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Présentée par

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Etude de l'existence globale et de la stabilisation de quelques problèmes d'évolution non linéaires avec retard

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Devant le jury composé de :

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THÉSE

présenté par:

Nadia MEZOUAR

Pour obtenir le Diplôme Doctorat en Sciences

Spécialité: Mathématiques

Option : Mathématique appliquées

Intitulé :

ETUDE DE L'EXISTENCE GLOBALE ET DE LA STABILISATION DE QUELQUES PROBLÈMES D'ÉVOLUTION NON LINÉAIRE AVEC RETARD.

Soutenu le :20/03/2018
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Dédicace

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Abstract

The present thesis is devoted to the study of global existence and asymptotic behavior in time of solutions to nonlinear of the Kirchhoff equations and Viscoelastic equations. This work consists of three chapters, will be devoted to the study of the global existence and asymptotic behavior of some evolution equations with nonlinear dissipative terms, viscoelastic term and delay term. In chapter 1, we consider a degenerate Kirchhoff equation with a weak frictional damping and delay terms in the internal feedback, we prove general stability estimates using some properties of convex functions, without imposing any growth condition at the frictional damping term. In chapter 2, we consider the non linear Petrovsky equation with general nonlinear dissipative terms and a delay term, we prove the existence of global solutions in suitable Sobolev and we prove general stability estimates. In chapter 3, we study a non-degenerate Kirchhoff equation with general nonlinear dissipation term and time varying delay term, we prove existence of global solution under condition on the weight of the delay term in the feedback and the weight of the term without delay and the speed of delay. Also we prove that the energy of the system decays to zero with an explicit decay rate estimate even if the nonlinear dissipation term has not a polynomial behavior in zero. In this PhD thesis, chapter 2 -3, we prove the existence of global solutions in suitable Sobolev spaces by using Faedo-Galarkin method combined with the energy estimate method. Furthermore, the general decay results of the energy are established via suitable Lyapunov functionals and by using some properties of convex functions.

Key words and phrases: Global existence, Delay term, General decay, Degenerate or non degenerate Kirchhoff equation, Petrovsky equation, Multiplier method, Lyapunov functional, Nonlinear feedback, Nonlinear dissipation, Relaxation function, Viscoelastic, Convexity.

Publications

1. N, Mezouar, M, Abdelli and A, Rachah. Existence of global solutions and decay estimates for a viscoelastic Petrovsky equation with a delay term in the nonlinear internal feedback. *Electron.J. Differ. Equations.*, No. 58, . (2017) , 1-25.

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Contents

Introduction

The mathematical description of transversal small vibrations of elastic string, fixed at the ends, is an old question. The first investigations on this problem were done by d'Alembert (1717-1793) and Euler (1707-1783). We consider an orthogonal Cartesian coordinate system (x, u) in $\mathbb{R}^2$. Suppose that the string, in the rest position, is on the x axis with fixed ends at the points M and N. If u(x, t) is the vertical displacement of a point X of the string, with coordinate x at time t. The mathematical model proposed by d'Alembert, in the modern notation, is:

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x},$$

where $c^2 = \frac{P_0}{m}$, with P_0 the initial tension and m the mass of the string MN. D'Alembert observed that the configurations of the displacement of the string are given by:

$$u(x,t) = \Phi(x + ct) + \Psi(x - ct),$$

where Φ and Ψ are arbitrary functions. To obtain the d'Alembert model we impose many restrictions on the physical problem.
Kirchhoff model **[23]** and Carrier **[12]** was proposed for the same physical problem of the vertical displacement of the elastics strings when the ends are fixed, but the tension is variable during the deformations of the string. It can be written as

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{\rho h} + \frac{E}{2L\rho} \int_0^L \left| \frac{\partial u}{\partial x}(x,t) \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1}$$

where the function $u = u(x, t)$ is the vertical displacement at the space coordinate x varying in the segment $[0, L]$ and the time $t > 0$, ρ is the mass density, h is the area of the cross section of the string and τ is the resistance modulus, P_0 is the initial tension on the string, L is the length of the string, E is the Young's modulus of the material. The nonlinear coefficient

$$C(t) = \int_0^L \left| \frac{\partial u}{\partial x}(x,t) \right|^2 dx,$$

is obtained by the variation of the tension during the deformation of the string. The model (1) is obtain as follow:
We denote by $S(t)$ the length of the deformation curve presente by $\Gamma(t)$. We represent by $\overrightarrow{P(t)}$ the tension at the point X of $\Gamma(t)$ and it's equal to P_0 at $t = 0$. Let consider a small vertical vibrations of a stretched elastic string in de section $[0, L]$. Thus, we take care only of the vertical component of the vector $\overrightarrow{P(t)}$, which is:

$$P(t) sin\theta,$$

where $P(t)$ is the module of $\overrightarrow{P(t)}$ and θ is the measure of the angle of the x axis with $\overrightarrow{P(t)}$. Since the vibrations are small, θ is small, so by the approximation:

$$\cos\theta \simeq 1 \text{ and } \sin\theta \simeq \tan\theta \simeq \frac{\partial u}{\partial x}. \tag{2}$$

The variation of the tension with respect to x produce a force on $\Gamma(t)$. By Newtons second law (Force=mass×acceleration) we get:

$$\frac{\partial}{\partial x}(P(t) sin\theta) = \rho h \frac{\partial v_t}{\partial t},$$

where v_t is the transverse speed in the direction of u(x,t) and it defined by:

$$v_t = \frac{\partial u}{\partial t}.$$

Then

$$\frac{\partial}{\partial x}(P(t) sin\theta) = \rho h \frac{\partial^2 u}{\partial t^2}. \tag{3}$$

In the other hand and by using (2) we get:

$$\frac{\partial}{\partial x}(P(t) sin\theta) = P(t) \frac{\partial}{\partial x}(\tan\theta) = P(t) \frac{\partial^2 u}{\partial x^2}. \tag{4}$$

Hence from (3) and (4) we obtain:

$$P(t)\frac{\partial^2 u}{\partial x^2} = \rho h \frac{\partial^2 u}{\partial t^2}. \tag{5}$$

By the Hooke's law we have

$$P(t) - P_0 = Eh\frac{S(t) - L}{L}, \tag{6}$$

where $S(t) = \int_0^L (1 + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}\, dx$.

Apply the Taylor's development in a neighborhood of zero we get:

$$S(t) \simeq L + \frac{1}{2}\int_0^L (\frac{\partial u}{\partial x})^2\, dx.$$

Then (6) become

$$P(t) = P_0 + \frac{Eh}{2L}\int_0^L (\frac{\partial u}{\partial x})^2\, dx. \tag{7}$$

Substituting (7) in (5) and dividing both sides by ρh, we get (1).

The natural generalization of the model (1) is given by the following nonlinear mixed problem

$$\begin{cases} u_{tt} - M\Big(x, t, \int_\Omega |\nabla u|^2\, dx\Big)\Delta u = 0 & \text{in } \Omega \times (0, T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0, T[, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & \text{in } \Omega, \end{cases} \tag{8}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, M is a positive real function on $\Omega \times (0,T) \times [0;\infty)$, $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and (u^0, u^1) are the initial data.

This problem which model the nonhomogeneous materials, has it's origin in the model (1) when the physic elements ρ, h and E are not constants, but depends on the point x in the string and the instant t .

In the Kirchhoff-Carrier model (1), the function $M : [0;\infty) \longrightarrow \mathbb{R}$ such that $M(r) = \frac{P_0}{\rho\ h} + \frac{E}{2L\rho} r$.

We say that a problem is:

1) Coercive iff $M(r) \geq \nu > 0$ for each $r \geq 0$.

2) Coercive at ∞ iff $M(r) \geq 0$ for each $r \geq 0$ and $\int_0^\infty M(r)\,dr = +\infty$.

3) Mildly degenerate iff $M(r) > 0$.

4) Really degenerate iff $M(r) = 0$.

The kirchhoff equation have two cases, we say that

1) Degenerate case if $M(0) = 0$. For (1) this is equivalent to $P_0 = 0$.

2) Non-degenerate case if $M \geq m_0 > 0$. For (1) this is equivalent to $P_0 \neq 0$.

We recall that viscoelastic materials exhibit natural damping, which is due to the special property of these materials to retain a memory of their past history. From the mathematical point of view, these damping effects are modeled by integro-differential operators. A simple example is the viscoelastic membrane equation

$$\begin{cases} u_{tt} - \Delta u - \int_0^t h(t-s)\Delta u(s)\,dx = 0 & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,T[, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & \text{in } \Omega, \end{cases}$$

in a bounded open domain $\Omega \subset \mathbb{R}^n$. The memory term, represented by the convolution term in the equation, expresses the fact that the stress at any instant t depends on the past history of strains which the material has undergone from time 0 up to t. Therefore, the dynamics of viscoelastic materials are of great importance and interest as they have wide applications in natural sciences. Models of Petrovsky type are of interest in applications in various areas in mathematical physics, as well as in geophysics and ocean acoustics [42], [49].

The big problem of a mathematician is to represent a really systems with sufficiently precision and a simple structure model. From a practical view's point, more particularly in the science of engineers, we note that the delay's phenomenons appear naturally in the physical processes. Among the main sources have delays, we cited catch or actuator reaction times, information transmission times, material transfer times or measurement times. So in order to get closer to the really process, better

modeling consists of designing delay systems, a better modeling consists in conceiving delay systems, in which differential equations of evolution intervene, depend not only by the valuer of their state variables at the instant present t but also depend by a part of theirs past valuer. More general time-delay effects arise in many practical problems, in most instances, physical, chemical, biological, thermal, and economic phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the control of partial differential equations with time delay effects has become an active area of research. In many cases it was shown that delay is a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used.
The problem of stabilization consists to determinate the asymptotic behaviour of the energy by E(t), to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero, they are several type of stabilization

1) Strong stabilization: $\lim_{t\to+\infty} E(t) = 0$.

2) Uniform stabilization: if the energy satisfies

$$E(t) \leq Cf(t),$$

where C depends on the norm of initials conditions and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continues decreasing function with $\lim_{t\to+\infty} f(t) = 0$.

3) Weak Stabilization: $(u(t), u'(t)) \rightharpoonup (0, 0)$ when $t \to +\infty$ in an Hilbert space.

One of the fundamental motivations for the study of evolution equations is to understand qualitatively the long-term behavior of their solutions, especially when these equations can not be solved explicitly.
In 1982, Lyapunov introduced an energy function that he used it to study the stability of some nonlinear systems without calculate explicitly their solutions. This method is known today by **Lyapunov's methode** and it played an important role in the stability theory of differential and ordinary equations.
This PhD Thesis consists of three chapters.

- The **chapter 1**

  We consider the initial boundary value problem for the nonlinear Kirchhoff equation with general nonlinear dissipation term

$$\begin{cases} (|u_t|^{l-2}u_t)_t - \Big(\int_\Omega |\nabla u|^2\,dx\Big)^\gamma \Delta u + \mu_1 g(u_t(x,t)) + \mu_2 g(u_t(x,t-\tau)) = 0 & \text{in } \Omega\times]0,+\infty[,\\ u(x,t) = 0 & \text{on } \partial\Omega\times[0,+\infty[,\\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & \text{in } \Omega,\\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau[, \end{cases} \tag{9}$$

  where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, $l \geq 2$, $\gamma \geq 0$, are given constants, μ_1 and μ_2 are positive real numbers, $\tau > 0$ is a time delay and (u^0, u^1, f_0) are in a suitable space.

  In this chapter, we well posed and determine the asymptotic behavior of the solutions. Then we show that the energy of solutions decays to zero with explicit decay rate estimate, we obtain a general stability estimates.

- The **chapter 2**

  We consider the existence and decay properties of global solutions for the initial boundary value problem of viscoelastic Petrovsky equation

$$\begin{cases} |u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-s)\Delta^2 u(s)\,ds & \\ +\mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0 & \text{in } \Omega\times]0,+\infty[,\\ u(x,t) = 0 & \text{on } \partial\Omega\times[0,+\infty[,\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega,\\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau[, \end{cases} \tag{10}$$

  where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, $\partial\Omega$ is a smooth boundary, $l > 0$, μ_1 and μ_2 are positive real numbers, h is a positive non-increasing function defined on $\mathbb{R}^+$, g_1 and g_2 are two functions, $\tau > 0$ is a time delay and (u_0, u_1, f_0) are the initial data in a suitable function space.

  We use the Faedo-Galerkin method combined with the energy estimate method to prove the existence of global solutions and we use some properties of convex functions to study the decay of the energy.

- The **chapter 3**

  It is devoted to study the global existence and decay properties of solutions for the initial boundary value problem of viscoelastic non-degenerate Kirchhoff equation of the form

$$\begin{cases} |u_t|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds & \\ +\mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau(t))) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in } \Omega\times]0,\tau(0)[, \end{cases} \tag{11}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, $l > 0$, μ_1 and μ_2 are positive real numbers, h is a positive function which decays exponentially, $\tau(t) > 0$ is a time varying delay, g_1 and g_2 are two functions, and the initial data (u_0, u_1, f_0) are in a suitable function space. $M(r) = a + br^\gamma$ is a C^1-function for $r \geq 0$, with $a, b > 0$, and $\gamma \geq 1$.

We prove the existence of global solutions in suitable Sobolev spaces by using Faedo-Galarkin approximations together with some energy estimates under condition on the weight of the delay term in the feedback and the weight of the term without delay and the speed of delay. Furthermore, we study a general stability estimates by using some properties of convex functions.

Chapter 1

Preliminaries

In this chapter, we will introduce and state without proofs some important materials needed in the proof of our results (See [11, 29]),

1.1 Banach Spaces-Definition and properties

We first review some basic facts from calculus in the most important class of linear spaces " Banach spaces".

Definition 1.1.1. *. A Banach space is a complete normed linear space X. Its dual space X' is the linear space of all continuous linear functional $f : X \to \mathbb{R}$.*

Proposition 1.1.1. *X' equipped with the norm $\|.\|_{X'}$ defined by*

$$\|f\|_{X'} = \sup\{|f(u)| : \|u\| \leq 1\}, \tag{1.1}$$

is also a Banach space. We shall denote the value of $f \in X'$ at $u \in X$ by either $f(u)$ or $\langle f, u\rangle_{X',X}$.

Remark 1.1.1. *From X'we construct the bidual or second dual $X'' = (X')'$. Furthermore, with each $u \in X$ we can define $\varphi(u) \in X''$ by $\varphi(u)(f) = f(u), f \in X'$. This satisfies clearly $\|\varphi(x)\| \leq \|u\|$. Moreover, for each $u \in X$ there is an $f \in X'$ with $f(u) = \|u\|$ and $\|f\| = 1$. So it follows that $\|\varphi(x)\| = \|u\|$.*

Definition 1.1.2. . *Since φ is linear we see that*

$$\varphi : X \to X'',$$

is a linear isometry of X onto a closed subspace of X'', we denote this by

$$X \hookrightarrow X''.$$

Definition 1.1.3. . *If φ is onto X'' we say X is reflexive, $X \cong X''$.*

Theorem 1.1.1. . *Let X be Banach space. Then X is reflexive, if and only if,*

$$B_X = \{x \in X : \|x\| \leq 1\},$$

is compact with the weak topology $\sigma(X, X')$. (See the next subsection for the definition of $\sigma(X, X')$).

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Definition 1.1.4. . *Let X be a Banach space, and let $(u_n)_{n\in N}$ be a sequence in X. Then u_n converges strongly to u in X if and only if*

$$\lim \|u_n - u\|_X = 0,$$

and this is denoted by $u_n \to u$, or $\lim_{n\to\infty} u_n = u$.

Definition 1.1.5. *The Banach space E is said to be separable if there exists a countable subset D of E which is dense in E, i.e. $\overline{D} = E$.*

Proposition 1.1.2. *If E is reflexive and if F is a closed vector subspace of E, then F is reflexive.*

Corollaire 1.1.1. *The following two assertions are equivalent:*
(i) E is reflexive;
(ii) E' is reflexive.

1.1.1 The weak and weak star topologies

Let X be a Banach space and $f \in X'$. Denote by

$$\begin{aligned} \varphi_f : X &\longrightarrow \mathbb{R} \\ x &\longmapsto \varphi_f(x), \end{aligned}$$

when f cover X', we obtain a family $(\varphi_f)_{f\in X'}$ of applications to X in $\mathbb{R}$.

Definition 1.1.6. *The weak topology on X, denoted by $\sigma(X, X')$, is the weakest topology on X for which every $(\varphi_f)_{f\in X'}$ is continuous.*

We will define the third topology on X', the weak star topology, denoted by $\sigma(X', X)$. For all $x \in X$. Denote by

$$\begin{aligned} \varphi_f : X' &\longrightarrow \mathbb{R} \\ f &\longmapsto \varphi_x(f) = \langle f, x\rangle_{X',X}, \end{aligned}$$

when x cover X, we obtain a family $(\varphi_x)_{x\in X'}$ of applications to X' in $\mathbb{R}$.

Definition 1.1.7. *. The weak star topology on X' is the weakest topology on X' for which every $(\varphi_x)_{x\in X'}$ is continuous.*

Remark 1.1.2. *Since $X \subset X''$, it is clear that, the weak star topology $\sigma(X', X)$ is weakest then the topology $\sigma(X', X'')$, and this later is weakest then the strong topology.*

Definition 1.1.8. *A sequence (u_n) in X is weakly convergent to x if and only if*

$$\lim_{n\to\infty} f(u_n) = f(u),$$

for every $f \in X'$, and this is denoted by $u_n \rightharpoonup u$

Remark 1.1.3. *1. If the weak limit exist, it is unique.*

2. If $u_n \to u \in X(strongly)$ then $u_n \rightharpoonup u(weakly)$.

3. If $dim X < +\infty$, then the weak convergent implies the strong convergent.

Proposition 1.1.3. *On the compactness in the three topologies in the Banach space X:*

1. *First, the unit ball*
$$B' \equiv \{x \in X : \|x\| \leq 1\}, \tag{1.2}$$
in X is compact if and only if $dim(X) < \infty$.

2. *Second, the unit ball B' in X'(The closed subspace of a product of compact spaces) is weakly compact in X' if and only if X is reflexive.*

3. *Third, B' is always weakly star compact in the weak star topology of X'.*

Proposition 1.1.4. *Let (f_n) be a sequence in X'. We have:*

1. $[f_n \rightharpoonup^* f \text{ in } \sigma(X', X)] \Leftrightarrow [f_n(x) \rightharpoonup^* f(x),\ \forall x \in X]$.

2. *If $f_n \to f$(strongly) then $f_n \rightharpoonup f$, in $\sigma(X', X'')$,*
If $f_n \rightharpoonup f$ in $\sigma(X', X'')$, then $f_n \rightharpoonup^* f$, in $\sigma(X', X)$.

3. *If $f_n \rightharpoonup^* f$ in $\sigma(X', X)$ then $\|f_n\|$ is bounded and $\|f\| \leq \liminf \|f_n\|$.*

4. *If $f_n \rightharpoonup^* f$ in $\sigma(X', X)$ and $x_n \to x$(strongly) in X, then $f_n(x_n) \to f(x)$.*

1.1.2 Hilbert spaces

Now, we give some important results on these spaces here.

Definition 1.1.9. *A Hilbert space H is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.*

Theorem 1.1.2. *(Riesz). If $(H; \langle ., .\rangle)$ is a Hilbert space, $\langle ., .\rangle$ being a scalar product on H, then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $x \in H$ such that $f = \langle x, .\rangle$ and $\|f\|'_H = \|x\|_H$.*

Remark 1.1.4. *: From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.*

Theorem 1.1.3. *. Let $(u_n)_{n\in N}$ is a bounded sequence in the Hilbert space H, it posses a subsequence which converges in the weak topology of H.*

Theorem 1.1.4. *. In the Hilbert space, all sequence which converges in the weak topology is bounded.*

Theorem 1.1.5. *. Let $(u_n)_{n\in N}$ be a sequence which converges to u, in the weak topology and $(\upsilon_n)_{n\in N}$ is an other sequence which converge weakly to υ, then*

$$\lim_{n\to\infty} \langle \upsilon_n, u_n\rangle = \langle \upsilon, u\rangle \tag{1.3}$$

Theorem 1.1.6. *. Let X be a normed space, then the unit ball*

$$B' \equiv \{x \in X : \|x\| \leq 1\}, \tag{1.4}$$

of X' is compact in $\sigma(X', X)$.

1.2 Functional Spaces

1.2.1 The $L^p(\Omega)$ spaces

Definition 1.2.1. *Let $1 \leq p \leq \infty$ and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |f(x)|^p dx < \infty \right\}. \tag{1.5}$$

Notation 1.2.1. *If $p = \infty$, we have*

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \text{ is measurable and there exists a constant } C \text{ such that } |f(x)| \leq C \; a.e \in \Omega\}.$$

Also, we denote by

$$\|f\|_\infty = \inf\{C,\ |f(x)| \leq C \; a.e \in \Omega\}. \tag{1.6}$$

Notation 1.2.2. *For $p \in \mathbb{R}$ and $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e. $\frac{1}{p} + \frac{1}{q} = 1$.*

Theorem 1.2.1. *$L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$.*

Remark 1.2.1. *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx, \tag{1.7}$$

is a Hilbert space.

Theorem 1.2.2. *For $1 < p < \infty$, $L^p(\Omega)$ is a reflexive space.*

1.2.2 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Theorem 1.2.3. *(Holder's inequality). Let* $1 \leq p \leq \infty$*. Assume that* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$*, then* $fg \in L^p(\Omega)$ *and*

$$\int_\Omega |fg| dx \leq \|f\|_p \|g\|_q.$$

Lemma 1.2.1. *(Young's inequality). Let* $f \in L^p(\mathbb{R})$ *and* $g \in L^g(\mathbb{R})$ *with* $1 < p < \infty$ *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$*. Then* $f * g \in L^r(\mathbb{R})$ *and*

$$\|f * g\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}.$$

Lemma 1.2.2. *. Let* $1 \leq p \leq r \leq q$*,* $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$*, and* $1 \leq \alpha \leq 1$*. Then*

$$\|u\|_{L^r} \leq \|u\|_{L^p}^{\alpha} \|u\|_{L^q}^{1-\alpha}.$$

Lemma 1.2.3. *If* $\mu(\Omega) < \infty$*,* $1 \leq p \leq q \leq \infty$*, then* $L^q \hookrightarrow L^p$ *and*

$$\|u\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p} - \frac{1}{q}} \|u\|_{L^q}.$$

1.2.3 The $W^{m,p}(\Omega)$ spaces

Proposition 1.2.1. *Let* Ω *be an open domain in* $\mathbb{R}^N$*. Then the distribution* $T \in D'(\Omega)$ *is in* $L^p(\Omega)$ *if there exists a function* $f \in L^p(\Omega)$ *such that*

$$\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x) dx, \text{ for all } \varphi \in D(\Omega),$$

where $1 \leq p \leq \infty$ *and it's well-known that* f *is unique.*

Now, we will introduce the Sobolev spaces: The Sobolev space $W^{k,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order k have a finite L^p norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \left\{ f \in L^p(\Omega); D^\alpha f \in L^p(\Omega). \ \ \forall \alpha; |\alpha| \leq k \right\}.$$

With this definition, the Sobolev spaces admit a natural norm:

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} , \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^\infty(\Omega)} , \text{ for } p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\| \ . \ \|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)}.$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H_0^m(\Omega)$ (respectively $W_0^{k,p}(\Omega)$).

Now, we introduce a space of functions with values in a space X (a separable Hilbert space).

The space $L^2(a, b; X)$ is a Hilbert space for the inner product

$$(f, g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X \, dt$$

We note that $L^\infty(a, b; X) = (L^1(a, b; X))'$. Now, we define the Sobolev spaces with values in a Hilbert space X. For $k \in \ N$, $p \in [1, \infty]$, we set:

$$W^{k,p}(a, b; X) = \left\{ v \in L^p(a, b; X); \frac{\partial v}{\partial x_i} \in L^p(a, b; X). \ \ \forall i \leq k \right\} ,$$

The Sobolev space $W^{k,p}(a, b; X)$ is a Banach space with the norm

$$\|f\|_{W^{k,p}(a,b;X)} = \left(\sum_{i=0}^{k} \| \frac{\partial f}{\partial x_i} \|_{L^p(a,b;X)}^p \right)^{1/p} , \quad \text{for } p < +\infty$$

and

$$\|f\|_{W^{k,\infty}(a,b;X)} = \sum_{i=0}^{k} \| \frac{\partial v}{\partial x_i} \|_{L^\infty(a,b;X)} , \quad \text{for } p = +\infty$$

The spaces $W^{k,2}(a,b;X)$ form a Hilbert space and it is noted $H^k(0,T;X)$. The $H^k(0,T;X)$ inner product is defined by:

$$(u,v)_{H^k(a,b;X)} = \sum_{i=0}^{k} \int_a^b \left(\frac{\partial u}{\partial x^i}, \frac{\partial v}{\partial x^i} \right)_X dt \ .$$

Theorem 1.2.4. *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

where p^* is given by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ (where $p = n, p^* = \infty$). Moreover there exists a constant $C = C(p,n)$ such that

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corollaire 1.2.1. *Let $1 \leq p < n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad \forall q \in [p, p^*]$$

with continuous imbedding.

For the case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad \forall q \in [n, +\infty[$$

Theorem 1.2.5. *Let $p > n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

with continuous imbedding.

Corollaire 1.2.2. *Let* Ω *a bounded domain in* $\mathbb{R}^n$ *of* C^1 *class with* $\Gamma = \partial\Omega$ *and* $1 \leq p \leq \infty$. *We have*

$$if\ 1 \leq p < \infty, \ \ then\ W^{1,p}(\Omega) \subset L^{p^*}(\Omega)\ where\ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

$$if\ p = n, \ \ then\ W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[.$$

$$if\ p > n, \ \ then\ W^{1,p}(\Omega) \subset L^\infty(\Omega)$$

with continuous imbedding. Moreover, if $p > n$ *we have:*

$$\forall u \in W^{1,p}(\Omega), \quad |u(x) - u(y)| \leq C|x - y|^\alpha \|u\|_{W^{1,p}(\Omega)}\ a.e\ x, y \in \Omega$$

with $\alpha = 1 - \frac{n}{p} > 0$ *and* C *is a constant which depend on* p, n *and* Ω. *In particular* $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Corollaire 1.2.3. *Let* Ω *a bounded domain in* $\mathbb{R}^n$ *of* C^1 *class with* $\Gamma = \partial\Omega$ *and* $1 \leq p \leq \infty$. *We have*

$$if\ p < n,\ then\ W^{1,p}(\Omega) \subset L^q(\Omega) \forall q \in [1, p^*[\ where\ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

$$if\ p = n,\ then\ W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[.$$

$$if\ p > n, \ \ then\ W^{1,p}(\Omega) \subset C(\overline{\Omega})$$

with compact imbedding.

Remark 1.2.2. *We remark in particular that*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

with compact imbedding for $1 \leq p \leq \infty$ *and for* $p \leq q < p^*$.

Corollaire 1.2.4.

$$if\ \frac{1}{p} - \frac{m}{n} > 0,\ then\ W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)\ where\ \frac{1}{q} = \frac{1}{p} - \frac{m}{n}.$$

$$if\ \frac{1}{p} - \frac{m}{n} = 0,\ then\ W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [p, +\infty[.$$

$$if\ \frac{1}{p} - \frac{m}{n} < 0,\ then\ W^{m,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

with continuous imbedding.

Lemma 1.2.4. *(Sobolev-Poincarés inequality)*

$$\textit{If} \quad 2 \le q \le \frac{2n}{n-2}, n \ge 3 \quad \textit{and} \quad q \ge 2, n = 1, 2,$$

then

$$\|u\|_q \le C(q, \Omega)\|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega).$$

Remark 1.2.3. *For all $\varphi \in H^2(\Omega)$, $\Delta\varphi \in L^2(\Omega)$ and for Γ sufficiently smooth, we have*

$$\|\varphi(t)\|_{H^2(\Omega)} \le C\|\Delta\varphi(t)\|_{L^2(\Omega)}.$$

Proposition 1.2.2. *(Green's formula). For all $u \in H^2(\Omega)$, $\upsilon \in H^1(\Omega)$ we have*

$$-\int_\Omega \Delta u \upsilon dx = \int_\Omega \nabla u \nabla \upsilon dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \upsilon d\sigma,$$

where $\frac{\partial u}{\partial \eta}$ is a normal derivation of u at Γ.

1.2.4 The $L^p(0, T, X)$ spaces

Let X be a Banach space, denote by $L^p(0, T, X)$ the space of measurable functions

Definition 1.2.2.

$$\begin{aligned} f :]0, T[&\longrightarrow X \\ t &\longmapsto f(t) \end{aligned}$$

such that

$$\left(\int_0^T \|f(t)\|_X^p dt\right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, \textit{for } 1 \le p < \infty. \tag{1.8}$$

If $p = \infty$,

$$\|f\|_{L^p(0,T,X)} = \sup_{t \in]0,T[} ess\|f(t)\|_X. \tag{1.9}$$

Theorem 1.2.6. . *The space $L^p(0, T, X)$ is complete.*

We denote by $D'(0, T, X)$ the space of distributions in $]0, T[$ which take its values in X and let us define

$$D'(0, T, X) = \pounds(D]0, T[, X),$$

where $\pounds(\phi, \varphi)$ is the space of the linear continuous applications of ϕ to φ. Since $u \in D'(0, T, X)$, we define the distribution derivation as

$$\frac{\partial u}{\partial t}(\varphi) = -u\left(\frac{d\varphi}{dt}\right), \quad \forall \varphi \in D(]0, T[),$$

and since $u \in L^p(0, T, X)$, we have

$$u(\varphi) = \int_0^T u(t)\varphi(t)dt, \quad \forall \varphi \in D(]0, T[),$$

We will introduce some basic results on the $L^p(0, T, X)$ space. These results, will be very useful in the other chapters of this thesis.

