To the memory of
Prof. Bénamar CHOUAF

Dedication

I dedicate my dissertation work to

The memory of my father, also to my beloved mother the one who encourages me and beleives in me with her unconditional love.

*My sister **Fatima Zohra** who has always being my side.*

*My brothers **Mohammed Amine** and **Abdelkrim** who have supported me all the way since the beginning of my study.*

*My darling **Fatima** and **Rajaa**.*

*My dear nephews **Ibrahim** and **Adam**.*

To those who have been deprived from their right to study and to all those who believe in the richness of learning.

"Science is the key to our futur, and if you don't beleive in science, then you're holding everybody back."

[Bill Nye]

"All science requires mathematics. The knowledge of mathematical things is almost innate in us. This is the easiest of sciences, a fact which is obvious in that no one's brain rejects it; for laymen and people who are utterly illiterate know how to count and reckon." [Roger Bacon]

[Platon]

This thesis is written in tex **LaTeX 2**$_\varepsilon$.

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"Mathematics is the art of giving the same name to differents things." (Henri Poincaré)

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Contents

Abstract

The research reported in this thesis deals with the problem of fractional stochastic differential equations and inclusions in Hilbert spaces. We have discussed the existence and uniqueness result for an impulsive fractional stochastic evolution equations involving Caputo fractional derivative and fractional partial neutral stochastic functional integro-differential inclusions with state-dependent delay and analytic resolvent operators. Sufficient conditions for the existence are established by using the nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan and the fractional power of operators. The main results are obtained by means of the theory of operators semi-group, fractional calculus, fixed point technique and stochastic analysis theory and methods adopted directly from deterministic fractional equations.

The approximate controllability has also been investigated for a class of fractional neutral stochastic functional integro-differential inclusions involving the Caputo derivative in Hilbert spaces. A new set of sufficient conditions are formulated and proved for the approximate controllability of fractional stochastic integrodifferential inclusions under the assumption that the associated linear part of system is approximately controllable. The main techniques rely on the fractional calculus, operator semigroups and Bohnenblust-Karlin's fixed point theorem. An example is given to illustrate the obtained theory.

The List Of Works

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Introduction

Differential equations or inclusions with fractional order have recently proved to be strong tools in the modeling of many phenomena in various fields of engineering, physics and economics. As a consequence there was an intensive development of the theory of differential equations and inclusions of fractional order. Due to this fact, the fractional order models are capable to describe more realistic situation than the integer order models. Many articles have been devoted to the existence of solutions for fractional differential equations and inclusions, for example [3, 42, 125, 126].

In many cases, deterministic models often fluctuate due to environmental noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic ones. Taking the disturbances into account, the theory of differential equations or inclusions has been generalized to stochastic case. The existence, uniqueness, stability, controllability and other quantitative and qualitative properties of solutions of stochastic evolution equations or inclusions have recently received a lot of attention (see [50, 82, 116, 76, 99, 101, 121] and the references therein).

Fractional differential equations have been used in many field like fractals, chaos, electrical engineering, medical science, etc. In recent years, we have seen considerable development on the topics of fractional differential equations. For instance, we refer to the monographs of Abbas et al. [1], Kilbas et al. [56], Miller and Ross [88], Podlubny [96], and the papers [5, 8, 126].

In particular, differential equations with impulsive conditions constitute an important field of research due to their numerous applications in ecology, medicine, biology, electrical engineering and other areas of science. Many physical phenomena in evolution processes are modeled as impulsive fractional differential equations and existence results for such equations have been studied by several authors [25, 110, 124]. One of the important problems in the qualitative theory of impulsive differential equations is the existence of almost periodic solutions. At the present time, many results on the existence, uniqueness and stability of these solutions have been obtained (see [7, 77, 112, 114] and the references therein). However, only few papers deal with the existence of almost periodic solutions for impulsive fractional differential equations. Recently, Debbouche et al. [28] studied the existence of almost periodic and optimal mild solutions of fractional evolution equations with analytic semigroup in a Banach space. El Borai et al. [31] established the existence and uniqueness of almost periodic solutions of a class of nonlinear fractional differential equations with analytic semigroup in Banach space, and very recently, Stamov et al. [115] studied the existence of almost periodic solutions for fractional differential equations with impulsive effects. Due to the importance of stochastic approach, the existence of almost periodic solutions for stochastic differential equations has been discussed in [15, 20, 97]. The existence of almost periodic solutions for impulsive stochastic evolution equations has been reported in [21, 78]. However, up to now the problem for the existence of almost periodic solutions for impulsive fractional stochastic evolution equations have not been considered in the literature. In order to fill this gap, one of the aims of this thesis is to study the existence of square-mean piecewise almost periodic solutions of the impulsive fractional stochastic differential equations.

The fractional differential inclusions arise in the mathematical modeling of certain problems in economics, optimal controls, etc., so the problem of existence of solutions of fractional differential inclusions has been studied by several authors for different kind of problems (see [2, 3, 4, 79] and the references therein). In particular, delay fractional differential inclusions arise in many physical and biological applications, but often demand the use of non-constant or state-dependent delays. These inclusions are frequently called inclusions with state-dependent delay. Recently, the analysis of fractional differential equations or inclusions with state-dependent delay has received much attention [5]. Very recently, the existence of solutions to fractional differential inclusions with state dependent delay has been established (see [14] and the references therein).

The theory of differential inclusions has been generalized to stochastic functional differential inclusions (see [13] and the references therein). As the generalization of classical integro-differential inclusions, neutral stochastic functional integro-differential inclusions have attracted the researchers great interest, and some works have done on the existence results of mild solutions for these equations (see [76] and the references therein). However, to the best of our knowledge, it seems that little is known about fractional neutral stochastic integro-differential inclusions, and our second aim of this thesis is to fill this gap. Recently, Yan and Zhang in [122] studied the existence of mild solutions for a class of impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay by using properties of the solution operator and fixed point technique. Guendouzi et al. [37] investigated the existence of mild solutions for a class of impulsive fractional stochastic differential inclusions with state-dependent delay in Hilbert spaces. Sufficient conditions for the existence of solutions are derived by the authors using the nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan. Motivated by the previously mentioned works [14, 37, 119, 122], in this thesis, we also consider the existence of a class of fractional partial neutral stochastic integro-differential inclusions with state-dependent delay.

On the other hand, controllability is one of the important fundamental concepts in mathematical control theory and plays an important role in both deterministic and stochastic control systems. Roughly speaking, controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Recently, controllability of stochastic dynamical control systems in infinite dimensional spaces is well-developed using different kind of approaches (see [12, 27, 83, 99, 107, 108] and the references therein). It should be noted that it is generally difficult to realize the conditions of exact controllability for infinite-dimensional systems and thus the approximate controllability becomes a very important topic for dynamical systems. At present, there are few works in approximate controllability problems for different kind of systems described by fractional deterministic and stochastic differential equations such as work done in [36, 102, 103, 123]. Recently, Sakthivel et al. [105] derived a new set of sufficient conditions for approximate controllability of fractional stochastic differential equations. The approximate controllability of neutral stochastic fractional integro-differential equation with infinite delay in a Hilbert space has been studied in [106], and more recently, Guendouzi [36] derived a set of sufficient conditions for the approximate controllability of impulsive fractional stochastic system by using the Krasnoselskii's fixed point theorem with stochastic analysis theory.

However, to the best of our knowledge, it seems that little is known about approximate controllability of fractional deterministic and stochastic differential inclusions, and the last aim of this thesis is to close the gap. Yan et al. [119] studied the approximate controllability of partial fractional neutral stochastic functional integro-differential inclusions with state-dependent delay. Sakthivel et al. [104] studied the approximate controllability for a class of fractional nonlinear differential inclusions in Banach spaces. Inspired by the above mentioned works [36, 104, 105, 106], in this thesis we will study the approximate controllability problem for a class of fractional neutral stochastic functional integro-differential inclusions with infinite delay in Hilbert spaces which are natural generalizations of controllability concepts well known in the theory of infinite dimensional deterministic control systems.

The study of fractional stochastic differential equations, fractional stochastic functional inclusion and their controllability problem have been the scene of intensive study, this is what we will focus on in this thesis.

This thesis divided into four chapters. In the first one, we introduce the definitions, the basic notation about the classical stochastic calculus, and we briefly recall some basic properties of the Brownian motion in Hilbert space, then we discuss integration with respect to this process. At the end of this chapter we will present the definitions and properties of semigroup. Secondly, we will introduce and develop the concept of p-th mean almost periodicity. In particular, it will be shown that each p-th mean almost periodic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is uniformly continuous and stochastically bounded [81]. Furthermore, the collection of all p-th mean almost periodic processes is a Banach space when it is equipped with its natural norm. Moreover, two composition results for p-th mean almost periodic processes are established. These two resuts play a crucial role in the study of the existence (and uniqueness) of p-th mean almost periodic solutions to various stochastic differential equations on $L_p(\Omega, \mathcal{H})$ where $\mathcal{H}$ is a real separable Hilbert space. The third chapter is devoted to the controllability theory. The main objective of this chapter is to review the major progress that has been made on controllability of dynamical systems over the past number of years and present without mathematical proofs a review of recent controllability problems for a wide class of dynamical systems. In the last chapter which is the core of our present study, the first section is devoted to give some basic definitions, notations and preliminary facts needed to establish our main results, and we recall some results concerning fractional calculus. Then, we introduce some basic definitions and results of multivalued maps. In the second section, we establish criteria of the existence of square mean piecewise almost periodic solutions for impulsive fractional stochastic differential equations and its exponential stability. In the third section, we give our main results about the existence of solutions for a class of fractional partial neutral stochastic integro-differential inclusions with state-dependent delay, an example is given to illustrate our results. The fourth section in this chapter, is devoted to the existence results for fractional neutral stochastic integro-differential inclusions with infinite delay and control. Precisely, we study the approximate controllability for a class of fractional neutral stochastic functional integro-differential inclusions under the assumption that the associated linear part of system is approximately controllable. The application of our theoretical results is also given.

Chapter 1

Preliminary Background

In this chapter we summarize basic definitions and facts about real valued stochastic processes and generalize some of these results to processes with values in Hilbert spaces. For more detail we refer the reader to [73, 51, 46, 72, 35, 17, 26].

1.1 Notations and definitions

In this section the basic notations of the theory of stochastic calculus are considered. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_s\}$ satisfying the *usual conditions*:

1. $\mathcal{F}_s = \bigcap_{t>s} \mathcal{F}_t$ for all $s \geq 0$;
2. All $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$ are contained in $\mathcal{F}_t$.

A family $(x(t),\, t \geq 0)$ of $\mathbb{R}^d$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *stochastic process*, this process is adapted if all $x(t)$ are $\mathcal{F}_t$-measurable. Denoting $\mathcal{B}$, the Borel σ-field on $[0, \infty)$. The process x is measurable if $(t, \omega) \mapsto x(t, \omega)$ is a $\mathcal{B} \bigotimes \mathcal{F}$-measurable mapping. We say that $(x(t),\, t \geq 0)$ is continuous if the trajectories $t \mapsto x(t, \omega)$ are continuous for all $\omega \in \Omega$. One can show that a process is progressively measurable if it is right-continuous ([51], Prop. 1.13).

1.2 Brownian Motion

Brownian motion is a process of tremendous practical and theoretical significance. It originated (a) as a model of the phenomenon observed by Robert Brown in 1828 that "pollen grains suspended in water perform a continual swarming motion," and (b) in Bachelier's (1900) work as a model of the stock market. These are just two of many systems that Brownian motion has been used to model. On the theoretical side, Brownian motion is a Gaussian Markov process with stationary independent increments. It lies in the intersection of three important classes of processes and is a fundamental example in each theory.

1.2.1 Definition of Brownian Motion (A standard one-dimensional)

A stochastic process is a measurable function $x(t, w)$ defined on the product space $[0, \infty) \times \Omega$. In particular,

(a) for each t, $x(t, \cdot)$ is a random variable,

(b) for each w, $x(\cdot, w)$ is a measurable function (called a sample path).

For convenience, the random variable $x(t, \cdot)$ will be written as $x(t)$ or x_t. Thus a stochastic process $x(t, w)$ can also be expressed as $x(t)(w)$ or simply as $x(t)$ or x_t.

Definition 1.2.1. *([73]). A stochastic process $B(t, w)$ is called a Brownian motion if it satisfies the following conditions:*

(1) $\mathbb{P}\{w;\, B(0, w) = 0\} = 1$.

(2) For any $0 \leq s < t$, the random variable $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$, i.e., for any $a < b$,

$$\mathbb{P}\{a \leq B(t) - B(s) \leq b\} = \frac{1}{\sqrt{2\Pi(t-s)}} \int_a^b e^{-x^2/2(t-s)} dx.$$

(3) $B(t, w)$ has independent increments, i.e, for any $0 \leq t_1 < t_2 < \ldots < t_n$, the random variables $B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$, are independent.

(4) Almost all sample paths of B(t, w) are continuous functions, i.e.,

$$\mathbb{P}\{w; B(\cdot, w) \text{ is continuous}\} = 1.$$

Remark 1.2.1.1.

- A Brownian motion is sometimes defined as a stochastic process $B(t, w)$ satisfying conditions (1), (2), (3) in Definition (1.2.1). Such a stochastic process always has a continuous realization, i.e., there exists Ω_0 such that $\mathbb{P}(\Omega_0) = 1$ and for any $w \in \Omega_0$, $B(t, w)$ is a continuous function of t. This fact can be easily checked by applying the Kolmogorov continuity theorem. Thus condition (4) is automatically satisfied.
- A Brownian motion is also called a Wiener process since, it is the canoncial process defined on the Wiener space.

1.2.2 Simple Properties of Brownian Motion

Let $B(t)$ be a fixed Brownian motion. We give below some simple properties that follow directly from the definition of Brownian motion.

- For any $t > 0$, $B(t)$ is normally distributed with mean 0 and variance t. For any $s, t \geq 0$, we have $E[B(s)B(t)] = \min\{s, t\}$.
- Translation invariance: For fixed $t_0 \geq 0$, the stochastic process $\widetilde{B}(t) = B(t + t_0) - B(t_0)$ is also a Brownian motion.
- Scaling invariance: For any real number $\lambda > 0$, the stochastic process $\widetilde{B}(t) = B(\lambda t)/\sqrt{\lambda}$ is also a Brownian motion.

1.2.3 Quadratic Variation of Brownian Motion

The quadratic variation of Brownian motion $[B, B](t)$ is defined as

$$[B, B](t) = [B, B]([0, t]) = \lim \sum_{i=1}^{n} |B(t_i^n - B(t_{i-1}^n)|^2, \tag{1.1}$$

where the limit is taken over all shrinking partitions of $[0, t]$, with $\delta_n = \max_i(t_{i+1}^n - t_i^n)$ as $n \to \infty$. It is remarkable that although the sums in the definition (1.1) are random, their limit is non-random, as the following result shows.

Proposition 1.2.1. *Quadratic variation of a Brownian motion over* $[0, t]$ *is* t.

Proof. We refer the reader to [72].

1.2.4 Brownian paths

An occurrence of Brownian motion observed from time 0 to time T, is a random function of t on the interval $[0, T]$. It is called a realization, a path or trajectory.

Proposition 1.2.2. *([51]). A Brownian motion has its paths a.s. locally* γ*-Hölder continuous for* $\gamma \in [0, 1/2)$.

Proposition 1.2.3. *([51]). The Brownian motion's sample paths are a.s. nowhere differentiable.*

1.2.5 Three Martingales of Brownian Motion

Three main martingales associated with Brownian motion are given.

Definition 1.2.2. *A stochastic process* $\{x(t), t \geq 0\}$ *is a martingale if for any* t *it is integrable,* $\mathbb{E}|x(t)| < \infty$, *and for any* $s > 0$

$$\mathbb{E}(x(t+s)\backslash\mathcal{F}_t) = x(t), \text{ a.s.}$$

where $\mathcal{F}_t$ *is the information about the process up to time* t, *and the equality holds almost surely.*

The martingale property means that if we know the values of the process up to time t, and $x(t) = x$ then the expected future value at any future time is x.

Proposition 1.2.4. *([72]). Let* $B(t)_{t \in \mathbb{R}+}$ *be Brownian Motion. Then*

1. $B(t)$ *is a martingale.*
2. $B^2(t) - t$ *is a martingale.*
3. *For any* u, $e^{uB(t) - \frac{u^2}{2}t}$ *is a martingale.*

Proof. We refer the reader to [72].

1.2.6 Markov Property of Brownian Motion

The Markov Property states that if we know the present state of the process; then the future behaviour of the process is independent of its past. The process $x(t)$ has the Markov property if the conditional distribution of $x(t+s)$ given $x(t) = x$, does not depend on the past values (but it may depend on the present value x). The process "does not remember" how it got to the present state x. Let $\mathcal{F}_t$ denote the σ-field generated by the process up to time t.

Definition 1.2.3. *x is a Markov process if for any t and $s > 0$, the conditional distribution of $x(t+s)$ given $\mathcal{F}_t$ is the same as the conditional distribution of $x(t+s)$ given $x(t)$, that is,*

$$P(x(t+s) \leq y \setminus \mathcal{F}_t) = P(x(t+s) \leq y \setminus x(t)), \ a.s. \tag{1.2}$$

Theorem 1.2.1. *Brownian motion $B(t)$ possesses Markov property.*

Proof. We refer the reader to [72].

1.3 Stochastic Integration with respect to Brownian Motion

1.3.1 Motivation

In applications, it is typical to characterize the current state of a physical system by a real function of time $x(t)$, $t \geq 0$, called the state. Generally, the behavior of a physical system based on an input $w(t)$ for $t \geq 0$, can be specified by a differential equation of the form

$$\frac{dx(t)}{dt} = \mu(x(t)) + \sigma(x(t))w(t), \ t \geq 0, \tag{1.3}$$

where the functions μ and σ depend on the system properties. In classical analysis, the study of the solutions of such an equation is based on the assumptions that the system properties and the input are perfectly known and deterministic.

Here, we generalize Eq.(1.3) by assuming that the input is a real stochastic process. Because the input is random, the state becomes a real stochastic process.

Now, let x denote the solution of (1.3) with w replaced by a stochastic process Z. It is customary to assume that Z is a "white noise" process for which $\mathbb{E}[Z(t)] = 0$ and $Cov(Z(s), Z(t)) = 1$ if $s = t$ and is zero otherwise. It is important to note that for $t_1 < t_2 < t_3$,

$$Cov\left[\int_{t_1}^{t_2} Z(s)ds, \int_{t_1}^{t_2} Z(s)ds\right] = 0 \tag{1.4}$$

whereas

$$Var\left[\int_0^t Z(s)ds\right] = t. \tag{1.5}$$

The Gaussian white noise process is often used. Such a stochastic process $\{Z(t), \, t \in \mathbb{R}\}$ has irregular sample paths and is very difficult to work with directly. As a result, it is easier to work with its integral. This suggests writing (1.3) in the form

$$x(t) = x(0) + \int_0^t \mu(x(s))ds + \int_0^t \sigma(x(s))Z(s)ds. \tag{1.6}$$

In this integrated version, we need to make mathematical sense of the stochastic integral involving the integrator $Z(s)ds$. From a notational standpoint, it is common to write

$$dx(t) = \mu(x(t))dt + \sigma(x(t))Z(t)dt. \tag{1.7}$$

Note that given a Brownian motion B, it is not difficult to verify that

$$Cov\Big[B(t_2) - B(t_1),\, B(t_3) - B(t_2)\Big] = 0 \text{ and } Var\Big[B(t) - B(0)\Big] = t.$$

Given the similarity with (1.4) and (1.5), the latter hints that B can be viewed as integrated white noise so that we can rigorously define $\int_0^t Z(t)dt$ to be $B(t)$. This is quite an oversimplification. To write $B(t) = \int_0^t Z(t)dt$ would require that B is differentiable almost everywhere (in time t). Unfortunately, this is not the case: B is non differentiable at t. This oversimplification comes from the fact that white noise does not exist as a well-defined stochastic process. On the other hand, Brownian motion is well defined, so this suggests that we should replace (1.6) with

$$x(t) = x_0 + \int_0^t \mu(x(s))ds + \int_0^t \sigma(x(s))dB(s). \tag{1.8}$$

and (1.7) with

$$dx(t) = \begin{cases} \mu(x(t))dt + \sigma(x(t))dB(t) \\ x(0) = x_0. \end{cases} \tag{1.9}$$

Note that in (1.6), the integral $\int_0^t \mu(x(s))ds$ can be defined via a standard Riemann approximation. On the other hand, $\int_0^t \sigma(x(s))dB(s)$ must be defined differently since the integrator is a non differentiable stochastic process.

This leads us to outline the construction of the so-called Itô integral.

1.3.2 Itô integrals

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space and let $B = \{B(t),\, t \geq 0\}$ be a one dimensional Brownian motion defined on this space.

Definition 1.3.1. *Let $0 \leq S < T < \infty$. Denote by $V([S,T];\mathbb{R})$ the space of all real valued measurable $(\mathcal{F}_t)$-adapted processes $\Phi = \{\Phi(t), t \geq 0\}$ such that*

$$\|\Phi\|_V^2 = \mathbb{E}\Big[\int_S^T |\Phi(t)|^2 dt\Big] < \infty.$$

We identify Φ and $\overline{\Phi}$ in $V([S,T];\mathbb{R})$ if $\|\Phi - \overline{\Phi}\|_V^2 = 0$. In this case we say that Φ and $\overline{\Phi}$ are equivalent and we write $\Phi = \overline{\Phi}$.

The space $V([S,T];\mathbb{R})$ equipped with the norm$\|\cdot\|_V$ is a Banach space. Furthermore, without loss of generality we may assume that every stochastic process $\Phi \in V([S,T];R)$ is predictable.

Since full details on the construction of the Itô integral $\int_S^T \Phi(t)dB(t)$ for stochastic processes $\Phi \in V([S,T];\mathbb{R})$ can be found in either [90], here we shall outline only its construction. The idea of construction is as follows. First define the integral $\int_S^T \Psi(t)dB(t)$ for a class of simple processes Ψ then we show that each $\Phi \in V([S,T];\mathbb{R})$ can be approximated by such simple processes $\Psi's$ and we define the limit of $\int_S^T \Psi(t)dB(t)$ as the integral $\int_S^T \Phi(t)dB(t)$.

Let us first introduce the concept of simple stochastic processes.

Definition 1.3.2. *A stochastic process $\Psi \in V([S,T];\mathbb{R})$ is called simple if it is of the form*

$$\Psi(w,t) = \alpha_0(w)1_{[t_0,t_1)}(t) + \sum_{i=0}^{k-1} \alpha_i(w)1_{[t_i,t_{i+1})}(t),$$

with a partition $S = t_0 < t_1 < ... < t_k = T$ of $[S,T]$ and bounded $\mathcal{F}_{t_i}$-measurable random variables $\alpha_i, 0 \le i \le k-1$.

For any simple stochastic process $\Psi \in V([S,T];\mathbb{R})$ we define

$$\int_S^T \Psi(t)dB(t) := \sum_{i=0}^{k-1} \alpha_i[B(t_{i+1}) - B(t_i)].$$

Obviously, the integral $\int_S^T \Psi(t)dB(t)$ is a well-defined random variable. Moreover, the following properties hold:

$$\mathbb{E}\Big[\int_S^T \Psi(t)dB(t)\Big] = 0, \tag{1.10}$$

$$\mathbb{E}\Big|\int_S^T \Psi(t)dB(t)\Big|^2 = \int_S^T \mathbb{E}|\Psi(t)|^2 dt.$$

Also, for any simple stochastic processes $\Psi_1, \Psi_2 \in V([S,T];\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$, we have

$$\int_S^T [c_1\Psi_1(t) + c_2\Psi_2(t)]dB(t) = c_1\int_S^T \Psi_1(t)dB(t) + c_2\int_S^T \Psi_2(t)dB(t)$$

We can now extend the Itô integral from simple stochastic processes to stochastic processes in $V([S,T];\mathbb{R})$. This is based on the following approximation result.

Lemma 1.3.1. *([17]). For any $\Phi \in V([S,T];\mathbb{R})$, there exists a sequence (Ψ_n) of simple stochastic processes such that*

$$\lim_{n\to\infty} \int_S^T \mathbb{E}|\Phi(t) - \Psi_n(t)|^2 dt = 0.$$

We are now prepared to outline the construction of the Itô integral for a stochastic process $\Phi \in V([S,T];\mathbb{R})$. By Lemma (1.3.1), there is a sequence (Ψ_n) of simple stochastic processes such that

$$\lim_{n\to\infty} \int_S^T \mathbb{E}|\Phi(t) - \Psi_n(t)|^2 dt = 0.$$

Thus, by property (1.10),

$$\mathbb{E}\Big|\int_S^T \Psi_n(t)dB(t) - \Psi_m(t)dB(t)\Big|^2 = \int_S^T \mathbb{E}|\Psi_n(t) - \Psi_m(t)|^2 dt \to 0 \text{ as } m,n \to \infty.$$

Hence, the sequence $\Big\{\int_S^T \Psi_n(t)dB(t),\ n \ge 1\Big\}$ is a Cauchy sequence in $L_2(\Omega;\mathbb{R})$ which, in turn, implies that it is convergent. This leads us to the following definition.

Definition 1.3.3. *([17]). Let $\Phi \in V([S,T];\mathbb{R})$. The Itô integral Φ with respect to $(B(t))$ is defined by*

$$\int_S^T \Phi(t)dB(t) = \lim_{n\to\infty} \int_S^T \Psi_n(t)dB(t) \text{ in } L_2(\Omega;\mathbb{R}),$$

where (Ψ_n) is a sequence of simple stochastic processes such that

$$\lim_{n\to\infty} \mathbb{E}\bigg[\int_S^T |\Phi(t)-\Psi_n(t)|^2 dt\bigg] = 0.$$

It is important to note that this integral does not depend on the choice of approximating sequence.

The main properties of the Itô integral

Proposition 1.3.1. *([17]). Let Φ, Ψ be stochastic processes in $V([S,T];\mathbb{R})$, and let $0 \le S < U < T$. Then*

(a) $\mathbb{E}\bigg[\int_S^T \Phi(t)dB(t)\bigg] = 0;$

(b) $\mathbb{E}\bigg|\int_S^T \Phi(t)dB(t)\bigg|^2 = \int_S^T \mathbb{E}|\Phi(t)|^2 dt$ *(Itô Isometry);*

(c) $\int_S^T (c\Psi(t)+\Psi(t))dB(t) = c\int_S^T \Psi(t)dB(t) + \int_S^T \Psi(t)dB(t)$ *(c constant);*

(d) $\int_S^T \Phi(t)dB(t) = \int_S^U \Phi(t)dB(t) + \int_U^T \Phi(t)dB(t)$

(e) $\int_S^T \Phi(s)dB(s)$ *is $\mathcal{F}_T$-measurable.*

1.4 Stochastic Calculus in a Hilbert Space

In the previous section, we presented the elements of Stochastic Calculus for real stochastic processes. These elements are also valid for stochastic processes taking their values in a separable Hilbert space. However, the extensions can be connected with some difficulties when we would be interested, for instance, in analytical properties of sample paths of such processes.

Of interest to us will be operator-valued random variables and their integrals. Let $\mathcal{K}$ and $\mathcal{H}$ be two separable Hilbert spaces with norms $\|\cdot\|_{\mathcal{K}}, \|\cdot\|_{\mathcal{H}}$ and inner products $\langle\cdot,\cdot\rangle_{\mathcal{K}}, \langle\cdot,\cdot\rangle_{\mathcal{H}}$, respectively. From now on, without further specification we always use the same symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces involved when no confusion is possible.

1.4.1 Operator valued random variables

Of great interest to us will be operator valued random variables and their integrals. Let $\mathcal{K}$ and $\mathcal{H}$ be two separable Hilbert spaces and denote by $L = L(\mathcal{K},\mathcal{H})$ the set of all linear bounded operators from $\mathcal{K}$ into $\mathcal{H}$. The set L is a linear space and, equipped with the operator norm, becomes a Banach space. However, if both spaces are infinite dimensional, then L is not a separable space. To see this we can assume that $\mathcal{K} = \mathcal{H} = L^2(\mathbb{R}^1)$. Define for arbitrary $t \in \mathbb{R}^1$ the isometry $M(t)$ from $\mathcal{H}$ onto $\mathcal{H}$

$$M(t)x(z) = x(z+t),\ z \in \mathbb{R}^1,\ x \in \mathcal{H}. \tag{1.11}$$

Assume that $t > s$, $x \in \mathcal{H}$, then

$$|(M(t) - M(s))x| = |M(s)(M(t-s)x - x)| = |M(t-s)x - x|.$$

If $x \in \mathcal{H}$ has support in the interval $((s-t)/2, (t-s)/2)$ then the functions x and $M(t-s)x$ have disjoint supports and therefore $|(M(t)-M(s))x|^2 = 2|x|^2$. Consequently $\|M(t)-M(s)\| \geq \sqrt{2}$ and $L = L(\mathcal{H}, \mathcal{H})$ is not separable.

The nonseparability of L has several consequences. First of all the corresponding Borel σ-field $\mathscr{B}(L)$ is very rich, to the extent that very simple L-valued functions turn out to be nonmeasurable. In particular, the function $M(.)$ defined by (1.11), considered as a mapping from $(\mathbb{R}^1, \mathscr{B}(\mathbb{R}^1))$ into $(L, \mathscr{B}(L))$, is not measurable. To see this fix a non-Borel subset μ of $\mathbb{R}^1$ and define an open subset D of L by the formula

$$D = \{G \in L : \|G - M(t)\| < \sqrt{2}/2, \text{ for some } t \in \Gamma\}.$$

Since $\{t \in \mathbb{R}^1 : M(t) \in D\} = \Gamma$, therefore $M(.)$ cannot be measurable.

The lack of separability of L implies also that Bochner's definition cannot be applied directly to the L-valued functions. To overcome these difficulties it is convenient to introduce a weaker concept of measurability

A function $\Phi(.)$ from Ω into L is said to be strongly measurable if for arbitrary $u \in \mathcal{K}$ the function $\Phi(.)\mathcal{K}$ is measurable as a mapping from $(\Omega, \mathcal{F})$ into $(\mathcal{H}, \mathscr{B}(\mathcal{H}))$. Let $\mathscr{L}$ be the smallest σ-field of subsets of L containing all sets of the form

$$\{\Phi \in L : \Phi u \in A\},\ u \in \mathcal{K},\ A \in \mathscr{B}(\mathcal{H}),$$

then $\Phi : \Omega \to L$ is a strongly measurable mapping from $(\Omega, \mathcal{F})$ into $(L, \mathscr{L})$. Elements of $\mathscr{L}$ are called strongly measurable. If $\mathbb{P}$ is a nonnegative (and not necessarily a normalized) measure on $\mathcal{F}$ then Φ is said to be Bochner integrable if for arbitrary u the function $\Phi(.)u$ is Bochner integrable and there exists a bounded linear operator $\Psi \in L(\mathcal{K}, H)$ such that

$$\int_\Omega \Phi(w)u\mathbb{P}(dw) = \Psi u,\ \forall u \in \mathcal{K}.$$

The operator Ψ is then denoted as

$$\Psi = \int_\Omega \Phi(w)\mathbb{P}(dw)$$

and is called the strong Bochner integral of Φ. If $\mathcal{K}$ and $\mathcal{H}$ are separable then $\|\Phi(.)\|$ is a measurable function and

$$\|\Psi\| \leq \int_\Omega \|\Phi(w)\|\mathbb{P}(dw).$$

It is obvious that the function $M(.)$ defined by (1.11) is Bochner integrable over arbitrary finite interval $[0, T]$ and that

$$\left\| \int_0^T M(t)dt \right\| \leq \int_0^T \|M(t)\|dt.$$

If we restrict our investigation to smaller spaces-the space $L_1(\mathcal{K}, \mathcal{H})$ of all nuclear operators from $\mathcal{K}$ into $\mathcal{H}$ or the space $L_2(\mathcal{K}, \mathcal{H})$ of all Hilbert-Schmidt operators from $\mathcal{K}$ into $\mathcal{H}$-then the non measurability problem mentioned above does not appear. See Appendix A for basic definitions and properties of nuclear and Hilbert-Schmidt operators. This is because the spaces $L_1(\mathcal{K}, \mathcal{H})$ and $L_2(\mathcal{K}, \mathcal{H})$ are separable Banach spaces ($L_2(\mathcal{K}, \mathcal{H})$ is a Hilbert space). It is useful to note that $L_1(\mathcal{K}, \mathcal{H})$ and $L_2(\mathcal{K}, \mathcal{H})$ are strongly measurable subsets of $L(\mathcal{K}, \mathcal{H})$.

We will need the following result on measurable decomposition of a $L_1(\mathcal{K}, \mathcal{K}) = L_1(\mathcal{K})$ valued random variable.

Proposition 1.4.1. *([26]). Let $\mathcal{K}$ be a separable Hilbert space and assume that Φ is an $(L_1(\mathcal{K}), \mathscr{B}(L_1(\mathcal{K})))$ random variable on $(\Omega, \mathcal{F})$ such that $\Phi(w)$ is a nonnegative symmetric operator for all $w \in \Omega$. Then there exists a decreasing sequence $\{\lambda_n\}$ of nonnegative random variables and a sequence $\{g_n\}$ of $\mathcal{K}$-valued random variables such that* [1]

$$\Phi(w) = \sum_{n=1}^{\infty} \lambda_n(w) g_n(w) \otimes g_n(w),$$

for $w \in \Omega$. Moreover the sequences $\{\lambda_n\}$ and $\{g_n\}$ can be chosen in such a way that

$$|g_n(w)| = \begin{cases} 1 & if\ \lambda_n(w) > 0 \\ 0 & if\ \lambda_n(w) = 0 \end{cases} \tag{1.12}$$

and

$$\langle g_n(w), g_m(w) \rangle = 0, \ \ \forall n \neq m \text{ and } \forall w \in \Omega. \tag{1.13}$$

Note that, since for each w the operator $\Phi(w)$ is compact and nonnegative, there exists a decreasing sequence $\{\lambda_n(w)\}$ of real numbers and a sequence $\{g_n(w)\}$ in $\mathcal{K}$ such that (1.12 − 1.13) hold.

If $X, Y \in L_2(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H})$ and $\mathcal{H}$ is a Hilbert space, with inner product $\langle ., . \rangle$, we define the covariance operator of X and the correlation operator of (X, Y) by the formulae

$$Cov(X) = \mathbb{E}(X - \mathbb{E}(X)) \otimes (X - \mathbb{E}(X))),$$

and

$$Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}(X)) \otimes (Y - \mathbb{E}(Y))].$$

$Cov(X)$ is a symmetric positive and nuclear operator (see Appendix A) and

$$Tr[Cov(X)] = \mathbb{E}(|X - E(X)|^2).$$

In fact if $\{e_k\}$ is a complete orthonormal basis in $\mathcal{H}$ and, for simplicity, $\mathbb{E}(X) = 0$, we have

$$\begin{aligned} Tr[Cov(X)] &= \sum_{h=1}^{\infty} \langle Cov(X) e_h, e_h \rangle \\ &+ \sum_{h=1}^{\infty} \int_{\Omega} |\langle X(w), e_h \rangle|^2 \mathbb{P}(dw) = \mathbb{E}|X|^2. \end{aligned}$$

1.4.2 Probability measures on Hilbert spaces

A probability measure μ on $(\mathcal{H}, \mathscr{B}(\mathcal{H}))$ is called Gaussian if for arbitrary $h \in \mathcal{H}$ there exist $m \in \mathbb{R}^1$, $q \geq 0$ such that,

$$\mu(\{x \in \mathcal{H}; \langle h, x \rangle \in A\}) = \mathscr{N}(m, q)(A), \ \forall A \in \mathcal{B}(\mathbb{R}^1).$$

In particular, if μ is Gaussian, the following functionals,

$$\mathcal{H} \to \mathbb{R}^1, \ h \to \int_H \langle h, x \rangle \mu(dx), \tag{1.14}$$

[1] For arbitrary $a, b \in \mathcal{H}$ we denote by $a \otimes b$ the linear operator defined by $(a \otimes b)h = a\langle b, h \rangle$, $h \in \mathcal{H}$.

$$\mathcal{H} \times \mathcal{H} \to \mathbb{R}^1, \ (h_1, h_2) \to \int_H \langle h_1, x\rangle\langle h_2, x\rangle \mu(dx), \tag{1.15}$$

are well defined. We show now that they are continuous. We need a lemma on general probability measures.

Lemma 1.4.1. *([26]). Let ν be a probability measure on $(\mathcal{H}, \mathscr{B}(\mathcal{H}))$. Assume that for some $k \in N$*

$$\int_H |\langle z, x\rangle|^k \nu(dx) < +\infty, \ \forall z \in \mathcal{H},$$

then there exists a constant $c > 0$ such that

$$\left| \int_H \langle h_1, x\rangle ... \langle h_k, x\rangle \nu(dx) \right| \leq c|h_1|...|h_k|, \ h_1...h_k \in \mathcal{H}.$$

For a proof, see, e.g., Da Prato and Zabczyk [26].

It follows from the lemma that if μ is Gaussian, then there exist an element $m \in H$ and a linear operator Q, such that

$$\int_H \langle h, x\rangle \mu(dx) = \langle m, h\rangle, \ \forall h \in \mathcal{H}, \tag{1.16}$$

$$\int_H \langle h_1, x - m\rangle\langle h_2, x - m\rangle \mu(dx) = \langle Qh_1, h_2\rangle, \ \forall h_1, h_2 \in H. \tag{1.17}$$

The vector m is called the mean and Q is called the covariance operator of μ. It is clear that the operator Q is symmetric. Moreover, since

$$\langle Qh, h\rangle = \int_H \langle h, x - m\rangle^2 \mu(dx) \geq 0, \ h \in H,$$

it is also nonnegative. It follows from (1.16) – (1.17) that a Gaussian measure μ on $\mathcal{H}$ with mean m and covariance Q has the following characteristic functional

$$\widehat{\mu}(\lambda) = \int_H e^{i\langle \lambda, x\rangle} \mu(dx) = e^{i\langle \lambda, m\rangle - \frac{1}{2}\langle Q\lambda, \lambda\rangle}, \ \lambda \in \mathcal{H}.$$

It is therefore uniquely determined by m and Q. It is denoted by $\mathscr{N}(m, Q)$.

It turns out that the covariance operator has to be nuclear (see Appendix A for the definition and basic properties).

Proposition 1.4.2. *([26]). Let μ be a Gaussian probability measure with mean 0 and covariance Q. Then Q has finite trace.*

For a proof, see, e.g., Da Prato and Zabczyk [26].

In the following considerations we denote by $\{e_k\}$ a complete orthonormal basis on $\mathcal{H}$ which diagonalizes Q, and by $\{\lambda_k\}$ the corresponding set of eigenvalues of Q. Moreover, for any $x \in \mathcal{H}$ we set $x_k = \langle x, e_k\rangle$, $k \in \mathbb{N}$. Note that the random variables $x_1, x_2, ..., x_n$ are independent, because the covariance matrix of the $\mathbb{R}^n$-valued random variable $(x_1, x_2, ..., x_n)$ is precisely $(\lambda_i \delta_{ij})$.

Proposition 1.4.3. *([26]). Let Q be a positive, symmetric, trace class operator in $\mathcal{H}$ and let $m \in \mathcal{H}$. Then there exists a Gaussian measure in $\mathcal{H}$ with mean m and covariance Q.*

Proof. We refer the reader to [26].

