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THESE

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Intitulé

CONTRIBUTIONS À L'ÉTUDE DE QUELQUES ÉQUATIONS DIFFÉRENTIELLES FLOUES D'ORDRE FRACTIONNAIRE

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Dédicaces

Aux mémoires de mon père et de ma chère soeur Amaria.

Publications

- M. Benchohra, **A. Boukenkoul**, An Existence and uniqueness theorem for fuzzy H-integral equations of fractional order, *Malaya J. Mat.* 2(2)(2014) 151-159.
- M. Benchohra, **A. Boukenkoul**, Fuzzy Partial Hyperbolic Differential Equations. Accepted. Journal of Fuzzy Mathematics.
- M. Benchohra, **A. Boukenkoul**, Fuzzy Fractional Differential Equations Involving Caputo Derivative. Submitted.
- M. Benchohra, **A. Boukenkoul**, On fuzzy fractional order derivatives and Darboux problem for implicit differential equations. Submitted.

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Contents

Chapter 0

Introduction

Many important phenomena are represented by unconventional and complex models. Conventional methods are usually unsatisfactory in the treatment of such problems due to such factors as unmanageably of several variables. With modern advances in computing capabilities the practicability of unconventional models has become feasible. Many significant problems are now formulated using fuzzy sets with applications in fields such as control theory, artificial intelligence, dynamic systems are ubiquitous, inverted pendulums, biped walking robots, nuclear reactors, power system networks ,missiles, biological neurons, flying birds [43, 44, 46, 48, 77, 59, 69, 79, 80, 89, 91, 92].

Formally, the fuzzy theory defines an interface between qualitative/quantitative and symbolic/numeric. Practically offers an elegant approach to solving multi-dimensional and complex problems characterized by high interactive parties, involving humans both as a sensor and as maker/actuator.

A fuzzy set is defined by Zadeh [91, 92] as a class of objects with a continuum of grades of membership. A characteristic function assigns to each object a grade of membership frequently chosen between zero and one. The significance of fuzzy sets follows from the fundamental relationship between classes, properties and information. That is to say, information can be used express properties, properties to define classes and conversely [29, 79].

Fractional models appear in different areas of applied sciences such as material theory, transport processes, fluid flow phenomena, earthquakes, solute transport, chemistry, wave propagation, signal theory, biology, electromagnetic theory, thermodynamics, mechanics, geology, astrophysics, economics, control theory, chaos and fractal (see [53]). A systematic study of the fractional differential equations can be found in [9, 10, 34, 35, 36, 49, 53, 61, 65]. For basic results related to fuzzy differential equations we refer the reader to [12, 18, 16, 17, 31, 58, 69, 90].

In recent years, there has been a significant development in fractional calculus techniques in ordinary and partial functional differential equations and inclusions, some recent contributions can be seen in the monographs of Kilbas [53], Lakshmikantham et al [56], Miller and Ross [61], Samko [70], the papers of Agarwal et al [12], Abbas et al. [1, 2, 5, 7, 8], Diethelm [34, 35, 36], Podlubny [67] and the references therein. This thesis is arranged as follows :

In Chapter 1, we introduce notations, definitions and some preliminary notions. In *Section 1.1*, we give definition and basic operations on fuzzy sets. *Section 1.2* is concerned to recall algebraic operations and compactness criteria of subsets of R^n. In *Section 1.3*, we give some properties of the espace E^n. *Section 1.4* is devoted properties of fuzzy set valued mapping of real variables(continuity and measurability,upper and lower semi continuity,differentiation and integration). In *Section 1.5*, we give fractional calculus(fuzzy fractional calculus,fuzzy partial fractional calculus). In *Section 1.6* we give same fixed point theorem and specials functions which are used throughout this thesis.