Lemma 1.2.5. *. Let $f \in L^p(0, T, X)$ and $\frac{\partial f}{\partial t} \in L^p(0, T, X)$, $(1 \leq p \leq \infty)$, then the function f is continuous from $[0, T]$ to X. i.e. $f \in C^1(0, T, X)$.*

Lemma 1.2.6. *. Let $\varphi =]0, T[\times \Omega$ an open bounded domain in $\mathbb{R} \times \mathbb{R}^n$, and g_μ, g are two functions in $L^q(]0, T[, L^q(\Omega))$, $1 < q < \infty$ such that*

$$\|g_\mu\|_{L^q(]0,T[,L^q(\Omega))} \leq C, \forall \mu \in \mathbb{N} \tag{1.10}$$

and $g_\mu \to g \quad in \quad \varphi$, then $g_\mu \to g \quad in \quad L^q(\varphi)$.

Theorem 1.2.7. *. $L^p(0, T, X)$ equipped with the norm $\|.\|_{L^q(]0,T[,X)}$, $1 \leq p \leq \infty$ is a Banach space.*

Proposition 1.2.3. *. Let X be a reflexive Banach space, X' it's dual, and $1 \leq p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $L^p(0, T, X)$ is identify algebraically and topologically with $L^q(0, T, X')$.*

Proposition 1.2.4. *Let X, Y be Banach space, $X \subset Y$ with continuous embedding, then we have*

$$L^p(0, T, X) \subset L^p(0, T, Y),$$

with continuous embedding.

The following compactness criterion will be useful for nonlinear evolution problem, especially in the limit of the nonlinear terms.

Lemma 1.2.7. *(Aubin -Lions lemma) Let B_0, B, B_1 be Banach spaces with $B_0 \subset B \subset B_1$. Assume that the embedding $B_0 \hookrightarrow B$ is compact and $B \hookrightarrow B_1$ are continuous. Let $1 < p, q < \infty$. Assume further that B_0 and B_1 are reflexive. Define*

$$W \equiv \{u \in L^p(0, T, B_0) : u' \in L^q(0, T, B_1)\}. \tag{1.11}$$

Then, the embedding $W \hookrightarrow L^p(0, T, B)$ is compact.

1.2.5 Some Algebraic inequalities

Since our study based on some known algebraic inequalities, we want to recall few of them here.

Lemma 1.2.8. *(The Cauchy-Schwartz's inequality) Every inner product satisfies the Cauchy-Schwartz's inequality*

$$\langle x_1, x_2 \rangle \leq \|x_1\| \|x_2\|. \tag{1.12}$$

The equality sign holds if and only if x_1 and x_1 are dependent.

Lemma 1.2.9. *(Young's inequalities). For all $a, b \in \mathbb{R}^+$, we have*

$$ab \leq \alpha a^2 + \frac{1}{4\alpha} b^2 \tag{1.13}$$

where α is any positive constant.

Lemma 1.2.10. *For $a, b \geq 0$, the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{1.14}$$

where, $\frac{1}{p} + \frac{1}{q} = 1$.

1.3 Existence Methods

1.3.1 Faedo-Galerkin's approximations

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle ., . \rangle$ and the associated norm $\|.\|$

.

$$\begin{cases} u''(t) + A(t)u(t) = f(t) & \text{t in } [0,T], \\ u(x,0) = u_0(x),\ u'(x,0) = u_1(x), \end{cases} \tag{1.15}$$

where u and f are unknown and given function, respectively, mapping the closed interval $[0,T] \subset \mathbb{R}$ into a real separable Hilbert space H. $A(t)$ $(0 \le t \le T\ )$ are linear bounded operators in H acting in the energy space $V \subset H$.

Assume that $\langle A(t)u(t), v(t)\rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t; ., .)$ is a bilinear continuous in V. The problem (1.15) can be formulated as: Found the solution $u(t)$ such that

$$\begin{cases} u \in C([0,T]; V), u' \in C([0,T]; H) \\ \langle u''(t), v\rangle + a(t; u(t), v) = \langle f, v\rangle & \text{t in } D'(]0,T[), \\ u_0 \in V\ ,\ u_1 \in H, \end{cases} \tag{1.16}$$

This problem can be resolved with the approximation process of Fadeo-Galerkin.

Let V_m a sub-space of V with the finite dimension d_m, and let $\{w_{jm}\}$ one basis of V_m such that .

1. $V_m \subset V(\dim V_m < \infty), \forall m \in \mathbb{N}$
2. $V_m \to V$ such that, there exist a dense subspace ϑ in V and for all $\upsilon \in \vartheta$ we can get sequence $\{u_m\}_{m\in\mathbb{N}} \in V_m$ and $u_m \to u$ in V.
3. $V_m \subset V_{m+1}$ and $\overline{\cup_{m\in\mathbb{N}} V_m} = V$.

we define the solution u_m of the approximate problem

$$\begin{cases} u_m(t) = \sum_{j=1}^{d_m} g_j(t) w_{jm}, \\ u_m \in C([0,T]; V_m), u'_m \in C([0,T]; V_m),\ u_m \in L^2(0,T; V_m) \\ \langle u''_m(t), w_{jm}\rangle + a(t; u_m(t), w_{jm}) = \langle f, w_{jm}\rangle,\ 1 \le j \le d_m \\ u_m(0) = \sum_{j=1}^{d_m} \xi_j(t) w_{jm}, u'_m(0) = \sum_{j=1}^{d_m} \eta_j(t) w_{jm}, \end{cases} \tag{1.17}$$

where

$$\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \longrightarrow u_0 \text{ in V as } m \longrightarrow \infty$$

$$\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \longrightarrow u_1 \text{ in V as } m \longrightarrow \infty$$

By virtue of the theory of ordinary differential equations, the system (1.17) has unique local solution which is extend to a maximal interval $[0, t_m[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$ to obtain one solution defined for all $t > 0$.

1.3.2 A priori estimation and convergence

Using the following estimation

$$\|u_m\|^2 + \|u_m'\|^2 \leq C \left\{ \|u_m(0)\|^2 + \|u_m'(0)\|^2 + \int_0^T \|f(s)\|^2 ds \right\};\ 0 \leq t \leq T$$

and the Gronwall lemma we deduce that the solution u_m of the approximate problem (1.17) converges to the solution u of the initial problem (1.15). The uniqueness proves that u is the solution.

1.3.3 Gronwall's lemma

Lemma 1.3.1. *Let $T > 0$, $g \in L^1(0, T)$, $g \geq 0$ a.e and c_1, c_2 are positives constants. Let $\varphi \in L^1(0, T)$ $\varphi \geq 0$ a.e such that $g\varphi \in L^1(0, T)$ and*

$$\varphi(t) \leq c_1 + c_2 \int_0^t g(s)\varphi(s) ds \quad a.e \ in \ (0, T).$$

then, we have

$$\varphi(t) \leq c_1 exp \left(c_2 \int_0^t g(s) ds \right) \quad a.e \ in \ (0, T).$$

Chapter 2

Energy decay for degenerate Kirchhoff equation with weakly nonlinear dissipation and delay term

In collaboration with Salim A. Messaoudi

2.1 Introduction

In this paper we consider the initial boundary value problem for the nonlinear Kirchhoff equation

$$\begin{cases} (|u_t|^{l-2}u_t)_t - \left(\int_\Omega |\nabla u|^2\,dx\right)^\gamma \Delta u + \mu_1 g(u_t(x,t)) + \mu_2 g(u_t(x,t-\tau)) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau[, \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$,, $n \in \mathbb{N}^*$, $l \geq 2$, $\gamma \geq 0$, are given constants, μ_1 and μ_2 are positive real numbers, $\tau > 0$ is a time delay and (u^0, u^1, f_0) are in a suitable space.

In the absence of delay, (2.1) takes the form

$$(|u_t|^{l-2}u_t)_t - \Big(\int_\Omega |\nabla u|^2\,dx\Big)^\gamma \Delta u + \alpha(t)g(u_t(x,t)) = 0, \tag{2.2}$$

where α is a positive and non increasing differentiable function. Equation (2.2) has been studied by Abdelli and Messaoudi **[4]**, they established an explicit and general decay rate result by using some properties of convex functions. In **[3]** Abdelli and Benaissa treated (2.2) for g having a polynomial growth near the origin and established energy decay results depending on α and g under suitable relationship between l and γ.

When $l = 2$ and $\gamma = 0$, problem (2.2) was treated by Mustafa and Massaoudi **[40]**, where they studied the decay property of the energy of (2.2) and used the same method as in **[4]**.

Time delay is the property of a physical system by which the response to an applied force is delayed in its effect. Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. Time delays so often arise in many physical, chemical, biological, and economical phenomena. In recent years, the control of PDEs with time delay effects has become an active area of research, see, for example **[50]** and the references therein. In **[19]**, the authors showed that a small delay in a boundary control could turn a well-behave hyperbolic system into a wild one and therefore, delay becomes a source of instability. However, sometimes it can also improve the performance of the system.

Benaissa and Louhibi **[9]** studied the problem (2.1), with $l = 2$ and $\gamma = 0$, and proved the existence and uniqueness of a global solution with initial data $(u_0, u_1, f_0) \in H^2(\Omega)\cap H_0^1(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega, H_0^1(0,1))$. They used the Faedo-Galerkin method and the multiplier method and some properties of convex functions to study the decay of the energy.

In this article, we use some technique from Mustafa and Massaoudi **[40]** to establish an explicit and general decay result. The proof is based on the multiplier method and makes use of some properties of convex functions, the general Young inequality and Jensen's inequality. These convexity arguments were introduced and developed by Lasiecka et al., **[25]**, **[27]**, **[28]** and used, with appropriate modications, by Liu and Zuazua **[30]**, Alabau-Boussouira **[5]** and others.

The plan of this paper is as follows: In section 2, we give some hypotheses. In section 3, we state and prove the energy estimates.

2.2 Assumptions

To state and prove our result, we need some assumptions.
We shall use the embedding

$$H_0^1(\Omega) \hookrightarrow L^k(\Omega), \text{ if } \begin{cases} 2 \le k \le \frac{2n}{n-2} & \text{if } n \ge 3 \\ 2 \le k \le +\infty & \text{if } n = 1, 2 \end{cases} \tag{2.3}$$

with the same embedding constant denoted by C_* ; i.e

$$\|u\|_k \le C_* \|\nabla u\|_2.$$

(A1) $g : \mathbb{R} \to \mathbb{R}$ is non decreasing function of class C^1 and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying

$$\begin{cases} H(0) = 0 \text{ and } H \text{ is linear on } [0, \varepsilon] \text{ or} \\ H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \varepsilon] \text{ such that} \\ c_1 |s|^{l-1} \le |g(s)| \le c_2 |s|^p \quad \text{if } |s| \ge \varepsilon \\ |s|^l + |g(s)|^{\frac{l}{l-1}} \le H^{-1}(sg(s)) \quad \text{if } |s| \le \varepsilon \end{cases} \tag{2.4}$$

where H^{-1} denotes the inverse function of H and p, ε, c_1, c_2 are positive constants, with p satisfying

$$\begin{cases} l - 1 \le p \le \frac{n+2}{n-2} & \text{if } n > 2 \\ l - 1 \le p < \infty & \text{if } n \le 2, \end{cases}$$

and

$$\begin{cases} \beta_1 sg(s) \le G(s) \le \beta_2 sg(s) \\ G(s) = \int_0^s g(r) dr, \end{cases} \tag{2.5}$$

where β_1 and β_2 are positive constants.
(A2) We also assume that

$$\beta_2 \mu_2 < \beta_1 \mu_1.$$

Remark 2.2.1. *Let us denote by* Φ^* *the conjugate function of the differentiable convex function* Φ, *i.e.,*

$$\Phi^*(s) = \sup_{t\in\mathbb{R}^+}(st - \Phi(t)).$$

Then Φ^* *is the Legendre transform of* Φ, *which is given by (see Arnold* **[7]**, *p. 61 -62)*

$$\Phi^*(s) = s(\Phi')^{-1}(s) - \Phi[(\Phi')^{-1}(s)], \quad \textit{if} \;\; s \in (0, \Phi'(r)],$$

and Φ^* *satisfies the generalized Young inequality*

$$AB \leq \Phi^*(A) + \Phi(B), \quad \textit{if} \;\; A \in (0, \Phi'(r)] \;\; B \in (0, r]. \tag{2.6}$$

We introduce, as in **[32]**, the new variable

$$z(x, \rho, t) = u_t(x, t - \rho\tau), \;\; x \in \Omega, \quad \rho \in (0, 1), t > 0. \tag{2.7}$$

Then, we have

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \;\text{ in } \Omega \times (0, 1) \times (0, +\infty). \tag{2.8}$$

Therefore, problem (2.1) is equivalent to

$$\begin{cases} (|u_t|^{l-2}u_t)' - \left(\int_\Omega |\nabla u|^2\, dx\right)^\gamma \Delta u & \\ +\mu_1 g(u_t(x,t)) + \mu_2 g(z(x,1,t)) = 0 & \text{in } \Omega\times]0, +\infty[, \\ \tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, & \text{in } \Omega\times]0, 1[\times]0, +\infty[\\ u(x, t) = 0, & \text{on } \partial\Omega \times [0, \infty[\\ z(x, 0, t) = u_t(x, t), & \text{on } \Omega \times [0, \infty[\\ u(x, 0) = u^0(x), u_t(x, 0) = u^1(x), & \text{in } \Omega \\ z(x, \rho, 0) = f_0(x, -\rho\tau), & \text{in } \Omega\times]0, 1[. \end{cases} \tag{2.9}$$

2.3 Uniform Decay of the Energy

In this section we study the asymptotic behavior of the solution of system (2.1). We define the energy associated to the solution of system (2.9) by

$$E(t) = \frac{l-1}{l}\|u_t\|_l^l + \frac{1}{2(1+\gamma)}\|\nabla_x u\|_2^{2(\gamma+1)} + \xi \int_\Omega \int_0^1 G(z(x, \rho, t)\, d\rho\, dx, \tag{2.10}$$

where ξ is a positive constant such that

$$\tau\frac{\mu_2-\beta_1\mu_2}{\beta_1}<\xi<\tau\frac{\mu_1-\beta_2\mu_2}{\beta_2}. \tag{2.11}$$

Theorem 2.3.1. *Assume That* **(A1)**, **(A2)** *hold and* $l\geq 2(\gamma+1)$. *Then, there exist positive constants* $w_1,\ w_2,\ w_3$ *and* ε_0 *such that the solution energy of (2.9) satisfies*

$$E(t)\leq w_3H_1^{-1}(w_1t+w_2)\quad \forall t\geq 0, \tag{2.12}$$

where

$$H_1(t)=\int_t^1\frac{1}{H_2(s)}\,ds\qquad \text{and}\quad H_2(t)=tH'(\varepsilon_0 t). \tag{2.13}$$

Here, H_1 *is strictly decreasing and convex on* $(0,1]$ *with* $\lim_{t\to 0}H_1(t)=+\infty$.

The proof of this result relies on the following simple Lemma

Lemma 2.3.1. *Let (u,z) be a solution of the problem (2.9). Then, the energy functional, defined by (2.10), satisfies*

$$E'(t)\leq-\alpha_1\int_\Omega u_tg(u_t)\,dx\ -\alpha_2\int_\Omega z(x,1,t)g(z(x,1,t)\,dx\ \leq 0, \tag{2.14}$$

where $\alpha_1=\mu_1-\mu_2\beta_2-\frac{\xi\beta_2}{\tau}$ *and* $\alpha_2=\frac{\xi}{\tau}\beta_1-\mu_2(1-\beta_1)$.

Proof. Multiplying the first equation in (2.9) by u_t and integrating over Ω, using integration by parts, we get

$$\begin{aligned}\frac{d}{dt}\Big(\frac{l-1}{l}\|u_t\|_l^l+\frac{1}{2(1+\gamma)}\|\nabla_x u\|_2^{2(\gamma+1)}\Big)&+\mu_1\int_\Omega u_t(x,t)g(u_t(x,t))\,dx\\ &+\mu_2\int_\Omega u_t(x,t)g(z(x,1,t))\,dx=0.\end{aligned} \tag{2.15}$$

We multiply the second equation in (2.9) by $\xi g(z)$ and integrate the result over $\Omega\times(0,1)$, to obtain

$$\begin{aligned}\xi\int_\Omega\int_0^1 z_t(x,\rho,t)g(z(x,\rho,t))\,d\rho\,dx&=-\frac{\xi}{\tau}\int_\Omega\int_0^1 z_\rho(x,\rho,t)g(z(x,\rho,t)\,d\rho\,dx\\ &=-\frac{\xi}{\tau}\int_\Omega\int_0^1\frac{\partial z(x,\rho,t)}{\partial\rho}\,g(z(x,\rho,t)\,d\rho\,dx\\ &=-\frac{\xi}{\tau}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}\,G(z(x,\rho,t)\,d\rho\,dx\\ &=-\frac{\xi}{\tau}\int_\Omega(G(z(x,1,t)-G(z(x,0,t))\,dx.\end{aligned}$$

Then, recalling (2.5), we have

$$\xi \frac{d}{dt}\int_{\Omega}\int_{0}^{1} G(z(x,\rho,t))\,d\rho\,dx = -\frac{\xi}{\tau}\int_{\Omega} G(z(x,1,t)\,dx + \frac{\xi}{\tau}\int_{\Omega} G(u_t(x,t))\,dx. \quad (2.16)$$

Combining (2.15) and (2.16), we obtain

$$\begin{aligned} E'(t) = &-\frac{\xi}{\tau}\int_{\Omega} G(z(x,1,t)\,dx + \frac{\xi}{\tau}\int_{\Omega} G(u_t(x,t))\,dx \\ &- \mu_1 \int_{\Omega} u_t(x,t) g(u_t(x,t))\,dx - \mu_2 \int_{\Omega} u_t(x,t) g(z(x,1,t))\,dx. \end{aligned} \quad (2.17)$$

From (2.5), we obtain

$$\begin{aligned} E'(t) \leq &-(\mu_1 - \frac{\xi\beta_2}{\tau})\int_{\Omega} u_t(x,t) g(u_t(x,t))\,dx - \frac{\xi}{\tau}\int_{\Omega} G(z(x,1,t))\,dx \\ &- \mu_2 \int_{\Omega} u_t(x,t) g(z(x,1,t))\,dx. \end{aligned} \quad (2.18)$$

Using (2.5) and Remark 2.2.1, we obtain

$$G^*(s) = s g^{-1}(s) - G(g^{-1}(s)), \quad \forall s \geq 0.$$

Hence

$$\begin{aligned} G^*(g(z(x,1,t))) &= z(x,1,t) g(z(x,1,t)) - G(z(x,1,t)) \\ &\leq (1-\beta_1) z(x,1,t) g(z(x,1,t)). \end{aligned} \quad (2.19)$$

Using (2.5), (2.6) with $A = g(z(x,1,t))$ and $B = u_t(x,t)$, we have from (2.18) that

$$\begin{aligned} E'(t) \leq &-(\mu_1 - \frac{\xi\beta_2}{\tau})\int_{\Omega} u_t(x,t) g(u_t(x,t))\,dx - \frac{\xi\beta_1}{\tau}\int_{\Omega} z(x,1,t) g(z(x,1,t))\,dx \\ &- \mu_2 \int_{\Omega} u_t(x,t) g(z(x,1,t))\,dx \\ &\leq -(\mu_1 - \frac{\xi\beta_2}{\tau})\int_{\Omega} u_t(x,t) g(u_t(x,t))\,dx - \frac{\xi\beta_1}{\tau}\int_{\Omega} z(x,1,t) g(z(x,1,t))\,dx \\ &+ \mu_2 \int_{\Omega} (G(u_t(x,t)) + G^*(g(z(x,1,t)))\,dx \\ &\leq -(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\beta_2)\int_{\Omega} u_t(x,t) g(u_t(x,t))\,dx - \frac{\xi\beta_1}{\tau}\int_{\Omega} z(x,1,t) g(z(x,1,t))\,dx \\ &+ \mu_2 \int_{\Omega} G^*(g(z(x,1,t))\,dx \\ &\leq -(\mu_1 - \frac{\xi\beta_2}{\tau} - \mu_2\beta_2)\int_{\Omega} u_t(x,t) g(u_t(x,t))\,dx \\ &- (\frac{\xi\beta_1}{\tau} - \mu_2(1-\beta_1))\int_{\Omega} z(x,1,t) g(z(x,1,t))\,dx. \end{aligned} \quad (2.20)$$

Then, by using (2.11), estimate (2.14) follows. □

To prove our main result, we construct a Lyapunov functional F equivalent to E. For this, we define some functionals which allow us to obtain the desired estimate.

Lemma 2.3.2. *Let (u,z) be a solution of the problem (2.9). Then, the functional*

$$I(t) = \int_\Omega \int_0^1 e^{-2\tau\rho} G(z(x,\rho,t)\, d\rho\, dx, \tag{2.21}$$

satisfies the estimate

$$\begin{aligned} I'(t) \leq -2e^{-2\tau} \int_\Omega \int_0^1 G(z(x,\rho,t))\, d\rho\, dx - \frac{e^{-2\tau}}{\tau}\beta_1 \int_\Omega z(x,1,t) g(z(x,1,t))\, dx \\ + \frac{\beta_2}{\tau} \int_\Omega u_t(x,t) g(u_t(x,t))\, dx. \end{aligned} \tag{2.22}$$

Proof. Differentiating (2.21) with respect to t and using (2.4), (2.8) we get

$$\begin{aligned} I'(t) &= \int_\Omega \int_0^1 e^{-2\tau\rho} \frac{d}{dt} G(z(x,\rho,t))\, d\rho\, dx \\ &= \int_\Omega \int_0^1 e^{-2\tau\rho} z_t(x,\rho,t) g(z(x,\rho,t))\, d\rho\, dx \\ &= \frac{-1}{\tau} \int_\Omega \int_0^1 e^{-2\tau\rho} z_\rho(x,\rho,t) g(z(x,\rho,t))\, d\rho\, dx \\ &= \frac{-1}{\tau} \int_\Omega \int_0^1 e^{-2\tau\rho} \frac{d}{d\rho} G(z(x,\rho,t))\, d\rho\, dx \\ &= \frac{-1}{\tau} \int_\Omega \int_0^1 \Big[\frac{d}{d\rho}\Big(e^{-2\tau\rho} G(z(x,\rho,t))\Big) + 2\tau e^{-2\tau\rho} G(z(x,\rho,t))\Big]\, d\rho\, dx \\ &= \frac{-1}{\tau} \int_\Omega \Big[e^{-2\tau} G(z(x,1,t)) - G(u_t(x,t))\Big]\, dx - 2\int_\Omega \int_0^1 e^{-2\tau\rho} G(z(x,\rho,t))\, d\rho\, dx \\ &= -2I(t) - \frac{e^{-2\tau}}{\tau} \int_\Omega G(z(x,1,t))\, dx + \frac{1}{\tau} \int_\Omega G(u_t(x,t))\, dx \\ &\leq -2I(t) - \frac{e^{-2\tau}}{\tau}\beta_1 \int_\Omega z(x,1,t) g(z(x,1,t))\, dx + \frac{\beta_2}{\tau} \int_\Omega u_t(x,t) g(u_t(x,t))\, dx. \end{aligned} \tag{2.23}$$

Since $e^{-2\tau\rho}$ is a decreasing function for $\rho \in [0,1]$, we deduce

$$I(t) \geq \int_\Omega \int_0^1 e^{-2\tau} G(z(x,\rho,t))\, d\rho\, dx.$$

Thus, our proof is completed. □

Now, for $M > 0$, we introduce the following functional

$$F(t) = ME(t) + I(t) + \int_\Omega u|u_t|^{l-2}u_t \, dx. \tag{2.24}$$

Lemma 2.3.3. *Let (u,z) be a solution of the problem (2.9). Assume that* **(A1)** *and* **(A2)** *hold and that* $l \geq 2(\gamma+1)$*. Then* $F(t)$ *satisfies, along the solution and for some positive constants* $m, c > 0$*, the following estimate*

$$F'(t) \leq -mE(t) + c\int_\Omega (|u_t|^l + |ug(u_t)| + |ug(z(x,1,t))|)\, dx, \tag{2.25}$$

and $F(t) \sim E(t)$*.*

Proof. By taking the time derivative of (2.24), we get

$$\begin{aligned} F'(t) &= ME'(t) + I'(t) + \int_\Omega |u_t|^l \, dx + \int_\Omega u(|u_t|^{l-2}u_t)_t \, dx \\ &= ME'(t) + I'(t) + \int_\Omega |u_t|^l \, dx \ + \Big(\int_\Omega |\nabla u|^2 \, dx\Big)^\gamma \int_\Omega u\Delta u \, dx \\ &- \mu_1 \int_\Omega ug(u_t(x,t))\, dx - \mu_2 \int_\Omega ug(z(x,1,t))\, dx. \end{aligned} \tag{2.26}$$

Recalling Lemmas 2.3.1 and 2.3.2, we have

$$\begin{aligned} F'(t) \leq &-\left(M\alpha_1 - \frac{\beta_2}{\tau}\right)\int_\Omega u_t(x,t)g(u_t(x,t))\, dx \\ &- \left(M\alpha_2 + \frac{\beta_1 e^{-2\tau}}{\tau}\right)\int_\Omega z(x,1,t)g(z(x,1,t))\, dx + \int_\Omega |u_t|^l \, dx - \mu_1 \int_\Omega ug_1(u_t(x,t))\, dx \\ &- \mu_2 \int_\Omega ug(z(x,1,t))\, dx - \|\nabla u\|_2^{2(\gamma+1)} - 2e^{-2\tau}\int_\Omega \int_0^1 G(z(x,\rho,t)\, d\rho \, dx. \end{aligned} \tag{2.27}$$

Now, let us choose M sufficiently large such that

$$\left(M\alpha_1 - \frac{\beta_2}{\tau}\right) > 0.$$

Thus (2.27) becomes

$$\begin{aligned} F'(t) \leq &\int_\Omega |u_t|^l \, dx - \mu_1 \int_\Omega u_1 g(u_t(x,t))\, dx - \mu_2 \int_\Omega ug(z(x,1,t))\, dx \\ &- \frac{1}{(1+\gamma)}(1+\gamma)\|\nabla u\|_2^{2(\gamma+1)} - \frac{2e^{-2\tau}}{\xi}\xi \int_\Omega \int_0^1 G(z(x,\rho,t)\, d\rho \, dx. \end{aligned}$$

Noting by $m = \min\{1+\gamma, \frac{e^{-2\tau}}{\xi}\}$, we obtain that

$$F'(t) \le \Big(1 + m\frac{l-1}{l}\Big)\|u_t\|_l^l - \mu_1 \int_\Omega u g(u_t(x,t))\, dx - \mu_2 \int_\Omega u g(z(x,1,t))\, dx - mE(t).$$

Hence

$$F'(t) \le -mE(t) + c\int_\Omega (|u_t|^l + |ug(u_t(x,t))| + |ug(z(x,1,t))|)\, dx.$$

To prove $F(t) \sim E(t)$, we show that there exist two positive constants λ_1 and λ_2 such that

$$\lambda_1 E(t) \le F(t) \le \lambda_2 E(t). \tag{2.28}$$

We use (2.10) and (2.3) with $k = l$ and Young's inequalities with exponents $\frac{l}{l-1}$ and l, we get

$$\begin{aligned}
\Big|\int_\Omega u|u_t|^{l-2}u_t\, dx\Big| &\le C_\varepsilon \int_\Omega |u|^l\, dx + \varepsilon \int_\Omega |u_t|^l\, dx \\
&\le C_\varepsilon \|\nabla u\|_2^l + \varepsilon \|u_t\|_l^l \\
&\le C_\varepsilon E^{\frac{l}{2(\gamma+1)}}(t) + c\varepsilon E(t) \\
&\le C_\varepsilon E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(t)E(t) + c\varepsilon E(t).
\end{aligned}$$

By noting that $l \ge 2(\gamma+1)$, we have

$$\Big|\int_\Omega u|u_t|^{l-2}u_t\, dx\Big| \le C_\varepsilon E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(0)E(t) + c\varepsilon E(t).$$

Also,

$$\begin{aligned}
|I(t)| &\le \int_\Omega \int_0^1 e^{-2\tau\rho} G(z(x,\rho,t)\, d\rho\, dx \\
&\le \int_\Omega \int_0^1 G(z(x,\rho,t)\, d\rho\, dx \le \frac{1}{\xi}E(t).
\end{aligned}$$

Therefore, we arrive at

$$\Big(M - C_\varepsilon E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(0) - (c\varepsilon + \frac{1}{\xi})\Big)E(t) \le F(t) \le \Big(M + C_\varepsilon E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(0) + (c\varepsilon + \frac{1}{\xi})\Big)E(t),$$

So, we can choose M large enough so that $\lambda_1 = M - (C_\varepsilon E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(0) + (c\varepsilon + \frac{1}{\xi})) > 0$ and $M\alpha_1 - \frac{\beta_2}{\tau} > 0$. Then (2.25) and (2.28) hold true. □

2.3.1 Proof of Theorem 2.3.1

As in Komornik **[24]**, we consider the following partition of Ω

$$\Omega_1 = \{x \in \Omega : \ |u_t| \leq \varepsilon\}, \quad \Omega_2 = \{x \in \Omega : \ |u_t| > \varepsilon\},$$

and

$$\Omega_1^* = \{x \in \Omega : \ |z(x,1,t)| \leq \varepsilon\}, \quad \Omega_2^* = \{x \in \Omega : \ |z(x,1,t)| > \varepsilon\}.$$

We use (2.3), with $k = p+1$, (2.4), (2.10) and Hölder's inequality, we get

$$\begin{aligned}
\int_{\Omega_2} |ug(u_t)|\, dx &\leq \Big(\int_{\Omega_2} |u|^{p+1}\, dx\Big)^{\frac{1}{p+1}} \Big(\int_{\Omega_2} |g(u_t)|^{1+\frac{1}{p}}\, dx\Big)^{\frac{p}{p+1}} \\
&\leq C_* \|\nabla u\|_2 \Big(\int_{\Omega_2} |g(u_t)|^{1+\frac{1}{p}}\, dx\Big)^{\frac{p}{p+1}} \\
&\leq C_* \|\nabla u\|_2 \Big(\int_{\Omega_2} u_t g(u_t)\, dx\Big)^{\frac{p}{p+1}} \\
&\leq c E^{\frac{1}{2(\gamma+1)}}(t) (-E'(t))^{\frac{p}{p+1}}.
\end{aligned}$$

