République Algérienne Démocratique et Populaire

Ministère de l'Enseignement Supérieuret de la Recherche Scientifique

Université Djillali Liabès Sidi-Bel-Abbes

Faculté des Sciences Exactes

Thèse de doctorat

Option: Système Dynamique et Application

Présentée par:

Bayour Benaoumeur

Spécialité: Mathématiques

Intitulée

# Contributions dans les échelles de temps

Soutenue le 00/00/2017 devant le jury composé de:

| | | |
|---|---|---|
| Président | BENCHOHRA Mouffak | Pr, Université de Sidi Bel-Abbès |
| Directeur de Thèse | HAMMOUDI Ahmed | Pr, CU d'Ain Témouchent |
| Co-Directeur de Thèse | TORRES Delfim F. M. | Pr, Université de Aveiro, Portugal |
| Examinateurs | BELMEKKI Mohammed | Pr, Uni. de Saida |
| | OUAHAB Abdelghani | Pr, Uni. de Sidi Bel-Abbès |
| | OUMANSOUR Abderrahmane | MCA, Uni. de Sidi Bel-Abbès |

Année Universitaire: 2016 – 2017

People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

University of Djillali Liabes Sidi-Bel-Abbes

Faculty of Exact Sciences

PhD Thesis

Option: Dynamic Systems and Applications

Presented by:

Bayour Benaoumeur

PhD Thesis in Mathematics

Entitled

# Contributions on Time Scales

PhD Thesis submitted: 00/00/2017

The jury:

| | | |
|---|---|---|
| President | BENCHOHRA Mouffak | Professor, Sidi Bel-Abbès University |
| Supervisor | HAMMOUDI Ahmed | Professor,Ain Témouchent University |
| Co-Supervisor | TORRES Delfim F. M. | Professor, University of Aveiro, Portugal |
| Examiners committe | BELMEKKI Mohammed | Professor, University of Saida |
| | OUAHAB Abdelghani | Professor, Sidi Bel-Abbès University |
| | OUMANSOUR Abderrahmane | Assistant Prof., Sidi Bel-Abbès University |

university year : 2016 – 2017

# Contents

# Acknowledgements

First of all, thanks Allah.

Then I would like to express my deepest sense of Gratitude to my co-supervisor, Professor Delfim F. M. Torres, for inviting me to his university, the University of Aveiro, in Portugal, who offered his continuous advice and encouragement throughout the course of this thesis, and his efficiency cooperation and supervision in the original works of this thesis (see 1-2-3-4 next page).

I would like to express my very sincere gratitude to my supervisor, Professor Ahmed Hamoudi, who communicated his passion, curiosity and its mathematical opening in all these years.

I am also grateful to Prof. Mouffak Benchohra, who accepted to chair this thesis committee.

I am also thankful to Prof. Ouahab Abdelghani, Prof. Belmekki Mohammed, and Dr. Oumansour Abderrahmane, who in the midst of all their activity, accepted to be members of this thesis committee.

Finally, I take this opportunity to express the profound gratitude from my deep heart to my family, my friends in University of Sidi Bel-Abbes and in other Universities.

# Publications

1. **B. Bayour**, A. Hammoudi and D. F. M. Torres, *Existence of solution to a nonlinear first-order dynamic equation on time scales*, Journal of Mathematical Analysis, Volume 7, Issue 1 (2016), pages 31–38.

2. **B. Bayour** and D. F. M. Torres, *Complex-valued fractional derivatives on time scales*, Springer Proceedings in Mathematics & Statistics, Volume 164 (2016), Springer, pages 79–87.

3. **B. Bayour** and D. F. M. Torres, *Existence of solution to a local fractional nonlinear differential equation*, Journal of Computational and Applied Mathematics 312 (2017), pages 127–133.

4. **B. Bayour**, A. Hammoudi and D. F. M. Torres, *A truly conformable calculus on time scales*, submitted November 2016 Revised 04-March and 03-April, 2017; Accepted 23-May-2017,Journal Global and Stochastic Analysis(GSA), ISSN 2248-9444

# Introduction

In this work, we prove existence of solution to the following systems:

$$\begin{cases} x^{\nabla}(t) = f(t, x(t)), \quad t \in \mathbb{T}_k, \\ x(b) = x_0, \end{cases} \tag{1}$$

and

$$\begin{cases} x^{\nabla}(t) = f(t, x(\rho(t)), \quad \nabla.a.e.t \in \mathbb{T}_k, \\ x \in (BC). \end{cases} \tag{2}$$

Here $\mathbb{T}$ is an arbitrary bounded time scales, where we denote $a := \min \mathbb{T}$, $b := \max \mathbb{T}$, $\mathbb{T}^{\circ} = \mathbb{T} \backslash \{a\}$, and in (2) we have that $f : \mathbb{T}^{\circ} \times \mathbb{R}^n \to \mathbb{R}^n$ is a $\nabla$-Carathéodory function and $(BC)$ denotes one of the following boundary conditions:

$$x(b) = x_0 \tag{3}$$

or

$$x(b) = x(a). \tag{4}$$

In (1) we suppose $f : \mathbb{T}^{\circ} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Problem (2) (resp. (1)) unifies continuous and discrete problems. We use the notion of tube solution for system (2) (resp. (1)), in the spirit of the works of Gilbert and Frigon [30, 32, 34]. This notion is useful to get existence results for systems of differential equations of first order, as a generalization of lower and upper solutions [20, 29, 46, 55]. Our main result provides existence of solution to the nonlinear nabla boundary value problem (2) (resp. (1)). A solution of this problem will be a function $x \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n)$ (resp. $x \in C^1_{\nabla}(\mathbb{T}, \mathbb{R}^n)$), satisfying (2)(resp,(1)).

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking noninteger order powers of the differentiation and/or integration operators. Even though the term "fractional" is a misnomer, it has been widely accepted for a long time: the term was coined by the famous mathematician Leibniz in 1695 in a letter to L'Hôpital [52]. In the paper *What is a fractional derivative?* [47], Ortigueira and Machado distinguish between local and nonlocal fractional derivatives. Here we are concerned with local operators only. Such local approach to the fractional calculus dates back at least to 1974, to the use of the fractional incremental ratio in [25]. For an overview and recent developments of the local approach to fractional calculus we refer the reader to [45, 49, 56, 57] and references therein. Recently, Khalil et al. introduced a new well-behaved definition of local fractional (noninteger order) derivative, called the conformable fractional derivative [43]. The new calculus is very interesting and is getting an increasing of interest – see [19, 28] and references therein. In [1], Abdeljawad proves chain rules, exponential functions, Gronwall's inequality, fractional integration by parts, Taylor power series expansions and Laplace transforms for the conformable fractional calculus. Furthermore, linear differential systems are discussed [1]. In [11], Batarfi et al. obtain the Green function for a conformable fractional linear problem and then introduce the study of nonlinear conformable fractional differential equations. See also [4] where, using the conformable fractional derivative, a second-order conjugate boundary value problem is investigated and utilizing the corresponding positive fractional Green's function and an appropriate fixed point theorem, existence of a positive solution is proved. For abstract Cauchy problems of conformable fractional systems see [2]. Here we are concerned with the following problem:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t)), \quad t \in [a,b], \quad a > 0, \\ x(a) = x_0, \end{cases} \tag{5}$$

where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of $x$ at $t$ of order $\alpha$, $\alpha \in (0,1)$. For the first time in the literature of conformable fractional calculus, we introduce the notion of tube solution. Such idea of tube solution has been used with success to investigate existence of solutions for ordinary differentiable equations [30, 31], delta and nabla differential equations on time scales [12, 32, 34], and dynamic inclusions [33]. Roughly speaking, the tube solution method generalizes the method of lower and upper solution [24, 26, 35, 51].

The study of fractional (noninteger) order derivatives on discrete, continuous and, more generally, on an arbitrary nonempty closed set (i.e., a time scale) is a well-known subject under strong current development. The subject is very rich and several different definitions and approaches are available, either in discrete [9], continuous [53], and time-scale settings [16]. In continuous time, i.e., for the time scale $\mathbb{T} = \mathbb{R}$, several definitions are based on the classical Euler Gamma function $\Gamma$. For the time scale $\mathbb{T} = \mathbb{Z}$, the Gamma function is nothing else than the factorial, while for the $q$-scale one has the $q$-Gamma function $\Gamma_q$ [38]. For the definition of Gamma function on an arbitrary time scale $\mathbb{T}$ see [21]. Similarly to [16, 17], here we introduce a new notion of fractional derivative on an arbitrary time scale $\mathbb{T}$ that does not involve Gamma functions. Our approach is, however, different from the ones available in the literature [16, 17, 18, 19]. In particular, while in [16, 17, 18, 19] the fractional derivative at a point is always a real number, here, in contrast, the fractional derivative at a point is, in general, a complex number. For example, the derivative of order $\alpha \in (0, 1]$ of the square function $t^2$ is always given by $t^\alpha + (\sigma(t))^\alpha$, where $\sigma(t)$ is the forward jump operator of the time scale, which is in general a complex number (e.g., for $\alpha = 1/2$ and $t < 0$) and a generalization of the Hilger derivative $(t^2)^\Delta = t + \sigma(t)$.

Local, limit-based definitions of a so-called conformable derivative on time scales, have been recently formulated in [19] by

$$T_\alpha(f)(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} t^{1-\alpha}, \quad \alpha \in (0, 1], \tag{6}$$

and then in [59] by

$$T_\alpha(f)(t) = \frac{f^\sigma(t) - f(t)}{\sigma(t)^\alpha - t^\alpha}, \quad \alpha \in (0,1]. \tag{7}$$

Note that if $f$ is $\Delta$-diferentiable at a right-scattered point $t \in \mathbb{T}^\kappa_{[0,\infty)}$ [22], then $f$ is $\alpha$-differentiable in both cases: for the first definition (6) we have

$$T_\alpha(f)(t) = t^{1-\alpha} f^\Delta(t) \tag{8}$$

while for the second definition (7) one has

$$T_\alpha(f^\Delta)(t) = \frac{\mu(t)}{\sigma^\alpha(t) - t^\alpha} f^\Delta(t), \tag{9}$$

where $f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}$. The conformable calculus in the time scale $\mathbb{T} = \mathbb{R}$ is now a well-developed subject: see, e.g., [1, 2, 44] and references therein. However, the adjective *conformable* may not be appropriate, because $T_0 f \neq f$, that is, letting $\alpha \to 0$ does not result in the identity operator. This is also the case for the recent results of [58]. Moreover, according to (8) and (9), the variable $t$ must satisfy $t \geq 0$. With this in mind, in this paper we extend the calculus of [6], by considering a truly conformable derivative of order $\alpha$, $0 \leq \alpha \leq 1$, on an arbitrary time scale $\mathbb{T}$.

# Notations

| Notation | Definition |
|---|---|
| $\mathbb{N}_0$ | $\mathbb{N}\cup\{0\}$ |
| $\backslash$ | Difference of sets |
| $[a,b]_{\mathbb{T}}$ | $[a,b]\cap\mathbb{T}$ |
| $\mathbb{T}^{\circ}$ | $\mathbb{T}\backslash\{a\} \quad where \quad a=\min\mathbb{T}$ |
| $a.e.$ | Almost everywhere |
| $\langle\cdot,\cdot\rangle$ | Inner product |
| $\mu_{\nabla}$ | The Lebesgue nabla-measure |
| $\Vert\cdot\Vert_X$ | norm in the space $X$ |
| $C([a,b])$ | Space of continuous functions on $[a,b]$ |
| $C^k([a,b])$ | Space of functions of class K on $[a,b]$ |
| $L^1([a,b])$ | $\left\{f:[a,b]\to\mathbb{R}: f \text{ measurable}, \int_a^b \vert f(t)\vert\, dt<\infty\right\}$ |
| $C^k_{ld}(\mathbb{T})$ | Space of functions of class K nabla-derivative on$\mathbb{T}$and$f^{\nabla^k}$is ld-continuous on $(\mathbb{T})_{\kappa^\kappa}$ |
| $L^1_{\nabla}\left(\mathbb{T}_{[a,b]}\right)$ | $\left\{f:\mathbb{T}_{[a,b]}\to\mathbb{R}: f \text{ nabla-measurable}, \int_{\mathbb{T}_{[a,b]}} \vert f(t)\vert\,\nabla t<\infty\right\}$ |
| $\vert x\vert$ | Absolute value of $x$ |
| $\overline{\Omega}$ | closed set of $\Omega$ |

# Chapter 1

# Preliminaries

## 1.1 Preliminaries on analysis

For further results and deeper explanations, see [36].

**Definition 1.1.1.** *A map is a continuous tronsformation.*

**Definition 1.1.2** (See p. 112 of [36])**.** *Let $X$, $Y$ be topological spaces. A map $f : X \to Y$ is called compact if $f(X)$ is contained in a compact subset of $Y$.*

**Theorem 1.1.1.** *(chauder) Let $E$ be a normed linear space and let $C \subset E$ be convex. Then any compact $F : C \to C$ has a fixed point.*

**Theorem 1.1.2** ( Arzelà-Ascoli theorem )**.** *A subset of $C(\bar{\Omega})$ is relatively compact if and only if it is bounded and equicontinuous.*

## 1.2 Preliminaries on time scales

A time scale $\mathbb{T}$ is defined to be any nonempty closed subset of $\mathbb{R}$. Then the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{t \in \mathbb{T} : s < t\}.$$

For $t \in \mathbb{T}$, we say that $t$ is left-scattered (respectively right-scattered) if $\rho(t) < t$ (respectively $\sigma(t) > t$); that $t$ is isolated if it is left-scattered and right-scattered. Similarly, if $t > \inf(\mathbb{T})$ and $\rho(t) = t$, then we say that $t$ is left-dense; if $t < \sup(\mathbb{T})$ and $\sigma(t) = t$, then we say that $t$ is right-dense. Points that are simultaneously left- and right-dense are called dense. If $\mathbb{T}$ has a right-scattered minimum $m$, then we define $\mathbb{T}_\kappa := \mathbb{T} - \{m\}$; otherwise, we set $\mathbb{T}_\kappa := \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum $M$, then we define $\mathbb{T}^\kappa := \mathbb{T} - \{M\}$; otherwise $\mathbb{T}^\kappa := \mathbb{T}$. Finally, the (backward) graininess function $\nu : \mathbb{T}_\kappa \to [0, +\infty[$ is defined by $\nu(t) := t - \rho(t)$. And the (jump) graininess function $\mu : \mathbb{T}_\kappa \to [0, +\infty[$ is defined by $\mu(t) = \sigma(t) - t$.

### 1.2.1 Nabla-derivative on time scales

**Definition 1.2.1.** *Assume $f : \mathbb{T} \to \mathbb{R}^n$ is a function and let $t \in \mathbb{T}_\kappa$. We say that $f$ is $\nabla$-differentiable at $t$ if there exists a vector $f^\nabla(x) \in \mathbb{R}^n$ such that for all $\epsilon > 0$, there exists a neighborhood $U$ of $t$, where*

$$\left\| f(s) - f\left(\rho(t)\right) - f^\nabla(t)\left(s - \rho(t)\right) \right\| \leq \epsilon |s - \rho(t)| \tag{1.1}$$

*for every $s \in U$. We call $f^\nabla(t)$ the $\nabla$-derivative of $f$ at $t$. If $f$ is $\nabla$-differentiable at $t$ for every $t \in \mathbb{T}_\kappa$, then $f^\nabla : \mathbb{T}_\kappa \to \mathbb{R}^n$ is called the $\nabla$-derivative of $f$ on $\mathbb{T}_\kappa$.*

**Theorem 1.2.1.** *Assume $f : \mathbb{T} \to \mathbb{R}^n$ is a function and let $t \in \mathbb{T}_\kappa$.Then, we have the following.*

**(i)** *If $f$ is $\nabla$-differentiable at $t$, then $f$ is continuous at $t$.*

**(ii)** *If $f$ is continuous at $t$ and if $t$ is left-scattered, then $f$ is $\nabla$-differentiable at $t$ and*

$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$

**(iii)** *If $t$ is left dense, then $f$ is $\nabla$-differentiable at $t$ if and only if*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

*exists in $\mathbb{R}^n$. In this case, $f^\nabla(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$.*

**(iv)** *If $f$ is $\nabla$-differentiable at $t$, then $f(\rho(t)) = f(t) - \nu(t)f^{\nabla}(t)$.*

**Theorem 1.2.2.** *If $f, g : \mathbb{T} \to \mathbb{R}$ are $\nabla$-differentiable at $t \in \mathbb{T}_\kappa$, then*

**(i)** *$f + g$ is $\nabla$-differentiable at $t$ and $(f + g)^{\nabla}(t) = f^{\nabla}(t) + g^{\nabla}(t)$;*

**(ii)** *$\alpha f$ is $\nabla$-differentiable at $t$ for every $\alpha \in \mathbb{R}$ and $(\alpha f)^{\nabla}(t) = \alpha(f)^{\nabla}(t)$;*

**(iii)** *$fg$ is $\nabla$-differentiable at $t$ and*

$$(fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f(\rho(t))g^{\nabla}(t) = f(t)g^{\nabla}(t) + f^{\nabla}(t)g(\rho(t));$$

**(iv)** *If $g(t)g(\rho(t)) \neq 0$, then $\frac{f}{g}$ is $\nabla$-differentiable at $t$ and $(\frac{f}{g})^{\nabla}(t) = \frac{f^{\nabla}(t)g(t) - f(t)g^{\nabla}(t)}{g(t)g(\rho(t))}$.*

**Theorem 1.2.3.** *Let $W$ be an open set of $\mathbb{R}^n$ and $t \in \mathbb{T}$ be a left-dense point. If $g : \mathbb{T} \to \mathbb{R}^n$ is nabla-differentiable at $t$ and $f : W \to \mathbb{R}$ is differentiable at $g(t) \in W$, then $f \circ g$ is nabla-differentiable at $t$ with $(f \circ g)^{\nabla}(t) = \langle f'(g(t)), g^{\nabla}(t)\rangle$.*

*Proof.* Let $\epsilon > 0$. We need to show that there exists a neighborhood $U$ of $t$ such that $\left|f(g(t)) - f(g(s)) - \langle f'(g(t)), g^{\nabla}(t)\rangle(t - s)\right| \leq \epsilon|t - s|$ for all $s \in U$. Let $k > 0$ be a constant and $\epsilon' = \frac{\epsilon}{k}$. By hypotheses, there exists a neighborhood $U_1$ of $t$ where $\left\|g(t) - g(s) - g^{\nabla}(t)(t - s)\right\| \leq \epsilon'|t - s|$ for all $s \in U_1$. In addition, there exists a neighborhood $V \subset W$ of $g(t)$ such that $|f(g(t)) - f(y) - \langle f'(g(t)), g(t) - y\rangle| \leq \epsilon'|g(t) - y|$ for all $y \in V$. Since function $g$ is $\nabla$-differentiable at $t$, it is also continuous at this point, and there exists a neighborhood $U_2$ of $t$ such that $g(s) \in V$ for all $s \in U_2$. Let $U := U_1 \cap U_2$. In this case $U$ is a neighborhood of $t$ and if $s \in U$, then

$$\begin{aligned}
&|f(g(t)) - f(g(s)) - \langle f'(g(t)), g^{\nabla}(t)\rangle(t - s)| \\
&\quad \leq |f(g(t)) - f(g(s)) - \langle f'(g(t)), g(t) - g(s)\rangle| \\
&\qquad + \left|\langle f'(g(t)), g(t) - g(s)\rangle - \langle f'(g(t)), g^{\nabla}(t)\rangle(t - s)\right| \\
&\quad \leq \epsilon' \|g(t) - g(s)\| + \left|\langle f'(g(t)), g(t) - g(s) - g^{\nabla}(t)(t - s)\rangle\right| \\
&\quad \leq \epsilon' \left(\epsilon'|t - s| + \|g^{\nabla}(t)(t - s)\|\right) + \|f'(g(t))\| \, \|g(t) - g(s) - g^{\nabla}(t)(t - s)\| \\
&\quad \leq \epsilon'(1 + \|g^{\nabla}(t)\| + \|f'(g(t))\|)|t - s|.
\end{aligned}$$

Put $k = 1 + \|g^{\nabla}(t)\| + \|f'(g(t))\|$ and the theorem is proved. $\square$

**Example 1.2.1.** *Assume $x : \mathbb{T} \to \mathbb{R}^n$ is nabla differentiable at $t \in \mathbb{T}$. We know that $\|\cdot\| : \mathbb{R}^n \backslash \{0\} \to [0, +\infty[$ is differentiable if $t = \rho(t)$. It follows from Theorem 1.2.3 that*

$$\|x(t)\|^{\nabla} = \frac{\langle x(t), x^{\nabla}(t) \rangle}{\|x(t)\|}.$$

**Definition 1.2.2.** *A function $f : \mathbb{T} \to \mathbb{R}^n$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}^n$ is denoted by $C_{ld}(\mathbb{T}, \mathbb{R}^n)$. The set of functions $f : \mathbb{T} \to \mathbb{R}^n$ that are nabla-differentiable and whose nabla-derivative is ld-continuous, is denoted by $C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$.*

### 1.2.2 Lebesgue nabla integration

In this section we define a theory of measure and integration for an arbitrary bounded time scales $\mathbb{T}$ where $a = \min \mathbb{T} < \max \mathbb{T} = b$. We recall the notion of $\nabla$-measure in time scales.

**Definition 1.2.3.** *Let $\mathcal{F}_1$ be a family of intervals of $\mathbb{T}$ of the form*

$$(c, d] = \{t \in \mathbb{T} : c < t \leq d\},$$

*where $c, d \in \mathbb{T}$ and $c \leq d$. We define an additive measure $m_1 : \mathcal{F}_1 \to \mathbb{R}$ by $m_1((c, d]) = d - c$.*

**Definition 1.2.4.** *An outer measure $m_1^* : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$ is defined as follows: for $E \subset \mathbb{T}$,*

$$m_1^*(E) = \begin{cases} \inf \left\{ \sum_{k=1}^{k=n} m_1(A_k) : E \subset \bigcup\limits_{k=1}^{k=m} A_k \text{ with } A_k \in \mathcal{F}_1 \right\} & \text{if} \quad a \notin E, \\ +\infty & \text{if} \quad a \in E. \end{cases}$$

**Definition 1.2.5.** *A set $A \subset \mathbb{T}$ is said to be $\nabla$-measurable if for every set $E \subset \mathbb{T}$*

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \backslash A)).$$

Now, denote

$$\mathcal{M}_\nabla(m_1^*) = \{A \subset \mathbb{T} : A \text{ is } \nabla \text{ mesurable}\}.$$

The Lebesgue $\nabla$-measure on $\mathcal{M}_\nabla(m_1^*)$, denoted by $\mu_\nabla$, is the restriction of $m_1^*$ to $\mathcal{M}_\nabla(m_1^*)$. We get a complete measurable space with $\left(\mathbb{T}, \mathcal{M}_\nabla(m_1^*), \mu_\nabla\right)$. With this definition of complete measurable space for a bounded time scale $\mathbb{T}$, we can define the notions of $\nabla$-measurability and $\nabla$-integrability for functions $f : \mathbb{T} \to \mathbb{R}$, following the same ideas of the theory of Lebesgue integration.

### 1.2.3 Nabla-measurability and nabla-integrability for functions on time scales

**Definition 1.2.6.** *Let $E \subset \mathbb{T}$ be a $\nabla$-measurable set and $f : \mathbb{T} \to \mathbb{R}$ be a $\nabla$-measurable function. We say that $f \in L^1_\nabla(E)$ provided*

$$\int_E |f(s)|\nabla s < \infty.$$

*We say that a $\nabla$-measurable function $f : \mathbb{T} \to \mathbb{R}^n$ is in the set $L^1_\nabla(E, \mathbb{R}^n)$, provided*

$$\int_E |f_i(s)|\nabla s < \infty$$

*for each of its components $f_i : \mathbb{T} \to \mathbb{R}$.*

**Proposition 1.2.1.** *Assume $f \in L^1_\nabla(E, \mathbb{R}^n)$. Then,*

$$\left\| \int_E f(s)\nabla s \right\| \le \int_E \|f(s)\| \nabla s.$$

Many results of integration theory are established for measurable functions $f : X \to \mathbb{R}$ where $(X, \tau, \mu)$ is a complete measurable space. These results are in particular true for the measurable space $\left(\mathbb{T}, \mathcal{M}_\nabla(m_1^*), \mu_\nabla\right)$. We recall two results of the theory of integration adapted to our situation.