In Chapter 2, we shall be concerned by fuzzy integral equation of Riemman-Liouville fractional order generalized H-integrability this equation takes the form

$$y(t) = f(t) + \frac{1}{\Gamma(1-q)} \int_0^t \frac{g(s, y(s))}{(t-s)^q} \, ds. \qquad (0.0.1)$$

where $f : [0, T] \to E^n$ and $g : [0, T] \times E^n \to E^n$.
In section 2 the definition of E^n, auxiliary fact and results are guiven which will be used later.In section 3, the Riemman-Liouville H- differentiability is proposed for fuzzy-valued function and the some of important results of it are provided.
In section 2 the main theorem on the existence and uniqueness of equation (3.1.1)

In Chapter 3, we shall be concerned by Fuzzy Differential Equations Involving Caputo's Derivative . In this Chapter,we investigate the solution of Caputo's fuzzy fractional differential equations.

$$^cD^q x(t) = f(t, x(t)), \quad t \in I := [0, a], \quad 0 < q < 1 \qquad (0.0.2)$$

$$x(0) = x_0, \qquad (0.0.3)$$

where $f : I \times E^n \to E^n$ is a given function satisfying suitable conditions, $x_0 \in E^n$, and E^n is the set of fuzzy numbers to be defined later.

In Section 2 we recall some basic knowledge of fuzzy calculus and fractional calculus. several basic concepts and properties of fuzzy fractional calculus are presented .
For an application of the above cited theorem to fuzzy differential equations see [11].

Our aim in this work is to study the existence of the solution for fuzzy Caputo's fractional differential equations with initial condition. A Banach fixed point theorem will be used to investigate the existence of fuzzy solutions .

In Chapter 4, we investigate the solution of following Fuzzy Partial Hyperbolic Differential Equations with nonlocal condition:

$$({}^cD_0^q u)(x, y) = f(x, y, u(x, y)), \text{ if } (x, y) \in J := [0, a] \times [0, b], \qquad (0.0.4)$$

$$u(x,0) = \varphi(x),\ x \in [0,a],\ u(0,y) = \psi(y),\ y \in [0,b], \tag{0.0.5}$$

where $a, b > 0$, ${}^cD_0^r$ is the Caputo's fractional derivative of order $q = (q_1, q_2) \in (0,1] \times (0,1]$, $f : J \times E^n \to E^n$ is a given continuous function, $\varphi : [0,a] \to E^n, \psi : [0,b] \to E^n$ are given absolutely continuous functions with $\varphi(x) = \phi(x,0),\ \psi(y) = \phi(0,y)$ for each $x \in [0,a],\ y \in [0,b]$ and $\phi(0,0) = \psi(0) = \varphi(0)$.

In Section 2 we recall some basic knowledge of fuzzy calculus and fractional calculus. In Section 3 several basic concepts and properties of fuzzy Partial Hyperbolic Differential Equations fractional calculus are presented. We use the fixed point approach. For an application of the above cited approach to fuzzy differential equations see [11].

The Banach contraction principle and a fixed point theorem for absolute retract spaces are used to investigate the existence of fuzzy solutions for precedent fractional differential equation with nonlocal condition

In Chapter 5 Our aim in this chapter is to study the existence of solution for a fuzzy fractional order derivatives and Darboux problem for implicit differential equations

$$({}^cD_0^q u)(x,y) = f(x,y,u(x,y),{}^cD_0^q u)(x,y)),\ \text{if } (x,y) \in J := [0,a] \times [0,b], \tag{0.0.6}$$

$$\begin{cases} u(x,0) = \varphi(x), & x \in [0,a], \\ u(0,y) = \psi(y), & y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{0.0.7}$$

where $a, b > 0$, ${}^cD_0^r$ is the Caputo's fractional derivative of order $q = (q_1, q_2) \in (0,1] \times (0,1]$, $f : J \times E^n \times E^n \to E^n$ is a given continuous function, $\varphi : [0,a] \to E^n, \psi : [0,b] \to E^n$ are given absolutely continuous functions with $\varphi(x) = \phi(x,0),\ \psi(y) = \phi(0,y)$ for each $x \in [0,a],\ y \in [0,b]$ and $\phi(0,0) = \psi(0) = \varphi(0)$. The Banach contraction principle and a fixed point theorem for absolute retract spaces are used to investigate the existence of fuzzy solutions for this fractional differential equation with nonlocal condition.

