}N_2 - 1\right) \int_{\mathbb{R}^n} v_t^2 dx - \left(\frac{\mu_0}{2}N_3 - \frac{\mu_0\varepsilon_1}{2}N_2 - \frac{\alpha}{2}N_4\right) \int_{\mathbb{R}^n} |\Delta u|^2 dx \\ & - \left(\frac{\mu_0}{2}N_3 - \frac{\mu_0\varepsilon_1}{2}N_2 - \frac{\mu(0)}{2\varepsilon_2}N_4\right) \int_{\mathbb{R}^n} |\Delta u_t|^2 dx.\end{aligned} \tag{V.55}$$

Where

$$\Lambda_1 = -\left(N_1\frac{k_1}{2} - \left[\left(\frac{\mu_0\mu_1}{2\varepsilon_1} + \mu_1 + \frac{\alpha\mu_1}{2\varepsilon_1} + \frac{k_0C}{2}\right)N_2 + \frac{\mu_1}{2\mu_0}N_3 + \left(\frac{k_2}{2\varepsilon_2} + \frac{\mu_1}{2\varepsilon_2} + \frac{k_2\mu_0}{\varepsilon_2}\right)N_4\right]\right),$$

and

$$\Lambda_2 = \left(1 - \frac{\alpha C\varepsilon_1}{2}N_2 - \left(\frac{1}{\mu_1} + \frac{\mu(0)+1}{2} + \frac{\mu_0}{\mu_1}\right)\varepsilon_2 N_4\right).$$

Now, choosing ε_1, ε_2 small enough such that $0 < \varepsilon_1 < \varepsilon_2 < \frac{\mu_1}{2}$ and $N_4 > 0$ large enough we get

$$\alpha > \frac{\mu_0\mu_1(\mu_1 - 2\varepsilon_1)}{N_4(\mu_0\mu_1(\mu_1 - 2\varepsilon_1) - 2\varepsilon_1(\varepsilon_1(\mu_0 + \mu_1 C)))} > 0$$

is small enough. Then taking

$$\begin{aligned}N_1 > \frac{2}{k_1}\Bigg[&\left(\frac{\mu_0\mu_1 + \alpha\mu_1}{2\varepsilon_1} + \mu_1 + \frac{k_0C}{2}\right) + \left(2\frac{(\varepsilon_1+\alpha)\mu_0\varepsilon_2 + \mu(0)\mu_1}{\mu_0\mu_1\varepsilon_2(\mu_1 - 2\varepsilon_1)} + \alpha\right) \\ &+ \frac{\mu_1^2(\varepsilon_1\varepsilon_2\alpha + \mu(0)\mu_1)}{2\mu_0(\mu_1 - 2\varepsilon_1)} + \frac{k_2 + \mu_1 + 2k_2\mu_0}{2\varepsilon_2}\Bigg] N_4\end{aligned}$$

$$N_2 > \max\left\{\left(\frac{2(\varepsilon_1\mu_0 + \mu_1 C\varepsilon_1)}{\mu_0\mu_1(\mu_1 - 2\varepsilon_1)} + \alpha\right)N_4, \left(\frac{(\varepsilon_1+\alpha)\mu_0\varepsilon_2 + \mu(0)\mu_1}{\mu_0\mu_1(\mu_1 - 2\varepsilon_1)} + \alpha\right)N_4\right\}$$

and

$$N_3 > \max\left\{\left(\frac{(\varepsilon_1\mu_0 + \mu_1 C\varepsilon_1)}{\mu_0\mu_1(\mu_1 - 2\varepsilon_1)}\right)N_4, \left(\frac{\varepsilon_1\varepsilon_2\alpha\mu_0 + \mu(0)\mu_1}{\mu_0\mu_1(\mu_1 - 2\varepsilon_1)}\right)N_4\right\}$$

we get that there is a positive constant $\omega > 0$ such that

$$\frac{d}{dt}\mathcal{L}(t) \leq -wE(t). \tag{V.56}$$

Therefore

$$w\int_0^t E(s)ds \leq \mathcal{L}(0) - \mathcal{L}(t), \quad \forall t > 0. \tag{V.57}$$