1.4.3 Wiener Process and Stochastic Integrals in a Hilbert Space

Hilbert Space valued Wiener Processes

Assume that $\mathcal{K}$ is a separable Hilbert space, with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{K}}$, and W is a K-valued Wiener process. Then, for each $u \in \mathcal{K}$, the process

$$\langle W(t), u\rangle,\ t \geq 0,$$

is a real valued Wiener process. This implies in particular that $L(W(t))$ is a Gaussian measure with mean vector 0. Note also that for arbitrary $u, v \in \mathcal{K}, t, s \geq 0$,

$$E[\langle W(t), u\rangle\langle W(s), u\rangle] = t \wedge sE[\langle W(1), u\rangle^2]$$

and

$$E[\langle W(t), u\rangle\langle W(s), u\rangle] = t \wedge sE[\langle W(1), u\rangle\langle W(1), v\rangle] = t \wedge s\langle Qu, v\rangle,$$

where Q is the covariance operator of the Gaussian measure $L(W(1))$, see Subsection 1.4.2. The operator Q is of trace class and it completely characterizes distributions of W.

Let Q be a trace class nonnegative operator on a Hilbert space $\mathcal{K}$.

Definition 1.4.1. *A $\mathcal{K}$-valued stochastic process $W(t),\ t \geq 0$, is called a Q-Wiener process if*

1. $W(0) = 0$,
2. *W has continuous trajectories,*
3. *W has independent increments,*
4. $\mathscr{L}(W(t) - W(s)) = \mathscr{N}(0, (t-s)Q),\ \ t \geq s \geq 0.$

Note that there exists a complete orthonormal system $\{e_k\}$ in $\mathcal{K}$ and a bounded sequence of nonnegative real numbers $\{\lambda_k\}$ such that

$$Qe_k = \lambda_k e_k,\ k \in \mathbb{N}.$$

and

$$TrQ = \sum_{i=1}^{\infty} \lambda_i < \infty.$$

Proposition 1.4.4. *([26]). Assume that $W(t)$ is a Q-Wiener process. Then the following statements hold.*

(i) *W is a Gaussian process on $\mathcal{K}$ and*

$$\mathbb{E}(W(t)) = 0,\ \ Cov(W(t)) = tQ,\ t \geq 0.$$

(ii) *For arbitrary $t \geq 0$, W has the expansion*

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} w_j(t) e_j$$

where

$$w_j(t) = \frac{1}{\sqrt{\lambda_j}}\langle W(t), e_j\rangle,\ j \in \mathbb{N},$$

are real valued Brownian motions mutually independent on $(\Omega, \mathcal{F}, P)$.

For a proof, see, e.g., Da Prato and Zabczyk [26].

Proposition 1.4.5. *([26]). For any trace class symmetric nonnegative operator Q on a separable Hilbert space $\mathcal{K}$ there exists a Q-Wiener process $W(t)$, $t \geq 0$.*

For a proof, see, e.g., Da Prato and Zabczyk [26].

In case the time set is $\mathbb{R}, W$ can be obtained as follows: let $\{W_i(t), t \in \mathbb{R}\}, i = 1, 2,$ be independent $\mathcal{K}$-valued Q-Wiener processes, then

$$W(t) = \begin{cases} W_1(t), & if\, t \geq 0, \\ W_2(-t), & if\, t \leq 0. \end{cases}$$

is a Q-Wiener process with $\mathbb{R}$ as time parameter and with values in $\mathcal{K}$.

Stochastic Integrals in a Hilbert Space

In order to define stochastic integrals with respect to the Q-Wiener process W, we introduce the subset $\mathcal{K}_0 = Q^{\frac{1}{2}}\mathcal{K}$, which is a Hilbert space equipped with the norm

$$\|u\|_{\mathcal{K}_0} = \|Q^{\frac{1}{2}}u\|_{\mathcal{K}},\, u \in \mathcal{K}_0,$$

and define a proper space of operators

$$L_0^2 = L_0^2(\mathcal{K}_0, \mathcal{H}) = \{\psi \in L(\mathcal{K}_0, \mathcal{H}) : Tr[(\psi Q^{\frac{1}{2}})(\psi Q^{\frac{1}{2}})^*] < \infty\},$$

the space of all Hilbert-Schmidt operators from $\mathcal{K}_0$ into $\mathcal{H}$. It turns out that L_0^2 is a separable Hilbert space with norm

$$\|\psi\|^2_{L_0^2} = Tr[(\psi Q^{\frac{1}{2}})(\psi Q^{\frac{1}{2}})^*] \text{ for any } \psi \in L_0^2.$$

Clearly, for any bounded linear operator $\psi \in L(\mathcal{K}, \mathcal{H})$, this norm reduces to

$$\|\psi\|^2_{L_0^2} = Tr[\psi Q \psi^*].$$

For any $T \geq 0$, let $\Phi = \{\Phi(t), t \in [0, T]\}$, be an $\mathcal{F}_t$-adapted, L_0^2-valued process, and for any $t \in [0, T]$, define the following norm:

$$\|\Phi\|_t = \left\{\mathbb{E}\int_0^t Tr[(\Phi Q^{\frac{1}{2}})(\Phi Q^{\frac{1}{2}})^*]ds\right\}^{\frac{1}{2}}.$$

In general, we denote all L_0^2-valued predictable processes Φ such that $\|\Phi\|_T < \infty$ by $U^2([0, T], L_0^2)$. The stochastic integral $\int_0^t \Phi(s)dW(s) \in \mathcal{H}$ may be well defined for all $\Phi \in U^2([0, T], L_0^2)$ by

$$\int_0^t \Phi(s)dW(s) = L^2 - \lim_{n\to\infty} \sum_{j=0}^n \int_0^t \Phi(s)\sqrt{\lambda_j}e_j dw_j(s), t \in [0, T],$$

where W is the Q-Wiener process defined above.

Proposition 1.4.6. *For arbitrary $T > 0$, let $\Phi \in U^2([0, T], L_0^2)$. Then the stochastic integral $\int_0^t \Phi(s)dW(s)$ is a continuous, square integrable, $\mathcal{H}$-valued martingale on $[0, T]$ and*

$$\mathbb{E}\|\int_0^t \Phi(s)dW(s)\|^2_{\mathcal{H}} = \|\Phi\|_t^2,\, t \in [0, T]. \tag{1.18}$$

In fact, the stochastic integral $\int_0^t \Phi(s)dW(s)$, $t \geq 0$ may be extended to any L_0^2-valued adapted process Φ satisfying

$$\mathbb{P}\left\{w : \int_0^t \|\Phi(w,s)\|_{L_0^2}^2 ds < \infty,\ 0 \leq t \leq T\right\} = 1$$

Moreover, we may deduce the following generalized relation of (1.18):

$$\mathbb{E}\|\int_0^t \Phi(t)dW(s)\|_{\mathcal{H}}^2 \leq \mathbb{E}\int_0^t \|\Phi(s)\|_{L_0^2}^2 ds,\ 0 \leq t \leq T. \tag{1.19}$$

Note that the equality holds if the right-hand side of this inequality is finite.

The following proposition is a particular case of the Burkholder-Davis-Gundy inequality.

Proposition 1.4.7. *For any $p \geq 2$ and for arbitrary L_0^2-valued predictable process $\Phi(t), t \in [0,T]$, one has*

$$\mathbb{E}\left[\sup_{s\in[0,t]} \|\int_0^s \Phi(s)dW(s)\|^p\right] \leq C_p\mathbb{E}\left[\int_0^t \|\Phi(s)\|_{L_0^2}^2 ds\right]^{p/2}. \tag{1.20}$$

for some constant $C_p > 0$.

For a proof, see, e.g., Da Prato and Zabczyk [26].

1.4.4 Generalized Wiener processes and Stochastic Integrals in a Hilbert Space

Generalized Wiener processes on a Hilbert space

Let $W(t)$, $t \geq 0$, be a Wiener process on a Hilbert space $\mathcal{K}$ and let Q be its covariance operator. For each $a \in \mathcal{K}$ define a real valued Wiener process $W_a(t)$, $t \geq 0$, by the formula

$$W_a(t) = \langle a, W(t)\rangle,\ \ t \geq 0. \tag{1.21}$$

The transformation $a \to W_a$ is linear from $\mathcal{K}$ to the space of stochastic processes. Moreover it is continuous in the following sense:

$$t \geq 0,\ \{a_n\} \subset \mathcal{K},\ \lim_{n\to\infty} a_n = a \Rightarrow \lim_{n\to\infty} \mathbb{E}|W_a(t) - W_{a_n}(t)|^2 = 0. \tag{1.22}$$

Any linear transformation $a \to W_a$ whose values are real valued Wiener processes on $[0, +\infty)$ satisfying (1.22) is called a generalized Wiener process.

From this definition it follows that there exists a bilinear form $K(a,b)$, a, $b \in \mathcal{K}$, such that

$$E[W_a(t)W_b(s)] = t \wedge s\, K(a,b),\ \ t, s \geq 0,\ a,\ b \in \mathcal{K}. \tag{1.23}$$

Condition (1.22) easily implies that K is a continuous bilinear form in $\mathcal{K}$ and therefore there exists $Q \in L(\mathcal{K})$ such that

$$E[W_a(t)W_b(s)] = t \wedge s\langle Qa, b\rangle,\ \ t, s \geq 0,\ a,\ b \in \mathcal{K}. \tag{1.24}$$

The operator Q is self-adjoint and positive definite; we call it the covariance of the generalized Wiener process $a \to W_a$. If the covariance Q is the identity operator I then the generalized Wiener process is called a **cylindrical Wiener process** in $\mathcal{K}$.

Denote by $\mathcal{K}_0$ the image $Q^{1/2}(\mathcal{K})$ with the induced norm. We call $Q^{1/2}(\mathcal{K})$ the reproducing kernel (see Appendix B) of the generalized Wiener process $a \to W_a$.

It is easy to construct, for an arbitrary self-adjoint and positive definite operator Q, a generalized Wiener process $a \to W_a$ satisfying (1.24). Let in fact $\{e_j\}$ be a complete and orthonormal basis in $\mathcal{K}$ and $\{w_j\}$ a sequence of independent real valued standard Wiener processes. Define

$$W_a(t) = \sum_{j=1}^{\infty} \langle Q^{1/2} e_j, a \rangle w_j(t), \; t \geq 0, \; a \in \mathcal{K}. \tag{1.25}$$

Since

$$\sum_{j=1}^{\infty} |\langle Q^{1/2} e_j, a \rangle|^2 = |Q^{1/2} a|^2 < +\infty, \tag{1.26}$$

for each $a \in \mathcal{K}$ there exists a version of W_a which is a Wiener process. Since

$$E[W_a(t) W_b(s)] = (t \wedge s) \sum_{j=1}^{\infty} \langle Q^{1/2} e_j, a \rangle \langle Q^{1/2} e_j, b \rangle = (t \wedge s) \langle Qa, b \rangle, \tag{1.27}$$

the result follows.

Proposition 1.4.8. *([26]). Let $\mathcal{K}_1$ be a Hilbert space such that $\mathcal{K}_0 = Q^{1/2}(\mathcal{K})$ is embedded into $\mathcal{K}_1$ with a Hillbert-Schmidt embedding J. Then the formula*

$$W(t) = \sum_{j=1}^{\infty} Q^{1/2} e_j w_j(t), \; t \geq 0,$$

defines a $\mathcal{K}_1$-valued Wiener process. Moreover, if Q_1 is the covariance of W then the spaces $Q_1^{1/2}(\mathcal{K}_1)$ and $Q^{1/2}(\mathcal{K})$ are identical.

Proof. We refer the reader to [26].

Thus with some abuse of language we can say that an arbitrary generalized Wiener process on $\mathcal{K}$ is a classical Wiener process in some larger Hilbert space $\mathcal{K}_1$.

Stochastic integral for generalized Wiener processes

The construction of the stochastic integral required the assumption that Q was a nuclear operator; only then does the Q-Wiener process have values in $\mathcal{K}$. We can, however, easily extend the definition of the integral to the case of generalized Wiener processes with a covariance operator Q not necessarily of trace class. One can perform this in several equivalent ways. The following simple proposition plays an important role in these extensions. As before we denote by $\mathcal{K}_0 = Q^{1/2}(\mathcal{K})$ (with the induced norm $\|u\|_0 = \|Q^{-1/2}(u)\|, \; u \in \mathcal{K}_0$) the reproducing kernel of W (see Appendix B). We shall use again the notation,

$$L_0^2 = L^2(\mathcal{K}_0, \mathcal{H}).$$

Proposition 1.4.9. *([26]). Assume that Z is a $\mathcal{K}$-valued random variable with mean 0 and covariance Q and that R is a Hilbert-Schmidt operator from $\mathcal{K}_0$ into $\mathcal{H}$. If $\{R_n\} \subset L_2(\mathcal{K}_0, \mathcal{H})$ is such that*

$$\lim_{n \to \infty} \|R - R_n\|_{L_2(\mathcal{K}_0, \mathcal{H})} = 0,$$

there exists a random variable $R\,Z$ such that

$$\lim_{n \to \infty} \mathbb{E} \|R\,Z - R_n\,Z\|^2_{L_2(\mathcal{K}_0, \mathcal{H})} = 0.$$

$R\,Z$ is independent of the sequence $\{R_n\}$.

Proof. We refer the reader to [26].

If now $W_a,\ a \in \mathcal{K}$, is a generalized Wiener process with covariance Q, then by Proposition 1.4.8 there exists a sequence $\{w_j\}$ of independent Wiener processes and an orthonormal basis $\{ej\}$ in $\mathcal{K}$ such that

$$W_a(t) = \sum_{j=1}^{\infty} \langle a, Q^{1/2} e_j \rangle w_j(t),\ a \in \mathcal{K}, t \geq 0.$$

Moreover the formula

$$W(t) = \sum_{j=1}^{\infty} Q^{1/2} e_j w_j(t),\ t \geq 0,$$

defines a Wiener process on any Hilbert space $\mathcal{K}_1 \supset \mathcal{K}_0$ with Hilbert-Schmidt embedding. If $\Phi \in L_0^2$ then the random variables $\Phi W(t),\ t \geq 0$, described in Proposition 1.4.9, are given by the formula,

$$\Phi W(t) = \sum_{j=1}^{\infty} \Phi Q^{1/2} e_j w_j(t),\ t \geq 0,$$

and in particular we have

$$W_a(t) = \langle a, W(t) \rangle,\ t \geq 0.$$

Thus the construction of the stochastic integral

$$\int_0^t \Phi(s) dW(s),\ t \geq 0,$$

can be done as in the case when $TrQ < +\infty$. It is enough to take into account that random variables of the form

$$\Phi_{t_j}(W_{t_{j+1}} - W_{t_j}),$$

are defined in a unique way provided $\Phi_{t_j} \in L_0^2$. The basic formula

$$\mathbb{E}\left| \int_0^t \Phi(s) dW(s) \right|^2 = \mathbb{E}\left(\int_0^t \|\Phi(s)\|^2_{L_0^2} ds \right),\ t \geq 0, \tag{1.28}$$

remains the same.

Equivalently one can repeat the definition of the stochastic integral for a $\mathcal{K}_1$-valued Wiener process W determined by $W_a,\ a \in \mathcal{K}$. Again, the space of integrands and formula (1.28) remain the same.

1.4.5 Stochastic Fubini theorem

Let (E, ξ) be a measurable space and let $\Phi(t, w, x) \to \varphi(t, w, x)$ be a measurable mapping from $(\Omega_T \times E, \mathscr{P}_T \times \mathscr{B}(E))$ into $(L_0^2, \mathscr{B}(L_0^2))$. Thus, in particular, for arbitrary $x \in E$, $\Phi(\cdot, \cdot, x)$ is a predictable L_0^2-valued process. Let in addition μ be a finite positive measure on (E, ξ).

The following stochastic version of the Fubini theorem will be frequently used. It generalizes a similar result due to [22], see [49] for the finite dimensional case. (Sometimes to simplify notation we will not indicate the dependence of Φ on w)

Theorem 1.4.1. *([26]). Assume that (E, ξ) is a measurable space and let*

$$\Phi : (t, w, x) \to \Phi(t, w, x)$$

be a measurable mapping from $(\Omega_T \times E, \mathscr{P}_T \times \mathscr{B}(E))$ *into* $(L_0^2, \mathscr{B}(L_0^2))$. *Assume moreover that*

$$\int_E \|\Phi(\cdot,\cdot,x)\|_T \mu(dx) < +\infty$$

then $\mathcal{P}$*-a.s.*

$$\int_E \left[\int_0^T \Phi(t,x)dW(t)\right]\mu(dx) = \int_0^T \left[\int_E \Phi(t,x)\mu(dx)\right]dW(t).$$

Proof. We refer the reader to [26].

1.5 Elements of Semigroup Theory

In this section we review the fundamentals of semigroup theory and refer the reader to [35].

Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be Banach spaces. Denote by $L(X,Y)$ the family of bounded linear operators from X to Y. $L(X,Y)$ becomes a Banach space when equipped with the norm

$$\|T\|_{L(X,Y)} = \sup_{x\in X, \|x\|_X=1} \|Tx\|_Y, \ \ T \in L(X,Y).$$

For brevity, $L(X)$ will denote the Banach space of bounded linear operators on X.

The identity operator on X is denoted by I.

Let X^* denote the dual space of all bounded linear functionals x^* on X. X^* is again a Banach space under the supremum norm

$$\|x^*\|_{X^*} = \sup_{x\in X, \|x\|_X=1} |\langle x, x^*\rangle|,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality on $X \times X^*$.

For $T \in L(X,Y)$, the adjoint operator $T^* \in L(Y^*, X^*)$ is defined by

$$\langle x, T^*y^*\rangle = \langle Tx, y^*\rangle, \ x \in X, \ y^* \in Y^*.$$

Definition 1.5.1. *A family* $S(t) \in L(X), t \geq 0$, *of bounded linear operators on a Banach space* X *is called a strongly continuous semigroup (or a* C_0*-semigroup, for short) if*

1. $S(0) = I$,
2. *(Semigroup property)* $S(t+s) = S(t)S(s)$ *for every* $t, s \geq 0$,
3. *(Strong continuity property)* $\lim_{t\to 0^+} S(t)x = x$ *for every* $x \in X$.

Let $S(t)$ be C_0-semigroup on a Banach space X. Then, there exist constants $\alpha \geq 0$ and $M \geq 1$ such that

$$\|S(t)\|_{L(X)} \leq Me^{\alpha t}, \ \ t \geq 0 \tag{1.29}$$

- If $M = 1$, then $S(t)$ is called a pseudo-contraction semigroup.
- If $\alpha = 0$, then $S(t)$ is called uniformly bounded, and if $\alpha = 0$ and $M = 1$ (i.e,$\|S(t)\|_{L(X)} \leq 1$), then $S(t)$ is called a semigroup of contractions.
- If for every $x \in X$, the mapping $t \to S(t)x$ is differentiable for $t > 0$, then $S(t)$ is called a differentiable semigroup.

- A semigroup of linear operators $\{S(t),\, t \geq 0\}$ is called compact if the operators $S(t)$, $t > 0$, are compact

Definition 1.5.2. *A semigroup $S(t)$ on x is called analytic whenever $t \to T(t)$ is analytic in $(0, \infty)$ with values in $L(x)$.*

Definition 1.5.3. *Let $S(t)$ be a C_0-semigroup on a Banach space X. The linear operator A with domain*

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{t \to 0^+} \frac{S(t)x - x}{t} \; exists \right\}$$

defined by

$$Ax = \lim_{t \to 0^+} \frac{S(t)x - x}{t}$$

is called the infinitesimal generator of the semigroup $S(t)$.

Theorem 1.5.1. *Let A be an infinitesimal generator of a C_0-semigroup $S(t)$ on a Banach space X. Then*

1. *For $x \in X$,*
$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S(t)xds = S(t)x.$$
2. *For $x \in \mathcal{D}(A)$, $S(t)x \in D(A)$ and*
$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax$$
3. *For $x \in X$, $\int_0^t S(s)xds \in \mathcal{D}(A)$, and*
$$A\left(\int_0^t S(s)xds \right) = S(t)x - x.$$
4. *If $S(t)$ is differentiable then for $n = 1, 2, ..., S(t) : X \to \mathcal{D}(A^n)$ and*
$$S^{(n)}(t) = A^n S(t) \in L(X).$$
5. *If $S(t)$ is compact then $S(t)$ is continuous in the operator topology for $t > 0$, i.e.,*
$$\lim_{s \to t, s, t > 0} \|S(s) - S(t)\|_{L(X)} = 0.$$
6. *For $x \in \mathcal{D}(A)$,*
$$S(t)x - S(s)x = \int_s^t S(u)Axdu = \int_s^t AS(u)xdu$$
7. *$\mathcal{D}(A)$ is dense in X, and A is a closed linear operator.*
8. *The intersection $\bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$ is dense in X.*
9. *Let X be a reflexive Banach space. Then the adjoint semigroup $S(t)^*$ of $S(t)$ is a C_0-semigroup whose infinitesimal generator is A^*, the adjoint of A.*

If $X = \mathcal{H}$, a real separable Hilbert space, then for $h \in \mathcal{H}$, define the graph norm

$$\|h\|_{\mathcal{D}(A)} = (\|h\|^2_{\mathcal{H}} + \|Ah\|^2_{\mathcal{H}})^{\frac{1}{2}}.$$

Then $(\mathcal{D}(A), \|\cdot\|_{\mathcal{D}(A)})$ is a real separable Hilbert space.

Let $\mathscr{B}(\mathcal{H})$ denote the Borel σ-field on $\mathcal{H}$. Then $\mathcal{D}(A) \in \mathscr{B}(\mathcal{H})$, and

$$A : (\mathcal{D}(A), \mathscr{B}(H)|\mathcal{D}(A)) \to (\mathcal{H}, \mathscr{B}(\mathcal{H})).$$

Consequently, the restricted Borel σ-field $\mathscr{B}(\mathcal{H})|\mathcal{D}(A)$ coincides with the Borel σ-field on the Hilbert space $(\mathcal{D}(A), \|\cdot\|_{\mathcal{D}(A)})$, and measurability of $\mathcal{D}(A)$-valued functions can be understood with respect to either Borel σ-field.

Theorem 1.1. *([35]). Let $f : [0, T] \to \mathcal{D}(A)$ be measurable, and let $\int_0^t \|f(s)\|_{\mathcal{D}(A)} < \infty$. Then*

$$\int_0^t f(s)ds \in \mathcal{D}(A) \text{ and } \int_0^t Af(s)ds = A\int_0^t f(s)ds$$

Definition 1.5.4. *([35]). The resolvent set $\rho(A)$ of a closed linear operator A on a Banach space X is the set of all complex numbers λ for which $\lambda I - A$ has a bounded inverse, i.e., the operator $(\lambda I - A)^{-1} \in L(X)$. The family of bounded linear operators*

$$R(\lambda, A) = (\lambda I - A)^{-1}, \quad \lambda \in \rho(A),$$

is called the resolvent of A

We note that $R(\lambda, A)$ is a one-to-one transformation of X onto $\mathcal{D}(A)$, i.e.,

$$(\lambda I - A)R(\lambda, A)x = x, \quad x \in X,$$
$$R(\lambda, A)(\lambda I - A)x = x, \quad x \in \mathcal{D}(A).$$

In particular,

$$AR(\lambda, A)x = R(\lambda, A)Ax, \quad x \in \mathcal{D}(A).$$

In addition, we have the following commutativity property:

$$R(\lambda_1, A)R(\lambda_2, A) = R(\lambda_2, A)R(\lambda_1, A), \quad \lambda_1, \lambda_2 \in \rho(A).$$

Proposition 1.5.1. *([35]). Let $S(t)$ be a C_0-semigroup with infinitesimal generator A on a Banach space X. If $\alpha_0 = \lim_{t\to\infty} t^{-1} \ln \|S(t)\|_{L(X)}$, then any real number $\lambda > \alpha_0$ belongs to the resolvent set $\rho(A)$, and*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t)x dt, \quad x \in X.$$

Furthermore, for each $x \in X$,

$$\lim_{\lambda\to\infty} \|\lambda R(\lambda, A)x - x\|_X = 0.$$

Theorem 1.2. *(Hille-Yosida) ([35]). Let $A : \mathcal{D}(A) \subset X \to X$ be a linear operator on a Banach pace X. Necessary and suffcient conditions for A to generate a C_0 semigroup $S(t)$ are*

(i) A is a closed operator and $\overline{\mathcal{D}(A)} = X$;

(ii) there exist real numbers M and α such that for every $\lambda > \alpha$, $\lambda \in \rho(A)$ (the resolvent set) and

$$\|(R(\lambda, A))^r\|_{L(X)} \le M(\lambda - \alpha)^{-r}, \quad r = 1, 2, ...$$

In this case $\|S(t)\|_{L(X)} \le Me^{\alpha t}$, $t \ge 0$.

Chapter 2

P-th Mean Almost Periodic Random Functions

The concept of almost periodicity is important in probability especially for investigations on stochastic processes. The interest in such a notion lies in its significance and applications arising in engineering, statistics, etc.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In this chapter, we introduce and develop the notion of p-th mean almost periodicity. Among others, it will be shown that each p-th mean almost periodic process is uniformly continuous and stochastically bounded [81]. Furthermore, the collection of all p-th mean almost periodic processes is a Banach space when it is equipped with its natural norm. Moreover, we also present two composition results for p-th mean almost periodic processes (Theorems (2.2.1) and (2.2.2)).

2.1 Almost Periodic Functions

First of all, let us mention that most of the material on almost periodic functions here is taken from the book of Diagana [30]. Obviously, there is a vast literature on almost periodic functions. Here we chose, for convenience, to use the concept of almost periodicity in the sense of H. Bohr (1887-1951), which is equivalent to the other classical definitions. For more almost periodic functions, we refer to the landmark books by Bohr [19], Corduneanu [23], and fink [32].

2.1.1 Basic Definitions

If $(\mathfrak{B}, \|\cdot\|)$ is a Banach space, then $\mathcal{C}(\mathbb{R}, \mathfrak{B})$ will stand for the collection of continuous functions from $\mathbb{R}$ in $\mathfrak{B}$. $\mathcal{BC}(\mathbb{R}, \mathfrak{B})$, is the space of all bounded continuous functions from $\mathbb{R}$ into $\mathfrak{B}$, will be equipped with the sup norm. Similarly, $\mathcal{BC}(\mathbb{R} \times \mathfrak{B})$ denotes the space of all bounded continuous functions from $\mathbb{R} \times \mathfrak{B}$ in $\mathfrak{B}$.

Definition 2.1.1. *A function $f \in \mathcal{C}(\mathbb{R}, \mathfrak{B})$ is called (Bohr) almost periodic if for each $\epsilon > 0$, there exists $T_0(\epsilon) > 0$ such that every interval of length $T_0(\epsilon)$ contains a number τ with the following property:*

$$\|f(t+\tau) - f(t)\| < \epsilon \text{ for each } t \in \mathbb{R}.$$

The number τ above is then called an ϵ-translation number of f, and the collection of such functions will be denoted $\mathcal{AP}(\mathfrak{B})$. It is well-known that if $f \in \mathcal{AP}(\mathfrak{B})$, then its mean defined by

$$\mathcal{M}(f) := \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} f(t)dt$$

exists [19]. Consequently, for every $\lambda \in \mathbb{R}$, the following limit

$$a(f,\lambda) := \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} f(t)e^{-i\lambda t}dt$$

exists and is called the Bohr transform of f.

The set defined by

$$\sigma_b(f) := \{\lambda \in \mathbb{R} : a(f,\lambda) \neq 0\}$$

is called the Bohr spectrum of f [17].

We also have the following properties of the mean ([17]):

Proposition 2.1.1. *([78]). Let $f, g : \mathbb{R} \to \mathbb{C}$ be almost periodic functions and let $\alpha \in \mathbb{C}$. Then*

(i) $\mathcal{M}(\overline{f}(t)) = \overline{\mathcal{M}(f(t))}$;

(ii) $\mathcal{M}(\alpha f(t)) = \alpha\mathcal{M}(f(t))$;

(iii) $\mathcal{M}(f(t)) \geq 0$ *whenever* $f \geq 0$;

(iv) $\mathcal{M}(f(t) + g(t)) = \mathcal{M}(f(t)) + \mathcal{M}(g(t))$.

Furthermore, if $(f_n(t))$ is a uniformly convergent sequence of almost periodic functions which converges to $f(t)$, then

$$\lim_{n\to\infty} \mathcal{M}(f_n(t)) = \mathcal{M}(f(t)).$$

Example 2.1.1. **(i)** Each periodic function $\varphi : \mathbb{R} \to \mathfrak{B}$ is almost periodic.

(ii) The function $f_\alpha(t) = \sin t + \sin t\alpha$ where $\alpha \in \mathbb{R} - \mathbb{Q}$, is a classical example of an almost periodic function on $\mathbb{R}$, which is not periodic.

Remark 2.1.1. Let $f, g : \mathbb{R} \to \mathfrak{B}$ be almost periodic functions and let $\alpha \in \mathbb{R}$. Then the following hold:

(i) $f + g$ is almost periodic; if f, g are $\mathbb{C}$-valued, then $f.g$ is also almost periodic.

(ii) $t \mapsto f(t+\alpha)$, $t \mapsto f(\alpha t)$, and $t \mapsto \alpha f(t)$ are almost periodic.

(iii) Each almost periodic function is bounded.

2.1.2 Properties of Almost Periodic Functions

1. If $f : R \to \mathfrak{B}$ is almost periodic, then f is uniformly continuous in $t \in \mathbb{R}$. Moreover, the range $R(f) = \{f(t) : t \in \mathbb{R}\}$ is precompact in $\mathfrak{B}$.

2. Let $f \in \mathcal{AP}(\mathbb{R})$. If $g \in L^1(\mathbb{R})$, then $f * g$, the convolution of f with g on $\mathbb{R}$, is almost periodic.

3. If $f, g : \mathbb{R} \to \mathbb{C}$ are almost periodic functions and if there exists a constant $m > 0$ such that

$$m \leq |g(t)|$$

for each $t \in \mathbb{R}$, then $(f/g)(t) = f(t)/g(t)$ is almost periodic.

4. Let $(f_n(t))_{n\in\mathbb{N}}$ be a sequence of almost periodic functions such that $f_n(t)$ converges $f(t)$ uniformly in $t \in \mathbb{R}$. Then f is almost periodic.

5. Let f be an almost periodic function such that f' is uniformly continuous on $\mathbb{R}$, then f' is also almost periodic.

Theorem 2.1.1. *([17]). (Bochner's Criterion) A function $f : \mathbb{R} \to \mathfrak{B}$ is almost periodic if and only if for every sequence of real numbers $(s_n)_{n\in\mathbb{N}}$ there exists a subsequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\{f(t+\sigma_n)\}_{n\in\mathbb{N}}$ converges uniformly in $t \in R$.*

Definition 2.1.2. *([17]). A normed vector space $(\mathfrak{B}, \|\cdot\|)$ is said to be uniformly convex if for every $0 < \varepsilon < 2$ there exists a number $\delta(\varepsilon) > 0$ such that if $x, y \in \mathfrak{B}$ satisfy*

$$\|x\| = \|y\| = 1 \, and \, \|x-y\| \geq \varepsilon,$$

then $\|(x+y)/2\| \leq 1 - \delta(\varepsilon)$.

Remark 2.1.2. (a) $\mathbb{R}^n$ equipped with the Euclidean norm is uniformly convex.

(b) Hilbert spaces are uniformly convex.

Proposition 2.1.2. *Suppose that the Banach space $\mathfrak{B}$ is uniformly convex. If $f : \mathbb{R} \to \mathfrak{B}$ is almost periodic, then its antiderivative*

$$F(t) = \int_0^t f(\sigma) d\sigma$$

is almost periodic if and only if it is bounded in $\mathfrak{B}$, i.e., $\sup_{t\in\mathbb{R}} \|F(t)\| < \infty$

Definition 2.1.3. *([17]). A function $F \in \mathcal{BC}(\mathbb{R}\times\mathfrak{B}), (t,x) \mapsto F(t,x)$ is called almost periodic in $t \in R$ uniformly in $x \in \Gamma(\Gamma \subset \mathfrak{B}$ being a compact subset) if for each $\varepsilon > 0$ there exists $T_0(\varepsilon) > 0$ such that every interval of length $T_0(\varepsilon) > 0$ contains a number τ with the following property:*

$$\|F(t+\tau, x) - F(t,x)\| < \varepsilon, \forall x \in \Gamma, \forall t \in \mathbb{R}.$$

Here again, the number τ above is called an ε-translation number of F, and the class of such functions will be denoted $\mathcal{AP}(\mathbb{R}\times\mathfrak{B})$.

Proposition 2.1.3. *Let $(\mathfrak{B}, \|\cdot\|)$ and $(\mathfrak{B}', \|\cdot\|')$ be two Banach spaces over the same field F. Let $f : \mathbb{R}\times\mathfrak{B} \to \mathfrak{B}', (t,x) \to f(t,x)$ be almost periodic in $t \in R$ uniformly in $x \in \mathfrak{B}$. Suppose that f is Lipschitz in $x \in \mathfrak{B}$ uniformly in $t \in \mathbb{R}$, i.e., there exists $L \geq 0$ such that*

$$\|f(t,x) - F(t,y)\|' \leq L.\|x-y\|, \forall x, y \in \mathfrak{B}, t \in \mathbb{R}.$$

If $\phi : \mathbb{R} \to \mathfrak{B}$ is almost periodic, then the function $h(t) = f(t, \phi(t)) : \mathbb{R} \to \mathfrak{B}'$ is also almost periodic.

Definition 2.1.4. *A function $f \in \mathcal{BC}(\mathbb{R}, \mathfrak{B})$ is called (Bochner) almost periodic if for any sequence $(\sigma'_n)_{n\in\mathbb{N}}$ of real numbers there exists a subsequence $(\sigma_n)_{n\in\mathbb{N}}$ of $(\sigma'_n)_{n\in\mathbb{N}}$ such that the sequence of functions $(f(t+\sigma_n))_{n\in\mathbb{N}}$ converges uniformly in $t \in \mathbb{R}$.*

Theorem 2.1.2. *([17]). A function $f \in \mathcal{BC}(\mathbb{R}, \mathfrak{B})$ is Bohr almost periodic if and only if it is Bochner almost periodic.*

2.2 p-th Mean Almost Periodic Processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For $p \geq 2$ the spaces $L_p(\Omega; \mathfrak{B})$ and $\mathcal{BUC}(\mathbb{R}; L_p(\Omega; \mathfrak{B}))$ are Banach spaces when they are equipped with their respective norms $\| \cdot \|_{L_p(\Omega;\mathfrak{B})}$ and $\| \cdot \|_\infty$.

Definition 2.2.1. *A stochastic process $x : \mathbb{R} \to L_p(\Omega; \mathfrak{B})$ is said to be continuous whenever*

$$\lim_{t \to \to s} \mathbb{E}\|x(t) - x(s)\|^p = 0.$$

Definition 2.2.2. *A continuous stochastic process $x : \mathbb{R} \to L_p(\Omega; \mathfrak{B})$ is said to be stochastically bounded whenever*

$$\lim_{N \to \to \infty} \sup_{t \in \mathbb{R}} \mathbb{P}\{\|x(t)\| > N\} = 0.$$

Definition 2.2.3. *A stochastic process $x : \mathbb{R} \to L_p(\Omega; \mathfrak{B})$ is said to be p-th mean almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which*

$$\sup_{t \in \mathbb{R}} \mathbb{E}\|x(t + \tau) - x(t)\|^p < \varepsilon$$

A continuous stochastic process x, which is 2-nd mean almost periodic will be called square-mean almost periodic.

Like for classical almost periodic functions, the number τ will be called an ε-translation of x.

The collection of all p-th mean almost periodic stochastic processes $x : \mathbb{R} \to L_p(\Omega; \mathfrak{B})$ will be denoted by $\mathcal{AP}(\mathbb{R}; L_p(\Omega; \mathfrak{B}))$.

The next lemma provides some properties of p-th mean almost periodic processes ([17]).

Lemma 2.2.1. *If X belongs to $\mathcal{AP}(\mathbb{R}; L_p(\Omega; \mathfrak{B}))$, then*

(i) *the mapping $t \to \mathbb{E}\|x(t)\|^p$ is uniformly continuous;*

(ii) *there exists a constant $M > 0$ such that $\mathbb{E}\|x(t)\|^p \leq M$, for each $t \in R$;*

(iii) *X is stochastically bounded.*

Let $\mathcal{CUB}(R; L_p(\Omega; \mathfrak{B}))$ denote the collection of all stochastic processes $x : \mathbb{R} \to L_p(\Omega; \mathfrak{B})$, which are continuous and uniformly bounded. It is then easy to check that $\mathcal{CUB}(R; L_p(\Omega; \mathfrak{B}))$ is a Banach space when it is equipped with the norm:

$$\|x\|_\infty = \sup_{t \in \mathbb{R}} (\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}}.$$

Lemma 2.2.2. *([17]). $\mathcal{AP}(\mathbb{R}; L_p(\Omega; \mathfrak{B})) \subset \mathcal{BUC}(\mathbb{R}; L_p(\Omega; \mathfrak{B}))$ is a closed subspace.*

In view of Lemma 2.2.2, it follows that the space $\mathcal{AP}(\mathbb{R}; L_p(\Omega; \mathfrak{B}))$ of p-th mean almost periodic processes equipped with the sup norm $\|.\|_\infty$ is a Banach space.

Let $(\mathfrak{B}_1, \| \cdot \|), (\mathfrak{B}_2, \| \cdot \|)$ be Banach spaces and let $L_p(\Omega; \mathfrak{B}_1)$ and $L_p(\Omega; \mathfrak{B}_2)$be their corresponding L_p-spaces, respectively.

Definition 2.2.4. *A function $F : \mathbb{R} \times L_p(\Omega; \mathfrak{B}_1) \to L_p(\Omega; \mathfrak{B}_2), (t, y) \to F(t, y)$, which is jointly continuous, is said to be p-th mean almost periodic in $t \in \mathbb{R}$ uniformly in $y \in K$ where $K \subset L_p(\Omega; \mathfrak{B}_1)$ is compact if for any $\varepsilon > 0$, there exists $l_\varepsilon(K) > 0$ such that any interval of length $l_\epsilon(K)$ contains at least a number τ for which*

$$\sup \mathbb{E}\|F(t + \tau, y) - F(t, y)\|_2^p < \varepsilon$$

for each stochastic process $y : \mathbb{R} \to K$.

2.2.1 Composition of p-th Mean Almost Periodic Processes

Theorem 2.2.1. *([17]). Let* $F : \mathbb{R} \times L_p(\Omega; \mathfrak{B}_1) \to L_p(\Omega; \mathfrak{B}_2), (t, y) \to F(t, y)$ *be a p-th mean almost periodic process in* $t \in \mathbb{R}$ *uniformly in* $y \in K$ *where* $K \subset L_p(\Omega; \mathfrak{B}_1)$ *is compact. Suppose that* F *is Lipschitzian in the following sense:*

$$\mathbb{E}\|F(t, y) - F(t, z)\|_p^2 \le M\mathbb{E}\|y - z\|_1^p$$

for all $y, z \in L_p(\Omega; \mathfrak{B}_1)$ *and for each* $t \in \mathbb{R}$*, where* $M > 0$*. Then for any p-th mean almost periodic process* $\Phi : \mathbb{R} \to L_p(\Omega; \mathfrak{B}_1)$*, the stochastic process* $t \to F(t, \Phi(t))$ *is p-th mean almost periodic.*

Proof. We refer the reader to [17].