We present two results for the problem (6.1.1)–(6.1.2), the first one is based on Banach contraction principle and the second one on the theorem for absolute retract spaces.

Key words and phrases: Fuzzy logic and fuzzy interference systems, fuzzy set, Zadeh's extension principle, The space E^n, Fuzzy set-valued Mapping of real variables, Fuzzy fractional calculus, Fuzzy partial fractional calculus, Upper and lower semi continuity, etc...

AMS(MOS)Subject Classifications : 03E72, 26E50, 54A40, 94D05, 47S40, 47H10, 26A33, 34A60, 35R10, 34A34, 34B15.

Chapter 1

Motivational Models

In this chapter, we present some common models in which fuzzy constructions arise.

Various formulations have been proposed to obtain solution of fuzzy differential equations and they have limitations like ever-expanding support in fuzzy solution trajectory in time evolution of the solution with fuzzy parameters.

Although in [20] the authors gave theoretical investigations on modeling FDEs with fuzzy initial parameters. Theoretically, it is possible to apply the extension principle to solve the FDEs with fuzzy conditions. However, the complexity in using the extension principle is high and intractable. Under this condition, the necessity of a topological methods in solving FDE with initial values problems.

1.1 Fuzzy Malthus continuous model [20, 59]

Let us suppose that the population obeys the Malthusian growth, according to the model: $u^{'}(t) = au(t), u(0) = u_0 \in E^1$ and $a \in \mathbb{R}$ Since u_0 is a fuzzy set, the field $F(u) = au$ is the extension's Zadeh of function $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = ax$. The concept of fuzzy differential equation can be see in Kaleva(1987) and Seikkala, (1987).
Let $[u(t)]^\alpha = [u_1^\alpha(t), u_2^\alpha(t)]$, According to Seikkala (1987), to get the solution previous equation we must solve the deterministic systems

$$\begin{cases} u_1^\alpha(t)' = au_1^\alpha(t),\ u_1^\alpha(0) = u_{01}^\alpha, \\ u_2^\alpha(t)' = au_2^\alpha(t), u_2^\alpha(0) = u_{02}^\alpha \quad a \geq 0, \end{cases} \tag{1.1.1}$$

$$\begin{cases} u_1^\alpha(t)' = au_2^\alpha(t), \quad u_1^\alpha(0) = u_{01}^\alpha, \\ u_2^\alpha(t)' = au_1^\alpha(t), \quad u_2^\alpha(0) = u_{02}^\alpha \quad a < 0, \end{cases} \tag{1.1.2}$$

for each $\alpha \in [0, 1]$

The solution of (1.1.1) and (1.1.2) are respectively given by:

$$\begin{cases} u_1^\alpha(t) = u_{01}^\alpha e^{at}, \\ u_2^\alpha(t) = u_{02}^\alpha e^{at}, \qquad\qquad a \geq 0, \end{cases}$$

and

$$\begin{cases} u_1^\alpha(t) = \frac{u_{01}^\alpha - u_{02}^\alpha}{2} e^{-at} + \frac{u_{01}^\alpha + u_{02}^\alpha}{2} e^{at}, \\ u_2^\alpha(t) = \frac{u_{02}^\alpha - u_{01}^\alpha}{2} e^{-at} + \frac{u_{01}^\alpha + u_{02}^\alpha}{2} e^{at}, \qquad\qquad a < 0, \end{cases}$$

In May (1976) explores the qualitative properties of the logistic discrete as a population growth model. The Zadeh's extension of this model will be considered in the next example which has many interesting properties that appear in general nonlinear systems.