**Theorem 1.2.4** (Lebesgue-dominated convergence theorem)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^1_\nabla(\mathbb{T}^\circ)$. If there exists a function $f : \mathbb{T}^\circ \to \mathbb{R}$ such that $f_n(t) \to f(t)$ is $\nabla$ a.e. in $\mathbb{T}^\circ$ and if there exists a function $g \in L^1_\nabla(\mathbb{T}^\circ)$ such that $\|f_n(t)\| \le g(t)\nabla$ a.e. in $t \in \mathbb{T}^\circ$,for every $n \in \mathbb{N}$. Then $f_n \to f$ in $L^1_\nabla(\mathbb{T}^\circ)$.*

**Theorem 1.2.5.** *The set $L^1_\nabla(\mathbb{T}^\circ)$ is a Banach space endowed with the norm $\|f\|_{L^1_\nabla} := \int_{\mathbb{T}^\circ} |f(s)| \nabla s$.*

**Lemma 1.2.1.** *The set of left-scattered points of $\mathbb{T}$ is at most countable: there is a set of indexes $J \subset \mathbb{N}$ and a set $\{t_i\}_{i\in J} \subset \mathbb{T}$ such that $\mathcal{L}_\mathbb{T} := \{t \in \mathbb{T} : \rho(t) < t\} = \{t_i\}_{i\in J}$.*

*Proof.* Let $g : [a, b] \to \mathbb{R}$ be defined by

$$g(t) = \begin{cases} t & \text{if } t \in \mathbb{T}, \\ \rho(s) & \text{if } t \in (\rho(s), s), \ s \in \mathbb{T}. \end{cases}$$

It is obvious that the function $g$ is monotone on $[a, b]$ and continuous over the set $[a, b] \setminus \{t \in [a, b]_\mathbb{T} : \rho(t) < t\}$. As the set of all points of discontinuity of a monotone function is countable, then the set $\mathcal{L}_\mathbb{T}$ is countable. □

**Proposition 1.2.2.** *Let $A \subset \mathbb{T}$. Then, $A$ is $\nabla$-measurable if and only if $A$ is Lebesgue measurable. In such a case, the following properties hold for every $\nabla$-measurable set $A$.*

**i-** *If $a$ is not in $A$, then*

$$\mu_\nabla(A) = \sum_{j\in J_A} (t_j - \rho(t_j)) + \mu_L(A);$$

**ii-** *$\mu_\nabla(A) = \mu_L(A)$ if and only if $a$ is not in $A$ and $A$ has no left-scattered points. Here, $J_A = \{j \in J : t_j \in \mathcal{L}_\mathbb{T} \cap A\}$.*

To establish the relation between $\nabla$-integration on $\mathbb{T}$ and Lebesgue integration on a real compact interval, the function $f : \mathbb{T} \to \mathbb{R}$ is extended to $[a, b]$ in the following way:

$$\hat{f}(t) := \begin{cases} f(t) & \text{if } t \in \mathbb{T}, \\ f(t_j) & \text{if } t \in (\rho(t_j), t_j), \ j \in J_\mathbb{T}. \end{cases}$$

**Theorem 1.2.6.** *Let $E \subset \mathbb{T}$ be a $\nabla$-measurable set such that $a$ is not in $E$ and let $\hat{E} = E \cup \bigcup_{j \in J_E} (\rho(t_j), t_j)$. Let $f : \mathbb{T} \to \mathbb{R}$ be a $\nabla$-measurable function and $\hat{f} : [a,b] \to \mathbb{R}$ its extension on $[a,b]$. Then, $f$ is $\nabla$-measurable on $E$ if and only if $\hat{f}$ is Lebesgue integrable on $\hat{E}$. In such a case, one has*

$$\int_E f(s)\nabla s = \int_{\hat{E}} \hat{f}(s)ds. \tag{1.2}$$

Function $f : \mathbb{T} \to \mathbb{R}$ can be extended on $[a,b]$ in another way. Define $\bar{f}(t) : [a,b] \to \mathbb{R}$ by

$$\bar{f}(t) := \begin{cases} f(t) & \text{if } t \in \mathbb{T}, \\ f(t_j) + \frac{f(t_j) - f(\rho(t_j))}{\nu(t_j)}(t_j - t) & \text{if } t \in (\rho(t_j), t_j),\ j \in J_{\mathbb{T}}. \end{cases}$$

**Definition 1.2.7.** *A function $f : \mathbb{T} \to \mathbb{R}$ is said to be absolutely continuous on $\mathbb{T}$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $\{(a_k, b_k]\}_{k=1}^n$ with $a_k, b_k \in \mathbb{T}$ is a finite pairwise disjoint family of subintervals of $\mathbb{T}$ satisfying $\sum_{k=1}^n (b_k - a_k) < \delta$, then $\sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon$.*

**Lemma 1.2.2.** *If $\bar{f}$ is differentiable at $t \in (a,b] \cap \mathbb{R}$, then $f$ is $\nabla$-differentiable at $t$ and $f^{\nabla}(t) = \bar{f}'(t)$.*

**Theorem 1.2.7.** *Consider a function $f : \mathbb{T} \to \mathbb{R}$ and its extension $\bar{f} : [a,b] \to \mathbb{R}$. Then, $f$ is absolutely continuous on $\mathbb{T}$ if and only if $\bar{f}$ is absolutely continuous on $[a,b]$.*

**Theorem 1.2.8.** *A function $f : \mathbb{T} \to \mathbb{R}$ is absolutely continuous on $\mathbb{T}$ if and only if $f$ is $\nabla$-differentiable $\nabla$ almost everywhere on $\mathbb{T}^\circ$, $f^\nabla \in L^1_\nabla(\mathbb{T}^\circ)$ and*

$$\int_{(t,b]\cap\mathbb{T}} f^\nabla(s)\nabla s = f(b) - f(t), \quad \forall t \in \mathbb{T}. \tag{1.3}$$

**Proposition 1.2.3.** *Let $E$ be a Banach space and $u : [a,b] \to E$ an absolutely continuous function. Then the measure of the set $\{t \in [a,b] : u(t) = 0 \, and \, u'(t) \neq 0\}$ is zero.*

**Proposition 1.2.4.** *Let $g \in L^1_\nabla(\mathbb{T}^\circ)$ and $G : \mathbb{T} \to \mathbb{R}$ the function defined by*

$$G(t) := \int_{[b,t)\cap\mathbb{T}} g(s)\nabla s. \tag{1.4}$$

*Then, $G^\nabla(t) = g(t)$ $\nabla$ almost everywhere on $\mathbb{T}^\circ$.*

**Proposition 1.2.5.** *Let $u : \mathbb{T} \to \mathbb{R}$ be an absolutely continuous function. Then the $\nabla$-measure of the set $\{t \in \mathbb{T}^\circ \setminus \mathcal{L}_{\mathbb{T}^\circ} : u(t) = 0 \quad and \quad u^\nabla(t) \neq 0\}$ is zero.*

**Definition 1.2.8.** *The function $p$ is $\nu$-regressive if*

$$1 - \nu(t)p(t) \neq 0 \quad \textit{for all } t \in \mathbb{T}_\kappa.$$

*Define the $\nu$-regressive class of functions on $T_\kappa$ to be*

$$\mathcal{R}_\nu = \{p : \mathbb{T} \to \mathbb{R} \quad is \quad Id-continuous \quad and \quad \nu-regressive\}.$$

**Definition 1.2.9** (See [23])**.** *For $\epsilon > 0$, the (nabla) exponential function $\hat{e}_\epsilon(\cdot, t_0) : \mathbb{T} \to \mathbb{R}$ is defined as the unique solution to the initial value problem*

$$x^\nabla(t) = \epsilon x(t), \quad x(t_0) = 1.$$

*More explicitly, the exponential function $\hat{e}_\epsilon(\cdot, t_0) : \mathbb{T} \to \mathbb{R}$ is given by the formula*

$$\hat{e}_\epsilon(t, t_0) = exp\left(\int_{t_0}^t \hat{\xi}_\epsilon(\nu(s))\nabla s\right),$$

*where for $h \geq 0$ we define $\hat{\xi}_\epsilon(h)$ as*

$$\hat{\xi}_\epsilon(h) = \begin{cases} \epsilon & \textit{if } h = 0, \\ -\frac{log(1-h\epsilon)}{h} & \textit{otherwise}, \end{cases}$$

*where* log *is the principal logarithm function.*

**Theorem 1.2.9** (See [23])**.** *Let $p \in \mathcal{R}_\nu$ and $s, t, r \in \mathbb{T}$. Then*

**(1)** *$\hat{e}_0(t, s) \equiv 1$ and $\hat{e}_p(t, t) \equiv 1$;*

**(2)** *$\hat{e}_p(\rho(t), s) = (1 - \nu(t)p(t))\hat{e}_p(t, s)$;*

**(3)** $\hat{e}_{\ominus p}(t,s) = \frac{1}{\hat{e}_p(t,s)}$;

**(4)** $\hat{e}_p(t,s) = \frac{1}{\hat{e}_p(s,t)} = \hat{e}_{\ominus p}(s,t)$;

**(5)** $\hat{e}_p(t,s)\hat{e}_p(s,r) = \hat{e}_p(t,r)$;

**(6)** $\left(\frac{1}{\hat{e}_p(t,s)}\right)^{\nabla} = \frac{-p(t)}{\hat{e}_p^{\rho}(t,s)}$.

**Definition 1.2.10.** *A function $f : \mathbb{T}_\kappa \times \mathbb{R}^n \to \mathbb{R}^n$ is called a $\nabla$-Carathéodory function if the three following conditions hold:*

**(i)** *the map $t \to f(t,x)$ is $\nabla$-measurable for every $x \in \mathbb{R}^n$;*

**(ii)** *the map $x \to f(t,x)$ is continuous $\nabla$ a.e. $t \in \mathbb{T}_\kappa$;*

**(iii)** *for every $R > 0$, there exists a function $h_R \in L^1_\nabla(\mathbb{T}_\kappa, [0,\infty))$ such that $\|f(t,x)\| \leq h_R(t) \quad \nabla$ a.e. $t \in \mathbb{T}_\kappa$ and for every $x \in \mathbb{R}^n$ such that $\|x\| \leq R$.*

We recall the notion of Sobolev's space for functions defined on a bounded time scale $\mathbb{T}$.

**Definition 1.2.11.** *We say that a function $u : \mathbb{T} \to \mathbb{R}$ belongs to $W^{1,1}_\nabla(\mathbb{T})$ if and only if $u \in L^1_\nabla(\mathbb{T}^\circ)$ and there exists a function $g : \mathbb{T}_\kappa \to \mathbb{R}$ such that $g \in L^1_\nabla(\mathbb{T}^\circ)$ and*

$$\int_{\mathbb{T}^\circ} u(s)\phi^{\nabla}(s)\nabla s = -\int_{\mathbb{T}^\circ} g(s)\phi(\sigma(s))\nabla s \quad \text{for every } \phi \in C^1_{0,ld}(\mathbb{T}) \tag{1.5}$$

*with*

$$C^1_{0,ld}(\mathbb{T}) := \{f : \mathbb{T} \to \mathbb{R}; f \in C^1_{ld}(\mathbb{T}), f(a) = 0 = f(b)\}. \tag{1.6}$$

*We say that a function $f : \mathbb{T} \to \mathbb{R}^n$ is in the set $W^{1,1}_\nabla(\mathbb{T},\mathbb{R}^n)$ if each of its components $f_i$ are in $W^{1,1}_\nabla(\mathbb{T})$.*

## 1.3 Preliminaries on a local fractional derivative

We consider fractional derivatives in the conformable sense [43].

**Definition 1.3.1** (Conformable fractional derivative [43])**.** *Let* $\alpha \in (0,1)$ *and* $f : [0,\infty) \to \mathbb{R}$*. The conformable fractional derivative of* $f$ *of order* $\alpha$ *is defined by* $T_\alpha(f)(t) := \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon}$ *for all* $t>0$*. Often, we write* $f^{(\alpha)}$ *instead of* $T_\alpha(f)$ *to denote the conformable fractional derivative of* $f$ *of order* $\alpha$*. In addition, if the conformable fractional derivative of* $f$ *of order* $\alpha$ *exists, then we simply say that* $f$ *is* $\alpha$*-differentiable. If* $f$ *is* $\alpha$*-differentiable in some* $t \in (0,a)$*,* $a>0$*, and* $\lim_{t\to 0^+} f^{(\alpha)}(t)$ *exists, then we define* $f^{(\alpha)}(0) := \lim_{t\to 0^+} f^{(\alpha)}(t)$*.*

**Theorem 1.3.1** (See [43])**.** *Let* $\alpha \in (0,1]$ *and assume* $f, g$ *to be* $\alpha$*-differentiable. Then,*

1. $T_\alpha(af+bg) = aT_\alpha(f) + bT_\alpha(g)$ *for all* $a, b \in \mathbb{R}$*;*

2. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$*;*

3. $T_\alpha(f/g) = \left(gT_\alpha(f) - fT_\alpha(g)\right)/g^2$*.*

*If, in addition,* $f$ *is differentiable at a point* $t>0$*, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$*.*

**Remark 1.3.1.** *From Theorem 1.3.1 it follows that if* $f \in C^1$*, then one has*

$$\lim_{\alpha\to 1} T_\alpha(f)(t) = f'(t)$$

*and*

$$\lim_{\alpha\to 0} T_\alpha(f)(t) = tf'(t). \tag{1.7}$$

*So* $T_\alpha(f)$ *is "conformable" in the sense it coincides with* $f'$ *in the case* $\alpha \to 1$ *and satisfies similar properties to the integer-order calculus. Note that the property* $\lim_{\alpha\to 0} T_\alpha(f) \neq f$ *is not uncommon in fractional calculus, both for local and nonlocal operators: see, e.g., the local fractional derivative of [40, 41], for which property* (1.7) *also holds [7]; and the classical nonlocal Marchaud fractional derivative, which* *is zero when $\alpha \to 0$ [52]. Note, however, that we only have $T_\alpha(f)(t) = t^{1-\alpha} f'(t)$ in case $f$ is differentiable. If one considers a function that is not differentiable at a point $t$, then the conformable derivative is not $t^{1-\alpha} f'(t)$. For applications we refer the reader to [28].*

**Example 1.3.1.** *Let $\alpha \in (0, 1]$. Functions $f(t) = t^p$, $p \in \mathbb{R}$, $g(t) \equiv \lambda$, $\lambda \in \mathbb{R}$, $h(t) = e^{ct}$, $c \in \mathbb{R}$, and $\beta(t) = e^{\frac{1}{\alpha} t^\alpha}$, are $\alpha$-differentiable with conformable fractional derivatives of order $\alpha$ given by*

1. $T_\alpha(f)(t) = pt^{p-\alpha}$;
2. $T_\alpha(g)(t) = 0$;
3. $T_\alpha(h)(t) = ct^{1-\alpha} e^{ct}$;
4. $T_\alpha(\beta)(t) = e^{\frac{1}{\alpha} t^\alpha}$.

**Remark 1.3.2.** *Differentiability implies $\alpha$-differentiability but the contrary is not true: a nondifferentiable function can be $\alpha$-differentiable. For a discussion of this issue see [43].*

**Definition 1.3.2** (Conformable fractional integral [43])**.** *Let $\alpha \in (0, 1)$ and $f : [a, \infty) \to \mathbb{R}$. The conformable fractional integral of $f$ of order $\alpha$ from $a$ to $t$, denoted by $I_\alpha^a(f)(t)$, is defined by*

$$I_\alpha^a(f)(t) := \int_a^t \frac{f(\tau)}{\tau^{1-\alpha}} d\tau,$$

*where the above integral is the usual improper Riemann integral.*

**Theorem 1.3.2** (See [43])**.** *If $f$ is a continuous function in the domain of $I_\alpha^a$, then $T_\alpha\left(I_\alpha^a(f)\right)(t) = f(t)$ for all $t \geq a$.*

**Notation 1.3.1.** *Let $0 < a < b$. We denote by ${}_\alpha\mathfrak{J}_a^b[f]$ the value of the integral $\int_a^b \frac{f(t)}{t^{1-\alpha}} dt$, that is, ${}_\alpha\mathfrak{J}_a^b[f] := I_\alpha^a(f)(b)$.*

**Proposition 1.3.1.** *Assume $f \in L^1([a, b], \mathbb{R})$, $0 < a < b$. Then $\left|{}_\alpha\mathfrak{J}_a^b[f]\right| \leq {}_\alpha\mathfrak{J}_a^b[|f|]$.*

*Proof.* Let $f \in L^1([a,b],\mathbb{R})$. Then,

$$\left|{}_{\alpha}\mathfrak{J}_a^b[f]\right| = \left|\int_a^b \frac{f(t)}{t^{1-\alpha}}dt\right| \leq \int_a^b \left|\frac{f(t)}{t^{1-\alpha}}\right| dt = \int_a^b \frac{|f(t)|}{t^{1-\alpha}}dt.$$

Therefore, $\left|{}_{\alpha}\mathfrak{J}_a^b[f]\right| \leq {}_{\alpha}\mathfrak{J}_a^b[|f|]$ and the proposition is proved. $\square$

**Notation 1.3.2.** *We denote by $C^{(\alpha)}([a,b],\mathbb{R})$, $0 < a < b$, $\alpha > 0$, the set of all real-valued functions $f : [a,b] \to \mathbb{R}$ that are $\alpha$-differentiable and for which the $\alpha$-derivative is continuous. We often abbreviate $C^{(\alpha)}([a,b],\mathbb{R})$ by $C^{(\alpha)}([a,b])$.*

The next lemma is a consequence of the conformable mean value theorem proved in [43] by noting the discussion under Definition 2.1 in [1]. Note that $r(b) - r(a) = I_\alpha^a \left(r^{(\alpha)}\right)(b)$ follows from Lemma 2.8 in [1].

**Lemma 1.3.1.** *Let $r \in C^{(\alpha)}([a,b])$, $0 < a < b$, such that $r^{(\alpha)}(t) < 0$ on $\{t \in [a,b] : r(t) > 0\}$. If $r(a) \leq 0$, then $r(t) \leq 0$ for every $t \in [a,b]$.*

*Proof.* Suppose the contrary. If there exists $t \in [a,b]$ such that $r(t) > 0$, then there exists $t_\circ \in [a,b]$ such that $r(t_\circ) = \max_{a \leq t \leq b}(r(t)) > 0$ because $r \in C^{(\alpha)}([a,b])$ and $r(t) > 0$. There are two cases. (i) If $t_\circ > a$, then there exists an interval $[t_1, t_\circ]$ included in $[a, t_\circ]$ such that $r(t) > 0$ for all $t \in [t_1, t_\circ]$. It follows from the assumption $r^{(\alpha)}(t) < 0$ for all $t \in [t_1, t_\circ]$ and Lemma 2.8 of [1] that $I_\alpha^{t_1}\left(r^{(\alpha)}\right)(t_\circ) = r(t_\circ) - r(t_1) < 0$, which contradicts the fact that $r(t_\circ)$ is a maximum. (ii) If $t_\circ = a$, then $r(t_\circ) > 0$ is impossible from hypothesis. $\square$

**Theorem 1.3.3.** *If $g \in L^1([a,b])$, then function $x : [a,b] \to \mathbb{R}$ defined by*

$$x(t) := e^{-\frac{1}{\alpha}\left(\frac{t}{a}\right)^\alpha}\left(e^{\frac{1}{\alpha}}x_0 + {}_{\alpha}\mathfrak{J}_a^t\left[\frac{g(s)}{e^{-\frac{1}{\alpha}\left(\frac{s}{a}\right)^\alpha}}\right]\right) \tag{1.8}$$

*is solution to problem*

$$\begin{cases} x^{(\alpha)}(t) + \frac{1}{a^\alpha}x(t) = g(t), \quad t \in [a,b], \quad a > 0, \\ x(a) = x_0. \end{cases}$$

*Proof.* Let $x : [a,b] \to \mathbb{R}$ be the function defined by (1.8). We know from Theorems 1.3.1 and 1.3.2 that

$$\begin{aligned} x^{(\alpha)}(t) &= t^{1-\alpha}\left(-\frac{1}{\alpha}\left(\frac{1}{a}\right)^{\alpha}\alpha t^{\alpha-1}\right)e^{-\frac{1}{\alpha}\left(\frac{t}{a}\right)^{\alpha}}\left(e^{\frac{1}{\alpha}}x_0 + {}_{\alpha}\mathfrak{J}_a^t\left[\frac{g(s)}{e^{-\frac{1}{\alpha}\left(\frac{s}{a}\right)^{\alpha}}}\right]\right) + e^{-\frac{1}{\alpha}\left(\frac{t}{a}\right)^{\alpha}}\left(\frac{g(t)}{e^{-\frac{1}{\alpha}\left(\frac{t}{a}\right)^{\alpha}}}\right) \\ &= -\left(\frac{1}{a}\right)^{\alpha}e^{-\frac{1}{\alpha}\left(\frac{t}{a}\right)^{\alpha}}\left(e^{\frac{1}{\alpha}}x_0 + {}_{\alpha}\mathfrak{J}_a^t\left[\frac{g(s)}{e^{-\frac{1}{\alpha}\left(\frac{s}{a}\right)^{\alpha}}}\right]\right) + g(t) \\ &= -\left(\frac{1}{a}\right)^{\alpha}x(t) + g(t). \end{aligned}$$

We just obtained that $x^{(\alpha)}(t) + \left(\frac{1}{a}\right)^{\alpha}x(t) = g(t)$. On the other hand,

$$x(a) = e^{-\frac{1}{\alpha}\left(\frac{a}{a}\right)^{\alpha}}\left(e^{\frac{1}{\alpha}}x_0 + {}_{\alpha}\mathfrak{J}_a^a\left[\frac{g(s)}{e^{-\frac{1}{\alpha}\left(\frac{s}{a}\right)^{\alpha}}}\right]\right) = e^{-\frac{1}{\alpha}}\left(e^{\frac{1}{\alpha}}x_0 + 0\right) = x_0$$

and the proof is complete. □

Theorem 1.3.3 is enough for our purposes. It should be mentioned, however, that it can be generalized by benefiting from Lemma 2.8 in [1] with its higher-order version [1, Proposition 2.9].

**Theorem 1.3.4.** *If $g \in L^1([a,b])$ and $p(t)$ is continuous on $[a,b]$, then the function $x : [a,b] \to \mathbb{R}$ defined by*

$$x(t) = \frac{1}{\mu(t)}\left(x(a)\mu(a) + I_{\alpha}^{a}(\mu g)(t)\right) \tag{1.9}$$

*is a solution to the linear conformable equation*

$$x^{(\alpha)}(t) + p(t)x(t) = g(t), \quad x(a) = x_0, \quad a > 0. \tag{1.10}$$

*Proof.* Consider the integrating factor function $\mu(t) = e^{I_{\alpha}^{a}(p)(t)}$. Then, by means of item (3) of Example 1.3.1 and the Chain Rule [1, Theorem 2.11], one can see that $\mu^{(\alpha)}(t) = p(t)\mu(t)$. Then, multiply (1.10) by function $\mu(t)$. By means of the product rule (item (2) of Theorem 1.3.1), (1.10) turns to

$$(x(t)\mu(t))^{(\alpha)} = \mu(t)g(t). \tag{1.11}$$

Apply $I_{\alpha}^{a}$ to (1.11) and use Lemma 2.8 in [1] to conclude that (1.9) holds. □

Theorem 1.3.3 follows as a corollary from Theorem 1.3.4 by putting $\mu(t) = e^{\frac{1}{\alpha}\left(\frac{t}{a}\right)^\alpha}$ and $p(t) = e^{\frac{1}{\alpha}}$.

**Proposition 1.3.2.** *If $x : (0, \infty) \to \mathbb{R}$ is $\alpha$-differentiable at $t \in [a, b]$, then* $|x(t)|^{(\alpha)} = \frac{x(t)\, x^\alpha(t)}{|x(t)|}$.

*Proof.* From Definition 1.3.1 we have

$$
\begin{aligned}
|x(t)|^{(\alpha)} &= \lim_{\epsilon\to 0} \frac{|x(t+\epsilon t^{1-\alpha})| - |x(t)|}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{x\left(t+\epsilon t^{1-\alpha}\right)^2 - x(t)^2}{\epsilon\left(\left|x\left(t+\epsilon t^{1-\alpha}\right)\right| + |x(t)|\right)} \\
&= \lim_{\epsilon\to 0} \left[\frac{x(t+\epsilon t^{1-\alpha})^2 - x(t)^2}{\epsilon} \cdot \frac{1}{|x(t+\epsilon t^{1-\alpha})| + |x(t)|}\right] \\
&= \left[x(t)^2\right]^{(\alpha)} \frac{1}{2|x(t)|} \\
&= 2x(t)x^{(\alpha)}(t)\frac{1}{2|x(t)|},
\end{aligned}
$$

which proves the intended relation. $\square$

# Chapter 2

# Existence of solution to a nonlinear first-order dynamic equation on time scales

In this chapter we prove existence of solution to the following systems:

$$\begin{cases} x^{\nabla}(t) = f(t, x(t)), \quad t \in \mathbb{T}_k, \\ x(b) = x(a) \end{cases} \tag{2.1}$$

and

$$\begin{cases} x^{\nabla}(t) = f(t, x(\rho(t)), \quad \nabla \ a.e. \ t \in \mathbb{T}_k, \\ x \in (BC). \end{cases} \tag{2.2}$$

Here $\mathbb{T}$ is an arbitrary bounded time scale, where we denote $a := \min \mathbb{T}$, $b := \max \mathbb{T}$, $\mathbb{T}^{\circ} = \mathbb{T} \setminus \{a\}$, and in (2.2) $f : \mathbb{T}^{\circ} \times \mathbb{R}^n \to \mathbb{R}^n$ is a $\nabla$-Carathéodory (resp. in (2.1) is a continuous) function and $(BC)$ denotes one of the following boundary conditions:

$$x(b) = x_0 \tag{2.3}$$

or

$$x(b) = x(a). \tag{2.4}$$

The original results of this chapter are published in [12].

## 2.1 Main Result

### 2.1.1 Existence of solution to the system (2.1).

In this section we prove existence of solution to problem (2.1). A solution of this problem is a function $x \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$ satisfying (2.1). Let us recall that $\mathbb{T}$ is bounded with $a = \min \mathbb{T}$ and $b = \max \mathbb{T}$.