Theorem 2.2.2. *([17]). Let* $F : \mathbb{R} \times L_p(\Omega; \mathfrak{B}_1) \to L_p(\Omega; \mathfrak{B}_2), (t, y) \to F(t, y)$ *be a p-th mean almost periodic process in* $t \in \mathbb{R}$ *uniformly in* $y \in K$ *where* $K \subset L_p(\Omega; \mathfrak{B}_1)$ *is any compact subset. Suppose that* $F(t, \cdot)$ *is uniformly continuous on bounded subsets* $K' \subset L_p(\Omega; \mathfrak{B}_1)$ *in the following sense: For all* $\varepsilon > 0$ *there exists* $\delta_\varepsilon > 0$ *such that* $x, y \in K'$ *and* $\mathbb{E}\|x - y\|_1^p < \delta_\varepsilon$*, then*

$$\mathbb{E}\|F(t, y) - F(t, z)\|_2^p \le \varepsilon, \forall t \in \mathbb{R}.$$

Then for any p-th mean almost periodic process $\Phi : \mathbb{R} \to L_p(\Omega; \mathfrak{B}_1)$*, the stochastic process* $t \to F(t, \Phi(t))$ *is p-th mean almost periodic.*

Proof. We refer the reader to [17].

Chapter 3

Stochastic Controlability

Control theory is an interdisciplinary branch of engineering and mathematics that deals with influence behavior of dynamical systems. Controllability is one of the fundamental concepts in mathematical control theory and plays an important role in both deterministic and stochastic control theories (Klamka, 1991; Klamka, 1993; Mahmudov, 2003; Mahmudov and Denker, 2000). Controllability is a qualitative property of dynamical control systems and it is of particular importance in control theory. Systematic study of controllability was started at the beginning of sixties in the last century, when the theory of controllability based on the state space description for both time-invariant and time-varying linear control systems was worked out.

Roughly speaking, controllability generally means, that it is possible to steer dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In the literature there are many different definitions of controllability for both linear (Klamka, 1991; Klamka, 1993; Mahmudov, 2001; Mahmudov and Denker, 2000) and nonlinear dynamic systems (Klamka, 2000; Mahmudov, 2002; Mahmudov, 2003; Mahmudov and Zorlu, 2003), which do depend on the class of dynamic control systems and the set of admissible controls (Klamka, 1991; Klamka, 1996). Therefore, for linear and nonlinear deterministic dynamic systems there exist many different necessary and sufficient conditions for global and local controllabilities (Klamka, 1991; Klamka, 1993; Klamka, 1996; Klamka, 2000).

In recent years various controllability problems for different types of linear semilinear and nonlinear dynamical systems have been considered in many publications and monographs. Moreover, it should be stressed, that the most literature in this direction has been mainly concerned with different controllability problems for dynamical systems with unconstrained controls and without delays in the state variables or in the controls.

For stochastic control systems (both linear and nonlinear) the situation is by far less satisfactory. In recent years the extensions of deterministic controllability concepts to stochastic control systems have been discussed only in a limited number of publications. In the papers (Bashirov and Kerimov, 1997; Bashirov and Mahmudov, 1999; Ehrhard and Kliemann, 1982; Mahmudov, 2001; Mahmudov and Denker, 2000; Zabczyk, 1991) different kinds of stochastic controllability were discussed for linear finite dimensional stationary and nonstationary control systems. The papers (Fernandez-Cara et al., 1999; Kim Jong Uhn, 2004; Mahmudov, 2001; Mahmudov, 2003) are devoted to a systematic study of approximate and exact stochastic controllability for linear infinite dimensional control systems defined in Hilbert spaces. Stochastic controllability for finite dimensional nonlinear stochastic systems was discussed in (Arapostathis et al., 2001; Balasubramaniam and Dauer, 2001; Mahmudov and Zorlu, 2003; Sunahara et al., 1974; Sunahara et al., 1975). Using the theory of bounded nonlinear operators and linear semi-groups, various different types of stochastic controllability for nonlinear infinite dimensional control systems defined in Hilbert spaces were considered in (Mahmudov, 2002; Mahmudov, 2003). In (Klamka and Socha, 1977; Klamka and Socha, 1980), Lyapunov techniques were used to formulate and prove sufficient conditions for stochastic controllability of nonlinear finite dimensional stochastic systems with point delays in state variables. Moreover, it should be pointed out that the functional analysis approach to stochastic controllability problems is also extensively discussed for both linear and nonlinear stochastic control systems in the papers (Fernandez-Cara et al., 1999; Kim Jong Uhn, 2004; Mahmudov, 2001; Mahmudov, 2002; Mahmudov, 2003; Subramaniam and Balachandran, 2002).

3.1 Controllability significance

Controllability plays an essential role in the development of modern mathematical control theory. There are various important relationships between controllability, stability and stabilizability of linear both finite-dimensional and infinite-dimensional control systems. Controllability is also strongly related with the theory of realization and so called minimal realization and canonical forms for linear time-invariant control systems such as the Kalmam canonical form, the Jordan canonical form or the Luenberger canonical form. It should be mentioned, that for many dynamical systems there exists a formal duality between the concepts of controllability and observability. Moreover, controllability is strongly connected with the minimum energy control problem for many classes of linear finite dimensional, infinite dimensional dynamical systems, and delayed systems both deterministic and stochastic.

Therefore, controllability criteria are useful in the following branches of mathematical control theory:

- Stabilizability conditions, canonical forms, minimum energy control and minimal realization for positive systems,
- Stabilizability conditions, canonical forms, minimum energy control and minimal realization for fractional systems,
- Minimum energy control problem for a wide class of stochastic systems with delays in control and state variables,
- Duality theorems, canonical forms and minimum energy control for infinite dimensional systems.

Controllability has many important applications not only in control theory and systems theory, but also in such areas as industrial and chemical process control, reactor control, control of electric bulk power systems, aerospce engineering and recently in quantum systems theory.

Systematic study of controllability was started at the beginning of the sixties in the 20-th century, when the theory of controllability based on the description in the form of state space for both time-invariant and time-varying linear control systems was worked out. The extensive list of these publications can be found for example in the monographs [62] and [63] or in the survey papers [64] and [69].

3.2 Nonlinear and semilinear dynamical systems

The last decades have seen a continually growing interest in controllability theory of dynamical systems. This is clearly related to the wide variety of theoretical results and possible applications. Up to the present time the problem of controllability for continuous-time and discrete-time linear dynamical systems has been extensively investigated in many papers (see e.g. [62, 63, 64, 111] for extensive list of references). However, this is not true for the nonlinear dynamical systems especially with different types of delays in control and state variables, and for nonlinear dynamical systems with constrained controls.

Similarly, only a few papers concern constrained controllability problems for continuous or discrete semi-linear dynamical systems. It should be pointed out, that in the proofs of controllability results for nonlinear and semi-linear dynamical systems linearization methods and generalization of open mapping theorem [57] are extensively used. The special case of nonlinear dynamical systems are semi-linear systems. Let us recall that semi-linear dynamical control systems contain linear and pure nonlinear parts in the differential state equations [70, 9, 95, 109].

3.3 Infinite-dimensional systems

Infinite-dimensional dynamical control systems plays a very important role in mathematical control theory. This class consists of both continuous-time systems and discrete-time systems [62, 63, 64, 69, 111]. Continuous-time infinite-dimensional systems include for example, a very wide class of so-called distributed parameter systems described by numerous types of partial differential equations defined in bounded or unbounded regions and with different boundary conditions.

For infinite-dimensional dynamical systems it is necessary to distinguish between the notions of approximate and exact controllability [62, 63].

Exact controllability: The exact controllability property is the possibility to steer the state of the system from any initial data to any target by choosing the control as a function of time in an appropriate way.

Approximate controllability: The approximate controllability property is the possibility to steer the state of the system from any initial data to a state arbitrarily close to a target by choosing a suitable control.

It follows directly from the fact that in infinite-dimensional spaces there exist linear subspaces which are not closed. On the other hand, for nonlinear dynamical systems there exist two fundamental concepts of controllability; namely local controllability and global controllability [62, 63]. Therefore, for nonlinear abstract dynamical systems defined in infinite-dimensional spaces the following four main kinds of controllability are considered: local approximate controllability, global approximate controllability, local exact controllability and global exact controllability [62, 63, 64, 69].

Controllability problems for finite-dimensional nonlinear dynamical systems and stochastic dynamical systems have been considered in many publications; see e.g. [62, 63, 69, 60], and [61], for review of the literature. However, there exist only a few papers on controllability problems for infinite-dimensional nonlinear systems [57].

Among the fundamental theoretical results, used in the proofs of the main results for nonlinear or semi-linear dynamical systems, the most important include:

- Generalized open mapping theorem,
- Spectral theory of linear unbounded operators,
- Linear semi-groups theory for bounded linear operators,

- Lie algebras and Lie groups,
- Fixed-point theorems such as Banach, Schauder, Schaefer and Nussbaum theorems,
- Theory of completely positive trace preserving maps,
- Mild solutions of abstract differential and evolution equations in Hilbert and Banach spaces.

3.3.1 Nonlinear neutral impulsive integro-differential evolution systems in Banach spaces

In various fields of science and engineering, many problems that are related to linear viscoelasticity, nonlinear elasticity and Newtonian or non-Newtonian fluid mechanics have mathematical models which are described by differential or integral equations or integro-differential equations. This part of the paper centers around the controllability for dynamical systems described by the integrodifferential models. Such systems are modelled by abstract delay differential equations. In particular abstract neutral differential equations arise in many areas of applied mathematics and, for this reason, this type of equation has been receiving much attention in recent years and they depend on the delays of state and their derivative. Related works of this kind can be found in [57].

The study of differential equations with traditional initial value problem has been extended in several directions. One emerging direction is to consider the impulsive initial conditions. The impulsive initial conditions are combinations of traditional initial value problems and short-term perturbations, whose duration can be negligible in comparison with the duration of the process. Several authors [57] have investigated controllability of the impulsive differential equations.

As far as the controllability problems associated with finite-dimensional systems modelled by ordinary differential equations are concerned, this theory has been extensively studied during the last decades. In the finite-dimensional context, a system is controllable if and only if the algebraic Kalman rank condition is satisfied. According to this property, when a system is controllable for some time, it is controllable for all time. But this is no longer true in the context of infinite-dimensional systems modelled by partial differential equations.

The large class of scientific and engineering problems modelled by partial differential and integrodifferential equations can be expressed in various forms of differential and integro-differential equations in abstract spaces. It is interesting to study the controllability problem for such models in Banach spaces. The controllability problem for first and second order nonlinear functional differential and integrodifferential systems in Banach spaces has been studied by many authors by using semigroup theory, cosine family of operators and various fixed point theorems for nonlinear operators [95] and [109] such as Banach theorem, Nussbaum theorem, Schaefer theorem, Schauder theorem, Monch theorem or Sadovski theorem.

In recent years, the theory of impulsive differential equations has provided a natural frame work for mathematical modelling of many real world phenomena, namely in control, biological and medical domains. In these models, the investigated simulating processes and phenomena are subjected to certain perturbations whose duration is negligible in comparison with the total duration of the process. Such perturbations can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. These process tend to be more suitably modelled by impulsive differential equations, which allow for discontinuities in the evolution of the state.

On the other hand, the concept of controllability is of great importance in mathematical control theory. The problem of controllability is to show the existence of a control function, which steers the solution of the system from its initial state to final state, where the initial and final states may vary over the entire space. Many authors have studied the controllability of nonlinear systems with and without impulses, see for instance [68, 64, 71, 65, 9, 10, 89, 109]

In recent years, significant progress has been made in the controllability of linear and nonlinear deterministic systems [11, 93, 75] and the nonlocal initial condition which in many cases, has much better effect in applications then the traditional initial condition. The nonlocal initial value problems can be more useful than the standard initial value problems to describe many physical phenomena of dynamical systems. It should be pointed out, that the study of Volterra-Fredholm integro-differential equations plays an important role for abstract formulation of many initial, boundary value problems of perturbed differential partial integro-differential equations.

Recently, many authors have studied about mixed type integro-differential systems without (or with) delay conditions. Moreover, controllability of impulsive functional differential systems with nonlocal conditions has been studied by using the measures of noncompactness and Monch fixed point theorem and some sufficient conditions for controllability have been established.

It should be mentioned, that without assuming the compactness of the evolution system the existence, uniqueness and continuous dependence of mild solutions for nonlinear mixed type integro-differential equations with finite delay and nonlocal conditions has been also established.. The results were obtained by using Banach fixed point theorem and semi-group theory. More recently, the existence of mild solutions for the nonlinear mixed type integro-differential functional evolution equations with nonlocal conditions was derived and the results were achieved by using Monch fixed point theorem and fixed point theory.

To the best of our knowledge, up to now no work reported on controllability of impulsive mixed Volterra- Fredholm functional integro-differential evolution differential system with a finite delay and nonlocal conditions.

3.3.2 Second order impulsive functional integro-differential systems in Banach spaces

Second order differential equations arise in many areas of science and technology whenever a relationship involving some continuously changing quantities and their rates of change are known. In particular, second order differential and integro-differential equations serve as an abstract formulation of many partial integro-differential equations which arise in problems connected with the transverse motion of an extensible beam, the vibration of hinged bars and many other physical phenomena. So it is quite significant to study the controllability problem for such systems in Banach spaces.

The concept of controllability involves the ability to move a system around in its entire configuration space using only certain admissible manipulations. The exact definition varies slightly within the framework of the type of models. In many cases, it is advantageous to treat the second order abstract differential equations directly rather than to convert them to first order systems. In the proofs of controllability criteria some basic ideas from the theory of cosine families of operators, which is related to the second order equations are often used.

Damping may be mathematically modelled as a force synchronous with the velocity of the object but opposite in direction to it. The occurrence of damped second order equations can be found in [93] and [75]. The branch of modern applied analysis known as "impulsive" differential equations furnishes a natural framework to mathematically describe some jumping processes.

The theory of impulsive integro-differential equations and their applications to the field of physics have formed a very active research topic since the theory provides a natural framework for mathematical modelling of many physical phenomena [10] and [95]. In spite of the great possibilities for applications, the theory of these equations has been developing rather slowly due to obstacles of theoretical and technical character. The study of the properties of their solutions has been of an ever growing interest.

Recently, most efforts have been focused on the problem of controllability for various kinds of impulsive systems using different approaches [34] and [44]. In neutral delay differential equations, the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. Neutral differential equations arise in many fields and they depend on the delays of state and its derivative. Related works of this kind of equation can be found in [89] and [45]. For the fundamental solution of second order evolution system, one can refer the paper [80].

3.4 Stochastic systems

Classical control theory generally is based on deterministic approaches. However, uncertainty is a fundamental characteristic of many real dynamical systems. Theory of stochastic dynamical systems is now a well-established topic of research, which is still in intensive development and offers many open problems. Important fields of application are economics problems, decision problems, statistical physics, epidemiology, insurance mathematics, reliability theory, risk theory and others methods based on stochastic equations. Stochastic modelling has been widely used to model the phenomena arising in many branches of science and industry such as biology, economics, mechanics, electronics and telecommunications. The inclusion of random effects in differential equations leads to several distinct classes of stochastic equations, for which the solution processes have differentiable or non-differentiable sample paths. Therefore, stochastic differential equations and their controllability require many different method of analysis.

The general theory of stochastic differential equations both finite-dimensional and infinite-dimensional and their applications to the field of physics and technique can be found in the many mathematical monographs and related papers. This theory formed a very active research topic since provides a natural framework for mathematical modelling of many physical phenomena.

Controllability, both for linear or nonlinear stochastic dynamical systems, has recently received the attention of many researchers and has been discussed in several papers and monographs, in which where many different sufficient or necessary and sufficient conditions for stochastic controllability were formulated and proved [59, 60, 61, 86, 87]. However, it should be pointed out that all these results were obtained only for unconstrained admissible controls, finite dimensional state space and without delays in state or control.

Stochastic controllability problems for stochastic infinite-dimensional semi-linear impulsive integro-differential dynamical systems with additive noise and with or without multiple time-varying point delays in the state variables are also discussed in the literature. The proofs of the main results are based on certain theorems taken from the theory of stochastic processes, linearization methods for stochastic dynamical systems, theory of semi-groups of linear operators, different fixed-point theorems as Banach, Schauder, Schaefer, or Nussbaum fixed-point theorems and on so-called generalized open mapping theorem [57].

3.5 Delayed systems

Up to the present time the problem of controllability in continuous and discrete time linear dynamical systems has been extensively investigated in many papers (see e.g. [62, 64, 65, 59, 66, 52]). However, this is not true for the nonlinear or semi-linear dynamical systems, especially with delays in control and with constrained controls. Only a few papers concern constrained controllability problems for continuous or discrete nonlinear or semi-linear dynamical systems with constrained controls [66, 67].

Dynamical systems with distributed [68] delays in control and state variable were also considered. Using some mapping theorems taken from functional analysis and linear approximation methods sufficient conditions for constrained relative and absolute controllability will be derived and proved.

Let us recall that semi-linear dynamical control systems with delays may contain different types of delays, both in pure linear and pure nonlinear parts, in the differential state equations. Sufficient conditions for constrained local relative controllability near the origin in a prescribed finite time interval for semi-linear dynamical systems with multiple variable point delays or distributed delays in the control and in the state variables, which nonlinear term is continuously differentiable near the origin are presented in [66] and [67].

In the above mentioned papers it is generally assumed that the values of admissible controls are in a given convex and closed cone with vertex at zero, or in a cone with nonemptyinterior. The proof of the main result are based on a so called generalized open mapping theorem presented in the paper [100]. Moreover, necessary and sufficient conditions for constrained global relative controllability of an associated linear dynamical system with multiple point delays in control are also discussed.

3.6 Positive systems

In recent years, the theory of positive dynamical systems has become a natural frame work for mathematical modelling of many real world phenomena, namely in control, biological and medical domains. Positive dynamical systems are of fundamental importance to numerous applications in different areas of science such as economics, biology, sociology and communication.

Positive dynamical systems both linear and nonlinear are dynamical systems with states, controls and outputs belonging to positive cones in linear spaces. Therefore, in fact positive dynamical systems are nonlinear systems. Among many important developments in control theory over last two decades, control theory of positive dynamical systems [52] has played an essential role.

Controllability, reachability and realization problems for finite dimensional positive both continuous-time and discretetime dynamical systems were discussed for example in monograph [52] and paper [58], using the results taken directly from the nonlinear functional analysis and especially from the theory of semi-groups of bounded operators and general theory of unbounded linear operators.

3.7 Fractional systems

The development of controllability theory both for continuous-time and discrete-time dynamical systems with fractional derivatives and fractional difference operators has seen considerable advances since the publication of papers [53]- [56] and monograph [92]. Although classic mathematical models are still very useful, large dynamical systems prompt the search for more refined mathematical models, which leads to better understanding and approximations of real processes.

The general theory of fractional differential equations and fractional impulsive integrodifferential equations and their applications to the field of physics and technique can be found in the monograph [92]. This theory formed a very active research topic since provides a natural framework for mathematical modelling of many physical phenomena. In particular, the fast development of this theory has allowed to solve a wide range of problems in mathematical modelling and simulation of certain kinds of dynamical systems in physics and electronics. Fractional derivative techniques provide useful exploratory tools, including the suggestion of new mathematical models and the validation of existing ones.

Mathematical fundamentals of fractional calculus and fractional differential and difference equations are given in the monographs [92], and in [57]. Most of the earliest work on controllability for fractional dynamical systems was related to linear continuous-time or discrete-time systems with limited applications of the real dynamical systems. In addition, the earliest theoretical work concerned time-invariant processes without delays in state variables or in control.

Using the results presented for linear fractional systems and applying linearization method the sufficient conditions for local controllability near the origin are formulated and proved in the paper [58]. Moreover, applying generalized open mapping theorem in Banach spaces [100] and linear semi-group theory in the paper [111] the sufficient conditions for approximation controllability in finite time with conically constrained admissible controls are formulated and proved.

Controllability problems for different types of dynamical systems require the application of numerous mathematical concepts and methods taken directly from differential geometry, functional analysis, topology, matrix analysis and theory of ordinary and partial differential equations and theory of difference equations. The state-space models of dynamical systems provides a robust and universal method for studying controllability of various classes of systems.

Finally, it should be stressed, that there are numerous open problems for controllability concepts for special types of dynamical systems. For example, it should be pointed out, that up to present time the most literature on controllability problems has been mainly concerned with unconstrained controls and without delays in the state variables or in the controls.

Chapter 4

Stochastic Evolutions Equations

4.1 Preliminaries And Basic Properties

In this section, we mention notations, definitions, lemmas and preliminary facts needed to establish our main results.

Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ and $(\mathcal{K}, \|\cdot\|_{\mathcal{K}})$ denote two real separable Hilbert spaces. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $\mathcal{H}$, $\mathcal{K}$ and $(\cdot,\cdot)$ to denote the inner product without any confusion. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). Let $\{e_j\}_{j=1}^{\infty}$ be a complete orthonormal basis of $\mathcal{K}$. Suppose that $W = (W_t)_{t\geq 0}$ is a cylindrical $\mathcal{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $Tr(Q) = \sum_{j=1}^{\infty} \lambda_j = \lambda < \infty$, which satisfies $Qe_j = \lambda_j e_j$. So, actually, $W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} w_j(t) e_j$, where $\{w_j(t)\}_{j=1}^{\infty}$ are mutually independent one-dimentional standard Wiener processes. We assume that $\mathcal{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by w and $\mathcal{F}_b = \mathcal{F}$.

Let $L(\mathcal{K}, \mathcal{H})$ denote the space of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$ equipped with the usual operator norm $\|\cdot\|_{L(\mathcal{K},\mathcal{H})}$ and we abbreviate this notations to $L(\mathcal{H})$ when $\mathcal{H} = \mathcal{K}$. For $\psi \in L(\mathcal{K}, \mathcal{H})$ we define

$$\|\psi\|_Q^2 = Tr(\psi Q \psi^*) = \sum_{j=1}^{\infty} \|\sqrt{\lambda_j} \psi e_j\|^2.$$

If $\|\psi\|_Q^2 < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. Let $L_Q(\mathcal{K}, \mathcal{H})$ denote the space of all Q-Hilbert-Schmidt operator ψ. The completion $L_Q(\mathcal{K}, \mathcal{H})$ of $L(\mathcal{K}, \mathcal{H})$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\psi\|_Q^2 = (\psi, \psi)$ is a Hilbert space with the above norm topology. For more details, we refer the reader to Da Prato and Zabezyk [26].

The collection of all strongly measurable, square integrable, $\mathcal{H}$-valued random variables, denoted by $L_2(\Omega, \mathcal{H})$ is a Banach space equipped with norm $\|x(\cdot)\|_{L_2} = (\mathbb{E}\|x(\cdot, w)\|_H^2)^{1/2}$, where $\mathbb{E}(\cdot)$ denotes the expectation with respect to the measure $\mathbb{P}$. Let $\mathcal{C}(J, L_2(\Omega, \mathcal{H}))$ be the Banach space of all continuous maps from J into $L_2(\Omega, \mathcal{H})$ satisfying $\sup_{t=J} \mathbb{E}\|x(t)\|_{\mathcal{H}}^2 < \infty$. $L_2^{\mathcal{F}_0}(\Omega, \mathcal{H})$ denote the family of all $\mathcal{F}_0$-measurable, $\mathcal{H}$-valued random variable $x(0)$.

4.1.1 Fractional Calculus

Let us now recall some basic definitions and results of fractional calculus. For more details see [56, 88, 96].

Definition 4.1.1. *The fractional integral of order α with the lower limit zero for a function f is defined as*

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds, \quad t>0,\ \alpha>0,$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt.$$

Definition 4.1.2. *The Riemann-Liouville fractional derivative of order $\alpha>0$, $n-1<\alpha<n$, $n\in\mathbb{N}$ is defined as*

$$^{(R-L)}D^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_0^t (t-s)^{n-\alpha-1}f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 4.1.3. *The Caputo derivative of order $\alpha>0$ for a function $f:[0,\infty)\to\mathbb{R}$ can be written as*

$$^{c}D^{\alpha}f(t) = D^{\alpha}\left(f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \quad t>0,\ n-1<\alpha<n.$$

Remark 4.1.1. i. If $f(t)\in\mathcal{C}^n[0,\infty)$, then

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds = I^{n-\alpha}f^{(n)}(t), \quad t>0,\ n-1<\alpha<n.$$

ii. The Caputo derivative of a constant is equal to zero.

iii. If f is an abstract function with values in $\mathcal{H}$, then integrals which appear in Definition 4.1.1 and 4.1.2 are taken in Bochners sense.

4.1.2 Multivalued Analysis

We introduce some basic definitions and results of multivalued maps. Fore more details on multivalued maps, see the books of Deimling [29], Hu and Papageorgiou [48].

Definition 4.1.4. *A multivalued map $G:\mathcal{H}\to 2^{\mathcal{H}}\backslash\{\emptyset\}$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x\in\mathcal{H}$. G is bounded on bounded sets if $G(B)=\cup_{x\in B}G(x)$ is bounded in $\mathcal{H}$ for any bouded set B of $\mathcal{H}$, i.e., $\sup_{x\in B}\{\sup\{\|y\| : y\in G(x)\}\}<\infty$.*

Definition 4.1.5. *G is called upper semicontinuous (u.s.c. for short) on $\mathcal{H}$ if for each $x_0\in\mathcal{H}$, the set $G(x_0)$ is a nonempty closed subset of $\mathcal{H}$, and if for each open set V of $\mathcal{H}$ containing $G(x_0)$, there exists an open neighborhood N of x_0 such that $G(N)\subseteq V$.*

Definition 4.1.6. *The multi-valued operator G is called compact if $\overline{G(\mathcal{H})}$ is a compact subset of $\mathcal{H}$. G is said to be completly continuous if $G(D)$ is relatively compact for every bounded subset D of $\mathcal{H}$.*

If the multivalued map G is completly continuous with nonempty values, then G is u.s.c., if and only if G has a closed graph, i.e., $x_n\to x_*$, $y_n\to y_*$, $y_n\in G(x_n)$ imply $y_*\in G(x_*)$.
G has a fixed point if there is $x\in\mathcal{H}$ such that $x\in G(x)$.

4.2 Almost Periodic Solutions For Impulsive Fractional Stochastic Evolution Equations

In this Section[1] we studies the existence of square-mean piecewise almost periodic solutions of the following impulsive fractional stochastic differential equations in the form

$$^{c}D_t^{\alpha}x(t)+Ax(t)=F(t,x(t))+\Sigma(t,x(t))\frac{dW(t)}{dt}+\sum_{k=-\infty}^{\infty}G_k(x(t))\delta(t-\tau_k),\quad t\in J=\mathbb{R}, \quad (4.1)$$

where the state $x(\cdot)$ takes values in the space $L_2(\Omega,\mathcal{H})$, $\mathcal{H}$ is a separable real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$; the fractional derivative $^{c}D^{\alpha},\alpha\in(0,1)$, is understood in the Caputo sense; $-A:\mathcal{D}(A)\subset L_2(\Omega,\mathcal{H})\to L_2(\Omega,\mathcal{H})$ is the infinitesimal generator of an analytic semigroup of a bounded linear operator $S(t)$, $t\geq 0$, on $L_2(\Omega,\mathcal{H})$ satisfying the exponential stability; $\{W(t):t\geq 0\}$ is a given $\mathcal{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q\geq 0$ defined on a filtered complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$, $\mathcal{K}$ is another separable Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{K}}$ and norm $\|\cdot\|_{\mathcal{K}}$; $G_k:\mathcal{D}(G_k)\subset L_2(\Omega,\mathcal{H})\to L_2(\Omega,\mathcal{H})$ are continuous impulsive operators, $\delta(\cdot)$ is Dirac's delta-function, $F(t,x):\mathbb{R}\times L_2(\Omega,\mathcal{H})\to L_2(\Omega,\mathcal{H})$ and $\Sigma(t,x):\mathbb{R}\times L_2(\Omega,\mathcal{H})\to L_2(\Omega,L_0^2(\mathcal{K},\mathcal{H}))$ are jointly continuous functions (here, $L_0^2(\mathcal{K},\mathcal{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathcal{K}$ into $\mathcal{H}$).

Let $L_0^2=L^2(Q^{\frac{1}{2}}\mathcal{K},\mathcal{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathcal{K}$ to $\mathcal{H}$ with the inner product $(\varphi,\phi)_{L_0^2}=Tr[\varphi Q\phi^*]$.

Let $\mathcal{B}=\{\{\tau_k\}:\tau_k\in\mathbb{R},\ \tau_k<\tau_{k+1},\ k\in\mathbb{Z}\}$ be the set of all sequences unbounded and strictly increasing. We consider the impulsive fractional differential equation (4.1), and denote by $x(t)=x(t;t_0,x_0)$, $t_0\in\mathbb{R}$, $x_0\in\mathcal{H}$, the solution of (4.1) with the initial condition

$$x(t_0)=x_0. \quad (4.2)$$

Definition 4.2.1. *([78]). A stochastic process $x:\mathbb{R}\to L_2(\Omega,\mathcal{H})$, is said to be stochastically bounded if there exists $N>0$ such that $\mathbb{E}\|x(t)\|^2\leq N$ for all $t\in\mathbb{R}$.*

Definition 4.2.2. *([78]). A stochastic process $x:\mathbb{R}\to L_2(\Omega,\mathcal{H})$, is said to be stochastically continuous in $s\in\mathbb{R}$, if $\lim_{t\to s}\mathbb{E}\|x(t)-x(s)\|^2=0$.*

For $\{\tau_k\}\in\mathcal{B}$ and $k\in\mathbb{Z}$, let $\mathcal{PC}(\mathbb{R},L_2(\Omega,\mathcal{H}))$ be the space consisting of all stochastically bounded functions $\phi:\mathbb{R}\to L_2(\Omega,\mathcal{H})$ such that $\phi(\cdot)$ is stochastically continuous at t for any $t\notin\{\tau_k\}$, $\tau_k\in\mathbb{R}$, $k\in\mathbb{Z}$ and $\phi(\tau_k^-)=\phi(\tau_k)$. In particular, we introduce the space $\mathcal{PC}(\mathbb{R}\times L_2(\Omega,\mathcal{H}),L_2(\Omega,\mathcal{H}))$ formed by all piecewise stochastically continuous stochastic processes $\phi:\mathbb{R}\times L_2(\Omega,\mathcal{H})\to L_2(\Omega,\mathcal{H})$ such that for any $x\in L_2(\Omega,\mathcal{H})$, $\phi(\cdot,x)$ is stochastically continuous at t for any $t\notin\{\tau_k\}$ and $\phi(\tau_k^-,x)=\phi(\tau_k,x)$ for all $k\in\mathbb{Z}$, and for any $t\in\mathbb{R}$, $\phi(t,\cdot)$ is stochastically continuous at $x\in L_2(\Omega,\mathcal{H})$.

Remark 4.2.1. *([78, 118]).* The solution $x(t)=x(t;t_0,x_0)$ of the problem (4.1)-(4.2) is a piecewise stochastically continuous, $\mathcal{F}_t$-adapted measurable process with points of discontinuity at the moments τ_k, $k\in\mathbb{Z}$, at which it is continuous from the left.

[1] The section is based on the paper [39].

Definition 4.2.3. *([114]). The set of sequences* $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}$, $\{\tau_k\} \in \mathcal{B}$ *is said to be equipotentially almost periodic, if for arbitrary* $\epsilon > 0$ *there exists a relatively dense set* B_ϵ *of* $\mathbb{R}$ *such that for each* $\kappa \in B_\epsilon$ *there is an integer* $q \in \mathbb{Z}$ *such that* $|\tau_{k+q} - \tau_k - \kappa| < \epsilon$ *for all* $k \in \mathbb{Z}$.

Definition 4.2.4. *([20]). A stochastic process* $x \in \mathcal{PC}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$, *is said to be square-mean piecewise almost periodic, if:*

(i) The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}$, $\{\tau_k\} \in \mathcal{B}$ *is equipotentially almost periodic.*

(ii) For any $\epsilon > 0$, *there exists a real number* $\delta > 0$ *such that if the points* t' *and* t'' *belong to one and the same interval of continuity of* $x(t)$ *and satisfy the inequality* $|t' - t''| < \delta$, *then* $\mathbb{E}\|x(t') - x(t'')\|_{\mathcal{H}}^2 < \epsilon$.

(iii) For any $\epsilon > 0$, *there exists a relatively dense set* T *such that if* $\tau \in T$, *then* $\mathbb{E}\|x(t+\tau) - x(t)\|_{\mathcal{H}}^2 < \epsilon$, *satisfying the condition* $|t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$. *The elements of* T *are called* ϵ*-translation number of* x.

We denote by $\mathcal{AP}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$ the collection of all square-mean piecewise almost periodic processes, if thus is a Banach space with the norm $\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|_{L_2} = \sup_{t\in\mathbb{R}} (\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}}$ for $x \in \mathcal{AP}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$.

Lemma 4.2.1. ([78]). *Let* $F \in \mathcal{AP}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$. *Then,* $R(F)$, *the range of* F *is a relatively compact set of* $L_2(\Omega, \mathcal{H})$.

Definition 4.2.5. *([21]).*

For $\{\tau_k\} \in \mathcal{B}$, $k \in \mathbb{Z}$, *the function* $F(t, x) \in \mathcal{PC}(\mathbb{R} \times L_2(\Omega, \mathcal{H}), L_2(\Omega, \mathcal{H}))$ *is said to be square-mean piecewise almost periodic in* $t \in \mathbb{R}$ *and uniform on compact subset of* $L_2(\Omega, \mathcal{H})$ *if for every* $\epsilon > 0$ *and every compact subset* $K \subseteq L_2(\Omega, \mathcal{H})$, *there exists a relatively dense subset* T *of* $\mathbb{R}$ *such that*

$$\mathbb{E}\|F(t+\tau, x) - F(t, x)\|^2 < \epsilon,$$

for all $x \in K$, $\tau \in T$, $t \in \mathbb{R}$ *satisfying* $|t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$. *The collection of all such processes is denoted* $\mathcal{AP}(\mathbb{R} \times L_2(\Omega, \mathcal{H}), L_2(\Omega, \mathcal{H}))$.

Lemma 4.2.2. *([78]). Suppose that* $F(t, x) \in \mathcal{AP}(\mathbb{R} \times L_2(\Omega, \mathcal{H}), L_2(\Omega, \mathcal{H}))$ *and* $F(t, \cdot)$ *is uniformly continuous on each compact subset* $K \subseteq L_2(\Omega, \mathcal{H})$ *uniformly for* $t \in \mathbb{R}$. *That is, for all* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $x, y \in \mathcal{K}$ *and* $\mathbb{E}\|x - y\|^2 < \delta$ *implies that* $\mathbb{E}\|F(t, x) - F(t, y)\|^2 < \epsilon$ *for all* $t \in \mathbb{R}$. *Then* $F(\cdot, x(\cdot)) \in \mathcal{AP}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$ *for any* $x \in \mathcal{AP}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$.

We obtain the following corollary as an immediate consequences of Lemma 4.2.2.

Corollaire 4.2.1. *Let* $F(t, x) \in \mathcal{AP}(\mathbb{R} \times L_2(\Omega, \mathcal{H}), L_2(\Omega, \mathcal{H}))$ *and* F *is Lipschitz, i.e., there is a number* $c > 0$ *such that*

$$\mathbb{E}\|F(t, x) - F(t, y)\|^2 < c\mathbb{E}\|x - y)\|^2,$$

for all $t \in \mathbb{R}$ *and* $x, y \in L_2(\Omega, \mathcal{H})$, *if for any* $x \in \mathcal{AP}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$, *then* $F(\cdot, x(\cdot)) \in \mathcal{AP}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$.

Definition 4.2.6. *A sequence* $x : \mathbb{Z} \to L_2(\Omega, \mathcal{H})$ *is called a square-mean almost periodic sequence if* ϵ*-traslation set of* x

$$\mathcal{I}(x; \epsilon) = \{\tau \in \mathbb{Z} : \mathbb{E}\|x(n+\tau) - x(t)\|^2 < \epsilon, \text{ for all } n \in \mathbb{Z}\}$$

is a relatively dense set in $\mathbb{Z}$ *for all* $\epsilon > 0$.

The collection of all square-mean almost periodic sequences $x : \mathbb{Z} \to L_2(\Omega, \mathcal{H})$ will be denoted by $\mathcal{AP}(\mathbb{Z}, L_2(\Omega, \mathcal{H}))$.

Remark 4.2.2. If $x(n) \in \mathcal{AP}(\mathbb{Z}, L_2(\Omega, \mathcal{H}))$, then $\{x(n) : n \in \mathbb{Z}\}$ is stochastically bounded.

Lemma 4.2.3. *([78]). Assume that $F \in \mathcal{AP}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$, the sequence $\{x_k : k \in \mathbb{Z}\}$ is almost periodic in $L_2(\Omega, \mathcal{H})$ and $\{\tau_k^j\}$, $j \in \mathbb{Z}$, is equipotentially almost periodic. Then for each $\epsilon > 0$ there are relatively dense sets T_{ϵ,F,x_k} of $\mathbb{R}$ and $\widehat{T}_{\epsilon,F,x_k}$ of $\mathbb{Z}$ such that the following conditions hold:*

(i) $\mathbb{E}\|F(t+\tau) - F(t)\|^2 < \epsilon$ *for all* $t \in \mathbb{R}$, $|t - \tau_k| > \epsilon$, $\tau \in T_{\epsilon,F,x_k}$ *and* $k \in \mathbb{Z}$.

(ii) $\mathbb{E}\|x_{k+q} - x_k\|^2 < \epsilon$ *for all* $q \in \widehat{T}_{\epsilon,F,x_k}$ *and* $k \in \mathbb{Z}$.

(iii) For every $\tau \in T_{\epsilon,F,x_k}$, *there exists at least one number* $q \in \widehat{T}_{\epsilon,F,x_k}$ *such that* $|\tau_k^q - \tau| < \epsilon$, $k \in \mathbb{Z}$.

Consider the linear fractional impulsive stochastic differential equation corresponding to (4.1)

$$^cD_t^\alpha x(t) + Ax(t) = f(t) + \sigma(t)\frac{dW(t)}{dt} + \sum_{k=-\infty}^{\infty} g_k \delta(t - \tau_k). \tag{4.3}$$

where $f \in \mathcal{PC}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$, $\sigma \in \mathcal{PC}(\mathbb{R}, L_2(\Omega, L_0^2))$ and $g_k : \mathcal{D}(g_k) \subset L_2(\Omega, \mathcal{H}) \to L_2(\Omega, \mathcal{H})$.

Let us introduce the following conditions.

(C1) The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}$, $\{\tau_k\} \in \mathcal{B}$ is equipotentially almost periodic and there exists $\theta > 0$ such that $\inf_k \tau_k^1 = \theta$.

(C2) The function f is in $\mathcal{AP}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$ and locally Hölder continuous with points of discontinuity at the moment τ_k, $k \in \mathbb{Z}$ at which it is continuous from the left.

(C3) The function σ is in $\mathcal{AP}(\mathbb{R}, L_2(\Omega, L_0^2))$ and locally Hölder continuous with points of discontinuity at the moment τ_k, $k \in \mathbb{Z}$ at which it is continuous from the left.

(C4) $\{g_k\}$, $k \in \mathbb{Z}$, of impulsive operators is a square-mean almost periodic sequence.

Lemma 4.2.4. *([77, 78]). Let the condition **(C1)** holds. Then*

(i) There exists a constant $p > 0$ *such that, for every* $t \in \mathbb{R}$

$$\lim_{T\to\infty} \frac{\iota(t, t+T)}{T} = p.$$

(ii) For each $p > 0$ *there exists a positive integer* N *such that each interval of length* p *has no more than* N *elements of the sequence* $\{\tau_k\}$*, That is,*

$$\iota(s, t) \leq N(t - s) + N,$$

where $\iota(s, t)$ *is the number of point* τ_k *in the interval* (s, t).