1.2 Life expectancy [20, 41, 19]

Let A be a group with $n(t)$ individuals, in the instant t. Supposing that there is not individual's birth in this group the dynamics of the number of the individuals is modelled by the differential equation:

$$n'(t) = \lambda n(t), \quad n(0) = n_0$$

The question is: in which way does the environment or even the individuales' way of living influence the group's life expectancy?
A possible answer will be given supposing that the enviroment acts in the group as a whole.That is, we will not take into account such individual characteristics as strength, race,color, etc.This is the main characteristic to adopt fuzziness only in the equation's parameters, originally deterministic, as to the example of environmental stochasticity (May 1974; Turelli, 1986). To incorporate the fuzziness in parameter λ we can suppose that $\lambda = \lambda_1 + u_k(r)\lambda_2$, where λ_1 is the natural mortality rate and $u_k(r)\lambda_2$ is the coefficient which represents the presence of poverty in the mortality rate,λ,of the group.The mortality rate is maximal and $\lambda_1 + \lambda_2$ when $u_k(r) = 1$.For the model we have chosen as a 'poor set 'the fuzzy set given by:

$$u_k(r) = \begin{cases} [1 - (\frac{r}{r_0})^2]^k & if \;\; 0 \le r < r_0; \\ 0 & if \;\; r \ge r_0; \end{cases}$$

which comments and interpretations about the parameters k and r_0 and variable r can be found in Bassanezi and Barros (1995).Supposing now that r is proportional to the salaries of the studied group: $r = cS^m$,with c and m being two constants, we have the fuzzy set:

$$v_k(S) = \begin{cases} [1 - (\frac{S}{S_0})^{2m}]^k & if \;\; 0 \le S < S_0; \\ 0 & if \;\; S \ge S_0; \end{cases}$$

where $S_0 = (\frac{r_0}{c})^{\frac{1}{m}}$
To obtain the values of λ_1, λ_2 and S_0 we used a life expectancy table based on distinct salary levels(Bassanezi and Barros, 1995). The values we were calculated as: $\lambda_1 = \frac{1}{54.4}, \lambda_2 = 6.618 \times 10^{-3}$

and $S_0 = 3.2$ According to the salary distribution of a working group from the same region, for which we defined λ_1, λ_2 and S_0, we have $k = 1.51$ and $m = 0.4435$,supposing $c = 1$.So,Eq.(7) can be solved by using the classical ordinary differential equations which solution is given by:

$$n(t) = n_0 e^{-[\lambda_1 + \lambda_2 v_k(S)]t}$$

for each value of S. By this way

$$n'(t) = -(\lambda_1 + \lambda_2 v_k(S))n(t), n(0) = n_0 \in R_+$$

which solution for each instant is a fuzzy set.We have that the α-levels of $v_k$and $n(t)$ are :

$$[v_k]^\alpha = [0, S_0(1-\alpha^{\frac{1}{k}})^{\frac{1}{2m}}] \quad and \quad [n]^\alpha = [n_1^\alpha, n_2^\alpha]$$

with solution given by its α-levels :

$$\begin{cases} n_1^\alpha = \frac{n_{01}^\alpha + \sqrt{\frac{b}{\lambda_1}} n_{02}^\alpha}{2} exp(-\sqrt{\lambda_1 b} t) - \frac{(-n_{01}^\alpha + \sqrt{\frac{b}{\lambda_1}} n_{02}^\alpha)}{2} exp(\sqrt{\lambda_1 b} t) \\ n_2^\alpha = \frac{n_{02}^\alpha + \sqrt{\frac{\lambda_1}{b}} n_{01}^\alpha}{2} exp(-\sqrt{\lambda_1 b} t) + \frac{(-\sqrt{\frac{\lambda_1}{b}} n_{02}^\alpha + n_{02}^\alpha)}{2} exp(\sqrt{\lambda_1 b} t).; \end{cases}$$

Chapter 2

Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this thesis.