**Proposition 2.1.1.** *If $g \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$, then the function $x : \mathbb{T} \to \mathbb{R}^n$ defined by*

$$x(t) = \hat{e}_1(t,b)\left[\frac{\hat{e}_1(a,b)}{\hat{e}_1(a,b)-1}\int_{(a,b]\cap\mathbb{T}}\frac{g(s)}{\hat{e}_1(\rho(s),b)}\nabla s - \int_{(t,b]\cap\mathbb{T}}\frac{g(s)}{\hat{e}_1(\rho(s),b)}\nabla s\right]$$

*is solution to the problem*

$$\begin{aligned} x^\nabla(t) - x(t) &= g(t), \quad t \in \mathbb{T}_\kappa, \\ x(a) &= x(b). \end{aligned} \tag{2.5}$$

*Proof.* We check (2.5) for each pair $(x_i, g_i)$, $i \in \{1, 2, \ldots, n\}$, by direct calculation. To simplify notation, we omit the indices $i$ and we write

$$k = \frac{\hat{e}_1(a,b)}{\hat{e}_1(a,b)-1}\int_{(a,b]\cap\mathbb{T}}\frac{g(s)}{\hat{e}_1(\rho(s),b)}\nabla s.$$

From Theorem 1.2.2, we have that

$$\begin{aligned} &x^\nabla(t) - x(t) \\ &= \hat{e}_1(t,b)k - \hat{e}_1(t,b)\int_{(a,b]\cap\mathbb{T}}\frac{g(s)}{\hat{e}_1(\rho(s),b)}\nabla s \\ &+ \hat{e}_1(\rho(t),b)\frac{g(t)}{\hat{e}_1(\rho(t),b)} - \hat{e}_1(t,b)k + \hat{e}_1(t,b)\int_{(a,b]\cap\mathbb{T}}\frac{g(s)}{\hat{e}_1(\rho(s),b)}\nabla s \\ &= g(t) \end{aligned}$$

for all $t \in \mathbb{T}_\kappa$. It is easy to verify that $x(a) = x(b)$. □

**Lemma 2.1.1.** *Let $r \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$ be a function such that $r^\nabla(t) < 0$ for all $t \in \{t \in \mathbb{T}_\kappa; r(t) > 0\}$. If $r(b) \geq r(a)$, then $r(t) \leq 0$ for all $t \in \mathbb{T}$.*

*Proof.* Suppose that there exists a $t \in \mathbb{T}$ such that $r(t) > 0$. Then there exists a $t_0 \in \mathbb{T}$ such that $r(t_0) = \max_{t \in \mathbb{T}}(r(t) > 0)$. If $\rho(t_0) < t_0$, then

$$r^{\nabla}(t_0) = \frac{r(\rho(t_0)) - r(t_0)}{\rho(t_0) - t_0} \geq 0,$$

which contradicts the hypothesis. If $t_0 > a$ and $t_0 = \rho(t_0)$, then there exists an interval $[t_1, t_0]$ such that $r(t) > 0$ for all $t \in [t_1, t_0]$. Thus

$$\int_{t_1}^{t_0} r^{\nabla}(s) \nabla s = r(t_0) - r(t_1) < 0,$$

which contradicts the maximality of $r(t_0)$. Finally, if $t_0 = a$, then by hypothesis $r(b) \geq r(a)$ gives $r(a) = r(b)$. Taking $t_0 = b$, one can check that $r(b) \leq 0$ by using previous steps of the proof. The lemma is proved. $\square$

We introduce the notion of tube solution for problem (2.1) as follows.

**Definition 2.1.1.** *Let $(v, M) \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n) \times C^1_{ld}(\mathbb{T}, [0, +\infty[)$. We say that $(v, M)$ is a tube solution of* (2.1) *if*

1. *$\langle x - v(t), f(t, x(t)) - v^{\nabla}(t) \rangle + M(t)\|x - v(t)\| \leq M(t)M^{\nabla}(t)$ for every $t \in \mathbb{T}_\kappa$ and for every $x \in \mathbb{R}^n$ such that $\|x - v(t)\| = M(t)$;*

2. *$v^{\nabla}(t) = f(t, v(t))$ and $\|x - v(t)\| - M^{\nabla}(t) < 0$ for every $t \in \mathbb{T}_\kappa$ such that $M(t) = 0$;*

3. *$\|v(a) - v(b)\| \leq M(a) - M(b)$.*

Let $\mathbf{T}(v, M) := \{x \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n) : \|x(t) - v(t)\| \leq M(t) \text{ for every } t \in \mathbb{T}\}$. We consider the following problem:

$$\begin{aligned} x^{\nabla}(t) - x(t) &= f(t, \hat{x}(t)) - \hat{x}(t), \quad t \in \mathbb{T}_\kappa, \\ x(a) &= x(b), \end{aligned} \tag{2.6}$$

where

$$\hat{x}(t) = \begin{cases} \frac{M(t)}{\|x(t) - v(t)\|}(x(t) - v(t)) + v(t) & \text{if } \|x(t) - v(t)\| > M(t), \\ x(t) & \text{otherwise.} \end{cases}$$

Let us define the operator $\mathbf{T}_{\hat{p}} : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n)$ by

$$\mathbf{T}_{\hat{p}}(x)(t) = \hat{e}_1(t,b)\left[\frac{\hat{e}_1(a,b)}{\hat{e}_1(a,b)-1}\int_{(a,b]\cap\mathbb{T}}\frac{f(s,\hat{x}(s))-\hat{x}(s)}{\hat{e}_1(\rho(s),b)}\nabla s\right.$$
$$\left.-\int_{(t,b]\cap\mathbb{T}}\frac{f(s,\hat{x}(s))-\hat{x}(s)}{\hat{e}_1(\rho(s),b)}\nabla s\right].$$

**Proposition 2.1.2.** *If* $(v, M) \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n) \times C^1_{ld}(\mathbb{T}, [0, +\infty[)$ *is a tube solution of* (2.1)*, then* $\mathbf{T}_{\hat{p}} : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n)$ *is compact.*

*Proof.* We first prove the continuity of the operator $\mathbf{T}_{\hat{p}}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $C(\mathbb{T}, \mathbb{R}^n)$ converging to $x \in C(\mathbb{T}, \mathbb{R}^n)$. By proposition 1.2.1,

$$\begin{aligned}&\|\mathbf{T}_{\hat{p}}(x_n)(t) - \mathbf{T}_{\hat{p}}(x)(t)\|\\ &\quad\le (1+c)\|\hat{e}_1(t,b)\|\left\|\int_{(a,b]\cap\mathbb{T}}\frac{f(s,\hat{x}_n(s))-f(s,\hat{x}(s))-(\hat{x}_n(s)-\hat{x}(s))}{\hat{e}_1(\rho(s),b)}\nabla s\right\|\\ &\quad\le \frac{k(1+c)}{M}\left(\int_{(a,b]\cap\mathbb{T}}\|f(s,\hat{x}_n(s))-f(s,\hat{x}(s))\|+\|\hat{x}_n(s)-\hat{x}(s)\|\nabla s\right),\end{aligned}$$

where $k := \max_{t\in\mathbf{T}}|\hat{e}_1(t,b)|$, $M := \min_{t\in\mathbf{T}}(\hat{e}_1(t,b))$, and $c := \|\frac{\hat{e}_1(a,b)}{\hat{e}_1(a,b)-1}\|$. Since there is a constant $R > 0$ such that $\|\hat{x}\|_{C(\mathbb{T},\mathbb{R}^n)} < R$, there exists an index $N$ such that $\|\hat{x}_n\|_{C(\mathbb{T},\mathbb{R}^n)} < R$ for all $n > N$. Thus $f$ is uniformly continuous on $\mathbb{T}_\kappa \times B_R(0)$. Therefore, for $\epsilon > 0$ given, there is a $\delta > 0$ such that for all $x, y \in \mathbb{R}^n$, where

$$\|x-y\| < \delta < \frac{\epsilon M}{2k(1+c)(b-a)},$$

one has

$$\|f(s,y)-f(s,x)\| < \frac{\epsilon M}{2k(1+c)(b-a)}.$$

By assumption, for all $s \in \mathbb{T}_\kappa$ it is possible to find an index $\hat{N} > N$ such that $\|\hat{x}_n - \hat{x}\|_{C(\mathbb{T},\mathbb{R}^n)} < \delta$ for $n > \hat{N}$. In this case,

$$\|\mathbf{T}_{\hat{p}}(x_n)(t) - \mathbf{T}_{\hat{p}}(x)(t)\| \le \frac{2k(1+c)}{M}\int_{[a,b)\cap\mathbb{T}}\frac{\epsilon M}{2k(1+c)(b-a)}\nabla s \le \epsilon.$$

This proves the continuity of $\mathbf{T}_{\hat{p}}$. We now show that the set $\mathbf{T}_{\hat{p}}(C(\mathbb{T}, \mathbb{R}^n))$ is relatively compact. Consider a sequence $\{y_n\}_{n\in\mathbb{N}}$ of $\mathbf{T}_{\hat{p}}(C(\mathbb{T}, \mathbb{R}^n))$ for all $n \in \mathbb{N}$. It exists $x_n \in C(\mathbb{T}, \mathbb{R}^n)$ such that $y_n = \mathbf{T}_{\hat{p}}(x_n)$. From Proposition 1.2.1 one has

$$\|\mathbf{T}_{\hat{p}}(x_n)(t)\| \le \frac{k(1+c)}{M}\left(\int_{[a,b)\cap\mathbb{T}}\|f(s,\hat{x}_n(s))\|\nabla s + \int_{[a,b)\cap\mathbb{T}}\|\hat{x}_n(s)\|\nabla s\right).$$

By definition, there is an $R>0$ such that $\|\hat{x}_n(s)\| \leq R$ for all $s \in \mathbb{T}$ and all $n \in \mathbb{N}$. Function $f$ is compact on $\mathbb{T}_\kappa \times B_R(0)$ and we deduce the existence of a constant $A>0$ such that $\|f(s, \hat{x}_n(s)\| \leq A$ for all $s \in \mathbb{T}_\kappa$ and all $n \in \mathbb{N}$. The sequence $\{y_n\}_n \in \mathbb{N}$ is uniformly bounded. Note also that

$$\begin{aligned}
&\|\mathbf{T}_{\hat{p}}(x_n)(t_2) - \mathbf{T}_{\hat{p}}(x_n)(t_1)\| \leq B\|\hat{e}_1(t_2,b) - \hat{e}_1(t_1,b)\| \\
&+ k\left\|\int_{(a,b]\cap\mathbb{T}} \frac{f(s,\hat{x}_n(s)) - \hat{x}_n(s)}{\hat{e}_1(\rho(s),b)}\nabla s\right\| < B\|\hat{e}_1(t_2,b) - \hat{e}_1(t_1,b)\| + \frac{k(A+R)}{M}|t_2 - t_1|
\end{aligned}$$

for $t_1, t_2 \in \mathbb{T}$, where $B$ is a constant that can be chosen such that it is higher than

$$\sup_{n\in\mathbb{N}} \left\| \frac{\hat{e}_1(a,b)}{\hat{e}_1(a,b)-1} \int_{(a,b]\cap\mathbb{T}} \frac{f(s,\hat{x}_n(s)) - \hat{x}_n(s)}{\hat{e}_1(\rho(s),b)}\nabla s + \int_{(t,b]\cap\mathbb{T}} \frac{f(s,\hat{x}_n(s)) - \hat{x}_n(s)}{\hat{e}_1(\rho(s),b)}\nabla s \right\|.$$

This proves that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is equicontinuous. It follows from the Arzelà–Ascoli theorem, adapted to our context, that $\mathbf{T}_{\hat{p}}(C(\mathbb{T},\mathbb{R}^n))$ is relatively compact. Hence $\mathbf{T}_{\hat{p}}$ is compact. □

**Theorem 2.1.1.** *If $(v, M) \in C^1_{ld}(\mathbb{T}, [0,+\infty[) \times C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$ is a tube solution of* (2.1), *then problem* (2.1) *has a solution $x \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n) \cap \mathbf{T}(v, M)$.*

*Proof.* By Proposition 2.1.2, $\mathbf{T}_{\hat{p}}$ is compact. It has a fixed point by Schauder's fixed point theorem. Proposition 2.1.1 implies that this fixed point is a solution to problem (2.6). Then it suffices to show that for every solution $x$ of (2.6) one has $x \in \mathbf{T}(v, M)$. Consider the set $A = \{t \in \mathbb{T}_\kappa : \|x(t) - v(t)\| > M(t)\}$. If $t \in A$ is left dense, then by virtue of Example 1.2.1 we have

$$(\| x(t) - v(t) \| - M(t))^\nabla = \frac{\langle x(t) - v(t), x^\nabla(t) - v^\nabla(t)\rangle}{\|x(t) - v(t)\|} - M^\nabla(t).$$

If $t \in A$ is left scattered, then

$$\begin{aligned}
(\|x(t) - v(t)\| - M(t))^\nabla &= \|x(t) - v(t)\|^\nabla - M^\nabla(t) \\
&= \frac{\|x(t) - v(t)\|^2 - \|x(t) - v(t)\|\|x(\rho(t)) - v(\rho(t))\|}{\nu(t)\|x(t) - v(t)\|} - M^\nabla(t) \\
&\leq \frac{\langle x(t) - v(t), x(t) - v(t) - x(\rho(t)) + v(\rho(t))\rangle}{\nu(t)\|x(t) - v(t)\|} - M^\nabla(t) \\
&= \frac{\langle x(t) - v(t), [f(t,\hat{x}(t)) - \hat{x}(t) + x(t)] - v^\nabla(t)\rangle}{\|x(t) - v(t)\|} - M^\nabla(t).
\end{aligned}$$

We will show that if $t \in A$, then $(\|x(t) - v(t)\| - M(t))^{\nabla} < 0$. If $t \in A$ and $M(t) > 0$, then

$$
\begin{aligned}
(\|x(t){-}v(t)\| - M(t))^{\nabla} &= \|x(t) - v(t)\|^{\nabla} - M^{\nabla}(t) \\
&= \frac{\|x(t) - v(t)\|^2 - \|x(t) - v(t)\|\|x(\rho(t)) - v(\rho(t))\|}{\nu(t)\|x(t) - v(t)\|} - M^{\nabla}(t) \\
&\leq \frac{\langle x(t) - v(t), x(t) - v(t) - x(\rho(t)) + v(\rho(t))\rangle}{\nu(t)\|x(t) - v(t)\|} - M^{\nabla}(t) \\
&= \frac{\langle x(t) - v(t), x^{\nabla}(t) - v^{\nabla}(t)\rangle}{\|x(t) - v(t)\|} - M^{\nabla}(t) \\
&= \frac{\langle x(t) - v(t), f(t, \hat{x}(t)) - v^{\nabla}(t)\rangle}{\|x(t) - v(t)\|} + \frac{\langle x(t) - v(t), -\hat{x}(t) + x(t)\rangle}{\|x(t) - v(t)\|} - M^{\nabla}(t) \\
&= \frac{\langle \hat{x}^{\nabla}(t) - v(t), f(t, \hat{x}(t)) - v^{\nabla}(t)\rangle}{M(t)} - M(t) + \|x(t) - v(t)\| - M^{\nabla}(t) \\
&\leq \frac{M(t)M^{\nabla}(t) - M(t)\|x(t) - v(t)\|}{M(t)} - M(t) + \|x(t) - v(t)\| - M^{\nabla}(t) \\
&= -M(t) < 0.
\end{aligned}
$$

In addition, if $M(t) = 0$, then

$$
\begin{aligned}
(\|x(t){-}v(t)\| - M(t))^{\nabla} &= \frac{\langle x(t) - v(t), f(t, \hat{x}(t)) + [x(t) - \hat{x}(t)] - v^{\nabla}(t)\rangle}{\|x(t) - v(t)\|} - M^{\nabla}(t) \\
&\leq \frac{\langle x(t) - v(t), f(t, v(t)) - v^{\nabla}(t)\rangle}{\|x(t) - v(t)\|} + \|x(t) - v(t)\| - M^{\nabla}(t) < 0.
\end{aligned}
$$

If we set $r(t) := \|x(t){-}v(t)\|{-}M(t)$, then $r^{\nabla}(t) < 0$ for every $t \in \{t \in \mathbb{T}_{\kappa}, r(t) > 0\}$. Moreover, since $(v, M)$ is a tube solution of (2.1), one has

$$r(a) - r(b) \leq \|v(a) - v(b)\| - (M(a) - M(b)) \leq 0$$

and thus the hypotheses of Lemma 2.1.1 are satisfied, which proves the theorem. □

**Example 2.1.1.** *Consider the following boundary value problem on time scales:*

$$
\begin{gathered}
x^{\nabla}(t) = a_1\|x(t)\|^2 x(t) - a_2 x(t) + a_3\varphi(t), \quad t \in \mathbb{T}_{\kappa}, \\
x(a) = x(b),
\end{gathered}
\tag{2.7}
$$

*where $a_1$, $a_2$, $a_3 \geq 0$ are nonnegative real constants chosen such that $a_2 \geq a_1 + a_3 + 1$ and $\varphi : \mathbb{T}_{\kappa} \to \mathbb{R}^n$ is a continuous function satisfying $\|\varphi(t)\| = 1$ for every $t \in \mathbb{T}_{\kappa}$. It* *is easy to check that* $(v, M) \equiv (0, 1)$ *is a tube solution. By Theorem 2.1.1, problem* (2.7) *has a solution* $x$ *such that* $\|x(t)\| \leq 1$ *for every* $t \in \mathbb{T}$.

In the following section we prove existence of solution to problem (2.2).

### 2.1.2 Existence of solution to the system (2.2)

We introduce the notion of tube solution for system (2.2).

**Definition 2.1.2.** *Let* $(v, M) \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) \times W^{1,1}_{\nabla}(\mathbb{T}, [0, \infty))$. *We say that* $(v, M)$ *is a tube solution of* (2.2) *if*

**(i)** $\langle x - v(\rho(t)), f(t, x) - v^{\nabla}(t)\rangle \leq M(t)M^{\nabla}(t)$ $\nabla$ *a.e.* $t \in \mathbb{T}^{\circ}$ *and for all* $x \in \mathbb{R}^n$ *such that* $\|x - v(\rho(t))\| = M(\rho(t))$;

**(ii)** $v^{\nabla}(t) = f(t, v(\rho(t)))$ $\nabla$ *a.e.* $t \in \mathbb{T}^{\circ}$ *such that* $M(\rho(t)) = 0$;

**(iii)** $M(t) = 0$ *for all* $t \in \mathbb{T}^{\circ}$ *such that* $M(\rho(t)) = 0$;

**(iv)** *if* $(BC)$ *denotes* (2.3), *then* $\|x_0 - v(a)\| \leq M(b)$; *if* $(BC)$ *denotes* (2.4), *then* $\|v(b) - v(a)\| \leq M(b) - M(a)$.

We denote

$$\mathrm{T}(v, M) = \{x \in W^{1,1}_{\nabla}(\mathbb{T}, \mathbb{R}^n) : \|x(t) - v(t)\| \leq M(t), \forall t \in \mathbb{T}\}. \tag{2.8}$$

If $\mathbb{T}$ is a real interval $[a, b]$, then our definition of tube solution is equivalent to the notion of tube solution introduced in [46]. We consider the following problem:

$$\begin{cases} x^{\nabla}(t) + x(\rho(t)) = f(t, \bar{x}(\rho(t))) + \bar{x}(\rho(t)), \nabla \text{ a.e. } t \in \mathbb{T}_{\kappa}, \\ x \in (BC), \end{cases} \tag{2.9}$$

where

$$\bar{x}(s) = \begin{cases} \frac{M(s)}{\|x - v(s)\|}(x - v(s)) + v(s) & if \quad \|x - v(s)\| > M(s) \\ x(s) & otherwise. \end{cases} \tag{2.10}$$

**Lemma 2.1.2.** *Assume $r \in W_\nabla^{1,1}(\mathbb{T})$ such that $r^\nabla(t) > 0$ $\nabla$ a.e. $t \in \{t \in \mathbb{T} : r(\rho(t)) > 0\}$. If one of the following conditions holds,*

**(i)** *$r(b) \leq 0$;*

**(ii)** *$r(b) \leq r(a)$;*

*then $r(t) \leq 0$ for every $t \in \mathbb{T}$.*

*Proof.* Suppose the conclusion is false. Then, there exists $t_0 \in \mathbb{T}$ such that $r(t_0) = \max_{t\in\mathbb{T}}\{r(t)\} > 0$, since r is continuous on $\mathbb{T}$. If $t_0 < \sigma(t_0)$, then $r^\nabla(\sigma(t_0))$ exists, since $\mu_\nabla(\sigma(t_0)) = \sigma(t_0) - t_0 > 0$ and because $r \in W_\nabla^{1,1}(\mathbb{T})$. Thus

$$r^\nabla(\sigma(t_0)) = \frac{r(\sigma(t_0)) - r(\rho(\sigma(t_0)))}{\sigma(t_0) - \rho(\sigma(t_0))} = \frac{r(\sigma(t_0)) - r(t_0)}{\sigma(t_0) - t_0} \leq 0, \tag{2.11}$$

which is a contradiction since $r(\rho(\sigma(t_0))) = r(t_0) > 0$. If $t_0 = \sigma(t_0) < b$, then there exists an interval $[\sigma(t_0), t_1]$ such that $r(\rho(t)) > 0$ for every $t \in [\sigma(t_0), t_1]_\mathbb{T}$. Therefore, $0 > r(t_1) - r(\sigma(t_0)) = \int_{[\sigma(t_0),t_1]_\mathbb{T}} r^\nabla(s)\nabla s > 0$ by hypothesis and by Theorem 1.2.8. Hence, we get a contradiction. The case $t_0 = b$ is impossible if hypothesis $(i)$ holds and if $r(b) \leq r(a)$, then we must have $r(a) = r(b)$. If we take $t_0 = a$, by using previous steps of this proof, one can check that $r(a) \leq 0$ and, then, the lemma is proved. □

**Lemma 2.1.3.** *Let $r \in W_\nabla^{1,1}(\mathbb{T})$ be a function such that $r^\nabla(t) < 0$ $\nabla$ a.e. $t \in \{t \in \mathbb{T}_\kappa : r(t) > 0\}$ if $r(b) \geq r(a)$. Then $r(t) \leq 0$ for all $t \in \mathbb{T}$.*

*Proof.* Suppose the conclusion is false. Then, there exists $t_0 \in \mathbb{T}$ such that $r(t_0) = \max_{t\in\mathbb{T}}\{r(t)\} > 0$, since r is continuous on $\mathbb{T}$. If $t_0 < b$ and $\rho(t_0) < t_0$, then $r^\nabla(\sigma(t_0))$ exists, since $\mu_\nabla(t_0) = t_0 - \rho(t_0) > 0$ and because $r \in W_\nabla^{1,1}(\mathbb{T})$. Then

$$r^\nabla(t_0) = \frac{r(t_0) - r(\rho(t_0))}{t_0 - \rho(t_0)} \geq 0, \tag{2.12}$$

which contradicts the hypothesis of the lemma, $t_0 > 0$ and $t_0 = \rho(t_0)$. There exists an interval $[t_1, t_0)$ such that $r(t) > 0$ for all $t \in [t_1, t_0)_\mathbb{T}$. Then

$$0 > \int_{[t_1,t_0)_\mathbb{T}} r^\nabla(s)\nabla s = r(t_0) - r(t_1) \tag{2.13}$$

by Theorem 1.2.8, which contradicts the fact that $r(t_0)$ is a maximum. If $t_0 = a$, then by hypothesis, we must have $r(a) = r(b)$. Thus, we can take $t_0 = b$, and by using the previous steps of this proof, one can check that $r(b) \leq 0$. $\square$

**Proposition 2.1.3.** *If $g \in L^1_\nabla(\mathbb{T}, \mathbb{R}^n)$, then the function $x : \mathbb{T} \to \mathbb{R}^n$ defined by*

$$x(t) = \frac{1}{\hat{e}_1(t,b)} \left[ \frac{1}{1-\hat{e}_1(a,b)} \int_{(a,b]_\mathbb{T}} g(s)\hat{e}_1(s,b)\nabla s - \int_{(t,b]_\mathbb{T}} g(s)\hat{e}_1(s,b)\nabla s \right] \tag{2.14}$$

*is a solution of the problem*

$$\begin{cases} x^\nabla(t) + x(\rho(t)) = g(t), \quad \nabla \text{ } a.e. \text{ } t \in \mathbb{T}_k, \\ x(a) = x(b). \end{cases} \tag{2.15}$$

*Proof.* Assume that

$$K = \frac{1}{1-\hat{e}_1(a,b)} \int\limits_{(a,b]_\mathbb{T}} g(s)\hat{e}_1(s,b)\nabla s$$

and

$$x(t) = \frac{1}{\hat{e}_1(t,b)} \left( K - \int\limits_{(t,b]_\mathbb{T}} g(s)\hat{e}_1\left(s,b\right)\nabla s \right).$$

Then

$$\begin{aligned}
&x^\nabla(t) + x(\rho(t)) \\
&= \left(\frac{1}{\hat{e}_1(t,b)}\right)^\nabla \left(K - \int_{(t,b]_\mathbb{T}} g(s)\hat{e}_1(s,b)\nabla s\right) + \frac{1}{\hat{e}_1(\rho(t),b)}\big(g(t)\hat{e}_1(t,b)\big) \\
&\quad + \frac{1}{\hat{e}_1(\rho(t),b)}\left(K - \int_{(\rho(t),b]_\mathbb{T}} g(s)\hat{e}_1(s,b)\nabla s\right) \\
&= \frac{-1}{\hat{e}_1\left(\rho(t),b\right)}\left(K - \int_{(t,b]_\mathbb{T}} g(s)\hat{e}_1(s,b)\nabla s\right) + \frac{g(t)}{1-\nu(t)} + \frac{1}{(1-\nu(t))\hat{e}_1(t,b)}\left(-\nu(t)g(t)\hat{e}_1(t,b)\right) \\
&\quad + \frac{1}{\hat{e}_1\left(\rho(t),b\right)}\left(K - \int_{(t,b]_\mathbb{T}} g(s)\hat{e}_1(s,b)\nabla s\right) \\
&= g(t).
\end{aligned}$$