The following Lemma is an immediate consequence of Lemma 4.2.3.

Lemma 4.2.5. *([39]). Let the conditions **(C1)-(C4)** hold. Then, for each* $\epsilon > 0$ *there are relatively dense sets* $T_{\epsilon,f,\sigma,g_k}$ *of* $\mathbb{R}$ *and* $\widehat{T}_{\epsilon,f,\sigma,g_k}$ *of* $\mathbb{Z}$ *such that the following relations hold:*

(i) $\mathbb{E}\|f(t+\tau)-f(t)\|^2<\epsilon,\ t\in\mathbb{R},\ \tau\in T_{\epsilon,f,\sigma,g_k},\ |t-\tau_k|>\epsilon,\ k\in\mathbb{Z}.$

(ii) $\mathbb{E}\|\sigma(t+\tau)-\sigma(t)\|^2<\epsilon,\ t\in\mathbb{R},\ \tau\in T_{\epsilon,f,\sigma,g_k},\ |t-\tau_k|>\epsilon,\ k\in\mathbb{Z}.$

(iii) $\mathbb{E}\|g_{k+q}-g_k\|^2<\epsilon,\ k\in\mathbb{Z},\ q\in\widehat{T}_{\epsilon,f,\sigma,g_k}.$

(iv) *For each* $\tau\in T_{\epsilon,f,\sigma,g_k}$, $\exists q\in\widehat{T}_{\epsilon,f,\sigma,g_k}$, *such that* $|\tau_{k+q}-\tau_k-\tau|<\epsilon,\ k\in\mathbb{Z}.$

Now we present the definition of mild solutions for the problem (4.2) − (4.3) based on the paper [113].

Definition 4.2.7. *A stochastic process* $x\in\mathcal{PC}(J,L_2(\Omega,\mathcal{H}))$, $J\subset\mathbb{R}$ *is called the mild solution of the problem* (4.2) − (4.3) *if*

(i) $x_0\in L_2^{\mathcal{F}_0}(\Omega,\mathcal{H})$;

(ii) $x_t\in L_2(\Omega,\mathcal{H})$ *has càdlàg paths on* $t\in J$ *a.s., and it satisfies the following integral equation*

$$x(t)=\begin{cases}\mathcal{T}(t-t_0)x_0+\int_{t_0}^t(t-s)^{\alpha-1}\mathcal{S}(t-s)f(s)ds\\ \quad+\int_{t_0}^t(t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma(s)dW(s), & t\in[t_0,\tau_1],\\ \mathcal{T}(t-t_0)x_0+\mathcal{T}(t-\tau_1)g_1+\int_{t_0}^t(t-s)^{\alpha-1}\mathcal{S}(t-s)f(s)ds\\ \quad+\int_{t_0}^t(t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma(s)dW(s), & t\in(\tau_1,\tau_2],\\ \vdots\\ \mathcal{T}(t-t_0)x_0+\sum_{t_0<\tau_k<t}\mathcal{T}(t-\tau_k)g_k+\int_{t_0}^t(t-s)^{\alpha-1}\mathcal{S}(t-s)f(s)ds\\ \quad+\int_{t_0}^t(t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma(s)dW(s), & t\in(\tau_k,\tau_{k+1}],\end{cases}\tag{4.4}$$

where $\mathcal{T}(\cdot)$ and $\mathcal{S}(\cdot)$ are called characteristic solution operators and given by

$$\mathcal{T}=\int_0^\infty\xi_\alpha(\theta)S(t^\alpha\theta)d\theta,\quad \mathcal{S}=\alpha\int_0^\infty\theta\xi_\alpha(\theta)S(t^\alpha\theta)d\theta,$$

and for $\theta\in(0,\infty)$

$$\xi_\alpha(\theta)=\frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha(\theta^{-\frac{1}{\alpha}})\geq 0,$$

$$\varpi_\alpha(\theta)=\frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),$$

ξ_α is a probability density function defined on $(0,\infty)$, that is

$$\xi_\alpha(\theta)\geq 0,\quad \theta\in(0,\infty),\quad \textit{and}\quad \int_0^\infty\xi_\alpha(\theta)d\theta=1.$$

Remark 4.2.3.
$$\int_0^\infty \theta^\nu \xi_\alpha(\theta)d\theta = \int_0^\infty \theta^{-\alpha\nu}\varpi_\alpha(\theta)d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}.$$

In this section, we will also assume that $\xi_\alpha^2 \in L^1((0,\infty))$.

Let the operator $-A$ in (4.1) and (4.3) be an infinitesimal generator of an analytic semigroup $S(t)$ in $L_2(\Omega,\mathcal{H})$ and $0 \in \rho(A)$, the resolvent set of A. For any $\beta > 0$, we define the fractional power $A^{-\beta}$ of the operator a by

$$A^{-\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}S(t)dt,$$

where $A^{-\beta}$ is bounded, bijective and $A^\beta = (A^{-\beta})^{-1}$, $\beta > 0$ a closed linear operator on its domain $\mathcal{D}(A^\beta)$ and such that $\mathcal{D}(A^\beta) = \mathcal{R}(A^{-\beta})$ where $\mathcal{R}(A^{-\beta})$ is the range of $A^{-\beta}$. Furthermore, the subspace $\mathcal{D}(A^\beta)$ is dense in $L_2(\Omega,\mathcal{H})$ and the expression

$$\|x\|_\beta = \|A^\beta x\|, \quad x \in \mathcal{D}(A^\beta),$$

defines a norm on $L_2(\Omega,\mathcal{H}_\beta) := \mathcal{D}(A^\beta)$. The following properties are well known.

Lemma 4.2.6. ([94]). *Suppose that the preceding conditions are satisfied. Then*

(i) $S(t) : L_2(\Omega,\mathcal{H}) \to \mathcal{D}(A^\beta)$ *for every* $t > 0$ *and* $\beta \geq 0$.

(ii) For every $x \in \mathcal{D}(A^\beta)$, *the following equality* $S(t)A^\beta x = A^\beta S(t)x$ *holds.*

(iii) For every $t > 0$, *the operator* $A^\beta S(t)$ *is bounded and*

$$\|A^\beta S(t)\| \leq K_\beta t^{-\beta}e^{-\lambda t}, \quad K_\beta > 0,\ \lambda > 0.$$

(iv) For $0 < \beta \leq 1$ *and* $x \in \mathcal{D}(A^\beta)$, *we have*

$$\|S(t)x - x\| \leq C_\beta t^\beta \|A^\beta x\|, \quad C_\beta > 0.$$

When $-A$ generates a semi-group with negative exponent, we deduce that if $x(t)$ is a bounded solution of (4.3) on $\mathbb{R}$, then we take the limit as $t_0 \to -\infty$ and using (4.4), we obtain (see[16])

$$x(t) = \int_{-\infty}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f(s)ds + \int_{-\infty}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma(s)dW(s) + \sum_{\tau_k<t}\mathcal{T}(t-\tau_k)g_k. \quad (4.5)$$

4.2.1 The Main Results

Now, we present and prove our main theorems.

Theorem 4.2.1. *Assume that conditions **(C1)-(C4)** are satisfied and* $-A$ *is the infinitesimal generator of an analytic semi-group* $S(t)$, *then system* (4.2) − (4.3) *has a square-mean piecewise almost periodic mild solution.*

Proof. First, we shall show that the right-hand side of (4.5) is well defined. From conditions **(C2)-(C4)**, it follows that $f(t), \sigma(t)$ and $\{g_k\}$ are stochastically bounded, and let

$$\max\left\{\mathbb{E}\|f(t)\|_{pc}^2, \mathbb{E}\|\sigma(t)\|_{pc}^2, \mathbb{E}\|g_k\|_{L_2(\Omega,\mathcal{H})}^2\right\} \leq N_0, \qquad N_0 > 0.$$

In view of Lemma 4.2.6 and the definition of the norm in $\mathcal{H}_\beta$, we obtain

$$\begin{aligned}
& \mathbb{E}\|x(t)\|_\beta^2 = \mathbb{E}\|A^\beta x(t)\|^2 \\
\leq\ & 3\mathbb{E}\left\|\int_{-\infty}^t (t-s)^{\alpha-1}A^\beta \mathcal{S}(t-s)f(s)ds\right\|^2 \\
& +3\mathbb{E}\left\|\int_{-\infty}^t (t-s)^{\alpha-1}A^\beta \mathcal{S}(t-s)\sigma(s)dW(s)\right\|^2 + 3\mathbb{E}\left\|\sum_{\tau_k<t} A^\beta \mathcal{T}(t-\tau_k)g_k\right\|^2 \\
\leq\ & 3\alpha^2\mathbb{E}\left\|\int_{-\infty}^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)A^\beta S((t-s)^\alpha\theta)f(s)d\theta ds\right\|^2 \\
& +3\alpha^2\mathbb{E}\left\|\int_{-\infty}^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)A^\beta S((t-s)^\alpha\theta)\sigma(s)d\theta dW(s)\right\|^2 \\
& +3\sum_{\tau_k<t}\mathbb{E}\left\|\int_0^\infty \xi_\alpha(\theta)A^\beta S((t-\tau_k)^\alpha\theta)g_k d\theta\right\|^2 \\
\leq\ & 3\alpha^2\mathbb{E}\left[\int_{-\infty}^t\int_0^\infty \|\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)A^\beta S((t-s)^\alpha\theta)f(s)\|d\theta ds\right]^2 \\
& +3\alpha^2 Tr(Q)\mathbb{E}\left[\int_{-\infty}^t\int_0^\infty \|\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)A^\beta S((t-s)^\alpha\theta)\sigma(s)\|_{L_0^2}^2 d\theta ds\right] \\
& +3\sum_{\tau_k<t}\mathbb{E}\left[\int_0^\infty \|\xi_\alpha(\theta)A^\beta S((t-\tau_k)^\alpha\theta)g_k\|d\theta\right]^2 \\
\leq\ & 3\alpha^2 K_\beta^2\int_0^\infty \xi_\alpha(\theta)\int_{-\infty}^t \theta^{1-\beta}(t-s)^{-\alpha\beta+\alpha-1}e^{-\lambda\theta(t-s)^\alpha}dsd\theta \\
& \quad\times\int_0^\infty \xi_\alpha(\theta)\int_{-\infty}^t \theta^{1-\beta}(t-s)^{-\alpha\beta+\alpha-1}e^{-\lambda\theta(t-s)^\alpha}\mathbb{E}\|f(s)\|^2 dsd\theta \\
& +3\alpha^2K_\beta^2Tr(Q)\int_{-\infty}^t\int_0^\infty \theta^{2(1-\beta)}\xi_\alpha^2(\theta)(t-s)^{2(\alpha-\alpha\beta-1)}e^{-2\lambda\theta(t-s)^\alpha}\mathbb{E}\|\sigma(s)\|_{L_0^2}^2 d\theta ds \\
& +3K_\beta^2\sum_{\tau_k<t}\mathbb{E}\left[\int_0^\infty \theta^{-\beta}\xi_\alpha(\theta)(t-\tau_k)^{-\alpha\beta}e^{-\lambda\theta(t-\tau_k)^\alpha}\|g_k\|d\theta\right]^2.
\end{aligned} \tag{4.6}$$

We have, for $\eta = t-s$,

$$\begin{aligned}
\mathbb{E}\|x(t)\|_\beta^2 \leq\ & 3\alpha^2K_\beta^2N_0\int_0^\infty \xi_\alpha(\theta)\int_0^\infty \theta^{1-\beta}\eta^{-\alpha\beta+\alpha-1}e^{-\lambda\theta\eta^\alpha}d\eta d\theta \\
& \quad\times\int_0^\infty \xi_\alpha(\theta)\int_0^\infty \theta^{1-\beta}\eta^{-\alpha\beta+\alpha-1}e^{-\lambda\theta\eta^\alpha}d\eta d\theta \\
& +3\alpha^2K_\beta^2N_0Tr(Q)\int_0^\infty \xi_\alpha^2(\theta)\int_0^\infty \theta^{2(1-\beta)}\eta^{2(\alpha-\alpha\beta-1)}e^{-2\lambda\theta\eta^\alpha}d\eta d\theta \\
& +3K_\beta^2N_0R(\theta),
\end{aligned} \tag{4.7}$$

where

$$R(\theta) = \Bigg(\int_0^\infty \xi_\alpha(\theta)\Big[\sum_{0<t-\tau_k\le 1}(\theta(t-\tau_k)^\alpha)^{-\beta}e^{-\lambda\theta(t-\tau_k)^\alpha} + \sum_{j=1}^{\infty}\sum_{j<t-\tau_k\le j+1}(\theta(t-\tau_k)^\alpha)^{-\beta}e^{-\lambda\theta(t-\tau_k)^\alpha}\Big]d\theta\Bigg)^2.$$

By a standard calculation, we can deduce that

$$\begin{aligned}&\left(\alpha\int_0^\infty \xi_\alpha(\theta)\int_0^\infty \theta^{1-\beta}\eta^{-\alpha\beta+\alpha-1}e^{-\lambda\theta\eta^\alpha}d\eta d\theta\right)^2\\ &=\left(\frac{1}{\lambda^{1-\beta}}\int_0^\infty \xi_\alpha(\theta)\int_0^\infty(\lambda\theta\eta^\alpha)^{-\beta}e^{-\lambda\theta\eta^\alpha}d\lambda\theta\eta^\alpha d\theta\right)^2\\ &=\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}}.\end{aligned}\tag{4.8}$$

Since $\xi_\alpha^2 \in L^1((0,\infty))$, we further derive that

$$\alpha^2\int_0^\infty \xi_\alpha^2(\theta)\int_0^\infty \theta^{2(1-\beta)}\eta^{2(\alpha-\alpha\beta-1)}e^{-2\lambda\theta\eta^\alpha}d\eta d\theta \le N_1\frac{\Gamma^2(1-2\beta)}{\lambda^{2-2\beta}},\tag{4.9}$$

where $N_1 = \sup_{\theta\ge 0}\xi_\alpha^2(\theta)$.
By the help of **(C1)** and Lemma 4.2.4, we have

$$\begin{aligned}R(\theta) &\le \left(\int_0^\infty \xi_\alpha(\theta)\left(\frac{2N}{N_2^\beta}+\frac{2N}{e^\beta-1}\right)d\theta\right)^2\\ &= 4N^2\left(\frac{1}{N_2^\beta}+\frac{1}{e^\beta-1}\right)^2.\end{aligned}\tag{4.10}$$

where $N_2 = \min\{\theta(t-\tau_k)^\alpha, 0<t-\tau_k\le 1\}$.
Recalling (4.7), from (4.8) – (4.10), we obtain

$$\mathbb{E}\|x(t)\|_\beta^2 \le 3K_\beta^2 N_0\left[\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}}+Tr(Q)N_1\frac{\Gamma^2(1-2\beta)}{\lambda^{2(1-\beta)}}+4N^2\left(\frac{1}{N_2^\beta}+\frac{1}{e^\beta-1}\right)^2\right],\tag{4.11}$$

and $x(t)\in\mathcal{PC}(\mathbb{R}, L_2(\Omega,\mathcal{H}))$.

Let $\epsilon>0$, $\tau\in T_{\epsilon,f,\sigma,g_k}$ and $q\in\widehat{T}_{\epsilon,f,\sigma,g_k}$, $k\in\mathbb{Z}$, where the sets $T_{\epsilon,f,\sigma,g_k}$ and $\widehat{T}_{\epsilon,f,\sigma,g_k}$ are defined as in Lemma 4.2.5. We have

$$\begin{aligned}x(t+\tau)-x(t) &= \Bigg(\int_{-\infty}^{t+\tau}(t+\tau-s)^{\alpha-1}\mathcal{S}(t+\tau-s)f(s)ds\\ &\quad+\int_{-\infty}^{t+\tau}(t+\tau-s)^{\alpha-1}\mathcal{S}(t+\tau-s)\sigma(s)dW(s)+\sum_{\tau_k<t}\mathcal{T}(t+\tau-\tau_k)g_k\Bigg)\\ &\quad-\Bigg(\int_{-\infty}^{t}(t-s)^{\alpha-1}\mathcal{S}(t-s)f(s)ds\\ &\quad+\int_{-\infty}^{t}(t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma(s)dW(s)+\sum_{\tau_k<t}\mathcal{T}(t-\tau_k)g_k\Bigg)\\ &= \int_{-\infty}^{t}(t-s)^{\alpha-1}\mathcal{S}(t-s)[f(s+\tau)-f(s)]ds\\ &\quad+\int_{-\infty}^{t}(t-s)^{\alpha-1}\mathcal{S}(t-s)[\sigma(s+\tau)-\sigma(s)]d\widetilde{W}(s)\\ &\quad+\sum_{\tau_k<t}^{-\infty}\mathcal{T}(t-\tau_k)[g_{k+q}-g_k],\end{aligned}$$

where $\widetilde{W}(s) = W(s+\tau) - W(\tau)$ is also a Brownian motion and has the same distribution as w.

Then

$$\begin{aligned}
\mathbb{E}\|x(t+\tau)-x(t)\|_\beta^2 &= \mathbb{E}\|A^\beta(x(t+\tau)-x(t))\|^2 \\
&\le 3\mathbb{E}\left\|\int_{-\infty}^t (t-s)^{\alpha-1}A^\beta\mathcal{S}(t-s)[f(s+\tau)-f(s)]ds\right\|^2 \\
&\quad +3\mathbb{E}\left\|\int_{-\infty}^t (t-s)^{\alpha-1}A^\beta\mathcal{S}(t-s)[\sigma(s+\tau)-\sigma(s)]d\widetilde{W}(s)\right\|^2 \qquad (4.12)\\
. &\quad +3\mathbb{E}\left\|\sum_{\tau_k<t}^{-\infty}\mathcal{T}(t-\tau_k)[g_{k+q}-g_k]\right\|^2 \\
&\le M_\beta\epsilon,
\end{aligned}$$

where $|t-\tau_k| > \epsilon$ and

$$M_\beta = K_\beta\left[\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}} + Tr(Q)N_1\frac{\Gamma^2(1-2\beta)}{\lambda^{2(1-\beta)}} + 4N^2\left(\frac{1}{N_2^\beta}+\frac{1}{e^\beta-1}\right)^2\right].$$

The last inequality implies that $x(t)$ is a square-mean piecewise almost periodic process, so system (4.2) – (4.3) has a square-mean piecewise almost periodic solution. The proof is complete. □

In order to obtain the existence of square-mean piecewise almost periodic solution to system (4.1) – (4.2), we introduce the following conditions:

(C5) $-A : \mathcal{D}(A) \subseteq L_2(\Omega,\mathcal{H}) \to L_2(\Omega,\mathcal{H})$ is the infinitesimal generator of an exponentially stable analytic semi-group $S(t)$, $t\in\mathbb{R}$, on $L_2(\Omega,\mathcal{H})$.

(C6) $F(t,x) \in \mathcal{AP}(\mathbb{R}\times L_2(\Omega,\mathcal{H}_\beta), L_2(\Omega,\mathcal{H}))$ with respect to $t\in\mathbb{R}$ uniformly in $x\in K$, for each compact set $K\subseteq L_2(\Omega,\mathcal{H})$), and there exist constants $\widetilde{c}>0$, $0<\kappa<1$, $0<\beta<1$, such that
$$\mathbb{E}\|F(t_1,x_1)-F(t_2,x_2)\|^2 \le \widetilde{c}(|t_1-t_2|^\kappa + \mathbb{E}\|x_1-x_2\|_\beta^2),$$
where $(t_i,x_i)\in\mathbb{R}\times L_2(\Omega,\mathcal{H}_\beta)$, $i=1,2$.

(C7) $\Sigma(t,x) \in \mathcal{AP}(\mathbb{R}\times L_2(\Omega,\mathcal{H}_\beta), L_2(\Omega,L_0^2))$ with respect to $t\in\mathbb{R}$ uniformly in $x\in K$, for each compact set $K\subseteq L_2(\Omega,\mathcal{H})$, and there exist constants $\widehat{c}>0$, $0<\kappa<1$, $0<\beta<1$, such that
$$\mathbb{E}\|\Sigma(t_1,x_1)-\Sigma(t_2,x_2)\|_{L_0^2}^2 \le \widehat{c}(|t_1-t_2|^\kappa + \mathbb{E}\|x_1-x_2\|_\beta^2),$$
$(t_i,x_i)\in\mathbb{R}\times L_2(\Omega,\mathcal{H}_\beta)$, $i=1,2$.

(C8) The sequences $\{G_k(x)\}$ is almost periodic in $k\in\mathbb{Z}$ uniformly in $x\in K\subseteq L_2(\Omega,\mathcal{H})$, and there exist constants $\overline{c}>0$, $0<\beta<1$, such that
$$\mathbb{E}\|G_k(x_1)-G_k(x_2)\|^2 \le \overline{c}\mathbb{E}\|x_1-x_2\|_\beta^2,$$
where $x_1,x_2\in L_2(\Omega,\mathcal{H}_\beta)$.

Theorem 4.2.2. *Assume that conditions **(C1)**, **(C5)-(C8)** are satisfied, then the impulsive fractional stochastic system (4.1)–(4.2) admits a unique square-mean piecewise almost periodic mild solution.*

Proof. Let B the set of all $x \in \mathcal{AP}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$ with discontinuities of the first type at the point τ_k, $k \in \mathbb{Z}$, $\{\tau_k\} \in \mathcal{B}$, satisfying the inequality $\mathbb{E}\|x\|^2 \leq r, r > 0$. Obviously, B is a closed set of $\mathcal{AP}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$.

Define the operator Θ in B by

$$\begin{aligned}
\Theta x(t) &= \int_{-\infty}^{t} (t-s)^{\alpha-1} A^{\beta} \mathcal{S}(t-s) F(s, A^{-\beta} x(s)) ds \\
&\quad + \int_{-\infty}^{t} (t-s)^{\alpha-1} A^{\beta} \mathcal{S}(t-s) \Sigma(s, A^{-\beta} x(s)) dW(s) \\
&\quad + \sum_{\tau_k < t}^{-\infty} A^{\beta} \tau(t-\tau_k) G_k(A^{-\beta} x(\tau_k)).
\end{aligned} \tag{4.13}$$

Proceeding in the same way as in the proof of theorem 4.2.1, from conditions **(C6)-(C8)** and Lemma 2.2 in [21], we can show that Θ is well defined and $\Theta x(t) \in \mathcal{PC}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$.

First, we shall show that $\Theta x(t) \in B$. We define

$$\begin{aligned}
\Theta_1 x(t) &= \int_{-\infty}^{t} (t-s)^{\alpha-1} A^{\beta} \mathcal{S}(t-s) F(s, A^{-\beta} x(s)) ds \\
\Theta_2 x(t) &= \int_{-\infty}^{t} (t-s)^{\alpha-1} A^{\beta} \mathcal{S}(t-s) \Sigma(s, A^{-\beta} x(s)) dW(s).
\end{aligned}$$

Let us show that $\Theta_1 x \in B$, let $x \in B$. Using condition **(C6)**, since A^{β} is closed and $F(t,x) \in \mathcal{AP}(\mathbb{R} \times L_2(\Omega, \mathcal{H}_{\beta}), L_2(\Omega, \mathcal{H}))$, we have from Corollary 4.2.1, that $A^{-\beta} x \in B$ and $F(\cdot, A^{-\beta} x(\cdot)) \in \mathcal{AP}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$. Therefore, it follows from Definition 4.2.4, and Lemma 4.2.3, that for any $\epsilon > 0$, there exists a relatively dense set T such that for $\tau \in T$ the following property

$$\mathbb{E}\|F(t+\tau, A^{-\beta} x(t+\tau)) - F(t, A^{-\beta} x(t))\|^2 < \frac{\epsilon \lambda^{2(1-\beta)}}{K_{\beta}^2 \Gamma^2(1-\beta)}$$

hold, satisfying the condition $|t - \tau_k| > \epsilon$, for each $t \in \mathbb{R}$ and $k \in \mathbb{Z}$.

By virtue of Lemma 4.2.6, we have

$$\begin{aligned}
& \mathbb{E}\|\Theta_1 x(t+\tau) - \Theta_1 x(t)\|^2 \\
= \ & \mathbb{E}\left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} A^{\beta} \mathcal{S}(t-s) [F(s+\tau, A^{-\beta} x(s+\tau)) - F(s, A^{-\beta} x(s))] ds \right\|^2 \\
\leq \ & \alpha^2 K_{\beta}^2 \int_0^{\infty} \xi_{\alpha}(\theta) \int_0^{\infty} \theta^{1-\beta} \eta^{-\alpha\beta+\alpha-1} e^{-\lambda\theta\eta^{\alpha}} d\eta d\theta \int_0^{\infty} \xi_{\alpha}(\theta) \int_0^{\infty} \theta^{1-\beta} \eta^{-\alpha\beta+\alpha-1} e^{-\lambda\theta\eta^{\alpha}} \\
& \times \mathbb{E}\|F(t+\tau-\eta, A^{-\beta} x(t+\tau-\eta)) - F(t-\eta, A^{-\beta} x(t-\eta))\|^2 d\eta d\theta \\
\leq \ & \alpha^2 K_{\beta}^2 \Big(\int_0^{\infty} \xi_{\alpha}(\theta) \int_0^{\infty} \theta^{1-\beta} \eta^{-\alpha\beta+\alpha-1} e^{-\lambda\theta\eta^{\alpha}} d\eta d\theta \Big)^2 \\
& \times \sup_{t \in \mathbb{R}} \mathbb{E}\|F(t+\tau, A^{-\beta} x(t+\tau)) - F(t, A^{-\beta} x(t))\|^2 \\
= \ & K_{\beta}^2 \frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}} \sup_{t \in \mathbb{R}} \mathbb{E}\|F(t+\tau, A^{-\beta} x(t+\tau)) - F(t, A^{-\beta} x(t))\|^2 \\
< \ & K_{\beta}^2 \frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}} \times \frac{\epsilon \lambda^{2(1-\beta)}}{K_{\beta}^2 \Gamma^2(1-\beta)} = \epsilon
\end{aligned}$$

Hence, $\Theta_1 x(\cdot) \in B$.

Similarly, by using condition **(C7)**, since A^{β} is closed and $\Sigma(t,x) \in \mathcal{AP}(\mathbb{R} \times L_2(\Omega, \mathcal{H}_{\beta}), L_2(\Omega, L_0^2))$, we have from Corollary 4.2.1 that $A^{-\beta} x \in B$ and $\Sigma(\cdot, A^{-\beta} x(\cdot)) \in \mathcal{AP}(\mathbb{R}, L_2(\Omega, L_0^2))$. Therefore, it follows from Definition 4.2.4, and Lemma 4.2.3, that for any $\epsilon > 0$, there exists a relatively dense set T such that for $\tau \in T$ the following property

$$\mathbb{E}\|\Sigma(t+\tau, A^{-\beta}x(t+\tau)) - \Sigma(t, A^{-\beta}x(t))\|_{L_0^2}^2 < \frac{\epsilon\lambda^{2-2\beta}}{N_1 K_\beta^2 Tr(Q)\Gamma(1-2\beta)}$$

hold, satisfying the condition $|t-\tau_k| > \epsilon$, for each $t \in \mathbb{R}$ and $k \in \mathbb{Z}$.

By virtue of Lemma 4.2.6, for $\widetilde{W}(t) := W(t+\tau) - W(\tau)$, we have

$$\begin{aligned}
& \mathbb{E}\|\Theta_2 x(t+\tau) - \Theta_2 x(t)\|^2 \\
= \;& \mathbb{E}\left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} A^\beta \mathcal{S}(t-s)[\Sigma(s+\tau, A^{-\beta}x(s+\tau)) - \Sigma(s, A^{-\beta}x(s))] d\widetilde{W}(s) \right\|^2 \\
\leq \;& \alpha^2 K_\beta^2 Tr(Q) \int_0^\infty \xi_\alpha^2(\theta) \int_0^\infty \theta^{2(1-\beta)} \eta^{2(\alpha-\alpha\beta-1)} e^{-2\lambda\theta\eta^\alpha} \\
& \times \mathbb{E}\|\Sigma(t+\tau-\eta, A^{-\beta}x(t+\tau-\eta)) - \Sigma(t-\eta, A^{-\beta}x(t-\eta))\|_{L_0^2}^2 d\eta d\theta \\
\leq \;& \alpha^2 K_\beta^2 Tr(Q) \int_0^\infty \xi_\alpha^2(\theta) \int_0^\infty \theta^{2(1-\beta)} \eta^{2(\alpha-\alpha\beta-1} e^{-2\lambda\theta\eta^\alpha} d\eta d\theta \\
& \times \sup_{t\in\mathbb{R}} \mathbb{E}\|\Sigma(t+\tau, A^{-\beta}x(t+\tau)) - \Sigma(t, A^{-\beta}x(t))\|_{L_0^2}^2 \\
\leq \;& K_\beta^2 Tr(Q) N_1 \frac{\Gamma^2(1-2\beta)}{\lambda^{2-2\beta}} \sup_{t\in\mathbb{R}} \mathbb{E}\|\Sigma(t+\tau, A^{-\beta}x(t+\tau)) - \Sigma(t, A^{-\beta}x(t))\|_{L_0^2}^2 \\
< \;& K_\beta^2 Tr(Q) N_1 \frac{\Gamma^2(1-2\beta)}{\lambda^{2-2\beta}} \times \frac{\epsilon\lambda^{2-2\beta}}{N_1 K_\beta^2 Tr(Q)\Gamma^2(1-2\beta)} = \epsilon
\end{aligned}$$

Thus, $\Theta_2 x(\cdot) \in B$. And in view of the above, it is clear that Θ maps B into itself.

Next, we show that Θ is a contracting operator on B. Let $x_1, x_2 \in B$. Then, we have

$$\begin{aligned}
& \mathbb{E}\|\Theta x_1(t) - \Theta x_2(t)\|^2 \\
\leq \;& 3\mathbb{E}\left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} A^\beta \mathcal{S}(t-s)[F(s, A^{-\beta}x_1(s)) - F(s, A^{-\beta}x_2(s))] ds \right\|^2 \\
& + 3\mathbb{E}\left\| \int_{-\infty}^{t} (t-s)^{\alpha-1} A^\beta \mathcal{S}(t-s)[\Sigma(s, A^{-\beta}x_1(s)) - \Sigma(s, A^{-\beta}x_2(s))] dW(s) \right\|^2 \\
& + 3\mathbb{E}\left\| \sum_{\tau_k<t} A^\beta \tau(t-\tau_k)[G_k(A^{-\beta}x_1(\tau_k)) - G_k(A^{-\beta}x_2(\tau_k))] \right\|^2 \\
\leq \;& 3K_\beta^2 c_* \Bigg[\alpha^2 \left(\int_0^\infty \xi_\alpha(\theta) \int_0^\infty \theta^{1-\beta} \eta^{-\alpha\beta+\alpha-1} e^{-\lambda\theta\eta^\alpha} d\eta d\theta \right)^2 + \alpha^2 Tr(Q) \int_0^\infty \xi_\alpha^2(\theta) \\
& \times \int_0^\infty \theta^{2(1-\beta)} \eta^{2(\alpha-\alpha\beta-1)} e^{-2\lambda\theta\eta^\alpha} d\eta d\theta + R(\theta) \Bigg] \sup_{t\in\mathbb{R}} \mathbb{E}\|x_1(t) - x_2(t)\|^2,
\end{aligned}$$

where $c_* = \max\{\widehat{c}, \widetilde{c}, \bar{c}\} > 0$ is sufficiently small and $R(\theta)$ is defined as in above.

By following similar arguments like those used in (4.7) , we have

$$\begin{aligned}
& \mathbb{E}\|\Theta x_1(t) - \Theta x_2(t)\|^2 \\
\leq \;& 3c_* K_\beta^2 \Bigg[\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}} \\
& + Tr(Q) N_1 \frac{\Gamma^2(1-2\beta)}{\lambda^{2(1-\beta)}} + 4N^2 \left(\frac{1}{N_2^\beta} + \frac{1}{e^\beta - 1} \right)^2 \Bigg] \sup_{t\in\mathbb{R}} \mathbb{E}\|x_1(t) - x_2(t)\|^2.
\end{aligned}$$

Therefore, if c_* is chosen in the form

$$c_* \leq \left(3K_\beta^2 \left[\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}} + Tr(Q) N_1 \frac{\Gamma^2(1-2\beta)}{\lambda^{2(1-\beta)}} + 4N^2 \left(\frac{1}{N_2^\beta} + \frac{1}{e^\beta - 1} \right)^2 \right] \right)^{-1}.$$

we have

$$\begin{aligned}&\mathbb{E}\|\Theta x_1(t)-\Theta x_2(t)\|^2\\ \leq\ &3c_*K_\beta^2\Big[\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}}+Tr(Q)N_1\frac{\Gamma^2(1-2\beta)}{\lambda^{2(1-\beta)}}+4N^2\Big(\frac{1}{N_2^\beta}+\frac{1}{e^\beta-1}\Big)^2\Big]\|x_1-x_2\|_\infty^2,\end{aligned}$$

implies that,

$$\|\Theta x_1-\Theta x_2\|_\infty\leq\sqrt{\Lambda}\|x_1-x_2\|_\infty,$$

$$\Lambda=3c_*K_\beta^2\Big[\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}}+Tr(Q)N_1\frac{\Gamma^2(1-2\beta)}{\lambda^{2(1-\beta)}}+4N^2\Big(\frac{1}{N_2^\beta}+\frac{1}{e^\beta-1}\Big)^2\Big].$$

Thus, Θ is a contracting operator on B. So by the contraction principle, we conclude that there exists a unique fixed point x for Θ in B, such that $x=\Theta x$, that is

$$\begin{aligned}x(t)\ =\ &\int_{-\infty}^t(t-s)^{\alpha-1}A^\beta\mathcal{S}(t-s)F(s,A^{-\beta}x(s))ds\\ &+\int_{-\infty}^t(t-s)^{\alpha-1}A^\beta\mathcal{S}(t-s)\Sigma(s,A^{-\beta}x(s))dW(s)\\ &+\sum_{\tau_k<t}A^\beta\mathcal{T}(t-\tau_k)G_k(A^{-\beta}x(\tau_k)).\end{aligned}\tag{4.14}$$

for all $t\in\mathbb{R}$.

Now, from conditions **(C6)-(C8)** and since A^β is closed, $G_k\in\mathcal{AP}(\mathbb{Z},L_2(\Omega,\mathcal{H}))$, $F(t,x)\in\mathcal{AP}(\mathbb{R}\times L_2(\Omega,\mathcal{H}_\beta),L_2(\Omega,\mathcal{H}))$, and $\Sigma(t,x)\in\mathcal{AP}(\mathbb{R}\times L_2(\Omega,\mathcal{H}_\beta),L_2(\Omega,L_0^2))$, we have from corollary 4.2.1 that $F(\cdot,A^{-\beta}x(\cdot))\in\mathcal{AP}(\mathbb{R},L_2(\Omega,\mathcal{H})),\Sigma(\cdot,A^{-\beta}x(\cdot))\in\mathcal{AP}(\mathbb{R},L_2(\Omega,L_0^2))$ and $G_k(A^{-\beta}x(\cdot))$ is square-mean almost periodic sequence. Therefore, by Lemma 2.2 in [21], and Remark 4.2.2, it follows that $F(\cdot,A^{-\beta}x(\cdot))$, $\Sigma(\cdot,A^{-\beta}x(\cdot))$ and $G_k(A^{-\beta}x(\cdot))$ are stochastically bounded, and $\mathbb{E}\|F(t,A^{-\beta}x(t))\|^2$, $\mathbb{E}\|\Sigma(t,A^{-\beta}x(t))\|_{L_0^2}^2$ are uniformly continuous in t. We also get

$$\begin{aligned}A^{-\beta}x(t)\ =\ &\int_{-\infty}^t(t-s)^{\alpha-1}\mathcal{S}(t-s)F(s,A^{-\beta}x(s))ds\\ &+\int_{-\infty}^t(t-s)^{\alpha-1}\mathcal{S}(t-s)\Sigma(s,A^{-\beta}x(s))dW(s)+\sum_{\tau_k<t}\mathcal{T}(t-\tau_k)G_k(A^{-\beta}x(\tau_k)).\end{aligned}$$

with $A^{-\beta}x$ is stochastically bounded in the sense that for $r>0$, for each $t\in\mathbb{R}$, $\mathbb{E}\|A^{-\beta}x(t)\|^2\leq r$. Hence, $A^{-\beta}x\in B$ is mild solution of the problem (4.1) – (4.2). □

Theorem 4.2.3. *Assume that the conditions **(C1), (C5)-(C8)** are satisfied, then the impulsive fractional stochastic system (4.1) – (4.2) has an exponentially stable almost periodic solution.*

Proof. Let $u(t)$ be the solution of the following integral equation

$$\begin{aligned}u(t)\ =\ &\int_{-\infty}^t(t-s)^{\alpha-1}A^\beta\mathcal{S}(t-s)F(s,A^{-\beta}u(s))ds\\ &+\int_{-\infty}^t(t-s)^{\alpha-1}A^\beta\mathcal{S}(t-s)\Sigma(s,A^{-\beta}u(s))dW(s)+\sum_{\tau_k<t}A^\beta\mathcal{T}(t-\tau_k)G_k(A^{-\beta}u(\tau_k)).\end{aligned}\tag{4.15}$$

Consider the equation

$$^{c}D_t^{\alpha}x(t) + Ax = F(t, A^{-\beta}u(t)) + \Sigma(t, A^{-\beta}u(t))\frac{dW(t)}{dt} + \sum_{t_0<\tau_k} G_k(A^{-\beta}u(\tau_k))\delta(t-\tau_k), \qquad t \in \mathbb{R}. \tag{4.16}$$

In view of Theorem 4.2.2, it follows that there exists a unique square-mean piecewise almost periodic solution in the form

$$\begin{aligned} \psi(t) &= \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)F(s, A^{-\beta}u(s))ds \\ &\quad + \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)\Sigma(s, A^{-\beta}u(s))dW(s) + \sum_{\tau_k<t} \mathcal{T}(t-\tau_k)G_k(A^{-\beta}u(\tau_k)). \end{aligned} \tag{4.17}$$

Then

$$\begin{aligned} A^{\beta}\psi(t) &= \int_{-\infty}^{t} A^{\beta}(t-s)^{\alpha-1}\mathcal{S}(t-s)F(s, A^{-\beta}u(s))ds \\ &\quad + \int_{-\infty}^{t} A^{\beta}(t-s)^{\alpha-1}\mathcal{S}(t-s)\Sigma(s, A^{-\beta}u(s))dW(s) \\ &\quad + \sum_{\tau_k<t}^{-\infty} A^{\beta}\tau(t-\tau_k)G_k(A^{-\beta}u(\tau_k)) \\ &= u(t). \end{aligned} \tag{4.18}$$

The last equality shows that $\psi(t) = A^{-\beta}u(t)$ is a solution of (4.1) − (4.2), and the uniqueness follows from the uniqueness of the solution of (4.15) from (4.14).