2.1 Definition and basic operations on fuzzy sets

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest.

The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in [0,1],with $u(x) = 0$ corresponding to non-membership, $0 < u(x) < 1$, to partial membership and $u(x) = 1$ to full membership.

According to Zadeh a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0, 1]$ for some function $u : X \to [0, 1]$. There function u itself is often used synonymously for the Fuzzy set.

Example 2.2 *the function* $u : \mathbb{R}^1 \to [0, 1]$ *with*

$$u(x) = \begin{cases} 0; \ \textit{if} \ \ x \leq 1, \\ \dfrac{1}{99}(x-1); \ \textit{if} \ \ 1 < x \leq 100, \\ 1; \ \textit{if} \ \ 100 < X, \end{cases}$$

provides an example of a fuzzy set of real numbers $x \geq 1$ (Fig2.1). There are of course many other reasonable choices of membership grade function .

The only membership possibilities for an ordinary or crisp subset A of X are non-membership and full membership. Such a set can thus be identified with the fuzzy set on X given by its characteristic function $\chi_A : X \to \{0, 1\}$

$$\chi_A(x) = \begin{cases} 0; \ \text{if} \ \ x \notin A, \\ 1; \ \text{if} \ \ x \in A, \end{cases}$$

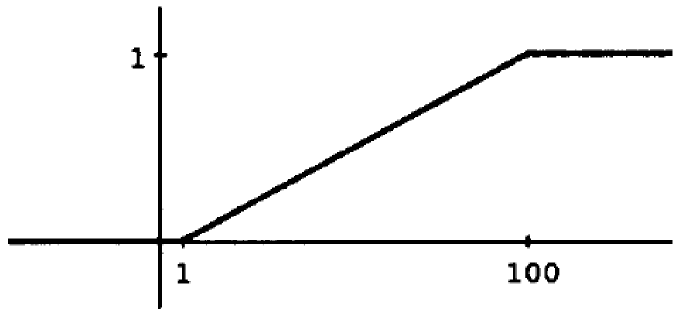

Figure 2.1: A fuzzy set of real numbers $x >> 1$

Figure (Fig 2.2) shows the characteristic function of the interval $1 \leq x \leq 2$.

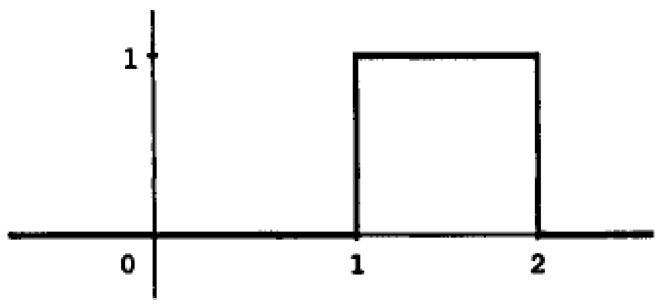

Figure 2.2: The fuzzy set $\chi_{[1,2]}$.

The Extension principle is introduced by Zadeh in 1965 [92] , and it provideds an universal way of extending mappings or functions from the crisp domain to the fuzzy domain

Theorem 2.1 *[40] (Zadeh's extension principle) If* $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is a function, then we can extend* $f : E^n \times E^n \longrightarrow E^n$ *by* $f(u,v)(z) = \sup_{z=f(u,v)} inf(u(x), v(y))$ *and it is also well known that* $[f(u,v)]^\alpha = f[[u]^\alpha, [v]^\alpha]$.

The union, intersection and complement of fuzzy sets can defined point wise in terms of their membership grades without using the extension principle.
Consider a function $u : X \to [0,1]$ as a fuzzy subset of a nonempty base space X and denote the totality of all such function or fuzzy sets by $F(X)$. The complement u^c of $u \in F(X)$, the union $u \bigcup v$ and the intersection $u \bigcap v$ of $u, v \in F(X)$ are defined as sequently, for each $t \in X$ we have

(i) $u^c(t) = 1 - u(t)$.