On the other hand, it is not difficult to prove that there exists a constant $\zeta > 0$ such that

$$\mathcal{L}(0) - \mathcal{L}(t) \leq \frac{\zeta}{w}(E(0) + E_2(0) + E_3(0)), \quad \forall t > 0. \tag{V.58}$$

From (V.57) and (V.58) we obtain

$$\int_0^t E(s)ds \leq \zeta(E(0) + E_2(0) + E_3(0)) \tag{V.59}$$

Finally, since

$$\frac{d}{dt}tE(t) = E(t) + t\frac{d}{dt}E(t) \leq E(t). \tag{V.60}$$

from (V.59) we get

$$E(t) \leq \frac{C}{t}(E(0) + E_2(0) + E_3(0)). \tag{V.61}$$

Where $C = \frac{\zeta}{w}$. Finally, we obtain (V.20), which completes the proof.

Bibliography

[1] Alabau-Boussouira, F. and Cannarsa, P., *A general method for proving sharp energy decay rates for memory-dissipative evolution equations*, C. R. Acad. Sci. Paris, Ser. I 347, 867-872 (2009).

[2] Agre, K. and Rammaha, M. A. *Systems of nonlinear wave equations with damping and source terms*, Diff .Integral Equations,19 , 1235-1270 (2007).

[3] Aassila, M., *Stability and asymptotic behaviour of solutions of the heat equation*, IMAJ. Appl. Math. 69(1) 93–109 (2004).

[4] Alabau, F., Cannarsa, P. and Komornik, V., *Indirect internal stabilization of weakly coupled evolution equations* Journal of Evolution Equations, 2, 127-150 (2002).

[5] Alabau, F., *Indirect boundary stabilization of weakly coupled systems,* SIAM J. Control Optimization, 41 , 511-541 (2002).

[6] Alabau, F., *Observabilité frontiere indirecte de systemes faiblement couples* C.R. Acad. Sci. Paris, tome 333 Serie I, 645-650 (2001).

[7] Arnold, V. I., *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, (1989).

[8] Ang, D. D. and Dinh, A. P. N., *Strong solutions of a quasilinear wave equation with non linear damping,* SIAM Journal on Mathematical Analysis, vol. 19, no. 2, pp. 337-347, (1988).

[9] Almeida, R.G.C. and Santos, M.L. *Lack of exponential decay of a coupled system of wave equations with memory* Nonlinear Analysis: Real World Applications 12 1023-1032 (2011).

[10] Beniani, A., Benaissa, A. and Zennir, K., *Polynomial Decay Of Solutions To The Cauchy Problem For A Petrowsky-Petrowsky System In* $\mathcal{R}^n$ Journal Acta Applicandae Mathematicae, Vol.146, Issue 1, pp. 67–79, (2016).

[11] Benaissa, A., Beniani, A. and Zennir, K., *General decay of solution for coupled system of viscoelastic wave equations of kirchhoff type with density in* $\mathcal{R}^n$*,* Journal Facta Universitatis. Ser. Math. Inform. Vol. 31, No 5, 1073–1090, (2016).

[12] Beniani, A., Zennir, K. and Benaissa, A., *Stability For The Lamé System With A Time Varying Delay Term In A Nonlinear Internal Feedback* Journal Clifford Analysis, Clifford Algebras and their applications. Vol. 5, No.4, PP. 287–298 , (2016).

[13] Benaissa, A. and Benazzouz, S., *Energy decay of solutions to the Cauchy problem for a nondissipative wave equation*, J.Math.Phys. 51, 123-504 (2010).