Let $u(t) = u(t; t_0, u_0)$ and $v(t) = v(t; t_0, v_0)$ be two solutions of equation (4.1), then

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)F(s, A^{-\beta}u(s))ds \\ &\quad + \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)\Sigma(s, A^{-\beta}u(s))dW(s) + \sum_{\tau_k<t} \mathcal{T}(t-\tau_k)G_k(A^{-\beta}u(\tau_k)), \end{aligned} \tag{4.19}$$

$$\begin{aligned} v(t) &= \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)F(s, A^{-\beta}v(s))ds \\ &\quad + \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)\Sigma(s, A^{-\beta}v(s))dW(s) + \sum_{\tau_k<t} \mathcal{T}(t-\tau_k)G_k(A^{-\beta}v(\tau_k)), \end{aligned} \tag{4.20}$$

and $z(t) = u(t) - v(t)$ is in $\mathcal{AP}(\mathbb{R}, L_2(\Omega, \mathcal{H}))$,

$$z = \mathcal{T}(t-t_0)z(t_0). \tag{4.21}$$

The proof follows from (4.21), the estimates from Lemma 4.2.6, and the fact that $\iota(t_0 - t) - p(t-t_0) = o(t)$ for $t \to \infty$. □

4.3 Existence Of Solutions For Fractional Partial Neutral Stochastic Functional Integro-Differential Inclusions With State-Dependent Delay and Analytic Resolvent Operator

In this Section[2] we consider the existence of a class of fractional partial neutral stochastic integro-differential inclusions with state-dependent delay of the form

$$ {}^cD_t^\alpha[x(t)-g(t,x_t)] \in Ax(t)+\int_0^t R(t-s)x(s)ds+F(t,x_{\rho(t,x_t)})\frac{dW(t)}{dt}, \qquad t\in J:=[0,b], \quad (4.22)$$

$$x_0=\varphi\in\mathcal{B}, \qquad x'(0)=0, \quad (4.23)$$

where the state $x(\cdot)$ takes values in a separable real Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$; ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha\in(1,2)$; A, $(R(t))_{t\geq 0}$ are closed linear operators defined on a common domain which is dense in $(\mathcal{H},\|\cdot\|_{\mathcal{H}})$; $D_t^\alpha\sigma(t)$ represents the Caputo derivative of order $\alpha>0$ defined by

$$D_t^\alpha\sigma(t)=\int_0^t \eta_{n-\alpha}(t-s)\frac{d^n}{ds^n}\sigma(s)ds,$$

n is the smallest integer greater than or equal to α and $\eta_\beta(t):=t^{\beta-1}/\Gamma(\beta)$, $t>0$, $\beta\geq 0$. The time history $x_t:(-\infty,0]\to\mathcal{H}$ is given by $x_t(\theta)=x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically. Let $\mathcal{K}$ be another separable Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{K}}$ and norm $\|\cdot\|_{\mathcal{K}}$. Suppose $\{W(t):t\geq 0\}$ is a given $\mathcal{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q>0$ defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is generated by the Wiener process W. The initial data $\varphi=\{\varphi(t):-\infty<t\leq 0\}$ is an $\mathcal{F}_0$-adapted, $\mathcal{B}$- valued random variable independent of the Wiener process W with finite second moments. $F,g,\varphi\in\mathcal{B}$ and ρ are given functions to be specified later.

Then, we will study only the existence of mild solutions for the fractional partial neutral stochastic integro-differential inclusion (4.22) – (4.23), which are natural genaralizations of the concept of mild solution for fractional evolution equations well known in the theory of infinite dimensional deterministic system. This is the difference between this work and the work mentioned in [119]. Specifically, sufficient conditions for the existence are given by means of the nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan with the analytic α-resolvent operator. The known results appeared in [5, 14] are generalized to the fractional stochastic multi-valued settings and the case of infinite delay.

In this section, A and $R(t)$, $t\geq 0$ are closed linear operators defined on a common domain $\mathcal{D}(A)$ which is dense in $\mathcal{H}$. The notation $[\mathcal{D}(A)]$ represents the domain of A endowed with the graph norm. We denote by $(-A)^\beta$ the fractional power of the operator $-A$ for $\beta\in(0,1]$. Furthermore, the subspace $\mathcal{D}((-A)^\beta)$ is dense in $\mathcal{H}$ and the expression $\|x\|_\beta=\|(-A)^\beta x\|, x\in\mathcal{D}((-A)^\beta)$, defines a norm on $\mathcal{D}((-A)^\beta)$. Hereafter, let $\mathcal{H}_\beta$ be the Banach space $\mathcal{D}((-A)^\beta)$ endowed with the norm $\|x\|_\beta$, which is equivalent to the graph norm of $(-A)^\beta$. For more details about the above preliminaries, we refer to [94].

In this work, we assume that the phase space $(\mathcal{B},\|.\|_{\mathcal{B}})$ is a seminormed linear space of $\mathcal{F}_0$-measurable functions mapping $(-\infty,0]$ into $\mathcal{H}_\beta$, and satisfying the following fundamental axioms due to Hale and Kato [41].

[2]The section is based on the paper [40].

(i) If $x : (-\infty, b) \to \mathcal{H}$, $b > 0$, is continuous on $[0, b)$ and x_0 in $\mathcal{B}$, then for every $t \in [0, a)$ the following conditions hold:

(a) x_t is in $\mathcal{B}$;

(b) $\|x(t)\|_\beta \leq \widetilde{H}\|x_t\|_{\mathcal{B}}$;

(c) $\|x_t\|_{\mathcal{B}} \leq K(t)\sup\{\|x(s)\|_\beta : 0 \leq s \leq t\} + M(t)\|x_0\|_{\mathcal{B}}$, where $\widetilde{H} \geq 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, K is continuous, M is locally bounded, and $\widetilde{H}, K, M$ are independent of $x(\cdot)$.

(ii) For the function $x(\cdot)$ in (i), x_t is a $\mathcal{B}$-valued function on $[0, a)$.

(iii) The space $\mathcal{B}$ is complete.

Now, we consider the closed subspace $\mathcal{Z}$ of all continuous processes x that belongs to the space $\mathcal{C}((-\infty, b], L_2^{\mathcal{F}}(\Omega, \mathcal{B}))$ consisting of measurable and $\mathcal{F}_t$-adapted processes such that $\varphi \in \mathcal{B}$ ad the restriction $x : J \to L_2^{\mathcal{F}}(\Omega, \mathcal{B})$ is continuous. Let $\|\cdot\|_Z$ be a seminorm in $\mathcal{Z}$ defined by

$$\|x\|_{\mathcal{Z}} = (\sup_{t\in J}\|x_t\|_{\mathcal{B}}^2)^{\frac{1}{2}}.$$

It is easy to verify that $\mathcal{Z}$ furnished with the norm topology as defined above, is a Banach space.

The following result is a consequence of the phase space axioms.

Lemma 4.3.1. *([120]). Let $x : (-\infty, b] \to \mathcal{H}_\beta$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \varphi(t) \in L_2^{\mathcal{F}_0}(\Omega, \mathcal{B})$ and the restriction $x : J \to L_2^{\mathcal{F}}(\Omega, \mathcal{B})$ is continuous, then*

$$\|x_s\|_{\mathcal{B}} \leq M_b\mathbb{E}\|\varphi\|_{\mathcal{B}} + K_b \sup_{0\leq s\leq b} \mathbb{E}\|x(s)\|_\beta,$$

where $K_b = \sup\{K(t) : t \in J\}$ and $M_b = \sup\{M(t) : t \in J\}$.

Throughout this section, we use the notation $\mathcal{P}(\mathcal{H})$ for the family of all nonempty subsets of $\mathcal{H}$. Let us introduce the following notations:

$\mathcal{P}_{cl}(\mathcal{H}) = \{Y \in \mathcal{P}(\mathcal{H}) : Y \text{ is closed }\}$, $\mathcal{P}_{bd}(\mathcal{H}) = \{Y \in \mathcal{P}(\mathcal{H}) : Y \text{ is bounded }\}$,
$\mathcal{P}_{cv}(\mathcal{H}) = \{Y \in \mathcal{P}(\mathcal{H}) : Y \text{ is convex }\}$, $\mathcal{P}_{cp}(\mathcal{H}) = \{Y \in \mathcal{P}(\mathcal{H}) : Y \text{ is compact }\}$,
$\mathcal{P}_{cd}(\mathcal{H}) = \{Y \in \mathcal{P}(\mathcal{H}) : Y \text{ is compact-acyclic }\}$.

For $x \in \mathcal{H}$ and $Y, Z \in \mathcal{P}_{bd,cl}(\mathcal{H})$, we denote by $D(x, Y) = \inf\{\|x - y\| : y \in Y\}$ and $\epsilon(Y, Z) = \sup_{a\in Y} D(a, Z)$, and the Hausdorff metric $H_d : \mathcal{P}_{bd,cl}(\mathcal{H}) \times \mathcal{P}_{bd,cl}(\mathcal{H}) \to \mathbb{R}^+$ by $H_d(A, B) = \max\{\epsilon(A, B), \epsilon(B, A)\}$.

A multi-valued map $G : J \to \mathcal{P}_{bd,cl,cv}(\mathcal{H})$ is measurable if for each $x \in \mathcal{H}$, the function $t \to D(x, G(t))$ is a measurable function on J.

Definition 4.3.1. *([33]). Let $G : \mathcal{H} \to \mathcal{P}_{bd,cl}(\mathcal{H})$ be a multi-valued map. Then G is called a multi-valued contraction if there exists a constant $0 < \kappa < 1$ such that for each $x, y \in \mathcal{H}$ we have*

$$H_d(G(x) - G(y)) \leq \kappa\|x - y\|.$$

The constant κ is called a contraction constant of G.

Now, we give knowledge on the α-resolvent operator which appeared in [6].

Definition 4.3.2. *A one-parameter family of bounded linear operators $(\mathcal{R}_\alpha(t))_{t\geq 0}$ on $\mathcal{H}$ is called an α-resolvent operator for*

$$^cD^\alpha x(t) = Ax(t) + \int_0^t R(t-s)x(s)ds, \tag{4.24}$$

$$x_0 = \varphi \in \mathcal{H}, \qquad x'(0) = 0, \tag{4.25}$$

if the following conditions are verified.

(i) The function $\mathcal{R}_\alpha(\cdot) : [0,\infty) \to L(\mathcal{H})$ is strongly continuous and $\mathcal{R}_\alpha(0)x = x$ for all $x \in \mathcal{H}$ and $\alpha \in (1,2)$.

(ii) For $x \in \mathcal{D}(A), \mathcal{R}_\alpha(\cdot)x \in \mathcal{C}([0,\infty),[\mathcal{D}(A)]) \cap \mathcal{C}^1((0,\infty),\mathcal{H})$ and

$$D_t^\alpha \mathcal{R}_\alpha(t)x = A\mathcal{R}_\alpha(t)x + \int_0^t R(t-s)\mathcal{R}_\alpha(s)xds,$$

$$D_t^\alpha \mathcal{R}_\alpha(t)x = \mathcal{R}_\alpha(t)Ax + \int_0^t \mathcal{R}_\alpha(t-s)R(s)xds$$

for every $t \geq 0$.

The consideration of this section is based on the following conditions:

(C1) The operator $A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ is a closed linear operator with $[\mathcal{D}(A)]$ dense in $\mathcal{H}$, and for some $\phi_0 \in (0, \frac{\Pi}{2}]$, for each $\phi < \phi_0$ there is a positive constant $C_0 = C_0(\phi)$ such that $\lambda \in \rho(A)$ for each

$$\lambda \in \Sigma_{0,\alpha\vartheta} = \{\lambda \in \boldsymbol{C}, \lambda \neq 0, |\arg(\lambda)| < \alpha\vartheta\},$$

where $1 < \alpha < 2, \vartheta = \phi + \frac{\Pi}{2}$ and $\|R(\lambda, A)\| \leq \frac{C_0}{|\lambda|}$ for all $\lambda \in \Sigma_{0,\alpha\vartheta}$.

(C2) For all $t \geq 0, R(t) : \mathcal{D}(R(t)) \subseteq \mathcal{H} \to \mathcal{H}$ is a closed linear operator, $\mathcal{D}(A) \subseteq \mathcal{D}(R(t))$ and $R(\cdot)x$ is strongly measurable on $(0,\infty)$ for each $x \in \mathcal{D}(A)$. There exists $b(\cdot) \in L^1_{loc}(\mathbb{R}^+)$ such that $\widehat{b}(\lambda)$ exists for $Re(\lambda) > 0$ and $\|R(t)x\|_{\mathcal{H}} \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in \mathcal{D}(A)$. Moreover, the operator-valued function $\widehat{R} : \sum_{0,\Pi/2} \to L(\mathcal{D}(A),\mathcal{H})$ has an analytical extension (still denoted by $\widehat{R}$) to $\sum_{0,\vartheta}$ such that $\|\widehat{R}(\lambda)x\|_{\mathcal{H}} \leq \|\widehat{R}(\lambda)\|_{\mathcal{H}}\|x\|_1$ for all $x \in \mathcal{D}(A)$, and $\|\widehat{R}(\lambda)\|_{\mathcal{H}} = o(1/|\lambda|)$, as $|\lambda| \to \infty$.

(C3) There exists a subspace $\widetilde{\mathcal{D}} \subseteq \mathcal{D}(A)$ dense in $[\mathcal{D}(A)]$ and a positive constant C_1 such that $A(\widetilde{\mathcal{D}}) \subseteq \mathcal{D}(A)$, $\widehat{R}(\lambda)\widetilde{\mathcal{D}} \subseteq \mathcal{D}(A)$, and $\|A\widehat{R}(\lambda)x\|_{\mathcal{H}} \leq C_1\|x\|_{\mathcal{H}}$ for every $x \in \widetilde{\mathcal{D}}$ and all $\lambda \in \Sigma_{0,\vartheta}$.

In the sequel, for $r > 0$ and $\frac{\Pi}{2} < \theta < \vartheta$,

$$\Sigma_{r,\theta} = \{\lambda \in \mathbf{C}, |\lambda| > r, |\arg(\lambda)| < \theta\},$$

$\Gamma^i_{r,\theta}, i = 1,2,3,$ are the paths

$\Gamma^1_{r,\theta} = \{te^{i\theta} : t \geq r\}, \quad \Gamma^2_{r,\theta} = \{te^{i\xi} : |\xi| \leq \theta\}, \quad \Gamma^3_{r,\theta} = \{te^{-i\theta} : t \geq r\},$
and $\Gamma_{r,\theta} = \cup_{i=1}^3 \Gamma^i_{r,\theta}$ oriented counterclockwise. In addition, $\rho_\alpha(G_\alpha)$ are the sets

$$\rho_\alpha(G_\alpha) = \{\lambda \in \mathbf{C} : G_\alpha(\lambda) := \lambda^{\alpha-1}(\lambda^\alpha I - A - \widehat{R}(\lambda))^{-1} \in L(\mathcal{H})\}.$$

We now define the operator family $(\mathcal{R}_\alpha(t))_{t\geq 0}$ by

$$\mathcal{R}_\alpha(t) := \begin{cases} \frac{1}{2\Pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} G_\alpha(\lambda)d\lambda, & t > 0, \\ I & , t = 0. \end{cases}$$

Definition 4.3.3. *An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, b] \to \mathcal{H}_\beta$ is called a mild solution of the problem (4.22) – (4.23) if $x_0 = \varphi$, $x_{\rho(s,x_s)} \in \mathcal{B}$ for every $s \in J$; the function $s \to A\mathcal{S}_\alpha(t-s)g(s,x_s)$ and $s \to \int_0^s R(s-\tau)\mathcal{S}_\alpha(s-\tau)g(\tau,x_\tau)d\tau$ is integrable on $[0,t)$ for all $t \in (0,b]$ and the restriction of $x(\cdot)$ to the interval $[0,b)$ is a continuous stochastic process, such that the following stochastic integral inclusion is verified:*

$$x(t) \in \mathcal{R}_\alpha(t)[\varphi(0) - g(0,\varphi)] + g(t,x_t) + \int_0^t A\mathcal{S}_\alpha(t-s)g(s,x_s)ds$$

$$+ \int_0^t \int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)g(\tau,x_\tau)d\tau ds + \int_0^t \mathcal{S}_\alpha(t-s)F(s,x_{\rho(s,x_s)})dW(s), \quad t \in J := [0,b].$$

Definition 4.3.4. *([6]). Let $\alpha \in (1,2)$, we define the family $(\mathcal{S}_\alpha(t))_{t\geq 0}$ by*

$$\mathcal{S}_\alpha(t) := \int_0^t g_{\alpha-1}(t-s)\mathcal{R}_\alpha(s)ds$$

for each $t \geq 0$.

Lemma 4.3.2. *([8]). There exists $r_1 > 0$ such that $\sum_{r_1,\vartheta} \subseteq \rho_\alpha(G_\alpha)$ and the function $G_\alpha : \sum_{r_1,\vartheta} \to L(\mathcal{H})$ is analytic. Moreover,*

$$G_\alpha(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\Big[I - \widehat{R}(\lambda)(\lambda^\alpha - A)^{-1}\Big]^{-1},$$

and there exist constants $\widetilde{M}_i, i = 1, 2$ such that

$$\|G_\alpha(\lambda)\| = \frac{\widetilde{M}_1}{|\lambda|}; \qquad \|AG_\alpha(\lambda)x\| = \frac{\widetilde{M}_2}{|\lambda|}\|x\|_1, \qquad x \in \mathcal{D}(A); \qquad \|AG_\alpha(\lambda)\| = \frac{\widetilde{M}_2}{|\lambda|^{1-\alpha}}$$

for every $\lambda \in \sum_{r_1,\vartheta}$

Lemma 4.3.3. *([6]). Assume that the condition $(C1) - (C3)$ are satisfied. Then there exists a unique α-resolvent operator for the problem (4.24) – (4.25).*

Lemma 4.3.4. *([6]). The function $\mathcal{R}_\alpha : [0,\infty) \to L(\mathcal{H})$ is strongly continuous and $\mathcal{R}_\alpha : (0,\infty) \to L(\mathcal{H})$ is uniformly continuous.*

Lemma 4.3.5. *([6]).*

(i) *If the function $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $L(\mathcal{H})$, then $\mathcal{S}_\alpha(\cdot)$ is exponentially bounded in $L(\mathcal{H})$.*

(ii) If the function $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $L(\mathcal{D}(A))$, then $\mathcal{S}_\alpha(\cdot)$ is exponentially bounded in $L(\mathcal{D}(A))$.

Lemma 4.3.6. *([6]). If $R(\lambda_0^\alpha, A) = (\lambda_0^\alpha - A)^{-1}$ is compact for some $\lambda_0 \in \rho(A)$, then $\mathcal{R}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are compact for all $t > 0$.*

Lemma 4.3.7. *([8]). Assume that the conditions $(C1) - (C3)$ are satisfied. Let $\alpha \in (1, 2)$ and $\beta \in (0, 1)$ such that $\alpha\beta \in (0, 1)$, then there exists a positive number M_β such that*

$$\|(-A)^\beta \mathcal{R}_\alpha(t)\| \leq M_\beta e^{rt} t^{-\alpha\beta}, \qquad \|(-A)^\beta \mathcal{S}_\alpha(t)\| \leq M_\beta e^{rt} t^{\alpha(1-\beta)-1}$$

for all $t > 0$. If $x \in [\mathcal{D}((-A)^\beta)]$, then

$$(-A)^\beta \mathcal{R}_\alpha(t)x = \mathcal{R}_\alpha(t)(-A)^\beta x, \qquad (-A)^\beta \mathcal{S}_\alpha(t)x = \mathcal{S}_\alpha(t)(-A)^\beta x.$$

At the end, we recall the nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan.

Lemma 4.3.8. *([91]). Let $\mathcal{H}$ be a Hilbert space with V an open convex subset of $\mathcal{H}$ and $y \in \mathcal{H}$. Suppose that*

(a) $\Phi : \overline{V} \to \mathcal{P}_{cd}(\mathcal{H})$ has closed graph;

(b) $\Phi : \overline{V} \to \mathcal{P}_{cd}(\mathcal{H})$ is a condensing map with $\Phi(\overline{V})$ a subset of a bounded set in $\mathcal{H}$ hold. Then either

(i) Φ has a fixed point in $\overline{V}$, or

(ii) There exist $y \in \partial V$ and $\lambda \in (0, 1)$ with $y \in \lambda\Phi(y) + (1 - \lambda)\{y_0\}$.

4.3.1 The Main Results

In this section, we shall present and prove our main result. Assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous. In addition, we make the following hypotheses: for some $\beta \in (0, 1)$

(**H1**) : The operator families $\mathcal{R}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are compact for all $t > 0$, and there exist constants M and M_1 such that $\|\mathcal{R}_\alpha(t)\|_{L(\mathcal{H})} \leq M$ and $\|\mathcal{S}_\alpha(t)\|_{L(\mathcal{H})} \leq M$ for every $t \in J$ and

$$\|(-A)^\beta \mathcal{S}_\alpha(t)\| \leq M_1 t^{\alpha(1-\beta)-1}, \ \ 0 < t \leq b.$$

(**H2**) : $R(\cdot)x \in \mathcal{C}(J, \mathcal{H})$ for every $x \in [\mathcal{D}((-A)^{1-\beta})]$, and there exist a positive integrable function $m \in L^1([0, b])$ and a constant M_2 such that

$$\|R(s)\mathcal{S}_\alpha(t)\|_{L([\mathcal{D}((-A)^\beta)], \mathcal{H})} \leq M_2 m(s) t^{\alpha\beta-1}, \ \ 0 \leq s < t \leq b.$$

(**H3**) : There exists a constant $\delta \in (0, 1)$ such that $g : J \times \mathcal{B} \to [\mathcal{D}((-A)^{\delta+\beta})]$ satisfies the Lipschitz condition, i.e., there exists a constant $L_g > 0$ such that

$$\mathbb{E}\|(-A)^{\delta+\beta} g(t, \psi_1) - (-A)^{\delta+\beta} g(t, \psi_2)\|^2 \leq L_g \|\psi_1 - \psi_2\|_{\mathcal{B}}^2, \ t \in J, \ \psi_1, \psi_2 \in \mathcal{B},$$

$$\mathbb{E}\|(-A)^{\delta+\beta} g(t, \psi)\|^2 \leq L_g \left(1 + \|\psi\|_{\mathcal{B}}^2\right), \ \ t \in J, \ \psi \in \mathcal{B}.$$

(**H4**) : The function $t \to \varphi_t$ is continuous from $\varepsilon(\rho^-) = \{\rho(s, \psi) \leq 0, (s, \psi) \in J \times \mathcal{B}\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \varepsilon(\rho^-) \to (0, \infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \leq J^\varphi(t) \|\varphi\|_{\mathcal{B}}$ for each $t \in \varepsilon(\rho^-)$.

(**H5**) : The multi-valued map $F : J \times \mathcal{B} \to \mathcal{P}_{bd,cl,cv}(L(\mathcal{K},\mathcal{H}))$; for each $t \in J$, the function $F(t,\cdot) : \mathcal{B} \to \mathcal{P}_{bd,cl,cv}(L(\mathcal{K},\mathcal{H}))$; is u.s.c. and for each $\psi \in \mathcal{B}$, the function $F(\cdot,\psi)$ is measurable; for each fixed $\psi \in \mathcal{B}$, the set

$$S_{F,\psi} = \{f \in L^2(J, L(\mathcal{K},\mathcal{H})) : f(t) \in F(t,\psi) \text{ for a.e.} \, t \in J\}$$

is nonempty.

(**H6**) : There exists a positive function $l : J \to \mathbb{R}^+$ such that the function $s \mapsto (t-s)^{2\alpha(1-\beta)-2}l(s)$ belongs to $L^1([0,t],\mathbb{R}^+)$, $t \in J$, and

$$\limsup_{\|\psi\|^2_{\mathcal{B}} \to \infty} \frac{\|F(t,\psi)\|^2}{l(t)\|\psi\|^2_{\mathcal{B}}} = \gamma$$

uniformly in $t \in J$ for a nonnegative constant γ, where

$$\|F(t,\psi)\|^2 = \sup\{\mathbb{E}\|f\|^2 : f(t) \in F(t,\psi)\}.$$

Lemma 4.3.9. *([43]). Let $x : (-\infty, b] \to \mathcal{H}$ such that $x_0 = \varphi$. If (**H4**) is satisfied, then*

$$\|x_s\|_{\mathcal{B}} \le (M_b + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \sup\{\|x(\theta)\|;\, \theta \in [0, \max[0,s]]\}, \quad s \in \varepsilon(\rho^-) \cup J,$$

where $J_0^{\varphi} = \sup_{t\in\varepsilon(\rho^-)} J^{\varphi}(t)$.

Remark 4.3.1. Let $\varphi \in \mathcal{B}$ and $t \le 0$. The notation φ_t represents the function defined by $\varphi_t = \varphi(t+\theta)$. Consequently, if the function $x(\cdot)$ in axiom (i), (fundamental axioms of phase space), is such that $x_0 = \varphi$, then $x_t = \varphi_t$. We observe that φ_t is well-defined for $t < 0$ since the domain of φ is $(-\infty, 0]$.

Lemma 4.3.10. *([74]). Let J be a compact interval and $\mathcal{H}$ be a Hilbert space. Let F be a multivalued map satisfying (**H5**) and let Γ be a linear continuous operator from $L^2(J,\mathcal{H})$ to $\mathcal{C}(J,\mathcal{H})$. Then, the operator $\Gamma o S_F : \mathcal{C}(J,\mathcal{H}) \to \mathcal{P}_{cp,cv}(\mathcal{C}(J,\mathcal{H}))$ is a closed graph in $\mathcal{C}(J,\mathcal{H}) \times \mathcal{C}(J,\mathcal{H})$.*

Theorem 4.3.1. *Let $\varphi \in L_2^{\mathcal{F}_0}(\Omega,\mathcal{H}_\beta)$. If the assumptions (**H1**) $-$ (**H6**) are satisfied and $\rho(t,\psi) \le t$ for every $(t,\psi) \in J \times \mathcal{B}$, then the system (4.22) $-$ (4.23) has at least one mild solution on J, provided that*

$$6L_g K_b^2 \|(-A)^{-(\delta+\beta)}\|^2 \left[1 + \|m\|_2^2 \frac{M_2^2 b^{2\alpha(\delta+\beta)-1}}{2\alpha(\delta+\beta)-1}\right] + 6L_g K_b^2 \frac{M_1^2 b^{2\alpha(\delta+\beta)}}{\alpha^2(\delta+\beta)^2} < 1. \tag{4.26}$$

Proof. Let $B' = \{x : (-\infty, b] \to \mathcal{H}_{\mathcal{B}}$ such that $x_0 = 0 \in \mathcal{B}, x \backslash J \in \mathcal{C}(J,\mathcal{H}_\beta)\}$ endowed with the uniform convergence topology. Let $\mathcal{Z}' = \mathcal{C}((-\infty,b], L_2(\Omega,\mathcal{B}'))$. Consider the multivalued map $\Phi : \mathcal{Z}' \to \mathcal{P}(\mathcal{Z}')$ defined by: Φx the set of $h \in \mathcal{Z}'$ such that

$$h(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0], \\ \mathcal{R}_\alpha(t)[\varphi(0) - g(0,\varphi)] + g(t,\overline{x}_t) + \int_0^t A\mathcal{S}_\alpha(t-s)g(s,\overline{x}_s)ds & \\ + \int_0^t \int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)g(\tau,\overline{x}_\tau)d\tau ds + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dW(s) & \text{for a.e. } t \in J, \end{cases}$$

where $f \in S_{F,\overline{x}_\rho} = \{f \in L^2(L(\mathcal{K},\mathcal{H})) : f(t) \in F(t, \overline{x}_\rho(t,\overline{x}_t)) \; for \; a.e. \; t \in J\}$ and $\overline{x} : (-\infty, 0] \to \mathcal{H}_\beta$ is such that $\overline{x}_0 = \varphi$ and $\overline{x} = x$ on J.

We shall show that the operator Φ has a fixed point, which is then a mild solution for the problem (4.22) $-$ (4.23). Let $\overline{\varphi} : (-\infty, 0) \to \mathcal{H}_\beta$ be the extension of $(-\infty, 0]$ such that

$\overline{\varphi}(\theta)=\varphi(0)=0$ on J and $J_0^\varphi=\sup\{J^\varphi(s): s\in\varepsilon(\rho^-)\}$. We now show that Φ satisfies all the conditions of Lemma 4.3.8. The proof will be given in several steps.

Step 1. We shall show that there exists an open set $V\subseteq\mathcal{Z}'$ with $x\in\lambda\Phi x$ for $0<\lambda<1$ and $x\notin\partial V$. Let $\lambda\in(0,1)$ and $x\in\lambda\Phi x$, then there exists $f\in S_{F,\overline{x}_\rho}$ such that

$$\begin{aligned}x(t)&=\lambda\mathcal{R}_\alpha(t)[\varphi(0)-g(0,\varphi)]+\lambda g(t,\overline{x}_t)+\lambda\int_0^t A\mathcal{S}_\alpha(t-s)g(s,\overline{x}_s)ds\\&+\lambda\int_0^t\int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)g(\tau,\overline{x}_\tau)d\tau ds+\lambda\int_0^t\mathcal{S}_\alpha(t-s)f(s)dW(s),\quad t\in J.\end{aligned}$$

It follows from assumption (**H6**) that there exist two nonnegative real numbers a_1 and a_2 such that for any $\psi\in\mathcal{B}$ and $t\in J$,

$$\|F(t,\psi)\|^2\le a_1 l(t)+a_2 l(t)\|\psi\|_{\mathcal{B}}^2. \tag{4.27}$$

Then, by (**H1**) – (**H3**) and (4.27), from the above equation, for $t\in J$, we have

$$\begin{aligned}&\mathbb{E}\|x(t)\|^2\\
\le\;&5\mathbb{E}\|\mathcal{R}_\alpha(t)[\varphi(0)-g(0,\varphi)]\|^2+5\mathbb{E}\|g(t,\overline{x}_t)\|^2+5\mathbb{E}\left\|\int_0^t A\mathcal{S}_\alpha(t-s)g(s,\overline{x}_s)ds\right\|^2\\
&+5\mathbb{E}\left\|\int_0^t\int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)g(\tau,\overline{x}_\tau)d\tau ds\right\|^2+5\mathbb{E}\left\|\int_0^t\mathcal{S}_\alpha(t-s)f(s)dW(s)\right\|^2\\
\le\;&10M^2\Big(\mathbb{E}\|\varphi(0)\|^2+\|(-A)^{-(\delta+\beta)}\|^2L_g\Big(1+\|\varphi\|_{\mathcal{B}}^2\Big)\Big)+5\|(-A)^{-(\delta+\beta)}\|^2\\
&\times L_g\Big(1+\|\overline{x}_t\|_{\mathcal{B}}^2\Big)+5M_1^2\frac{b^{\alpha(\beta-\delta)}}{\alpha(\beta-\delta)}\int_0^t(t-s)^{\alpha(\beta-\delta)-1}L_g\Big(1+\|\overline{x}_s\|_{\mathcal{B}}^2\Big)ds\\
&+5M_2^2\|(-A)^{-(\delta+\beta)}\|^2\int_0^t\int_0^s m^2(t-\tau)(t-s)^{2(\alpha(\delta+\beta)-1)}L_g\Big(1+\|\overline{x}_\tau\|_{\mathcal{B}}^2\Big)d\tau ds\\
&+5M_1^2\|(-A)^{-\beta}\|^2Tr(Q)\int_0^t(t-s)^{2(\alpha(1-\beta)-1)}[a_1l(s)+a_2l(s)\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathcal{B}}^2]ds\\
\le\;&10M^2\Big(\widetilde{H}^2\mathbb{E}\|\varphi\|_{\mathcal{B}}^2+\|(-A)^{-(\delta+\beta)}\|^2L_g\Big(1+\|\varphi\|_{\mathcal{B}}^2\Big)\Big)+5\|(-A)^{-(\delta+\beta)}\|^2\\
&\times L_g\Big(1+\|\overline{x}_t\|_{\mathcal{B}}^2\Big)+5M_1^2\frac{b^{\alpha(\beta-\delta)}}{\alpha(\beta-\delta)}\int_0^t(t-s)^{\alpha(\beta-\delta)-1}L_g\Big(1+\|\overline{x}_s\|_{\mathcal{B}}^2\Big)ds\\
&+5M_2^2\|(-A)^{-(\delta+\beta)}\|^2\int_0^t\int_0^s m^2(t-\tau)(t-s)^{2(\alpha(\delta+\beta)-1)}L_g\Big(1+\|\overline{x}_\tau\|_{\mathcal{B}}^2\Big)d\tau ds\\
&+5M_1^2\|(-A)^{-\beta}\|^2Tr(Q)a_1\int_0^t(t-s)^{2(\alpha(1-\beta)-1)}l(s)ds\\
&+5M_1^2\|(-A)^{-\beta}\|^2Tr(Q)a_2\int_0^t(t-s)^{2(\alpha(1-\beta)-1)}l(s)\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathcal{B}}^2ds\\
\le\;&\widetilde{M}+5\|(-A)^{-(\delta+\beta)}\|^2L_g\|\overline{x}_t\|_{\mathcal{B}}^2+5M_1^2\frac{b^{\alpha(\beta-\delta)}}{\alpha(\beta-\delta)}L_g\int_0^t(t-s)^{\alpha(\beta-\delta)-1}\|\overline{x}_s\|_{\mathcal{B}}^2ds\\
&+5M_2^2\|(-A)^{-(\delta+\beta)}\|^2L_g\int_0^t\int_0^s m^2(t-\tau)(t-s)^{2(\alpha(\delta+\beta)-1)}\|\overline{x}_\tau\|_{\mathcal{B}}^2d\tau ds\\
&+5M_1^2\|(-A)^{-\beta}\|^2Tr(Q)a_2\int_0^t(t-s)^{2(\alpha(1-\beta)-1)}l(s)\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathcal{B}}^2ds,\end{aligned}$$

where

$$\widetilde{M} = 10M^2\bigg(\widetilde{H}^2\mathbb{E}\|\varphi\|_{\mathcal{B}}^2 + \|(-A)^{-(\delta+\beta)}\|^2 L_g\bigg(1+\|\varphi\|_{\mathcal{B}}^2\bigg)\bigg) + 5\|(-A)^{-(\delta+\beta)}\|^2 L_g + 5M_1^2 \times \frac{b^{2\alpha(\beta-\delta)}}{\alpha^2(\beta-\delta)^2} L_g + 5M_2^2\|(-A)^{-(\delta+\beta)}\|^2 L_g \int_0^t \int_0^s m^2(t-\tau)(t-s)^{2(\alpha(\delta+\beta)-1)} d\tau ds + 5M_1^2\|(-A)^{-\beta}\|^2 Tr(Q) a_1 \int_0^t (t-s)^{2(\alpha(1-\beta)-1)} l(s) ds.$$

By Lemma 4.3.1, it follows that $\rho(s,\overline{x}_s) \le s$, $s\in[0,t]$, $t\in[0,b]$ and

$$\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathcal{B}}^2 \le 2[(M_b+J_0^{\varphi})\mathbb{E}\|\varphi\|_{\mathcal{B}}]^2 + 2K_b^2 \sup_{0\le s\le b}\mathbb{E}\|x(s)\|^2. \tag{4.28}$$

Then, for each $t\in[0,b]$, we have

$$\begin{aligned}\mathbb{E}\|x(t)\|^2 \;\le\;& \widehat{M} + 10\|(-A)^{-(\delta+\beta)}\|^2 L_g K_b^2 \sup_{0\le t\le b}\mathbb{E}\|x(t)\|^2 \\ &+10M_1^2\frac{b^{\alpha(\beta-\delta)}}{\alpha(\beta-\delta)} L_g K_b^2 \int_0^t (t-s)^{\alpha(\beta-\delta)-1} \sup_{0\le v\le s}\mathbb{E}\|x(v)\|^2 ds \\ &+10M_2^2\|(-A)^{-(\delta+\beta)}\|^2 L_g K_b^2 \int_0^t\int_0^s m^2(t-\tau)(t-s)^{2(\alpha(\delta+\beta)-1)} \sup_{0\le v\le s}\mathbb{E}\|x(v)\|^2 d\tau ds \\ &+10M_1^2\|(-A)^{-\beta}\|^2 Tr(Q) a_2 K_b^2 \int_0^t (t-s)^{2(\alpha(1-\beta)-1)} l(s) \sup_{0\le v\le s}\mathbb{E}\|x(v)\|^2 ds,\end{aligned}$$

where

$$\begin{aligned}\widehat{M} =\;& \widetilde{M} + 5\|(-A)^{-(\delta+\beta)}\|^2 L_g\widehat{C} + 5M_1^2\frac{b^{2\alpha(\beta-\delta)}}{\alpha^2(\beta-\delta)^2} L_g\widehat{C} \\ &+5M_2^2\|(-A)^{-(\delta+\beta)}\|^2 L_g\widehat{C}\int_0^t\int_0^s m^2(t-\tau)(t-s)^{2(\alpha(\delta+\beta)-1)} d\tau ds \\ &+5M_1^2\|(-A)^{-\beta}\|^2 Tr(Q) a_2\widehat{C}\int_0^b (b-s)^{2(\alpha(1-\beta)-1)} l(s) ds,\end{aligned}$$

$\widehat{C} = 2[(M_b+J_0^{\varphi})\mathbb{E}\|\varphi\|_{\mathcal{B}}]^2$.
Hence by the condition (4.26), we have

$$\begin{aligned}&\sup_{0\le t\le b}\mathbb{E}\|x(t)\|^2 \\ \le\;& \frac{\widehat{M}}{1-\widetilde{C}} + \int_0^b \bigg[K_1(t-s)^{\alpha(\beta-\delta)-1} + \int_0^s K_2 m^2(t-\tau)(t-s)^{2(\alpha(\delta+\beta)-1)} d\tau \\ &+K_3(t-s)^{2(\alpha(1-\beta)-1)} l(s)\bigg] \sup_{0\le v\le s}\mathbb{E}\|x(v)\|^2 ds,\end{aligned}$$

where

$$K_1 = \frac{10M_1^2 L_g K_b^2 b^{\alpha(\beta-\delta)}}{(1-\widetilde{C})\alpha(\beta-\delta)}, \qquad K_2 = \frac{10M_2^2\|(-A)^{-(\delta+\beta)}\|^2 L_g K_b^2}{1-\widetilde{C}},$$

$$K_3 = \frac{10M_1^2\|(-A)^{-\beta}\|^2 a_2 K_b^2 Tr(Q)}{1-\widetilde{C}}, \qquad \text{and } \widetilde{C} = 10\|(-A)^{-(\delta+\beta)}\|^2 L_g K_b^2 < 1.$$

Applying Gronwall's inequality in the above expression, we obtain

$$\begin{aligned}&\sup_{0\le t\le b}\mathbb{E}\|x(t)\|^2 \\ \le\;& \frac{\widehat{M}}{1-\widetilde{C}} \exp\left\{\frac{K_1 b^{\alpha(\beta-\delta)}}{\alpha(\beta-\delta)} + \frac{K_2\|m\|_2^2 b^{2\alpha(\delta+\beta)-1}}{2\alpha(\delta+\beta)-1} + K_3\int_0^b (t-s)^{2(\alpha(1-\beta)-1)} l(s) ds\right\}\end{aligned}$$

and therefore

$$\|x\|_{\mathcal{Z}}^2 \leq \frac{\widehat{M}}{1-\widetilde{C}} \exp\left\{\frac{K_1 b^{\alpha(\beta-\delta)}}{\alpha(\beta-\delta)} + \frac{K_2\|m\|_2^2 b^{2\alpha(\delta+\beta)-1}}{2\alpha(\delta+\beta)-1} + K_3\int_0^b (t-s)^{2(\alpha(1-\beta)-1)} l(s)ds\right\} < \infty$$

then there exists r^* such that $\|x\|_{\mathcal{Z}}^2 \neq r^*$. Set $V = \{x \in \mathcal{Z}' : \|x\|_{\mathcal{Z}}^2 < r^*\}$. From the choice of V, there is no $x \in \partial V$ such that $x \in \lambda\Phi x$ for $0 < \lambda < 1$.