Now, for the second condition, we have

$$\begin{aligned}
x(a) &= \frac{1}{\hat{e}_1\,(a,b)}\left[\frac{1}{1-\hat{e}_1(a,b)}\int_{(a,b]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s - \int_{(a,b]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s\right] \\
&= \frac{1}{\hat{e}_1(a,b)}\left(\frac{1}{1-\hat{e}_1(a,b)}-1\right)\int_{(a,b]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s \\
&= \frac{1}{1-\hat{e}_1(a,b)}\int_{(a,b]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s \\
&= x(b).
\end{aligned}$$

□

**Proposition 2.1.4.** *If $g \in L^1_\nabla(\mathbb{T},\mathbb{R}^n)$, then the function $x : \mathbb{T} \to \mathbb{R}^n$ defined by*

$$x(t) = \frac{1}{\hat{e}_1(t,b)}\left[x_0 - \int_{(t,b]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s\right] \tag{2.16}$$

*is a solution of the problem*

$$\begin{cases} x^\nabla(t) + x(\rho(t)) = g(t) \quad \nabla \ a.e. \ t \in \mathbb{T}_k, \\ x(b) = x_0. \end{cases} \tag{2.17}$$

*Proof.* Let $x(t) = \frac{1}{\hat{e}_1(t,b)}\left[x_0 - \int_{(t,b]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s\right]$. We calculate $x^\nabla(t)$. Then we obtain

$$
\begin{aligned}
& x^{\nabla}(t) + x\left(\rho(t)\right) \\
& = \left(\frac{1}{\hat{e}_1(t,b)}\right)^{\nabla}\left(x_0 - \int_{(t,b]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s\right) \\
& + \frac{1}{\hat{e}_1(\rho(t),b)}\left(\hat{e}_1(t,b)g(t)\right) + \frac{1}{\hat{e}_1\left(\rho(t),b\right)}\left(x_0 - \int_{(\rho(t),b]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s\right) \\
& = \frac{-1}{\hat{e}(\rho(t),b)}\left(x_0 - \int_{(t,b]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s\right) \\
& + \frac{1}{\hat{e}_1\left(\rho(t),b\right)}\hat{e}_1(t,b)g(t) + \frac{1}{\hat{e}_1(\rho(t),b)}\left(x_0 - \int_{(\rho(t),t]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s - \int_{(t,b]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s\right) \\
& = \frac{-1}{\hat{e}(\rho(t),b)}\left(x_0 - \int_{(t,b]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s\right) \\
& + \frac{1}{(1-\nu(t))\hat{e}(t,b)}g(t)\hat{e}_1(t,b) + \frac{1}{\hat{e}(\rho(t),b)}\left(x_0 - \int_{(\rho(t),t)_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s - \int_{(t,b]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s\right) \\
& = \frac{1}{(1-\nu(t))\hat{e}(t,b)}g(t)\hat{e}_1(t,b) + \frac{1}{(1-\nu(t))\hat{e}(t,b)}\left(-\nu(t)g(t)\hat{e}_1(t,b)\right) \\
& = g(t).
\end{aligned}
$$

For the second condition, we have

$$
x(b) = \hat{e}_1\left(b,b\right)\left(x_0 - \int_{(b,b]_{\mathbb{T}}} g(s)\hat{e}_1(s,b)\nabla s\right) = x_0.
$$

□

Let us define the operator $\mathrm{T}_1 : C(\mathbb{T},\mathbb{R}^n) \to C(\mathbb{T},\mathbb{R}^n)$ by

$$
\mathrm{T}_1(x)(t) = \hat{e}_1(t,b)\left[x_0 - \int_{(t,b]\cap\mathbb{T}} \hat{e}_1(s,b)\left(f\left(s,\bar{x}\left(\rho(s)\right)\right) + \bar{x}\left(\rho(s)\right)\right)\nabla s\right]. \tag{2.18}
$$

**Proposition 2.1.5.** *If $(v,M) \in W^{1,1}_{\nabla}(\mathbb{T},\mathbb{R}^n) \times W^{1,1}_{\nabla}(\mathbb{T},[0,\infty))$ is a tube solution of* (2.2), (2.3), *then $\mathrm{T}_1 : C(\mathbb{T},\mathbb{R}^n) \to C(\mathbb{T},\mathbb{R}^n)$ is compact.*

*Proof.* We first observe that from Definitions 1.2.10 and 2.1.2 there exists a function $h \in L^1_{\nabla}(\mathbb{T}_\kappa,[0,\infty))$ such that $\|f(t,\bar{x}(\rho(t))) + \bar{x}(\rho(t))\| \leq h(t)$ $\nabla$ a.e. $t \in \mathbb{T}_\kappa$ for all $x \in C(\mathbb{T},\mathbb{R}^n)$. Let $(x_n)_{n\in\mathbb{N}} \subset C(\mathbb{T},\mathbb{R}^n)$ be a sequence of $C(\mathbb{T},\mathbb{R}^n)$ converging to

$x \in C(\mathbb{T}, \mathbb{R}^n)$. By Proposition 1.2.1,

$$
\begin{aligned}
&\big\|\mathrm{T}_1(x_n(t)) - \mathrm{T}_1(x(t))\big\| \\
&= \Big\|\hat{e}_1(t,b)\big[x_0 - \int_{(t,b]\cap\mathbb{T}} \hat{e}_1(s,b)(f(s,\bar{x}_n(\rho(s))) + \bar{x}_n(\rho(s)))\nabla s\big] \\
&- \hat{e}_1(t,b)\big[x_0 - \int_{(t,b]\cap\mathbb{T}} \hat{e}_1(s,b)(f(s,\bar{x}(\rho(s))) + \bar{x}(\rho(s)))\nabla s\big]\Big\| \\
&= \Big\|\hat{e}_1(t,b)\int_{(t,b]\cap\mathbb{T}} \hat{e}_1(s,b)[(f(s,\bar{x}_n(\rho(s))) + \bar{x}_n(\rho(s))) - (f(s,\bar{x}(\rho(s))) + \bar{x}(\rho(s)))]\nabla s\Big\| \\
&\leq K\big[\int_{(t,b]\cap\mathbb{T}} \|(f(s,\bar{x}_n(\rho(s))) + \bar{x}_n(\rho(s)) - (f(s,\bar{x}(\rho(s))) + \bar{x}(\rho(s))\|\nabla s\big],
\end{aligned}
$$

where $K := \max_{t,t_1\in\mathbb{T}} |\hat{e}_1^2(t_1,t)|$. Then, we must show that the sequence $(g_n)_{n\in\mathbb{N}}$ defined by

$$g_n(s) = \big[f(s,\bar{x}_n(\rho(s))) + \bar{x}_n(\rho(s))\big] \tag{2.19}$$

converges to the function $g$ in $L^1_\nabla(\mathbb{T}, \mathbb{R}^n)$ where

$$g(s) = f(s,\bar{x}(\rho(s))) + \bar{x}(\rho(s)). \tag{2.20}$$

We can easily check that $\bar{x}_n(t) \to \bar{x}(t)$ for every $t \in \mathbb{T}$ and, then, by item (iii) in Definition 1.2.10, $g_n(s) \to g(s)$, $\nabla$ a.e., $s \in \mathbb{T}$. Using also the fact that $\|g_n(s)\| \leq h(s)$, $\nabla$ a.e., $s \in \mathbb{T}$, we deduce that $g_n \to g$ in $L^1_\nabla(\mathbb{T}, \mathbb{R}^n)$. This proves the continuity of $\mathrm{T}_1$.

For the second part of the proof, we have to show that the set $\mathrm{T}_1(C(\mathbb{T}, \mathbb{R}^n))$ is relatively compact. Let $y = \mathrm{T}_1(x) \in \mathrm{T}_1(\mathbb{T}, \mathbb{R}^n)$. Therefore,

$$
\begin{aligned}
\|\mathrm{T}_1(x)(t)\| &\leq K\left(\|x_0\| + \left\|\int_{(t,b]\cap\mathbb{T}} \hat{e}_1(s,b)\left(f\left(s,\bar{x}\left(\rho(s)\right)\right) + \bar{x}\left(\rho(s)\right)\right)\nabla s\right\|\right) \\
&\leq K\left(\|x_0\| + \|h\|_{L^1_\nabla(\mathbb{T})}\right).
\end{aligned}
\tag{2.21}
$$

So, $\mathrm{T}_1(C(\mathbb{T}, \mathbb{R}^n))$ is uniformly bounded. This set is also equicontinuous since for every $t_1, t_2 \in \mathbb{T}$,

$$
\begin{aligned}
&\big\|\mathrm{T}(x)(t_2) - \mathrm{T}(x)(t_1)\big\| \\
&= \Big\|\hat{e}_1(t_2, b)[x_0 - \int_{(t_2,b]\cap\mathbb{T}} \hat{e}_1(s,b)\big(f(s,\bar{x}(\rho(s))) + \bar{x}(\rho(s))\big)\nabla s] \\
&- \hat{e}_1(t_1, b)[x_0 - \int_{(t_1,b]\cap\mathbb{T}} \hat{e}_1(s,b)\big(f(s,\bar{x}(\rho(s))) + \bar{x}(\rho(s))\big)\nabla s]\Big\| \\
&\leq \|[\hat{e}_1(t_2,b) - \hat{e}_1(t_1,b)]x_0\| + \|\hat{e}_1(t_2,b)\int_{(t_2,b]\cap\mathbb{T}} \hat{e}_1(s,b)\big(f(s,\bar{x}(\rho(s))) + \bar{x}(\rho(s))\big)\nabla s \\
&- \hat{e}_1(t_1,b)\int_{(t_1,b]\cap\mathbb{T}} \hat{e}_1(s,b)(f(s,\bar{x}(\rho(s))) + \bar{x}(\rho(s)))\nabla s\| \\
&\leq \|x_0\|\|\hat{e}_1(t_2,t_1)\| + M\|\int_{(t_1,t_2)} \big(f(s,\bar{x}(\rho(s))) + \bar{x}(\rho(s))\big)\nabla s\| \\
&\leq \|x_0\|\|\hat{e}_1(t_2,t_1)\| + M\int_{(t_2,t_1)} h(s)\nabla s
\end{aligned}
$$

where $M = \max_{t\in\mathbb{T}}\{\hat{e}_1(t,b)\}$. By an analogous version of the Arzelà-Ascoli Theorem adapted to our context, $\mathrm{T}_1(C(\mathbb{T},\mathbb{R}^n))$ is relatively compact. Hence, $\mathrm{T}_1$ is compact. □

We now define the operator $\mathrm{T}_2 : C(\mathbb{T},\mathbb{R}^n) \to C(\mathbb{T},\mathbb{R}^n)$ by

$$\mathrm{T}_2(x)(t) = \frac{1}{\hat{e}_1(t,b)}\left[\frac{1}{\hat{e}_1(a,b)-1}\int_{(a,b]_\mathbb{T}} g(s)\hat{e}_1(s,b)\nabla s + \int_{(t,b]_\mathbb{T}} g(s)\hat{e}_1(s,b)\nabla s\right].$$

**Proposition 2.1.6.** *If* $(v, M) \in W^{1,1}_\nabla(\mathbb{T},\mathbb{R}^n) \times W^{1,1}_\nabla(\mathbb{T},[0,\infty))$ *is a tube solution of* (2.2), (2.4), *then* $\mathrm{T}_2 : C(\mathbb{T},\mathbb{R}^n) \to C(\mathbb{T},\mathbb{R}^n)$ *is compact.*

**Theorem 2.1.2.** *If* $(v, M) \in W^{1,1}_\nabla(\mathbb{T},\mathbb{R}^n) \times W^{1,1}_\nabla(\mathbb{T},[0,\infty))$ *is a tube solution of* (2.2), *then problem* (2.2) *has a solution* $x \in W^{1,1}_\nabla(\mathbb{T},\mathbb{R}^n) \cap \mathbf{T}(v,M)$.

*Proof.* By Proposition 2.1.5 (resp. Proposition 2.1.6) $\mathrm{T}_1$ (resp. $\mathrm{T}_2$) is compact. It has a fixed point by the Schauder fixed-point theorem. Proposition 2.1.3 (resp., Proposition 2.1.4) implies that this fixed point is a solution for the system (2.9). Then, it suffices to show that for every solution $x$ of (2.2), $x \in \mathbf{T}(v,M)$. Consider the set $A = \{t \in \mathbb{T}_\kappa : \|x(\rho(t)) - v(\rho(t))\| > M(\rho(t))\}$. By Example 1.2.1 $\nabla$ a.e. on the set $\tilde{A} = \{t \in A : t = \rho(t)\}$, we have

$$\begin{aligned}(\|x(t) - v(t)\| - M(t))^{\nabla} &= \frac{\langle x(t) - v(t), x^{\nabla}(t) - v^{\nabla}(t)\rangle}{\|x(t) - v(t)\|} - M^{\nabla}(t) \\ &= \frac{\langle x(\rho(t)) - v(\rho(t)), x^{\nabla}(t) - v^{\nabla}(t)\rangle}{\|x(t) - v(t)\|} - M^{\nabla}(t).\end{aligned}$$

If $t \in A$ is left scattered, then $\mu_{\nabla}\{t\} > 0$ and

$$\begin{aligned}&(\|x(t) - v(t)\| - M(t))^{\nabla} \\ &= \frac{\|x(t) - v(t)\| - \|x(\rho(t)) - v(\rho(t))\|}{\nu(t)} - M^{\nabla}(t) \\ &= \frac{\|x(t) - v(t)\|\|x(\rho(t)) - v(\rho(t))\| - \|x(\rho(t)) - v(\rho(t))\|^2}{\nu(t)\|x(\rho(t)) - v(\rho(t))\|} - M^{\nabla}(t) \\ &= \frac{\langle x(\rho(t)) - v(\rho(t)), x(t) - v(t)\rangle - \langle x(\rho(t)) - v(\rho(t)), x(\rho(t)) - v(\rho(t))\rangle}{\nu(t)x(\rho(t)) - v(\rho(t))} - M^{\nabla}(t) \\ &= \frac{\langle x(\rho(t)) - v(\rho(t)), \big(x(t) - x(\rho(t))\big) - \big(v(t) - v(\rho(t))\big)\rangle}{\nu(t)\|x(\rho(t)) - v(\rho(t)\|} - M^{\nabla}(t) \\ &= \frac{\langle x(\rho(t)) - v(\rho(t)), x^{\nabla}(t) - v^{\nabla}(t)\rangle}{\big(x(\rho(t)) - v(\rho(t))\big)} - M^{\nabla}(t).\end{aligned}$$

Therefore, since $(v, M)$ is a tube solution of (2.2), we have $\nabla$ a.e. $t \in \{t \in A : M(\rho(t)) > 0\}$ that

$$\begin{aligned}&(\|x(t) - v(t)\| - M(t))^{\nabla} \\ &\leq \frac{\langle x(\rho(t)) - v(\rho(t)), f(t, \bar{x}(\rho(t))) + \bar{x}(\rho(t)) - x(\rho(t)) - v^{\nabla}(t)\rangle}{\|x(\rho(t)) - v(\rho(t)\|} - M^{\nabla}(t) \\ &= \frac{\langle \bar{x}(\rho(t)) - v(\rho(t)), f(t, \bar{x}(\rho(t))) - v^{\nabla}(t)\rangle}{M(\rho(t))} + \big(M(\rho(t)) - \|x(\rho(t)) - v(\rho(t)\|\big) - M^{\nabla}(t) \\ &< \frac{M(\rho(t))M^{\nabla}(t)}{M(\rho(t))} - M^{\nabla}(t) = 0.\end{aligned}$$

On the other hand, we have $\nabla$ a.e. on $\{t \in A : M(\rho(t)) = 0\}$ that

$$
\begin{aligned}
&(\|x(t) - v(t)\| - M(t))^{\nabla} \\
&= \frac{\langle x(\rho(t)) - v(\rho(t)), f\left(t, \bar{x}\left(\rho(t)\right)\right) - \bar{x}\left(\rho(t)\right) - x(\rho(t)) - v^{\nabla}(t)\rangle}{\|x(\rho(t)) - v(\rho(t))\|} - M^{\nabla}(t) \\
&= \frac{\langle x(\rho(t)) - v(\rho(t)), f(t, \bar{x}(\rho(t))) - v^{\nabla}(t)\rangle}{\|x(\rho(t)) - v(\rho(t)\|} + \frac{\langle x(\rho(t)) - v(\rho(t)), \bar{x}(\rho(t)) - x(\rho(t))\rangle}{\|x(\rho(t)) - v(\rho(t)\|} - M^{\nabla}(t) \\
&< \frac{\langle x(\rho(t)) - v(\rho(t)), f(t, v(\rho(t))) - v^{\nabla}(t)\rangle}{\|x(\rho(t)) - v(\rho(t)\|} - M^{\nabla}(t) \\
&= -\|x(\rho(t)) - v(\rho(t)\| - M^{\nabla}(t) \\
&< -M^{\nabla}(t) = 0.
\end{aligned}
$$

This last equality follows from item (iii) of Definition 2.1.2 and Proposition 1.2.3. If we set $r(t) = \|x(t) - v(t)\| - M(t)$, then $r^{\nabla}(t) < 0$ $\nabla$ a.e. $t \in \{t \in \mathbb{T}_{\kappa}, r(\rho(t)) > 0\}$. Moreover, since $(v, M)$ is a tube solution of (2.2) and $x$ satisfies (2.3) (resp. $x$ satisfies (2.4)), then $r(b) \leq 0$ (resp. $r(b) - r(a) \leq x(b) - v(a) - M(a) - M(b) \leq 0$. Lemma 2.1.2(resp.lemma 2.1.3) implies that $A = \emptyset$. Therefore, $x \in \mathbf{T}(v, M)$ and, hence, the theorem is proved. $\square$

# Chapter 3

# Existence of solution to a local fractional differential equation

We prove existence of solution to a local fractional nonlinear differential equation with initial condition:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t)), \quad t \in [a, b], \quad a > 0, \\ x(a) = x_0, \end{cases} \tag{3.1}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of $x$ at $t$ of order $\alpha$, $\alpha \in (0, 1)$. For that we introduce the notion of tube solution. The original results of this chapter are published in [14].

## 3.1 Main Result

We begin by introducing the notion of tube solution to problem (3.1).

**Definition 3.1.1.** *Let* $(v, M) \in C^{(\alpha)}([a, b], \mathbb{R}) \times C^{(\alpha)}([a, b], [0, \infty))$. *We say that* $(v, M)$ *is a tube solution to problem* (3.1) *if*

*(i)* $(y - v(t))\left(f(t, y) - v^{(\alpha)}\right) \leq M(t)M^{(\alpha)}(t)$ *for every* $t \in [a, b]$ *and every* $y \in \mathbb{R}$ *such that* $|y - v(t)| = M(t)$;

*(ii)* $v^{(\alpha)}(t) = f(t, v(t))$ *and* $M^{(\alpha)}(t) = 0$ *for all* $t \in [a,b]$ *such that* $M(t) = 0$*;*

*(iii)* $|x_0 - v(a)| \leq M(a)$.

**Notation 3.1.1.** *We introduce the following notation:*

$$\mathbf{T}(v, M) := \left\{x \in C^{(\alpha)}([a,b], \mathbb{R}) : |x(t) - v(t)| \leq M(t),\ t \in [a,b]\right\}.$$

Consider the following problem:

$$\begin{cases} x^{(\alpha)}(t) + \frac{1}{a^\alpha} x(t) = f(t, \widetilde{x}(t)) + \frac{1}{a^\alpha}\widetilde{x}(t), \quad t \in [a,b], \quad a > 0, \\ x(a) = x_0, \end{cases} \tag{3.2}$$

where

$$\widetilde{x}(t) := \begin{cases} \frac{M(t)}{|x(t)-v(t)|}(x(t) - v(t)) + v(t) & \text{if } |x(t) - v(t)| > M(t), \\ x(t) & \text{otherwise.} \end{cases} \tag{3.3}$$

Let us define the operator $\mathbf{N} : C([a,b]) \to C([a,b])$ by

$$\mathbf{N}(x)(t) := e^{-\frac{1}{\alpha}(\frac{t}{a})^\alpha} \left( e^{\frac{1}{\alpha}} x_0 + {}_\alpha\mathfrak{J}_a^t \left[ \frac{f(s, \widetilde{x}(s)) + \frac{1}{a^\alpha}\widetilde{x}(s)}{e^{-\frac{1}{\alpha}(\frac{s}{a})^\alpha}} \right] \right).$$

In the proof of Proposition 3.1.1, we use the concept of compact function. Compact operators occur in many problems of classical analysis. Note that operator $N$ is nonlinear because $f$ is nonlinear. In the nonlinear case, the first comprehensive research on compact operators was due to Schauder [36, p. 137]. In this context, the Arzelà–Ascoli theorem asserts that a subset is relatively compact if and only if it is bounded and equicontinuous [36, p. 607].

**Proposition 3.1.1.** *If* $(v, M) \in C^{(\alpha)}([a,b], \mathbb{R}) \times C^{(\alpha)}([a,b], [0, \infty))$ *is a tube solution to* (3.1)*, then* $\mathbf{N} : C([a,b]) \to C([a,b])$ *is compact.*

*Proof.* Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $C([a,b], \mathbb{R})$ converging to $x \in C([a,b], \mathbb{R})$. By

Proposition 1.3.1,

$$
\begin{aligned}
|\mathbf{N}(x_n(t)) - \mathbf{N}(x(t))| &= \Bigg| e^{-\frac{1}{\alpha}(\frac{t}{a})^\alpha} \left( e^{\frac{1}{\alpha}} x_0 + {}_\alpha\mathfrak{J}_a^t \left[ \frac{f(s,\widetilde{x}_n(s)) + \frac{1}{a^\alpha}\widetilde{x}_n(s)}{e^{-\frac{1}{\alpha}(\frac{s}{a})^\alpha}} \right] \right) \\
&\quad - e^{-\frac{1}{\alpha}\left(\frac{t}{a}\right)^\alpha} \left( e^{\frac{1}{\alpha}} x_0 + {}_\alpha\mathfrak{J}_a^t \left[ \frac{f(s,\widetilde{x}(s)) + \frac{1}{a^\alpha}\widetilde{x}(s)}{e^{-\frac{1}{\alpha}\left(\frac{s}{a}\right)^\alpha}} \right] \right) \Bigg| \\
&\le \frac{K}{C} {}_\alpha\mathfrak{J}_a^t \left[ \left| \left( f(s,\widetilde{x}_n(s)) + \frac{1}{a^\alpha}\widetilde{x}_n(s) \right) - \left( f(s,\widetilde{x}(s)) + \frac{1}{a^\alpha}\widetilde{x}(s) \right) \right| \right] \\
&\le \frac{K}{C} \left( {}_\alpha\mathfrak{J}_a^t \left[ |f(s,\widetilde{x}_n(s)) - f(s,\widetilde{x}(s))| \right] + \frac{1}{a^\alpha} {}_\alpha\mathfrak{J}_a^t \left[ |\widetilde{x}_n(s) - \widetilde{x}(s)| \right] \right),
\end{aligned}
$$

where $K := \max_{a\le s\le b}\{e^{-\frac{1}{\alpha}(\frac{s}{a})^\alpha}\}$ and $C := \min_{a\le s\le b}\{e^{-\frac{1}{\alpha}(\frac{s}{a})^\alpha}\}$. We need to show that the sequence $\{g_n\}_{n\in\mathbb{N}}$ defined by $g_n(s) := f(s,\widetilde{x}_n(s)) + \frac{1}{a^\alpha}\widetilde{x}_n(s)$ converges in $C([a,b])$ to function $g(s) = f(s,\widetilde{x}(s)) + \frac{1}{a^\alpha}\widetilde{x}(s)$. Since there is a constant $R > 0$ such that $\|\widetilde{x}\|_{C([a,b],\mathbb{R})} < R$, there exists an index $N$ such that $\|\widetilde{x}_n\|_{C([a,b],\mathbb{R})} \le R$ for all $n > N$. Thus, $f$ is uniformly continuous on $[a,b] \times B_R(0)$. Therefore, for $\epsilon > 0$ given, there is a $\delta > 0$ such that

$$|y - x| < \delta < \frac{C\epsilon\alpha a^\alpha}{2K(b^\alpha - a^\alpha)}$$

for all $x, y \in \mathbb{R}$;

$$|f(s,y) - f(s,x)| < \frac{C\epsilon\alpha}{2K(b^\alpha - a^\alpha)}$$

for all $s \in [a,b]$. By assumption, one can find an index $\hat{N} > N$ such that $\|\widetilde{x}_n - \widetilde{x}\|_{C([a,b],\mathbb{R})} < \delta$ for $n > \hat{N}$. In this case,

$$
\begin{aligned}
|\mathbf{N}(x_n)(t) - \mathbf{N}(x)(t)| &< \frac{K}{C} \left( {}_\alpha\mathfrak{J}_a^b \left[ \frac{C\epsilon\alpha}{2k(b^\alpha - a^\alpha)} \right] + \frac{1}{a^\alpha} {}_\alpha\mathfrak{J}_a^b \left[ \frac{C\epsilon\alpha a^\alpha}{2k(b^\alpha - a^\alpha)} \right] \right) \\
&= \frac{2KC\epsilon\alpha}{2kC(b^\alpha - a^\alpha)} {}_\alpha\mathfrak{J}_a^b[1] \\
&= \frac{\epsilon\alpha}{b^\alpha - a^\alpha} \frac{b^\alpha - a^\alpha}{\alpha} \\
&= \epsilon.
\end{aligned}
$$

This proves the continuity of $\mathbf{N}$. We now show that the set $\mathbf{N}(C([a,b]))$ is relatively compact. Consider a sequence $\{y_n\}_{n\in\mathbb{N}}$ of $\mathbf{N}(C([a,b]))$ for all $n \in \mathbb{N}$. It exists