[14] Benaissa, A. and Guesmia, A. *Energy decay for wave equations of* ϕ*-Laplacian type with weakly nonlinear dissipation,* Electron. J. Differ. Equations, (2008).

[15] Benaissa, A. and Mokeddem, S., *Global existence and energy decay of solutions to the Cauchy problem for a wave equation with a weakly nonlinear dissipation*, Abstr. Appl. Anal, 11 935-955 (2004).

[16] Benaissa, A. and Messaoudi, S. A., *Blow-up of solutions for the Kirchhoff equation of q-Laplacian type with nonlinear dissipation*, Colloq.Math. 94 ,103-109 (2002).

[17] Beyrath, A., *Indirect internal stabilization of coupled systems with locally distributed damping,* C.R. Acad. Sci. Paris, tome 333 Serie I , 451-456 (2001).

[18] Brown, K. J. and Stavrakakis, N. M. *Global bifurcation results for a semilinear elliptic equation on all of* $\mathbb{R}^n$, Duke Math. J., 85 , 77-94 (1996).

[19] Brezis, H. *Analyse fonctionnelle et application,* Ed, Masson, (1983).

[20] Caraballo, T., Real, J. and Shaikhet, L. *Method of Lyapunov functionals construction in stability of delay evolution equations,* J.Math.Anal.Appl. 334 1130-1145 (2007).

[21] Cavalcanti, M.M., Domingos Cavalcanti, V.N., Guesmia, A. *Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory*. J. Differ. Equ. 259(12), 7540-7577 (2015).

[22] Cavalcanti, M.M. and Oquendo, H.P. *Frictional versus viscoelastic damping in a semilinear wave equation*, SIAM J. Control Optim. 42 (4) 1310-1324 (2003).

[23] Dafermos, C. M and Oquendo, H.P., *Asymptotic stability in viscoelasticity*, Arch. Ration. Mech. Anal. 37, 297-308 (1970).

[24] Dai, Y., Cai, Y. Z. and Xu, X. M. *Synchronization and exponential estimates of complex networks with mixed time-varying coupling delays,* Int.J.Autom.Comput.6 (3) 301-307 (2009).

[25] Datko, R., *Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks,* SIAMJ.Control Optim.26 687-713 (1988).

[26] Hsinchu, Taiwan, *Recent Advances in Nonlinear Analysis*, world Scientific, (2006).

[27] Haraux, A., *Two remarks on dissipative hyperbolic problems,* Research Notes in Math- ematics, vol. 122. Pitman:Boston, MA, 161-179 (1985).

[28] Huang, F., *Characteristic condition for exponential stability of linear dynamical systems in Hilbert space.* Ann. Differ. Equ. 1(1), 43-56 (1985).

[29] Guesmia, A., *Asymptotic behavior for coupled abstract evolution equations with one infinite memory.* Appl. Anal. 94(1), 184-217 (2015)

[30] Guesmia, A., *Quelques Résultats De Stabilisation Indirecte Des Systèmes Couplés Non Dissipatifs,* Math. Soc. Simon Stevin 15 , 479-497 (2008).

[31] Guesmia, A. and Messaoudi, S.A., *On the control of solutions of a viscoelastic equation.* Appl. Math. Comput. 206 , No.2,589-597 (2008).

[32] Guesmia, A., and Messaoudi, S. *On the boundary stabilization of a compactly coupled system of nonlinear wave equations,* Intern. J. Evolut. Equa., 1 , 211-224 (2006).

[33] Guesmia, A., *Inégalités intégrales et applicationà la stabilisation des systèmes distribués non dissipatifs, HDR thesis,* Paul Verlaine-Metz Univeristy,(2006).

[34] Guesmia, A., *A new approach of stabilization of nondissipative distributed systems,* SIAM J.Control Optimization 42,24-52 (2003).

[35] Guesmia, A., *Nouvelles inégalités intégrales et application à la stabilisation des systèmes distribu és non dissipatifs,* C.R.Math.Acad.Sci.Paris, 336, 801-804 (2003).