Step 2. Φ has closed graph. Let $x^{(n)} \to x^*, h_n \in \Phi x^{(n)}, x^{(n)} \in \overline{V} = B_r(0, \mathcal{Z}')$ and $h_n \to h_*$. From Axiom (i), it is easy to see that $\left(\overline{x^{(n)}}\right)_s \to \overline{x^*}_s$ uniformly for $s \in (-\infty, b]$ as $n \to \infty$. We prove that $h_* \in \Phi\overline{x^*}$. Now, $h_n \in \Phi\overline{x^{(n)}}$ means that there exists $f_n \in S_{F,\overline{x^{(n)}}_\rho}$ such that for each $t \in [0, b]$,

$$h_n(t) = \mathcal{R}_\alpha(t)[\varphi(0) - g(0,\varphi)] + g(t, (\overline{x^{(n)}})_t) + \int_0^t A\mathcal{S}_\alpha(t-s)g(s, (\overline{x^{(n)}})_s)ds + \int_0^t\int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)g(\tau, (\overline{x^{(n)}})_\tau)d\tau ds + \int_0^t \mathcal{S}_\alpha(t-s)f_n(s)dW(s).$$

We must prove that there exists $f_* \in S_{F,\overline{x^*}_\rho}$ such that for each $t \in [0, b]$,

$$h_*(t) = \mathcal{R}_\alpha(t)[\varphi(0) - g(0,\varphi)] + g(t, (\overline{x^*})_t) + \int_0^t A\mathcal{S}_\alpha(t-s)g(s, (\overline{x^*})_s)ds + \int_0^t\int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)g(\tau, (\overline{x^*})_\tau)d\tau ds + \int_0^t \mathcal{S}_\alpha(t-s)f_*(s)dW(s).$$

For every $t \in [0, b]$, we have

$$\Bigg\|\Bigg(h_n(t) - \mathcal{R}_\alpha(t)[\varphi(0) - g(0,\varphi)] - g(t, (\overline{x^{(n)}})_t) - \int_0^t A\mathcal{S}_\alpha(t-s)g(s, (\overline{x^{(n)}})_s)ds - \int_0^t\int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)g(\tau, (\overline{x^{(n)}})_\tau)d\tau ds\Bigg) - \Bigg(h_*(t) - \mathcal{R}_\alpha(t)[\varphi(0) - g(0,\varphi)] - g(t, (\overline{x^*})_t) - \int_0^t A\mathcal{S}_\alpha(t-s)g(s, (\overline{x^*})_s)ds - \int_0^t\int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)g(\tau, (\overline{x^*})_\tau)d\tau ds\Bigg)\Bigg\|_{\mathcal{Z}}^2 \to 0 \quad as \quad n \to \infty.$$

Consider the linear continuous operator $\Theta : L^2(L(\mathcal{K}, \mathcal{H})) \to \mathcal{Z}$,

$$\Theta(f)(t) = \int_0^t \mathcal{S}_\alpha(t-s)f(s)dW(s).$$

From Lemma 4.3.10, it follows that $\Theta \circ S_F$ is a closed graph operator. Moreover,

$$\Bigg(h_n(t) - \mathcal{R}_\alpha(t)[\varphi(0) - g(0,\varphi)] - g(t, (\overline{x^{(n)}})_t) - \int_0^t A\mathcal{S}_\alpha(t-s)g(s, (\overline{x^{(n)}})_s)ds - \int_0^t\int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)g(\tau, (\overline{x^{(n)}})_\tau)d\tau ds\Bigg) \in \Gamma(S_{F,\overline{x^{(n)}}}).$$

Since $\overline{x^{(n)}} \to \overline{x^*}$ for some $f_* \in S_{F,\overline{x^*}_\rho}$, it follows from Lemma 4.3.10 that

$$\begin{aligned}
& h_*(t) - \mathcal{R}_\alpha(t)[\varphi(0) - g(0,\varphi)] - g(t,(\overline{x^*})_t) - \int_0^t A\mathcal{S}_\alpha(t-s)g(s,(\overline{x^*})_s)ds \\
& - \int_0^t\int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)g(\tau,(\overline{x^*})_\tau)d\tau ds \\
& = \int_0^t \mathcal{S}_\alpha(t-s)f_*(s)dW(s).
\end{aligned}$$

Therefore, Φ has a closed graph.

Step 3. We show that the operator Φ is condensing. For this purpose, we decompose Φ as $\Phi = \Phi_1 + \Phi_2$, where the map $\Phi_1 : \overline{V} \to \mathcal{P}(\mathcal{Z}')$ be defined by $\Phi_1 x$ the set of $h_1 \in \mathcal{Z}'$ such that

$$h_1(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0], \\ -\mathcal{R}_\alpha(t) - g(0,\varphi) + g(t,\overline{x}_t) + \int_0^t A\mathcal{S}_\alpha(t-s)g(s,\overline{x}_s)ds & \\ + \int_0^t\int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)g(\tau,\overline{x}_\tau)d\tau ds & \text{for a.e. } t \in J, \end{cases}$$

and the map $\Phi_2 : \overline{V} \to \mathcal{P}(\mathcal{Z}')$ be defined by $\Phi_2 x$ the set of $h_2 \in \mathcal{Z}'$ such that

$$h_2(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0], \\ \mathcal{R}_\alpha(t)\varphi(0) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dW(s) & \text{for a.e. } t \in J, \end{cases}$$

We will verify that Φ_1 is a contraction while Φ_2 is a completly continuous operator.

We first show that Φ_1 is a contraction on $\mathcal{Z}'$. Let $t \in J$ and $y^*, y^{**} \in \mathcal{Z}'$. From (**H3**) and Lemma 4.3.1, we have

$$\begin{aligned}
& \mathbb{E}\|(\Phi_1 y^*)(t) - (\Phi_1 y^{**})(t)\|^2 \\
\leq\ & 3\mathbb{E}\left\|(-A)^{-(\delta+\beta)}[(-A)^{\delta+\beta}g(t,\overline{y^*}_t) - (-A)^{\delta+\beta}g(t,\overline{y^{**}}_t)]\right\|^2 \\
& +3\mathbb{E}\left\|\int_0^t (-A)^{1-\delta-\beta}\mathcal{S}_\alpha(t-s)[(-A)^{\delta+\beta}g(s,\overline{y^*}_s) - (-A)^{\delta+\beta}g(s,\overline{y^{**}}_s)]ds\right\|^2 \\
& +3\mathbb{E}\left\|\int_0^t\int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)(-A)^{-\delta-\beta}[(-A)^{\delta+\beta}g(\tau,\overline{y^*}_\tau) - (-A)^{\delta+\beta}g(\tau,\overline{y^{**}}_\tau)]d\tau ds\right\|^2 \\
\leq\ & 3\|(-A)^{-\delta-\beta}\|^2 L_g\|\overline{y^*}_t - \overline{y^{**}}_t\|^2_{\mathcal{B}} \\
& +3M_1^2\frac{b^{\alpha(\delta+\beta)}}{\alpha(\delta+\beta)}\int_0^t (t-s)^{\alpha(\delta+\beta)-1}\mathbb{E}\|(-A)^{\delta+\beta}g(s,\overline{y^*}_s) - (-A)^{\delta+\beta}g(s,\overline{y^{**}}_s)ds\|^2 \\
& +3M_2^2\|(-A)^{-\delta-\beta}\|^2\int_0^t\int_0^s m^2(s)(t-s)^{2\alpha(\delta+\beta)-2} \\
& \times\mathbb{E}\|(-A)^{\delta+\beta}g(\tau,\overline{y^*}_\tau) - (-A)^{\delta+\beta}g(\tau,\overline{y^{**}}_\tau)\|^2 d\tau ds \\
\leq\ & 3\|(-A)^{-\delta-\beta}\|^2 L_g\|\overline{y^*}_t - \overline{y^{**}}_t\|^2_{\mathcal{B}} + 3M_1^2 L_g\frac{b^{\alpha(\delta+\beta)}}{\alpha(\delta+\beta)} \\
& \times\int_0^t (t-s)^{\alpha(\delta+\beta)-1}\|\overline{y^*}_s - \overline{y^{**}}_s\|^2_{\mathcal{B}}ds \\
& +3M_2^2 L_g\|(-A)^{-\delta-\beta}\|^2\int_0^t\int_0^s m^2(s)(t-s)^{2\alpha(\delta+\beta)-2}\|\overline{y^*}_\tau - \overline{y^{**}}_\tau\|^2_{\mathcal{B}}d\tau ds
\end{aligned}$$

$$\begin{aligned}
&\leq 6L_gK_b^2\left[\|(-A)^{-\delta-\beta}\|^2 + \frac{M_1^2 b^{2\alpha(\delta+\beta)}}{\alpha^2(\delta+\beta)^2} + \|(-A)^{-\delta-\beta}\|^2\|m\|_2^2\frac{M_1^2 b^{2\alpha(\delta+\beta)-1}}{2\alpha(\delta+\beta)-1}\right] \\
&\quad\times \sup_{s\in[0,b]} \mathbb{E}\|\overline{y^*}(s) - \overline{y^{**}}(s)\|^2 \\
&= 6L_gK_b^2\left[\|(-A)^{-\delta-\beta}\|^2 + \frac{M_1^2 b^{2\alpha(\delta+\beta)}}{\alpha^2(\delta+\beta)^2} + \|(-A)^{-\delta-\beta}\|^2\|m\|_2^2\frac{M_1^2 b^{2\alpha(\delta+\beta)-1}}{2\alpha(\delta+\beta)-1}\right] \\
&\quad\times \sup_{s\in[0,b]} \mathbb{E}\|y^*(s) - y^{**}(s)\|^2 \qquad (\textit{since } \overline{y} = y \textit{ on } J).
\end{aligned}$$

Taking supermum over t, we get

$$\|\Phi_1 y^* - \Phi_1 y^{**}\|_{\mathcal{Z}}^2 \leq C^*\|y^* - y^{**}\|_{\mathcal{Z}}^2,$$

where

$$C^* = 6L_gK_b^2\left[\|(-A)^{-\delta-\beta}\|^2 + \frac{M_1^2 b^{2\alpha(\delta+\beta)}}{\alpha^2(\delta+\beta)^2} + \|(-A)^{-\delta-\beta}\|^2\|m\|_2^2\frac{M_2^2 b^{2\alpha(\delta+\beta)-1}}{2\alpha(\delta+\beta)-1}\right] < 1.$$

Thus Φ_1 is a contraction on $\mathcal{Z}'$.

Now, we prove that $\Phi_2 x$ is convex for each $x \in \overline{V}$. In fact, if h_2^1, h_2^2 belong to $\Phi_2 x$, then there exist $f_1, f_2 \in S_{F,\overline{x}_\rho}$ such that

$$h_2^i(t) = \mathcal{R}_\alpha(t)\varphi(0) + \int_0^t \mathcal{S}_\alpha(t-s)f_i(s)dW(s), \qquad t \in J,\ i = 1, 2.$$

Let $0 \leq \lambda \leq 1$. For each $t \in J$, we have

$$(\lambda h_2^1 + (1-\lambda)h_2^2)(t) = \mathcal{R}_\alpha(t)\varphi(0) + \int_0^t \mathcal{S}_\alpha(t-s)[\lambda f_1(s) + (1-\lambda)f_2(s)]dW(s).$$

Since $S_{F,\overline{x}_\rho}$ is convex (because F has convex values), we have $(\lambda h_2^1 + (1-\lambda)h_2^2) \in \Phi_2 x$.

Next, we show that the operator $\Phi_2(\overline{V})$ is completely continuous. We first show that $\Phi_2(\overline{V})$ is equicontinuous. if $x \in \overline{V}$, from Lemma 4.3.1, it follows that

$$\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathcal{B}}^2 \leq [(M_b + J_0^\varphi)\|\varphi\|_{\mathcal{B}}]^2 + 2K_b^2 r^* := r'.$$

Let $0 < \tau_1 < \tau_2 \leq b$ and $\epsilon > 0$ be small. For each $x \in \overline{V}$, $h_2 \in \Phi_2 x$, there exists $f \in S_{F,\overline{x}_\rho}$ such that

$$h_2(t) = \mathcal{R}_\alpha(t)\varphi(0) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dW(s). \tag{4.29}$$

Then

$$\begin{aligned}
&\mathbb{E}\|h_2(\tau_2) - h_2(\tau_1)\|^2 \\
&\leq 4\mathbb{E}\|[\mathcal{R}_\alpha(\tau_2) - \mathcal{R}_\alpha(\tau_1)\varphi(0)]\|^2 + 4\mathbb{E}\left\|\int_0^{\tau_1-\epsilon}[\mathcal{S}_\alpha(\tau_2-s) - \mathcal{S}_\alpha(\tau_1-s)]f(s)dW(s)\right\|^2 \\
&\quad+4\mathbb{E}\left\|\int_{\tau_1-\epsilon}^{\tau_1}[\mathcal{S}_\alpha(\tau_2-s) - \mathcal{S}_\alpha(\tau_1-s)]f(s)dW(s)\right\|^2 + 4\mathbb{E}\left\|\int_{\tau_1}^{\tau_2}\mathcal{S}_\alpha(\tau_2-s)f(s)dW(s)\right\|^2 \\
&\leq 4\mathbb{E}\|[\mathcal{R}_\alpha(\tau_2) - \mathcal{R}_\alpha(\tau_1)\varphi(0)]\|^2 + 4\|(-A)^{-\beta}\|^2(a_1 + a_2r')(\tau_1-\epsilon)^{2(1-\alpha(1-\beta))}Tr(Q) \\
&\quad\times\int_0^{\tau_1-\epsilon}\|(-A)^{\beta}[\mathcal{S}_\alpha(\tau_2-s) - \mathcal{S}_\alpha(\tau_1-s)]\|^2(\tau_1-\epsilon-s)^{2(\alpha(1-\beta)-1)}l(s)ds \\
&\quad+4M_1^2\|(-A)^{-\beta}\|^2(a_1 + a_2r')Tr(Q)\int_{\tau_1-\epsilon}^{\tau_1}[(\tau_2-s)^{2(\alpha(1-\beta)-1)}l(s) + (\tau_1-s)^{2(\alpha(1-\beta)-1)}l(s)]ds \\
&\quad+4M_1^2\|(-A)^{-\beta}\|^2(a_1 + a_2r')Tr(Q)\int_{\tau_1}^{\tau_2}[(\tau_2-s)^{2(\alpha(1-\beta)-1)}l(s)ds.
\end{aligned}$$

The right-hand side of the above inequalities tends to zero independent of $x \in \overline{V}$, as $\tau_2 - \tau_1 \to 0$, with ϵ sufficiently small, since the compactness of $\mathcal{R}_\alpha(t)$, $\mathcal{S}_\alpha(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus the set $\{\Phi_2 x : x \in \overline{V}\}$ is equicontinuous. The equicontinuity for the other cases $\tau_1 < \tau_2 \leq 0$ or $\tau_1 \leq 0 \leq \tau_2 \leq b$ is very simple.

It remains to prove that $\Phi_2(\overline{V})(t) = \{h_2(t) : h_2(t) \in \Phi_2(\overline{V})\}$ is relatively compact in $\mathcal{H}_\beta$ for every $t \in J$. Let $0 < t \leq s \leq b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For each $x \in \overline{V}$, we define

$$h_{2,\epsilon}(t) = \mathcal{R}_\alpha(t)\varphi(0) + \int_0^{t-\epsilon} \mathcal{S}_\alpha(t-s)f(s)dW(s),$$

where $f \in S_{F,\overline{x}_\rho}$. Using the compactness of $\mathcal{S}_\alpha(t)$ for $t > 0$, we deduce that the set $U_\epsilon(t) = \{h_{2,\epsilon}(t) : x \in \overline{V}\}$ is relatively compact in $\mathcal{H}_\beta$ for every $\epsilon, 0 < \epsilon < t$. Moreover, for every $x \in \overline{V}$ we have

$$\begin{aligned} \mathbb{E}\|h_2(t) - h_{2,\epsilon}(t)\|^2 &\leq \mathbb{E}\left\| \int_{t-\epsilon}^t \mathcal{S}_\alpha(t-s)f(s)dW(s) \right\|^2 \\ &\leq M_1^2\|(-A)^{-\beta}\|^2(a_1 + a_2 r')Tr(Q) \int_{t-\epsilon}^t (t-s)^{2(\alpha(1-\beta)-1)} l(s)ds. \end{aligned}$$

The right-hand side of the above inequality tends to zero as $\epsilon \to 0$. Since there are relatively compact sets arbitrarily close to the set $U(t) = \{h_2(t) : x \in \overline{V}\}$, hence the set $U(t)$ is relatively compact in $\mathcal{H}_\beta$. By Arzelá-Ascoli theorem, we conclude that $\Phi_2(\overline{V})$ is completely continuous.

As a consequence of the above Steps 1-3, we conclude that Φ is a condensing map. All of the conditions of Lemma 4.3.8. are satisfied, we deduce that Φ has a fixed point $x \in \mathcal{Z}'$, which is a mild solution of the problem (4.22)−(4.23). The proof is complete. □

4.3.2 An Example

Consider the following fractional partial neutral stochastic functional integro-differential inclusions of the form

$$\begin{aligned} &D_t^\alpha\left[z(t,x) - \int_{-\infty}^t \int_0^\Pi \mu_1(t-s,\tau,x)z(s,\tau)d\tau ds\right] \\ &\in \frac{\partial^2}{\partial x^2}z(t,x) + \int_0^t (t-s)^\gamma e^{-\eta(t-s)}\frac{\partial^2}{\partial x^2}z(s,x)ds \\ &+ \int_{-\infty}^t \mu_2(t,s-t,x,z(s-\rho_1(t)\rho_2(\|z(t)\|,x))dW(s), \quad 0 \leq t \leq b,\ 0 \leq x \leq \Pi, \end{aligned} \tag{4.30}$$

$$z(t,0) = z(t,\Pi) = 0, \quad 0 \leq t \leq b, \tag{4.31}$$

$$z_t(0,x) = 0, \quad 0 \leq x \leq \Pi, \tag{4.32}$$

$$z(\tau,x) = \varphi(\tau,x), \quad -\infty < \tau \leq 0, \quad 0 \leq x \leq \Pi, \tag{4.33}$$

where D_t^α is a Caputo fractional partial derivative of order $\alpha \in (0,1)$, γ and η are positive numbers, φ is continuous and $W(t)$ denotes a standard cylindrical Wiener process in $\mathcal{H}$ defined on a stochastic space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{H} = L_2([0,\pi])$ with the norm $\|\cdot\|$ and Define the operator $A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ by $Aw = w''$ with the domain

$$\mathcal{D}(A) = \{w \in \mathcal{H};\ w, w' \text{ are absolutely continuous, } w'' \in \mathcal{H} \text{ and } w(0) = w(\pi) = 0\}.$$

It is well known that then A generates a strongly continuous semigroup $T(\cdot)$ which is compact, analytic, and self-adjoint. Furthermore, A has a discrete spectrum; the eigenvalues are $-n^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors $w_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. We also use the following properties:

i. If $w \in \mathcal{D}(A)$, then $Aw = \sum_{n=1}^{\infty} n^2\langle w, w_n\rangle w_n$.

ii. For each $w \in \mathcal{H}, A^{-1/2}w = \sum_{n=1}^{\infty} \frac{1}{n}\langle w, w_n\rangle w_n$. In particular, $\|A^{-1/2}\| = 1$.

iii. The operator $A^{1/2}$ is given by $A^{1/2}w = \sum_{n=1}^{\infty} n\langle w, w_n\rangle w_n$ on the space $\mathcal{D}(A^{1/2}) = \{w(\cdot) \in \mathcal{H}, \sum_{n=1}^{\infty} n\langle w, w_n\rangle w_n \in \mathcal{H}\}$.

Hence, A is sectorial-type operator and (C1) is satisfied. The operator $R(t) : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}, t \geq 0, R(t)x = t^{\gamma}e^{-\eta t}x''$ for $x \in \mathcal{D}(A)$. Moreover, it is easy to see that conditions (C2) and (C3) are satisfied with $b(t) = t^{\gamma}e^{-\eta t}$ and $\mathcal{D}(A) = \mathcal{C}_0^{\infty}([0, \Pi])$, where $\mathcal{C}_0^{\infty}([0, \Pi])$ is the space of infinitely differentiable functions that vanish at $x = 0$ and $x = \Pi$.

Let $\mathcal{H}_{\frac{1}{2}} := (\mathcal{D}((-A)^{\frac{1}{2}}, \|\cdot\|_{\frac{1}{2}})$, where $\|\cdot\|_{\frac{1}{2}} := \|(-A)^{\frac{1}{2}}x\|$ for each $x \in \mathcal{D}((-A)^{\frac{1}{2}})$.

Let $r \geq 0, 1 \leq p < 1$ and let $h : (-\infty, -r) \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (h-5), (h-6) in the terminology of Hino et al. [47]. Briefly, this means that h is locally integrable and there is a non-negative, locally bounded function ϑ on $(-\infty, 0]$ such that $h(\xi + \tau) \leq \vartheta(\xi)h(\tau)$ for all $\xi \leq 0$ and $\theta \in (-\infty, -r)\backslash N_{\xi}$, where $N_{\xi} \subseteq (-\infty, -r)$ is a set whose Lebesgue measure zero. We denote by $\mathcal{Z}_r \times L^p(h, \mathcal{H}_{\frac{1}{2}})$ the set consisting of all classes of functions $\varphi(-\infty, 0] \to \mathcal{H}_{\frac{1}{2}}$ such that $\varphi_{|[-r,0]} \in \mathcal{Z}([-r, 0], \mathcal{H}_{\frac{1}{2}}), \varphi(\cdot)$ is Lebesgue measurable on $(-\infty, -r)$ and $h\|\varphi\|_{\frac{1}{2}}^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm is given by

$$\|\varphi\|_{\mathcal{B}} = \sup_{-r\leq\tau\leq 0} \|\varphi(\tau)\|_{\frac{1}{2}} + \left(\int_{-\infty}^{-r} h(\tau)\|\varphi(\tau)\|_{\frac{1}{2}}^p d\tau\right)^{\frac{1}{p}}.$$

The space $\mathcal{B} = \mathcal{Z}_r \times L^p(h, \mathcal{H}_{\frac{1}{2}})$ satisfies axioms (i), (ii). Moreover, when $r = 0$ and $p = 2$, we can take $\widetilde{H} = 1, M(t) = \vartheta(-t)^{\frac{1}{2}}$ and $K(t) = 1 + (\int_{-t}^{0} h(\tau)d\tau)^{\frac{1}{2}}$ for $t \geq 0$ (see [47], Theorem 1.3.8 for details.)

In addition, we choose $\delta = \frac{1}{2}$ and assume that the following conditions hold:

(h-1) the functions $\rho_i : [0, \infty) \to [0, \infty), i = 1, 2$, are continuous.

(h-2) the functions $\mu_1(s, \tau, x), \frac{\partial\mu_1(s,\tau,x)}{\partial x}, \frac{\partial^2\mu_1(s,\tau,x)}{\partial x^2}$ are measurable, $\mu_1(s, \tau, \Pi) = \mu_1(s, \tau, 0) = 0$ for every $(-\infty, 0] \times [0, \Pi]$ and

$$L_g = \max\left\{\left(\int_0^{\Pi}\int_{-\infty}^{0}\int_0^{\Pi} \frac{1}{h(s)}\left(\frac{\partial^i\mu_1(s, \tau, x)}{\partial x^i}\right)^2 d\tau ds dx\right)^{\frac{1}{2}} : i = 0, 1, 2\right\} < \infty.$$

(h-3) the function $\mu_2 : \mathbb{R}^4 \to \mathbb{R}$ is continuous and there exist continuous functions $a_1, a_2 : \mathbb{R} \to \mathbb{R}$ such that

$$|\mu_2(t, s, x, y)| \leq a_1(t)a_2(s)|y|, \quad (t, s, x, y) \in \mathbb{R}^4$$

with $L_F = (\int_{-\infty}^{0} \frac{(a_2(s))^2}{h(s)} ds)^{\frac{1}{2}} < \infty$.

In the sequel, $\mathcal{B}$ will be the phase space $\mathcal{Z}_0 \times L^2(h, \mathcal{H}_{\frac{1}{2}})$ with $\varphi(s)(\tau) = \varphi(s, \tau)$. Let $g : [0, b] \times \mathcal{B} \to \mathcal{H}, F : [0, b] \times \mathcal{B} \to \mathcal{P}_{bd,cl,cv}(\mathcal{H})$ be the operators defined by

$$g(t, \varphi)(x) = \int_{-\infty}^{0}\int_0^{\Pi} \mu_1(-s, v, x)\varphi(s, v)dvds,$$

$$F(t,\varphi)(x) = \int_{-\infty}^{0} \mu_2(t,-s,x,\varphi(s,x))ds, \quad \rho(t,\varphi) = \rho_1(t)\rho_2(\|\varphi(0)\|).$$

Using these definitions, we can represent the system (4.30) − (4.33) in the abstract form (4.22) − (4.23). Moreover, using (h-2), we can prove that g is $\mathcal{D}(A)$-valued and $\|Ag(t,\varphi)\| \leq L_g\|\varphi\|_{\mathcal{B}}$. Similarly, using (h-3), we see that F is continuous and $\|F(t,\varphi)\| \leq a_t\|\varphi\|_{\mathcal{B}}$ for all $(t,\varphi) \in [0,b] \times \mathcal{B}$, where $a(t) = L_F a_1(t), t \in [0,b]$. Furthermore, we can impose some suitable conditions on the above-defined functions to verify the assumptions on Theorem 4.3.1, we can conclude that system (4.30) − (4.33) has at least one mild solution on $[0,b]$.

4.4 Approximate Controllability of Fractional Neutral Stochastic Functional Integro-Differential Inclusions with Infinite Delay

In this Section[3], we investigate the approximate controllability for a class of fractional neutral stochastic functional integro-differential inclusions with infinite delay of the form

$$\begin{cases} {}^cD_t^\alpha[x(t)-g(t,x_t)] \in Ax(t)+Bu(t)+f\left(t,x_t,\int_0^t H(t,s,x_s)ds\right) \\ \quad +\Sigma\left(t,x_t,\int_0^t K(t,s,x_s)ds\right)\frac{dW(t)}{dt}, \qquad t\in J:=[0,b], \\ x_0=\phi\in\mathcal{B}_h, \qquad t\in(-\infty,0], \end{cases} \tag{4.34}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of order $0<\alpha<1$; $x(.)$ takes value in the Hilbert space $\mathcal{H}$; A is the infinitesimal generator of a strongly continuous semigroup of a bounded linear operator $\{T(t),\, t\geq 0\}$ on $\mathcal{H}$; $W=\{W(t)\,:\,t\geq 0\}$ is a given $\mathcal{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q\geq 0$ defined on the filtered complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$; the control function $u(\cdot)$ is given in $L_2(J,\mathcal{U})$ of admissible control functions, $\mathcal{U}$ is a Hilbert space; B is a bounded linear operator from $\mathcal{U}$ into $\mathcal{H}$. The histories $x_t:(-\infty,0]\to\mathcal{H}$ defined by $x_t(\theta)=x(t+\theta)$, $\theta\leq 0$, belongs to an abstract phase space $\mathcal{B}_h$; $g:J\times\mathcal{B}_h\to\mathcal{H}$, $f:J\times\mathcal{B}_h\times\mathcal{H}\to\mathcal{H}$, $\Sigma:J\times\mathcal{B}_h\times\mathcal{H}\to BCC(L(\mathcal{K},\mathcal{H}))$ and $H,K:J\times J\times\mathcal{B}_h\to\mathcal{H}$ are appropriate functions to be specified later. The initial data $\phi=\{\phi(t)\,:\,t\in(-\infty,0]\}$ is an $\mathcal{F}_0$ mesurable, $\mathcal{B}_h$-valued random variable independent of W with finite second moments.

In this section, we assume that $A:\mathcal{D}(A)\subset\mathcal{H}\to\mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup of a bounded linear operator $T(t)$, $t\geq 0$ on $\mathcal{H}$. That is to say, $\|T(t)\|\leq M$ for some constant $M\geq 1$ and every $t\geq 0$. Without loss of generality, we assume that $0\in\rho(A)$, the resolvent set of A. Then it is possible to define the fractional power A^α for $0<\alpha\leq 1$, as a closed linear operator on its domain $\mathcal{D}(A^\alpha)$ with inverse A^{-1}.

The following are basic properties of A^α :

i. $\mathcal{H}_\alpha=\mathcal{D}(A^\alpha)$ is a Hilbert space with the norm $\|x\|_\alpha=\|A^\alpha x\|$ for $x\in\mathcal{D}(A^\alpha)$.

ii. $T(t):\mathcal{H}\to\mathcal{H}_\alpha$ for $t\geq 0$.

iii. $A^\alpha T(t)x=T(t)A^\alpha x$ for each $x\in\mathcal{D}(A)$ and $t\geq 0$.

iv. for every $t>0$, $A^\alpha T(t)$ is bounded on $\mathcal{H}$ and there exists $M_\alpha>0$ such that

$$\|A^\alpha T(t)\|\leq\frac{M_\alpha}{t^\alpha}.$$

v. $A^{-\alpha}$ is a bounded linear operator for $0\leq\alpha\leq 1$ in $\mathcal{H}$.

Now we present the abstract phase space $\mathcal{B}_h$, which has been used in [24, 98]. Assume that $h:(-\infty,0]\to(0,+\infty)$ is a continuous function with $l=\int_{-\infty}^0 h(t)dt<+\infty$. Define the phase

[3] The section is based on the paper [38].

space $\mathcal{B}_h$ by

$$\mathcal{B}_h = \Big\{\phi : (-\infty, 0] \to \mathcal{H}, \text{ for any } a > 0, \ (\mathbb{E}|\phi(\theta)|^2)^{1/2} \text{ is bounded and measurable function on } [-a, 0] \text{ with } \int_{-\infty}^{0} h(s) \sup_{s\le\theta\le 0} (\mathbb{E}|\phi(\theta)|^2)^{1/2} ds < \infty \Big\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s\le\theta\le 0} (\mathbb{E}|\phi(\theta))|^2)^{1/2} ds, \quad \phi \in \mathcal{B}_h,$$

then $(\mathcal{B}_h, \|.\|_{\mathcal{B}_h})$ is a Banach space.

Now we consider the space

$$\mathcal{B}'_h = \Big\{x : (-\infty, b] \to \mathcal{H} \text{ such that } x|_J \in \mathcal{C}(J, \mathcal{H}), \ x_0 = \phi \in \mathcal{B}_h\Big\}.$$

Set $\|\cdot\|_b$ be a seminorm in $\mathcal{B}'_h$ defined by

$$\|x\|_b = \|\phi\|_{\mathcal{B}_h} + \sup_{s\in[0,b]} (\mathbb{E}\|x(s)\|^2)^{1/2}, \quad x \in \mathcal{B}'_h.$$

Lemma 4.4.1. *([99]). Assume that $x \in \mathcal{B}'_h$, then for $t \in J$, $x_t \in \mathcal{B}_h$. Moreover,*

$$l(\mathbb{E}\|x(l)\|^2)^{1/2} \le \|x_t\|_{\mathcal{B}_h} \le l \sup_{s\in[0,t]} (\mathbb{E}\|x(s)\|^2)^{1/2} + \|\phi\|_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^{0} h(s) ds < \infty$.

In the following, $BCC(\mathcal{H})$ denotes the set of all nonempty bounded, closed and convex subset of $\mathcal{H}$.

Definition 4.4.1. *A multi-valued map $G : J \to BCC(\mathcal{H})$ is said to be measurable if, for each $x \in \mathcal{H}$, the function $\vartheta : J \to \mathbb{R}$, defined by*

$$\vartheta(t) = d(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\}.$$

belong to $L^1(J, \mathbb{R})$.

Definition 4.4.2. *The multi-valued map $\Sigma : J \times \mathcal{H} \to BCC(\mathcal{H})$ is said to be L^2-Carathéodory if*

i. $t \mapsto \Sigma(t, x)$ is measurable for each $x \in \mathcal{H}$;

ii. $x \mapsto \Sigma(t, x)$ is u.s.c. for almost all $t \in J$;

iii. for each $r > 0$, there exists $l_r \in L^1(J, \mathbb{R})$ such that

$$\|\Sigma(t, x)\|^2 := \sup_{\sigma\in\Sigma(t,x)} \mathbb{E}\|\sigma\|^2 \le l_r(t), \ \ \textit{for all} \ \ \|x\|_b^2 \le r \textit{ and for a.e. } t \in J.$$

We have the following lemma due to Lasota and Opial [74]

Lemma 4.4.2. *([74]). Let J be a compact real interval, $BCC(\mathcal{H})$ be the set of all nonempty, bounded, closed, and convex subset of $\mathcal{H}$ and Σ be a L^2-Carathéodory multi-valued map $S_{\Sigma,x} \neq \emptyset$ and let Γ be a linear continuous mapping from $L^2(J, \mathcal{H})$ to $\mathcal{C}(J, \mathcal{H})$. Then, the operator*

$$\Gamma \circ S_\Sigma : \mathcal{C}(J, \mathcal{H}) \to BCC(\mathcal{C}(J, \mathcal{H})), \quad x \mapsto (\Gamma \circ S_\Sigma)(x) := \Gamma(S_{\Sigma,x}),$$

is a closed graph operator in $\mathcal{C}(J, \mathcal{H}) \times \mathcal{C}(J, \mathcal{H})$, where $S_{\Sigma,x}$ is known as the selectors set from Σ, is given by

$$\sigma \in S_{\Sigma,x} = \{\sigma \in L^2(L(\mathcal{K}, \mathcal{H})) : \sigma(t) \in \Sigma(t, x), \ \textit{for a.e. } t \in J\}.$$

Now, we present the definition of mild solutions for the system (4.34) based on the papers [24, 117].

Definition 4.4.3. *An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, b] \to \mathcal{H}$ is called a mild solution of the system (4.34) if $x_0 = \phi \in \mathcal{B}_h$ on $(-\infty, 0]$ satisfying $x_0 \in L_2^{\mathcal{F}_0}(\Omega, \mathcal{H})$ and the following integral inclusion*

$$\begin{aligned} x(t) \in \mathcal{T}(t)[\phi(0) - g(0, \phi(0))] + g(t, x_t) + \int_0^t (t-s)^{\alpha-1} A\mathcal{S}(t-s) g(s, x_s) ds \\ + \int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) Bu(s) ds + \int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) f\Big(s, x_s, \int_0^s H(s, \tau, x_\tau) d\tau\Big) ds \\ + \int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) \Sigma\Big(s, x_s, \int_0^s K(s, \tau, x_\tau) d\tau\Big) dW(s), \qquad t \in J, \end{aligned} \tag{4.35}$$

is satisfied, where $\mathcal{T}(\cdot)$ and $\mathcal{S}(\cdot)$ are the characteristic solution operators.

The following properties of $\mathcal{T}(t)$ and $\mathcal{S}(t)$ appeared in [126] are useful.

Lemma 4.4.3. *The operator $\mathcal{T}(t)$ and $\mathcal{S}(t)$ have the following properties:*

i. For any fixed $t \geq 0$, $\mathcal{T}$ and $\mathcal{S}$ are linear and bounded operators, that is, for any $x \in \mathcal{H}$,

$$\|\mathcal{T}(t)x\| \leq M\|x\|, \quad \|\mathcal{S}(t)x\| \leq \frac{\alpha M}{\Gamma(\alpha+1)}\|x\|;$$

ii. $\{\mathcal{T}(t), t \geq 0\}$ and $\{\mathcal{S}(t), t \geq 0\}$ are strongly continuous;

iii. for every $t \geq 0$, $\{\mathcal{T}(t), t \geq 0\}$ and $\{\mathcal{S}(t), t \geq 0\}$ are also compact if $\{T(t), t \geq 0\}$ is compact;

iv. For any $x \in \mathcal{H}$, β, $\delta \in (0, 1)$, we have $AT_\alpha(t)x = A^{1-\beta}T_\alpha(t)A^\beta x$ and

$$\|A^\delta T_\alpha(t)\| \leq \frac{\alpha C_\delta \Gamma(2-\delta)}{t^{\alpha\delta}\Gamma(1+\alpha(1-\delta))}, \quad t \in (0, b].$$

At the end, we recall the fixed point theorem of Bohnenblust-Karlin's [18] which is used to establish the existence of the mild solution to the system (4.34).

Lemma 4.4.4. *Let D be a nonempty subset of $\mathcal{H}$, which is bounded, closed, and convex. Suppose $G : D \to 2^{\mathcal{H}} \backslash \{\emptyset\}$ is u.s.c. with closed, convex values, and such that $G(D) \subseteq D$ and $G(D)$ is compact. Then G has a fixed point.*

4.4.1 The Main Results

In this subsection, we shall formulate and prove sufficient conditions for the approximate controllability of the system (4.34). To do this, we first prove the existence of solutions for fractional control system. Then, we show that under certain assumptions, the approximate controllability of the fractional stochastic control system (4.34) is implied by the approximate controllability of the associated linear part.

Definition 4.4.4. *Let $x_b(\phi; u)$ be the state value of (4.34) at the terminal time b corresponding to the control u and the initial value ϕ. Introduce the set*

$$\mathcal{R}(b,\phi) = \{x_b(\phi; u)(0) : u(\cdot) \in L_2(J,\mathcal{U})\},$$

which is called the reachable set of (4.34) at the terminal time b and its closure in $\mathcal{H}$ is denoted by $\overline{\mathcal{R}(b,\phi)}$. The system (4.34) is said to be approximately controllable on the interval J if $\overline{\mathcal{R}(b,\phi)} = \mathcal{H}$; that is, given an arbitrary $\epsilon > 0$, it is possible to steer from the point $\pi(0)$ to within a distance ϵ from all points in the state space $\mathcal{H}$ at time b.

In order to a study the approximate controllability for the fractional control system (4.34), we consider its fractional linear part

$$\begin{aligned} D_t^\alpha x(t) &\in Ax(t) + (Bu)(t), \qquad t \in [0,b] \\ x(0) &= \phi \in \mathcal{B}_h. \end{aligned} \tag{4.36}$$

It is convenient at this point to introduce the controllability and resolvent operators associated with (4.36) as

$$\begin{aligned} L_0^b &= \int_0^b (b-s)^{\alpha-1}\mathcal{S}(b-s)Bu(s)ds : L^2(J,\mathcal{U}) \to \mathcal{H}, \\ \Gamma_0^b &= L_0^b(L_0^b)^* = \int_0^b (b-s)^{\alpha-1}\mathcal{S}(b-s)BB^*\mathcal{S}^*(b-s)ds : \mathcal{H} \to \mathcal{H}, \end{aligned} \tag{4.37}$$

respectively, where B^* denotes the adjoint of B and $\mathcal{S}^*(t)$ is the adjoint of $\mathcal{S}(t)$. It is straightforward that the operator Γ_0^b is a linear bounded operator.

In order to establish the existence result, we need the following hypothesis:

(H1) : A generates a strongly continuous compact semigroup $\{T(t) : t \geq 0\}$ in $\mathcal{H}$.