$x_n \in C([a,b])$ such that $y_n = \mathbf{N}(x_n)$. Observe that from Proposition 1.3.1 we have

$$\begin{aligned}|\mathbf{N}(x_n)(t)| &= \left| e^{-\frac{1}{\alpha}(\frac{t}{a})^\alpha} \left( e^{\frac{1}{\alpha}} x_0 + {}_\alpha\mathfrak{J}_a^t \left[ \frac{f(s,\widetilde{x}_n(s)) + \frac{1}{a^\alpha}\widetilde{x}_n(s)}{e^{-\frac{1}{\alpha}\left(\frac{s}{a}\right)^\alpha}} \right] \right) \right| \\ &\leq K \left( e^{\frac{1}{\alpha}} |x_0| + \frac{1}{C} {}_\alpha\mathfrak{J}_a^b \left[ \left| f(t,\widetilde{x}_n(s)) + \frac{1}{a^\alpha}\widetilde{x}_n(s) \right| \right] \right) \\ &\leq K \left( e^{\frac{1}{\alpha}} |x_0| + \frac{1}{C} {}_\alpha\mathfrak{J}_a^b \left[ |f\left(t,\widetilde{x}_n(s)\right)| \right] + \frac{1}{Ca^\alpha} {}_\alpha\mathfrak{J}_a^b \left[ |\widetilde{x}_n(s)| \right] \right).\end{aligned}$$

By definition, there is an $R > 0$ such that $|\widetilde{x}_n(s)| \leq R$ for all $s \in [a,b]$ and all $n \in \mathbb{N}$. The function $f$ is compact on $[a,b] \times B_R(0)$ and we can deduce the existence of a constant $A > 0$ such that $|f(s,\widetilde{x}_n(s))| \leq A$ for all $s \in [a,b]$. The sequence $\{y_n\}_{n\in\mathbb{N}}$ is uniformly bounded for all $n \in \mathbb{N}$. Observe also that for $t_1, t_2 \in [a,b]$ we have

$$\begin{aligned}&|\mathbf{N}(x_n)(t_2) - \mathbf{N}(x_n)(t_1)| \\ &\qquad \leq B \left| e^{-\frac{1}{\alpha}(\frac{t_1}{a})^\alpha} - e^{-\frac{1}{\alpha}(\frac{t_2}{a})^\alpha} \right| + \frac{K(A+\acute{R})}{C} \left| {}_\alpha\mathfrak{J}_{t_1}^{t_2}[1] \right| \\ &\qquad < B \left| e^{-\frac{1}{\alpha}(\frac{t_1}{a})^\alpha} - e^{-\frac{1}{\alpha}(\frac{t_2}{a})^\alpha} \right| + \frac{K(A+\acute{R})}{C} \frac{1}{\alpha} \left| t_1^\alpha - t_2^\alpha \right|,\end{aligned}$$

where $B := e^\alpha x_0$, $\acute{R} := \frac{R}{a^\alpha}$, $K := \max_{a\leq t\leq b}\{e^{-\frac{1}{\alpha}(\frac{t}{a})^\alpha}\}$, and $C := \min_{a\leq t\leq b}\{e^{-\frac{1}{\alpha}(\frac{t}{a})^\alpha}\}$. This proves that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is equicontinuous. By the Arzelà–Ascoli theorem, $\mathbf{N}(C([a,b]))$ is relatively compact and hence $\mathbf{N}$ is compact. $\square$

**Theorem 3.1.1.** *If* $(v,M) \in C^{(\alpha)}([a,b],\mathbb{R}) \times C^{(\alpha)}([a,b],[0,\infty))$ *is a tube solution to* (3.1)*, then problem* (3.1) *has a solution* $x \in C^{(\alpha)}([a,b],\mathbb{R}) \cap \mathrm{T}(v,M)$*.*

*Proof.* By Proposition 3.1.1, the operator $\mathbf{N}$ is compact. It has a fixed point by the Schauder fixed point theorem (see p. 137 of [36]). Therefore, Theorem 1.3.3 implies that such fixed point is a solution to problem (3.2)–(3.3) and it suffices to show that for every solution $x$ to problem (3.2)–(3.3), $x \in \mathbf{T}(v,M)$. Consider the set $A := \{t \in [a,b] : |x(t) - v(t)| > M(t)\}$. If $t \in A$, then by virtue of Proposition 1.3.2 we have

$$(|x(t)-v(t)| - M(t))^{(\alpha)} = \frac{(x(t)-v(t))\left(x^{(\alpha)}(t) - v^{(\alpha)}(t)\right)}{|x(t)-v(t)|} - M^{(\alpha)}(t).$$

Therefore, since $(v, M)$ is a tube solution to problem (3.1), we have on $\{t \in A : M(t) > 0\}$ that

$$
\begin{aligned}
&(|x(t) - v(t)| - M(t))^{(\alpha)} \\
&\quad= \frac{(x(t) - v(t))\left(x^{(\alpha)}(t) - v^{(\alpha)}(t)\right)}{|x(t) - v(t)|} - M^{(\alpha)}(t) \\
&\quad= \frac{(x(t) - v(t))\left(f(t, \widetilde{x}(t)) + \left(\frac{1}{a^\alpha}\widetilde{x}(t) - \frac{1}{a^\alpha}x(t)\right) - v^{(\alpha)}(t)\right)}{|x(t) - v(t)|} - M^{(\alpha)}(t) \\
&\quad= \frac{(\widetilde{x}(t) - v(t))\left(f(t, \widetilde{x}(t)) - v^{(\alpha)}(t)\right)}{M(t)} + \frac{(\widetilde{x}(t) - v(t))\,(\widetilde{x}(t) - x(t))}{a^\alpha M(t)} - M^{(\alpha)}(t) \\
&\quad= \frac{(\widetilde{x}(t) - v(t))\left(f(t, \widetilde{x}(t)) - v^{(\alpha)}(t)\right)}{M(t)} + \left[\frac{M(t)}{|x(t) - v(t)|} - 1\right] \frac{|x(t) - v(t)|^2}{a^\alpha\,|x(t) - v(t)|} - M^{(\alpha)}(t) \\
&\quad= \frac{(\widetilde{x}(t) - v(t))\left(f(t, \widetilde{x}(t)) - v^{(\alpha)}(t)\right)}{M(t)} + \left[\frac{M(t)}{a^\alpha} - \frac{|x(t) - v(t)|}{a^\alpha}\right] - M^{(\alpha)}(t) \\
&\quad\leq \frac{M(t)M^\alpha(t)}{M(t)} + \frac{1}{a^\alpha}\left[M(t) - |x(t) - v(t)|\right] - M^{(\alpha)}(t) \\
&\quad< 0.
\end{aligned}
$$

On the other hand, we have on $t \in \{\tau \in A : M(\tau) = 0\}$ that

$$
\begin{aligned}
(|x(t) - v(t)| - M(t))^{(\alpha)} &= \frac{(x(t) - v(t))\left(f(t, \widetilde{x}(t)) + \left(\frac{1}{a^\alpha}\widetilde{x}(t) - \frac{1}{a^\alpha}x(t)\right) - v^{(\alpha)}(t)\right)}{|x(t) - v(t)|} - M^{(\alpha)}(t) \\
&= \frac{(x(t) - v(t))\left(f(t, \widetilde{x}(t)) - v^{(\alpha)}(t)\right)}{|x(t) - v(t)|} - \frac{1}{a^\alpha}|x(t) - v(t)| - M^{(\alpha)}(t) \\
&< -M^{(\alpha)}(t) \\
&= 0.
\end{aligned}
$$

The last equality follows from Definition 3.1.1. If we set $r(t) := |x(t) - v(t)| - M(t)$, then $r^{(\alpha)} < 0$ on $A := \{t \in [a, b] : r(t) > 0\}$. Moreover, since $(v, M)$ is a tube solution to problem (3.1) and $x$ satisfies $|x_0 - v(a)| \leq M(a)$, we know that $r(a) \leq 0$ and Lemma 1.3.1 implies that $A = \emptyset$. Therefore, $x \in \mathrm{T}(v, M)$ and the theorem is proved. $\square$

## 3.2 An Example

Consider the conformable noninteger order system

$$\begin{cases} x^{(\frac{1}{2})}(t) = a\frac{\sqrt{t}}{1+t}x^3(t) + bx(t)e^{cx(t)}, \quad t \in [1,2], \\ x(1) = 0, \end{cases} \tag{3.4}$$

where $a, b \in (\infty, 0]$ and $c$ is a real constant. According to Definition 3.1.1, $(v, M) \equiv (0, 1)$ is a tube solution. It follows from our Theorem 3.1.1 that problem (3.4) has a solution $x$ such that $|x(t)| \leq 1$ for every $t \in [1, 2]$.

# Chapter 4

# Complex-valued fractional derivatives on time scales

We introduce a notion of fractional (noninteger order) derivative on an arbitrary nonempty closed subset of the real numbers (on a time scale). Main properties of the new operator are proved and several illustrative examples given. The original results of this chapter are published in [13].

## 4.1 Main Result

Let $f : \mathbb{T} \to \mathbb{R}$ with $\mathbb{T}$ a given time scale. We introduce here a new definition of fractional (noninteger) delta derivative of order $\alpha \in (0, 1]$ at a point $t \in \mathbb{T}^\kappa$.

**Definition 4.1.1** (The delta fractional derivative of order $\alpha$)**.** *Assume $f : \mathbb{T} \to \mathbb{R}$ with $\mathbb{T}$ a time scale. Let $t \in \mathbb{T}^\kappa$ and $\alpha \in (0, 1]$. We define $f^{\Delta^\alpha}(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$ there is a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$\left|[f^\alpha(\sigma(t)) - f^\alpha(s)] - f^{\Delta^\alpha}(t)[\sigma(t)^\alpha - s^\alpha]\right| \leq \epsilon \left|\sigma(t)^\alpha - s^\alpha\right| \qquad (4.1)$$

*for all $s \in U$. We call $f^{\Delta^\alpha}(t)$ the delta derivative of order $\alpha$ of $f$ at $t$ or the delta fractional (noninteger order) derivative of $f$ at $t$. Moreover, we say that $f$ is delta* *differentiable of order $\alpha$ on $\mathbb{T}^\kappa$ provided $f^{\Delta^\alpha}(t)$ exists for all $t \in \mathbb{T}^\kappa$. Function $f^{\Delta^\alpha} : \mathbb{T}^\kappa \to \mathbb{C}$ is then called the delta derivative of order $\alpha$ of $f$ on $\mathbb{T}^\kappa$.*

**Remark 4.1.1.** *In (4.1) we use $f^\alpha$ to denote the power $\alpha$ of $f$. It is clear that the new derivative coincides with the standard Hilger derivative in the integer order case $\alpha = 1$. Differently from $\alpha = 1$, in general $f^{\Delta^\alpha}(t)$ is a complex number.*

**Theorem 4.1.1.** *Assume $f : \mathbb{T} \to \mathbb{R}$ with $\mathbb{T}$ a time scale. Let $t \in \mathbb{T}^\kappa$ and $\alpha \in \mathbb{R}$. Then the following properties hold:*

1. *If $f$ is continuous at $t$ and $t$ is right-scattered, then $f$ is delta differentiable of order $\alpha$ at $t$ with*
$$f^{\Delta^\alpha}(t) = \frac{f^\alpha(\sigma(t)) - f^\alpha(t)}{\sigma^\alpha(t) - t^\alpha}. \tag{4.2}$$

2. *If $t$ is right-dense, then $f$ is delta differentiable of order $\alpha$ at $t$ if and only if the limit*
$$\lim_{s \to t} \frac{f^\alpha(t) - f^\alpha(s)}{t^\alpha - s^\alpha}$$
*exists as a finite number. In this case*
$$f^{\Delta^\alpha}(t) = \lim_{s \to t} \frac{f^\alpha(t) - f^\alpha(s)}{t^\alpha - s^\alpha}. \tag{4.3}$$

3. *If $f$ is delta differentiable of order $\alpha$ at $t$, then*
$$f^\alpha(\sigma(t)) = f^\alpha(t) + (\sigma(t)^\alpha - t^\alpha) f^{\Delta^\alpha}(t).$$

*Proof.* 1. Assume $f$ is continuous at $t$ and $t$ is right scattered. By continuity,
$$\lim_{s \to t} \frac{f^\alpha(\sigma(t)) - f^\alpha(s)}{\sigma^\alpha(t) - s^\alpha} = \frac{f^\alpha(\sigma(t)) - f^\alpha(t)}{\sigma^\alpha(t) - t^\alpha}.$$
Hence, given $\epsilon > 0$, there is a neighborhood $U$ of $t$ such that
$$\left| \frac{f^\alpha(\sigma(t)) - f^\alpha(s)}{\sigma^\alpha(t) - s^\alpha} - \frac{f^\alpha(\sigma(t)) - f^\alpha(t)}{\sigma^\alpha(t) - t^\alpha} \right| \leq \epsilon$$
for all $s \in U$. It follows that
$$\left| f^\alpha(\sigma(t)) - f^\alpha(s) - \frac{f^\alpha(\sigma(t)) - f^\alpha(t)}{\sigma^\alpha(t) - t^\alpha} [\sigma^\alpha(t) - s^\alpha] \right| \leq \epsilon \left| \sigma^\alpha(t) - s^\alpha \right|$$
for all $s \in U$. Hence, we get the desired result (4.2).

2. Assume $f$ is differentiable at $t$ and $t$ is right-dense. Let $\epsilon > 0$ be given. Since $f$ is differentiable at $t$, there is a neighborhood $U$ of $t$ such that

$$\mid [f^{\alpha}(\sigma(t)) - f^{\alpha}(t)] - f^{\Delta^{\alpha}}(t)[\sigma^{\alpha}(t) - s^{\alpha}] \mid \leq \epsilon \mid \sigma^{\alpha}(t) - s^{\alpha} \mid$$

for all $s \in U$. Since $\sigma(t) = t$, we have that

$$\mid [f^{\alpha}(\sigma(t)) - f^{\alpha}(t)] - f^{\Delta^{\alpha}}(t)[t^{\alpha} - s^{\alpha}] \mid \leq \epsilon \mid \sigma^{\alpha}(t) - s^{\alpha} \mid$$

for all $s \in U$. It follows that $\left|\frac{f^{\alpha}(t)-f^{\alpha}(s)}{t^{\alpha}-s^{\alpha}} - f^{\Delta^{\alpha}}(t)\right| \leq \epsilon$ for all $s \in U$, $s \neq t$, and we get the desired equality (4.3). Assume $\lim_{s\to t} \frac{f^{\alpha}(t)-f^{\alpha}(s)}{t^{\alpha}-s^{\alpha}}$ exists and is equal to $X$ and $\sigma(t) = t$. Let $\epsilon > 0$. Then there is a neighborhood $U$ of $t$ such that

$$\left|\frac{f^{\alpha}(\sigma(t)) - f^{\alpha}(s)}{t^{\alpha} - s^{\alpha}} - X\right| \leq \epsilon$$

for all $s \in U$. Because $\mid f^{\alpha}(\sigma(t)) - f^{\alpha}(s) - X(t^{\alpha} - s^{\alpha}) \mid \leq \epsilon|t^{\alpha} - s^{\alpha}|$ for all $s \in U$,

$$f^{\Delta^{\alpha}}(t) = X = \lim_{s\to t} \frac{f^{\alpha}(t) - f^{\alpha}(s)}{t^{\alpha} - s^{\alpha}}.$$

3. If $\sigma(t) = t$, then $\sigma^{\alpha}(t) - t^{\alpha} = 0$ and

$$f^{\alpha}(\sigma(t)) = f^{\alpha}(t) = f^{\alpha}(t) + (\sigma^{\alpha}(t) - t^{\alpha})f^{\Delta^{\alpha}}(t).$$

On the other hand, if $\sigma(t) > t$, then by item 1

$$\begin{aligned} f^{\alpha}(\sigma(t)) &= f^{\alpha}(t) + (\sigma^{\alpha}(t) - t^{\alpha})\frac{f^{\alpha}(\sigma(t)) - f^{\alpha}(t)}{\sigma(t)^{\alpha} - t^{\alpha}} \\ &= f^{\alpha}(t) + (\sigma^{\alpha}(t) - t^{\alpha})f^{\Delta^{\alpha}}(t) \end{aligned}$$

and the proof is complete.

□

**Example 4.1.1.** *If $\mathbb{T} = \mathbb{R}$, then* (4.3) *yields that $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable of order $\alpha$ at $t \in \mathbb{R}$ if and only if $f^{\Delta^{\alpha}}(t) = \lim_{s\to t} \frac{f^{\alpha}(t)-f^{\alpha}(s)}{t^{\alpha}-s^{\alpha}}$ exists, i.e., if and only if $f$ is fractional differentiable at $t$. In this case we get the derivative $f^{(\alpha)}$ of [39].*

**Example 4.1.2.** *If $\mathbb{T} = \mathbb{Z}$, then item 1 of Theorem 4.1.1 yields that $f : \mathbb{Z} \to \mathbb{R}$ is delta-differentiable of order $\alpha$ at $t \in \mathbb{Z}$ with*

$$f^{\Delta^\alpha}(t) = \frac{f^\alpha(\sigma(t)) - f^\alpha(t)}{\sigma^\alpha(t) - t^\alpha} = \frac{f^\alpha(t+1) - f^\alpha(t)}{(t+1)^\alpha - t^\alpha}.$$

**Example 4.1.3.** *If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) \equiv \lambda \in \mathbb{R}$, then $f^{\Delta^\alpha}(t) \equiv 0$. Indeed, if $t$ is right-scattered, then by item 1 of Theorem 4.1.1 $f^{\Delta^\alpha}(t) = \frac{f^\alpha(\sigma(t)) - f^\alpha(t)}{\sigma^\alpha(t) - t^\alpha} = \frac{\lambda^\alpha - \lambda^\alpha}{\sigma^\alpha(t) - t^\alpha} = 0$; if $t$ is right-dense, then by (4.3) we get $f^{\Delta^\alpha}(t) = \lim_{s \to t} \frac{\lambda^\alpha - \lambda^\alpha}{t^\alpha - s^\alpha} = 0$.*

**Example 4.1.4.** *If $f : \mathbb{T} \to \mathbb{R}$, $t \mapsto t$, then $f^{\Delta^\alpha} \equiv 1$ because if $\sigma(t) > t$ (i.e., $t$ is right-scattered), then $f^{\Delta^\alpha}(t) = \frac{f^\alpha(\sigma(t)) - f^\alpha(t)}{\sigma^\alpha(t) - t^\alpha} = \frac{\sigma^\alpha(t) - t^\alpha}{\sigma^\alpha(t) - t^\alpha} = 1$; if $\sigma(t) = t$ (i.e., $t$ is right-dense), then $f^{\Delta^\alpha} = \lim_{s \to t} \frac{f^\alpha(t) - f^\alpha(s)}{t^\alpha - s^\alpha} = \frac{t^\alpha - s^\alpha}{t^\alpha - s^\alpha} = 1$.*

**Example 4.1.5.** *Let $g : \mathbb{T} \to \mathbb{R}$, $t \mapsto \frac{1}{t}$. We have $g^{\Delta^\alpha}(t) = -\frac{1}{(t\sigma(t))^\alpha}$. Indeed, if $\sigma(t) = t$, then $g^{\Delta^\alpha}(t) = -\frac{1}{t^{2\alpha}}$; if $\sigma(t) > t$, then*

$$g^{\Delta^\alpha}(t) = \frac{g^\alpha(\sigma(t)) - g^\alpha(t)}{\sigma^\alpha(t) - t^\alpha} = \frac{\left(\frac{1}{\sigma(t)}\right)^\alpha - \left(\frac{1}{t}\right)^\alpha}{\sigma^\alpha(t) - t^\alpha} = \frac{\frac{t^\alpha - \sigma^\alpha(t)}{t^\alpha \sigma^\alpha(t)}}{t^\alpha - \sigma^\alpha(t)} = -\frac{1}{t^\alpha \sigma^\alpha(t)}.$$

**Example 4.1.6.** *Let $h : \mathbb{T} \to \mathbb{R}$, $t \mapsto t^2$. We have $h^{\Delta^\alpha}(t) = \sigma^\alpha(t) + t^\alpha$. Indeed, if $t$ is right-dense, then $h^{\Delta^\alpha}(t) = \lim_{s \to t} \frac{t^{2\alpha} - s^{2\alpha}}{t^\alpha - s^\alpha} = 2t^\alpha$; if $t$ is right-scattered, then*

$$h^{\Delta^\alpha}(t) = \frac{h^\alpha(\sigma(t)) - h^\alpha(t)}{\sigma^\alpha(t) - t^\alpha} = \frac{\sigma^{2\alpha}(t) - t^{2\alpha}}{\sigma^\alpha(t) - t^\alpha} = \sigma^\alpha(t) + t^\alpha.$$

**Example 4.1.7.** *Consider the time scale $\mathbb{T} = h\mathbb{Z}$, $h > 0$. Let $f$ be the function defined by $f : h\mathbb{Z} \to \mathbb{R}$, $t \mapsto (t - c)^2$, $c \in \mathbb{R}$. The fractional derivative of order $\alpha$ of $f$ at $t$ is*

$$\begin{aligned} f^{\Delta^\alpha}(t) &= \frac{f^\alpha(\sigma(t)) - f^\alpha(t)}{\sigma^\alpha(t) - t^\alpha} = \frac{((\sigma(t) - c)^2)^\alpha - ((t - c)^2)^\alpha}{\sigma^\alpha(t) - t^\alpha} \\ &= \frac{(t + h - c)^{2\alpha} - (t - c)^{2\alpha}}{(t + h)^\alpha - t^\alpha}. \end{aligned}$$

**Remark 4.1.2.** *Examples 4.1.5, 4.1.6 and 4.1.7 show that in general $f^{\Delta^\alpha}(t)$ is a complex number (for instance, choose $\alpha = \frac{1}{2}$ and $t < 0$).*

**Theorem 4.1.2.** *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are continuous and delta differentiable of order $\alpha$ at $t \in \mathbb{T}^\kappa$. Then the following proprieties hold:*

1. *For any constant $\lambda$, function $\lambda f : \mathbb{T} \to \mathbb{R}$ is delta differentiable of order $\alpha$ at $t$ with $(\lambda f)^{\Delta^\alpha} = \lambda^\alpha f^{\Delta^\alpha}$.*

2. *The product $fg : \mathbb{T} \to \mathbb{R}$ is delta differentiable of order $\alpha$ at $t$ with*

$$(fg)^{\Delta^\alpha}(t) = f^{\Delta^\alpha}(t)g^\alpha(t) + f^\alpha(\sigma(t))g^{\Delta^\alpha}(t) = f^{\Delta^\alpha}(t)g^\alpha(\sigma(t)) + f^\alpha(t)g^{\Delta^\alpha}(t).$$

3. *If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is delta differentiable of order $\alpha$ at $t$ with*

$$\left(\frac{1}{f}\right)^{\Delta^\alpha}(t) = \frac{-f^{\Delta^\alpha}(t)}{f^\alpha(\sigma(t))f^\alpha(t)}.$$

4. *If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is delta differentiable of order $\alpha$ at $t$ with*

$$\left(\frac{f}{g}\right)^{\Delta^\alpha}(t) = \frac{f^{\Delta^\alpha}(t)g^\alpha(t) - f^\alpha(t)g^{\Delta^\alpha}(t)}{g^\alpha(\sigma(t))g^\alpha(t)}.$$

*Proof.* 1. Let $\epsilon \in (0,1)$. Define $\epsilon^* = \frac{\epsilon}{|\lambda|^\alpha} \in (0,1)$. Then there exists a neighborhood $U$ of $t$ such that $|f^\alpha(\sigma(t)) - f^\alpha(s) - f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)| \leq \epsilon^*|\sigma^\alpha(t) - s^\alpha|$ for all $s \in U$. It follows that

$$\begin{aligned}
&|(\lambda f)^\alpha(\sigma(t)) - (\lambda f)^\alpha(s) - \lambda^\alpha f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)| \\
&\quad = |\lambda|^\alpha \mid f^\alpha(\sigma(t)) - f^\alpha(s) - f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha) \mid \\
&\quad \leq \epsilon^*|\lambda|^\alpha|\sigma^\alpha(t) - s^\alpha| \leq \frac{\epsilon}{|\lambda|^\alpha}|\lambda|^\alpha|\sigma^\alpha(t) - s^\alpha| = \epsilon|\sigma^\alpha(t) - s^\alpha|
\end{aligned}$$

for all $s \in U$. Thus $(\lambda f)^{\Delta^\alpha}(t) = \lambda^\alpha f^{\Delta^\alpha}(t)$ holds.