[36] Guesmia, A., *Energy Decay for a Damped Nonlinear Coupled System,* J. Math. Anal. Appl. 239 , 38-48 (1999).

[37] Georgiev, V. and Todorova,G., *Existence of a solution of the wave equation with nonlinear damping and source term*, J. Diff .Eq.,109 ,295-308 (1994).

[38] Kafini, M., *uniforme decay of solutions to Cauchy viscoelastic problems with density*, Elec. J. Diff. Equ Vol.2011 No. 93, pp. 1-9 (2011).

[39] Kafini, M., Messaoudi, S. A. and Tatar Nasser-eddine, *Decay rate of solutions for a Cauchy viscoelastic evolution equation*, Indagationes Mathematicae 22 103-115 (2011).

[40] Ammari, K., Nicaise, S. and Pignotti, Cristina., *Feedback boundary stabilization of wave equations with interior delay,* Systems and Control Letters 59, 623–628 (2010).

[41] Kafini, M., Messaoudi, S. A., *On the uniform decay in viscoelastic problem in* $\mathbb{R}^n$, Applied Mathematics and Computation 215, 1161-1169 (2009).

[42] karachalios, N. I. and Stavrakakis, N.M *Existence of global attractor for semilinear dissipative wave equations on* $\mathbb{R}^n$, J. Diff. Equ 157, 183-205 (1999).

[43] Karahalios, N. I. and Stavrakakis, N. M, *Global Existence and Blow-Up Results for Some Nonlinear Wave Equations on* $\mathbb{R}^n$ *,* Adv. Diff. Equations, 6, 155-174 (2001).

[44] Komornik, V. and Rao, B., *Rapid boundary stabilization of compactly coupled wave equations,* Asymptotic Anal. 14 , 339-359 (1997).

[45] Komornik, V. *Rapid boundary stabilization of linear distributed systems.* SIAM J. Control Optimization 35, 1591-1613 (1997).

[46] Komornik, V. and Zuazua, E., *A direct method for the boundary stabilization of the wave equation.* J. Math. Pures Appl. 69:33–54, (1990).

[47] Lasiecka, I. and Toundykov, D. *Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms.* Nonlin Anal. 64, 1757-1797 (2006).

[48] De Lima Santos, M. *Decay rates for solutions of a system of wave equations with memory.* Electron. J. Dif. Equations, 38 :1-17, (2002).

[49] Levine, H.A., Park, S. R. and Serrin, J., *Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation,* Journal of Mathematical Analysis and Applications, Vo 228, no. 1, pp. 181-205, (1998).

[50] Levine, H.A. and Serrin, J., *Global nonexistence theorems for quasilinear evolution equations with dissipation,* Archive for Rational Mechanics and Analysis, vo l37, no. 4, pp. 341-361, (1997).

[51] Lasiecka, I. and Tataru, D. *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary dampin,* Diff. Inte. Equa., 6, 507-533 (1993).

[52] Lasiecka, I S. A. Messaoudi, and Muhammad I. Mustafa, *Note on intrinsic decay rates for abstract wave equations with memory*, J. Math. Phys. 031504 (2013).

[53] Lagnese, J. *Decay of solutions of the wave equation in a bounded region with boundary dissipation.* J. Differential Equations, 50:163–182, (1983).

[54] Lions, J. L., *quelques méthodes de résolution des problemes aux limites non lineaires*, Dunod, Gaulthier-Villars, Paris (1969).

[55] Lions, J. L. and Strauss, W. *Some nonlinear evolution equations,* Bull. Soc. Math. France, Vol. 93, pp. 43-96 (1965).

[56] Mustafa, Muhammad I., *Well posedness and asymptotic behavior of a coupled system of nonlinear viscoelastic equations*, Nonlinear Analysis 13 452–463 (2012).