Then, we use Young's inequality and recall the fact that $p \geq l-1 \geq 2\gamma+1$, we have, for any $\delta > 0$,

$$\begin{aligned}
\int_{\Omega_2} |ug(u_t)|\, dx &\leq c\delta E^{\frac{p+1}{2(\gamma+1)}}(t) + C_\delta(-E'(t)) \\
&\leq c\delta E^{\frac{p+1}{2(\gamma+1)}}(t) - C_\delta E'(t) \\
&\leq c\delta E^{\frac{p-(2\gamma+1)}{2(\gamma+1)}}(0) E(t) - C_\delta E'(t).
\end{aligned} \tag{2.29}$$

and

$$\int_{\Omega_2^*} |ug(z(x,1,t))|\, dx \leq c\delta E^{\frac{p-(2\gamma+1)}{2(\gamma+1)}}(0) E(t) - C_\delta E'(t). \tag{2.30}$$

So (2.4), (2.29) and (2.30) yield

$$\begin{aligned}
&\int_{\Omega_2} [|u_t|^l + |ug(u_t)|]\, dx + \int_{\Omega_2^*} |ug(z(x,1,t))|\, dx \\
&\leq c \int_{\Omega_2} u_t g(u_t)\, dx + c\delta E^{\frac{p-(2\gamma+1)}{2(\gamma+1)}}(0) E(t) - C_\delta E'(t) \\
&\leq -cE'(t) + c\delta E^{\frac{p-(2\gamma+1)}{2(\gamma+1)}}(0) E(t) - C_\delta E'(t) \\
&\leq c\delta E^{\frac{p-(2\gamma+1)}{2(\gamma+1)}}(0) E(t) - C_\delta E'(t).
\end{aligned} \tag{2.31}$$

Exploiting (2.10) and Young's inequality, we get, for any $\delta > 0$,

$$\begin{aligned}\int_{\Omega_1} [|u_t|^l + |ug(u_t)|]\,dx &\le \int_{\Omega_1} |u_t|^l\,dx + \delta \int_{\Omega_1} |u|^l\,dx + C_\delta \int_{\Omega_1} |g(u_t)|^{\frac{l}{l-1}}\,dx \\ &\le \int_{\Omega_1} |u_t|^l\,dx + c\delta E^{\frac{l}{2(\gamma+1)}}(t) + C_\delta \int_{\Omega_1} |g(u_t)|^{\frac{l}{l-1}}\,dx,\end{aligned} \tag{2.32}$$

and

$$\int_{\Omega_1^*} |ug(z(x,1,t))|\,dx \le c\delta E^{\frac{l}{2(\gamma+1)}}(t) + C_\delta \int_{\Omega_1^*} |g(z(x,1,t))|^{\frac{l}{l-1}}\,dx. \tag{2.33}$$

Now, for δ small enough, the function $L(t) = F(t) + C_\delta E(t)$ satisfies

$$\begin{aligned}L'(t) \le &\Big(-m + c\delta E^{\frac{p-(2\gamma+1)}{2(\gamma+1)}}(0) + c\delta E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(0)\Big)E(t) + C_\delta \int_{\Omega_1^*} |g(z(x,1,t))|^{\frac{l}{l-1}}\,dx \\ &+ \int_{\Omega_1} |u_t|^l\,dx + C_\delta \int_{\Omega_1} |g(u_t)|^{\frac{l}{l-1}}\,dx \\ &\le -dE(t) + c\int_{\Omega_1} \Big(|u_t|^l + |g(u_t)|^{\frac{l}{l-1}}\Big)\,dx + C\int_{\Omega_1^*} |g(z(x,1,t))|^{\frac{l}{l-1}}\,dx \\ &\le -dE(t) + c\int_{\Omega_1} \Big(|u_t|^l + |g(u_t)|^{\frac{l}{l-1}}\Big)\,dx + C\int_{\Omega_1^*} \Big(|z(x,1,t)|^l + |g(z(x,1,t))|^{\frac{l}{l-1}}\Big)\,dx,\end{aligned} \tag{2.34}$$

and

$$L(t) \sim E(t).$$

- Case 1. H is linear on $[0,\varepsilon]$, Using (2.4) and Lemma 2.3.1, we deduce that

$$\begin{aligned}L'(t) &\le -dE(t) + c\Big[\alpha_1 \int_{\Omega_1} u_t(x,t)g_1(u_t(x,t))\,dx + \alpha_2 \int_{\Omega_1^*} z(x,1,t)g(z(x,1,t))dx\Big] \\ &\le -dE(t) - cE'(t).\end{aligned}$$

Thus $R = L + cE$ satisfies

$$R'(t) \le -dE(t) \le -c'R(t).$$

So,

$$R(t) \le R(0)e^{-c't}.$$

Hence,

$$E(t) \le C(E(0))e^{-c't} = C(E(0))H_1^{-1}(-c't).$$

- Case 2. H is nonlinear on $[0,\varepsilon]$ so, we define

$$I_1(t) = \frac{1}{|\Omega_1|}\int_{\Omega_1} u_t g(u_t)\,dx,$$

and

$$I_2(t) = \frac{1}{|\Omega_1^*|}\int_{\Omega_1^*} z(x,1,t)g(z(x,1,t))\,dx,$$

and exploit Jensen's inequality and the concavity of H^{-1} to obtain

$$H^{-1}(I_1(t)) \geq c\int_{\Omega_1} H^{-1}(u_t g(u_t))\,dx,$$

and

$$H^{-1}(I_2(t)) \geq c\int_{\Omega_1^*} H^{-1}(z(x,1,t)g(z(x,1,t)))\,dx.$$

Using (2.4), we obtain

$$\int_{\Omega_1}\Big(|u_t|^l + |g(u_t)|^{\frac{l}{l-1}}\Big)\,dx \leq c\int_{\Omega_1} H^{-1}(u_t g(u_t))\,dx \leq cH^{-1}(I_1(t)), \tag{2.35}$$

and

$$\begin{aligned}\int_{\Omega_1^*} |g(z(x,1,t))|^{\frac{l}{l-1}}\,dx &\leq \int_{\Omega_1^*}\Big(|z(x,1,t)|^l + |g(z(x,1,t))|^{\frac{l}{l-1}}\Big)\,dx \\ &\leq \int_{\Omega_1^*} H^{-1}(z(x,1,t)g(z(x,1,t))\,dx \\ &\leq cH^{-1}(I_2(t)).\end{aligned} \tag{2.36}$$

For $\varepsilon_0 < \varepsilon$ and $w_0 > 0$, we define L_0 by

$$L_0(t) = H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)L(t) + w_0 E(t).$$

Then, we easily see that, For $a_1,\ a_2 > 0$

$$a_1 L_0(t) \leq E(t) \leq a_2 L_0(t), \tag{2.37}$$

By recalling that $E' \leq 0$, $H' > 0$, $H'' > 0$ on $(0,\varepsilon]$ and using (2.34), (2.35) and (2.36), we obtain

$$\begin{aligned}L_0'(t) &= \varepsilon_0\frac{E'(t)}{E(0)}H''\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)L(t) + H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)L'(t) + w_0E'(t) \\ &\leq -dE(t)H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) + cH'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)H^{-1}(I_1(t)) \\ &+ cH'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)H^{-1}(I_2(t)) + w_0E'(t),\end{aligned}$$

using Remark 2.2.1 with H^*, the convex conjugate of H in the sense of Young, we obtain

$$\begin{aligned} L_0'(t) &\le -dE(t)H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) + cH^*\Big(H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)\Big) + cI_1(t) + cI_2(t) + w_0E'(t) \\ &\le -dE(t)H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) + c\varepsilon_0\frac{E(t)}{E(0)}H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) - w_1E'(t) + w_0E'(t), \end{aligned}$$

where w_1 is a positive constant depending of Ω_1, Ω_1^*, α_1, α_2. By taking ε_0 small enough and $w_0 > w_1$, we obtain

$$L_0'(t) \le -w\frac{E(t)}{E(0)}H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) = -wH_2\Big(\frac{E(t)}{E(0)}\Big), \tag{2.38}$$

where $H_2(t) = tH'(\varepsilon_0 t)$. Since

$$H_2'(t) = H'(\varepsilon_0 t) + \varepsilon_0 tH''(\varepsilon_0 t),$$

and H is convex on $(0,\varepsilon]$, we find that $H_2'(t) > 0$ and $H_2(t) > 0$ on $(0,1]$. By setting $L_1(t) = \frac{a_1L_0(t)}{E(0)}$, we easily see that, by (2.37), we have

$$L_1(t) \sim E(t). \tag{2.39}$$

Using (2.38), we arrive at

$$L_1'(t) \le -w_1H_2(L_1(t)).$$

By recalling (2.13), we deduce $H_2(t) = \frac{-1}{H_1'(t)}$, hence

$$L_1'(t) \le w_1\frac{1}{H_1'(L_1(t))},$$

which gives

$$\Big[H_1(L_1(t))\Big]' = L_1'(t)H_1'(L_1(t)) \le w_1.$$

A simple integration leads to

$$H_1(L_1(t)) \le w_1t + H_1(L_1(0)).$$

Consequently

$$L_1(t) \le H_1^{-1}(w_1t + w_2). \tag{2.40}$$

Using (2.39) and (2.40), we obtain (2.12). The proof is completed.

Remark 2.3.1. *If g satisfies*

$$g_0(|s|) \leq |g(s)| \leq g_0^{-1}(|s|) \qquad \text{for all } |s| \leq \varepsilon, \tag{2.41}$$

and

$$c_1|s|^{l-1} \leq |g(s)| \leq c_2|s|^p \qquad \text{for all } |s| \geq \varepsilon, \tag{2.42}$$

(This kind of hypotheses, with $\varepsilon = 1$, l=2 and p=1, was considered by Liu and Zuazua [30], and Alabau-Boussouira [5]) for a function $g_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is convex and strictly increasing of the class $C^1(\mathbb{R}_+) \cap C^2(]0,\infty))$ with $g_0(0) = 0$, and for positive constants c_1, c_2, ε, then the condition (A1) is satisfied for $H(s) = s^{\frac{1}{l}} g_0(s^{\frac{1}{l}})$.

2.3.2 Exemples

We give some examples to illustrate the energy decay rates given by Theorem 2.3.1. Here we assume that g satisfies (2.42) near the origin with the following various examples for g_0 :

Example 2.3.1. *If $g_0(s) = s^p(-\ln s)^q$, where $p \geq 1$ and $q \geq 0$. Then $H(s) = cs^{\frac{p+1}{l}}(-\ln s^{\frac{1}{l}})^q$ satisfies (A1) in a neighborhood of zero. Then, using Theorem 2.3.1, we have*

$$H'(s) = cs^{\frac{p+1-l}{l}}(-\ln s^{\frac{1}{l}})^{q-1}\Big(\frac{p+1}{l}(-\ln s^{\frac{1}{l}}) - \frac{q}{l}\Big),$$

with $\varepsilon_0 = 1$, we have

$$H_2(s) = cs^{\frac{p+1}{l}}(-\ln s^{\frac{1}{l}})^{q-1}\Big(\frac{p+1}{l}(-\ln s^{\frac{1}{l}}) - \frac{q}{l}\Big),$$

and

$$H_1(t) = \int_t^1 \frac{1}{cs^{\frac{p+1}{l}}(-\ln s^{\frac{1}{l}})^{q-1}\Big(\frac{p+1}{l}(-\ln s^{\frac{1}{l}}) - \frac{q}{l}\Big)}\, ds.$$

Setting $z = s^{-1/l}$, then

$$H_1(t) = c\int_1^{t^{-\frac{1}{l}}} \frac{z^{p-l}}{(\ln z)^{q-1}\Big(\frac{p+1}{l}(\ln z) - \frac{q}{l}\Big)}\, dz, \quad \text{when } t \text{ is near 0.}$$

We have tree cases:

- *Case 1. If* $p = l - 1$ *and* $q = 1$

$$H_1(t) = c\ln(-\ln t), \quad \textit{near zero},$$

we deduce that

$$H_1^{-1}(t) = e^{-e^{c't}},$$

then

$$E(t) \leq ce^{-e^{c't}}.$$

- *Case 2. If* $p = l - 1$ *and* $q < 1$. *Then*

$$H_1(t) = c'(-\ln t)^{1-q}.$$

Hence

$$H_1^{-1}(t) = e^{-ct^{\frac{1}{1-q}}},$$

then

$$E(t) \leq ce^{-c't^{\frac{1}{1-q}}}.$$

- *Case 3. If* $\frac{p+1}{l} > 1 \Rightarrow p > l - 1$.
Applying Lemma (6.1) (see **[10]***), we obtain*

$$H_1(t) \sim c'' \frac{1}{t^{\frac{p-1+1}{l}}(-\ln t)^q} \quad as \quad t \to 0,$$

we deduce that

$$H_1^{-1}(t) \sim ct^{-\frac{l}{p-l+1}}(\ln t)^{-\frac{ql}{p-l+1}}.$$

Then

$$E(t) \leq ct^{-\frac{l}{p-l+1}}(\ln t)^{-\frac{ql}{p-l+1}}.$$

Example 2.3.2. *If* $g_0(s) = e^{-\frac{1}{s}}$*, then* $H(s) = s^{\frac{1}{l}} e^{-s^{-l}}$ *satisfies (A1). We have*

$$\int_0^t s^{\frac{1}{l}} e^{-s^{-l}} ds = \int_{\frac{1}{t}}^{+\infty} \frac{1}{z^{\frac{1}{l}+2}} e^{-z^l} dz.$$

Applying Lemma (6.1)(see **[10]***), we obtain*

$$H_2(t) = ct^{\frac{l^2+1}{l}+1} e^{-t^{-l}}.$$

Also, we have

$$H_1(t) = c \int_t^1 \frac{e^{s^{-l}}}{s^{\frac{l^2+1}{l}+1}} ds = c \int_1^{\frac{1}{t}} z^{\frac{l^2+1}{l}-1} e^{z^l} dz,$$

where we use the following change of variable $s = \dfrac{1}{z}$*. Applying Lemma (6.1) (see* **[10]***), we obtain*

$$H_1(t) \sim ct^{-\frac{1}{l}} e^{t^{-l}}, \quad as \quad t \to 0,$$

and we deduce that

$$H_1^{-1}(t) \sim (\ln t)^{-\frac{1}{l}}.$$

Then

$$E(t) \leq c \left(\ln\left(c't + c''\right)\right)^{-l}.$$

where Ω *is a bounded domain in* $\mathbb{R}^n$*.*

Chapter 3

Existence of global solutions and decay estimates for a viscoelastic Petrovsky equation with a delay term in the non-linear internal feedback

In collaboration with Amira Rachah

3.1 Introduction

3.1.1 The model

In this article we consider the existence and decay properties of global solutions for the initial boundary value problem of viscoelastic Petrovsky equation

$$\begin{cases} |u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-s)\Delta^2 u(s)\, ds & \\ +\mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau[, \end{cases} \tag{3.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, $\partial\Omega$ is a smooth boundary, $l > 0$, μ_1 and μ_2 are positive real numbers, h is a positive non-increasing function defined on

$\mathbb{R}^+$, g_1 and g_2 are two functions, $\tau > 0$ is a time delay and (u_0, u_1, f_0) are the initial data in a suitable function space. Cavalcanti et al. **[14]** studied the following nonlinear viscoelastic problem with strong damping

$$|u_t|^l u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds - \gamma \Delta u_t = 0, \;\; x \in \Omega, \;\; t > 0. \tag{3.2}$$

Under the assumptions $0 < l \leq \frac{2}{n-2}$ if $n \geq 3$ or $l > 0$ if $n = 1, 2$ and h decays exponentially, they obtained the global existence of weak solutions for $\gamma \geq 0$ and the uniform exponential decay rates of the energy for $\gamma > 0$. In the case of $\gamma = 0$ when a source term competes with the dissipation induced by the viscoelastic term, Messaoudi and Tatar **[36]** studied the equation

$$|u_t|^l u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds + b|u|^{p-2}u = 0, \;\; x \in \Omega, \;\; t > 0.$$

They used the potential well method to show that the damping induced by the viscoelastic term is enough to ensure global existence and uniform decay of solutions provided that the initial data are in some stable set. Han and Wang **[20]**, investigated a related problem with linear damping

$$|u_t|^l u_{tt} - \Delta u - \Delta u_{tt} - \int_0^t h(t-s)\Delta u(s)\,ds + u_t = 0, \;\; x \in \Omega, \;\; t > 0.$$

Using the Faedo-Galerkin method, they showed the global existence of weak solutions and obtained uniform exponential decay of solutions by introducing a perturbed energy functional. Recently, these results have been extended by Wu **[47]** to a general case where a source term and a nonlinear damping term are present. In the presence of the source term, problem (3.2) has been discussed by many authors, and related results concerning local or global existence, asymptotic behavior and blow-up of solution have been recently established (see **[6]**, **[31]**, **[37]**).
Park and Kang **[41]** studied the following nonlinear viscoelastic problem with damping

$$|u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + u_t = 0, \;\; x \in \Omega, \;\; t > 0.$$

Santos et al. **[43]** considered the existence and uniform decay for the following nonlinear beam equation in a non-cylindrical domain:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|_2^2)\Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + \alpha u_t = 0, \;\; \text{in } \widehat{Q},$$

where $\widehat{Q} = \bigcup_{0 \leq t \leq \infty} \Omega_t \times \{t\}$.

Benaissa, Benguessoum and Messaoudi **[8]** proved the existence of global solution, as well as, a general stability result for the following equation

$$u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)\, ds + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0, \quad x \in \Omega, \ t > 0. \tag{3.3}$$

when h is decays at a certain rate. In the absence of the viscoelastic term (i.e. if $h \equiv 0$), problem (3.3) has been studied by many authors. It is well known that in the further absence of a damping mechanism, the delay term causes instability of the system (see, for instance, Datko et al., **[19]**). On the contrary, in the absence of the delay term, the damping term assures global existence for arbitrary initial data and energy decay is estimated depending on the rate of growth of g_1 (see Alabau-Boussouira, **[5]**, Benaissa and Guesmia **[10]**, Haraux **[21]**, Komornik **[24]**, Lasiecka and Tataru **[26]**). Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see Shinskey **[45]**). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. Time delays so often arise in many physical, chemical, biological, and economical phenomena. In recent years, the control of PDEs with time delay effects has become an active area of research (see Abdallah et al **[2]**, Suh and Bien **[46]** and Zhong **[?]**). To stabilise a hyperbolic system involving delay terms, additional control terms are necessary (see Nicaise and Pignotti **[32]**, Nicaise and Pignotti **[33]**, Xu et al., **[48]**). In Nicaise and Pignotti **[32]**, the authors examined the problem (P) in the linear situation (i.e. if $g_1(s) = g_2(s) = s$ for all $s \in \mathbb{R}$) and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they found a sequence of delays for which the corresponding solution of (3.3) will be instable if $\mu_2 \geq \mu_1$. The main approach used in Nicaise and Pignotti (2006) is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay were acting in the boundary domain. We also recall the result by Xu et al. **[48]** , where the authors proved the same result as in Nicaise and Pignotti (2006) for the one space dimension by adopting the spectral analysis approach. Very recently, Benaissa and

Louhibi **[9]** extended the result of Nicaise and Pignotti **[32]** to the non-linear case. Datko et al. In **[19]** showed that a small delay in a boundary control could turn such well-behave hyperbolic system into a wild one and therefore, delay becomes a source of instability. However, sometimes it can also improve the performance of the systems (see Suh and Bien **[46]**).

The main purpose of this paper is to prove global solvability and energy decay estimates of the solutions of problem (3.1) when h is of exponential decay rate and $g_1,\ g_2$ are non-linear. We would like to see the influence of frictional and viscoelastic damping on the rate of decay of solutions in the presence of non-linear degenerate delay term. Of course, the most interesting case occurs when we have delay term and simultaneous and complementary damping mechanisms.

To obtain global solutions of problem (3.1), we use the Galerkin approximation scheme (see Lions **[29]**) together with the energy estimate method. The technique based on the theory of non-linear semi-groups used in Nicaise and Pignotti **[32]** does not seem to be applicable in the non-linear case. To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by Cavalcanti et al. **[13]**, Daoulatli et al. **[16]**, Lasiecka and Doundykov **[27]** and Lasiecka and Tataru **[26]** , and used by Liu and Zuazua **[30]**, Eller et al. **[18]** and Alabau-Boussouira **[5]**.

3.1.2 Statement of results

We use the Sobolev spaces $H^4(\Omega),\ H_0^2(\Omega)$ and the Hilbert space $L^p(\Omega)$ with their usual scalar products and norms.

The prime $'$ and the subscript t will denote time differentiation and we denote by $(.,.)$ the inner product in $L^2(\Omega)$.

The constant C denotes a general positive constant, which may be different in different estimates.

Now we introduce, as in the work of in Nicaise and Pignotti **[32]**, the new variable

$$z(x,\rho,t) = u_t(x,t-\tau\rho),\ \ x\in\Omega,\quad \rho\in(0,1), t>0.$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \text{ in } \Omega \times (0,1) \times (0,+\infty). \tag{3.4}$$

Therefore, the problem (3.1) is equivalent to

$$\begin{cases} |u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-s)\Delta^2 u(s)\,ds & \\ +\mu_1 g_1(u_t(x,t)) + \mu_2 g_2(z(x,1,t)) = 0 & \text{in } \Omega\times]0,+\infty[, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & \text{in } \Omega\times]0,1[\times]0,+\infty[\\ u(x,t) = 0, & \text{on } \partial\Omega \times [0,\infty[\\ z(x,0,t) = u_t(x,t), & \text{on } \Omega \times [0,\infty[\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega \\ z(x,\rho,0) = f_0(x,-\rho\tau), & \text{in } \Omega\times]0,1[. \end{cases} \tag{3.5}$$

To state and prove our result, we need some assumptions.

(A1) Assume that l satisfies

$$\begin{cases} 0 < l \le \frac{2}{n-2} & \text{if } n \ge 3 \\ 0 < l < \infty & \text{if } n = 1,2. \end{cases}$$

(A2) $g_1 : \mathbb{R} \to \mathbb{R}$ is non decreasing function of class C^1 and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and of class $C^1(\mathbb{R}_+) \cap C^2(]0,+\infty[)$ satisfying

$$\begin{cases} H(0) = 0 \text{ and } H \text{ is linear on } [0,\varepsilon] \text{ or} \\ H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0,\varepsilon] \text{ such that} \\ |g_1(s)| \le c_2|s| \quad \text{if } |s| \ge \varepsilon \\ g_1^2(s) \le H^{-1}(sg_1(s)) \quad \text{if } |s| \le \varepsilon, \end{cases} \tag{3.6}$$

where H^{-1} denotes the inverse function of H and ε, c_2 are positive constants. $g_2 : \mathbb{R} \to \mathbb{R}$ is an odd no decreasing function of class $C^1(\mathbb{R})$ such that there exist $c_3, \alpha_1, \alpha_2 > 0$,

$$|g_2'(s)| \le c_3, \tag{3.7}$$

$$\alpha_1 s g_2(s) \le G(s) \le \alpha_2 s g_1(s), \tag{3.8}$$

where

$$G(s) = \int_0^s g_2(r)dr,$$

(A3)

$$\alpha_2\mu_2 < \alpha_1\mu_1.$$

(A4) For the relaxation function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^1 function such that

$$\int_0^\infty h(s)\,ds = \beta < 1, \tag{3.9}$$

and we assume that there exist a positive constant ζ verifying

$$h'(t) \le -\zeta h(t). \tag{3.10}$$

We define the energy associated to the solution of system (3.5) by

$$\begin{aligned} E(t) = & \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{1}{2}\Big(1 - \int_0^t h(s)\,ds\Big)\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}(h\circ\Delta u)(t) \\ & + \xi\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx, \end{aligned} \tag{3.11}$$

where ξ is a positive constant such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1} < \xi < \tau\frac{\mu_1 - \alpha_2\mu_2}{\alpha_2},$$

and

$$(h\circ v)(t) = \int_0^t h(t-s)\|v(t) - v(s)\|_2^2\,ds.$$

Now we have the existence of a Global solution

Theorem 3.1.1. *Let $u_0 \in H^4(\Omega)\cap H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega)$ and $f_0 \in H_0^2(\Omega, H^2(0,1))$ satisfy the compatibility condition*

$$f(.,0) = u_1.$$

Assume that **(A1)**-**(A4)** *hold. Then the problem (3.1) admits a weak solution*

$$u \in L^\infty([0,\infty); H^4(\Omega)\cap H_0^2(\Omega)),\ u_t \in L^\infty([0,\infty); H_0^2(\Omega)), u_{tt} \in L^2([0,\infty);\ H_0^1(\Omega)).$$

Also we have a uniform decay rates for the energy.

Theorem 3.1.2. *Assume that* **(A1)**-**(A4)** *hold. Then, there exist a positive constants* $w_1,\ w_2,\ w_3$ *and* ε_0 *such that the solution energy of (3.1) satisfies*

$$E(t) \leq w_3 H_1^{-1}(w_1 t + w_2) \quad \forall t > 0,$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\, ds \tag{3.12}$$

$$H_2(t) = \begin{cases} t & \text{If } H \text{ is linear on } [0,\varepsilon] \\ tH'(\varepsilon_0 t) & \text{If } H'(0) = 0 \text{ and } H'' > 0, \text{ on }]0,\varepsilon], \end{cases}$$

here, H_1 *is strictly decreasing and convex on* $(0,1]$ *with* $\lim_{t\to 0} H_1(t) = +\infty$.

3.2 Preliminaries

Let λ_1 be the first eigenvalue of the spectral Dirichlet problem

$$\Delta^2 u = \lambda_1 u, \text{ in } \Omega, \quad u = \frac{\partial u}{\partial \eta} = 0 \text{ in } \Gamma,$$

$$\|\nabla u\|_2 \leq \frac{1}{\sqrt{\lambda_1}} \|\Delta u\|_2. \tag{3.13}$$

Next we have a Sobolev-Poincarès inequality **[1]**.

Lemma 3.2.1. *Let* q *be a number with*

$$2 \leq q < +\infty (n = 1, 2) \text{ or } 2 \leq q \leq 2n/(n-2)(n \geq 3),$$

then there exists a constant $C_s = C_s(\Omega, q)$ *such that*

$$\|u\|_q \leq C_s \|\nabla u\|_2 \quad for \quad u \in H_0^1(\Omega).$$

Lemma 3.2.2. *[41] For* $h, \Psi \in C^1([0,+\infty[, \mathbb{R})$ *we have*

$$\int_\Omega h * \Psi \Psi_t\, dx = -\frac{1}{2} h(t)\|\Psi(t)\|_2^2 + \frac{1}{2}(h' \circ \Psi)(t) - \frac{1}{2}\frac{d}{dt}\Big[(h \circ \Psi)(t) - \Big(\int_0^t h(s)\, ds\Big)\|\Psi\|_2^2\Big].$$

Remark 3.2.1. *Let us denote by Φ^* the conjugate function of the differentiable convex function Φ, i.e.,*

$$\Phi^*(s) = \sup_{t\in\mathbb{R}^+}(st - \Phi(t)).$$

Then Φ^* is the Legendre transform of Φ, which is given by (see Arnold **[7]***, p. 61 -62)*

$$\Phi^*(s) = s(\Phi')^{-1}(s) - \Phi[(\Phi')^{-1}(s)], \quad \textit{if} \;\; s \in (0, \Phi'(r)],$$

and Φ^* satisfies the generalized Young inequality

$$AB \leq \Phi^*(A) + \Phi(B), \quad \textit{if} \;\; A \in (0, \Phi'(r)] \;\; B \in (0, r]. \tag{3.14}$$

Lemma 3.2.3. *Let (u,z) be a solution of the problem (3.5). Then, the energy functional defined by (3.11) satisfies*

$$\begin{aligned} E'(t) \leq & -\beta_1 \int_\Omega u_t g_1(u_t)\, dx - \beta_2 \int_\Omega z(x,1,t) g_2(z(x,1,t))\, dx \\ & - \frac{1}{2} h(t)\|\Delta u(t)\|^2 + \frac{1}{2}(h' \circ \Delta u)(t) \leq 0, \end{aligned} \tag{3.15}$$

where $\beta_1 = \mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2$ and $\beta_2 = \frac{\xi\alpha_1}{\tau} - \mu_2(1-\alpha_1)$.