2. Let $\epsilon \in (0,1)$. Define $\epsilon^* = \epsilon[1 + |f^\alpha(t)| + |g^\alpha(\sigma(t))| + |g^{\Delta^\alpha}(\sigma(t))|]^{-1}$. Then $\epsilon^* \in (0,1)$ and there exist neighborhoods $U_1, U_2$ and $U_3$ of $t$ such that

$$|f^\alpha(\sigma(t)) - f^\alpha(s) - f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)| \leq \epsilon^*|\sigma^\alpha(t) - s^\alpha|$$

for all $s \in U_1$, $|g^\alpha(\sigma(t)) - g^\alpha(s) - g^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)| \leq \epsilon^*|\sigma^\alpha(t) - s^\alpha|$ for all $s \in U_2$ and such as $f$ is continuous. Then $|f(t) - f(s)| \leq \epsilon^*$ for all $s \in U_3$. Define $U = U_1 \cap U_2 \cap U_3$ and let $s \in U$. It follows that

$$
\begin{aligned}
&|(fg)^\alpha(\sigma(t)) - (fg)^\alpha(s) - [g^{\Delta^\alpha}(t)f^\alpha(t) + g^\alpha(\sigma(t))f^{\Delta^\alpha}(t)][\sigma^\alpha(t) - s^\alpha]| \\
&= |[f^\alpha(\sigma(t)) - f^\alpha(s) - f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)](g^\alpha(\sigma(t))) + g^\alpha(\sigma(t))f^\alpha(s) \\
&\quad + g^\alpha(\sigma(t))f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha) - f^\alpha(s)g^\alpha(s) \\
&\quad - [g^{\Delta^\alpha}(t)f^\alpha(t) + g^\alpha(\sigma(t))f^{\Delta^\alpha}(t)][\sigma^\alpha(t) - s^\alpha]| \\
&= |[f^\alpha(\sigma(t)) - f^\alpha(s) - f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)](g^\alpha(\sigma(t))) \\
&\quad + [g^\alpha(\sigma(t)) - g^\alpha(s) - g^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)](f^\alpha(t)) \\
&\quad + [g^\alpha(\sigma(t)) - g^\alpha(s) - g^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)](f^\alpha(s) - f^\alpha(t)) + f^\alpha(s)g^\alpha(s) \\
&\quad + g^{\Delta^\alpha}(t)f^\alpha(s)(\sigma^\alpha(t) - s^\alpha) + g^\alpha(\sigma(t))f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha) - g^\alpha(s)f^\alpha(s) \\
&\quad + g^{\Delta^\alpha}(t)f^\alpha(s)(\sigma^\alpha(t) - s^\alpha) + g^\alpha(\sigma(t))f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha) - f^\alpha(s)g^\alpha(s) \\
&\quad - [g^{\Delta^\alpha}(t)f^\alpha(t) + g^\alpha(\sigma(t))f^{\Delta^\alpha}(t)][\sigma^\alpha(t) - s^\alpha]| \\
&\leq |f^\alpha(\sigma(t)) - f^\alpha(s) - f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)||(g^\alpha(\sigma(t)))| \\
&\quad + |g^\alpha(\sigma(t)) - g^\alpha(s) - g^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)||(f^\alpha(t))| \\
&\quad + |g^\alpha(\sigma(t)) - g^\alpha(s) - g^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)||f^\alpha(s) - f^\alpha(t)| \\
&\quad + |g^{\Delta^\alpha}(t)||f^\alpha(t) - f^\alpha(s)||\sigma^\alpha(t) - s^\alpha| \\
&= \epsilon^*|(g^\alpha(\sigma(t)))||\sigma^\alpha(t) - s^\alpha| \\
&\quad + \epsilon^*|(f^\alpha(t))||\sigma^\alpha(t) - s^\alpha| + \epsilon^*|\sigma^\alpha(t) - s^\alpha|\epsilon^* + \epsilon^*|g^{\Delta^\alpha}(t)||\sigma^\alpha(t) - s^\alpha| \\
&\leq \epsilon^*|\sigma^\alpha(t) - s^\alpha|(\epsilon^* + |(f^\alpha(t))| + |g^{\Delta^\alpha}(t)| + |g^{\Delta^\alpha}(t)|) \\
&\leq \epsilon^*|\sigma^\alpha(t) - s^\alpha|(1 + |(f^\alpha(t))| + |g^{\Delta^\alpha}(t)| + |g^{\Delta^\alpha}(t)|) = \epsilon|\sigma^\alpha(t) - s^\alpha|.
\end{aligned}
$$

Thus $(fg)^{\Delta^\alpha}(t) = f^\alpha(t)g^{\Delta^\alpha}(t) + f^{\Delta^\alpha}(t)g^\alpha(\sigma(t))$ holds at $t$. The other product rule follows from this last equality by interchanging functions $f$ and $g$.

3. We use the delta derivative of a constant (Example 4.1.3). Since $\left(f \cdot \frac{1}{f}\right)^{\Delta^\alpha}(t) = 0$, it follows from item 2 that $\left(\frac{1}{f}\right)^{\Delta^\alpha}(t)f^\alpha(\sigma(t)) + f^{\Delta^\alpha}(t)\frac{1}{f^\alpha(t)} = 0$. Because we are assuming $f(t)f(\sigma(t)) \neq 0$, one has $\left(\frac{1}{f}\right)^{\Delta^\alpha}(t) = \frac{-f^{\Delta^\alpha}(t)}{f^\alpha(\sigma(t))f^\alpha(t)}$.

4. For the quotient formula we use items 2 and 3 to compute

$$\begin{aligned}\left(\frac{f}{g}\right)^{\Delta^\alpha}(t) &= \left(f\cdot\frac{1}{g}\right)^{\Delta^\alpha}(t) = f^\alpha(t)\left(\frac{1}{g}\right)^{\Delta^\alpha}(t) + f^{\Delta^\alpha}(t)\frac{1}{g^\alpha(\sigma(t))}\\ &= -f^\alpha(t)\frac{g^{\Delta^\alpha}(t)}{g^\alpha(\sigma(t))g^\alpha(t)} + f^{\Delta^\alpha}(t)\frac{1}{g^\alpha(\sigma(t))}\\ &= \frac{f^{\Delta^\alpha}(t)g^\alpha(t) - f^\alpha(t)g^{\Delta^\alpha}(t)}{g^\alpha(\sigma(t))g^\alpha(t)}.\end{aligned}$$

This concludes the proof.

□

**Remark 4.1.3.** *The delta derivative of order $\alpha$ of the sum $f+g:\mathbb{T}\to\mathbb{R}$ does not satisfy the usual property, that is, in general $(f+g)^{\Delta^\alpha}(t)\neq(f)^{\Delta^\alpha}(t)+(g)^{\Delta^\alpha}(t)$. For instance, let $\mathbb{T}$ be an arbitrary time scale and $f,g$ be functions defined by $f:\mathbb{T}\to\mathbb{R}$, $t\mapsto t$, and $g:\mathbb{T}\to\mathbb{R}$, $t\mapsto 2t$. One can easily find that $(f+g)^{\Delta^\alpha}(t)=\sqrt{3}\neq f^{\Delta^\alpha}(t)+g^{\Delta^\alpha}(t)=1+\sqrt{2}$.*

**Proposition 4.1.1.** *Let $\alpha\in\mathbb{R}$ and $m\in\mathbb{N}$, $m>1$. For $g$ defined by $g(t)=t^m$ we have*

$$g^{\Delta^\alpha}(t)=\sum_{k=0}^{m-1}(t^\alpha)^{m-k-1}(\sigma^\alpha)^k(t). \tag{4.4}$$

*Proof.* We prove the formula by induction. If $m=2$, then $g(t)=t^2$ and from Example 4.1.6 we know that $g^{\Delta^\alpha}(t)=\sum_{k=0}^{1}(t^\alpha)^{1-k}(\sigma^\alpha)^k(t)=t^\alpha+\sigma^\alpha(t)$. Now assume

$$g^{\Delta^\alpha}(t)=\sum_{k=0}^{m-1}(t^\alpha)^{m-k-1}(\sigma^\alpha)^k(t)$$

holds for $g(t)=t^m$ and let $G(t)=t^{m+1}=t\cdot g(t)$. We use the product rule of Theorem 4.1.2 to obtain

$$\begin{aligned}G^{\Delta^\alpha}(t) &= g^\alpha(t)+\sigma^\alpha(t)g^{\Delta^\alpha}(t)=(t^\alpha)^m+\sigma^\alpha(t)\sum_{k=0}^{m-1}(t^\alpha)^{m-k-1}(\sigma^\alpha)^k(t)\\ &= (t^\alpha)^m+\sum_{k=0}^{m-1}(t^\alpha)^{m-k-1}(\sigma^\alpha)^{k+1}(t)=(t^\alpha)^m+\sum_{k=1}^{m-1}(t^\alpha)^{m-k}(\sigma^\alpha)^k(t)\\ &= \sum_{k=0}^{m}(t^\alpha)^{m-k}(\sigma^\alpha)^k(t).\end{aligned}$$

Hence, by mathematical induction, (4.4) holds. □

**Example 4.1.8.** *Choose $m = 3$ in Proposition 4.1.1. Then $(t^3)^{\Delta^\alpha} = t^{2\alpha} + (t\sigma(t))^\alpha + \sigma^{2\alpha}(t)$.*

The notion of fractional derivative here introduced can be easily extended to any arbitrary real order $\alpha$.

**Definition 4.1.2.** *Let $\alpha > 0$ and $N \in \mathbb{N}_0$ be such that $N < \alpha \leq N + 1$. Then we define $f^{\Delta^\alpha} = \left(f^{\Delta^N}\right)^{\Delta^{\alpha - N}}$, where $f^{\Delta^N}$ is the usual Hilger derivative of order $N$.*

# Chapter 5

# A Truly Conformable Calculus on Time Scales

We introduce the definition of conformable derivative on time scales and develop its calculus. Fundamental properties of the conformable derivative and integral on time scales are proved. Linear conformable differential equations with constant coefficients are investigated as well as hyperbolic and trigonometric functions. The original results of this chapter were publication [15]

## 5.1 Main results

We begin by introducing the notion of conformable differential operator of order $\alpha \in [0, 1]$ on an arbitrary time scale $\mathbb{T}$.

**Definition 5.1.1.** *Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. We define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$\big|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\big| \leq \epsilon\, |\sigma(t) - s|$$

*for all $s \in U$. We call $f^\Delta(t)$ the delta derivative of $f$ at $t$.*

**Definition 5.1.2** (Conformable delta differential operator of order $\alpha$)**.** *Let $\mathbb{T}$ be a time scale and let $\alpha \in [0,1]$. An operator $\Delta^\alpha$ is conformable if and only if $\Delta^0$ is the identity operator and $\Delta^1$ is the standard differential operator on $\mathbb{T}$. Precisely, operator $\Delta^\alpha$ is conformable if and only if for a differentiable function $f$ in the sense of Definition 5.1.1, one has $\Delta^0 f = f$ and $\Delta^1 f = f^\Delta$.*

Proposition 5.1.1 gives an extension of [6] to time scales $\mathbb{T}$: for $\mathbb{T} = \mathbb{R}$, (5.2) subject to (5.1) gives Definition 1.3 of [6].

**Definition 5.1.3.** *A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $k : [0,1] \times \mathbb{T} \to [0,\infty)$ is rd-continuous if $k(\alpha,\cdot) : \mathbb{T} \to [0,\infty)$ is rd-continuous for all $\alpha \in [0,1]$ and $k(\cdot,t) : [0,1] \to [0,\infty)$ is continuous for all $t \in \mathbb{T}$.*

**Proposition 5.1.1** (A conformable derivative $\Delta^\alpha$ on time scales)**.** *Let $\mathbb{T}$ be a time scale, $\alpha \in [0,1]$, and $\kappa_0, \kappa_1 : [0,1] \times \mathbb{T} \to [0,\infty)$ be rd-continuous functions such that*

$$\begin{gathered}
\lim_{\alpha \to 0^+} \kappa_1(\alpha,t) = 1, \quad \lim_{\alpha \to 0^+} \kappa_0(\alpha,t) = 0, \\
\lim_{\alpha \to 1^-} \kappa_1(\alpha,t) = 0, \quad \lim_{\alpha \to 1^-} \kappa_0(\alpha,t) = 1, \\
\kappa_1(\alpha,t) \neq 0, \quad \kappa_0(\alpha,t) \neq 0, \quad \alpha \in (0,1],
\end{gathered} \tag{5.1}$$

*for all $t \in \mathbb{T}$. Then, the differential operator $\Delta^\alpha f$, defined by*

$$\Delta^\alpha f(t) = \kappa_1(\alpha,t) f(t) + \kappa_0(\alpha,t) f^\Delta(t) \tag{5.2}$$

*in the class of $\Delta$-differentiable functions $f$, is conformable in the sense of Definition 5.1.2.*

*Proof.* The result is a trivial consequence of (5.1)–(5.2): $\Delta^0 f = f$ and $\Delta^1 f = f^\Delta$. □

**Remark 5.1.1.** *Let $\alpha \in (0,1]$, $\kappa_1(\alpha,t) \equiv 0$, and $\kappa_0(\alpha,t) = t^{1-\alpha}$. Then, formally, we recover* (8) *from* (5.2)*. However, such choice of $\kappa_0$ and $\kappa_1$ is not allowed by* (5.1) *because* (8) *is not conformable in agreement with Definition 5.1.2.*

Many examples of conformable derivatives on time scales are easily obtained from Proposition 5.1.1.

**Example 5.1.1.** *One can take* $\kappa_1 \equiv (1-\alpha)\omega^{\alpha}$ *and* $\kappa_0 \equiv \alpha\omega^{1-\alpha}$ *for* $\omega \in (0,\infty)$ *or* $\kappa_1(\alpha,t) = (1-\alpha)|t|^{\alpha}$ *and* $\kappa_0(\alpha,t) = \alpha|t|^{1-\alpha}$ *on* $\mathbb{T}\setminus\{0\}$ *in Proposition 5.1.1. In this last case,*

$$\Delta^{\alpha} f(t) = (1-\alpha)|t|^{\alpha} f(t) + \alpha|t|^{1-\alpha} f^{\Delta}(t).$$

**Example 5.1.2.** *Similarly to Example 5.1.1,*

$$\Delta^{\alpha} f(t) = \cos\left(\frac{\alpha\pi}{2}\right)|t|^{\alpha} f(t) + \sin\left(\frac{\alpha\pi}{2}\right)|t|^{1-\alpha} f^{\Delta}(t)$$

*is a conformable derivative.*

**Lemma 5.1.1.** *Let* $\alpha,\beta \in [0,1]$ *and the functions* $\kappa_i, i = 0,1$ *are* $\Delta_t$*- differentiable and continuous compared to* $\alpha$ *. Note that, in general,* $\Delta^{\alpha}\Delta^{\beta} \neq \Delta^{\beta}\Delta^{\alpha}$.

*Proof.* Let $\Delta^{\alpha} f(t) = \kappa_1(\alpha,t)f(t) + \kappa_0(\alpha,t)f^{\Delta}(t)$ and $\Delta^{\beta} f(t) = \kappa_1(\beta,t)f(t) + \kappa_0(\beta,t)f^{\Delta}(t)$ we calculated $\Delta^{\beta}\Delta^{\alpha} f(t)$ we have

$\Delta^{\beta}\Delta^{\alpha} f(t)$

$= \kappa_1(\beta,t)\left(\kappa_1(\alpha,t)f(t) + \kappa_0(\alpha,t)f^{\Delta}(t)\right) + \kappa_0(\beta,t)\left(\kappa_1(\alpha,t)f(t) + \kappa_0(\alpha,t)f^{\Delta}(t)\right)^{\Delta}$

$= \kappa_1(\beta,t)\kappa_1(\alpha,t)f(t) + \kappa_1(\beta,t)\kappa_0(\alpha,t)f^{\Delta}(t) + \kappa_0(\beta,t)\left[\kappa_1^{\Delta}(\alpha,t)f^{\sigma}(t) + \kappa_1(\alpha,t)f^{\Delta}(t)\right.$

$\left.+ \kappa_0^{\Delta}(\alpha,t)f^{\Delta^{\sigma}}(t) + \kappa_0(\alpha,t)f^{\Delta^2}(t)\right]$

$= \kappa_1(\beta,t)\kappa_1(\alpha,t)f(t) + \kappa_1(\beta,t)\kappa_0(\alpha,t)f^{\Delta}(t) + \kappa_0(\beta,t)\kappa_1^{\Delta}(\alpha,t)f^{\sigma}(t) + \kappa_0(\beta,t)\kappa_1^{\Delta}(\alpha,t)f^{\Delta}(t)$

$+ \kappa_0(\beta,t)\kappa_0^{\Delta}(\alpha,t)f^{\Delta^{\sigma}}(t) + \kappa_0(\beta,t)\kappa_0(\alpha,t)f^{\Delta^2}(t)$

We follow the same calculation we find

$\Delta^{\alpha}\Delta^{\beta} f(t)$

$= \kappa_1(\alpha,t)\kappa_1(\beta,t)f(t) + \kappa_1(\alpha,t)\kappa_0(\beta,t)f(t)f^{\Delta}(t) + \kappa_0(\alpha,t)\kappa_1^{\Delta}(\beta,t)f^{\sigma}(t) + \kappa_0(\alpha,t)\kappa_1(\beta,t)f^{\Delta}(t)$

$+ \kappa_0(\alpha,t)\kappa_0^{\Delta}(\beta,t)f^{\Delta^{\sigma}}(t) + \kappa_0(\alpha,t)\kappa_0(\beta,t)f^{\Delta^2}(t)$

since $\kappa_0(\alpha,t)\kappa_1^{\Delta}(\beta,t) \neq \kappa_0(\beta,t)\kappa_1^{\Delta}(\alpha,t)$ and $\kappa_0(\beta,t)\kappa_0^{\Delta}(\alpha,t) \neq \kappa_0(\alpha,t)\kappa_0^{\Delta}(\beta,t)$ and hence

$\Delta^{\beta}\Delta^{\alpha} \neq \Delta^{\alpha}\Delta^{\beta}$

$\square$

**Example 5.1.3.** *:One can take $\kappa_1 \equiv (1-\alpha)\omega^\alpha$ and $\kappa_0 \equiv \alpha\omega^{1-\alpha}$ for $\omega \in (0,\infty)_{\mathbb{T}}$*

*Let $\alpha = \frac{1}{2}, \beta = 1$ we have $\Delta\Delta^{\frac{1}{2}} f(t) = \frac{1}{2}\left[\frac{\sigma^{\frac{1}{2}}(t) - t^{\frac{1}{2}}}{\mu(t)}\left(f^\sigma(t) + f^{\Delta^\sigma}(t)\right) + t^{\frac{1}{2}}\left(f^\Delta(t) + f^{\Delta^2}(t)\right)\right]$ on the other hand $\Delta^{\frac{1}{2}}\Delta f(t) = \frac{1}{2}t^{\frac{1}{2}}\left(f^\Delta(t) + f^{\Delta^2}(t)\right)$*

*if $\sigma(t) \neq t$ then*

$\Delta\Delta^{\frac{1}{2}} f(t) \neq \Delta^{\frac{1}{2}}\Delta f(t)$

**Definition 5.1.4** (Conformable exponential function on time scales)**.** *Let $\alpha \in (0,1]$, $s, t \in \mathbb{T}$ with $s \leq t$ and let function $p : \mathbb{T} \to \mathbb{R}$ be rd-continuous. Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{T} \to [0,\infty)$ be rd-continuous and satisfy (5.1) with $1 + \mu(t)\frac{p(t) - \kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ for all $t \in \mathbb{T}^\kappa$. Then, the conformable exponential function on the time scale $\mathbb{T}$ with respect to $\Delta^\alpha$ in (5.2) is defined to be*

$$E_p(t,s) = e_{\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s), \quad E_0(t,s) = e_{\frac{-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s), \tag{5.3}$$

*where $e_{q(t)}(t,s)$ denotes the exponential function on the time scale $\mathbb{T}$ – see definition (2.30) in [22].*

Note that if $\mathbb{T} = \mathbb{R}$, then $E_p(t,s) = e^{\int_s^t \frac{p(\tau)-\kappa_1(\alpha,\tau)}{\kappa(\alpha,\tau)} d\tau}$ and $E_0(t,s) = e^{-\int_s^t \frac{\kappa_1(\alpha,\tau)}{\kappa(\alpha,\tau)} d\tau}$.

### 5.1.1 Fundamental properties of the conformable operators

Using (5.2) and (5.3), we begin by proving several basic but important results.

**Theorem 5.1.1** (Basic properties of conformable derivatives)**.** *Let the conformable differential operator $\Delta^\alpha$ be given as in (5.2), where $\alpha \in [0,1]$. Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{T} \to [0,\infty)$ be rd-continuous and satisfy (5.1) with $1 + \mu(t)\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0} \neq 0$ for all $t \in \mathbb{T}^\kappa$. Assume functions $f$ and $g$ are differentiable, as needed. Then,*

*(i)* $\Delta^\alpha(af + bg) = a\Delta^\alpha(f) + b\Delta^\alpha(g)$ *for all* $a, b \in \mathbb{R}$*;*

*(ii)* $\Delta^\alpha c = c\kappa_1(\alpha, \cdot)$ *for all constants* $c \in \mathbb{R}$*;*

*(iii)* $\Delta^\alpha(fg) = f\Delta^\alpha(g) + g^\sigma\Delta^\alpha(f) - fg^\sigma\kappa_1(\alpha, \cdot)$*;*

*(iv)* $\Delta^\alpha\left(\frac{f}{g}\right) = \frac{g\Delta^\alpha(f) - f\Delta^\alpha(g)}{gg^\sigma} + \frac{f}{g}\kappa_1(\alpha, \cdot)$*;*

*(v)* $\Delta^\alpha E_p(t,s) = p(t)E_p(t,s)$ *for all* $\alpha \in (0,1]$*;*

*(vi)* $\Delta^\alpha \left( \int_a^t \frac{f(s)E_0(t,s)}{\kappa_0(\alpha,s)} \Delta s \right) = f(t)$ *for all* $\alpha \in (0,1]$.

*Proof.* Relations (i) and (ii) are obvious. From (5.2), it also follows (iii)–(vi):

(iii)

$$\begin{aligned}
\Delta^\alpha(fg) &= \kappa_0(fg^\Delta + f^\Delta g^\sigma) + \kappa_1(fg) \\
&= f\kappa_0 g^\Delta + g^\sigma \kappa_0 f^\Delta + \kappa_1(fg) \\
&= f(\kappa_0 g^\Delta + \kappa_1 g) + g^\sigma(\kappa_0 f^\Delta + \kappa_1 f) - g^\sigma \kappa_1 f \\
&= f\Delta^\alpha g + g^\sigma \Delta^\alpha f - g^\sigma \kappa_1 f;
\end{aligned}$$

(iv)

$$\begin{aligned}
\Delta^\alpha \left(\frac{f}{g}\right) &= \kappa_0 \left(\frac{f^\Delta g - f g^\Delta}{g g^\sigma}\right) + \kappa_1 \left(\frac{f}{g}\right) \\
&= \frac{\kappa_0(f^\Delta g - f g^\Delta)}{g g^\sigma} + \kappa_1 \left(\frac{f}{g}\right) \\
&= \frac{(\kappa_0 f^\Delta + \kappa_1 f)g - \kappa_1 f g - \left(\kappa_0 g^\Delta + \kappa_1 g\right) f + f\kappa_1 g}{g g^\sigma} + \kappa_1 \left(\frac{f}{g}\right) \\
&= \frac{g\Delta^\alpha f - f\Delta^\alpha g}{g g^\sigma} + \kappa_1 \left(\frac{f}{g}\right);
\end{aligned}$$

(v) $\Delta^\alpha E_{p(t)}(t,s)$

$$\begin{aligned}
&= \kappa_1(\alpha,t) e_{\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s) + \kappa_0(\alpha,t) \left( e_{\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s) \right)^\Delta \\
&= \kappa_1(\alpha,t) \left( e_{\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s) \right) + \kappa_0(\alpha,t) \left( \frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} e_{\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s) \right) \\
&= p(t) e_{\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s) \\
&= p(t) E_p(t,s);
\end{aligned}$$

(vi) $\Delta^{\alpha} \int_a^t \frac{f(s)E_0(t,s)}{\kappa_0(\alpha,s)} \Delta s$

$$\begin{aligned}
&= \kappa_1(\alpha,t) \int_a^t \frac{f(s)E_0(t,s)}{\kappa_0(\alpha,s)} \Delta s \\
&\quad + \kappa_0(\alpha,t) \left( \frac{f(t)E_0(t,t)}{\kappa_0(\alpha,t)} + \int_a^t \frac{f(s)(-\frac{\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)})E_0(t,s)}{\kappa_0(\alpha,s)} \Delta s \right) \\
&= \kappa_1(\alpha,t) \int_a^t \frac{f(s)E_0(t,s)}{\kappa_0(\alpha,s)} \Delta s + f(t) - \kappa_1(\alpha,t) \int_a^t \frac{f(s)E_0(t,s)}{\kappa_0(\alpha,s)} \Delta s \\
&= f(t).
\end{aligned}$$

The proof is complete. $\square$

**Definition 5.1.5** (Conformable integrals of order $\alpha$)**.** *Let $\alpha \in (0,1]$ and let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{T} \to [0,\infty)$ be rd-continuous and satisfy* (5.1) *with $1 + \mu(t)\frac{\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ for all $t \in \mathbb{T}^{\kappa}$ and $t_0 \in \mathbb{T}$. In light of* (5.3) *and items (v) and (vi) of Theorem 5.1.1, we define the conformable antiderivative of order $\alpha$ by*

$$\int \Delta^{\alpha} f(t) \Delta^{\alpha} t = f(t) + cE_0(t,t_0), \quad c \in \mathbb{R}.$$

*The conformable $\alpha$-integral of $f$ over $\mathbb{T}_{[a,t]}$ is defined by*

$$\int_a^t f(s)E_0(t,s)\Delta^{\alpha} s := \int_a^t \frac{f(s)E_0(t,s)}{\kappa_0(\alpha,s)} \Delta s, \tag{5.4}$$

*where on the right-hand side we have the standard $\Delta$-integral of time scales [22, 23].*

**Theorem 5.1.2** (Basic properties of the conformable $\alpha$-integral)**.** *Let the conformable differential operator on time scales $\Delta^{\alpha}$ be given as in* (5.2)*; the integral be given as in* (5.4)*; with $\alpha \in (0,1]$. Let functions $\kappa_0, \kappa_1$ be rd-continuous and satisfy* (5.1) *with $1 + \mu(t)\frac{\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ for all $t \in \mathbb{T}^{\kappa}$ and let $f$ and $g$ be $\Delta$-differentiable, as needed. Then,*