(H2) : The function $g : J \times \mathcal{B}_h$ is continuous and there exists a constant $M_g > 0$, $0 < \beta < 1$ such that g is $\mathcal{H}_\beta$ valued and

$$\|A^\beta g(t,\psi) - A^\beta g(t,\varphi)\|^2 \leq M_g\|\psi-\varphi\|^2_{\mathcal{B}_h}, \qquad \psi,\, \varphi \in \mathcal{B}_h,\, t \in J := [0,b],$$
$$\|A^\beta g(t,\psi)\|^2 \leq M_g(1 + \|\psi\|^2_{\mathcal{B}_h}).$$

(H3) : The function $f : J \times \mathcal{B}_h \times \mathcal{H} \to \mathcal{H}$ satisfies the following:

i. $f(.,\psi,x) : J \to \mathcal{H}$ is measurable for each $(\psi,x) \in \mathcal{B}_h \times \mathcal{H}$ and $f(t,\cdot,\cdot) : \mathcal{B}_h \times \mathcal{H} \to \mathcal{H}$ is continuous for a.e. $t \in J$, and for $\psi \in \mathcal{B}_h$, $f(\cdot,\cdot,\psi) : J \times \mathcal{H} \to \mathcal{H}$ is strongly measurable;

ii. There is a positive integrable function $n \in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\Xi_f : [0,\infty) \to (0,\infty)$ such that for every $(t,\psi,x) \in J \times \mathcal{B}_h \times \mathcal{H}$, we have

$$\mathbb{E}\|f(t,\psi,x)\|^2 \leq n(t)\Xi_f(\|\psi\|^2_{\mathcal{B}_h} + \mathbb{E}\|x\|^2), \qquad \liminf_{r\to\infty} \frac{\Xi_f(r)}{r} = \Upsilon.$$

(**H4**) : the multi-valued map $\Sigma : J \times \mathcal{B}_h \times \mathcal{H} \to BCC(L(\mathcal{K}, \mathcal{H}))$ is an L^2-Carathéodory function satisfies the following conditions:

i. For each $t \in J$, the function $\Sigma(t, \cdot, \cdot) : \mathcal{B}_h \times \mathcal{H} \to BCC(L(\mathcal{K}, \mathcal{H}))$ is u.s.c; and for each $(\psi, x) \in \mathcal{B}_h \times \mathcal{H}$, the function $\Sigma(\cdot, \psi, x)$ is measurable. And for each fixed $(\psi, x) \in \mathcal{B}_h \times \mathcal{H}$, the set
$$S_{\Sigma,\psi,x} = \{\sigma \in L^2(L(\mathcal{K}, \mathcal{H})) : \sigma(t) \in \Sigma(t, \psi, x) \text{ for a.e. } t \in J\}$$
is nonempty;

ii. There exists a positive function $l_r : J \to \mathbb{R}^+$ such that
$$\sup\{\mathbb{E}\|\sigma\|^2 : \sigma(t) \in \Sigma(t, \psi, x)\} \leq l_r(t)$$
for a.e. $t \in J$ and the function $s \mapsto (t-s)^{2(\alpha-1)} l_r(s)$ belongs to $L^1([0,t], \mathbb{R}^+)$ such that
$$\liminf_{r\to\infty} \frac{\int_0^t (t-s)^{2(\alpha-1)} l_r(s) ds}{r} = \Lambda < +\infty.$$

(**H5**) : The function $H : J \times J \times \mathcal{B}_h \to \mathcal{H}$ satisfies:

i. For each $(t, s) \in J \times J$, the function $H(t, s, \cdot) : \mathcal{B}_h \to \mathcal{H}$ is continuous, and for each $\psi \in \mathcal{B}_h$, the function $H(\cdot, \cdot, \psi) : J \times J \to \mathcal{H}$ is strongly measurable.

ii. There exists a constant $M_0 > 0$ such that hat $\mathbb{E}\|H(t, s, \psi)\|^2 \leq M_0(1 + \|\psi\|^2_{\mathcal{B}_h})$, for all $t, s \in J$ and $\psi \in \mathcal{B}_h$.

(**H6**) : The function $K : J \times J \times \mathcal{B}_h \to \mathcal{H}$ satisfies:

i. For each $(t, s) \in J \times J$, the function $K(t, s, \cdot) : \mathcal{B}_h \to \mathcal{H}$ is continuous, and for each $\psi \in \mathcal{B}_h$, the function $K(\cdot, \cdot, \psi) : J \times J \to \mathcal{H}$ is strongly measurable.

The following lemma is required to define the control function.

Lemma 4.4.5. *([84]). For any $\widetilde{x}_b \in L_2(\mathcal{F}_b, \mathcal{H})$ there exists $\widetilde{\phi} \in L_2^{\mathcal{F}}(\Omega, L^2(J, L(\mathcal{K}, \mathcal{H})))$ such that $\widetilde{x}_b = \mathbb{E}\widetilde{x}_b + \int_0^b \widetilde{\phi}(s) dW(s)$.*

Now for any $\epsilon > 0$, $\widetilde{x}_b \in L_2(\mathcal{F}_b, \mathcal{H})$ and for $\sigma \in S_{\Sigma,\psi,x}$, we define the control function
$$\begin{aligned} u^\epsilon(t,x) &= B^* \mathcal{S}^*(b-t)(\epsilon I + \Gamma_0^b)^{-1} \left\{ \mathbb{E}\widetilde{x}_b + \int_0^b \widetilde{\phi}(s) dW(s) - \mathcal{T}(b)[\phi(0) - g(0, \phi(0))] - g(b, x_b) \right\} \\ &\quad - B^* \mathcal{S}^*(b-t) \int_0^b (\epsilon I + \Gamma_s^b)^{-1} (b-s)^{\alpha-1} A \mathcal{S}(b-s) g(s, x_s) ds \\ &\quad - B^* \mathcal{S}^*(b-t) \int_0^b (\epsilon I + \Gamma_s^b)^{-1} (b-s)^{\alpha-1} \mathcal{S}(b-s) f\left(s, x_s, \int_0^s H(s, \tau, x_\tau) d\tau\right) ds \\ &\quad - B^* \mathcal{S}^*(b-t) \int_0^b (\epsilon I + \Gamma_s^b)^{-1} (b-s)^{\alpha-1} \mathcal{S}(b-s) \sigma(s) dW(s). \end{aligned}$$

Let us now explain and prove the following theorem about the existence of solution for the fractional system (4.34).

Theorem 4.4.1. *Assume that the assumptions (**H1**) − (**H6**) hold. Then for each $\epsilon > 0$, the system (4.34) has a mild solution on J provided that*

$$\Bigg[4M_g\|A^{-\beta}\|^2l^2 + \frac{4l^2M_gC_{1-\beta}^2\Gamma^2(1+\beta)b^{2\alpha\beta}}{\beta^2\Gamma^2(1+\alpha\beta)} + \frac{4l^2(1+M_0b)\Upsilon M^2b^{2\alpha}}{\Gamma^2(1+\alpha)}\sup_{s\in J} n(s)$$

$$+\frac{M^2\alpha^2\Lambda Tr(Q)}{\Gamma^2(1+\alpha)}\Bigg] \times \left(6 + \left(\frac{MM_Bb^\alpha}{\Gamma(1+\alpha)}\right)^2\frac{42}{\epsilon^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^2\right) < 1.$$

Proof. In order to prove the existence of mid solutions for system (4.34) transform it into a fixed point problem. For any $\epsilon > 0$, we consider the operator $\Phi^\epsilon : \mathcal{B}'_h \to 2^{\mathcal{B}'_h}$ defined by $\Phi^\epsilon x$ the set of $z \in \mathcal{B}'_h$ such that

$$z(t) = \begin{cases} \phi(t), \qquad t \in (-\infty, 0], \\ \mathcal{T}(t)[\phi(0) - g(0,\phi(0))] + g(t,x_t) + \int_0^t (t-s)^{\alpha-1}A\mathcal{S}(t-s)g(s,x_s)ds \\ \quad + \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)Bu^\epsilon(s,x)ds + \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f(s,x_s,\int_0^s H(s,\tau,x_\tau)d\tau)ds \\ \quad + \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma(s)dW(s), \qquad t\in J, \end{cases} \tag{4.38}$$

Where $\sigma \in S_{\Sigma,\psi,x}$.

For $\phi \in \mathcal{B}_h$, define

$$\widehat{\phi}(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ \mathcal{T}(t)\phi(0), \qquad t \in J. \end{cases}$$

then $\widehat{\phi} \in \mathcal{B}'_h$. Let $x(t) = y(t) + \widehat{\phi}(t), -\infty < t \le b$. It is easy to see that x satisfies (4.35) if and only if y satisfies $y_0 = 0$ and

$$y(t) \in -\mathcal{T}(t)g(0,\phi(0)) + g(t, y_t + \widehat{\phi}_t) + \int_0^t (t-s)^{\alpha-1}A\mathcal{S}(t-s)g(s, y_s + \widehat{\phi}_s)ds$$

$$+ \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)Bu^\epsilon(s, y + \widehat{\phi})ds$$

$$+ \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f\left(s, y_s + \widehat{\phi}_s, \int_0^s H(s,\tau,y_\tau + \widehat{\phi}_\tau)d\tau\right)ds$$

$$+ \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)\Sigma\left(s, y_s + \widehat{\phi}_s, \int_0^s K(s,\tau,y_\tau + \widehat{\phi}_\tau)d\tau\right)dW(s), \qquad t \in J,$$

where for $\sigma \in S_{\Sigma,\psi,x}$,

$$\begin{aligned} &u^\epsilon(s, y+\widehat{\phi}) \\ &\quad = B^*\mathcal{S}^*(b-t)(\epsilon I + \Gamma_0^b)^{-1}\left\{\mathbb{E}\widetilde{x}_b + \int_0^b \widetilde{\phi}(s)dW(s) - \mathcal{T}(b)[\phi(0) - g(0,\phi(0))] - g(b, y_b + \widehat{\phi}_b)\right\} \\ &\quad\quad -B^*\mathcal{S}^*(b-t)\int_0^b (\epsilon I + \Gamma_s^b)^{-1}(b-s)^{\alpha-1}A\mathcal{S}(b-s)g(s, y_s + \widehat{\phi}_s)ds \\ &\quad\quad -B^*\mathcal{S}^*(b-t)\int_0^b (\epsilon I + \Gamma_s^b)^{-1}(b-s)^{\alpha-1}\mathcal{S}(b-s)f\left(s, y_s + \widehat{\phi}_s, \int_0^s H(s,\tau,y_\tau + \widehat{\phi}_\tau)d\tau\right)ds \\ &\quad\quad -B^*\mathcal{S}^*(b-t)\int_0^b (\epsilon I + \Gamma_s^b)^{-1}(b-s)^{\alpha-1}\mathcal{S}(b-s)\sigma(s)dW(s). \end{aligned}$$

Let $\mathcal{B}''_h = \{y \in \mathcal{B}'_h : y_0 = 0 \in \mathcal{B}_h\}$. For any $y \in \mathcal{B}''_h$, we have

$$\|y\|_b = \|y_0\|_{\mathcal{B}_h} + \sup_{s\in[0,b]} (\mathbb{E}\|y(s)\|^2)^{\frac{1}{2}} = \sup_{s\in[0,b]} (\mathbb{E}\|y(s)\|^2)^{\frac{1}{2}},$$

thus$(\mathcal{B}''_h, \|\cdot\|_b)$ is a Banach space. Set $B_r = \{y \in \mathcal{B}''_h : \|y\|_b^2 \leq r\}$ for some $r > 0$, then B_r is clearly a bounded closed convex set in $\mathcal{B}''_h$, and for $y \in B_r$, from Lemma 4.4.1, we have

$$\begin{aligned} \|y_t + \widehat{\phi}_t\|^2_{\mathcal{B}_h} &\leq 2(\|y_t\|^2_{\mathcal{B}_h} + \|\widehat{\phi}_t\|^2_{\mathcal{B}_h}) \\ &\leq 4\left(l^2 \sup_{s\in[0,t]} \mathbb{E}\|y(s)\|^2 + \|y_0\|^2_{\mathcal{B}_h} + l^2 \sup_{s\in[0,t]} \mathbb{E}\|\widehat{\phi}(s)\|^2 + \|\widehat{\phi}_0\|^2_{\mathcal{B}_h}\right) \\ &\leq 4l^2(r + M^2\mathbb{E}\|\phi(0)\|^2_{\mathcal{H}}) + 4\|\phi\|^2_{\mathcal{B}_h} = r'. \end{aligned} \tag{4.39}$$

Define the multi-valued map $\Psi : \mathcal{B}''_h \to 2^{\mathcal{B}''_h}$ by Ψy the set of $\overline{z} \in \mathcal{B}''_h$ and there exists $\sigma \in L^2(L(\mathcal{K}, \mathcal{H}))$ such that $\sigma \in S_{\Sigma,\psi,x}$ and

$$\overline{z}(t) = \begin{cases} 0, \qquad t \in (-\infty, 0], \\ -\mathcal{T}(t)g(0, \phi(0)) + g(t, y_t + \widehat{\phi}_t) + \int_0^t (t-s)^{\alpha-1} A\mathcal{S}(t-s)g(s, y_s + \widehat{\phi}_s)ds \\ + \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)Bu^\epsilon(s, y + \widehat{\phi})ds \\ + \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f(s, y_s + \widehat{\phi}_s, \int_0^s H(s, \tau, y_\tau + \widehat{\phi}_\tau)d\tau)ds \\ + \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma(s)dW(s), \qquad t \in J, \end{cases} \tag{4.40}$$

Obviously, the operator Φ^ϵ has a fixed point is equivalent to Ψ has on. So, our aim is to show that Ψ has a fixed point. For the sake of convenience, we subdivise the proof into in several steps.

Step 1 Ψ is convex for each $y \in B_r$. In fact, if $\overline{z}_1, \overline{z}_2$ belong to ψy, then there exist $\sigma_1, \sigma_2 \in S_{\Sigma,\psi,x}$ such that

$$\begin{aligned} \overline{z}_j(t) &= -\mathcal{T}(t)g(0, \phi(0)) + g(t, y_t + \widehat{\phi}_t) + \int_0^t (t-s)^{\alpha-1} A\mathcal{S}(t-s)g(s, y_s + \widehat{\phi}_s)ds \\ &+ \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)Bu^\epsilon(s, y + \widehat{\phi})ds \\ &+ \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f\left(s, y_s + \widehat{\phi}_s, \int_0^s H(s, \tau, y_\tau + \widehat{\phi}_\tau)d\tau\right)ds \\ &+ \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma_j(s)dW(s), \qquad t \in J. \end{aligned}$$

Let $0 \leq \lambda \leq 1$. Then for each $t \in J$, we have

$$(\lambda\overline{z}_1+(1-\lambda)\overline{z}_2)(t) = -\mathcal{T}(t)g(0,\phi(0)) + g(t,y_t+\widehat{\phi}_t) + \int_0^t (t-s)^{\alpha-1}A\mathcal{S}(t-s)g(s,y_s+\widehat{\phi}_s)ds$$

$$+\int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)ds$$

$$+\int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)BB^*\mathcal{S}(b-t)$$

$$\times\Bigg[(\epsilon I+\Gamma_0^b)^{-1}\left\{\mathbb{E}\widetilde{x}_b+\int_0^b \widetilde{\phi}(s)dW(s)-\mathcal{T}(b)[\phi(0)-g(0,\phi(0))]-g(b,y_b+\widehat{\phi}_b)\right\}$$

$$-\int_0^b (\epsilon I+\Gamma_s^b)^{-1}(b-s)^{\alpha-1}A\mathcal{S}(b-s)g(s,y_s+\widehat{\phi}_s)ds$$

$$-\int_0^b (\epsilon I+\Gamma_s^b)^{-1}(b-s)^{\alpha-1}\mathcal{S}(b-s)f\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)ds$$

$$-\int_0^b (\epsilon I+\Gamma_s^b)^{-1}(b-s)^{\alpha-1}\mathcal{S}(b-s)[\lambda\sigma_1(s)+(1-\lambda)\sigma_2(s)]dW(s)\Bigg]ds$$

$$+\int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)[\lambda\sigma_1(s)+(1-\lambda)\sigma_2(s)]dW(s).$$

It is easy to see that $S_{\Sigma,\psi,x}$ is convex since Σ has convex values. So, $\lambda\sigma_1+(1-\lambda)\sigma_2 \in S_{\Sigma,\psi,x}$. Thus, $(\lambda\overline{z}_1+(1-\lambda)\overline{z}_2)\in\Psi y$.

Step 2 We show that there exists some $r>0$ such that $\Psi(B_r)\subseteq B_r$. If it is not true, then there exists $\epsilon>0$ such that for every positive number r and $t\in J$, there exists a function $y^r(.)\in B_r$, but $\Psi(y^r)\notin B_r$, that is, $\mathbb{E}\|(\Psi y^r)(t)\|^2 \equiv \{\|\overline{Z}^r\|_b^2 : \overline{z}^r\in(\Psi y^r)\}\geq r$. For such $\epsilon>0$, an elementary inequality can show that

$$\begin{aligned} r &\leq \mathbb{E}\|(\Psi y^r)(t)\|^2 \\ &\leq 6\mathbb{E}\|-\mathcal{T}(t)g(0,\phi(0))\|^2 \\ &\quad +6\mathbb{E}\|g(t,y_t^r+\widehat{\phi}_t)\|^2 + 6\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}A\mathcal{S}(t-s)g(s,y_s^r+\widehat{\phi}_s)ds\right\|^2 \\ &\quad +6\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f\left(s,y_s^r+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau^r+\widehat{\phi}_\tau)d\tau\right)ds\right\|^2 \\ &\quad +6\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)Bu^\epsilon(s,y^r+\widehat{\phi})ds\right\|^2 + 6\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma^r(s)dW(s)\right\|^2 \\ &= 6\sum_{i=1}^6 I_i, \end{aligned} \tag{4.41}$$

for some $\sigma^r\in S_{\Sigma,\psi,x}$.

Let us estimate each term above $I_i, i=1,...,6$. By Lemma 4.4.1 and assumptions (**H1**)-(**H2**), we have

$$I_1 \leq M^2\|A^{-\beta}\|^2\mathbb{E}\|A^\beta g(0,\phi)\|^2 \leq M^2\|A^{-\beta}\|^2 M_g(1+\|\phi\|^2_{\mathcal{B}_h}), \tag{4.42}$$

$$\begin{aligned} I_2 &\leq \|A^{-\beta}\|^2\mathbb{E}\|A^\beta g(t, y_t^r+\widehat{\phi}_t)\|^2 \leq M_g\|A^{-\beta}\|^2(1+\|y_t^r+\widehat{\phi}_t\|^2_{\mathcal{B}_h}) \\ &\leq M_g\|A^{-\beta}\|^2(1+r'). \end{aligned} \tag{4.43}$$

By a standard calculation involving Lemma 4.4.3, assumption (**H2**), Eq. (4.39) and the Hölder inequality, we can deduce that

$$\begin{aligned} I_3 &\leq \mathbb{E}\Big[\int_0^t \|(t-s)^{\alpha-1}A^{1-\beta}\mathcal{S}(t-s)A^\beta g(s, y_s^r+\widehat{\phi}_s)\|ds\Big]^2 \\ &\leq K_1(\alpha,\beta)\int_0^t (t-s)^{\alpha\beta-1}ds\int_0^t (t-s)^{\alpha\beta-1}\mathbb{E}\|A^\beta g(s,y_s^r+\widehat{\phi}_s)\|^2 ds \\ &\leq \frac{K_1(\alpha,\beta)b^{\alpha\beta}}{\alpha\beta}\int_0^t (t-s)^{\alpha\beta-1}\mathbb{E}\|A^\beta g(s,y_s^r+\widehat{\phi}_s)\|^2 ds \\ &\leq \frac{K_1(\alpha,\beta)b^{\alpha\beta}}{\alpha\beta}\int_0^t (t-s)^{\alpha\beta-1}M_g(1+\|y_s^r+\widehat{\phi}_s\|^2_{\mathcal{B}_h})ds \\ &\leq \frac{K_1(\alpha,\beta)b^{2\alpha\beta}}{(\alpha\beta)^2}M_g(1+r'), \end{aligned} \tag{4.44}$$

where $K_1(\alpha,\beta)=\frac{\alpha^2C^2_{1-\beta}\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}$.

Together with assumption (**H3**), (**H5**) and (4.39), we have

$$\begin{aligned} I_4 &\leq \mathbb{E}\Big[\int_0^t \|(t-s)^{\alpha-1}S(t-s)f\left(s, y_s^r+\widehat{\phi}_s, \int_0^s H(s,\tau,y_\tau^r+\widehat{\phi}_\tau)d\tau\right)\|ds\Big]^2 \\ &\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2\int_0^t (t-s)^{\alpha-1}ds\int_0^t (t-s)^{\alpha-1}\mathbb{E}\left\|f\left(s, y_s^r+\widehat{\phi}_s, \int_0^s H(s,\tau,y_\tau^r+\widehat{\phi}_\tau)d\tau\right)\right\|^2 ds \\ &\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}\mathbb{E}\left\|f\left(s, y_s^r+\widehat{\phi}_s, \int_0^s H(s,\tau,y_\tau^r+\widehat{\phi}_\tau)d\tau\right)\right\|^2 ds \\ &\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}n(s)\Xi_f\left(\|y_t^r+\widehat{\phi}_t\|^2_{\mathcal{B}_h}+\mathbb{E}\|\int_0^s H(s,\tau,y_\tau^r+\widehat{\phi}_\tau)d\tau\|^2\right)ds \\ &\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}n(s)\Xi_f(r'+M_0(1+r')b)ds \\ &\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^{2\alpha}}{\alpha^2}\Xi_f(r^{'}+M_0(1+r')b)\sup_{s\in J} n(s). \end{aligned} \tag{4.45}$$

A similar argument involves Lemma 4.4.3 and assumptions (**H4**), (**H6**); we obtain

$$\begin{aligned} I_5 &\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 Tr(Q)\int_0^t (t-s)^{2(\alpha-1)}\mathbb{E}\|\sigma^r(s)\|^2 ds \\ &\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 Tr(Q)\int_0^t (t-s)^{2(\alpha-1)} l_r(s) ds. \end{aligned} \tag{4.46}$$

Now, we have

$$\begin{aligned} I_6 &\leq \mathbb{E}\bigg[\int_0^t \|(t-s)^{\alpha-1}\mathcal{S}(t-s)Bu^\epsilon(s,y^r+\widehat{\phi})\|ds\bigg]^2 \\ &\leq \left(\frac{\alpha M M_B}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}\mathbb{E}\|u^\epsilon(s,y^r+\widehat{\phi})\|^2 ds, \end{aligned}$$

where $M_B = \|B\|$. By using (**H2**)-(**H6**) Hölder's inequality, Eq.(4.39) Lemma 4.4.3, for some $\sigma^r \in S_{\Sigma,\psi,x}$, we get

$$\begin{aligned} \mathbb{E}\|u^\epsilon(s,y^r+\widehat{\phi})\|^2 &\leq \frac{1}{\epsilon^2}M_B^2\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2\bigg\{7\|\mathbb{E}\widetilde{x}_b + \int_0^b \widetilde{\phi}(s)dW(s)\|^2 + 7\mathbb{E}\|\tau(b)\phi(0)\|^2 \\ &+7\mathbb{E}\|\tau(b)g(0,\phi(0))\|^2 + 7\mathbb{E}\|g(b,y_b^r+\widehat{\phi}_b)\|^2 \\ &+7\mathbb{E}\bigg\|\int_0^b (b-s)^{\alpha-1}A\mathcal{S}(b-s)g(s,y_s^r+\widehat{\phi}_s)ds\bigg\|^2 \\ &+7\mathbb{E}\bigg\|\int_0^b (b-s)^{\alpha-1}\mathcal{S}(b-s)f\left(s,y_s^r+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau^r+\widehat{\phi}_\tau)d\tau\right)ds\bigg\|^2 \\ &+7\mathbb{E}\bigg\|\int_0^b (b-s)^{\alpha-1}\mathcal{S}(b-s)\sigma^r(s)dW(s)\bigg\|^2\bigg\} \\ &\leq \frac{7}{\epsilon^2}\left(\frac{\alpha M M_B}{\Gamma(1+\alpha)}\right)^2\bigg[2\|\mathbb{E}\widetilde{x}_b\|^2 + 2\int_0^b \mathbb{E}\|\widetilde{\phi}(s)\|^2 ds + M^2\|\phi(0)\|^2 \\ &+M^2M_g\|A^{-\beta}\|^2M_g(1+\|\phi\|_{\mathcal{B}_h}^2) + M_g\|A^{-\beta}\|^2(1+r') + \frac{K_1(\alpha,\beta)b^{2\alpha\beta}}{(\alpha\beta)^2}M_g(1+r') \\ &+\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^{2\alpha}}{\alpha^2}\Xi_f(r'+M_0(1+r')b)\sup_{s\in J} n(s) \\ &+\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 Tr(Q)\int_0^t (t-s)^{2(\alpha-1)}l_r(s)ds\bigg]. \end{aligned}$$

Thus,

$$I_6 \leq \left(\frac{MM_Bb^\alpha}{\Gamma(1+\alpha)}\right)^2\frac{7}{\epsilon^2}\left(\frac{\alpha M M_B}{\Gamma(1+\alpha)}\right)^2\widehat{M} \tag{4.47}$$

where

$$\widehat{M} = \Bigg[2\|\mathbb{E}\widetilde{x}_b\|^2 + 2\int_0^b \mathbb{E}\|\widetilde{\phi}(s)\|^2 ds + M^2\|\phi(0)\|^2 + M^2 M_g\|A^{-\beta}\|^2(1+\|\phi\|^2_{\mathcal{B}_h})$$
$$+M_g\|A^{-\beta}\|^2(1+r^{'})\frac{K_1(\alpha,\beta)b^{2\alpha\beta}}{(\alpha\beta)^2}M_g(1+r')$$
$$+\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^{2\alpha}}{\alpha^2}\Xi_f(r'+M_0(1+r')b)\sup_{s\in J} n(s).$$
$$+\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 Tr(Q)\int_0^t (t-s)^{2(\alpha-1)} l_r(s)ds\Bigg].$$

Combining these estimate yields (4.41) − (4.47)

$$\begin{aligned}
r &\leq \mathbb{E}\|(\Psi y^r)(t)\|^2 \\
&\leq L_0 + 24M_g\|A^{-\beta}\|^2 l^2\left[1+\left(\frac{MM_Bb^\alpha}{\Gamma(1+\alpha)}\right)^2\frac{7}{\epsilon^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^2\right]r \\
&\quad+\frac{24l^2M_gC^2_{1-\beta}\Gamma^2(1+\beta)b^{2\alpha\beta}}{\beta^2\Gamma^2(1+\alpha\beta)}\left[1+\left(\frac{MM_Bb^\alpha}{\Gamma(1+\alpha)}\right)^2\frac{7}{\epsilon^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^2\right]r \\
&\quad+6\left(\frac{Mb^\alpha}{\Gamma(1+\alpha)}\right)^2\left[1+\left(\frac{MM_Bb^\alpha}{\Gamma(1+\alpha)}\right)^2\frac{7}{\epsilon^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^2\right] \\
&\quad\times\Xi_f\Bigg(M_0b+4l^2(1+M_0b)r+4l^2M^2\mathbb{E}\|\phi(0)\|^2_{\mathcal{H}}(1+M_0b) \\
&\quad+4\|\phi\|^2_{\mathcal{B}_h}(1+M_0b)\Bigg)\sup_{s\in J} n(s)+6\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \\
&\quad\times\left[1+\left(\frac{MM_Bb^\alpha}{\Gamma(1+\alpha)}\right)^2\frac{7}{\epsilon^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^2\right]Tr(Q)\int_0^t (t-s)^{2(\alpha-1)} l_r(s)ds\Bigg],
\end{aligned} \tag{4.48}$$

where

$$\begin{aligned}
L_0 &= 6M^2\|A^{-\beta}\|^2M_g(1+\|\phi\|^2_{\mathcal{B}_h})+6M_g\|A^{-\beta}\|^2\left[1+\left(\frac{MM_Bb^\alpha}{\Gamma(1+\alpha)}\right)^2\frac{7}{\epsilon^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^2\right] \\
&\quad\times(1+4l^2M^2\mathbb{E}\|\phi(0)\|^2_{\mathcal{H}}+4\|\phi\|^2_{\mathcal{B}_h})+\frac{6l^2M_gC^2_{1-\beta}\Gamma^2(1+\beta)b^{2\alpha\beta}}{\beta^2\Gamma^2(1+\alpha\beta)} \\
&\quad\times\left[1+\left(\frac{MM_Bb^\alpha}{\Gamma(1+\alpha)}\right)^2\frac{7}{\epsilon^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^2\right](1+4l^2M^2\mathbb{E}\|\phi(0)\|^2_{\mathcal{H}}+4\|\phi\|^2_{\mathcal{B}_h}) \\
&\quad+6\left(\frac{MM_Bb^\alpha}{\Gamma(1+\alpha)}\right)^2\frac{7}{\epsilon^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^2\Bigg(2\|\mathbb{E}\widetilde{x}_b\|^2+2\int_0^b\mathbb{E}\|\widetilde{\phi}(s)\|^2ds+M^2\|\phi(0)\|^2 \\
&\quad+M^2M_g\|A^{-\beta}\|^2(1+\|\phi\|^2_{\mathcal{B}_h})\Bigg)
\end{aligned}$$

Dividing both sides of (4.48) by r and taking $r \to \infty$, we obtain that

$$\left(4M_g\|A^{-\beta}\|^2 l^2 + \frac{4l^2 M_g C_{1-\beta}^2 \Gamma^2(1+\beta) b^{2\alpha\beta}}{\beta^2\Gamma^2(1+\alpha\beta)} + \frac{4l^2(1+M_0 b)\Upsilon M^2 b^{2\alpha}}{\Gamma^2(1+\alpha)} \sup_{s\in J} n(s) + \frac{M^2\alpha^2\Lambda Tr(Q)}{\Gamma^2(1+\alpha)}\right) \times \left[6 + \left(\frac{MM_B b^\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{42}{\epsilon^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^2\right] \geq 1,$$

which is a contradiction to our assumptions. Thus for $\alpha > 0$, for some positive number r and some $\sigma^r \in S_{\Sigma,\psi,x}$, $\Psi(B_r) \subseteq B_r$.

Step 3 $\Psi(B_r)$ is equicontinuous. Indeed, let $\epsilon > 0$ small, $0 < t_1 < t_2 \leq b$. For each $y \in B_r$ and $\overline{z}$ belong to $\Psi_1 y$, there exists $\sigma \in S_{\Sigma,\psi,x}$ such that for each $t \in J$, we have

$$\begin{aligned}
&\mathbb{E}\|\overline{z}(t_2) - \overline{z}(t_1)\|^2 \\
&\leq 22\mathbb{E}\| - (\mathcal{T}(t_2) - \mathcal{T}(t_1))g(0,\phi)\|^2 + 22\mathbb{E}\|g(t_2, y_{t_2} + \widehat{\phi}_{t_2}) - g(t_1, y_{t_1} + \widehat{\phi}_{t_1})\|^2 \\
&+22\mathbb{E}\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}A\mathcal{S}(t_2-s)g(s, y_s + \widehat{\phi}_s)ds\right\|^2 \\
&+22\mathbb{E}\left\|\int_{t_1-\epsilon}^{t_1}(t_2-s)^{\alpha-1}A[\mathcal{S}(t_2-s) - \mathcal{S}(t_1-s)]g(s, y_s + \widehat{\phi}_s)ds\right\|^2 \\
&+22\mathbb{E}\left\|\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]A\mathcal{S}(t_1-s)g(s, y_s + \widehat{\phi}_s)ds\right\|^2 \\
&+22\mathbb{E}\left\|\int_{0}^{t_1-\epsilon}(t_2-s)^{\alpha-1}A[\mathcal{S}(t_2-s) - \mathcal{S}(t_1-s)]g(s, y_s + \widehat{\phi}_s)ds\right\|^2 \\
&+22\mathbb{E}\left\|\int_{0}^{t_1-\epsilon}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]A\mathcal{S}(t_1-s)g(s, y_s + \widehat{\phi}_s)ds\right\|^2 \\
&+22\mathbb{E}\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathcal{S}(t_2-s)Bu^\epsilon(s, y + \widehat{\phi})ds\right\|^2 \\
&+22\mathbb{E}\left\|\int_{t_1-\epsilon}^{t_1}(t_2-s)^{\alpha-1}[\mathcal{S}(t_2-s) - \mathcal{S}(t_1-s)]Bu^\epsilon(s, y + \widehat{\phi})ds\right\|^2 \\
&+22\mathbb{E}\left\|\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]\mathcal{S}(t_1-s)Bu^\epsilon(s, y + \widehat{\phi})ds\right\|^2 \\
&+22\mathbb{E}\left\|\int_{0}^{t_1-\epsilon}(t_2-s)^{\alpha-1}[\mathcal{S}(t_2-s) - \mathcal{S}(t_1-s)]Bu^\epsilon(s, y + \widehat{\phi})ds\right\|^2 \\
&+22\mathbb{E}\left\|\int_{0}^{t_1-\epsilon}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]\mathcal{S}(t_1-s)Bu^\epsilon(s, y + \widehat{\phi})ds\right\|^2
\end{aligned}$$

$$+22\mathbb{E}\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathcal{S}(t_2-s)f\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)ds\right\|^2$$

$$+22\mathbb{E}\left\|\int_{t_1-\epsilon}^{t_1}(t_2-s)^{\alpha-1}[\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)]f\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)ds\right\|^2$$

$$+22\mathbb{E}\left\|\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathcal{S}(t_1-s)f\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)ds\right\|^2$$

$$+22\mathbb{E}\left\|\int_0^{t_1-\epsilon}(t_2-s)^{\alpha-1}[\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)]f\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)ds\right\|^2$$

$$+22\mathbb{E}\left\|\int_0^{t_1-\epsilon}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathcal{S}(t_1-s)f\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)ds\right\|^2$$

$$+22\mathbb{E}\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathcal{S}(t_2-s)\sigma(s)dW(s)\right\|^2$$

$$+22\mathbb{E}\left\|\int_{t_1-\epsilon}^{t_1}(t_2-s)^{\alpha-1}[\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)]\sigma(s)dW(s)\right\|^2$$

$$+22\mathbb{E}\left\|\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathcal{S}(t_1-s)\sigma(s)dW(s)\right\|^2$$

$$+22\mathbb{E}\left\|\int_0^{t_1-\epsilon}(t_2-s)^{\alpha-1}[\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)]\sigma(s)dW(s)\right\|^2$$

$$+22\mathbb{E}\left\|\int_0^{t_1-\epsilon}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathcal{S}(t_1-s)]\sigma(s)dW(s)\right\|^2.$$

Applying Lemma 4.4.3 and the Hölder inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\|\overline{z}(t_2)-\overline{z}(t_1)\|^2\\
&\le 22\|(\mathcal{T}(t_2)-\mathcal{T}(t_1))\|^2\mathbb{E}\|g(0,\phi)\|^2+22\mathbb{E}\|g(t_2,y_{t_2}+\widehat{\phi}_{t_2})-g(t_1,y_{t_1}+\widehat{\phi}_{t_1})\|^2\\
&+22\frac{K_1(\alpha,\beta)(t_2-t_1)^{\alpha\beta}}{\alpha\beta}\int_{t_1}^{t_2}(t_2-s)^{\alpha\beta-1}\mathbb{E}\|A^\beta g(s,y_s+\widehat{\phi}_s)\|^2ds\\
&+22\epsilon^2\int_{t_1-\epsilon}^{t_1}(t_2-s)^{\alpha\beta-1}\|A^{1-\beta}\|^2\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\|^2\mathbb{E}\|A^\beta g(s,y_s+\widehat{\phi}_s)\|^2ds\\
&+22K_1(\alpha,\beta)\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds\\
&\times\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathbb{E}\|A^\beta g(s,y_s+\widehat{\phi}_s)\|^2ds\\
&+22(t-\epsilon)^2\int_0^{t_1-\epsilon}(t_2-s)^{\alpha\beta-1}\|A^{1-\beta}\|^2\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\|^2\mathbb{E}\|A^\beta g(s,y_s+\widehat{\phi}_s)\|^2ds
\end{aligned}$$

$$+22K_1(\alpha,\beta)\int_0^{t_1-\epsilon}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds$$

$$\times\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathbb{E}\|A^\beta g(s,y_s+\widehat{\phi}_s)\|^2ds$$

$$+22N(\alpha)M_B^2\frac{(t_2-t_1)^\alpha}{\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{E}\|u^\epsilon(s,y+\widehat{\phi})\|^2ds$$

$$+22\frac{\epsilon^\alpha}{\alpha}M_B^2\int_{t_1-\epsilon}^{t_1}(t_2-s)^{\alpha-1}\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\|^2\mathbb{E}\|u^\epsilon(s,y+\widehat{\phi})\|^2ds$$

$$+22N(\alpha)M_B^2\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds$$

$$\times\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathbb{E}\|u^\epsilon(s,y+\widehat{\phi})\|^2ds$$

$$+22\frac{(t_1-\epsilon)^\alpha}{\alpha}M_B^2\int_0^{t_1-\epsilon}(t_2-s)^{\alpha-1}\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\|^2\mathbb{E}\|u^\epsilon(s,y+\widehat{\phi})\|^2ds$$

$$+22N(\alpha)M_B^2\int_0^{t_1-\epsilon}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds$$

$$\times\int_0^{t_1-\epsilon}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathbb{E}\|u^\epsilon(s,y+\widehat{\phi})\|^2ds$$

$$+22N(\alpha)\frac{(t_2-t_1)^\alpha}{\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{E}\left\|f\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)\right\|^2ds$$

$$+22\frac{\epsilon^\alpha}{\alpha}\int_{t_1-\epsilon}^{t_1}(t_2-s)^{\alpha-1}\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\|^2\mathbb{E}\left\|f\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)\right\|^2ds$$

$$+22N(\alpha)\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds$$

$$\times\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathbb{E}\left\|f\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)\right\|^2ds$$

$$+22\frac{(t_1-\epsilon)^\alpha}{\alpha}\int_0^{t_1-\epsilon}(t_2-s)^{\alpha-1}\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\|^2\mathbb{E}\left\|f\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)\right\|^2ds$$

$$+22N(\alpha)\int_0^{t_1-\epsilon}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds$$

$$\times\int_0^{t_1-\epsilon}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathbb{E}\left\|f\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)\right\|^2ds$$

$$+22N(\alpha)Tr(Q)\int_{t_1}^{t_2}[(t_2-s)^{2(\alpha-1)}l_r(s)ds$$

$$+22Tr(Q)\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{2(\alpha-1)}\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\|^2 l_r(s)ds$$

$$+22N(\alpha)Tr(Q)\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]^2 l_r(s)ds$$

$$+22Tr(Q)\int_{0}^{t_1-\epsilon}[(t_2-s)^{2(\alpha-1)}\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\|^2 l_r(s)ds$$

$$+22N(\alpha)Tr(Q)\int_{0}^{t_1-\epsilon}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]^2 l_r(s)ds.$$

with $K_1(\alpha,\beta)=\frac{\alpha^2 C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}$ and $N(\alpha)=(\frac{M\alpha}{\Gamma(1+\alpha)})^2$. Therefore, for ϵ sufficiently small, we can verify that the right-hand side of the above inequality tends to zero as $t_2\to t_1$. On the other hand, the compactness of $\mathcal{S}(t)$ (Lemma 3.4 in [126]),implies the continuity in the uniform operator topology. Thus Ψ maps B_r into an equicontinuous family of functions.