Proof. By multiplying the first equation in (3.5) by u_t, integrating over Ω and using integration by parts, we obtain

$$\begin{aligned} \frac{d}{dt}\Big[\frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2\Big] + \mu_1 \int_\Omega u_t(x,t) g_1(u_t(x,t))\, dx \\ + \mu_2 \int_\Omega u_t(x,t) g_2(z(x,1,t))\, dx = \int_\Omega \int_0^t h(t-s)\Delta u(s)\Delta u_t(t)\, ds\, dx. \end{aligned} \tag{3.16}$$

By applying the Lemma 3.2.2, the term on the right-hand side of (3.16) can be rewritten as

$$\begin{aligned} \int_\Omega \int_0^t & h(t-s)\Delta u(s)\Delta u_t(t)\, ds\, dx + \frac{1}{2} h(t)\|\Delta u(t)\|_2^2 \\ & = \frac{1}{2}\frac{d}{dt}\Big[\int_0^t h(s)\, ds \|\Delta u(t)\|_2^2 - (h \circ \Delta u)(t)\Big] + \frac{1}{2}(h' \circ \Delta u)(t). \end{aligned}$$

Consequently,(3.16) becomes

$$\begin{aligned}&\frac{d}{dt}\Big[\frac{1}{l+2}\|u_t\|_{l+2}^{l+2}+\frac{1}{2}\Big(1-\int_0^t h(s)\,ds\Big)\|\Delta u\|_2^2+\frac{1}{2}\|\nabla u_t\|_2^2+\frac{1}{2}(h\circ\Delta u)(t)\Big]\\&=-\mu_1\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx-\mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\,dx\\&-\frac{1}{2}h(t)\|\Delta u(t)\|_2^2+\frac{1}{2}(h'\circ\Delta u)(t).\end{aligned}\tag{3.17}$$

We multiply the second equation in (3.5) by $\xi g_2(z)$, we integrate the result over $\Omega\times(0,1)$, to obtain

$$\begin{aligned}\xi\int_\Omega\int_0^1 z_t(x,\rho,t)g_2(z(x,\rho,t))\,d\rho\,dx&=-\frac{\xi}{\tau}\int_\Omega\int_0^1 z_\rho(x,\rho,t)g_2(z(x,\rho,t))\,d\rho\,dx\\&=-\frac{\xi}{\tau}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}\Big(G(z(x,\rho,t))\Big)\,d\rho\,dx\\&=-\frac{\xi}{\tau}\int_\Omega(G(z(x,1,t))-G(z(x,0,t)))\,dx.\end{aligned}$$

Hence

$$\xi\frac{d}{dt}\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx=-\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx+\frac{\xi}{\tau}\int_\Omega G(u_t(x,t))\,dx.\tag{3.18}$$

By combining (3.17) and (3.18), we obtain

$$\begin{aligned}E'(t)=&-\frac{1}{2}h(t)\|\Delta u(t)\|_2^2+\frac{1}{2}(h'\circ\Delta u)(t)-\mu_1\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx\\&-\mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\,dx-\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx+\frac{\xi}{\tau}\int_\Omega G(u_t(x,t))\,dx,\end{aligned}$$

and by recalling (3.8), we obtain

$$\begin{aligned}E'(t)\le&-(\mu_1-\frac{\xi\alpha_2}{\tau})\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx-\frac{1}{2}h(t)\|\Delta u(t)\|_2^2+\frac{1}{2}(h'\circ\Delta u)(t)\\&-\mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\,dx-\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx.\end{aligned}\tag{3.19}$$

From the definition of G and by using the remark 3.2.1, we obtain

$$G^*(s)=sg_2^{-1}(s)-G(g_2^{-1}(s)),\quad\forall s\ge0.$$

Hence

$$\begin{aligned}G^*(g_2(z(x,1,t)))&=z(x,1,t)g_2(z(x,1,t))-G(z(x,1,t))\\&\le(1-\alpha_1)z(x,1,t)g_2(z(x,1,t)).\end{aligned}$$

By using of (3.8) and (3.14) with $A = g_2(z(x,1,t))$ and $B = u_t(x,t)$, we obtain from (3.19)

$$\begin{aligned}
E'(t) &\le -\Big(\mu_1 - \frac{\xi\alpha_2}{\tau}\Big)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx - \frac{1}{2}h(t)\|\Delta u(t)\|_2^2 + \frac{1}{2}(h'\circ\Delta u)(t)\\
&\quad + \mu_2\int_\Omega (G(u_t(x,t)) + G^*(g_2(z(x,1,t)))\,dx - \frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx\\
&\quad \le -\Big(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\Big)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx\\
&\quad - \Big(\frac{\xi\alpha_1}{\tau} - \mu_2(1-\alpha_1)\Big)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx\\
&\quad - \frac{1}{2}h(t)\|\Delta u(t)\|_2^2 + \frac{1}{2}(h'\circ\Delta u)(t) \le 0.
\end{aligned}$$

This completes the proof. □

3.3 Proofs of main results

3.3.1 Proof of Theorem 3.1.1

Throughout this section we assume $u_0 \in H^4(\Omega)\cap H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega)$ and $f_0 \in H_0^2(\Omega, H^2(0,1))$. We will use the Faedo-Galerkin method to prove the existence of global solution. Let $T > 0$ be fixed and let $\{w^k\}, k \in \mathbb{N}$ be a basis of $H_0^2(\Omega)$, V_k the space generated by $w^1, w^2 ....w^k$. Now, we define, for $1 \le j \le k$, the sequence $\phi^j(x,\rho)$ as follows:

$$\phi^j(x,0) = w^j.$$

Then, we may extend $\phi^j(x,0)$ by $\phi^j(x,\rho)$ over $L^2(\Omega\times(0,1))$ such that $(\phi^j)_j$ forms a base of $L^2(\Omega, H^2(0,1))$ and denote Z_k the space generated by $\{\phi^k\}$. We construct approximate solutions (u^k, z^k), $k = 1,2,3,...,$ in the form

$$u^k(t) = \sum_{j=1}^{k} c^{jk}(t)w^j(x),$$

$$z^k(t) = \sum_{j=1}^{k} d^{jk}(t)\phi^j,$$

where c^{jk} and $d^{jk}(j=1,2,...,k)$ are determined by the ordinary differential equations

$$\begin{cases} (|u_t^k(t)|^l u_{tt}^k(t), w^j) + (\Delta_x u^k(t), \Delta_x w^j) + (\nabla_x u_{tt}^k, \nabla_x w^j) \\ -\int_0^t h(t-s)(\Delta u^k(s), \Delta w^j)\, ds + \mu_1(g_1(u_t^k), w^j) + \mu_2(g_2(z^k(.,1)), w^j) = 0, \\ z^k(x,0,t) = u_t^k(x,t), \end{cases} \tag{3.20}$$

$$u^k(0) = u_0^k = \sum_{j=1}^{k}(u_0, w^j)w^j \to u_0, \text{ in } H^4(\Omega)\cap H_0^2(\Omega) \text{ as } k \to +\infty, \tag{3.21}$$

$$u_t^k(0) = u_1^k = \sum_{j=1}^{k}(u_1, w^j)w^j \to u_1, \text{ in } H_0^2(\Omega) \text{ as } k \to +\infty, \tag{3.22}$$

and

$$\begin{cases} (\tau z_t^k + z_\rho^k, \phi^j) = 0, \\ 1 \le j \le k, \end{cases} \tag{3.23}$$

$$z^k(\rho,0) = z_0^k = \sum_{j=1}^{k}(f_0, \phi^j)\phi^j \to f_0, \text{ in } H_0^2(\Omega, H^2(0,1)) \text{ as } k \to +\infty. \tag{3.24}$$

Since $0 < l \le \frac{2}{n-2}$ if $n \ge 3$, according to Sobolev embeddings, we have

$$H_0^2(\Omega) \hookrightarrow L^{2(l+1)}(\Omega)$$

and the same occurs for $n = 1, 2$ where $l > 0$. Noting that $\frac{l}{2(l+1)} + \frac{1}{2(l+1)} + \frac{1}{2} = 1$, from the generalized Hölder inequality, the nonlinear term $(|u_t^k(t)|^l u_{tt}^k(t), w_j)$ in (3.20) makes sense. The standard theory of ODE guarantees that the system (3.20)-(3.24) has an unique solution in $[0, t_k)$, with $0 < t_k < T$, by Zorn lemma since the nonlinear terms in (3.20) are locally Lipschitz continuous. Note that $u^k(t)$ is of class $\mathcal{C}^2$.

In the next step, we obtain a priori estimates for the solution of the system (3.20)-(3.24), so that it can be extended outside $[0, t_k)$ to obtain one solution defined for all $t > 0$, using a standard compactness argument for the limiting procedure.

The first estimate. Since the sequences u_0^k, u_1^k and z_0^k converge and from Lemma 3.2.3, we can find a positive constant C_1 independent of k such that

$$\begin{aligned} E^k(t) - E^k(0) &\le -\beta_1 \int_0^t \int_\Omega u_t^k g_1(u_t^k)\, dx\, ds - \beta_2 \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\, dx\, ds \\ &\quad - \frac{1}{2}\int_0^t h(s)\|\Delta u^k(s)\|^2\, ds + \frac{1}{2}\int_0^t (h' \circ \Delta u^k)(s)\, ds \\ &\le -\beta_1 \int_0^t \int_\Omega u_t^k g_1(u_t^k)\, dx\, ds - \beta_2 \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\, dx\, ds. \end{aligned}$$

As h is a positive non increasing function, so we obtain

$$E^k(t) + \beta_1 \int_0^t \int_\Omega u_t^k g_1(u_t^k)\, dx\, ds + \beta_2 \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\, dx\, ds \le E^k(0) \le C_1, \tag{3.25}$$

where

$$E^k(t) = \frac{1}{l+2}\|u_t^k\|_{l+2}^{l+2} + \frac{1}{2}\Big(1 - \int_0^t h(s)\, ds\Big)\|\Delta u^k\|_2^2 + \frac{1}{2}\|\nabla u_t^k\|_2^2 + \frac{1}{2}(h \circ \Delta u^k)(t)$$
$$+ \xi \int_\Omega \int_0^1 G(z^k(x,\rho,t))\, d\rho\, dx,$$

and C_1 is a positive constant depending only on $\|u_0\|_{H_0^2}$ and $\|u_1\|_{H_0^1}$. Noting (3.9) and (3.25), we obtain the first estimate:

$$\begin{aligned} \|u_t^k\|_{l+2}^{l+2} + \|\Delta u^k\|_2^2 + \|\nabla u_t^k\|_2^2 + (h \circ \Delta u^k)(t) + \int_\Omega \int_0^1 G(z^k(x,\rho,t))\, d\rho \\ + \int_0^t \int_\Omega u_t^k g_1(u_t^k)\, dx\, ds + \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\, dx\, ds \le C_2, \end{aligned} \tag{3.26}$$

where C_2 is a positive constant depending only on $\|u_0\|_{H_0^2}$, $\|u_1\|_{H_0^1}$, l, β, ξ, β_1 and β_2 . These estimates imply that the solution (u^k, z^k) exists globally in $[0, +\infty)$.
Estimate (3.26) yields that

$$u^k \text{ is bounded in } L^\infty_{loc}(0,\infty,H_0^2(\Omega)), \tag{3.27}$$

$$u_t^k \text{ is bounded in } L^\infty_{loc}(0,\infty,H_0^1(\Omega)), \tag{3.28}$$

$$G(z^k(x,\rho,t)) \text{ is bounded in } L^\infty_{loc}(0,\infty,L^1(\Omega \times (0,1))), \tag{3.29}$$

$$u_t^k(t) g_1(u_t^k(t)) \text{ is bounded in } L^1(\Omega \times (0,T)), \tag{3.30}$$

$$z^k(x,1,t) g_2(z^k(x,1,t)) \text{ is bounded in } L^1(\Omega \times (0,T))). \tag{3.31}$$

The second estimate. Replacing w^j by $-\Delta_x w^j$ in (3.20), multiplying by c_t^{jk} and summing over j from 1 to k, it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big[\|\nabla \Delta u^k\|_2^2 + \|\Delta_x u_t^k\|_2^2\Big] - \int_\Omega |u_t^k(t)|^l u_{tt}^k(t) \Delta_x u_t^k\, dx \\ &- \int_0^t h(t-s) \int_\Omega \nabla \Delta u^k(s) \nabla \Delta u_t^k(s)\, dx\, ds + \mu_1 \int_\Omega |\nabla_x u_t^k|^2 g_1'(u_t^k)\, dx \\ &+ \mu_2 \int_\Omega \nabla_x u_t^k \nabla_x z^k(x,1,t) g_2'(z^k(x,1,t))\, dx = 0. \end{aligned} \tag{3.32}$$

Using the Green's formula, we have

$$-\int_\Omega |u_t^k(t)|^l u_{tt}^k(t)\Delta_x u_t^k\,dx = \frac{d}{dt}\int_\Omega |u_t^k(t)|^l|\nabla_x u_t^k|^2\,dx - (l+1)\int_\Omega |u_t^k|^l\nabla u_{tt}^k(t)\nabla_x u_t^k\,dx. \tag{3.33}$$

Replacing ϕ^j by $-\Delta_x\phi^j$ in (3.23), multiplying by d^{jk} and summing over j from 1 to k, it follows that

$$\tau\int_\Omega \nabla_x z_t^k\nabla_x z^k\,dx + \int_\Omega \nabla_x z_\rho^k\nabla_x z^k\,dx = 0.$$

Then, we obtain

$$\frac{\tau}{2}\frac{d}{dt}\|\nabla z^k\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|\nabla_x z^k\|_2^2 = 0.$$

We integrate over $(0,1)$ to find

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1 \|\nabla_x z^k(x,\rho,t)\|_2^2\,d\rho + \frac{1}{2}\|\nabla_x z^k(x,1,t)\|_2^2 - \frac{1}{2}\|\nabla_x u_t^k(t)\|_2^2 = 0. \tag{3.34}$$

Combining (3.32)-(3.34) and using Lemma 3.2.2, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\Big(1-\int_0^t h(s)\,ds\Big)\|\nabla\Delta u^k\|_2^2 + \|\Delta_x u_t^k\|_2^2 + (h\circ\nabla\Delta u^k) + \tau\int_0^1 \|\nabla_x z^k(x,\rho,t)\|_2^2\,d\rho\\
&+2\int_\Omega |u_t^k(t)|^l|\nabla_x u_t^k|^2\,dx\Big] + \frac{1}{2}\|\nabla_x z^k(x,1,t)\|_2^2\\
&= (l+1)\int_\Omega |u_t^k|^l\nabla u_{tt}^k(t)\nabla_x u_t^k\,dx - \mu_1\int_\Omega |\nabla_x u_t^k|^2 g_1'(u_t^k)\,dx\\
&-\mu_2\int_\Omega \nabla_x u_t^k\nabla_x z^k(x,1,t)g_2'(z^k(x,1,t))\,dx + \frac{1}{2}\|\nabla_x u_t^k\|_2^2 - \frac{1}{2}h(t)\|\nabla\Delta u^k\|_2^2 + \frac{1}{2}(h'\circ\nabla\Delta u^k).
\end{aligned} \tag{3.35}$$

From the first estimate (3.26) and Young's inequality, we obtain

$$\begin{aligned}
(l+1)\int_\Omega |u_t^k|^l\nabla u_{tt}^k(t)\nabla_x u_t^k\,dx &\le (l+1)C_2^{l/(l+2)+1/2}\|\nabla u_{tt}^k\|_2\\
&\le \eta\|\nabla u_{tt}^k\|_2^2 + \frac{(l+1)^2C_2^{2l/(l+2)+1}}{4\eta}, \quad \eta>0.
\end{aligned} \tag{3.36}$$

By using (3.7), (3.26) and Young's inequality, we obtain

$$\begin{aligned}
\mu_2\int_\Omega \nabla_x u_t^k\nabla_x z^k(x,1,t)g_2'(z^k(x,1,t))\,dx &\le \eta\|\nabla_x z^k(x,1,t)\|_2^2 + \frac{(\mu_2c_3)^2}{4\eta}\|\nabla_x u_t^k\|_2^2\\
&\le \eta\|\nabla_x z^k(x,1,t)\|_2^2 + \frac{(\mu_2c_3)^2C_2}{4\eta}, \quad \eta>0.
\end{aligned} \tag{3.37}$$

Taking into account (3.36), (3.37) into (3.35) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\Big(1-\int_0^t h(s)\,ds\Big)\|\nabla\Delta u^k\|_2^2+\|\Delta_x u_t^k\|_2^2+(h\circ\nabla\Delta u^k)+\tau\int_0^1\|\nabla_x z^k(x,\rho,t)\|_2^2\,d\rho \\
&\quad+2\int_\Omega |u_t^k(t)|^l|\nabla_x u_t^k|^2\,dx\Big]+\mu_1\int_\Omega|\nabla_x u_t^k|^2 g_1'(u_t^k)\,dx+(\frac{1}{2}-\eta)\|\nabla_x z^k(x,1,t)\|_2^2 \\
&\quad\le\eta\|\nabla u_{tt}^k\|_2^2-\frac{1}{2}h(t)\|\nabla\Delta u^k\|_2^2+\frac{1}{2}(h'\circ\nabla\Delta u^k)+C_2(\eta).
\end{aligned} \tag{3.38}$$

Multiplying (3.20) by c_{tt}^{jk} and summing over j from 1 to k, it follows that

$$\begin{aligned}
\int_\Omega |u_t^k|^l|u_{tt}^k|^2\,dx+\|\nabla u_{tt}^k\|_2^2=&-\int_\Omega\Delta^2u^k u_{tt}^k\,dx+\int_0^t h(t-s)\int_\Omega\Delta u^k(s)\Delta u_{tt}^k(t)\,dx\,ds \\
&-\mu_1\int_\Omega u_{tt}^k g_1(u_t^k)\,dx-\mu_2\int_\Omega u_{tt}^k g_2(z^k(x,1,t))\,dx.
\end{aligned} \tag{3.39}$$

Differentiating (3.23) with respect to t, we obtain

$$(\tau z_{tt}^k+z_{t\rho}^k,\phi^j)=0,$$

Multiplying by d_t^{jk} and summing over j from 1 to k, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\|z_t^k\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_t^k\|_2^2=0,$$

Integrating over $(0,1)$ with respect to ρ, we obtain

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1\|z_t^k\|_2^2\,d\rho+\frac{1}{2}\|z_t^k(x,1,t)\|_2^2-\frac{1}{2}\|u_{tt}^k(x,t)\|_2^2=0. \tag{3.40}$$

Summing (3.39) and (3.40), we obtain

$$\begin{aligned}
&\int_\Omega|u_t^k|^l|u_{tt}^k|^2\,dx+\|\nabla u_{tt}^k\|_2^2+\frac{\tau}{2}\frac{d}{dt}\int_0^1\|z_t^k\|_2^2\,d\rho+\frac{1}{2}\|z_t^k(x,1,t)\|_2^2 \\
&\quad=-\int_\Omega\Delta^2u^k u_{tt}^k\,dx+\int_0^t h(t-s)\int_\Omega\Delta u^k(s)\Delta u_{tt}^k(t)\,dx\,ds \\
&\quad+\frac{1}{2}\|u_{tt}^k(x,t)\|_2^2-\mu_1\int_\Omega u_{tt}^k g_1(u_t^k)\,dx-\mu_2\int_\Omega u_{tt}^k g_2(z^k(x,1,t))\,dx.
\end{aligned} \tag{3.41}$$

By using Young's inequality, the right hand side of (3.41) can be estimated as follows:

$$\int_\Omega\Delta^2u^k u_{tt}^k\,dx\le\eta\|\nabla u_{tt}^k\|_2^2+\frac{1}{4\eta}\|\nabla\Delta u^k\|_2^2,\quad\eta>0, \tag{3.42}$$

and

$$\begin{aligned}
&\int_0^t h(t-s)\int_\Omega \Delta u^k(s)\Delta u_{tt}^k(t)\,dx\,ds\\
&=-\int_0^t h(t-s)\int_\Omega \nabla\Delta u^k(s)\nabla u_{tt}^k(t)\,dx\,ds\\
&\le \eta\|\nabla u_{tt}^k\|_2^2+\frac{1}{4\eta}\int_\Omega\Big(\int_0^t h(t-s)|\nabla\Delta u^k(s)|\,ds\Big)^2\,dx\\
&\le \eta\|\nabla u_{tt}^k\|_2^2+\frac{1}{4\eta}\int_\Omega\Big(\int_0^t h(t-s)(|\nabla\Delta u^k(s)-\nabla\Delta u^k(t)|+|\nabla\Delta u^k(t)|)\,ds\Big)^2\,dx,
\end{aligned}\tag{3.43}$$

Then we use Young's inequality to obtain, for any $\eta>0$

$$\begin{aligned}
&\int_\Omega\Big(\int_0^t h(t-s)(|\nabla\Delta u^k(s)-\nabla\Delta u^k(t)|+|\nabla\Delta u^k(t)|)\,ds\Big)^2\,dx\\
&\quad\le \int_\Omega\Big(\int_0^t h(t-s)|\nabla\Delta u^k(s)-\nabla\Delta u^k(t)|\,ds\Big)^2\,dx+\int_\Omega\Big(\int_0^t h(t-s)|\nabla\Delta u^k(t)|\,ds\Big)^2\,dx\\
&\quad+2\int_\Omega\Big(\int_0^t h(t-s)|\nabla\Delta u^k(s)-\nabla\Delta u^k(t)|\,ds\Big)\Big(\int_0^t h(t-s)|\nabla\Delta u^k(t)|\,ds\Big)\,dx\\
&\quad\le (1+\eta)\int_\Omega\Big(\int_0^t h(t-s)|\nabla\Delta u^k(t)|\,ds\Big)^2\,dx\\
&\quad+(1+\frac{1}{\eta})\int_\Omega\Big(\int_0^t h(t-s)|\nabla\Delta u^k(s)-\nabla\Delta u^k(t)|\,ds\Big)^2\,dx,
\end{aligned}$$

Using (3.9), we obtain

$$\begin{aligned}
&\int_\Omega\Big(\int_0^t h(t-s)|\nabla\Delta u^k(s)-\nabla\Delta u^k(t)|+|\nabla\Delta u^k(t)|\,ds\Big)^2\,dx\\
&\qquad\le \beta^2(1+\eta)\|\nabla\Delta u^k(t)\|_2^2+\beta(1+\frac{1}{\eta})(h\circ\nabla\Delta u^k).
\end{aligned}\tag{3.44}$$

By Young's inequality, we obtain

$$\begin{aligned}
\mu_1\int_\Omega u_{tt}^k g_1(u_t^k)\,dx &\le \eta\int_\Omega |u_{tt}^k|^2\,dx+\frac{\mu_1^2}{4\eta}\int_\Omega |g_1(u_t^k)|^2\,dx\\
&\le \eta C_s^2\|\nabla u_{tt}^k\|_2^2+\frac{\mu_1^2}{4\eta}\int_\Omega |g_1(u_t^k)|^2\,dx
\end{aligned}\tag{3.45}$$

$$\mu_2\int_\Omega u_{tt}^k g_2(z^k(x,1,t))\,dx\le \eta C_s^2\|\nabla u_{tt}^k\|_2^2+\frac{\mu_2^2}{4\eta}\int_\Omega |g_2(z^k(x,1,t))|^2\,dx.\tag{3.46}$$

Taking into account (3.42)- (3.46) into (3.41) yields

$$\begin{aligned}
&\int_\Omega |u_t^k|^l |u_{tt}^k|^2\,dx + \Big(1 - 2\eta(1+C_s^2) - \frac{C_s^2}{2}\Big)\|\nabla u_{tt}^k\|_2^2 + \frac{\tau}{2}\frac{d}{dt}\int_0^1 \|z_t^k\|_2^2\,d\rho + \frac{1}{2}\|z_t^k(x,1,t)\|_2^2 \\
&\qquad \le \frac{\beta^2(1+\eta)}{4\eta}\|\nabla\Delta u^k\|_2^2 + \frac{\beta}{4\eta}(1+\frac{1}{\eta})(h\circ\nabla\Delta u^k) \\
&\qquad + \frac{\mu_1^2}{4\eta}\int_\Omega |g_1(u_t^k)|^2\,dx + \frac{\mu_2^2}{4\eta}\int_\Omega |g_2(z^k(x,1,t))|^2\,dx
\end{aligned} \tag{3.47}$$

Thus, from (3.38) and (3.47), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\Big(1 - \int_0^t h(s)\,ds\Big)\|\nabla\Delta u^k\|_2^2 + \|\Delta_x u_t^k\|_2^2 + (h\circ\nabla\Delta u^k) + \tau\int_0^1 \|\nabla_x z^k(x,\rho,t)\|_2^2\,d\rho \\
&\quad + 2\int_\Omega |u_t^k(t)|^l |\nabla_x u_t^k|^2\,dx + \tau\int_0^1 \|z_t^k\|_2^2\,d\rho\Big] + \mu_1\int_\Omega |\nabla_x u_t^k|^2 g_1'(u_t^k)\,dx + c_2'\|\nabla_x z^k(x,1,t)\|_2^2 \\
&\quad + \int_\Omega |u_t^k|^l |u_{tt}^k|^2\,dx + \Big(1 - \eta(3+2C_s^2) - \frac{C_s^2}{2}\Big)\|\nabla u_{tt}^k\|_2^2 + \frac{1}{2}\|z_t^k(x,1,t)\|_2^2 \\
&\quad \le -\frac{1}{2}h(t)\|\nabla\Delta u^k\|_2^2 + \frac{1}{2}(h'\circ\nabla\Delta u^k) + \frac{\beta^2(1+\eta)}{4\eta}\|\nabla\Delta u^k\|_2^2 + \frac{\beta}{4\eta}(1+\frac{1}{\eta})(h\circ\nabla\Delta u^k) \\
&\quad + \frac{\mu_1^2}{4\eta}\int_\Omega |g_1(u_t^k)|^2\,dx + \frac{\mu_2^2}{4\eta}\int_\Omega |g_2(z^k(x,1,t))|^2\,dx + C_2(\eta)
\end{aligned}$$

By choosing η small enough such that $1 - \eta(3+2C_s^2) - \frac{C_s^2}{2} > 0$, integrating over $(0,t)$ and using (3.10), we obtain

$$\begin{aligned}
&\Big(1 - \int_0^t h(s)\,ds\Big)\|\nabla\Delta u^k\|_2^2 + \|\Delta_x u_t^k\|_2^2 + (h\circ\nabla\Delta u^k) + \tau\int_0^1 \|\nabla_x z^k(x,\rho,t)\|_2^2\,d\rho \\
&\quad + 2\int_\Omega |u_t^k(t)|^l |\nabla_x u_t^k|^2\,dx + \tau\int_0^1 \|z_t^k\|_2^2\,d\rho + \mu_1\int_0^t\int_\Omega |\nabla_x u_t^k|^2 g_1'(u_t^k)\,dx\,ds \\
&\quad + c_2'\int_0^t \|\nabla_x z^k(x,1,t)\|_2^2\,ds + \int_0^t\int_\Omega |u_t^k|^l |u_{tt}^k|^2\,dx\,ds \\
&\quad + \Big(1 - \eta(3+2C_s^2) - \frac{C_s^2}{2}\Big)\int_0^t \|\nabla u_{tt}^k\|_2^2\,ds + \frac{1}{2}\int_0^t \|z_t^k(x,1,s)\|_2^2\,ds \\
&\quad \le \frac{\beta^2(1+\eta)}{4\eta}\int_0^t \|\nabla\Delta u^k\|_2^2\,ds + \frac{\beta}{4\eta}(1+\frac{1}{\eta})\int_0^t (h\circ\nabla\Delta u^k)\,ds \\
&\quad + \frac{\mu_1^2}{4\eta}\int_0^t\int_\Omega |g_1(u_t^k)|^2\,dx\,ds + \frac{\mu_2^2}{4\eta}\int_0^t\int_\Omega |g_2(z^k(x,1,t))|^2\,dx\,ds + C_2(\eta)T
\end{aligned}$$

Using (3.6), Jensen's inequality and the concavity of H^{-1}, we obtain

$$\begin{aligned}\int_\Omega |g_1(u_t^k)|^2\,dx &\le \int_{|u_t^k|\ge\varepsilon} |g_1(u_t^k)|^2\,dx + \int_{|u_t^k|\le\varepsilon} |g_1(u_t^k)|^2\,dx\\ &\le \int_{|u_t^k|\ge\varepsilon} u_t^k g_1(u_t^k)\,dx + \int_\Omega H^{-1}(u_t^k g_1(u_t^k))\,dx\\ &\le \int_{|u_t^k|\ge\varepsilon} u_t^k g_1(u_t^k)\,dx + cH^{-1}\Big(\int_\Omega u_t^k g_1(u_t^k)\,dx\Big)\end{aligned}$$

$$\begin{aligned}\int_\Omega |g_1(u_t^k)|^2\,dx &\le \int_{|u_t^k|\ge\varepsilon} u_t^k g_1(u_t^k)\,dx + c'H^*(1) + c''\int_\Omega u_t^k g_1(u_t^k)\,dx\\ &\le c'H^*(1) + c'\int_\Omega u_t^k g_1(u_t^k)\,dx\\ &\le c'H^*(1) + c(-E')\end{aligned}\tag{3.48}$$

and

$$\int_\Omega |g_2(z^k(x,1,t))|^2\,dx \le c'\int_\Omega z^k(x,1,t)g_2(z^k(x,1,t)\,dx \le c(-E')$$

Using Gronwall' Lemma, we obtain

$$\begin{aligned}\|\nabla\Delta u^k\|_2^2 + \|\Delta_x u_t^k\|_2^2 + (h\circ\nabla\Delta u^k) + \int_0^1 \|\nabla_x z^k(x,\rho,t)\|_2^2\,d\rho + \int_0^1 \|z_t^k\|_2^2\,d\rho\\ + \int_0^t \|\nabla u_{tt}^k(s)\|_2^2\,ds \le C_3\end{aligned}\tag{3.49}$$

We observe that the estimate (3.26) and (3.49) that there exists a subsequence $\{u^m\}$ of $\{u^k\}$ and a function u such that

$$u^m \rightharpoonup u \text{ weakly star in } L^\infty(0,T,H^4(\Omega)\cap H_0^2(\Omega))\tag{3.50}$$

$$u_t^m \rightharpoonup u_t \text{ weakly star in } L^\infty(0,T,H_0^2(\Omega))\tag{3.51}$$

$$g_1(u_t^m) \rightharpoonup \chi \text{ weakly star in } L^2(\Omega\times(0,T))\tag{3.52}$$

$$u_{tt}^m \rightharpoonup u_{tt} \text{ weakly star in } L^2(0,T,H_0^1(\Omega))\tag{3.53}$$

$$z^m \rightharpoonup z \text{ weakly star in } L^\infty(0,T,H_0^1(\Omega,L^2(0,1)))\tag{3.54}$$

$$z_t^m \rightharpoonup z_t \text{ weakly star in } L^\infty(0,T,L^2(\Omega\times(0,1)))\tag{3.55}$$

$$g_2(z^m(x,1,t)) \rightharpoonup \psi \text{ weakly star in } L^2(\Omega \times (0,T)) \tag{3.56}$$

From the first estimate (3.49) and Lemma 3.2.1, we deduce

$$\begin{aligned} \| |u_t^k|^l u_t^k \|_{L^2(0,T,L^2(\Omega)} &= \int_0^T \|u_t^k\|_{2(l+1)}^{2(l+1)}\, dt \\ &\leq \Big(\frac{C_s}{\sqrt{\lambda}}\Big)^{2(l+1)} \int_0^T \|\Delta u_t^k\|_2^{2(l+1)}\, dt \leq \Big(\frac{C_s}{\sqrt{\lambda}}\Big)^{2(l+1)} C_3^{2(l+1)} T. \end{aligned}$$

On the other hand, from Aubin-Lions theorem, (see Lions **[29]**), we deduce that there exists a subsequence $\{u^m\}$ of $\{u^k\}$ such that

$$u_t^m \to u_t \text{ strongly in } L^2(0,T,L^2(\Omega)) \tag{3.57}$$

which implies

$$u_t^m \to u_t \text{ almost everywhere in } \mathcal{A}. \tag{3.58}$$

Hence

$$|u_t^m|^l u_t^m \to |u_t|^l u_t \text{ almost everywhere in } \quad \mathcal{A} \tag{3.59}$$

where $\mathcal{A} = \Omega \times (0,T)$. Thus, using (3.57), (3.59) and Lions Lemma, we derive

$$|u_t^m|^l u_t^m \rightharpoonup |u_t|^l u_t \text{ weakly in } \quad L^2(0,T,L^2(\Omega)) \tag{3.60}$$

and

$$z^m \to z \text{ strongly in } L^2(0,T,L^2(\Omega))$$

which implies $z^m \to z$ almost everywhere in $\mathcal{A}$.