*(i) the derivative of the definite integral of $f$ is given by*

$$\Delta^{\alpha} \left( \int_a^t f(s)E_0(t,s)\Delta^{\alpha} s \right) = f(t);$$

*(ii) the definite integral of the derivative of $f$ is given by*

$$\int_a^t \Delta^\alpha[f(s)]E_0(t,s)\Delta^\alpha s = f(s)E_0(t,s)\Big|_{s=a}^{s=t} = f(t) - f(a)E_0(t,a);$$

*(iii) an integration by parts formula is given by*

$$\int_a^b f(t)\Delta^\alpha[g(t)]E_0(b,t)\Delta^\alpha t = f(t)g(t)E_0(b,t)\mid_{t=a}^{t=b} - \int_a^b g(t)\left(\Delta^\alpha[f(t)] - \kappa_1(\alpha,t)f(t)\right)E_0(b,t)\Delta^\alpha t;$$

*(iv) a version of the Leibniz rule for differentiation of an integral is given by*

$$\Delta^\alpha\left[\int_a^t f(t,s)E_0(t,s)\Delta^\alpha s\right] = f(t,t) + \int_a^t \left(\Delta_t^\alpha[f(t,s)] - \kappa_1(\alpha,t)f(t,s)\right)E_0(t,s)\Delta^\alpha s,$$

*where the derivative inside the last integral is with respect to $t$, or*

$$\Delta^\alpha\left[\int_a^t f(t,s)\Delta^\alpha s\right] = f(t,t) + \int_a^t \Delta^\alpha[f(t,s)]\Delta^\alpha s,$$

*if $E_0$ is absent.*

*Proof.* The proof of (i) is clear. Note that (ii) is a special case of (iii). Now we prove (iii):

$$\begin{aligned}
&\int_a^b f(t)\Delta^\alpha(g)(t)E_0(b,t)\Delta^\alpha t \\
&\quad= \int_a^b \left[\Delta^\alpha(fg)(t) - g^\sigma(t)\Delta^\alpha f(t) + f(t)g^\sigma(t)\kappa_1(\alpha,t)\right]E_0(b,t)\Delta^\alpha t \\
&\quad= \int_a^b \Delta^\alpha(fg)(t)E_0(b,t)\Delta^\alpha t - \int_a^b g^\sigma(t)\left[\Delta^\alpha(f)(t) - f(t)\kappa_1(\alpha,t)\right]E_0(b,t)\Delta^\alpha t \\
&\quad= f(t)g(t)E_0(b,t)|_a^b - \int_a^b g^\sigma(t)[\Delta^\alpha(f)(t) - f(t)\kappa_1(\alpha,t)]E_0(b,t)\Delta^\alpha t.
\end{aligned}$$

For the first part of (iv), we have:

$$
\begin{aligned}
&\Delta^\alpha \left[\int_a^t f(t,s)E_0(t,s)\Delta^\alpha s\right]\\
&= \Delta^\alpha \int_a^t \frac{f(t,s)E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s\\
&= \kappa_0(\alpha,t)\left[\int_a^t f(t,s)E_0(t,s)\Delta^\alpha s\right]^\Delta + \kappa_1(\alpha,t)\int_a^t f(t,s)E_0(t,s)\Delta^\alpha s\\
&= \kappa_0(\alpha,t)\left[\frac{f(t,t)}{\kappa_0(\alpha,t)} + \int_a^t \left(\frac{f^\Delta(t,s)E_0(t,s)}{\kappa_0(\alpha,s)} - f(t,s)\frac{\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}\frac{E_0(t,s)}{\kappa_0(\alpha,s)}\right)\Delta s\right]\\
&\quad + \kappa_1(\alpha,t)\int_a^t \frac{f(t,s)E_0(t,s)}{\kappa_0(\alpha,t)}\Delta s\\
&= f(t,t) + \kappa_0(\alpha,t)\int_a^t \left[\frac{f^\Delta(t,s)E_0(t,s)}{\kappa_0(\alpha,s)} - \frac{f(t,s)\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}\frac{E_0(t,s)}{\kappa_0(\alpha,s)}\right]\Delta s\\
&\quad + \kappa_1(\alpha,t)\int_a^t \frac{f(t,s)E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s\\
&= f(t,t) + \int_a^t \left[\Delta_t^\alpha f(t,s) - \kappa_1(\alpha,t)f(t,s)\right]E_0(t,s)\Delta^\alpha s.
\end{aligned}
$$

For the second expression in (iv), in the case $E_0(t,s)$ is absent from the integral expression, one has

$$
\begin{aligned}
&\Delta^\alpha \left[\int_a^t f(t,s)\Delta^\alpha s\right]\\
&= \Delta^\alpha \left[\int_a^t \frac{f(t,s)}{\kappa_0(\alpha,s)}\Delta s\right]\\
&= \kappa_0(\alpha,t)\left(\int_a^t \frac{f(t,s)}{\kappa_0(\alpha,s)}\Delta s\right)^\Delta + \kappa_1(\alpha,t)\int_a^t \frac{f(t,s)}{\kappa_0(\alpha,s)}\Delta s\\
&= \kappa_0(\alpha,t)\frac{f(t,t)}{\kappa_0(\alpha,t)} + \kappa_0(\alpha,t)\int_a^t \left(\frac{f(t,s)}{\kappa_0(\alpha,s)}\right)^\Delta \Delta s + \kappa_1(\alpha,t)\int_a^t \frac{f(t,s)}{\kappa_0(\alpha,s)}\Delta s\\
&= f(t,t) + \int_a^t \Delta_t^\alpha f(t,s)\Delta^\alpha s.
\end{aligned}
$$

Note that we use $\Delta_t^\alpha f(t,s)$ to denote the derivative of $f$ with respect to $t$. $\square$

### 5.1.2 Conformable polynomials

Let functions $\kappa_0, \kappa_1 : [0,1]\times\mathbb{T} \to [0,\infty)$ be rd-continuous such that the conditions in (5.1) are satisfied with $1+\mu(t)\frac{\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ for all $t \in \mathbb{T}^\kappa$. We define, recursively, the functions $h_n : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $n \in \mathbb{N}_0$, by

$$h_0(t,s) \equiv 1, \quad h_n(t,s) = \int_s^t h_{n-1}(\tau,s)\Delta^\alpha \tau, \quad n \in \mathbb{N}, \tag{5.5}$$

for all $t, s \in \mathbb{T}$.

**Remark 5.1.2.** *If $\alpha = 1$ and $n \in \mathbb{N}_0$, then $h_n(t,s) = \frac{1}{n!}(t-s)^n$.*

In the follow examples, we explore the new functions $h_n(t,s)$ given by (5.5).

**Example 5.1.4.** *Let $\alpha \in (0,1]$, $\mathbb{T} = \mathbb{Z}$, $\omega_0, \omega_1 \in (0,\infty)$, and $\kappa_1$ satisfy (5.1). Take $\kappa_0(\alpha,t) \equiv \alpha\omega_0^{1-\alpha}$. Because $\Delta^\alpha \tau = \frac{1}{\kappa_0(\alpha,\tau)}\Delta\tau = \frac{1}{\alpha\omega_0^{1-\alpha}}\Delta\tau$, letting $h_0(t,s) \equiv 1$, we calculate $h_1$ via (5.5), getting*

$$h_1(t,s) = \int_s^t h_0(\tau,s)\Delta^\alpha\tau = \int_s^t \frac{\Delta\tau}{\alpha\omega_0^{1-\alpha}} = \frac{1}{\alpha\omega_0^{1-\alpha}}\int_s^t \Delta\tau = \frac{1}{\alpha\omega_0^{1-\alpha}}(t-s)$$

*and*

$$h_2(t,s) = \int_s^t \frac{\tau - s}{\alpha\omega_0^{1-\alpha}}\Delta^\alpha\tau = \frac{1}{(\alpha\omega_0^{1-\alpha})^2}\left[\frac{(\tau-s)^2}{2}\right]_s^t = \frac{1}{(\alpha\omega_0^{1-\alpha})^2}\frac{(t-s)^2}{2}.$$

*The factorial function $t^{\underline{k}}$ for $n \in \mathbb{N}_0$ is defined (see [42]) by*

$$t^{\underline{0}} = 1, \quad t^{\underline{k}} = t(t-1)(t-2)\cdots(t-k+1), \quad k \in \mathbb{N}.$$

*We claim that*

$$h_k(t,s) = \frac{1}{(\alpha\omega_0^{1-\alpha})^k}\frac{(t-s)^{\underline{k}}}{k!} = \frac{1}{(\alpha\omega_0^{1-\alpha})^k}\binom{k-s}{k} \tag{5.6}$$

*for $t, s \in \mathbb{T}$, $t \geq s$, and $k \in \mathbb{N}$. Assume (5.6) holds for $k$ replaced by $m$. Then,*

$$h_{m+1}(t,s) = \int_s^t h_m(\tau,s)\Delta\tau = \frac{1}{(\alpha\omega_0^{1-\alpha})^m}\int_s^t \frac{(\tau-s)^{\underline{m}}}{m!}\Delta^\alpha\tau = \frac{(t-s)^{\underline{m+1}}}{(\alpha\omega_0^{1-\alpha})^{m+1}}.$$

**Example 5.1.5.** *Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$, $\kappa_0(\alpha,t) = \alpha\omega^{1-\alpha}$, $\alpha \in (0,1]$, $t, s \in \mathbb{T}$, $t \geq s$, $k \in \mathbb{N}$. Again, starting with $h_0(t,s) \equiv 1$, we see that*

$$h_1(t,s) = \int_s^t h_0(\tau,s)\Delta^\alpha\tau = \int_s^t 1\Delta^\alpha\tau = \int_s^t \frac{\Delta\tau}{\alpha\omega^{1-\alpha}} = \frac{1}{\alpha\omega^{1-\alpha}}(t-s),$$

$$
\begin{aligned}
h_2(t,s) &= \int_s^t h_1(\tau,s)\Delta^\alpha \tau \\
&= \int_s^t \frac{1}{\alpha\omega^{1-\alpha}}(\tau - s)\Delta^\alpha \tau \\
&= \frac{1}{\alpha\omega^{1-\alpha}} \int_s^t \frac{\tau - s}{\alpha\omega^{1-\alpha}}\Delta\tau \\
&= \frac{1}{(\alpha\omega^{1-\alpha})^2} \int_s^t (\tau - s)\Delta\tau \\
&= \frac{1}{(\alpha\omega^{1-\alpha})^2} \frac{(t-s)(t-h-s)}{2},
\end{aligned}
$$

$$
\begin{aligned}
h_3(t,s) &= \int_s^t h_2(\tau,s)\Delta^\alpha \tau \\
&= \frac{1}{(\alpha\omega^{1-\alpha})^2} \int_s^t \frac{(\tau-s)(\tau-h-s)}{2\alpha\omega^{1-\alpha}}\Delta\tau \\
&= \frac{1}{(\alpha\omega^{1-\alpha})^3} \frac{(t-s)(t-h-s)(t-2h-s)}{3!}.
\end{aligned}
$$

*We claim that for $k \in \mathbb{N}_0$ we have*

$$
h_k(t,s) = \frac{1}{(\alpha\omega^{1-\alpha})^k} \frac{\prod_{i=0}^{k-1}(t - ih - s)}{k!}. \tag{5.7}
$$

*Assume* (5.7) *holds for $k$ replaced by $m$. Then,*

$$
\begin{aligned}
h_{m+1}(t,s) &= \int_s^t \frac{1}{(\alpha\omega^{1-\alpha})^m} \frac{\prod_{i=0}^{m-1}(\tau - ih - s)}{m!}\Delta^\alpha \tau \\
&= \frac{1}{(\alpha\omega^{1-\alpha})^m} \int_s^t \frac{\prod_{i=0}^{m-1}(\tau - ih - s)}{m!} \frac{\Delta\tau}{\alpha\omega^{1-\alpha}} \\
&= \frac{1}{(\alpha\omega^{1-\alpha})^{m+1}} \frac{\prod_{i=0}^{m}(\tau - ih - s)}{(m+1)!}.
\end{aligned}
$$

**Example 5.1.6.** *Let $\mathbb{T} = q^{\overline{\mathbb{Z}}}$, $q > 1$, $q \in \mathbb{Z}$, $t, s \in \mathbb{T}$, $t \geq s$, $k \in \mathbb{N}_0$, and $\kappa_0 = \alpha\omega^{1-\alpha}$. From $h_0(t,s) \equiv 1$, it follows that*

$$
h_k(t,s) = \int_s^t h_{k-1}(\tau,s)\Delta^\alpha \tau = \frac{1}{(\alpha\omega^{1-\alpha})^k} \prod_{m=0}^{k-1} \left( \frac{t - q^m s}{\sum_{j=0}^m q^j} \right).
$$

**Proposition 5.1.2.** *Assume $h_n$ is a function satisfying* (5.5). *Then,*

$$
\Delta^\alpha h_n(t,s) = h_{n-1}(t,s) + \kappa_1(\alpha,t)h_n(t,s).
$$

*Proof.* By Theorem 5.1.1 (ii) and Theorem 5.1.2 (iv), the following relationship holds:

$$\begin{aligned}\Delta_t^\alpha h_n(t,s) &= \Delta_t^\alpha \int_s^t h_{n-1}(\tau,s)\Delta^\alpha\tau \\ &= \kappa_1(\alpha,t)\int_s^t h_{n-1}(\tau,s)\Delta^\alpha\tau + \kappa_0(\alpha,t)\frac{h_{n-1}(t,s)}{\kappa_0(\alpha,t)} \\ &= h_{n-1}(t,s) + \kappa_1(\alpha,t)h_n(t,s).\end{aligned}$$

The proof is complete. □

### 5.1.3 Linear second-order conformable differential equations on time scales

Let $\mathbb{T}$ be an arbitrary time scale, $\alpha \in [0,1]$, and let $\kappa_0, \kappa_1 : [0,1]\times\mathbb{T} \to [0,\infty)$ be rd-continuous functions such that (5.1) holds Considering that the functions $\kappa_i(\alpha,t), i = 0,1$ are $\Delta_t$- differentiable and continuous compared to $\alpha$ with $1+\mu(t)\frac{\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ for all $t \in \mathbb{T}^\kappa$. In addition, let $\Delta^\alpha$ be as in (5.2), and let $t_0 \in \mathbb{T}$. In this section we are concerned with the following linear second-order conformable dynamic equation on time scales with constant coefficients:

$$\Delta^\alpha\Delta^\alpha y(t) + a\Delta^\alpha y(t) + by(t) = f(t), \quad t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}, \tag{5.8}$$

where we assume $a, b \in \mathbb{R}$, $f \in C_{rd}$. Introduce the operator $L_{2\Delta^\alpha} : C^2_{rd} \to C_{rd}$ by

$$L_{2\Delta^\alpha}(y)(t) = \Delta^\alpha\Delta^\alpha y(t) + a\Delta^\alpha y(t) + by(t) \tag{5.9}$$

for all $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$.

**Lemma 5.1.2.** *The operator $L_{2\Delta^\alpha}$ defined by (5.9) is a linear operator, i.e.,*

$$L_{2\Delta^\alpha}(py_1 + qy_2) = pL_{2\Delta^\alpha}(y_1) + qL_{2\Delta^\alpha}(y_2),$$

*where $p, q \in \mathbb{R}$ and $y_1, y_2 \in C^2_{rd}$. If $y_1$ and $y_2$ solve the homogeneous equation*

$$L_{2\Delta^\alpha}y = 0,$$

*then so does $y = py_1 + qy_2$, $p, q \in \mathbb{R}$.*

*Proof.* Using (i) of Theorem 5.1.1, we find that

$$\begin{aligned}
&L_{2\Delta^\alpha}(py_1+qy_2)(t)\\
&\quad= \Delta^\alpha\Delta^\alpha(py_1(t)+qy_2(t)) + a\Delta^\alpha(py_1(t)+qy_2(t)) + b(py_1(t)+qy_2(t))\\
&\quad= pL_{2\Delta^\alpha}(y_1)(t) + qL_{2\Delta^\alpha}(y_2)(t) = 0
\end{aligned}$$

for all $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$ and all $p, q \in \mathbb{R}$. □

**Definition 5.1.6.** *Let $a, b \in \mathbb{R}$ and $f \in C_{rd}$. Equation* (5.8) *is called regressive if*

$$\kappa_0^2 - \mu\kappa_0(a+2\kappa_1) + \mu^2(b + a\kappa_1 + \kappa_1^2) \neq 0$$

*for all $t \in \mathbb{T}^\kappa$.*

**Theorem 5.1.3** (uniqueness of solution)**.** *Let $t_0 \in \mathbb{T}^\kappa$ the functions $\kappa_i(\alpha, t), i = 0, 1$ are $\Delta_t$- differentiable and continuous compared to $\alpha$ . Assume that the dynamic equation* (5.8) *is regressive.if the equation $L_{2\Delta^\alpha}y(t) = 0$ admitting a solutions $y_1, y_2$, with $y_1(t)\Delta^\alpha y_2(t) - \Delta^\alpha y_1(t)y_2(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$, Then the initial value problem*

$$L_{2\Delta^\alpha}y(t) = 0, \quad y(t_0) = y_0, \quad \Delta^\alpha y(t_0) = y_0^\alpha, \tag{5.10}$$

*where $y_0$ and $y_0^\alpha$ are given constants, has a unique solution defined on $\mathbb{T}_{[t_0,\infty)}$.*

*Proof.* If $y_1, y_2$ are two solutions of $L_{2\Delta^\alpha}y(t) = 0$, then $y(t) = py_1(t)+qy_2(t)$, $p, q \in \mathbb{R}$, is a solution of $L_{2\Delta^\alpha}y(t) = 0$. Therefore, we want to see if we can pick $p$ and $q$ so that $y_0 = y(t_0) = py_1(t_0) + qy_2(t_0)$, $y_0^\alpha = p\Delta^\alpha y_1(t_0) + q\Delta^\alpha y_2(t_0)$. Let

$$\mathbf{M} = \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ \Delta^\alpha y_1(t_0) & \Delta^\alpha y_2(t_0) \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} p \\ q \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} y_0 \\ y_0^\alpha \end{pmatrix}.$$

System $M \times X = B$ has a unique solution provided matrix $M$ is invertible. □

**Definition 5.1.7.** *For two $\Delta^\alpha$-differentiable functions on $\mathbb{T}_{[t_{0,\infty})}$ $y_1$ and $y_2$, we define the Wronskian $W = W(y_1, y_2)$ by*

$$W(t) = \det\begin{pmatrix} y_1(t) & y_2(t) \\ \Delta^\alpha y_1(t) & \Delta^\alpha y_2(t) \end{pmatrix}, \quad t \in \mathbb{T}_{[t_{0,\infty})}.$$

*We say that two solutions $y_1$ and $y_2$ of $L_{2\Delta^\alpha}y = 0$ form a fundamental set of solutions for $L_{2\Delta^\alpha}y = 0$ provided $W(y_1, y_2)(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$.*

**Theorem 5.1.4.** *If the pair of functions $y_1$ and $y_2$ form a fundamental system of solutions for $L_{2\Delta^\alpha}y = 0$, $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$, then*

$$y(t) = py_1(t) + qy_2(t), \quad p, q \in \mathbb{R}, \tag{5.11}$$

*is a general solution of $L_{2\Delta^\alpha}y = 0$, $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$. In particular, the solution of the initial value problem* (5.10) *is given by*

$$y(t) = \frac{\Delta^\alpha y_2(t_0)y_0 - y_2(t_0)y_0^\alpha}{W(y_1, y_2)(t_0)} y_1(t) + \frac{y_1(t_0)y_0^\alpha - \Delta^\alpha y_1(t_0)y_0}{W(y_1, y_2)(t_0)} y_2(t).$$

**Remark 5.1.3.** *By general solution we mean that every function of form* (5.11) *is a solution and every solution is of this form.*

*Proof.* The proof is similar to the one of Theorem 3.7 of [22]. □

### 5.1.4 Hyperbolic and trigonometric functions

Now we consider the linear second-order homogeneous dynamic conformable equation with constant coefficients

$$\Delta^\alpha \Delta^\alpha y(t) + a\Delta^\alpha y(t) + by(t) = 0, \quad a, b \in \mathbb{R}, \quad t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}. \tag{5.12}$$

We assume (5.12) to be regressive, i.e., $\kappa_0 - \mu(a + 2\kappa_1) + \mu^2(b + a\kappa_1 + \kappa_1^2) \neq 0$, $t \in \mathbb{T}^\kappa$. Let $\lambda \in \mathbb{C}$ be such that $1 + \mu(t)\frac{\lambda - \kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$, $t \in \mathbb{T}^\kappa$, and $y(t) = E_\lambda(t, t_0)$, $t \in \mathbb{T}^\kappa_{[t_0,\infty)}$, be a solution of (5.12). If $y(t) = E_\lambda(t, t_0)$, then

$$\Delta^\alpha \Delta^\alpha y(t) + a\Delta^\alpha y(t) + by(t) = (\lambda^2 + a\lambda + b)E_\lambda(t, t_0)$$

and, because $E_\lambda(t, t_0) \neq 0$, $y(t) = E_\lambda(t, t_0)$ is a solution of (5.12) if and only if $\lambda$ satisfies the characteristic equation of (5.12):

$$\lambda^2 + a\lambda + b = 0. \tag{5.13}$$

The solutions $\lambda_1$ and $\lambda_2$ of (5.13) are given by

$$\lambda_1 = \frac{-a-\sqrt{a^2-4b}}{2} \quad \text{and} \quad \lambda_2 = \frac{-a+\sqrt{a^2-4b}}{2} \tag{5.14}$$

and, since (5.12) is regressive, $1+\mu(t)\frac{\lambda_1-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ and $1+\mu(t)\frac{\lambda_2-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ for all $t \in \mathbb{T}^\kappa$.

**Theorem 5.1.5.** *Suppose equation* (5.12) *is regressive and* $a^2-4b \neq 0$. *Then,* $E_{\lambda_1}(\cdot,t_0)$ *and* $E_{\lambda_2}(\cdot,t_0)$ *form a fundamental system of* (5.12)*, where* $t_0 \in \mathbb{T}$ *and* $\lambda_1$ *and* $\lambda_2$ *are given as in* (5.14)*, and the solution of the initial value problem*

$$\Delta^\alpha\Delta^\alpha y(t) + a\Delta^\alpha y(t) + by(t) = 0, \quad y(t_0) = y_0, \quad \Delta^\alpha y_{t_0} = y_0^\alpha, \tag{5.15}$$

*is given by*

$$y_0(t) = \frac{E_{\lambda_1}(t,t_0)+E_{\lambda_2}(t,t_0)}{2} + \frac{ay_0+2y_0^\alpha}{\sqrt{a^2-4b}}\frac{E_{\lambda_2}(t,t_0)-E_{\lambda_1}(t,t_0)}{2},$$

$t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$.

*Proof.* Since $\lambda_1$ and $\lambda_2$, given by (5.14), are solutions of the characteristic equation (5.13), we know that both $E_{\lambda_1}(\cdot,t_0)$ and $E_{\lambda_2}(\cdot,t_0)$ are solutions of (5.12). Moreover,

$$\begin{aligned} W\left(E_{\lambda_1}(t,t_0),E_{\lambda_2}(t,t_0)\right) &= \det\begin{pmatrix} E_{\lambda_1}(t,t_0) & E_{\lambda_2}(t,t_0) \\ \lambda_1 E_{\lambda_1}(t,t_0) & \lambda_2 E_{\lambda_2}(t,t_0)\end{pmatrix} \\ &= \lambda_2 E_{\lambda_1}(t,t_0)E_{\lambda_2}(t,t_0) - \lambda_1 E_{\lambda_1}(t,t_0)E_{\lambda_2}(t,t_0) \\ &= (\lambda_2-\lambda_1)E_{\lambda_1}(t,t_0)E_{\lambda_2}(t,t_0) \\ &= \sqrt{a^2-4b}E_{\lambda_1}(t,t_0)E_{\lambda_2}(t,t_0), \end{aligned}$$

so that $W(E_{\lambda_1}(t,t_0)E_{\lambda_2}(t,t_0)) \neq 0$ for all $t \in \mathbb{T}^\kappa_{[t_0,\infty)}$, unless $a^2-4b=0$. Having obtained a fundamental system $y_1 = E_{\lambda_1}(\cdot,t_0)$ and $y_2 = E_{\lambda_2}(\cdot,t_0)$ of (5.12), now we obtain a solution of (5.15), namely $y(t) = c_1y_1(t)+c_2y_2(t)$. For that we solve the linear system of equations

$$\begin{cases} y_0 = c_1y_1(t_0)+c_2y_2(t_0) \\ \Delta^\alpha y(t_0) = \lambda_1c_1y_1(t_0)+\lambda_2c_2y_2(t_0) \end{cases}$$

in the unknowns $c_1$ and $c_2$, obtaining $c_1 = \frac{y_0}{2} - \frac{ay_0+2y_0^\alpha}{2\sqrt{a^2-4b}}$ and $c_2 = \frac{y_0}{2} + \frac{ay_0+2y_0^\alpha}{2\sqrt{a^2-4b}}$. □

Hyperbolic functions are associated with the case $a = 0$ and $b < 0$.