Next, we prove that $V(t)=\{(\Psi_1 y)(t): y\in B_r\}$ is relatively compact in $\mathcal{H}$. Obviously, $V(t)$ is relatively compact in $\mathcal{B}_h''$ for $t=0$. Let $0<t\le b$ be fixed and ϵ be a real number satisfying $0<\epsilon<t$. For $\delta>0$ and $y\in B_r$, define an operator $\Psi^{\epsilon,\delta}$ on B_r by $\Psi^{\epsilon,\delta}y$ the set of $\overline{z}^{\epsilon,\delta}\in\mathcal{B}_h''$ such that

$$\begin{aligned}
\overline{z}^{\epsilon,\delta} &= -\int_{\delta}^{\infty}\xi_\alpha(\theta)T(t^\alpha\theta)g(0,\phi(0))d\theta+g(t-\epsilon,y_{t-\epsilon}+\widehat{\phi}_{t-\epsilon})\\
&\quad+\alpha\int_0^{t-\epsilon}(t-s)^{\alpha-1}A\left(\int_\delta^\infty\theta\xi_\alpha(\theta)T((t-s)^\alpha\theta)d\theta\right)g(s,y_s+\widehat{\phi}_s)ds\\
&\quad+\alpha\int_0^{t-\epsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)u^\epsilon(s,y+\widehat{\phi})d\theta ds\\
&\quad+\alpha\int_0^{t-\epsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)d\theta ds\\
&\quad+\alpha\int_0^{t-\epsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)\sigma(s)d\theta dW(s)\\
&= -T(\epsilon^\alpha\delta)\int_\delta^\infty\xi_\alpha(\theta)T(t^\alpha\theta-\epsilon^\alpha\delta)g(0,\phi(0))d\theta+g(t-\epsilon,y_{t-\epsilon}+\widehat{\phi}_{t-\epsilon})\\
&\quad+T(\epsilon^\alpha\delta)\alpha\int_0^{t-\epsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)AT((t-s)^\alpha\theta-\epsilon^\alpha\delta)g(s,y_s+\widehat{\phi}_s)d\theta ds\\
&\quad+T(\epsilon^\alpha\delta)\alpha\int_0^{t-\epsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta-\epsilon^\alpha\delta)u^\epsilon(s,y+\widehat{\phi})d\theta ds\\
&\quad+T(\epsilon^\alpha\delta)\alpha\int_0^{t-\epsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta-\epsilon^\alpha\delta)f\\
&\quad\times\left(s,y_s+\widehat{\phi}_s,\int_0^s H(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)d\theta ds\\
&\quad+T(\epsilon^\alpha\delta)\alpha\int_0^{t-\epsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta-\epsilon^\alpha\delta)\sigma(s)d\theta dW(s),
\end{aligned}$$

$\sigma\in S_{\Sigma,\psi,x}$. Since $T(\epsilon^\alpha\delta),(\epsilon^\alpha\delta>0)$, is a compact operator, then the set $V^{\epsilon,\delta}(t)=\{(\Psi_1^{\epsilon,\delta}y)(t): y\in B_r\}$ is relatively compact in $\mathcal{H}$ for every $\epsilon, 0<\epsilon<t$ and for all $\delta>0$. Moreover, for every

$y \in B_r$, we have

$$\begin{aligned}
&\mathbb{E}\|\overline{z}(t) - \overline{z}^{\epsilon,\delta}(t)\|^2 \\
&\leq 10\mathbb{E}\Bigg\| \int_0^\delta \xi_\alpha(\theta)T(t^\alpha\theta)g(0,\phi(0))d\theta\Bigg\|^2 + 10\mathbb{E}\Bigg\| g(t, y_t + \widehat{\phi}_t) - g(t-\epsilon, y_{t-\epsilon} + \widehat{\phi}_{t-\epsilon})\Bigg\|^2 \\
&\quad + 10\alpha^2\mathbb{E}\Bigg\| \int_0^t \int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)AT((t-s)^\alpha\theta)g(s, y_s + \widehat{\phi}_s)d\theta ds\Bigg\|^2 \\
&\quad + 10\alpha^2\mathbb{E}\Bigg\| \int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)AT((t-s)^\alpha\theta)g(s, y_s + \widehat{\phi}_s)d\theta ds\Bigg\|^2 \\
&\quad + 10\alpha^2\mathbb{E}\Bigg\| \int_0^t \int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)u^\epsilon(s, y + \widehat{\phi})d\theta ds\Bigg\|^2 \\
&\quad + 10\alpha^2\mathbb{E}\Bigg\| \int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)u^\epsilon(s, y + \widehat{\phi})d\theta ds\Bigg\|^2 \\
&\quad + 10\alpha^2\mathbb{E}\Bigg\| \int_0^t \int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f\left(s, y_s + \widehat{\phi}_s, \int_0^s H(s,\tau,y_\tau + \widehat{\phi}_\tau)d\tau\right)d\theta ds\Bigg\|^2 \\
&\quad + 10\alpha^2\mathbb{E}\Bigg\| \int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f\left(s, y_s + \widehat{\phi}_s, \int_0^s H(s,\tau,y_\tau + \widehat{\phi}_\tau)d\tau\right)d\theta ds\Bigg\|^2 \\
&\quad + 10\alpha^2\mathbb{E}\Bigg\| \int_0^t \int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)\sigma(s)d\theta dW(s)\Bigg\|^2 \\
&\quad + 10\alpha^2\mathbb{E}\Bigg\| \int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)\sigma(s)d\theta dW(s)\Bigg\|^2 \\
&\quad = \sum_{i=1}^{10} J_i
\end{aligned} \tag{4.49}$$

A similar argument as before can show that

$$\begin{aligned}
J_1 &\leq 10M^2\|A^{-\beta}\|^2\mathbb{E}\|A^\beta g(0,\phi)\|^2\left(\int_0^\delta \xi_\alpha(\theta)d\theta\right)^2 \\
&\leq 10M^2\|A^{-\beta}\|^2 M_g(1 + \|\phi\|^2_{\mathcal{B}_h})\left(\int_0^\delta \xi_\alpha(\theta)d\theta\right)^2.
\end{aligned} \tag{4.50}$$

$$\begin{aligned}
J_2 &\leq 10\|A^{-\beta}\|^2\mathbb{E}\|A^\beta g(t, y_t + \widehat{\phi}_t) - A^\beta g(t-\epsilon, y_{t-\epsilon} + \widehat{\phi}_{t-\epsilon})\|^2 \\
&\leq 10\|A^{-\beta}\|^2 M_g(\epsilon + \|(y_t - y_{t-\epsilon}) + (\widehat{\phi}_t - \widehat{\phi}_{t-\epsilon})\|^2_{\mathcal{B}_h}).
\end{aligned} \tag{4.51}$$

$$\begin{aligned}
J_3 &\leq 10\alpha^2\mathbb{E}\Big[\int_0^t\int_0^\delta \|\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)A^{1-\beta}T((t-s)^\alpha\theta)A^\beta g(s,y_s+\widehat{\phi}_s)ds\|d\theta ds\Big]^2\\
&\leq 10\alpha^2M_{1-\beta}^2\int_0^t(t-s)^{\alpha+\alpha\beta-2}ds\int_0^t(t-s)^{\alpha+\alpha\beta-2}\mathbb{E}\|A^\beta g(s,y_s+\widehat{\phi}_s)\|^2ds\Big(\int_0^\delta\xi_\alpha(\theta)d\theta\Big)^2\\
&\leq 10\Big(\frac{\alpha M_{1-\beta}}{\alpha+\alpha\beta-1}\Big)^2b^{2\alpha+2\alpha\beta-2}M_g(1+r')\Big(\int_0^\delta\xi_\alpha(\theta)d\theta\Big)^2.
\end{aligned} \tag{4.52}$$

$$\begin{aligned}
J_4 &\leq 10\alpha^2M_{1-\beta}^2\int_{t-\epsilon}^t(t-s)^{\alpha+\alpha\beta-2}ds\int_{t-\epsilon}^t(t-s)^{\alpha+\alpha\beta-2}\mathbb{E}\|A^\beta g(s,y_s+\widehat{\phi}_s)\|^2ds\Big(\int_0^\infty\xi_\alpha(\theta)d\theta\Big)^2\\
&\leq 10\Big(\frac{\alpha M_{1-\beta}}{(\alpha+\alpha\beta-1)\Gamma(1+\alpha)}\Big)^2\epsilon^{2\alpha+2\alpha\beta-2}M_g(1+r').
\end{aligned} \tag{4.53}$$

Where we have used the equality given in Remark 4.2.3.

$$\begin{aligned}
J_5 &\leq 10\alpha^2(MM_B)^2\int_0^t(t-s)^{\alpha-1}ds\int_0^t(t-s)^{\alpha-1}\mathbb{E}\|u^\epsilon(s,y+\widehat{\phi})\|^2ds\Big(\int_0^\delta\xi_\alpha(\theta)d\theta\Big)^2\\
&\leq 10\alpha(MM_B)^2b^\alpha\int_0^t(t-s)^{\alpha-1}\frac{7}{\epsilon^2}\Big(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\Big)^2\widehat{M}ds\Big(\int_0^\delta\xi_\alpha(\theta)d\theta\Big)^2.
\end{aligned} \tag{4.54}$$

$$\begin{aligned}
J_6 &\leq 10\alpha^2(MM_B)^2\int_{t-\epsilon}^t(t-s)^{\alpha-1}ds\int_{t-\epsilon}^t(t-s)^{\alpha-1}\mathbb{E}\|u^\epsilon(s,y+\widehat{\phi})\|^2ds\Big(\int_0^\infty\xi_\alpha(\theta)d\theta\Big)^2\\
&\leq \frac{10\alpha(MM_B)^2\epsilon^\alpha}{\Gamma^2(1+\alpha)}\int_{t-\epsilon}^t(t-s)^{\alpha-1}\frac{7}{\epsilon^2}\Big(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\Big)^2\widehat{M}ds.
\end{aligned} \tag{4.55}$$

$$\begin{aligned}
J_7 &\leq 8\alpha^2M^2\int_0^t(t-s)^{\alpha-1}ds\int_0^t(t-s)^{\alpha-1}\mathbb{E}\Big\|f\left(s,y_s+\widehat{\phi}_s,\int_0^sH(s,\tau,y_\tau+\widehat{\phi}_\tau)d\tau\right)\Big\|^2ds\\
&\quad\times\Big(\int_0^\delta\xi_\alpha(\theta)d\theta\Big)^2\\
&\leq 8\alpha M^2b^\alpha\int_0^t(t-s)^{\alpha-1}n(s)\mathbb{E}_f(r'+M_0(1+r')b)ds\Big(\int_0^\delta\xi_\alpha(\theta)d\theta\Big)^2.
\end{aligned} \tag{4.56}$$

$$J_8 \leq 8\alpha^2 M^2 \int_{t-\epsilon}^{t} (t-s)^{\alpha-1} ds \int_{t-\epsilon}^{t} (t-s)^{\alpha-1} \mathbb{E} \left\| f\left(s, y_s + \widehat{\phi}_s, \int_0^s H(s,\tau,y_\tau + \widehat{\phi}_\tau)d\tau\right) \right\|^2 ds$$
$$\times \left(\int_0^\infty \xi_\alpha(\theta) d\theta \right)^2$$
$$\leq \frac{8\alpha M^2 \epsilon^\alpha}{\Gamma^2(1+\alpha)} \int_{t-\epsilon}^{t} (t-s)^{\alpha-1} n(s) \mathbb{E}_f (r' + M_0(1+r')b) ds. \tag{4.57}$$

$$J_9 \leq 8\alpha^2 M^2 Tr(Q) \int_0^t (t-s)^{2(\alpha-1)} \mathbb{E}\|\sigma(s)\|^2 ds \left(\int_0^\delta \xi_\alpha(\theta) d\theta \right)^2$$
$$\leq 8\alpha^2 M^2 Tr(Q) \int_0^t (t-s)^{2(\alpha-1)} l_r(s) ds \left(\int_0^\delta \xi_\alpha(\theta) d\theta \right)^2. \tag{4.58}$$

$$J_{10} \leq 8\alpha^2 M^2 Tr(Q) \int_{t-\epsilon}^{t} (t-s)^{2(\alpha-1)} \mathbb{E}\|\sigma(s)\|^2 ds \left(\int_0^\delta \xi_\alpha(\theta) d\theta \right)^2$$
$$\leq \frac{8\alpha^2 M^2}{\Gamma^2(1+\alpha)} Tr(Q) \int_{t-\epsilon}^{t} (t-s)^{2(\alpha-1)} l_r(s) ds. \tag{4.59}$$

Recalling (4.49), from (4.50) – (4.59), we see that for each $y \in B_r$,

$$\mathbb{E}\|\overline{z}_1(t) - \overline{z}_1^{\epsilon,\delta}(t)\|^2 \to 0 \qquad as \qquad \epsilon \to 0^+, \qquad \delta \to 0^+.$$

Therefore, there are relative compact sets arbitrary close to the set $V(t), t > 0$. Hence, the set $V(t), t > 0$ is also relatively compact in $\mathcal{H}$.
Step 4 Ψ has a closed graph. Let $y_n \to y_*$ as $n \to \infty$, $\overline{z}_n \in \Psi y_n$ for each $y_n \in B_r$, and $\overline{z}_n \to \overline{z}_*$ as $n \to \infty$. We shall show that $\overline{z}_* \in \Psi y_*$. Since $\overline{z}_n \in \Psi y_n$, then there exists $\sigma_n \in S_{\Sigma,\psi,y_n}$ such that

$$\overline{z}_n(t) = \begin{cases} 0, \quad t \in (-\infty, 0], \\ -\mathcal{T}(t)g(0,\phi(0)) + g(t,(y_n)_t + \widehat{\phi}_t) + \int_0^t (t-s)^{\alpha-1} A\mathcal{S}(t-s) g(s,(y_n)_s + \widehat{\phi}_s) ds \\ + \int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) B u^\epsilon(s,(y_n) + \widehat{\phi}) ds \\ + \int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) f(s,(y_n)_s + \widehat{\phi}_s, \int_0^s H(s,\tau,(y_n)_\tau + \widehat{\phi}_\tau) d\tau) ds \\ + \int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) \sigma_n(s) dW(s), \quad t \in J, \end{cases} \tag{4.60}$$

We must prove that there exists $\sigma_* \in S_{\Sigma,\psi,y_*}$ such that

$$\overline{z}_*(t) = \begin{cases} 0, \quad t \in (-\infty, 0], \\ -\mathcal{T}(t)g(0,\phi(0)) + g(t,(y_*)_t + \widehat{\phi}_t) + \int_0^t (t-s)^{\alpha-1} A\mathcal{S}(t-s) g(s,(y_*)_s + \widehat{\phi}_s) ds \\ + \int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) B u^\epsilon(s,(y_*) + \widehat{\phi}) ds \\ + \int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) f(s,(y_*)_s + \widehat{\phi}_s, \int_0^s H(s,\tau,(y_*)_\tau + \widehat{\phi}_\tau) d\tau) ds \\ + \int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) \sigma_*(s) dW(s), \quad t \in J, \end{cases} \tag{4.61}$$

Now, for every $t \in J$, since g is continuous, and from the definition of u^ϵ we get

$$\Bigg\|\Bigg(\overline{z}_n(t) + \mathcal{T}(t)g(0,\phi(0)) - g(t,(y_n)_t + \widehat{\phi}_t) - \int_0^t (t-s)^{\alpha-1} A\mathcal{S}(t-s)g(s,(y_n)_s + \widehat{\phi}_s)ds$$

$$- \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)Bu^\epsilon(s,(y_n) + \widehat{\phi})ds$$

$$- \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f\Bigg(s,(y_n)_s + \widehat{\phi}_s, \int_0^s H(s,\tau,(y_n)_\tau + \widehat{\phi}_\tau)d\tau\Bigg)ds\Bigg)$$

$$-\Bigg(\overline{z}_*(t) + \mathcal{T}(t)g(0,\phi(0)) - g(t,(y_*)_t + \widehat{\phi}_t) - \int_0^t (t-s)^{\alpha-1} A\mathcal{S}(t-s)g(s,(y_*)_s + \widehat{\phi}_s)ds$$

$$- \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)Bu^\epsilon(s,(y_*) + \widehat{\phi})ds$$

$$- \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f\Bigg(s,(y_*)_s + \widehat{\phi}_s, \int_0^s H(s,\tau,(y_*)_\tau + \widehat{\phi}_\tau)d\tau\Bigg)ds\Bigg)\Bigg\|_b^2$$

$$\to 0 \quad as \quad n \to \infty.$$

Consider the linear contiuous operator $\Theta : L^2(J,\mathcal{H}) \to \mathcal{C}(J,\mathcal{H})$,

$$\begin{aligned}\sigma \mapsto (\Theta\sigma)(t) &= \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma(s)dW(s) - \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)BB^*S^*(b-t) \\ &\quad \times\Bigg(\int_0^b (\epsilon I + \Gamma_s^b)^{-1}(b-s)^{\alpha-1}\mathcal{S}(b-s)\sigma(s)dW(s)\Bigg)ds.\end{aligned}$$

We can see that the operator Θ is linear and continuous. Moreover, on has

$$\|\Theta\sigma\|^2 \le \left(\frac{Mb^\alpha}{\Gamma(1+\alpha)}\right)^2 Tr(Q)\|\sigma\|^2_{L^2(L(\mathcal{K},\mathcal{H}))}\left[2 + \left(\frac{MM_Bb^\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{2}{\epsilon^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)^2\right].$$

From Lemma 4.4.2, it follows that $\Theta \circ S_\Sigma$ is a closed graph operator. Also, from the definition of Θ, we have that

$$\Bigg(\overline{z}_n(t) + \mathcal{T}(t)g(0,\phi(0)) - g(t,(y_n)_t + \widehat{\phi}_t) - \int_0^t (t-s)^{\alpha-1} A\mathcal{S}(t-s)g(s,(y_n)_s + \widehat{\phi}_s)ds$$

$$- \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)u^\epsilon(s,(y_n) + \widehat{\phi})ds$$

$$- \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f\Bigg(s,(y_n)_s + \widehat{\phi}_s, \int_0^s H(s,\tau,(y_n)_\tau + \widehat{\phi}_\tau)d\tau\Bigg)ds\Bigg) \in \Theta(S_{\Sigma,\psi,y_n}).$$

Since $y_n \to y_*$, for some $y_* \in S_{\Sigma,\psi,y_*}$, it follows from Lemma 4.4.2 that

$$\Big(\overline{z}_*(t) + \mathcal{T}(t)g(0,\phi(0)) - g(t,(y_*)_t + \widehat{\phi}_t) - \int_0^t (t-s)^{\alpha-1} A\mathcal{S}(t-s)g(s,(y_*)_s + \widehat{\phi}_s)ds$$

$$- \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)u^{\epsilon}(s,(y_*) + \widehat{\phi}_s)ds$$

$$- \int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f\Big(s,(y_*)_s + \widehat{\phi}_s, \int_0^s H(s,\tau,(y_*)_\tau + \widehat{\phi}_\tau)d\tau\Big)ds\Big) \in \Theta(S_{\Sigma,\psi,y_*}).$$

therefore Ψ has a closed graph.

As a consequence of step 1 to step 4 with the Arzela-Ascoli theorem, we conclude that Ψ is a compact multivalued map, u.s.c. with convex closed values. As a consequence of Lemma 4.4.4, we can deduce that Ψ has a fixed point x which is a mild solution of system (4.34). □

Further, in order to prove the approximate controllability result, the following additional assumption is required;
(**H7**) : The linear fractional inclusion (4.36) is approximately controllable.
(**H8**) : The functions $g(t,\psi) : J \times \mathcal{B}_h \to \mathcal{H}_\beta, f(t,\psi,x) : J \times \mathcal{B}_h \times \mathcal{H} \to \mathcal{H}$ and $\Sigma(t,\psi,x)$: $J \times \mathcal{B}_h \times \mathcal{H} \to BCC(L(\mathcal{K},\mathcal{H}))$ are uniformly bounded for all $t \in J, \psi \in \mathcal{B}_h$ and $x \in \mathcal{H}$.

Remark 4.4.1. In view of [85], (**H7**) is equivalent to $\epsilon R(\epsilon,\Gamma_0^b) := \epsilon(\epsilon I + \Gamma_0^b)^{-1} \to 0$, as $\epsilon \to 0$ in the strong operator.

Theorem 4.4.2. *Assume that the assumptions of Theorem 4.4.1 hold and in addition, hypothesis (**H7**) and (**H8**) are satisfied. Then, the fractional stochastic differential inclusion (4.34) is approximately controllable on J.*

Proof. Let $x^\epsilon \in B_r$ be a fixed point of the operator Φ^ϵ. By Theorem 4.4.1, any fixed point of Φ^ϵ is a mild solution of (4.34) under the control function $u(s,x^\epsilon)$ and satisfies, by the stochastic Fubini theorem, that for some $\sigma^\epsilon \in S_{\Sigma,\psi,x^\epsilon}$

$$\begin{aligned} x^\epsilon(b) \;=\; & \widetilde{x}_b - \epsilon(\epsilon I + \Gamma_0^b)^{-1}\Big[\mathbb{E}\widetilde{x}_b + \int_0^b \widetilde{\phi}(s)dW(s) - \mathcal{T}(b)[\phi(0) - g(0,\phi(0))] - g(b,x_b^\epsilon)\Big] \\ & + \epsilon\int_0^b (\epsilon I + \Gamma_s^b)^{-1}(b-s)^{\alpha-1}A\mathcal{S}(b-s)g(s,x_s^\epsilon)ds \\ & + \epsilon\int_0^b (\epsilon I + \Gamma_s^b)^{-1}(b-s)^{\alpha-1}\mathcal{S}(b-s)f\left(s,x_s^\epsilon,\int_0^s H(s,\tau,x_\tau^\epsilon)d\tau\right)ds \\ & + \epsilon\int_0^b (\epsilon I + \Gamma_s^b)^{-1}(b-s)^{\alpha-1}\mathcal{S}(b-s)\sigma^\epsilon(s)dW(s). \end{aligned} \tag{4.62}$$

Moreover, by the boundedness of Σ, g and f (assumption (**H8**)) and Dunford-Pettis Theorem, we have that the sequences $\{\sigma^\epsilon(s)\}, \{A^\beta g(s,x_s^\epsilon)\}$ and $\{f(s,x_s^\epsilon,\int_0^s H(s,\tau,x_\tau^\epsilon)d\tau)\}$ are weakly compact in $L^2(L(\mathcal{K},\mathcal{H})), L^2([0,b];\mathcal{H}_\beta)$ and $L^2([0,b];\mathcal{H})$, so there are a subsequences still denoted by $\{A^\beta g(s,x_s^\epsilon)\}, \{f(s,x_s^\epsilon,\int_0^s H(s,\tau,x_\tau^\epsilon)d\tau)\}$ and $\{\sigma^\epsilon(s)\}$, that weakly converge to, say, g, f and σ respectively in $L^2([0,b];\mathcal{H}_\beta), L^2([0,b];\mathcal{H})$, and $L^2(L(\mathcal{K},\mathcal{H}))$. Thus, from the above equation, we have

$$\mathbb{E}\|x^{\epsilon}(b)-\widetilde{x}_b\|^2 \le 9\|\epsilon(\epsilon I+\Gamma_0^b)^{-1}[\mathbb{E}\widetilde{x}_b-\mathcal{T}(b)(\phi(0)-g(0,\phi(0)))]\|^2$$

$$+9\mathbb{E}\bigg(\int_0^b \|\epsilon(\epsilon I+\Gamma_0^b)^{-1}\widetilde{\phi}(s)\|^2_{L^2(\mathcal{K},\mathcal{H})}ds\bigg)+9\mathbb{E}\|\epsilon(\epsilon I+\Gamma_0^b)^{-1}g(b,x_b^{\epsilon})\|^2$$

$$+9\mathbb{E}\bigg(\int_0^b (b-s)^{\alpha-1}\|\epsilon(\epsilon I+\Gamma_s^b)^{-1}A\mathcal{S}(b-s)[g(s,x_s^{\epsilon})-g(s)]\|ds\bigg)^2$$

$$+9\mathbb{E}\bigg((b-s)^{\alpha-1}\|\epsilon(\epsilon I+\Gamma_s^b)^{-1}A\mathcal{S}(b-s)g(s)\|ds\bigg)^2$$

$$+9\mathbb{E}\bigg(\int_0^b (b-s)^{\alpha-1}\|\epsilon(\epsilon I+\Gamma_s^b)^{-1}\mathcal{S}(b-s)[f\left(s,x_s^{\epsilon},\int_0^s H(s,\tau,x_\tau^{\epsilon})d\tau\right)-f(s)]ds\bigg)^2$$

$$+9\mathbb{E}\bigg(\int_0^b (b-s)^{\alpha-1}\|\epsilon(\epsilon I+\Gamma_s^b)^{-1}\mathcal{S}(b-s)f(s)\|ds\bigg)^2$$

$$+9\mathbb{E}\bigg(\int_0^b (b-s)^{\alpha-1}\|\epsilon(\epsilon I+\Gamma_s^b)^{-1}\mathcal{S}(b-s)[\sigma^{\epsilon}(s)-\sigma(s)]\|^2_{L^2}ds\bigg)$$

$$+9\mathbb{E}\bigg(\int_0^b (b-s)^{\alpha-1}\|\epsilon(\epsilon I+\Gamma_s^b)^{-1}\mathcal{S}(b-s)\sigma(s)\|^2_{L^2(\mathcal{K},\mathcal{H})}ds\bigg)$$

On the other hand, by assumption (**H7**) for all $0\le s\le b$, the operator $\epsilon(\epsilon I+\Gamma_0^b)^{-1}\to 0$ strongly as $\epsilon\to 0^+$, and moreover $\|\epsilon(\epsilon I+\Gamma_0^b)^{-1}\|\le 1$ Thus, by the Lebesgue dominated convergence theorem and the compactness of $\mathcal{S}(t)$ we obtain $\mathbb{E}\|x^{\epsilon}(b)-\widetilde{x}_b\|^2\to 0$ as $\epsilon\to 0^+$. This proves the approximate cotrollability of (4.34). □

4.4.2 An Application

As an application to theorem 4.4.2, we study the following simple example. Consider a control system governed by the fractional order neutral functional stochastic integro-differential inclusion of the form

$$\begin{aligned}&{}^cD_t^{\alpha}\left[z(t,\eta)+\int_{-\infty}^0 b(\theta,\eta)z(t,\theta)d\theta\right]\in\frac{\partial^2}{\partial\eta^2}z(t,\eta)+\widehat{\mu}(t,\eta)\\ &\quad+\mu\bigg(t,\int_{-\infty}^t\mu_1(s-t)z(s,\eta)ds,\int_0^t\int_{-\infty}^0\mu_2(s,\eta,\tau-s)z(\tau,\eta)d\tau ds\bigg)\\ &\quad+\nu\bigg(t,\int_{-\infty}^t\mu_1(s-t)z(s,\eta)ds,\int_0^t\int_{-\infty}^0\mu_3(s,\eta,\tau-s)z(\tau,\eta)d\tau ds\bigg)\frac{dW(t)}{dt},\end{aligned}\tag{4.63}$$

$$\begin{aligned}&\eta\in[0,\pi],\quad t\in[0,b],\\ &z(t,0)=z(t,\pi)=0,\quad t\ge 0,\\ &(0,\eta)=\psi(t,\eta),\qquad 0\le\eta\le\pi,\quad t\in(-\infty,0],\end{aligned}$$

$\beta(t)$ is a standard cylindrical wiener process in $\mathcal{H}$ defined on a stochastic space $(\Omega,\{\mathcal{F}_t\},\mathcal{F},\mathbb{P})$; the fractional derivative ${}^cD_t^{\alpha}, 0<\alpha<1$ is understood in the Caputo sense; $\psi(t,\eta),\mu,\nu,\mu_2$ and μ_3 are continuous.

To rewrite this system into the abstract form (4.34), let $\mathcal{H} = L_2([0, \pi])$ with the norm $\|\cdot\|$. Define $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ by $Ay = y''$ with the domain

$$\mathcal{D}(A) = \{y \in \mathcal{H};\ y, y' \text{ are absolutely continuous},\ y'' \in \mathcal{H} \text{ and } y(0) = y(\pi) = 0\}.$$

Then A is an infinitesimal generator of strongly continuous semigroup $(T(t))_{t\geq 0}$ which is compact, analytic and self-adjoint. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n = 0, 1, 2...$ and corresponding normalized eigenfunctions are given by $z_n(\eta) = \sqrt{\frac{2}{\pi}}\sin(n\eta)$. We also use the following properties:

i. If $y \in \mathcal{D}(A)$, then $Ay = \sum_{n=1}^{\infty} n^2\langle y, z_n\rangle z_n$.

ii. For each $y \in \mathcal{H}, A^{-1/2}y = \sum_{n=1}^{\infty} \frac{1}{n}\langle y, z_n\rangle z_n$. In particular, $\|A^{-1/2}\| = 1$.

iii. The operator $A^{1/2}$ is given by $A^{1/2}y = \sum_{n=1}^{\infty} n\langle y, z_n\rangle z_n$ on the space $\mathcal{D}(A^{1/2}) = \{y(\cdot) \in \mathcal{H}, \sum_{n=1}^{\infty} n\langle y, z_n\rangle z_n \in \mathcal{H}\}$.

Now, we present a special phase space $\mathcal{B}_h$. Let $h(s) = e^{2s}, s < 0$; then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2}$. Let

$$\|\varphi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} (\mathbb{E}\|\varphi(\theta)\|)^{\frac{1}{2}};$$

then $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a banach space.

For $(t, \varphi) \in [0, b] \times \mathcal{B}_h$, where $\varphi(\theta)(\eta) = \psi(\theta, \eta), (\theta, \eta) \in (-\infty, 0] \times [0, \pi]$, let $z(t) = z(t, \cdot)$, that is $z(t)(\eta) = z(t, \eta)$.

Define an infinite-dimensional space $\mathcal{U}$ by $\mathcal{U} = \{u\backslash u = \sum_{n=2}^{\infty} u_n v_n, \text{ with } \sum_{n=2}^{\infty} \mathcal{U}_n^2 < \infty\}$ for each $v \in \mathcal{H}$. The norm in $\mathcal{U}$ is defined by $\|u\|_{\mathcal{U}}^2 = \sum_{n=2}^{\infty} \mathcal{U}_n^2$. Now, we define a continuous linear mapping B from $\mathcal{H}$ into $\mathcal{H}$ as $Bu = 2u_2v_1 + \sum_{n=2}^{\infty} u_n v_n$ for $u = \sum_{n=2}^{\infty} u_n v_n \in \mathcal{U}$.

Define the bounded linear operator $B : \mathcal{U} \to \mathcal{H}$ by $(Bu)(t)(\eta) = \widehat{\mu}(t, \eta), 0 \leq \eta \leq \pi, u \in \mathcal{U}, g : J \times \mathcal{B}_h \to L_2([0, \pi]), f : J \times \mathcal{B}_h \times L_2([0, \pi]) \to L_2([0, \pi])$ and $\Sigma : J \times \mathcal{B}_h \times L_2([0, \pi]) \to L(L_2([0, \pi]), L_2([0, \pi]))$ by

$$g(t, \varphi)(\eta) = \int_{-\infty}^{0} b(\theta)\varphi(\theta)(\eta)d\theta,$$

$$f\left(t, \varphi, \int_0^t g_1(s, t)ds\right)(\eta) = \mu\left(t, \int_{-\infty}^{0} \mu_1(\theta)\varphi(\theta)d\theta, \int_0^t \int_{-\infty}^{0} \mu_2(s, \eta, \theta)\varphi(\theta, \eta)d\theta ds\right)$$

$$\Sigma\left(t, \varphi, \int_0^t g_2(s, t)ds\right)(\eta) = \nu\left(t, \int_{-\infty}^{0} \mu_1(\theta)\varphi(\theta)d\theta, \int_0^t \int_{-\infty}^{0} \mu_3(s, \eta, \theta)\varphi(\theta, \eta)d\theta ds\right).$$

On the other hand, the linear system corresponding to (4.63) is approximately controllable (but not exactly controllable). Thus, with the above choices, the system (4.63) can be written in the abstract form of (4.34) and all the conditions of Theorem 4.4.2, are satisfied. Further, we can impose some suitable conditions on the above-defined functions to verify the assumptions on Theorem 4.4.2, we can conclude that system (4.63) is approximately controllable on $[0, b]$.

Conclusion

The main goal of this thesis is to investigate the subject of fractional stochastic differential equations and inclusions in Hilbert spaces. We have discussed the existence and uniqueness result for an impulsive fractional stochastic evolution equations involving Caputo fractional derivative and fractional partial neutral stochastic functional integro-differential inclusions with state-dependent delay and analytic resolvent operators. Sufficient conditions for the existence are established by using the nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan and the fractional power of operators. The main results are obtained by means of the theory of operators semi-group, fractional calculus, fixed point technique and stochastic analysis theory and methods adopted directly from deterministic fractional equations.

In the same line of thought, we have explored some results about the approximate controllability of fractional neutral stochastic functional integro-differential inclusions in Hilbert spaces by using the natural assumption that the corresponding linear system is approximately controllable. With the use of the fractional calculus and stochastic analysis technique, control function has been constructed. Moreover, the control function, together with operator semigroup, has helped us to obtain sufficient conditions for the approximate controllability of the control system via Bohnenblust-Karlin's fixed point theorem. An application is provided to illustrate the applicability of the new result.

Our future work will try to make some the above results and study the approximate controllability for impulsive fractional neutral stochastic functional integro-differential inclusions with state-dependent delay.

Appendix A

Nuclear and Hilbert-Schmidt Operators

Let E, G be Banach spaces and let $L(E, G)$ be the Banach spaces of all linear bounded operators from E into G endowed with the usual supremum norm. We write $L_1(E)$ instead of $L_1(E, E)$. We denote by E^* and G^* the dual space of E and G respectively. An element $T \in L(E, G)$ is said to be a nuclear or trace class operator if there exist two sequences $\{a_j\} \subset G$, $\{\varphi_j\} \subset E^*$ such that

$$\sum_{i=1}^{\infty} \|a_j\| \|\varphi_j\| < +\infty \tag{4.64}$$

and T has the representation

$$Tx = \sum_{i=1}^{\infty} a_j \varphi_j, \ \ x \in E.$$

The spaces of all nuclear operators from E into G, endowed with the norm

$$\|T\|_1 = \inf \left\{ \sum_{j=1}^{\infty} \|a_j\| \|\varphi_j\| : Tx = \sum_{j=1}^{\infty} a_j \varphi_j \right\}$$

is a Banach space (see ([26])), and will be denoted as $L_1(E, G)$. Let K be another Banach space; it is clear that if $T \in L_1(E, G)$ and $S \in L(G, K)$ then $TS \in L_1(E, K)$ and $\|TS\|_1 \leq \|T\| \|S\|_1$. Let H be a separable Hilbert space and let $\{e_k\}$ be a complete orthonormal system in H. If $T \in L_1(H, H)$ then we define

$$TrT = \sum_{j=1}^{\infty} \langle Te_j, e_j \rangle.$$

Proposition 4.4.1. *If $T \in L_1(H)$ then $Tr\,T$ is a well defined number independent of the choice of the orthonormal basis $\{e_k\}$.*

Proof. We refer the reader to [26].
Note also that

$$|Tr\,T| \leq \|T\|_1, \ \forall T \in L_1(H)$$

Corollaire 4.1. *If $T \in L_1(H)$ and $S \in L(H)$, then $TS \in L_1(H)$ and*

$$Tr\,TS = Tr\,ST \leq \|T\|_1 \|S\|.$$

Proposition 4.4.2. *A nonnegative operator $T \in L(H)$ is of trace class if and only if for an orthonormal basis $\{e_k\}$ on H*

$$\sum_{i=1}^{\infty} \langle Te_j, e_j \rangle < +\infty.$$

Moreover in this case $Tr\,T = \|T\|_1$.

Proof. We refer the reader to [26].

Let E and F be two separable Hilbert spaces with complete orthonormal bases $\{e_k\} \subset H$, $\{f_j\} \subset F$, respectively. A linear bounded operator $T : H \to E$ is said to be Hilbert-Schmidt if

$$\sum_{k=1}^{\infty} |Te_k|^2 < \infty.$$

Since

$$\sum_{k=1}^{\infty} |Te_k|^2 = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |\langle Te_k, f_j\rangle|^2 = \sum_{k=1}^{\infty} |T^* f_j|^2,$$

the definition of Hilbert-Schmidt operator, and the number $\|T\|_2 = (\sum_{k=1}^{\infty} |Te_k|^2)^{1/2}$, is independent of the choice of the basis $\{e_k\}$. Moreover $\|T\|_2 = \|T^*\|_2$.

One can check easily that the set $L_2(E, F)$ of all Hilbert-Schmidt operators from E into F, equipped with the norm

$$\|T\|_2 = \left(\sum_{k=1}^{\infty} |Te_k|^2\right)^{1/2}$$

is a separable Hilbert space, with the scalar product

$$\langle S, T\rangle_2 = \sum_{k=1}^{\infty} \langle Se_k, Te_k\rangle.$$

The double sequence of operators $\{f_j \otimes e_k\}_{j,k\in N}$ is a complete orthonormal basis in $L_2(E, F)$. [4]

Proposition 4.4.3. *Let E, F, G be separable Hilbert spaces. If $T \in L_2(E, F)$ and $S \in L_2(F, G)$ then $ST \in L_1(E, G)$ and*

$$\|ST\|_1 \leq \|S\|_2 \|T\|_2.$$

Proof. We refer the reader to [26].

[4] For arbitrary $b \in E$, $a \in F$ we denoteby $b \otimes a$ the linear operator defined by $(b \otimes a).h = \langle a, h\rangle b$, $h \in F$.

Appendix B

Reproducing Kernels

Let μ be a symmetric Gaussian measure on a Banach space E. A linear subspace $H \subset E$ equipped with a Hilbert norm $|.|_H$ is said to be a reproducing kernel space for μ if H is complete, continuously embedded in E and such that for arbitrary $\varphi \in E^*$

$$\mathscr{L}(\varphi) = \mathscr{N}(0, |\varphi|_H^2),$$

where

$$|\varphi|_H = \sup_{|h|_H \leq 1} |\varphi(h)|.$$

Theorem 4.4.3. *For an arbitrary symmetric Gaussian measure μ on a separable Banach space, there exists a unique reproducing kernel space $(H, |.|_H)$.*

Proof. We refer the reader to [26].

The reproducing kernel space of μ will be denoted by H_μ. In a sense it is independent of the Banach space E.

Proposition 4.4.4. *Assume that a Banach space E_1 is continuously and as a Borel set imbedded in E. If the measure μ is symmetric and Gaussian on E and E_1, then the reproducing kernel space calculated with respect to E or E_1 is the same.*

Proof. We refer the reader to [26].

Note that if $h = J_\varphi$, $\varphi \in E^*$, then we have

$$\langle h, x \rangle_H = \varphi(x), \;\; x \in H. \tag{4.65}$$

Therefore the functional $\langle h, x \rangle_H$ can be naturally extended to the whole space E. If h is an arbitrary element on H, then there exists a sequence $\{h_n\} = \{J\varphi_n\}$, $\varphi_n \in E^*$, $n \in \mathbb{N}$, such that

$$\lim_{n \to \infty} |h - h_n| = 0.$$

Moreover, for some $\varphi \in L^2$, $\varphi_n \to \varphi$. Functional φ does not belong in general to E^* but is defined in a unique way as an element of $L_2(E, \mathscr{B}(E), \mu)$. In this way definition 4.65 can be extended to all remaining $h \in H$ and μ-almost all $x \in E$. We have the following reproducing kernel formula:

$$\int_E \langle h, x \rangle_H \langle g, x \rangle_H \mu(dx) = \langle h, g \rangle_H, \;\; h, g \in H.$$

The following result will be used to check that a given measure μ on a separable Banach space is Gaussian and to identify its reproducing kernel.

Proposition 4.4.5. *Let μ be a measure in a separable Banach space E and M a linear subspace of E^* generating the Borel s-field $\mathscr{B}(E)$.*

(i) *If arbitrary $\varphi \in M$ has a symmetric Gaussian law then μ is symmetric Gaussian.*

(ii) *If in addition H_0 is a Hilbert space continuously embedded into E and such that $\mathscr{L}(\varphi) = \mathscr{N}(0, |\varphi|_0^2)$ for arbitrary $\varphi \in M$, then H_0 is the reproducing kernel of μ.*

Proof. We refer the reader to [26].

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