[57] Mustafa, Muhammad I. and Messaoudi, S. A. *General stability result for viscoelastic wave equations*, Journal of Mathematical Physics 53, 053702 (2012).

[58] Messaoudi, S.A. and Said-Houari, B., *Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms*, J. Math. Anal. Appl. 365 277-287 (2010).

[59] S. A. Messaoudi and Nasser-eddine Tatar, *Uniform stabilization of solutions of a nonlinear system of viscoelastic equations*, App. Anal. 87 247-263 (2008).

[60] Messaoudi, S. A., *Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation*, J. Math .Anal. Appl. 320, 902–915 (2006).

[61] Matsuyama, T. *Asyumptotics for the nonlinear dissipative wave equation,* Trans. Amer. Math. Soc, Vol. 355 (2003), pp. 865-899.

[62] Martinez, P., *A new method to obtain decay rate estimates for dissipative systems*, ESAIM Control Optimal. Calc. Var. 4, 419-444 (1999).

[63] Milla, M. and Medeiros, L. A, *On the existence of global solutions of a coupled nonlinear klein-gordon equations,* Funkcial. Ekvac. 30(1),147–161(1987).

[64] Nakao, M. *Energy decay to the Cauchy problem for a generalised nonlinear Klien-Gordon equation with a nonlinear dissipative term,* J. Dynamical Systems and Differentail Equations, Vol.3, pp.349-362 (2011).

[65] Nicaise, S., Pignotti, C. and Valein. J., *Exponential stability of the wave equation with boundary time-varying delay.* Discrete Contin. Dyn. Syst. Ser. S, 4:693 – 722, (2011).

[66] Nicaise, S. and Valein, J., *Stabilization of second order evolution equations with unbounded feedback with delay*, ESAIM Control Optim. Calc. Var.16, 420–456 (2010).

[67] Nicaise, S., Valein, J. and Fridman, E., *Stability of the heat and of the wave equations with boundary time-varying delays.* Discrete Contin. Dyn. Syst. Ser. S, 2: 559–581, (2009).

[68] Nicaise, S. and Pignotti, C., *Stabilization of the wave equation with boundary or internal distributed delay*, Differential Integral Equations 21,935–958 (2008).

[69] Nicaise, S. and Pignotti, C., *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim. 45, 1561-1585 (2006).

[70] Nakao, M. and Ono, K., *Global existence to the cauchy problem of the semilinear wave equation with a nonlinear dissipation,* Funkcialaj Ekvacioj, 38, 417Ű431 (1995).

[71] Nakao, M. *Remarks on the existence and uniqueness of global decaying solutions of the non-linear dissipative wave equations,* Math. Z, 206, 265-2276 (1991).

[72] Nakao, M., *A difference inequality and its applications to nonlinear evolution equations,* J. Math. Soc. Japan, 30, 747-762 (1978).

[73] Papadopulos, P.G. Stavrakakies, *Global existence and blow-up results for an equations of Kirchhoff type on* $\mathbb{R}^n$, Methods in Nolinear Analysis 17, 91-109, (2001).

[74] Reed, M. and Simon, B., *Methods of Mathematical Physics III: Scattering Theory*, Academic Press, New York, (1979).

[75] Rudin, W. *Real and complex Analysis,* (2nd edition), McGraw-Hill, New York, (1974).

[76] R.C.A.M Van Der Vorst, *Best constant for the embedding of the space* $H^2 \cap H_0^1(\Omega)$ *into* $L^{2n/n-4}(\Omega)$, Diff. Int. Eq. 6(2), 259-276, (1993).

[77] Munoz Rivera, J. E., *Global solution on a quasilinear wave equation with memory*, Boll. Unione Mat. Ital. B (7) 8, no. 2, 289-303 (1994).

[78] Said-Houari, B., Messaoudi, S. A. and Guesmia, A. *General decay of solutions of a nonlinear system of viscoelastic wave equations*, Nonlinear Difffer. Equ. Appl. Springer Basel AG (2011).