Lemma 3.3.1. *For each $T > 0$, $g_1(u_t)$, $g_2(z(x,1,t)) \in L^1(\mathcal{A})$ and $\|g_1(u')\|_{L^1(\mathcal{A})}$, $\|g_2(z(x,1,t))\|_{L^1(\mathcal{A})} \leq K$, where K is a constant independent of t.*

Proof. By **(A2)** and (3.58), we have

$$g_1(u_t^m(x,t)) \to g_1(u_t(x,t)) \text{ almost everywhere in } \mathcal{A},$$

$$0 \leq u_t^k(x,t) g_1(u_t^m(x,t)) \to u_t(x,t) g_1(u_t(x,t)) \text{ almost everywhere in } \mathcal{A}.$$

Hence, by (3.30) and Fatou's Lemma, we have

$$\int_0^T \int_\Omega u_t(x,t) g_1(u_t(x,t))\, dx\, dt \leq K_1 \text{ for } T > 0 \tag{3.61}$$

Now, we can estimate $\int_0^T \int_\Omega |g_1(u_t(x,t))|\,dx\,dt$. By Cauchy-Schwarz inequality and using (3.48) (3.61), we have

$$\begin{aligned}\int_0^T \int_\Omega |g_1(u_t(x,t))|\,dx\,dt &\le c|\mathcal{A}|^{\frac{1}{2}}\Big(\int_0^T \int_\Omega u_t(x,t) g_1(u_t(x,t))\,dx\,dt\Big)^{\frac{1}{2}}\\ &\le c|\mathcal{A}|^{\frac{1}{2}} K_1^{\frac{1}{2}} \equiv K.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_0^T \int_\Omega |g_2(z(x,1,t)))|\,dx\,dt &\le c|\mathcal{A}|^{\frac{1}{2}}\Big(\int_0^T \int_\Omega z(x,1,t) g_2(z(x,1,t))\,dx\,dt\Big)^{\frac{1}{2}}\\ &\le c|\mathcal{A}|^{\frac{1}{2}} K_1^{\frac{1}{2}} \equiv K.\end{aligned}$$

□

Lemma 3.3.2. *$g_1(u_t^k) \to g_1(u_t)$ in $L^1(\Omega \times (0,T))$ and $g_2(z^k) \to g_2(z)$ in $L^1(\Omega \times (0,T))$*

Proof. Let $E \subset \Omega \times [0,T]$ and set

$$E_1 = \Big\{(x,t) \in E,\ |g_1(u_t^k(x,t))| \le \frac{1}{\sqrt{|E|}}\Big\},\ E_2 = E \backslash E_1,$$

where $|E|$ is the measure of E. If $M(r) = \inf\{|s|,\ s \in \mathbb{R} \text{ and } |g(s)| \ge r\}$

$$\int_E |g_1(u_t^k)|\,dx\,dt \le c\sqrt{|E|} + \Big(M\Big(\frac{1}{\sqrt{|E|}}\Big)\Big)^{-1} \int_{E_2} |u_t^k g_1(u_t^k)|\,dx\,dt.$$

By applying (3.30) we deduce that $\sup_k \int_E |g_1(u_t^k)|\,dx\,dt \to 0$ as $|E| \to 0$. From Vitali's convergence theorem we deduce that

$$g_1(u_t^k) \to g_1(u_t) \quad \text{in} \quad L^1(\Omega \times (0,T)).$$

Similarly, we have

$$g_2(z^k) \to g_2(z) \quad \text{in} \quad L^1(\Omega \times (0,T)).$$

This completes the proof. □

Hence

$$g_1(u_t^k) \rightharpoonup g_1(u_t) \ \text{ weak in } L^2(\Omega \times (0,T)), \tag{3.62}$$

and

$$g_2(z^k) \rightharpoonup g_2(z) \ \text{ weak in } L^2(\Omega \times (0,T)). \tag{3.63}$$

By multiplying (3.20) by $\theta(t) \in \mathcal{D}(0,T)$ and by integrating over $(0,T)$, it follows that

$$\begin{aligned}
-\frac{1}{l+1}\int_0^T (|u_t^k(t)|^l u_t^k(t), w^j)\theta'(t)\,dt + \int_0^T (\Delta_x u^k(t), \Delta_x w^j)\theta(t)\,dt + \int_0^T (\nabla_x u_{tt}^k, \nabla_x w^j)\theta(t)\,dt \\
- \int_0^T \int_0^t h(t-s)(\Delta u^k(s), \Delta w^j)\theta(t)\,ds\,dt + \mu_1 \int_0^T (g_1(u_t^k), w^j)\theta(t)\,dt \\
+ \mu_2 \int_0^T (g_2(z^k(.,1)), w^j)\theta(t)\,dt = 0
\end{aligned} \tag{3.64}$$

and multiplying (3.23) by $\theta(t) \in \mathcal{D}(0,T)$ and integrating over $(0,T)\times(0,1)$, it follows that

$$\int_0^T \int_0^1 (\tau z_t^k + z_\rho^k, \phi^j)\theta(t)\,dt\,d\rho = 0. \tag{3.65}$$

The convergence of (3.50)- (3.56), (3.60), (3.62) and (3.63) are sufficient to pass to the limit in (3.64) and (3.65) in order to obtain

$$\begin{aligned}
-\frac{1}{l+1}\int_0^T (|u_t|^l u_t, w)\theta'(t)\,dt + \int_0^T (\Delta_x u, \Delta_x w)\theta(t)\,dt + \int_0^T (\nabla_x u_{tt}, \nabla_x w)\theta(t)\,dt \\
- \int_0^T \int_0^t h(t-s)(\Delta u(s), \Delta w)\theta(t)\,ds\,dt + \mu_1 \int_0^T (g_1(u_t), w)\theta(t)\,dt \\
+ \mu_2 \int_0^T (g_2(z(.,1)), w)\theta(t)\,dt = 0,
\end{aligned}$$

and

$$\int_0^T \int_0^1 (\tau z_t + z_\rho, \phi)\theta(t)\,dt\,d\rho = 0,$$

By integrating, we have

$$\begin{aligned}
\int_0^T \Big(|u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-s)\Delta^2 u(s)\,ds \\
+ \mu_1 g_1(u_t) + \mu_2 g_2(z(.,1)), w\Big)\theta(t)\,dt = 0,
\end{aligned}$$

This completes the proof of Theorem 3.1.1.

3.3.2 Proof of Theorem 3.1.2

To prove our main result, we define some functionals

$$\psi(t) = \int_\Omega \int_0^1 e^{-2\tau\rho} G(z(x,\rho,t))\,d\rho\,dx, \tag{3.66}$$

$$\phi(t) = \frac{1}{l+1}\int_\Omega |u_t|^l u_t u\,dx + \int_\Omega \nabla u_t \nabla u\,dx \tag{3.67}$$

$$\varphi(t) = \int_\Omega \Big(\Delta u_t - \frac{1}{l+1}|u_t|^l u_t\Big)\int_0^t h(t-s)(u(t)-u(s))\,ds\,dx \tag{3.68}$$

Set

$$F(t) = ME(t) + \varepsilon_1\psi(t) + \varepsilon_2\phi(t) + \varphi(t), \tag{3.69}$$

where M , ε_1 and ε_2 are suitable positive constants to be determined later.

Lemma 3.3.3. *There exist two positive constants κ_0 and κ_1 depending on ε_1 , ε_2 and M such that for all $t > 0$*

$$\kappa_0 E(t) \le F(t) \le \kappa_1 E(t). \tag{3.70}$$

Proof. Using (3.11), we have

$$|\psi(t)| \le \frac{1}{\xi}E(t). \tag{3.71}$$

From Young's inequality and Lemma 3.2.1, we deduce

$$\begin{aligned}
|\phi(t)| &\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\|u\|_{l+2}^{l+2} + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 \\
&\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\|\Delta u\|_2^{l+2} + \frac{1}{2\lambda_1}\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 \\
&\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \Big\{\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{2E(0)}{1-\beta}\Big)^{\frac{l}{2}} + \frac{1}{2\lambda_1}\Big\}\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2,
\end{aligned} \tag{3.72}$$

Integrating by parts, we have

$$\begin{aligned}
\varphi(t) = &-\int_\Omega \nabla u_t \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx \\
&-\int_\Omega \frac{1}{l+1}|u_t|^l u_t \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx,
\end{aligned}$$

we use Young's inequality applied with the conjugate exponents $\frac{l+2}{l+1}$ and $l+2$, the second term in the right hand side can be estimated as

$$\begin{aligned}
&\Big| -\int_\Omega \frac{1}{l+1}|u_t|^l u_t \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx\Big| \\
&\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\int_\Omega \Big(\int_0^t h(t-s)|u(t)-u(s)|\,ds\Big)^{l+2}dx \\
&\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\int_\Omega \Big(\int_0^t h(s)\,ds\Big)^{l+1}\int_0^t h(t-s)|u(t)-u(s)|^{l+2}\,ds\,dx \\
&\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\beta^{l+1}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{4E(0)}{1-\beta}\Big)^{\frac{l}{2}}(h\circ\Delta u)(t)
\end{aligned} \tag{3.73}$$

and

$$
\begin{aligned}
&\Big| -\int_\Omega \nabla u_t \int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \Big| \\
&\le \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\int_\Omega \Big(\int_0^t h(t-s)|\nabla u(t)-\nabla u(s)|\,ds\Big)^2 dx \\
&\le \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{\beta}{2\lambda_1}(h\circ\Delta u)(t).
\end{aligned} \tag{3.74}
$$

By combining (3.73) and (3.74), we deduce that

$$
|\varphi(t)| \le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{1}{2}\|\nabla u_t\|_2^2 + \Big\{\frac{(l+1)^{-1}}{l+2}\beta^{l+1}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{4E(0)}{1-\beta}\Big)^{\frac{l}{2}} + \frac{\beta}{2\lambda_1}\Big\}(h\circ\Delta u)(t). \tag{3.75}
$$

By combining (3.71), (3.72) and (3.75), we have

$$
\begin{aligned}
F(t) &\le (M+\frac{\varepsilon_1}{\xi})E(t) + \frac{\varepsilon_2+1}{l+2}\|u_t\|_{l+2}^{l+2} + \varepsilon_2\Big\{\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{2E(0)}{1-\beta}\Big)^{\frac{l}{2}} + \frac{1}{2\lambda_1}\Big\}\|\Delta u\|_2^2 \\
&+ \frac{\varepsilon_2+1}{2}\|\nabla u_t\|_2^2 + \Big\{\beta^{l+1}\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{4E(0)}{1-\beta}\Big)^{\frac{l}{2}} + \frac{\beta}{2\lambda_1}\Big\}(h\circ\Delta u)(t) \\
&\le \kappa_1 E(t).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
F(t) &\ge (M-\frac{\varepsilon_1}{\xi})E(t) - \frac{\varepsilon_2+1}{l+2}\|u_t\|_{l+2}^{l+2} - \varepsilon_2\Big\{\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{2E(0)}{1-\beta}\Big)^{\frac{l}{2}} + \frac{1}{2\lambda_1}\Big\}\|\Delta u\|_2^2 \\
&- \frac{\varepsilon_2+1}{2}\|\nabla u_t\|_2^2 - \Big\{\beta^{l+1}\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{4E(0)}{1-\beta}\Big)^{\frac{l}{2}} + \frac{\beta}{2\lambda_1}\Big\}(h\circ\Delta u)(t) \\
&\ge \frac{1}{l+2}\Big(M - \Big\{\frac{\varepsilon_1}{\xi}+\varepsilon_2+1\Big\}\Big)\|u_t\|_{l+2}^{l+2} + \frac{1}{2}\Big(M - \Big\{\frac{\varepsilon_1}{\xi}+\varepsilon_2+1\Big\}\Big)\|\nabla u_t\|_2^2 \\
&+ (M\xi-\varepsilon_1)\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx \\
&+ \Big(\frac{1}{2}\Big(M-\frac{\varepsilon_1}{\xi}\Big)\Big(1-\int_0^t h(s)\,ds\Big) - \varepsilon_2\Big\{\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{2E(0)}{1-\beta}\Big)^{\frac{l}{2}} + \frac{1}{2\lambda_1}\Big\}\Big)\|\Delta u\|_2^2 \\
&+ \Big(\frac{1}{2}\Big(M-\frac{\varepsilon_1}{\xi}\Big) - \Big\{\beta^{l+1}\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{4E(0)}{1-\beta}\Big)^{\frac{l}{2}} + \frac{\beta}{2\lambda_1}\Big\}\Big)(h\circ\Delta u)(t) \\
&\ge \kappa_0 E(t),
\end{aligned}
$$

for M large enough. □

Lemma 3.3.4. *Let (u,z) be the solution to (3.5). Then, it holds*

$$
\psi'(t) \le -2\psi(t) - \frac{\alpha_1 e^{-2\tau}}{\tau}\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \; + \frac{\alpha_2}{\tau}\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx. \tag{3.76}
$$

Proof. By differentiating (3.66) with respect to t and using (3.4) and (3.8), we get

$$\begin{aligned}
\psi'(t) &= -\frac{1}{\tau}\int_\Omega\int_0^1 e^{-2\tau\rho}\frac{\partial}{\partial\rho}G(z(x,\rho,t))\,d\rho\,dx \\
&= -\frac{1}{\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\Big(e^{-2\tau\rho}G(z(x,\rho,t))\Big)\,dx + 2\tau e^{-2\tau\rho}G(z(x,\rho,t))\Big]\,d\rho\,dx \\
&= -\frac{1}{\tau}\int_\Omega \Big[e^{-2\tau}G(z(x,1,t) - G(u_t(x,t))\Big]\,dx - 2\int_\Omega\int_0^1 e^{-2\tau\rho}G(z(x,\rho,t))\,d\rho\,dx \\
&= -\frac{1}{\tau}\int_\Omega e^{-2\tau}G(z(x,1,t)\,dx + \frac{1}{\tau}\int_\Omega G(u_t(x,t))\,dx - 2\int_\Omega\int_0^1 e^{-2\tau\rho}G(z(x,\rho,t))\,d\rho\,dx \\
&= -2\psi(t) + \frac{1}{\tau}\int_\Omega G(u_t(x,t))\,dx - \frac{e^{-2\tau}}{\tau}\int_\Omega G(z(x,1,t))\,dx \\
&\le -2\psi(t) + \frac{\alpha_2}{\tau}\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx - \frac{\alpha_1 e^{-2\tau}}{\tau}\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx.
\end{aligned}$$

Thus, our proof is completed. □

Lemma 3.3.5. *Let (u,z) be a solution of the problem (3.5). Then, for any $\eta > 0$, it holds*

$$\begin{aligned}
\phi'(t) \le{}& \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|_2^2 - \Big(1-\beta-\eta-\frac{\eta C_s^2}{\lambda_1}(\mu_1+\mu_2)\Big)\|\Delta u\|_2^2 + \frac{\beta}{4\delta}(h\circ\Delta u)(t) \\
&+ \frac{\mu_1}{4\eta}\int_\Omega |g_1(u_t)|^2\,dx + \frac{\mu_2}{4\eta}\int_\Omega |g_2(z(x,1,t))|^2\,dx.
\end{aligned} \tag{3.77}$$

Proof. Differentiating (3.67) with respect to t and using the first equation of (3.5), we obtain

$$\begin{aligned}
\phi'(t) &= \frac{1}{l+1}\int_\Omega (|u_t|^l u_t)' u\,dx + \frac{1}{l+1}\int_\Omega |u_t|^{l+2}\,dx + \int_\Omega \nabla u_{tt}\nabla u\,dx + \int_\Omega \nabla u_t\nabla u_t\,dx \\
&= \int_\Omega |u_t|^l u_{tt}u\,dx + \frac{1}{l+1}\|u_t\|^{l+2} - \int_\Omega \Delta u_{tt}u\,dx + \|\nabla u_t\|_2^2 \\
&= \int_\Omega \Big(|u_t|^l u_{tt} - \Delta u_{tt}\Big)u\,dx + \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|_2^2 \\
&= \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|^2 \\
&\quad - \int_\Omega \Big(\Delta^2 u + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(z(x,1,t)) - \int_0^t h(t-s)\Delta^2 u(s)\,ds\Big)u\,dx \\
&= \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|_2^2 - \|\Delta u\|_2^2 + \int_\Omega \Delta u(t)\int_0^t h(t-s)\Delta u(s)\,ds\,dx \\
&\quad - \mu_1\int_\Omega u g_1(u_t(x,t))\,dx - \mu_2\int_\Omega u g_2(z(x,1,t))\,dx
\end{aligned}$$

By using Young's inequality and Sobolev embedding, we can estimate the fourth term in the right side as follow:

$$
\begin{aligned}
&\int_\Omega \Delta u(t) \int_0^t h(t-s)\Delta u(s)\, ds\, dx \\
&\leq \int_0^t h(s)\, ds \|\Delta u(t)\|_2^2 + \int_\Omega \int_0^t h(t-s)|\Delta u(t)||\Delta u(s) - \nabla u(t)\, ds\, dx \\
&\leq \int_0^t h(s)\, ds \|\Delta u(t)\|_2^2 + \eta \|\Delta u(t)\|_2^2 + \frac{\beta}{4\eta}(h \circ \Delta u)(t) \\
&\leq (\beta + \eta)\|\Delta u(t)\|_2^2 + \frac{\beta}{4\eta}(h \circ \Delta u)(t)
\end{aligned}
$$

Since

$$
\int_\Omega u g_1(u_t)\, dx \leq \frac{\eta C_s^2}{\lambda_1}\|\Delta u\|_2^2 + \frac{1}{4\eta}\int_\Omega |g_1(u_t)|^2\, dx \tag{3.78}
$$

$$
\int_\Omega u g_2(z(x,1,t))\, dx \leq \frac{\eta C_s^2}{\lambda_1}\|\Delta u\|_2^2 + \frac{1}{4\eta}\int_\Omega |g_2(z(x,1,t))|^2\, dx \tag{3.79}
$$

This completes the proof. □

Lemma 3.3.6. *Let (u,z) be a solution of the problem (3.5). Then, for any $\delta > 0$, it holds*

$$
\begin{aligned}
&\varphi'(t) \leq \delta(2\beta^2+1)\|\Delta u(t)\|_2^2 + \Big(\delta + \frac{\delta a_0}{l+1} - \int_0^t h(s)\, ds\Big)\|\nabla u_t\|_2^2 \\
&+ \beta\Big(2\delta + \frac{1}{2\delta} + \frac{\mu_1 C_s^2}{4\delta\lambda_1} + \frac{\mu_2 C_s^2}{4\delta\lambda_1}\Big)(h \circ \nabla u)(t) + \mu_1\delta\|g_1(u_t(x,t))\|_2^2 \\
&- \frac{h(0)}{4\delta\lambda_1}\Big(1 + \frac{C_s^2}{(l+1)}\Big)(h' \circ \nabla u)(t) + \mu_2\delta\|g_2(z(x,1,t))\|_2^2 - \frac{1}{l+1}\int_0^t h(s)\, ds\|u_t\|_{l+2}^{l+2}.
\end{aligned} \tag{3.80}
$$

Proof. By using the Liebnitz formula, and the first equation of (3.5), we have

$$\begin{aligned}
\varphi'(t) = & -\int_\Omega \Big(\int_0^t h(t-s)\Delta u(s)\,ds\Big)\Big(\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))\,ds\Big)\,dx \\
& + \int_\Omega \Delta u(t)\Big(\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))\,ds\Big)\,dx \\
& + \mu_1 \int_\Omega g_1(u_t(x,t)) \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx \\
& + \mu_2 \int_\Omega g_2(z(x,1,t)) \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx \\
& - \int_\Omega \nabla u_t \int_0^t h'(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
& - \frac{1}{l+1}\int_\Omega |u_t|^l u_t \int_0^t h'(t-s)(u(t)-u(s))\,ds\,dx \\
& - \int_0^t h(s)\,ds\|\nabla u_t(t)\|_2^2 - \frac{1}{l+1}\int_0^t h(s)\,ds\|u_t(t)\|_{l+2}^{l+2} \\
= & \; I_1+I_2+I_3+I_4+I_5+I_6 - \int_0^t h(s)\,ds\|\nabla u_t(t)\|_2^2 - \frac{1}{l+1}\int_0^t h(s)\,ds\|u_t(t)\|_{l+2}^{l+2},
\end{aligned} \tag{3.81}$$

In what follows we will estimate $I_1, ..., I_6$. So for $\delta > 0$, we have

$$\begin{aligned}
|I_1| \le & \; \delta \int_\Omega \Big(\int_0^t h(t-s)|\Delta u(s)|\,ds\Big)^2 dx + \frac{1}{4\delta}\int_\Omega \Big(\int_0^t h(t-s)|\Delta u(t)-\Delta u(s)|\,ds\Big)^2 dx \\
\le & \; \delta \int_\Omega \Big(\int_0^t h(t-s)(|\Delta u(s)-\Delta u(t)| + |\Delta u(t)|)\,ds\Big)^2 dx + \frac{1}{4\delta}\Big(\int_0^t h(s)\,ds\Big)(h\circ\Delta u)(t) \\
\le & \; 2\delta\Big(\int_0^t h(t)\,ds\Big)^2 \|\Delta u(t)\|_2^2 + \Big(2\delta + \frac{1}{4\delta}\Big)\int_0^t h(s)\,ds\Big(h\circ\Delta u\Big)(t) \\
\le & \; 2\delta\beta^2\|\Delta u(t)\|_2^2 + \beta\Big(2\delta+\frac{1}{4\delta}\Big)\Big(h\circ\Delta u\Big)(t).
\end{aligned} \tag{3.82}$$

Similarly,

$$|I_2| \le \delta\|\Delta u(t)\|_2^2 + \frac{\beta}{4\delta}(h\circ\Delta u)(t) \tag{3.83}$$

$$|I_3| \le \delta\mu_1\|g_1(u_t(x,t))\|_2^2 + \frac{\mu_1\beta C_s^2}{4\delta\lambda_1}(h\circ\Delta u)(t) \tag{3.84}$$

$$|I_4| \le \delta\mu_2\|g_2(z(x,1,t))\|_2^2 + \frac{\mu_2\beta C_s^2}{4\delta\lambda_1}(h\circ\Delta u)(t) \tag{3.85}$$

$$
\begin{aligned}
|I_5| &\leq \delta \int_\Omega |\nabla u_t|^2\,dx + \frac{1}{4\delta}\int_\Omega \Big(\int_0^t |h'(t-s)||\nabla u(t)-\nabla u(s)|\,ds\Big)^2\,dx \\
&\leq \delta\|\nabla u_t\|^2 + \frac{1}{4\delta}\int_\Omega\int_0^t -h'(s)\,ds\int_0^t -h'(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx \\
&\leq \delta\|\nabla u_t\|^2 - \frac{h(0)}{4\delta\lambda_1}(h'\circ\Delta u)(t)
\end{aligned} \tag{3.86}
$$

$$
\begin{aligned}
|I_6| &\leq \frac{1}{l+1}\Big[\delta\int_\Omega ||u_t|^l u_t|^2\,dx + \frac{1}{4\delta}\int_\Omega\Big(\int_0^t h'(t-s)(u(t)-u(s))\,ds\Big)^2\,dx\Big] \\
&\leq \frac{1}{l+1}\Big[\delta\|\ u_t\|_{2(l+1)}^{2(l+1)} - \frac{h(0)C_s^2}{4\delta\lambda_1}(h'\circ\Delta u)(t)\Big] \\
&\leq \frac{\delta C_s^{2(l+1)}}{l+1}\|\ \nabla u_t\|_2^{2(l+1)} - \frac{h(0)C_s^2}{4\delta\lambda_1(l+1)}(h'\circ\Delta u)(t) \\
&\leq \frac{\delta a_0}{l+1}\|\ \nabla u_t\|_2^2 - \frac{h(0)C_s^2}{4\delta\lambda_1(l+1)}(h'\circ\Delta u)(t),
\end{aligned} \tag{3.87}
$$

where $a_0 = C_s^{2(l+1)}(2E(0))^l$. □

Lemma 3.3.7. *Let (u,z) be a solution of the problem (3.5). Assume that* **(A1)**- **(A4)** *hold, then $F(t)$ satisfies the following estimate, along the solution and for some positive constants $m, a_6 > 0$,*

$$F'(t) \leq -mE(t) + a_6\|g_1(u_t(x,t))\|_2^2. \tag{3.88}$$

Proof. From (3.15), (3.69), (3.76) and (3.77), we conclude that for any $t \geq t_0 > 0$,

$$
\begin{aligned}
F'(t) &= ME'(t) + \varepsilon_1\psi'(t) + \varepsilon_2\phi'(t) + \varphi'(t) \\
&\leq -(M\beta_1 - \varepsilon_1\frac{\alpha_2}{\tau})\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx \\
&- \Big(M\beta_2 - c_3\mu_2\{\delta + \frac{\varepsilon_2}{4\delta}\} - \varepsilon_1\frac{\alpha_1 e^{-2\tau}}{\tau}\Big)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \\
&- 2\varepsilon_1\psi(t) - \frac{1}{l+1}(h_0-\varepsilon_2)\|u_t\|_{l+2}^{l+2} - \Big(h_0 - \varepsilon_2 - \delta\Big(1+\frac{a_0}{l+1}\Big)\Big)\|\nabla u_t\|_2^2 \\
&- \Big(\frac{Mh_1}{2} + \varepsilon_2\{1-\beta-\delta-\frac{\delta C_s^2}{\lambda_1}(\mu_1+\mu_2)\} - \delta\Big(2\beta^2+1\Big)\Big)\|\Delta u\|_2^2 \\
&+ \Big(\frac{M}{2} - \frac{h(0)}{4\delta\lambda_1}\Big\{1+\frac{C_s^2}{l+1}\Big\}\Big)(h'\circ\Delta u)(t) \\
&+ \Big(\frac{\beta\varepsilon_2}{4\delta} + 2\beta\delta + \frac{\beta}{2\delta} + \frac{C_s^2\beta}{2\delta\lambda_1}\{\mu_1+\mu_2\}\Big)(h\circ\Delta u)(t) \\
&+ \mu_1(\delta + \frac{\varepsilon_2}{4\delta})\|g_1(u_t(x,t))\|_2^2
\end{aligned}
$$

where $h_0 = \int_0^{t_0} h(s)\,ds > 0$ and $h_1 = \min\{h(t)|\forall t \geq t_0\}$. We take $\varepsilon_2 < h_0$ and $\delta > 0$ sufficiently small such that

$$a_1 = \frac{1}{l+1}(h_0 - \varepsilon_2) > 0, \;\; a_2 = h_0 - \varepsilon_2 - \delta\Big(1 + \frac{a_0}{l+1}\Big) > 0.$$

We choose M large enough such that

$$a_3 = \frac{Mh_1}{2} + \varepsilon_2\{1 - \beta - \delta - \frac{\delta C_s^2}{\lambda_1}(\mu_1 + \mu_2)\} - \delta\Big(2\beta^2 + 1\Big) > 0,$$

and

$$a_4 = \zeta\Big(\frac{M}{2} - \frac{h(0)}{4\delta\lambda_1}\Big\{1 + \frac{C_s^2}{l+1}\Big\}\Big) - \Big(\frac{\beta\varepsilon_2}{4\eta} + 2\beta\delta + \frac{\beta}{2\delta} + \frac{C_s^2}{2\delta\lambda_1}\{\mu_1 + \mu_2\}\Big) > 0$$

$$M\beta_1 - \varepsilon_1\frac{\alpha_2}{\tau} > 0, \quad M\beta_2 - c_3\mu_2\{\delta + \frac{\varepsilon_2}{4\delta}\} - \varepsilon_1\frac{\alpha_1 e^{-2\tau}}{\tau} > 0.$$

Then

$$F'(t) \leq -a_1\|u_t\|_{l+2}^{l+2} - a_2\|\nabla u_t\|_2^2 - a_3\|\Delta u\|_2^2 - a_4(h \circ \Delta u)(t) - a_5\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho$$
$$+ \frac{\varepsilon_1}{\xi}E(t) + a_6\|g_1(u_t(x,t))\|_2^2$$

where $a_5 = 2\varepsilon_1$ and $a_6 = \mu_1(\delta + \frac{\varepsilon_2}{4\eta})$. □