**Definition 5.1.8** (Hyperbolic functions)**.** *Let $\mathbb{T}$ be a time scale and $t_0 \in \mathbb{T}$. If $p \in C_{rd}$ and $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(-p^2 + \kappa_1^2) \neq 0$ for all $t \in \mathbb{T}^\kappa$, then we define the hyperbolic functions $\cosh_{p\Delta^\alpha}(\cdot, t_0)$ and $\sinh_{p\Delta^\alpha}(\cdot, t_0)$ on $\mathbb{T}_{[t_0,\infty)}$ by*

$$\cosh_{p\Delta^\alpha}(\cdot, t_0) = \frac{E_p(\cdot, t_0) + E_{-p}(\cdot, t_0)}{2} \quad \text{and} \quad \sinh_{p\Delta^\alpha}(\cdot, t_0) = \frac{E_p(\cdot, t_0) - E_{-p}(\cdot, t_0)}{2}.$$

**Remark 5.1.4.** *The condition $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(-p^2 + \kappa_1^2) \neq 0$ of Definition 5.1.8 is equivalent to $1 + \mu(t)\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ and $1 - \mu(t)\frac{p(t)+\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$.*

**Lemma 5.1.3.** *Let $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(-p^2 + \kappa_1^2) \neq 0$ for all $t \in \mathbb{T}^\kappa$. Then,*

$$\Delta^\alpha \cosh_{p\Delta^\alpha}(\cdot, t_0) = p \sinh_{p\Delta^\alpha}(\cdot, t_0),$$
$$\Delta^\alpha \sinh_{p\Delta^\alpha}(\cdot, t_0) = p \cosh_{p\Delta^\alpha}(\cdot, t_0),$$
$$\cosh^2_{p\Delta^\alpha}(\cdot, t_0) - \sinh^2_{p\Delta^\alpha}(\cdot, t_0) = E_p(\cdot, t_0)E_{-p}(\cdot, t_0),$$

*for all $t \in \mathbb{T}_{[t_0,\infty)}$.*

*Proof.* The first two formulas are trivially verified. The last relation follows from

$$\begin{aligned}\left(\cosh^2_{p\Delta^\alpha} - \sinh^2_{p\Delta^\alpha}\right)(\cdot, t_0) &= \left(\frac{E_p(\cdot, t_0) + E_{-p}(\cdot, t_0)}{2}\right)^2 - \left(\frac{E_p(\cdot, t_0) - E_{-p}(\cdot, t_0)}{2}\right)^2 \\ &= \frac{E_p^2(\cdot, t_0) + 2E_p(\cdot, t_0)E_{-p}(\cdot, t_0) + E_{-p}^2(\cdot, t_0)}{4} \\ &\qquad - \frac{E_p^2(\cdot, t_0) - 2E_p(\cdot, t_0)E_{-p}(\cdot, t_0) + E_{-p}^2(\cdot, t_0)}{4} \\ &= E_p(\cdot, t_0)E_{-p}(\cdot, t_0)\end{aligned}$$

for all $t \in \mathbb{T}_{[t_0,\infty)}$. □

**Theorem 5.1.6.** *If $\gamma \in \mathbb{R}\backslash\{0\}$, $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(-\gamma^2 + \kappa_1^2) \neq 0$, and $t_0 \in \mathbb{T}^\kappa$, then*

$$y(t) = c_1 \cosh_{\gamma\Delta^\alpha}(t, t_0) + c_2 \sinh_{\gamma\Delta^\alpha}(t, t_0)$$

*is a general solution of*

$$\Delta^\alpha \Delta^\alpha y - \gamma^2 y = 0 \tag{5.16}$$

*on $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$, where $c_1$ and $c_2$ are arbitrary constants.*

*Proof.* One can easily prove that $\cosh_{\gamma\Delta^\alpha}(\cdot,t_0)$ and $\sinh_{\gamma\Delta^\alpha}(\cdot,t_0)$ are solutions of (5.16). We prove that they form a fundamental set of solutions for (5.16):

$$\begin{aligned}
W\left(\cosh_{\gamma\Delta^\alpha}(t,t_0),\sinh_{\gamma\Delta^\alpha}(t,t_0)\right) &= \det\begin{pmatrix} \cosh_{\gamma\Delta^\alpha}(t,t_0) & \sinh_{\gamma\Delta^\alpha}(t,t_0)) \\ \gamma\sinh_{\gamma\Delta^\alpha}(t,t_0)) & \gamma\cosh_{\gamma\Delta^\alpha}(t,t_0) \end{pmatrix} \\
&= \gamma\cosh^2_{\gamma\Delta^\alpha}(t,t_0) - \gamma\sinh^2_{\gamma\Delta^\alpha}(t,t_0) \\
&= \gamma\left(\cosh^2_{\gamma\Delta^\alpha}(t,t_0) - \sinh^2_{\gamma\Delta^\alpha}(t,t_0)\right) \\
&= \gamma E_\gamma(t,t_0)E_{-\gamma}(t,t_0)
\end{aligned}$$

is different from zero for all $t\in\mathbb{T}^\kappa_{[t_0,\infty)}$, unless $\gamma=0$. □

**Example 5.1.7.** *Let $\mathbb{T}$ be a time scale, $t_0\in\mathbb{T}^\kappa$. If $\kappa_0^2-2\mu\kappa_0\kappa_1+\mu^2(-\gamma^2+\kappa_1^2)\neq 0$, with $\gamma\in\mathbb{R}\backslash\{0\}$, then the solution of the initial value problem*

$$\Delta^\alpha\Delta^\alpha y(t)-\gamma^2 y(t)=0,\quad y(t_0)=y_0,\quad \Delta^\alpha y(t_0)=y_0^\alpha,$$

*is given by*

$$y(t)=y_0\cosh_{\gamma\Delta^\alpha}(t,t_0)+\frac{y_0^\alpha}{\gamma}\sinh_{\gamma\Delta^\alpha}(t,t_0)$$

*for all $t\in\mathbb{T}^{\kappa^2}_{[t_0,\infty)}$.*

**Definition 5.1.9** (Trigonometric functions)**.** *Let $\mathbb{T}$ be a time scale, $p\in C_{rd}$, $t_0\in\mathbb{T}$, and $\kappa_0^2-2\mu\kappa_0\kappa_1+\mu^2(p^2+\kappa_1^2)\neq 0$ for all $t\in\mathbb{T}^\kappa$. Then we define the trigonometric functions $\cos_{p\Delta^\alpha}(\cdot,t_0)$ and $\sin_{p\Delta^\alpha}(\cdot,t_0)$ by*

$$\cos_{p\Delta^\alpha}(\cdot,t_0)=\frac{E_{ip}(\cdot,t_0)+E_{-ip}(\cdot,t_0)}{2}\quad\text{and}\quad \sin_{p\Delta^\alpha}(\cdot,t_0)=\frac{E_{ip}(\cdot,t_0)-E_{-ip}(\cdot,t_0)}{2i}.$$

**Remark 5.1.5.** *The condition $\kappa_0^2-2\mu\kappa_0\kappa_1+\mu^2(p^2+\kappa_1^2)\neq 0$ of Definition 5.1.9 is equivalent to $1+\mu(t)\left(\frac{ip(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}\right)\neq 0$ and $1-\mu(t)\frac{ip(t)+\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}\neq 0$.*

**Lemma 5.1.4.** *Let $p\in C_{rd}$. If $\kappa_0^2-2\mu\kappa_0\kappa_1+\mu^2(p^2+\kappa_1^2)\neq 0$ for all $t\in\mathbb{T}^\kappa$, then*

$$\Delta^\alpha\cos_{p\Delta^\alpha}(\cdot,t_0)=-p\sin_{p\Delta^\alpha}(\cdot,t_0),$$

$$\Delta^\alpha\sin_{p\Delta^\alpha}(\cdot,t_0)=p\cos_{p\Delta^\alpha}(\cdot,t_0),$$

$$\cos^2_{p\Delta^\alpha}(\cdot,t_0)+\sin^2_{p\Delta^\alpha}(\cdot,t_0)=E_{ip}(\cdot,t_0)E_{-ip}(\cdot,t_0).$$

**Remark 5.1.6.** *If $\alpha = 1$, then $E_{ip}(\cdot, t_0) = e_{ip}(\cdot, t_0) = \cos_p(\cdot, t_0) + i\sin_p(\cdot, t_0)$.*

*Proof.* Similarly to Lemma 5.1.3, the first two formulas are easily verified. We have

$$\begin{aligned}
&\cos^2_{p\Delta^\alpha}(\cdot, t_0) + \sin^2_{p\Delta^\alpha}(\cdot, t_0) \\
&\quad = \left(\frac{E_{ip}(\cdot, t_0) + E_{-ip}(\cdot, t_0)}{2}\right)^2 + \left(\frac{E_{ip}(\cdot, t_0) - E_{-ip}(\cdot, t_0)}{2i}\right)^2 \\
&\quad = \frac{E^2_{ip}(\cdot, t_0) + 2E_{ip}(\cdot, t_0)E_{-ip}(\cdot, t_0) + E^2_{-ip}(\cdot, t_0)}{4} \\
&\qquad - \frac{E^2_{ip}(\cdot, t_0) - 2E_{ip}(\cdot, t_0)E_{-ip}(\cdot, t_0) + E^2_{-ip}(\cdot, t_0)}{4} \\
&\quad = E_{ip}(\cdot, t_0)E_{-ip}(\cdot, t_0)
\end{aligned}$$

and the last relation also holds. □

**Example 5.1.8.** *Let $\mathbb{T} = \mathbb{R}$, $\gamma \in \mathbb{R}$, and $t_0 \in \mathbb{T}$. Then, the conformable trigonometric functions cosine and sine are given by*

$$\begin{aligned}
\cos_{\gamma\Delta^\alpha}(t, t_0) &= \frac{E_{i\gamma}(t, t_0) + E_{-i\gamma}(t, t_0)}{2} \\
&= \frac{e^{\int_{t_0}^t \frac{i\gamma - \kappa_1(s,t_0)}{\kappa_0(s,t_0)}ds} + e^{\int_{t_0}^t \frac{-i\gamma - \kappa_1(s,t_0)}{\kappa_0(s,t_0)}ds}}{2} \\
&= \frac{e^{i\int_{t_0}^t \frac{\gamma}{\kappa_0(s,t_0)}ds} e^{-\int_{t_0}^t \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)}ds} + e^{-i\int_{t_0}^t \frac{\gamma}{\kappa_0(s,t_0)}ds} e^{-\int_{t_0}^t \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)}ds}}{2} \\
&= \frac{e^{-\int_{t_0}^t \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)}ds}\left(2\cos(\int_{t_0}^t \frac{\gamma}{\kappa_0(s,t_0)}ds)\right)}{2} \\
&= e^{-\int_{t_0}^t \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)}ds} \cos\left(\int_{t_0}^t \frac{\gamma}{\kappa_0(s, t_0)}ds\right)
\end{aligned}$$

*and*

$$\begin{aligned}
\sin_{\gamma\Delta^\alpha}(t, t_0) &= \frac{E_{i\gamma}(t, t_0) - E_{-i\gamma}(t, t_0)}{2i} \\
&= \frac{e^{\int_{t_0}^t \frac{i\gamma - \kappa_1(s,t_0)}{\kappa_0(s,t_0)}ds} - e^{\int_{t_0}^t \frac{-i\gamma - \kappa_1(s,t_0)}{\kappa_0(s,t_0)}ds}}{2i} \\
&= \frac{e^{i\int_{t_0}^t \frac{\gamma}{\kappa_0(s,t_0)}ds} e^{-\int_{t_0}^t \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)}ds} - e^{-i\int_{t_0}^t \frac{\gamma}{\kappa_0(s,t_0)}ds} e^{-\int_{t_0}^t \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)}ds}}{2i} \\
&= e^{-\int_{t_0}^t \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)}ds} \sin\left(\int_{t_0}^t \frac{\gamma}{\kappa_0(s, t_0)}ds\right).
\end{aligned}$$

**Theorem 5.1.7.** *Let $\mathbb{T}$ be a time scale and $t_0 \in \mathbb{T}^\kappa$. If $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(\gamma^2 + \kappa_1^2) \neq 0$, $\gamma \in \mathbb{R}\backslash\{0\}$, then $y(t) = c_1 \cos_{\gamma\Delta^\alpha}(t, t_0) + c_2 \sin_{\gamma\Delta^\alpha}(t, t_0)$ is a general solution of*

$$\Delta^\alpha \Delta^\alpha y + \gamma^2 y = 0, \quad t \in \mathbb{T}^{\kappa^2}. \tag{5.17}$$

*Proof.* One can easily show that $\cos_{\gamma\Delta^\alpha}(\cdot, t_0)$ and $\sin_{\gamma\Delta^\alpha}(\cdot, t_0)$ are solutions of (5.17). We prove that they form a fundamental set of solutions for (5.17): for $\gamma \neq 0$,

$$W\left(\cos_{\gamma\Delta^\alpha}(t, t_0), \sin_{\gamma\Delta^\alpha}(t, t_0)\right) = \det \begin{pmatrix} \cos_{\gamma\Delta^\alpha}(t, t_0) & \sin_{\gamma\Delta^\alpha}(t, t_0)) \\ -\gamma \sin_{\gamma\Delta^\alpha}(t, t_0)) & \gamma \cos_{\gamma\Delta^\alpha}(t, t_0) \end{pmatrix}$$
$$= \gamma \left(\cos^2_{\gamma\Delta^\alpha}(t, t_0) + \sin^2_{\gamma\Delta^\alpha}(t, t_0)\right)^2 = \gamma E_{ip}(t, t_0) E_{-ip}(t, t_0) \neq 0$$

for all $t \in \mathbb{T}^\kappa_{[t_0,\infty)}$. We conclude that $y(t) = c_1 \cos_{\gamma\Delta^\alpha}(t, t_0) + c_2 \sin_{\gamma\Delta^\alpha}(t, t_0)$, $t \in \mathbb{T}_{[t_0,\infty)}$, is a general solution of (5.17). $\square$

# Bibliography


[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. **279** (2015), 57–66.

[2] T. Abdeljawad, M. Al Horani and R. Khalil, *Conformable fractional semigroups of operators*, Journal of Semigroup Theory and Applications **2015** (2015), Art. ID 7, 9 pp.

[3] R. P. Agarwal and M. Bohner, *Basic calculus on time scales and some of its applications*, Results Math. **35** (1999), no. 1-2, 3–22.

[4] D. R. Anderson and R. I. Avery, *Fractional-order boundary value problem with Sturm-Liouville boundary conditions*, Electron. J. Differential Equations **2015** (2015), no. 29, 10 pp.

[5] D. Anderson, J. Bullock, L. Erbe, A. Peterson and H. Tran, *Nabla dynamic equations on time scales*, Panamer. Math. J. **13** (2003), no. 1, 1–47.

[6] D. R. Anderson and D. J. Ulness, *Newly defined conformable derivatives*, Adv. Dyn. Syst. Appl. **10** (2015), no. 2, 109–137.

[7] D. R. Anderson and D. J. Ulness, *Properties of the Katugampola fractional derivative with potential application in quantum mechanics*, J. Math. Phys. **56** (2015), no. 6, 063502, 18 pp.

[8] F. M. Atıcı and G. Sh. Guseinov, *On Green's functions and positive solutions for boundary value problems on time scales*, J. Comput. Appl. Math. **141** (2002), no. 1-2, 75–99.

[9] F. M. Atıcı and M. Uyanik, *Analysis of discrete fractional operators*, Appl. Anal. Discrete Math. **9** (2015), no. 1, 139–149.

[10] P. Ballesteros, J. Batista and S. Bell, *Taylor Monomials on Time Scales*, preprint, Department of Mathematics, Kansas State University, 2012.

[11] H. Batarfi, J. Losada, J. J. Nieto and W. Shammakh, *Three-Point Boundary Value Problems for Conformable Fractional Differential Equations*, J. Funct. Spaces **2015** (2015), Art. ID 706383, 6 pp.

[12] B. Bayour, A. Hammoudi and D. F. M. Torres, *Existence of solution to a nonlinear first-order dynamic equation on time scales*, J. Math. Anal. **7** (2016), no. 1, 31–38.

[13] B. Bayour and D. F. M. Torres, *Complex-valued fractional derivatives on time scales*. In: Differential and Difference Equations with Applications, Springer Proceedings in Mathematics & Statistics, Volume 164 (2016), 79–87.

[14] B. Bayour and D. F. M. Torres, *Existence of solution to a local fractional nonlinear differential equation*, J. Comput. Appl. Math. **312** (2017), 127–133.

[15] B. Bayour, A. Hammoudi and D. F. M. Torres, *A Truly Conformable Calculus On Time Scales*, Global and Stochastic Analysis(GSA),issn 2248-9444,Accepted 23-May-2017.

[16] N. Benkhettou, A. M. C. Brito da Cruz and D. F. M. Torres, *A fractional calculus on arbitrary time scales: fractional differentiation and fractional integration*, Signal Process. **107** (2015), 230–237.

[17] N. Benkhettou, A. M. C. Brito da Cruz and D. F. M. Torres, *Nonsymmetric and symmetric fractional calculi on arbitrary nonempty closed sets*, Math. Methods Appl. Sci. **39** (2016), no. 2, 261–279.

[18] N. Benkhettou, A. Hammoudi and D. F. M. Torres, *Existence and uniqueness of solution for a fractional Riemann–Liouville initial value problem on time scales*, J. King Saud Univ. Sci. **28** (2016), no. 1, 87–92.

[19] N. Benkhettou, S. Hassani and D. F. M. Torres, *A conformable fractional calculus on arbitrary time scales*, J. King Saud Univ. Sci. **28** (2016), no. 1, 93–98.

[20] C. Bereanu and J. Mawhin, *Existence and multiplicity results for periodic solutions of nonlinear difference equations*, J. Difference Equ. Appl. **12** (2006), no. 7, 677–695.

[21] M. Bohner and B. Karpuz, *The gamma function on time scales*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **20** (2013), no. 4, 507–522.

[22] M. Bohner and A. Peterson, *Dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2001.

[23] M. Bohner and A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2003.

[24] M. Bohner and C. C. Tisdell, *Second order dynamic inclusions*, J. Nonlinear Math. Phys. **12** (2005), suppl. 2, 36–45.

[25] P. L. Butzer and U. Westphal, *An access to fractional differentiation via fractional difference quotients, in Fractional calculus and its applications* (Proc. Internat. Conf., Univ. New Haven, West Haven, Conn., 1974), 116–145. Lecture Notes in Math., 457, Springer, Berlin, 1975.

[26] A. Cabada, M. R. Grossinho and F. Minhós, *Extremal solutions for third-order nonlinear problems with upper and lower solutions in reversed order*, Nonlinear Anal. **62** (2005), no. 6, 1109–1121.

[27] A. Cabada and D. R. Vivero, *Expression of the Lebesgue $\Delta$-integral on time scales as a usual Lebesgue integral: application to the calculus of $\Delta$-antiderivatives*, Math. Comput. Modelling **43** (2006), no. 1-2, 194–207.

[28] W. S. Chung, *Fractional Newton mechanics with conformable fractional derivative*, J. Comput. Appl. Math. **290** (2015), 150–158.

[29] Q. Dai and C. C. Tisdell, *Existence of solutions to first-order dynamic boundary value problems*, Int. J. Difference Equ. **1** (2006), no. 1, 1–17.

[30] M. Frigon, *Boundary and periodic value problems for systems of nonlinear second order differential equations*, Topol. Methods Nonlinear Anal. **1** (1993), no. 2, 259–274.

[31] M. Frigon and H. Gilbert, *Existence theorems for systems of third order differential equations*, Dynam. Systems Appl. **19** (2010), no. 1, 1–23.

[32] M. Frigon and H. Gilbert, *Boundary value problems for systems of second-order dynamic equations on time scales with $\Delta$-Carathéodory functions*, Abstr. Appl. Anal. **2010** (2010), Art. ID 234015, 26 pp.

[33] M. Frigon and H. Gilbert, *Systems of first order inclusions on time scales*, Topol. Methods Nonlinear Anal. **37** (2011), no. 1, 147–163.

[34] H. Gilbert, *Existence theorems for first-order equations on time scales with $\Delta$-Carathéodory functions*, Adv. Difference Equ. **2010** (2010), Art. ID 650827, 20 pp.

[35] J. R. Graef, L. Kong, F. M. Minhós and J. Fialho, *On the lower and upper solution method for higher order functional boundary value problems*, Appl. Anal. Discrete Math. **5** (2011), no. 1, 133–146.

[36] A. Granas and J. Dugundji, *Fixed point theory*, Springer Monographs in Mathematics, Springer, New York, 2003.

[37] R. J. Higgins and A. Peterson, Cauchy functions and Taylor's formula for time scales $\mathbb{T}$, in *Proceedings of the Sixth International Conference on Difference Equations*, 299–308, CRC, Boca Raton, FL, 2004.

[38] V. Kac and P. Cheung, *Quantum calculus*, Universitext, Springer, New York, 2002.

[39] A. Karci and A. Karadogan, *Fractional order derivative and relationship between derivative and complex functions*, Math. Sci. Appl. E-Notes **2** (2014), no. 1, 44–54.

[40] U. N. Katugampola, *A new approach to generalized fractional derivatives*, Bulletin of Mathematical Analysis and Applications. Volume 6 Issue 4 (2014), Pages 1-15.

[41] U. N. Katugampola, *A new fractional derivative with classical properties*, preprint, 2014. `arXiv:1410.6535`

[42] W. G. Kelley and A. C. Peterson, *Difference equations*, second edition, Harcourt/Academic Press, San Diego, CA, 2001.

[43] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math. **264** (2014), 65–70.

[44] M. J. Lazo and D. F. M. Torres, *Variational calculus with conformable fractional derivatives*, IEEE/CAA Journal of Automatica Sinica, in press. DOI: 10.1109/JAS.2016.7510160

[45] R. L. Magin, *Fractional calculus in Bioengineering*, CR in Biomedical Engineering **32** (2004), no. 1, 1–104.

[46] B. Mirandette, *Résultats d'existence pour des systèmes d'équations différentielles du premier ordre avec tube-solution*, M.Sc. thesis, University of Montréal, 1996.

[47] M. D. Ortigueira and J. A. Tenreiro Machado, *What is a fractional derivative?*, J. Comput. Phys. **293** (2015), 4–13.

[48] Picard, E., Sur lapplication des mthodes dapproximations successives ltude de certaines quations diffrentielles ordinaires, J. de Math. 9(1893), 217271.

[49] D. Prodanov, *Fractional variation of Hölderian functions*, Fract. Calc. Appl. Anal. **18** (2015), no. 3, 580–602.

[50] Rusnák, J., Constructions of lower and upper solutions for a nonlinear boundary value problem of the third order and their applications.Math. Slovaca40(1990), no. 1, 101110.

[51] M. Ruyun, Z. Jihui and F. Shengmao, *The method of lower and upper solutions for fourth-order two-point boundary value problems*, Journal Of Mathematical Analysis And Applications 215, 415 - 422 (1997).

[52] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives*, Gordon and Breach, Yverdon, 1993.

[53] G. S. Taverna and D. F. M. Torres, *Generalized fractional operators for nonstandard Lagrangians*, Math. Methods Appl. Sci. **38** (2015), no. 9, 1808–1812.

[54] Scorza Dragoni, G.,Il problema dei valori ai limiti studiato in grande per gli integrali di una equazione differenzile del secondo ordine. Giornale Mat. (Battaglini)69 (1931), 77-112.

[55] C. C. Tisdell and A. Zaidi, *Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling*, Nonlinear Anal. **68** (2008), no. 11, 3504–3524.

[56] X. J. Yang, *Advanced local fractional calculus and its applications*, World Science Publisher, New York, 2012.

[57] X.-J. Yang, D. Baleanu and J. A. T. Machado, *Application of the local fractional Fourier series to fractal signals, in Discontinuity and complexity in nonlinear physical systems*, 63–89, Nonlinear Syst. Complex, Springer, Cham, 2014.

[58] D. Zhao and T. Li, *On conformable delta fractional calculus on time scales*, J. Math. Computer Sci. **16** (2016), 324–335.

[59] D. Zhao and X. You, *A new fractional derivative on time scales*, Advances in Applied Mathematical Analysis Volume 11, Number 1 (2016), pp. 19.

# Abstract

Our PhD thesis consists with two part, the common denominator between them is the time scales. The first part devoted for the existence of solution. To get the result We introduce the notion of tube solution. The Part 1 of thesis to riders are separated at two points. In the first points we study a non-linear first order dynamic equation on arbitrary compact time scales. In the second points we study the existence of Solution to a local fractional differential equation. In the part 2 of thesis we are interested to introduce two new notions. The first notion is a Complex-valued fractional derivatives on time scales here we prove the properties of this derivative. Finally we introduce the concept of conformable derivative on time scales and we developed its calculs..



**Keywords and phrases:** nabla dynamic equations; existence; tube solution; fractional differential equations; initial value problems; conformable fractional derivatives.

**AMS (MOS) Subject Classifications:** 34B15; 34N05; 26A33; 34A12; 26E70.

# Résumé

Notre Thèse de doctorat se compose en deux parties, le dénominateur commun entre eux est l'échelles de temps. La premier partie est consacrée a l'existence de solutions. Pour obtenir le rsultat Nous introduisons la notion de tube solution. La part 1 de thèse se consiste de deux points. Dans le premier point nous étudions un problème d'équations dynamiques de premier ordre non-linaire sur un arbitraire compacts échelles de temps. Dans la deuxième point nous étudions un problème d'équations différentielles fractionnaire local. Dans la part 2 de thèse nous sommes intéressés a présenter deux nouvelles notions. La première notion est la dérivée fractionnaire complexe sur l'échelles de temps ici nous démontrons les propriétés de cette dérivation. Et enfin, nous introduisons le concept de dérivée fractionnaire conforme sur l'échelles de temps et développer leur calcul.

**Mots et Phrases Clefs:** nabla équations dynamiques; non linéaires des problèmes de valeur limite; existence; tube solution; équations diffrentielles fractionnaires; problèmes de valeur initiale; dérivées fractionnaires conforme.

**Classification AMS:** 34B15; 34N05; 26A33; 34A12; 26E70.