[79] Santos, M. L., *Decay rates for solutions of a system of wave equations with memory*, Elec. J. Diff. Equ. 38, 1-17 (2002).

[80] Todorova, G. and Yordanov, B., *Critical exponent for a nonlinear wave equation with damping,* J. Diferential Equations, 174, 464-489 (2001).

[81] Todorova., G., *Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms,* Comptes Rendus de l'académie des Sciences. Série 1, vo 326, no. 2, pp. 191-196, (1998).

[82] Vitillaro, E., *Global nonexistence theorems for a class of evolution equations with dissipative,*Archive for Rational Mechanics and Analysis,vo 149, no. 2, pp. 155-182, (1999).

[83] zennir, K., *General decay of solutions for damped wave equation of Kirchhoff type with density in* $\mathbb{R}^n$, Ann Univ Ferrara. DOI 10.1007/s11565-015-0223 (2015).

[84] Zhen-Hu Ning and Qing-Xu Yan, *Stabilization of the wave equation with variable coefficients and a delay in dissipative boundary feedback,* J.Math.Anal.Appl.367, 167–173 (2010).

[85] Zuazua, E., *Exponential decay for the semi-linear wave equation with locally distributed damping.* Comm. Partial Differential Equations, 15:205–235, (1990).

[86] Zeidler, E.,*Nonlinear Functional Analysis and its Applications,* Vol. II, Monotone Operators, Springer-Verlag, Berlin, (1990).

[87] Zhou, Yong, *A blow-up result for a nonlinear wave equation with damping and vanishing initial energy in* $\mathbb{R}^n$, Appl. Math. Lett. 18, 281-286 (2005).

Abstract

The present thesis is devoted to the study of global existence and asymptotic behavior in time of solution of Timoshenko system and coupled system .

This work consists of five chapters, will be devoted to the study of the global existence and asymptotic behaviour of some evolution equation with linear, nonlinear dissipative terms and viscoelastic equation. In chapter 1, we recall of some fundamental inequalities. In chapter 2, we consider the Cauchy problem for a coupled system of wave equation, we prove polynomial decay of solution. In chapter 3, we study the Cauchy problem for a coupled system of a viscoelastic wave equation, we prove exponential stability of the solution. In chapter 4, we study the Cauchy problem for a coupled system of a nonlinear weak viscoelastic wave equations, we prove existence and uniqueness of global solution and prove exponential stability of the solution. In this work, the proof an existence and uniqueness for global solution is based on stable set for small data combined with Faedo-Galerkin. The proof an decay estimate is based on multiplier method, Lyapunov functional for some perturbed energy. In chapter 5, we consider a system of viscoelastic wave equations of Petrowsky-Petrowsky type, we use a spaces weighted by density function to establish a very general decay rate of solution.

Résume

Dans cette thèse, nous avons considéré quelques problèmes aux dérivées partielles de type hyperbolique (Equations et systèmes d'équations) avec la présence de différents mécanismes de dissipation, d'amortissement de différent point de vue. Sous quelques hypothèses sur les données initiales et aux bords, conditions sur les termes de dissipation, d'amortissement et les termes source, nous avons concentré notre étude sur l'existence et le comportement asymptotique des solutions où nous avons obtenu plusieurs résultats sur la décroissance de l'énergie, la croissance exponentielle.

ملخص

في هذه الأطروحة اقترحنا بعض المسائل الرياضية لمعادلات و جمل معادلات قطوع زائدة بوجود آليات مختلفة للتبديد بعدة أشكال غير خطية من زوايا مختلفة. تحت بعض الفرضيات على الشروط الابتدائية و الشروط الحدية، ركزنا دراستنا على وجود الحلول ودراسة السلوك المقارب للحلول الموجودة عند اللانهاية الزمنية أين توصلنا لإيجاد عدة نتائج حول طريقة تناقص الطاقة، التزايد الآسي .