Proof of Theorem 3.1.2 As in Komornik **[24]**, we consider the following partition of Ω

$$\Omega_1 = \{x \in \Omega : \; |u_t| > \varepsilon\}, \quad \Omega_2 = \{x \in \Omega : \; |u_t| \leq \varepsilon\}$$

By using (3.6), we have

$$\int_{\Omega_1} |g_1(u_t)|^2\,dx \leq c_2\int_{\Omega_1} u_t g_1(u_t)\,dx \leq -cE'(t). \tag{3.89}$$

Case 1. H is linear on $[0,\varepsilon]$. In this case, one can easily check that there exists $c_1 > 0$, such that $|g_1(s)| \leq c_1 s$ for all $s \leq \varepsilon$, and thus,

$$\int_{\Omega_2} |g_1(u_t)|^2\,dx \leq c_1\int_{\Omega_2} u_t g_1(u_t)\,dx \leq -cE'(t). \tag{3.90}$$

$$(F(t) + cE(t))' \leq -mH_2(E(t)). \tag{3.91}$$

Case 2. $H'(0) = 0$ and $H'' > 0$ on $]0,\varepsilon]$ we define

$$I(t) = \frac{1}{|\Omega_2|}\int_{\Omega_2} u_t g(u_t)\,dx,$$

and exploit Jensen's inequality and the concavity of H^{-1} to obtain

$$H^{-1}(I(t)) \geq c \int_{\Omega_2} H^{-1}(u_t g(u_t))\, dx,$$

by using (3.6), we obtain

$$\begin{aligned}\int_{\Omega_2} |g_1(u_t)|^2\, dx &\leq c \int_{\Omega_2} H^{-1}(u_t g(u_t))\, dx \\ &\leq cH^{-1}(I(t)) \leq cH^{-1}(-cE'(t)).\end{aligned} \tag{3.92}$$

A combination of (3.88), (3.89) and (3.92) yields

$$(F(t) + cE(t))' \leq -mE(t) + cH^{-1}(-cE'(t)), \quad t \geq t_0. \tag{3.93}$$

By recalling that $E' \leq 0,\ H' > 0,\ H'' > 0$ on $(0, \varepsilon]$ and using (3.93), we obtain

$$\begin{aligned}&\Big(H'(\varepsilon_0 E(t))\{F(t) + cE(t)\} + cE(t)\Big)' \\ &= \varepsilon_0 E'(t) H''(\varepsilon_0 E(t))(F(t) + cE(t)) + H'(\varepsilon_0 E(t))(F(t) + cE(t))' + cE'(t) \\ &\leq -mH'(\varepsilon_0 E(t))E(t) + cH'(\varepsilon_0 E(t))H^{-1}(-cE'(t)) + cE'(t),\end{aligned} \tag{3.94}$$

by using Remark 3.2.1 with H^*, the convex conjugate of H in the sense of Young, we obtain

$$\begin{aligned}&\Big(H'(\varepsilon_0 E(t))\{F(t) + cE(t)\} + cE(t)\Big)' \\ &\qquad\leq -mH'(\varepsilon_0 E(t))E(t) + cH^*(H'(\varepsilon_0 E(t))) \\ &\qquad\leq -mH'(\varepsilon_0 E(t))E(t) + c\varepsilon_0 H'(\varepsilon_0 E(t))E(t) \\ &\qquad\leq -cH'(\varepsilon_0 E(t))E(t) \\ &\qquad= -cH_2(E(t)).\end{aligned} \tag{3.95}$$

Let

$$\widetilde{F}(t) = \begin{cases} F(t) + cE(t) & \text{If } H \text{is linear on } [0, \varepsilon], \\ \\ H'(\varepsilon_0 E(t))\{F(t) + cE(t))\} + cE(t) & \text{If } H'(0) = 0 \text{ and } H'' > 0 \ \ ]0, \varepsilon], \end{cases} \tag{3.96}$$

From (3.91) and (3.95), it follows that

$$\frac{d}{dt}\widetilde{F}(t) \leq -cH_2(E(t)), \ \ \forall t \geq t_0.$$

On the other hand, after choosing $M > 0$ larger if needed, we can observe from Lemma 3.3.3 that $F(t)$ is equivalent to $E(t)$. So, $\widetilde{F}(t)$ is also equivalent to $E(t)$, for some positive constants $\widetilde{\epsilon_1}$ and $\widetilde{\epsilon_2}$

$$\widetilde{\epsilon_1} E(t) \leq \widetilde{F}(t) \leq \widetilde{\epsilon_2} E(t). \tag{3.97}$$

By setting

$$L(t) = \epsilon \widetilde{F}(t), \quad \text{for} \;\; \epsilon < \frac{1}{\widetilde{\epsilon_2}},$$

we easily see that, by (3.97), we have

$$L(t) \sim E(t)$$

$$\begin{aligned} L'(t) \leq \epsilon \widetilde{F}(t) &\leq -\epsilon c H_2(E(t)) \\ &\leq -\epsilon c H_2\Big(\frac{1}{\widetilde{\epsilon_2}} \widetilde{F}(t)\Big) \\ &\leq -\epsilon c H_2\Big(\epsilon \; \widetilde{F}(t)\Big) \\ &\leq -\epsilon c H_2(L(t)), \end{aligned}$$

then

$$\frac{L'(t)}{H_2(L(t))} \leq -\epsilon c_5 \tag{3.98}$$

By recalling (3.12), we deduce $H_2(t) = -1/H_1'(t)$, hence

$$L'(t) H_1'(L(t)) \geq \epsilon c, \quad \forall t \geq t_0.$$

A simple integration over (t_0, t) yields

$$H_1(L(t)) \geq H_1(L(t_0)) + \epsilon c(t - t_0).$$

By choosing $\epsilon > 0$ sufficiently small such that $H_1(L(t_0)) - \epsilon c t_0 > 0$, and exploiting the fact that H_1^{-1} is decreasing, we infer that

$$L(t)) \leq H_1^{-1}(\epsilon c t + H_1(L(t_0)) - \epsilon c t_0). \tag{3.99}$$

Consequently, the equivalence of F, $\widetilde{F}$, L and E yields the estimate

$$E(t) \leq w_3 H_1^{-1}(w_1 t + w_2),$$

where $w_1 = \epsilon c$ and $w_2 = H_1(F(t_0)) - \epsilon c t_0$.
This completes the proof of Theorem 3.1.2.

Chapter 4

Global existence and energy decay of solutions to a viscoelastic non-degenerate Kirchhoff equation with a time varying delay term

4.1 Introduction

4.1.1 The model

In this paper we consider global existence and decay properties of solutions for the initial boundary value problem of viscoelastic non-degenerate Kirchhoff equation of the form

$$\begin{cases} |u_t|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds & \\ +\mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau(t))) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \partial\Omega\times[0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in } \Omega\times]0,\tau(0)[, \end{cases} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, $l > 0$, μ_1 and μ_2 are positive real numbers, h is a positive function which decays exponentially, $\tau(t) > 0$ is a time varying delay, g_1 and g_2 are two functions, and the initial data (u_0, u_1, f_0) are in a suitable function space. $M(r) = a + br^\gamma$ is a C^1-function for $r \geq 0$, with $a, b > 0$, and $\gamma \geq 1$.

In the absence of delay term (i.e.$\mu_2 = 0$), Hang and Wang **[20]** considered the following nonlinear viscoelastic equation with damping:

$$|u_t|^l u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\, ds + u_t(x,t) = 0, \text{ in } \Omega\times]0,+\infty[.$$

They proved the global existence and established uniform decay results. Time delay is often present in applications and practical problems . In recent years, the control of PDEs with time delay effects has become an active area of research, see, for example **[35]**,**[50]** and the references therein. In **[19]**, the authors showed that a small delay in a boundary control could turn a well-behave hyperbolic system into a wild one and therefore, delay becomes a source of instability. However, sometimes it can also improve the performance of the system. Shun-Tang Wu **[44]** treated the problem (4.1) for $g_1(x) = g_2(x) = x$ and τ is a constant time delay. He proved the local existence result by Faedo-Galerkin method and established the decay result by suitable Lyapunov functionals under appropriate conditions on μ_1, μ_2 and on the kernel h.
In the case when $l = 0$ and $M(r) = 1$ and τ is a constant time delay, Benaissa et al **[8]** proved the global existence and uniform decay for the following problem:

$$u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)\, ds + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0, \text{ in } \Omega\times]0,+\infty[. \tag{4.2}$$

Also, the problem (4.2) was treated by Kirane and Said Houari **[22]** for $g_1(x) = g_2(x) = x$ and τ is a constant time delay. Daewook **[15]** studied the following viscoelastic kirchhoff equation with varying time delay and nonlinear source term:

$$u_{tt} - M(x,t,\|\nabla u\|^2)\Delta u + \int_0^t h(t-s)div(a(x)\nabla u(s))\, ds + |u|^m u$$
$$+\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau(t)) = 0, \text{ in } \Omega\times]0,+\infty[,$$

which is a description of axially moving viscoelastic materials. Under the smallness condition with respect to Kirchhoff coefficient and the relaxation function and by summing $0 \le m \le \frac{2}{n-2}$ if $n > 2$ or $0 \le m$ if $n \le 2$, he obtained the uniform decay rate of the Kirchhoff type energy.

The main of this paper is to give a global solvability in Sobolev spaces and energy decay estimate of the solution to problem (P) for a weakly nonlinear damping and in the presence

of a time varying delay term. We proof the existence of global solutions in suitable Sobolev spaces by combining the energy method with Fadeo-Galerkin procedure and we use a perturbed energy method and some proprieties of convex functions to establish an explicit and general decay result.These convexity arguments were introduced and developed by Lasiecka et al., **[25]**, **[27]**, **[28]** and used, with appropriate modications, by Liu and Zuazua **[30]**, Alabau-Boussouira **[5]** and others.
This paper is organized as follows: In section 2, we give some hypotheses and state our main result. In section 3, we prove the global existence of weak solutions. In section 4, we derive the uniform decay of the energy.

4.1.2 Formulation of the results

We use the Sobolev spaces $H^2(\Omega)$, $H_0^1(\Omega)$ and the Hilbert space $L^p(\Omega)$ with their usual scalar products and norms.
The prime $'$ and the subscript t will denote time differentiation.
The constant C denotes a general positive constant, which may be different in different estimates.
Now we introduce, as in **[32]**, the new variable

$$z(x,\rho,t) = u_t(x,t-\rho\tau(t)),\ x\in\Omega,\quad \rho\in(0,1), t>0.$$

Then, we have

$$\tau(t)z_t(x,\rho,t) + (1-\rho\tau'(t))z_\rho(x,\rho,t) = 0,\ \text{in } \Omega\times(0,1)\times(0,+\infty). \tag{4.3}$$

Therefore, problem (4.1) is equivalent to

$$\begin{cases} |u_t|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds & \\ +\mu_1 g_1(u_t(x,t)) + \mu_2 g_2(z(x,1,t)) = 0 & \text{in } \Omega\times]0,+\infty[, \\ \tau(t)z_t(x,\rho,t) + (1-\rho\tau'(t))z_\rho(x,\rho,t) = 0, & \text{in } \Omega\times]0,1[\times]0,+\infty[\\ u(x,t) = 0, & \text{on } \partial\Omega\times[0,\infty[\\ z(x,0,t) = u_t(x,t), & \text{on } \Omega\times[0,\infty[\\ u(x,0) = u_0(x),\quad u_t(x,0) = u_1(x), & \text{in } \Omega \\ z(x,\rho,0) = f_0(x,-\rho\tau(0)), & \text{in } \Omega\times]0,1[. \end{cases} \tag{4.4}$$

To state and prove our result, we need some assumptions.

(A1) Assume that l satisfies

$$\begin{cases} 0 < l \leq \frac{2}{n-2} & \text{if } n > 2, \\ 0 < l < \infty & \text{if } n \leq 2. \end{cases}$$

(A2) For the relaxation function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^1 function such that

$$a - \int_0^\infty h(s)\,ds = k > 0,$$

and suppose that there exist a positive constant ζ verifying

$$h'(t) \leq -\zeta h(t).$$

(A3) $g_1 : \mathbb{R} \to \mathbb{R}$ is non decreasing function of class C^1 and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying

$$\begin{cases} H(0) = 0 \text{ and } H \text{ is linear on } [0, \varepsilon] \text{ or} \\ H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \varepsilon] \text{ such that} \\ c_1|s| \leq |g_1(s)| \leq c_2|s| \quad \text{if } |s| \geq \varepsilon \\ s^2 + g_1(s)^2 \leq H^{-1}(sg_1(s)) \quad \text{if } |s| \leq \varepsilon \end{cases} \tag{4.5}$$

where H^{-1} denotes the inverse function of H and ε, c_1, c_2 are positive constants.
$g_2 : \mathbb{R} \to \mathbb{R}$ is an odd non decreasing function of class $C^1(\mathbb{R})$ such that there exist $c_3, \alpha_1, \alpha_2 > 0$,

$$|g_2'(s)| \leq c_3, \tag{4.6}$$

and

$$\begin{cases} \alpha_1 s g_2(s) \leq G(s) \leq \alpha_2 s g_1(s), \\ G(s) = \int_0^s g_2(r)dr. \end{cases} \tag{4.7}$$

(A4) τ is a function in $W^{2,+\infty}([0,T])$, $T > 0$, such that

$$\begin{cases} 0 < \tau_0 \leq \tau(t) \leq \tau_1, \quad \forall t > 0 \\ \tau'(t) \leq d < 1, \quad \forall t > 0. \end{cases}$$

Where τ_0 and τ_1 are positive numbers.

(A5) We also assume that

$$\mu_2 < \frac{\alpha_1(1-d)}{\alpha_2(1-\alpha_1 d)}\mu_1.$$

We define the energy associated to the solution of system (4.4) by

$$\begin{aligned} E(t) = \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{b}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} + \frac{1}{2}\Big(a - \int_0^t h(s)ds\Big)\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 \\ + \frac{1}{2}(h\,o\,\nabla u)(t) + \xi\tau(t)\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx, \end{aligned} \tag{4.8}$$

where ξ is a positive constant such that

$$\frac{\mu_2(1-\alpha_1)}{\alpha_1(1-d)} < \xi < \frac{\mu_1 - \alpha_2\mu_2}{\alpha_2}, \tag{4.9}$$

and

$$(h\,o\,v)(t) = \int_0^t h(t-s)\|v(.,t) - v(.,s)\|^2\,ds.$$

Theorem 4.1.1. (Global existence) *Let* $(u_0, u_1, f_0) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega, H^1(0,1))$ *satisfy the compatibility condition*

$$f_0(.,0) = u_1.$$

Assume That **(A1)-(A5)** *hold.Then the problem (4.1) admits a weak solution*

$$u \in L^\infty([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)), u_t \in L^\infty([0,\infty); H_0^1(\Omega)), u_{tt} \in L^\infty([0,\infty); L^2(\Omega)).$$

Theorem 4.1.2. (Uniform decay rates of energy) *Assume That* **(A1)-(A5)** *hold. Then, there exist positive constants* w_1, w_2, w_3 *and* ε_0 *such that the solution energy of (4.1) satisfies*

$$E(t) \le w_3 H_1^{-1}(w_1 t + w_2) \quad \forall t \ge 0, \tag{4.10}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\,ds \quad \text{and} \quad H_2(t) = tH'(\varepsilon_0 t). \tag{4.11}$$

Here, H_1 *is strictly decreasing and convex on* $(0,1]$ *with* $\lim_{t\to 0} H_1(t) = +\infty$.

4.2 Preliminaries

The following lemma states an important property of the convolution operator.

Lemma 4.2.1. (Sobolev-Poincarès inequality). *Let* q *be a number with*

$$2 \le q < +\infty (n = 1,2) \;\; or \;\; 2 \le q \le 2n/(n-2) (n \ge 3),$$

then there exists a constant $C_s = C_s(\Omega, q)$ *such that*

$$\|u\|_q \le C_s\|\nabla u\|_2 \;\; for \;\; u \in H_0^1(\Omega).$$

Lemma 4.2.2. *[41]For $h, \varphi \in C^1([0,+\infty[, \mathbb{R})$ we have*

$$\int_\Omega h * \varphi\varphi_t \, dx = -\frac{1}{2}h(t)\|\varphi(t)\|^2 + \frac{1}{2}(h' o \, \varphi)(t) - \frac{1}{2}\frac{d}{dt}[(h \, o \, \varphi)(t) - (\int_0^t h(s)\, ds)\|\varphi\|^2].$$

Remark 4.2.1. *Let us denote by Φ^* the conjugate function of the differentiable convex function Φ, i.e.,*

$$\Phi^*(s) = \sup_{t\in\mathbb{R}^+}(st - \Phi(t)).$$

Then Φ^* is the Legendre transform of Φ, which is given by (see Arnold **[7]***, p. 61 -62)*

$$\Phi^*(s) = s(\Phi')^{-1}(s) - \Phi[(\Phi')^{-1}(s)], \quad \textit{if} \;\; s \in (0, \Phi'(r)],$$

and Φ^* satisfies the generalized Young inequality

$$AB \leq \Phi^*(A) + \Phi(B), \quad \textit{if} \;\; A \in (0, \Phi'(r)] \;\;\; B \in (0, r]. \tag{4.12}$$

Lemma 4.2.3. *Let (u,z) be a solution of the problem (4.4). Then, the energy functional defined by (4.8) satisfies*

$$E'(t) \leq -\lambda\int_\Omega u_t g_1(u_t)\, dx - \beta\int_\Omega z(x,1,t)g_2(z(x,1,t))\, dx - \frac{1}{2}h(t)\|\nabla u(t)\|^2 + \frac{1}{2}(h' o\nabla u)(t) \leq 0,$$

where $\lambda = \mu_1 - \xi\alpha_2 - \mu_2\alpha_2$ and $\beta = \xi(1-d)\alpha_1 - \mu_2(1-\alpha_1)$

Proof. Multiplying the first equation in (4.4) by u_t, integrating over Ω and using integration by parts, we get

$$\begin{aligned} &\frac{d}{dt}\Big[\frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{b}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} + \frac{1}{2}a\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2\Big] \\ &+ \int_\Omega\int_0^t h(t-s)\nabla u(s)\nabla u_t(t)\, ds\, dx + \mu_1\int_\Omega u_t(x,t)g_1(u_t(x,t))\, dx \\ &+ \mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\, dx = 0. \end{aligned} \tag{4.13}$$

Consequently, by applying the Lemma 4.2.2, equation (4.13) becomes

$$\begin{aligned} &\frac{d}{dt}\Big[\frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{b}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} + \frac{1}{2}\Big(a - \int_0^t h(s)ds\Big)\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}(ho\nabla u)(t)\Big] \\ &+ \frac{1}{2}h(t)\|\nabla u(t)\|^2 - \frac{1}{2}(h' o\nabla u)(t) + \mu_1\int_\Omega u_t(x,t)g_1(u_t(x,t))\, dx \;\; + \mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\, dx = 0. \end{aligned} \tag{4.14}$$

We multiply the second equation in (4.4) by $\xi g_2(z)$ and integrate the result over $\Omega\times(0,1)$, to obtain

$$\xi\tau(t)\int_\Omega\int_0^1 z_t(x,\rho,t)g_2(z(x,\rho,t))\,d\rho\,dx=-\xi\int_\Omega\int_0^1(1-\rho\tau'(t))\frac{\partial}{\partial\rho}\,G(z(x,\rho,t))d\rho\,dx.$$

Consequently

$$\begin{aligned}\frac{d}{dt}\Big(\xi\tau(t)\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx\Big)&=\xi\tau'(t)\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx\\&-\xi\int_\Omega\int_0^1(1-\rho\tau'(t))\frac{\partial}{\partial\rho}\,G(z(x,\rho,t))\,d\rho\,dx\\&=-\xi\int_\Omega\int_0^1\frac{\partial}{\partial\rho}\Big((1-\rho\tau'(t))G(z(x,\rho,t))\Big)\,d\rho\,dx\\&=-\xi(1-\tau'(t))\int_\Omega G(z(x,1,t))\,dx+\xi\int_\Omega G(u_t(x,t))\,dx.\end{aligned}\tag{4.15}$$

Combining (4.14) and (4.15), we obtain

$$\begin{aligned}E'(t)=&-\xi(1-\tau'(t))\int_\Omega G(z(x,1,t))\,dx+\xi\int_\Omega G(u_t(x,t))\,dx-\frac{1}{2}h(t)\|\nabla u(t)\|^2\\&+\frac{1}{2}(h'o\nabla u)(t)-\mu_1\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx-\mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\,dx.\end{aligned}$$

From (4.7) and $(A4)$, we get

$$\begin{aligned}E'(t)\le&-(\mu_1-\xi\alpha_2)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx-\xi(1-d)\alpha_1\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx\\&-\mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\,dx-\frac{1}{2}h(t)\|\nabla u(t)\|^2+\frac{1}{2}(h'o\nabla u)(t).\end{aligned}\tag{4.16}$$

Using (4.7) and Remark 4.2.1, we obtain

$$G^*(s)=sg_2^{-1}(s)-G(g_2^{-1}(s)),\quad\forall s\ge0.$$

Hence

$$\begin{aligned}G^*(g_2(z(x,1,t)))&=z(x,1,t)g_2(z(x,1,t))-G(z(x,1,t))\\&\le(1-\alpha_1)z(x,1,t)g_2(z(x,1,t)).\end{aligned}$$

Using (4.7), (4.12) with $A = g_2(z(x,1,t))$ and $B = u_t(x,t)$, we have from (4.16) that

$$\begin{aligned}
E'(t) &\le -(\mu_1 - \xi\alpha_2)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx - \xi(1-d)\alpha_1\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \\
&+ \mu_2\int_\Omega (G(u_t(x,t)) + G^*(g_2(z(x,1,t)))\,dx - \frac{1}{2}h(t)\|\nabla u(t)\|^2 + \frac{1}{2}(h'o\nabla u)(t) \\
&\le -(\mu_1 - \xi\alpha_2 - \mu_2\alpha_2)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx \\
&- (\xi(1-d)\alpha_1 - \mu_2(1-\alpha_1))\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \\
&- \frac{1}{2}h(t)\|\nabla u(t)\|^2 + \frac{1}{2}(h'o\nabla u)(t).
\end{aligned}$$

This completes the proof.

□

4.3 Global existence-Proof of Theorem4.1.1

Throughout this section we assume $u_0 \in H^2(\Omega) \cap H_0^1(\Omega), u_1 \in H_0^1(\Omega)$ and $f_0 \in H_0^1(\Omega, H^1(0,1))$. We will use Faedo-Galerkin method to prove the existence of global solution. Let $T > 0$ be fixed and let $w^k, k \in \mathbb{N}$ be a basis of $H^2(\Omega) \cap H_0^1(\Omega)$, V_k the space generated by w^k. Now, we define, for $1 \le j \le k$, the sequence $\phi^j(x,\rho)$ as follows:

$$\phi^j(x,0) = w^j.$$

Then, we may extend $\phi^j(x,0)$ by $\phi^j(x,\rho)$ over $L^2(\Omega\times(0,1))$ such that $(\phi^j)_j$ forms a basis of $L^2(\Omega, H^1(0,1))$ and denote Z_k the space generated by $\{\phi^k\}$. We construct approximate solutions (u^k, z^k),$k = 1,2,3,...$, in the form

$$u^k(t) = \sum_{j=1}^{k} c^{jk}(t)w^j,$$

$$z^k(t) = \sum_{j=1}^{k} d^{jk}(t)\phi^j,$$

where c^{jk} and $d^{jk}(j = 1,2,...,k)$ are determined by the following ordinary differential equations:

$$\begin{cases}
(|u_t^k|^l u_{tt}^k, w^j) + M(\|\nabla u^k(t)\|^2)(\nabla u^k, \nabla w^j) + (\nabla u_{tt}^k, \nabla w^j) \\
-\int_0^t h(t-s)(\nabla u^k(s), \nabla w^j)\,ds + \mu_1(g_1(u_t^k), w^j) + \mu_2(g_2(z^k(.,1)), w^j) = 0, \\
1 \le j \le k, \\
z^k(x,0,t) = u_t^k(x,t),
\end{cases} \tag{4.17}$$

$$u^k(0) = u_0^k = \sum_{j=1}^{k}(u_0, w^j)w^j \to u_0, \text{ in } H^2(\Omega)\cap H_0^1(\Omega) \,\text{as}\, k \to +\infty, \tag{4.18}$$

$$u_t^k(0) = u_1^k = \sum_{j=1}^{k}(u_1, w^j)w^j \to u_1, \text{ in } H_0^1(\Omega) \,\text{as}\, k \to +\infty, \tag{4.19}$$

and

$$\begin{cases} (\tau(t)z_t^k + (1-\rho\tau'(t))z_\rho^k, \phi^j) = 0, \\ 1 \le j \le k, \end{cases} \tag{4.20}$$

$$z^k(\rho, 0) = z_0^k = \sum_{j=1}^{k}(f_0, \phi^j)\phi^j \to f_0, \text{ in } H_0^1(\Omega, H^1(0,1)) \,\text{as}\, k \to +\infty, \tag{4.21}$$

Here we denote by (.,.) the inner product in $L^2(\Omega)$.
Noting that $\frac{l}{2(l+1)} + \frac{1}{2(l+1)} + \frac{1}{2} = 1$, from the generalized Hölder inequality, we obtain

$$(|u_t^k|^l u_{tt}^k, w_j) = \int_\Omega |u_t^k|^l u_{tt}^k w_j \, dx \le \Big(\int_\Omega |u_t^k|^{2(l+1)} \, dx\Big)^{\frac{l}{2(l+1)}} \|u_{tt}^k\|_{2(l+1)} \|w_j\|_2$$

Since $(A1)$ holds, according to Sobolev embedding the nonlinear term $(|u_t^k|^l u_{tt}^k, w_j)$ in (4.17) makes sense.
The standard theory of ODE guarantees that the system (4.17)-(4.21) has a unique solution in $[0, t_k)$, with $0 < t_k < T$, by Zorn lemma since the nonlinear terms in (4.17) are locally Lipschitz continuous. Note that $u^k(t)$ is of class C^2.
In the next step, we obtain a priori estimates for the solution of the system (4.17)-(4.21), so that it can be extended outside $[0, t_k)$ to obtain one solution defined for all $t > 0$, using a standard compactness argument for the limiting procedure.