(ii) $u \bigcup v(t) = u(t) \bigcup v(t)$.

(iii) $u \bigcap v(t) = u(t) \bigcap v(t)$.

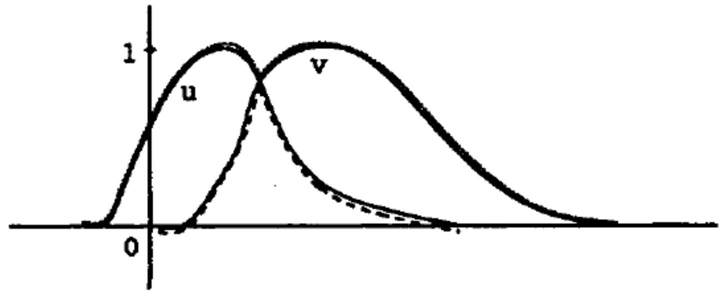

Figure 2.3: Union and intersection of two fuzzy sets u and v, $u\bigcup v$ is shown as the dotted curve, while $u\bigcap v$ is indicated by the dashed curve

Clearly: $u^c, u\bigcup v, u\bigcap v \in F(X)$. The Zadeh extension principle allows a crisp mapping $f : X_1 \times X_2 \to Y$where X_1, X_2 and Y are nonempty sets, to be extended to a mapping on fuzzy sets

$\widetilde{f} : F(X_1) \times F(X_2) \to F(Y)$ where

$$\widetilde{f}(u_1, u_2)(y) = \begin{cases} \sup\limits_{(t_1,t_2)\in f^{-1}(y)} u_1(t_1) \wedge u_2(t_2); \text{ if } f^{-1}(y) \neq \varnothing, \\ 0; \text{ if } f^{-1}(y) = \varnothing \end{cases}$$

for $y \in Y$ Here $f^{-1}(y) = \{(t_1, t_2) \in X_1 \times X_2 : f(t_1, t_2) = y\}$ may be empty or contain one more points.

The obvious generalization holds for mappings defined on N-tuple $X_1 \times X_2 \times ... \times X_N$ where $N \geq 1$, with the $\wedge$ operator being superfluous when $N = 1$.
The definitions of addition and scalar multiplication of fuzzy sets in $F(X_1)$ involve the extension principle and require the base set X to be linear space. For the addition of two fuzzy sets $u, v \in F(X)$ the Zadeh extension principle is applied to the function $f : X \times X \to X$ defined by $f(X_1, X_2) = X_1 + X_2$ to give

$$\widetilde{u+v}(x) = \sup_{x_1+x_2=x} u(x_1) \wedge u(x_2),$$

for all $x \in X$.
While for scalar multiplication of $x \in F(X)$ by a non zero scalar c the function $f : X \to X$ defined by $f(x) = cx$ is extended to $\widetilde{cu}(x) = u(\frac{x}{c})$ for $x \in X$. Obviously both $\widetilde{u+v}$ and $\widetilde{cu}$ belong to $F(X)$.

2.3 Algebraic operations and compactness criteria of Subsets of $\mathbb{R}^n$

We shall mainly the following three spaces of nonempty subsets of $\mathbb{R}^n$:

(i) C^n consisting of all nonempty closed subsets of $\mathbb{R}^n$.

(ii) K^n consisting of all nonempty compact (i.e. closed and bounded) subsets of $\mathbb{R}^n$.

(iii) K_c^n consisting of all nonempty compact convex subsets of $\mathbb{R}^n$.

Thus we have the strict inclusions

$$K_c^n \subset K^n \subset C^n.$$

Recall that a nonempty subset A of R^n is convex if and only if for all $x, y \in A$ and all $\lambda \in [0,1]$the point

$$a = \lambda x + (1-\lambda)y$$

belongs to A. For any nonempty subset A of $\mathbb{R}^n$ we denote by coA its convex hull, that is the totality of points a of the form(1,1) or, equivalently, the smallest convex subset containing A. Clearly, then

$$A \subseteq coA = co(coA).$$

with $A = coA$ if A is convex.Moreover coA is closed (compact) if A is closed (compact).