A. The first estimate.

Since the sequences u_0^k, u_1^k and z_0^k converge and from lemma 4.2.3, we can find a positive constant C_1 independent of k such that

$$\begin{aligned} E^k(t) - E^k(0) &\le -\lambda \int_0^t \int_\Omega u_t^k g_1(u_t^k)\, dx\, ds - \beta \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\, dx\, ds \\ &- \frac{1}{2}\int_0^t h(s)\|\nabla u^k(s)\|^2\, ds + \frac{1}{2}\int_0^t (h' o \nabla u^k)(s)\, ds \\ &\le -\lambda \int_0^t \int_\Omega u_t^k g_1(u_t^k)\, dx\, ds - \beta \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\, dx\, ds \end{aligned}$$

As h is a positive non increasing function, so we get

$$E^k(t) + \lambda \int_0^t \int_\Omega u_t^k g_1(u_t^k)\, dx\, ds + \beta \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\, dx\, ds \le E^k(0) \le C_1 \tag{4.22}$$

Where

$$E^k(t) = \frac{1}{l+2}\|u_t^k\|_{l+2}^{l+2} + \frac{b}{2(\gamma+1)}\|\nabla u^k\|_2^{2(\gamma+1)} + \frac{1}{2}\Big(a - \int_0^t h(s)ds\Big)\|\nabla u^k\|_2^2 + \frac{1}{2}\|\nabla u_t^k\|_2^2$$
$$+ \frac{1}{2}(h\, o\, \nabla u^k)(t) + \xi\tau(t)\int_\Omega \int_0^1 G(z^k(x,\rho,t))\, d\rho\, dx,$$

Noting $(A1)$ and (4.22), we obtain the first estimate:

$$\begin{aligned} \|u_t^k\|_{l+2}^{l+2} + \|\nabla u^k\|_2^2 + \|\nabla u_t^k\|_2^2 + (h \circ \nabla u^k)(t) + \int_\Omega \int_0^1 G(z^k(x,\rho,t))\, d\rho\, dx \\ + \int_0^t \int_\Omega u_t^k g_1(u_t^k)\, dx\, ds + \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\, dx\, ds \le C_2, \end{aligned} \tag{4.23}$$

where C_2 is a positive constant depending only on $\|u_0\|_{H_0^1},\ \|u_1\|_{H_0^1},\ l,\ \gamma,\ \xi,\ \tau_1,\ \lambda$ and β . These estimate imply that the solution (u^k, z^k) exists globally in $[0, +\infty)$.
Estimate (4.23) yields

$$u^k \text{ is bounded in } L^\infty_{loc}(0,\infty, H_0^1(\Omega)), \tag{4.24}$$

$$u_t^k \text{ is bounded in } L^\infty_{loc}(0,\infty, L^2(\Omega)), \tag{4.25}$$

$$G(z^k(x,\rho,t)) \text{ is bounded in } L^\infty_{loc}(0,\infty, L^1(\Omega\times(0,1))), \tag{4.26}$$

$$u_t^k(t) g_1(u_t^k(t)) \text{ is bounded in } L^1(\Omega\times(0,T)), \tag{4.27}$$

$$z^k(x,1,t) g_2(z^k(x,1,t)) \text{ is bounded in } L^1(\Omega\times(0,T))). \tag{4.28}$$

B. The second estimate.

Replacing w^j by $-\Delta w^j$ in (4.17), multiplying by c_t^{jk}, summing over j from 1 to k, it follows that

$$\int_\Omega |u_t^k(t)|^l u_{tt}^k(t)(-\Delta u_t^k)\, dx + \int_\Omega M(\|\nabla u^k\|^2)\Delta u^k \Delta u_t^k\, dx + \int_\Omega \Delta u_{tt}^k \Delta u_t^k\, dx$$
$$- \int_0^t h(t-s)\int_\Omega \Delta u^k \Delta u_t^k\, dx\, ds - \mu_1 \int_\Omega \Delta u_t^k g_1(u_t^k)\, dx - \mu_2 \int_\Omega \Delta u_t^k g_2(z^k(x,1,t))\, dx = 0.$$

Noting that $M(\|\nabla u^k\|^2) \geq a$ and by using the lemma 4.2.2, we obtain

$$
\begin{aligned}
&-\int_\Omega |u_t^k(t)|^l u_{tt}^k(t)\Delta u_t^k\,dx + \frac{1}{2}\frac{d}{dt}\Big[\Big(a-\int_0^t h(s)\,ds\Big)\|\Delta u^k\|_2^2 + \|\Delta u_t^k\|_2^2 + (ho\Delta u^k)\Big]\\
&\quad + \frac{1}{2}h(t)\|\Delta u^k\|_2^2 - \frac{1}{2}(h'o\Delta u^k) + \mu_1\int_\Omega |\nabla u_t^k|^2 g_1'(u_t^k)\,dx \\
&\quad + \mu_2\int_\Omega \nabla u_t^k \nabla z^k(x,1,t) g_2'(z^k(x,1,t))\,dx \leq 0.
\end{aligned}
\tag{4.29}
$$

By using the Green formula, we have

$$
-\int_\Omega |u_t^k(t)|^l u_{tt}^k \Delta u_t^k\,dx = \frac{d}{dt}\Big[\int_\Omega |u_t^k(t)|^l |\nabla u_t^k|^2\,dx\Big] - (l+1)\int_\Omega |u_t^k(t)|^l \nabla u_{tt}^k \nabla u_t^k\,dx.
$$

Consequently l'equation (4.29) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[2\int_\Omega |u_t^k(t)|^l|\nabla u_t^k|^2\,dx + \Big(a-\int_0^t h(s)\,ds\Big)\|\Delta u^k\|_2^2 + \|\Delta u_t^k\|_2^2 + (ho\Delta u^k)\Big]\\
&\quad - (l+1)\int_\Omega |u_t^k(t)|^l \nabla u_{tt}^k \nabla u_t^k\,dx + \frac{1}{2}h(t)\|\Delta u^k\|_2^2 - \frac{1}{2}(h'o\Delta u^k)\\
&\quad + \mu_1\int_\Omega |\nabla u_t^k|^2 g_1'(u_t^k)\,dx + \mu_2\int_\Omega \nabla u_t^k \nabla z_k(x,1,t) g_2'(z_k(x,1,t))\,dx \leq 0.
\end{aligned}
\tag{4.30}
$$

Replacing ϕ^j by $-\Delta\phi^j$ in (4.20), multiplying by d^{jk}, summing over j from 1 to k, it follows that

$$
\frac{\tau(t)}{1-\tau'(t)\rho}\int_\Omega \nabla z_t^k \nabla z^k\,dx + \int_\Omega \nabla z_\rho^k \nabla z^k\,dx = 0.
$$

Then, we get

$$
\frac{1}{2}\Big[\frac{d}{dt}\Big(\frac{\tau(t)}{1-\tau'(t)\rho}\|\nabla z^k\|_2^2\Big) - \big(\frac{\tau(t)}{1-\tau'(t)\rho}\big)'\|\nabla z^k\|_2^2\Big] + \frac{1}{2}\frac{d}{d\rho}\|\nabla z^k\|_2^2 = 0.
$$

We integrate over $(0,1)$, we find

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|\nabla z^k(x,\rho,t)\|_2^2\,d\rho + \frac{1}{2}\|\nabla z^k(x,1,t)\|_2^2\\
&= \frac{1}{2}\int_0^1 \Big(\frac{\tau(t)}{1-\tau'(t)\rho}\Big)'\|\nabla z^k(x,\rho,t)\|_2^2\,d\rho + \frac{1}{2}\|\nabla u_t^k(t)\|_2^2.
\end{aligned}
\tag{4.31}
$$

Combining (4.30) , (4.31) and using **(A2)**, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[2\int_{\Omega}|u_t^k(t)|^l|\nabla u_t^k|^2\,dx+\Big(a-\int_0^t h(s)\,ds\Big)\|\Delta u^k\|_2^2+\|\Delta u_t^k\|_2^2+(ho\Delta u^k)\\
&\quad+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|\nabla z^k(x,\rho,t)\|_2^2\,d\rho\Big]+\mu_1\int_{\Omega}|\nabla u_t^k|^2g_1'(u_t^k)\,dx+\frac{1}{2}\|\nabla z^k(x,1,t)\|_2^2\\
&\quad\le (l+1)\int_{\Omega}|u_t^k(t)|^l\nabla u_{tt}^k\nabla u_t^k\,dx-\mu_2\int_{\Omega}\nabla u_t^k\nabla z^k(x,1,t)g_2'(z^k(x,1,t))\,dx\\
&\quad+\frac{1}{2}\int_0^1(\frac{\tau(t)}{1-\tau'(t)\rho})'\|\nabla z^k(x,\rho,t)\|_2^2\,d\rho+\frac{1}{2}\|\nabla u_t^k(t)\|_2^2.
\end{aligned}\tag{4.32}$$

From the first estimate (4.23) and Young's inequality, we get

$$\begin{aligned}
\int_{\Omega}|u_t^k|^l\nabla u_{tt}^k(t)\nabla u_t^k\,dx&\le C_2^{l/(l+2)+1/2}\|\nabla u_{tt}^k\|_2\\
&\le \eta\|\nabla u_{tt}^k\|_2^2+\frac{C_2^{2l/(l+2)+1}}{4\eta},\quad \eta>0.
\end{aligned}\tag{4.33}$$

Using (4.6), Chaucy-Schwarz inequality, we obtain

$$\Big|\int_{\Omega}\nabla u_t^k\nabla z^k(x,1,t)g_2'(z^k(x,1,t))\,dx\Big|\le c_3^2\frac{\varepsilon}{2}\int_{\Omega}|\nabla u_t^k|^2\,dx+\frac{\varepsilon}{2}\int_{\Omega}|\nabla z^k(x,1,t)|^2\,dx. \tag{4.34}$$

Taking into account (4.33), (4.34) into (4.32) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[2\int_{\Omega}|u_t^k(t)|^l|\nabla u_t^k|^2\,dx+(a-\int_0^t h(s)\,ds)\|\Delta u^k\|_2^2+\|\Delta u_t^k\|_2^2+(ho\Delta u^k)\\
&\quad+\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|\nabla z^k(x,\rho,t)\|_2^2\,d\rho\Big]+\mu_1\int_{\Omega}|\nabla u_t^k|^2g_1'(u_t^k)\,dx+c\|\nabla z^k(x,1,t)\|_2^2\\
&\quad\le (l+1)\eta\|\nabla u_{tt}^k\|_2^2+C_\eta'C_1+c''\int_0^1(\frac{\tau(t)}{1-\tau'(t)\rho})'\|\nabla z^k(x,\rho,t)\|_2^2\,d\rho+c'\|\nabla u_t^k(t)\|_2^2.
\end{aligned}\tag{4.35}$$

Multiplying (4.17) by c_{tt}^{jk}, summing over j from 1 to k, it follows that

$$\begin{aligned}
&\int_{\Omega}|u_t^k|^l|u_{tt}^k|^2\,dx+\|\nabla u_{tt}^k\|_2^2=-\int_{\Omega}M(\|\nabla u^k\|^2)\nabla u^k\nabla u_{tt}^k\,dx\\
&+\int_0^t h(t-s)\int_{\Omega}\nabla u^k(s)\nabla u_{tt}^k(t)\,dx\,ds\\
&-\mu_1\int_{\Omega}u_{tt}^kg_1(u_t^k)\,dx-\mu_2\int_{\Omega}u_{tt}^kg_2(z^k(x,1,t))\,dx.
\end{aligned}\tag{4.36}$$

Differentiating (4.20) with respect to t, we get

$$((\frac{\tau(t)}{(1-\rho\tau'(t))})'z_t^k+\frac{\tau(t)}{(1-\rho\tau'(t))}z_{tt}^k+z_{t\rho}^k,\phi^j)=0.$$

Multiplying by d_t^{jk}, summing over j from 1 to k, it follows that

$$\Big(\frac{\tau(t)}{(1-\rho\tau'(t))}\Big)'\|z_t^k\|_2^2+\frac{1}{2}\frac{\tau(t)}{(1-\rho\tau'(t))}\frac{d}{dt}\|z_t^k\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_t^k\|_2^2=0,$$

then we have

$$\frac{1}{2}(\frac{\tau(t)}{(1-\rho\tau'(t))})'\|z_t^k\|_2^2+\frac{1}{2}\frac{d}{dt}\Big(\frac{\tau(t)}{(1-\rho\tau'(t))}\|z_t^k\|_2^2\Big)+\frac{1}{2}\frac{d}{d\rho}\|z_t^k\|_2^2=0.$$

Integrate over $(0,1)$ with respect to ρ, we obtain

$$\begin{aligned}&\frac{1}{2}\int_0^1(\frac{\tau(t)}{(1-\rho\tau'(t))})'\|z_t^k\|_2^2\,d\rho+\frac{1}{2}\frac{d}{dt}\int_0^1\frac{\tau(t)}{(1-\rho\tau'(t))}\|z_t^k\|_2^2\,d\rho\\&+\frac{1}{2}\|z_t^k(x,1,t)\|_2^2-\frac{1}{2}\|u_{tt}^k(x,t)\|_2^2=0.\end{aligned}\tag{4.37}$$

Summing (4.36) and (4.37), we get

$$\begin{aligned}&\int_\Omega|u_t^k|^l|u_{tt}^k|^2\,dx+\|\nabla u_{tt}^k\|_2^2+\frac{1}{2}\frac{d}{dt}\int_0^1\frac{\tau(t)}{(1-\rho\tau'(t))}\|z_t^k\|_2^2\,d\rho+\frac{1}{2}\|z_t^k(x,1,t)\|_2^2\\&=\frac{1}{2}\|u_{tt}^k(x,t)\|_2^2-\frac{1}{2}\int_0^1(\frac{\tau(t)}{(1-\rho\tau'(t))})'\|z_t^k\|_2^2\,d\rho-\int_\Omega M(\|\nabla u^k\|^2)\nabla u^k\nabla u_{tt}^k\,dx\\&+\int_0^t h(t-s)\int_\Omega\nabla u^k(s)\nabla u_{tt}^k(t)\,dx\,ds-\mu_1\int_\Omega u_{tt}^k g_1(u_t^k)\,dx-\mu_2\int_\Omega u_{tt}^k g_2(z^k(x,1,t))\,dx.\end{aligned}\tag{4.38}$$

By Cauchy-Schwarz,Sobolev and Young's inequalities, the right hand side of (4.38) can be estimated as follows:

$$\begin{aligned}|\int_\Omega M(\|\nabla u^k\|^2)\nabla u^k\nabla u_{tt}^k\,dx|&\le(a+b\|\nabla u^k\|^{2\gamma})\int_\Omega\nabla u^k\nabla u_{tt}^k\,dx\\&\le(a+bE(0)^{\frac{2\gamma}{2(\gamma+1)}})\int_\Omega\nabla u^k\nabla u_{tt}^k\,dx\\&\le\eta\|\nabla u_{tt}^k\|^2+\frac{m_0^2}{4\eta}\|\nabla u^k\|^2\end{aligned}\tag{4.39}$$

$$\begin{aligned}
&\Big|\int_0^t h(t-s)\int_\Omega \nabla u^k(s)\nabla u^k_{tt}(t)\,dx\,ds\Big| \\
&\qquad \le \eta\|\nabla u^k_{tt}\|^2 + \frac{1}{4\eta}\int_\Omega\Big(\int_0^t h(t-s)\nabla u^k(s)\,ds\Big)^2 dx \\
&\qquad \le \eta\|\nabla u^k_{tt}\|^2 + \frac{1}{4\eta}\int_0^t h(s)\,ds\int_\Omega\int_0^t h(t-s)|\nabla u^k(s)|^2\,ds\,dx \\
&\qquad \le \eta\|\nabla u^k_{tt}\|^2 + \frac{1}{4\eta}(a-k)\int_\Omega\int_0^t h(t-s)|\nabla u^k(s)|^2\,ds\,dx \\
&\qquad \le \eta\|\nabla u^k_{tt}\|^2 + \frac{1}{4\eta}(a-k)\int_0^t h(t-s)\|\nabla u^k(s)\|^2\,ds \\
&\qquad \le \eta\|\nabla u^k_{tt}\|^2 + \frac{1}{4\eta}(a-k)h(0)\int_0^t \|\nabla u^k(s)\|^2\,ds,
\end{aligned} \tag{4.40}$$

and from (4.5)

$$\begin{aligned}
\Big|\int_\Omega u^k_{tt}g_1(u^k_t)\,dx\Big| &\le \frac{1}{2}\int_\Omega |u^k_{tt}|^2\,dx + \frac{1}{2}\int_\Omega |g_1(u^k_t)| \\
&\le \frac{1}{2}\int_\Omega |u^k_{tt}|^2\,dx + \frac{1}{2}\int_{|u^k_t|\ge\varepsilon} |g_1(u^k_t)|^2\,dx + \frac{1}{2}\int_{|u^k_t|\le\varepsilon}|g_1(u^k_t)|^2\,dx \\
&\le \frac{1}{2}\int_\Omega |u^k_{tt}|^2\,dx + \frac{1}{2}\int_{|u^k_t|\ge\varepsilon} u^k_t g_1(u^k_t)\,dx + \frac{1}{2}\int_\Omega H^{-1}(u^k_t g_1(u^k_t))\,dx.
\end{aligned}$$

Using lemma4.2.3, Jensen's inequality and the concavity of H^{-1} , we obtain

$$\begin{aligned}
\Big|\int_\Omega u^k_{tt}g_1(u^k_t)\,dx\Big| &\le \frac{1}{2}\int_\Omega |u^k_{tt}|^2\,dx + \frac{1}{2}\int_{|u^k_t|\ge\varepsilon} u^k_t g_1(u^k_t)\,dx + cH^{-1}\Big(\int_\Omega u^k_t g_1(u^k_t)\,dx\Big) \\
&\le \frac{1}{2}\int_\Omega |u^k_{tt}|^2\,dx + \frac{1}{2}\int_{|u^k_t|\ge\varepsilon} u^k_t g_1(u^k_t)\,dx + c'H^*(1) + c''\int_\Omega u^k_t g_1(u^k_t)\,dx \\
&\le \frac{1}{2}\|u^k_{tt}\|_2^2\,dx + c'H^*(1) + c''\int_\Omega u^k_t g_1(u^k_t)\,dx \\
&\le C_s\|\nabla u^k_{tt}\|_2^2 + c'H^*(1) + c(-E').
\end{aligned} \tag{4.41}$$

From(4.6) (that is $|g(s)| \le c|s| \forall s \in \mathbb{R}$) we get

$$\begin{aligned}
\Big|\int_\Omega u^k_{tt}g_2(z^k(x,1,t))\,dx\Big| &\le \frac{1}{2}\int_\Omega |u^k_{tt}|^2\,dx + \frac{1}{2}\int_\Omega |g_2(z^k(x,1,t))|^2\,dx \\
&\le \frac{1}{2}\|u^k_{tt}\|_2^2 + c_3\int_\Omega z^k(x,1,t)g_2(z^k(x,1,t))\,dx \\
&\le C_s\|\nabla u^k_{tt}\|_2^2 + c'(-E').
\end{aligned} \tag{4.42}$$

Taking into account (4.39)-(4.42) into (4.38) yields

$$\begin{aligned}
&\int_\Omega |u_t^k|^l |u_{tt}^k|^2\,dx + \|\nabla u_{tt}^k\|^2 + \frac{1}{2}\frac{d}{dt}\int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)}\|z^k\|_2^2\,d\rho + \frac{1}{2}\|z^k(x,1,t)\|_2^2 \\
&\le (2\eta + 3C_s)\|\nabla u_{tt}^k\|^2 + \frac{a^2}{4\eta}\|\nabla u^k\|^2 + \frac{1}{4\eta}(a-k)h(0)\int_0^t \|\nabla u^k(s)\|^2\,ds \\
&+ c(-E') - \frac{1}{2}\int_0^1 \Big(\frac{\tau(t)}{1-\rho\tau'(t)}\Big)'\|z^k\|_2^2\,d\rho.
\end{aligned} \tag{4.43}$$

Combining (4.35) and (4.43), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[2\int_\Omega |u_t^k(t)|^l|\nabla u_t^k|^2\,dx + (a - \int_0^t h(s)\,ds)\|\Delta u^k\|_2^2 + \|\Delta u_t^k\|_2^2 + (ho\Delta u^k) \\
&+ \int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|\nabla z^k(x,\rho,t)\|_2^2\,d\rho + \int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)}\|z_t^k\|_2^2\,d\rho + cE(t)\Big] + \int_\Omega |u_t^k|^l|u_{tt}^k|^2\,dx \\
&+ \frac{1}{2}\|z_t^k(x,1,t)\|_2^2 + (1-(l+3)\eta - 3C_s)\|\nabla u_{tt}^k\|^2 + \mu_1\int_\Omega |\nabla u_t^k|^2 g_1'(u_t^k)\,dx + c\|\nabla z^k(x,1,t)\|_2^2 \\
&\le C_\eta' C_1 + c''\int_0^1 \Big(\frac{\tau(t)}{1-\rho\tau'(t)}\Big)'\|\nabla z^k(x,\rho,t)\|_2^2\,d\rho + c'\|\nabla u_t^k(t)\|_2^2 + \frac{a^2}{4\eta}\|\nabla u^k\|^2 \\
&+ \frac{1}{4\eta}(a-k)h(0)\int_0^t \|\nabla u^k(s)\|^2\,ds - \frac{1}{2}\int_0^1 \Big(\frac{\tau(t)}{1-\rho\tau'(t)}\Big)'\|z_t^k\|_2^2\,d\rho.
\end{aligned} \tag{4.44}$$

Then from (4.23) and by integration over (0,t), (4.44) yields

$$\begin{aligned}
&(a - \int_0^t h(s)\,ds)\|\Delta u^k\|_2^2 + \|\Delta u_t^k\|_2^2 + (ho\Delta u^k) + \int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|\nabla z^k(x,\rho,t)\|_2^2\,d\rho \\
&+ \int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)}\|z_t^k\|_2^2\,d\rho + cE(t) + 2\int_\Omega |u_t^k(t)|^l|\nabla u_t^k|^2\,dx + 2\int_0^t\int_\Omega |u_t^k|^l|u_{tt}^k|^2\,dx\,ds \\
&+ c\int_0^t \|z_t^k(x,1,t)\|_2^2\,ds + 2(1-(l+2)\eta - C_s)\int_0^t \|\nabla u_{tt}^k\|^2\,ds + c_*\int_0^t \|\nabla z^k(x,1,t)\|_2^2\,ds \\
&\le (C_\eta' C_1 + c' + \frac{a^2}{2\eta} + \frac{1}{2\eta}(a-k)h(0)T)T + E(0) + C_2 \\
&+ c_*''\int_0^t\int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|\nabla z^k(x,\rho,t)\|_2^2\,d\rho\,ds + c''\int_0^t\int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)}\|z_t^k\|_2^2\,d\rho\,ds.
\end{aligned}$$

For a suitable η, we get

$$\begin{aligned}
&\|\Delta u^k\|_2^2 + \|\Delta u_t^k\|_2^2 + (ho\Delta u^k) + \int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|\nabla z^k(x,\rho,t)\|_2^2\,d\rho \\
&+ \int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)}\|z_t^k\|_2^2\,d\rho + \int_0^t \|\nabla u_{tt}^k\|^2\,ds \\
&\leq (C'_\delta C_1 + +C'_1)T + C_2 + c'' \int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)}\|z_t^k\|_2^2\,d\rho \\
&+ c''_* \int_0^t \int_0^1 \Big(\frac{\tau(t)}{1-\rho\tau'(t)}\Big)' \|\nabla z^k(x,\rho,t)\|_2^2\,d\rho\,ds.
\end{aligned} \tag{4.45}$$

Using Gronwall' lemma, we obtained

$$\begin{aligned}
&\|\Delta u^k\|_2^2 + \|\Delta u_t^k\|_2^2 + (ho\Delta u^k) + \int_0^1 \frac{\tau(t)}{1-\tau'(t)\rho}\|\nabla z^k(x,\rho,t)\|_2^2\,d\rho \\
&+ \int_0^1 \frac{\tau(t)}{(1-\rho\tau'(t))}\|z_t^k\|_2^2\,d\rho + \int_0^t \|\nabla u_{tt}^k\|^2\,ds \leq C_3.
\end{aligned} \tag{4.46}$$

We observe from the estimate (4.23) and (4.46) that there exists a subsequence u^m of u^k and a functions u, z, χ, ψ such that

$$u^m \rightharpoonup u \text{ weakly star in } L^\infty(0,T,H^2(\Omega)\cap H_0^1(\Omega)), \tag{4.47}$$

$$u_t^m \rightharpoonup u_t \text{ weakly star in } L^\infty(0,T,H_0^2(\Omega)), \tag{4.48}$$

$$g_1(u_t^m) \rightharpoonup \chi \text{ weakly star in } L^2(\Omega\times(0,T)), \tag{4.49}$$

$$u_{tt}^m \rightharpoonup u_{tt} \text{ weakly star in } L^2(0,T,H_0^1(\Omega)), \tag{4.50}$$

$$z^m \rightharpoonup z \text{ weakly star in } L^\infty(0,T,H_0^1(\Omega,L^2(0,1))), \tag{4.51}$$

$$z_t^m \rightharpoonup z_t \text{ weakly star in } L^\infty(0,T,L^2(\Omega\times(0,1))), \tag{4.52}$$

$$g_2(z^m(x,1,t)) \rightharpoonup \psi \text{ weakly star in } L^2(\Omega\times(0,T)). \tag{4.53}$$

In the follow, we have to show that u is the solution of (4.1).

Firstly we will treat the nonlinear terms. From the first estimate (4.23) and Lemma 2.3, we deduce

$$\begin{aligned}
\||u_t^k|^l u_t^k\|_{L^2(0,T,L^2(\Omega)} &= \int_0^T \|u_t^k\|_{2(l+1)}^{2(l+1)}\,dt \\
&\leq C_s^{2(l+1)} \int_0^T \|\nabla u_t^k\|_2^{2(l+1)}\,dt \leq C_s^{2(l+1)} C_2^{2(l+1)} T.
\end{aligned}$$

On the other hand, from Aubin-Lions theorem, (see Lions **[29]**), we deduce that there exists a subsequence of $\{u^m\}$,still denoted by $\{u^m\}$ such that

$$u_t^m \to u_t \text{ strongly in } L^2(0,T,L^2(\Omega)), \tag{4.54}$$

which implies

$$u_t^m \to u_t \text{ almost everywhere in } \mathcal{A}. \tag{4.55}$$

Hence

$$|u_t^m|^l u_t^m \to |u_t|^l u_t \text{ almost everywhere in } \quad \mathcal{A}, \tag{4.56}$$

where $\mathcal{A} = \Omega \times (0,T)$. Thus, using (4.54), (4.56) and Lions Lemma, we derive

$$|u_t^m|^l u_t^m \rightharpoonup |u_t|^l u_t \text{ weakly in } \quad L^2(0,T,L^2(\Omega)), \tag{4.57}$$

and

$$z^m \to z \text{ strongly in } L^2(0,T,L^2(\Omega)),$$

which implies $z^m \to z$ almost everywhere in $\mathcal{A}$.

Lemma 4.3.1. *For each $T > 0$, $g_1(u_t)$, $g_2(z(x,1,t)) \in L^1(\mathcal{A})$ and $\|g_1(u')\|_{L^1(\mathcal{A})}$, $\|g_2(z(x,1,t))\|_{L^1(\mathcal{A})} \leq K$, where K is a constant independent of t.*

Proof. By **(A2)** and (4.55), we have

$$g_1(u_t^m(x,t)) \to g_1(u_t(x,t)) \text{ almost everywhere in } \mathcal{A},$$

$$0 \leq u_t^k(x,t) g_1(u_t^m(x,t)) \to u_t(x,t) g_1(u_t(x,t)) \text{ almost everywhere in } \mathcal{A}.$$

Hence, by (4.27) and Fatou's Lemma, we have

$$\int_0^T \int_\Omega u_t(x,t) g_1(u_t(x,t))\, dx\, dt \leq K_1 \text{ for } T > 0. \tag{4.58}$$

By Cauchy-Schwarz inequality and using (4.41) (4.58), we have

$$\begin{aligned} \int_0^T \int_\Omega |g_1(u_t(x,t))|\, dx\, dt &\leq c|\mathcal{A}|^{\frac{1}{2}} \Big(\int_0^T \int_\Omega u_t(x,t) g_1(u_t(x,t))\, dx\, dt \Big)^{\frac{1}{2}} \\ &\leq c|\mathcal{A}|^{\frac{1}{2}} K_1^{\frac{1}{2}} \equiv K. \end{aligned}$$

□

Lemma 4.3.2. $g_1(u_t^k) \rightharpoonup g_1(u_t)$ *weak in* $L^2(\Omega \times (0,T))$ *and* $g_2(z^k) \rightharpoonup g_2(z)$ *weak in* $L^2(\Omega \times (0,T))$

Proof. Let $E \subset \Omega \times [0,T]$ and set

$$E_1 = \Big\{(x,t) \in E,\ |g_1(u_t^k(x,t))| \leq \frac{1}{\sqrt{|E|}}\Big\},\ E_2 = E \backslash E_1,$$

where $|E|$ is the measure of E. If $M(r) = \inf\{|s|,\ s \in \mathbb{R} \text{ and } |g(s)| \geq r\}$

$$\int_E |g_1(u_t^k)|\,dx\,dt \leq c\sqrt{|E|} + \Big(M\Big(\frac{1}{\sqrt{|E|}}\Big)\Big)^{-1} \int_{E_2} |u_t^k g_1(u_t^k)|\,dx\,dt.$$

By applying (4.27) we deduce that $\sup_k \int_E |g_1(u_t^k)|\,dx\,dt \to 0$ as $|E| \to 0$. From Vitali's convergence theorem we deduce that

$$g_1(u_t^k) \to g_1(u_t) \quad \text{in } L^1(\Omega \times (0,T)).$$

Hence

$$g_1(u_t^k) \rightharpoonup g_1(u_t) \ \text{ weak in } L^2(\Omega \times (0,T)). \tag{4.59}$$

Similarly, we have

$$g_2(z^k) \rightharpoonup g_2(z) \ \text{ weak in } L^2(\Omega \times (0,T)). \tag{4.60}$$

□

By multiplying (4.17) and (4.20)) by $\theta(t) \in \mathcal{D}(0,T)$ and by integrating over $(0,T)$,it follows that

$$\begin{cases} -\frac{1}{l+1}\int_0^T (|u_t^k(t)|^l u_t^k(t), w^j)\theta'(t)\,dt + \int_0^T M(\|\nabla u^k(t)\|^2)(\nabla u^k(t), \nabla w^j)\theta(t)\,dt \\ +\int_0^T (\nabla u_{tt}^k, \nabla w^j)\theta(t)\,dt - \int_0^T\int_0^t h(t-s)(\nabla u^k(s), \nabla w^j)\theta(t)\,ds\,dt \\ +\mu_1 \int_0^T (g_1(u_t^k), w^j)\theta(t)\,dt + \mu_2\int_0^T (g_2(z^k(.,1)), w^j)\theta(t)\,dt = 0, \\ \int_0^T\int_0^1\int_\Omega (\tau(t) z_t^k + (1-\rho\tau'(t))z_\rho^k)\phi^j\theta(t)\,dx\,d\rho\,dt = 0, \end{cases} \tag{4.61}$$

For all $j = 1...k$. The convergence of (4.47)- (4.53), (4.57), (4.59) and (4.60) are sufficient to pass to the limit in (4.61) in order to obtain

$$\begin{cases} \int_0^T\int_\Omega \Big(|u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-s)\Delta^2 u(s)\,ds + \mu_1 g_1(u_t) \\ +\mu_2 g_2(z(.,1))\Big) w\theta(t)\,dx\,dt = 0, \\ \int_0^T\int_0^1\int_\Omega (\tau(t) z_t + (1-\rho\tau'(t))z_\rho)\phi^j\theta(t)\,dx\,d\rho\,dt = 0, \end{cases}$$

This completes the proof of Theorem 4.1.1.

4.4 Uniform Decay of the Energy-Proof of Theorem4.1.2

In this section we study the solution's asymptotic behavior of system (4.1).
To prove our main result, we construct a Lyapunov functional F equivalent to E. For this, we define some functionals which allow us to obtain the desired estimate.

Lemma 4.4.1. *Let (u,z) be a solution of the problem (4.4). Then, the functional*

$$I(t) = \tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}G(z(x,\rho,t))\,d\rho\,dx, \tag{4.62}$$

satisfies the estimate
i) $|I(t)| \leq \frac{1}{\xi}E(t)$
ii) $I'(t) \leq -2\tau(t)e^{-2\tau_1}\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx - \alpha_1(1-d)e^{-2\tau_1}\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx + \alpha_2\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx.$

Proof. ii)Differentiating (4.62) with respect to t and using (4.5), (4.3),($A4$) we get

$$\begin{aligned}
\frac{d}{dt}\,I(t) &= \tau'(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}G(z(x,\rho,t))\,d\rho\,dx\\
&+ \tau(t)\int_\Omega\int_0^1[e^{-2\tau(t)\rho}\frac{\partial G(z(x,\rho,t))}{\partial t} - 2\tau'(t)\rho e^{-2\tau(t)\rho}G(z(x,\rho,t))]\,d\rho\,dx\\
&= \int_\Omega\int_0^1 e^{-2\tau(t)\rho}\Big[\tau'(t)G(z(x,\rho,t)) + \tau(t)\frac{\partial G(z(x,\rho,t))}{\partial t}\Big]\,d\rho\,dx\\
&- 2\int_\Omega\int_0^1 \tau(t)\tau'(t)\rho e^{-2\tau(t)\rho}G(z(x,\rho,t))\,d\rho\,dx\\
&= -\int_\Omega\int_0^1 e^{-2\tau(t)\rho}\frac{\partial}{\partial\rho}\Big((1-\rho\tau'(t))G(z(x,\rho,t))\Big)\,d\rho\,dx\\
&- 2\int_\Omega\int_0^1 \tau(t)\tau'(t)\rho e^{-2\tau(t)\rho}G(z(x,\rho,t))\,d\rho\,dx\\
&= -\int_\Omega\int_0^1 \Big(\frac{\partial}{\partial\rho}\Big(e^{-2\tau(t)\rho}(1-\tau'(t)\rho)G(z(x,\rho,t))\Big)\\
&+ 2\tau(t)e^{-2\tau(t)\rho}(1-\tau'(t)\rho)G(z(x,\rho,t))\Big)\,d\rho\,dx - 2\tau(t)\tau'(t)\int_\Omega\int_0^1 \rho e^{-2\tau(t)\rho}G(z(x,\rho,t))\,d\rho\,dx\\
&= \int_\Omega G(u_t(x,t))\,dx - e^{-2\tau(t)}(1-\tau'(t))\int_\Omega G(z(x,1,t))\,dx\\
&- 2\tau(t)\int_\Omega\int_0^1[(1-\tau'(t)\rho) + \tau'(t)\rho]e^{-2\tau(t)\rho}G(z(x,\rho,t))\,d\rho\,dx\\
&\leq -2I(t) + \alpha_2\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx - e^{-2\tau(t)}(1-d)\alpha_1\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx.
\end{aligned}$$

Since $e^{-2\tau(t)\rho}$ is a decreasing function for $\rho \in [0,1]$ and $\tau(t) \in [\tau_0, \tau_1]$, we deduce

$$I(t) \geq \tau(t) \int_\Omega \int_0^1 e^{-2\tau_1} G(z(x,\rho,t))\, d\rho\, dx.$$

Thus, our proof is completed. □

Lemma 4.4.2. *Let (u,z) be a solution of the problem (4.4). Then, the functional*

$$\phi(t) = \frac{1}{l+1} \int_\Omega |u_t|^l u_t u\, dx + \int_\Omega \nabla u_t \nabla u\, dx,$$

satisfies the estimate
i) $|\phi(t)| \leq \frac{1}{l+2}\|u_t\|^{l+2} + \left(\frac{(l+1)^{-1}}{(l+2)} c_s^{l+2} (\frac{2E(0)}{a})^{\frac{l}{2}} + \frac{1}{2}\right)\|\nabla u\|^2 + \frac{1}{2}\|\nabla u_t\|^2.$
ii) $\phi'(t) \leq \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} - M(\|\nabla u\|^2)\|\nabla u\|^2 + (1+\eta)(a-k)\|\nabla u\|^2 + \frac{1}{4\eta}(ho\nabla u)(t) + \|\nabla u_t\|^2 - \mu_1 \int_\Omega u(x,t) g_1(u_t(x,t))\, dx - \mu_2 \int_\Omega u(x,t) g_2(z(x,1,t))\, dx.$
Where $\eta > 0$ and c_s is the sobolev embedding constant.

Proof. i) From Young's inequality ,the Sobolov embedding and lemma 4.2.3, we deduce

$$\begin{aligned}
|\phi(t)| &\leq \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\|u\|^{l+2} + \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 \\
&\leq \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2} c_s^{l+2}\|\nabla u\|^{l+2} + \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 \\
&\leq \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + (\frac{(l+1)^{-1}}{l+2} c_s^{l+2} (\frac{2E(0)}{a})^{\frac{l}{2}} + \frac{1}{2})\|\nabla u\|^2 + \frac{1}{2}\|\nabla u_t\|^2.
\end{aligned}$$

ii)Differentiating $\phi(t)$ with respect to t and using the first equation of (4.4), we get

$$\begin{aligned}
\phi'(t) &= \frac{1}{l+1}\int_\Omega (|u_t|^l u_t)' u\, dx + \frac{1}{l+1}\int_\Omega |u_t|^{l+2}\, dx + \int_\Omega \nabla u_{tt} \nabla u\, dx + \int_\Omega \nabla u_t \nabla u_t\, dx \\
&= \int_\Omega \Big[|u_t|^l u_{tt}\Big] u\, dx + \frac{1}{l+1}\|u_t\|^{l+2} - \int_\Omega \Delta u_{tt} u\, dx + \|\nabla u_t\|^2 \\
&= \frac{1}{l+1}\|u_t\|^{l+2} + \int_\Omega \Big[|u_t|^l u_{tt} - \Delta u_{tt}\Big] u\, dx + \|\nabla u_t\|^2 \\
&= \frac{1}{l+1}\|u_t\|^{l+2} + \int_\Omega \Big[M(\|\nabla u\|^2)\Delta u - \int_0^t h(t-s)\Delta u(s)\, ds - \mu_1 g_1(u_t(x,t)) \\
&\quad - \mu_2 g_2(z(x,1,t))\Big] u\, dx + \|\nabla u_t\|^2 \\
&= \frac{1}{l+1}\|u_t\|^{l+2} - M(\|\nabla u\|^2)\|\nabla u\|^2 + \int_\Omega \nabla u(t) \int_0^t h(t-s)\nabla u(s)\, ds\, dx \\
&\quad - \mu_1 \int_\Omega u g_1(u_t(x,t))\, dx - \mu_2 \int_\Omega u g_2(z(x,1,t))\, dx + \|\nabla u_t\|^2.
\end{aligned} \tag{4.63}$$

By use of Young's inequality and Sobolev embedding, we can estimate the third term in the right side as follow:

$$\begin{aligned}\int_\Omega \nabla u(t)\int_0^t h(t-s)\nabla u(s)\,ds\,dx &\le \int_0^t h(t-s)\int_\Omega \Big|\nabla u(t)(\nabla u(s)-\nabla u(t))\Big|\,dx\,ds\\ &+\|\nabla u(t)\|^2\int_0^t h(t-s)\,ds\\ &\le \|\nabla u(t)\|^2\int_0^t h(s)\,ds+\eta\|\nabla u(t)\|^2\int_0^t h(s)\,ds\\ &+\frac{1}{4\eta}\int_0^t h(t-s)\|\nabla u(s)-\nabla u(t))\|^2\,ds\\ &\le (1+\eta)(a-k)\|\nabla u(t)\|^2+\frac{1}{4\eta}(ho\nabla u)(t).\end{aligned}$$

Thus, our proof is completed. □

Lemma 4.4.3. *Let (u,z) be a solution of the problem (4.4). Then, the functional*

$$\psi(t)=\int_\Omega\Big(\Delta u_t-\frac{1}{l+1}|u_t|^l u_t\Big)\int_0^t h(t-s)(u(t)-u(s))\,ds\,dx,$$

satisfies the estimates

i) $|\psi(t)|\le\frac{1}{2}\|\nabla u_t\|^2+\Big(\frac{1}{2}(a-k)+\frac{(l+1)^{-1}}{(l+2)}(a-k)^{l+2}c_s^{l+2}(\frac{4E(0)}{a})^{\frac{l}{2}}\Big)(ho\nabla u)(t)+\frac{1}{l+2}\|u_t\|^{l+2}$.

ii) $\psi'(t)\le\delta(a-k)M(\|\nabla u\|^2)\|\nabla u\|^2+2\delta(a-k)^2\|\nabla u\|^2$
$+\Big(\frac{M_0}{4\delta}+(2\delta+\frac{1}{4\delta}+(\mu_1+\mu_2)\frac{c_s^2}{4\delta})(a-k)\Big)(ho\nabla u)(t)-\frac{h(0)}{4\delta}\Big(1+\frac{c_s^2}{l+1})\Big)(h'o\nabla u)(t)$
$+(\delta+\frac{\delta a_0}{l+1}-\int_0^t h(s)\,ds)\|\nabla u_t\|^2+\mu_1\delta\|g_1(u_t(x,t))\|^2+\mu_2\delta\|g_2(z(x,1,t))\|^2$
$-\frac{1}{l+1}\int_0^t h(s)\,ds\|u_t\|_{l+2}^{l+2},$.

Where $M_0=a+b(\frac{2E(0)}{a})^\gamma$,$a_0=c_s^{2(l+1)}(\frac{2E(0)}{a})^l$,$\eta>0$ *and* c_s *is the sobolev embedding constant.*

Proof. i)

$\psi(t)=-\int_\Omega\nabla u_t\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx-\int_\Omega\frac{1}{l+1}|u_t|^l u_t\int_0^t h(t-s)(u(t)-u(s))\,ds\,dx.$

We use Young's and Hölder's inequality with the conjugate exponents $p=\frac{l+2}{l+1}$ and $q=$

$l+2$,the second term in the right hand side can be estimated as

$$\begin{aligned}
&\Big| -\int_\Omega \frac{1}{l+1}|u_t|^l u_t \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx \Big| \\
&\le \frac{1}{l+2}\|u_t\|^{l+2} + \frac{(l+1)^{-1}}{l+2}\int_\Omega \Big[\int_0^t \Big(h(t-s)\Big)^{\frac{l+1}{l+2}}\Big((h(t-s))^{\frac{1}{l+2}}|u(t)-u(s)|\Big)\,ds\Big]^{l+2} dx \\
&\le \frac{1}{l+2}\|u_t\|^{l+2} + \frac{(l+1)^{-1}}{l+2}\Big(\int_0^t h(t-s)\,ds\Big)^{l+1}\int_\Omega\int_0^t h(t-s)|u(t)-u(s)|^{l+2}\,ds\,dx \\
&\le \frac{1}{l+2}\|u_t\|^{l+2} + \frac{(l+1)^{-1}}{l+2}(a-k)^{l+1}c_s^{l+2}(4\frac{E(0)}{a})^l(ho\nabla u)(t).
\end{aligned} \tag{4.64}$$

We get the last inequality from (4.8) and lemma 4.2.3.

The same , we use Young's and Hölder's inequalities with $p=q=2$, we get

$$\begin{aligned}
\Big| -\int_\Omega \nabla u_t \int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \Big| &\le \frac{1}{2}\int_\Omega \Big(\int_0^t h(t-s)|\nabla u(t)-\nabla u(s)|\,ds\Big)^2 dx \\
&+ \frac{1}{2}\|\nabla u_t\|^2 \\
&\le \frac{1}{2}(a-k)(ho\nabla u)(t) + \frac{1}{2}\|\nabla u_t\|^2,
\end{aligned} \tag{4.65}$$

Combining (4.64) and (4.65), we deduce

$$|\psi(t)| \le \frac{1}{2}\|\nabla u_t\|^2 + \Big(\frac{1}{2}(a-k) + \frac{(l+1)^{-1}}{(l+2)}(a-k)^{l+2}c_s^{l+2}(\frac{4E(0)}{a})^{\frac{l}{2}}\Big)(ho\nabla u)(t) + \frac{1}{l+2}\|u_t\|^{l+2}.$$

ii)We use the Liebnitz formula and the first equation of (4.4), we have

$$
\begin{aligned}
\psi'(t) &= \int_\Omega \Big(\Delta u_{tt} - |u_t|^l u_{tt}\Big) \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx \\
&+ \int_\Omega \Big(\Delta u_t - \frac{1}{l+1}|u_t|^l u_t\Big)\Big(\int_0^t (h'(t-s)(u(t)-u(s)) + h(t-s)u_t(t))\,ds\Big)\,dx \\
&= \int_\Omega M(\|\nabla u\|^2)\nabla u(t) \int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&- \int_\Omega \int_0^t h(t-s)\nabla u(s)\,ds \int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&+ \mu_1 \int_\Omega g_1(u_t(x,t)) \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx \\
&+ \mu_2 \int_\Omega g_2(z(x,1,t)) \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx \\
&- \int_\Omega \nabla u_t \int_0^t h'(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&- \frac{1}{l+1}\int_\Omega |u_t|^l u_t \int_0^t h'(t-s)(u(t)-u(s))\,ds\,dx - \|\nabla u_t\|^2 \int_0^t h(s)\,ds \\
&- \frac{1}{l+1}\|u_t\|^{l+2}\int_0^t h(s)\,ds \\
&= I_1 + I_2 + \mu_1 I_3 + \mu_2 I_4 + I_5 + I_6 - \|\nabla u_t\|^2 \int_0^t h(s)\,ds - \frac{1}{l+1}\|u_t\|^{l+2}\int_0^t h(s)\,ds.
\end{aligned} \tag{4.66}
$$

In what follows we will estimate $I_1, ..., I_6$

For I_1, we use Hölder's and Young's inequalities with $p = q = 2$, we get

$$
\begin{aligned}
|I_1| &\le M(\|\nabla u\|^2) \int_\Omega |\nabla u(t)| \Big(\int_0^t h(s)\,ds\Big)^{\frac{1}{2}} \Big(\int_0^t h(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\Big)^{\frac{1}{2}}\,dx \\
&\le M(\|\nabla u\|^2)\Big[\delta \int_\Omega |\nabla u(t)|^2 \int_0^t h(s)\,ds\,dx + \frac{1}{4\delta}\int_\Omega \int_0^t h(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\Big] \\
&\le M(\|\nabla u\|^2)\Big(\delta \|\nabla u(t)\|^2 \int_0^t h(s)\,ds + \frac{1}{4\delta}(ho\nabla u)(t)\Big) \\
&\le \delta M(\|\nabla u\|^2)\|\nabla u(t)\|^2 (a-k) + \frac{M_0}{4\delta}(ho\nabla u)(t).
\end{aligned} \tag{4.67}
$$

Where $M_0 = a + b(\frac{2E(0)}{a})^\gamma$ obtained by recalling (4.8)and lemma 4.2.3.

Similarly,

$$
\begin{aligned}
|I_2| &\le \delta \int_\Omega \Big(\int_0^t h(t-s)|\nabla u(s)|\,ds \Big)^2 dx + \frac{1}{4\delta} \int_\Omega \Big(\int_0^t h(t-s)|\nabla u(t) - \nabla u(s)|\,ds \Big)^2 dx \\
&\le \delta \int_\Omega \Big(\int_0^t h(t-s)(|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|)\,ds \Big)^2 dx + \frac{1}{4\delta}(\int_0^t h(s)\,ds)(ho\nabla u)(t) \\
&\le 2\delta \|\nabla u(t)\|^2 (\int_0^t h(t)\,ds)^2\,dx + (2\delta + \frac{1}{4\delta})(\int_0^t h(s)\,ds)(ho\nabla u)(t) \\
&\le 2\delta \|\nabla u(t)\|^2 (a-k)^2\,dx + (2\delta + \frac{1}{4\delta})(a-k)(ho\nabla u)(t).
\end{aligned}
\tag{4.68}
$$

$$
|I_3| \le \delta \|g_1(u_t(x,t))\|^2 + \frac{c_s^2}{4\delta}(a-k)(ho\nabla u)(t). \tag{4.69}
$$

$$
|I_4| \le \delta \|g_2(z(x,1,t))\|^2 + \frac{c_s^2}{4\delta}(a-k)(ho\nabla u)(t). \tag{4.70}
$$

$$
\begin{aligned}
|I_5| &\le \delta \int_\Omega |\nabla u_t|^2\,dx + \frac{1}{4\delta} \int_\Omega \Big(\int_0^t |h'(t-s)||\nabla u(t) - \nabla u(s)|\,ds \Big)^2 dx \\
&\le \delta \|\nabla u_t\|^2 + \frac{1}{4\delta} \int_0^t |h'(t-s)|\,ds \int_\Omega \int_0^t |h'(t-s)||\nabla u(t) - \nabla u(s)|^2\,ds\,dx.
\end{aligned}
$$

As h is a positive decreasing function so $|h'(t-s)| = -h'(t-s)$ and then

$$
\begin{aligned}
|I_5| &\le \delta \|\nabla u_t\|^2 + \frac{1}{4\delta} \int_0^t (-h'(t-s))\,ds \int_\Omega \int_0^t (-h'(t-s))|\nabla u(t) - \nabla u(s)|^2\,ds\,dx \\
&\le \delta \|\nabla u_t\|^2 - \frac{h(0)}{4\delta}(h'o\nabla u)(t).
\end{aligned}
\tag{4.71}
$$

$$
\begin{aligned}
|I_6| &\le \frac{1}{l+1}\Big[\delta \int_\Omega ||u_t|^l u_t|^2\,dx + \frac{1}{4\delta} \int_\Omega \Big(\int_0^t |h'(t-s)||u(t) - u(s)|\,ds \Big)^2 dx \Big] \\
&\le \frac{1}{l+1}\Big[\delta \|\ u_t\|_{2(l+1)}^{2(l+1)} - \frac{h(0)c_s^2}{4\delta}(h'o\nabla u)(t) \Big] \\
&\le \frac{\delta c_s^{2(l+1)}}{l+1} \|\ \nabla u_t\|_2^{2(l+1)} - \frac{h(0)c_s^2}{4\delta(l+1)}(h'o\nabla u)(t) \\
&\le \frac{\delta a_0}{l+1} \|\ \nabla u_t\|_2^2 - \frac{h(0)c_s^2}{4\delta(l+1)}(h'o\nabla u)(t).
\end{aligned}
\tag{4.72}
$$

Where $a_0 = c_s^{2(l+1)}(\frac{2E(0)}{a})^l$ obtained by recalling (4.8) and lemma 4.2.3.
Combining (4.66)and (4.67)-(4.72), we finish the proof. □

Now, for $M, \varepsilon_1 > 0$, we introduce the following functional

$$F(t) = ME(t) + \varepsilon_1 \phi(t) + \psi(t) + I(t). \tag{4.73}$$

Lemma 4.4.4. *Let (u,z) be a solution of the problem (4.4). Assume that* **(A1)**- **(A5)** *hold and that Then $F(t)$ satisfies, along the solution and for some positive constants $m, c > 0$, the following estimate*

$$\begin{aligned} F'(t) \leq -mE(t) + c\Big[&\|g_1(u_t(x,t))\|^2 + \|g_2(z(x,1,t))\|^2 \\ &+ \int_\Omega |u(x,t)g_1(u_t(x,t))|\,dx + \int_\Omega |u(x,t)g_2(z(x,1,t))|\,dx\Big], \end{aligned} \tag{4.74}$$

and $F(t) \sim E(t)$.

Proof. By recalling ii) of lemmas 4.2.3-4.4.3 and by **(A2)**, we deduce that for $t \geq t_0 > 0$

$$\begin{aligned} &F'(t) = ME'(t) + \varepsilon_1 \phi'(t) + \psi'(t) + I'(t) \\ &\leq -(M\lambda - \alpha_2) \int_\Omega u_t(x,t) g_1(u_t(x,t))\,dx \\ &- (M\beta + \alpha_1(1-d)e^{-2\tau_1}) \int_\Omega z(x,1,t) g_2(z(x,1,t))\,dx \\ &- 2\tau(t)e^{-2\tau_1} \int_\Omega \int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx - (\varepsilon_1 - \delta(a-k))M(\|\nabla u\|^2)\|\nabla u\|^2 \\ &- \frac{1}{l+1}(h_0 - \varepsilon_1)\|u_t\|_{l+2}^{l+2} - (h_0 - \delta(1 + \frac{a_0}{l+1}) - \varepsilon_1)\|\nabla u_t\|^2 \\ &- (\frac{Mh_1}{2} - \varepsilon_1(1+\eta)(a-k) - 2\delta(a-k)^2)\|\nabla u(t)\|^2 \\ &- \Big[\zeta(\frac{M}{2} - \frac{h(0)}{4\delta}\Big(1 + \frac{c_s^2}{l+1}\Big)) - \Big(\frac{\varepsilon_1}{4\eta} + \frac{M_0}{4\delta} + (2\delta + \frac{1}{4\delta} + (\mu_1 + \mu_2)\frac{c_s^2}{4\delta})(a-k)\Big)\Big](ho\nabla u)(t) \\ &+ \mu_1\delta\|g_1(u_t(x,t))\|^2 + \mu_2\delta\|g_2(z(x,1,t))\|^2 - \varepsilon_1\mu_1 \int_\Omega u(x,t)g_1(u_t(x,t))\,dx \\ &- \varepsilon_1\mu_2 \int_\Omega u(x,t)g_2(z(x,1,t))\,dx. \end{aligned}$$

Where $h_0 = \int_0^{t_0} h(s)ds$ and $h_1 = \min\{h(t), t \geq t_0 > 0\}$. We take $h_0 > \varepsilon_1$ and $\delta > 0$ sufficiently small such that
$a_4 = h_0 - \delta(1 + \frac{a_0}{l+1}) - \varepsilon_1 > 0$ and $a_2 = \varepsilon_1 - \delta(a-k) > 0$.
As long as ε_1 and δ are fixed,we choose M large enough such that

$$a_1 = M\lambda - \alpha_2 > 0, \quad a_5 = \frac{Mh_1}{2} - \varepsilon_1(1+\eta)(a-k) - 2\delta(a-k)^2 > 0,$$

and

$$a_6 = \zeta(\frac{M}{2} - \frac{h(0)}{4\delta}\Big(1 + \frac{c_s^2}{l+1}\Big)) - \Big(\frac{\varepsilon_1}{4\eta} + \frac{M_0}{4\delta} + (2\delta + \frac{1}{4\delta} + (\mu_1 + \mu_2)\frac{c_s^2}{4\delta})(a-k)\Big) > 0.$$

Thus

$$\begin{aligned} F'(t) &\leq -a_3\frac{1}{l+1}\|u_t\|_{l+2}^{l+2} - a_2 M(\|\nabla u\|^2)\|\nabla u\|^2 - a_4\|\nabla u_t\|^2 - a_6(ho\nabla u)(t) \\ &- 2\tau(t)e^{-2\tau_1}\int_\Omega\int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx \\ &+ c\Big[\|g_1(u_t(x,t))\|^2 + \|g_2(z(x,1,t))\|^2 + \int_\Omega |u(x,t)g_1(u_t(x,t))|\,dx \\ &+ \int_\Omega |u(x,t)g_2(z(x,1,t))|\,dx\Big] \\ &\leq -mE(t) + c\Big[\|g_1(u_t(x,t))\|^2 + \|g_2(z(x,1,t))\|^2 + \int_\Omega |u(x,t)g_1(u_t(x,t))|\,dx \\ &+ \int_\Omega |u(x,t)g_2(z(x,1,t))|\,dx\Big]. \end{aligned}$$

Where $m = min\{2a_2, 2e^{-2\tau_1}\xi, 2a_4, a_3\}$.

To prove $F(t) \sim E(t)$, we show that there exist two positive constants κ_1 and κ_2 such that

$$\kappa_1 E(t) \leq F(t) \leq \kappa_2 E(t). \tag{4.75}$$

From i) of lemmas 4.4.1-4.4.3, we get $\kappa > 0$ depending the $\varepsilon_1, a, l, c_s, E(0), k, \xi$ such that

$$\Big|\varepsilon_1\phi(t) + \psi(t) + I(t)\Big| \leq \kappa E(t).$$

For a choose of M large enough such that $\kappa_1 = M - \kappa > 0$, we get our result. □

4.4.1 Proof of Theorem 4.1.2

As in Komornik **[24]**, we consider the following partition of Ω

$$\Omega_1 = \{x \in \Omega : \ |u_t| \leq \varepsilon\}, \quad \Omega_2 = \{x \in \Omega : \ |u_t| > \varepsilon\}.$$

We use Young's inequality (with $p = q = 2$) , (2.10) and lemma 3.2.1,we have

$$\begin{aligned} \int_\Omega |ug_1(u_t)|\,dx + \|g_1(u_t)\|_2^2 &\leq \delta\|u\|_2^2 + (\frac{1}{4\delta} + 1)\|g_1(u_t)\|_2^2 \\ &\leq \delta C_s^2\|\nabla u\|_2^2 \\ &+ (\frac{1}{4\delta} + 1)(\int_{\Omega_1} H^{-1}(u_t g_1(u_t))\,dx + c_2\int_{\Omega_2} u_t g(u_t)\,dx) \\ &\leq \frac{2\delta C_s^2}{a}E(t) + c_\delta\int_{\Omega_1} H^{-1}(u_t g_1(u_t))\,dx - C_\delta E'(t). \end{aligned} \tag{4.76}$$

Similarly and by application of (4.6), we obtain

$$\begin{aligned}\int_\Omega |ug_2(z(x,1,t))|\,dx + \|g_2(z(x,1,t))\|_2^2 &\leq \delta C_s^2\|\nabla u\|_2 \\ &\quad + (\frac{1}{4\delta}+1)c_3\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \qquad (4.77)\\ &\leq \frac{2\delta C_s^2}{a}E(t) - C'_\delta E'(t).\end{aligned}$$

Combining (4.76) and (4.77) the (4.74) becomes

$$F'(t) \leq -(m - \frac{4\delta C_s}{a})E(t) - C''_\delta E'(t) + c_\delta \int_{\Omega_1} H^{-1}(u_t g_1(u_t))\,dx.$$

Now, for δ small enough, the function $L(t) = F(t) + C''_\delta E(t)$ satisfies

$$L'(t) \leq -dE(t) + c\int_{\Omega_1} H^{-1}(u_t g_1(u_t))\,dx, \qquad (4.78)$$

and

$$L(t) \sim E(t).$$

- Case 1. H is linear on $[0,\varepsilon]$, Using (4.5) and Lemma 4.2.3, we deduce that

$$L'(t) \leq -dE(t) - cE'(t).$$

  Thus $R = L + cE \sim E$ satisfies

$$R(t) \leq R(0)e^{-c't}$$

  Hence,

$$E(t) \leq C(E(0))e^{-c't}.$$

- Case 2. H is nonlinear on $[0,\varepsilon]$ so, we exploit Jensen's inequality and the concavity of H^{-1} to obtain

$$H^{-1}(\frac{1}{|\Omega_1|}\int_{\Omega_1} u_t g(u_t)\,dx) \geq c\int_{\Omega_1} H^{-1}(u_t g(u_t))\,dx.$$

  Then (4.78) becomes

$$L'(t) \leq -dE(t) + cH^{-1}(\frac{1}{|\Omega_1|}\int_{\Omega_1} u_t g(u_t)\,dx). \qquad (4.79)$$

  For $\varepsilon_0 < \varepsilon$ and $w_0 > 0$, we define L_0 by

$$L_0(t) = H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)L(t) + w_0 E(t).$$

Then, we easily see that, For b_1, $b_2 > 0$

$$b_1 L_0(t) \leq E(t) \leq b_2 L_0(t). \tag{4.80}$$

By recalling that $E' \leq 0$, $H' > 0$, $H'' > 0$ on $(0, \varepsilon]$ and using (4.79) , we obtain

$$\begin{aligned} L_0'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)} H''\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) L(t) + H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) L'(t) + w_0 E'(t) \\ &\leq -dE(t) H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) + cH'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) H^{-1}\big(\frac{1}{|\Omega_1|} \int_{\Omega_1} u_t g(u_t)\, dx\big) + w_0 E'(t), \end{aligned}$$

using Remark 4.2.1 with H^*, the convex conjugate of H in the sense of Young, we obtain

$$\begin{aligned} L_0'(t) &\leq -dE(t) H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) + cH^*\Big(H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big)\Big) + \frac{c}{|\Omega_1|} \int_{\Omega_1} u_t g(u_t)\, dx) + w_0 E'(t) \\ &\leq -dE(t) H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) + c\varepsilon_0 \frac{E(t)}{E(0)} H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) - w_1 E'(t) + w_0 E'(t), \end{aligned}$$

where w_1 is a positive constant depending of Ω_1, α_2. By taking ε_0 small enough and $w_0 > w_1$, we obtain

$$L_0'(t) \leq -w \frac{E(t)}{E(0)} H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) = -wH_2\Big(\frac{E(t)}{E(0)}\Big), \tag{4.81}$$

where $H_2(t) = tH'(\varepsilon_0 t)$ a positive increasing function on $(0, 1]$.
By setting $L_1(t) = \frac{b_1 L_0(t)}{E(0)}$, we easily see that, by (4.80), we have

$$L_1(t) \sim E(t). \tag{4.82}$$

Using (4.81), we arrive at

$$L_1'(t) \leq -w_1 H_2(L_1(t)).$$

By recalling (2.13), we deduce

$$L_1'(t) \leq w_1 \frac{1}{H_1'(L_1(t))},$$

which gives

$$\Big[H_1(L_1(t))\Big]' \leq w_1.$$

A simple integration leads to

$$H_1(L_1(t)) \leq w_1 t + H_1(L_1(0)),$$

consequently

$$L_1(t) \leq H_1^{-1}(w_1 t + w_2), \tag{4.83}$$

using (4.82) and (4.83), we obtain (4.10). The proof is completed.

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Abstract

In this thesis, we considered the problem with partial differential equations of hyperbolic type (wave equations, Kirchhoff equations and petrovsky equations) with the presence of different mechanisms of dissipation, damping and for more general forms of nonlinearities, addressed from a different angle. Under some assumptions on initial data, conditions on damping, delays and viscoelastic terms, we proved the global existence and asymptotic behavior of the solutions.

Résume

Dans cette thèse, nous avons considéré le problème aux dérivées partielles de type hyperbolique (Equations des Ondes, Equations de Kirchhoff et Equations de Petrovsky) avec la présence de différents mécanismes de dissipation, d'amortissement de différent point de vue. Sous quelques hypothèses sur les données initiales, conditions sur les termes de dissipation, les termes de retards et viscoélastiques, nous avons montré l'existence globale et le comportement asymptotique des solutions.

ملخص

في هذه الأطروحة قمنا بدراسة بعض المعادلات الرياضية من نوع قطوع زائدة (معادلة الموجات، معادلة كيرشوف ومعادلة بيتروفسكي) مع وجود آليات للتبديد، في ظل بعض الفرضيات على البيانات الاولية وعلى شروط التبديد والتأخير. وقد أثبتنا وجود حلول عامة لهذه المعادلات وحددنا معدل انخفاض الطاقة المرتبطة بهذه الحلول.