Definition 2.4 *Let A, B be two nonempty subsets of R^n and $\lambda \in R$. We define (Minkowski) addition and scalar multiplication by:*

$$A + B = \{a + b, a \in A, b \in B\}$$

and

$$\lambda A = \{\lambda a, a \in A\}$$

Proposition 2.5 *[40] C^n, K^n and K_c^n are closed under the operations of addition and scalar multiplication.*

Proposition 2.6 *[40] (C^n, d_H) is a complete separable metric space in which K^n and K_c^n are closed subsets hence, $(K^n, d_H)$and (K_c^n, d_H) are also complete separable metric spaces.*

Proposition 2.7 *[40] If $A, A', B, B' \in K^n$ then*

(i) $d_H(tA, tB) = td_H(A, B)$ for all $t \geq 0$

(ii) $d_H(A + B, A' + B') \leq d_H(A, A') + d_H(B, B')$

(iii) $d_H(coA, coB) \leq d_H(A, B)$

Proposition 2.8 *[40] If $A, B \in K_c^n$ and $c \in K^n$ then $d_H(A + C, B + C) = d_H(A, B)$.*

Proposition 2.9 *[40] A nonempty subset Aof the metric space (K^n, d_H) or (K_c^n, d_H) , is compact if and only if it is closed and uniformly bounded.*

2.10 The space E^n

Definition 2.11 *E^n is a familly of fuzzy sets $u : \mathbb{R}^n \longrightarrow [0,1]$ satisfing the fellow property :*

1) u est s.c.s (upper semi continuous).

2) $L_0(u)$ is a compact.

3) The set $L_1(u) = \{x \in \mathbb{R}^n \ : u(x) = 1\}$ is nonempty .

i.e; u is normal : $\exists x \in \mathbb{R}^n$ such that $u(x) = 1$.

Property 2.12 *For all $0 \leq \alpha \leq \beta \leq 1$*

$$[u]^\beta \subseteq [u]^\alpha \subseteq [u]^0.$$

Proposition 2.13 *[40] $[u]^\alpha \neq \varnothing$ for all $\alpha \in I$.*

Assumption 2.14 *$[u]^0$ is a bounded subset of $\mathbb{R}^n$.*

Assumption 2.15 *u is upper semi continuous.*

Proposition 2.16 *[40] $[u]^\alpha$ is a compact subset of $\mathbb{R}^n$ for all $\alpha \in I$.*

Another immediate consequence of Assumption 2.15.

Proposition 2.17 *[40] For any nondecreasing sequence $\alpha_i \to \alpha$ in I we have*

$$[u]^\alpha = \bigcap_{i \geq 1} [u]^{\alpha_i}.$$

Proposition 2.18 *[40] u is said to be fuzzy convex if*

$$u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}$$

for all $x, y \in [u]^0$ and $\lambda \in [0,1]$

We denote by E^n the espace of all fuzzy sets u of $\mathbb{R}^n$with satisfy assumptions 1..4 that is normal,fuzzy convex,uppersemicontinuous fuzzy sets with bounded support.

Proposition 2.19 *[40] Let $u \in E^n$ and write $c_\alpha = [u]^\alpha$ for $\alpha \in I$. Then*

(i) C_α is a nonempty compact convex subset of $\mathbb{R}^n$ for each $\alpha \in I$.

(ii) $C_\beta \subseteq C_\alpha$ for $0 \leq \alpha \leq \beta \leq 1$ and

(iii) $C_\alpha = \bigcap_{i=1}^{\infty} C_{\alpha_i}$ *for any non decreasing* $\alpha_i \to \alpha$ *in* I. *Or equivalently*

$$d_H(C_{\alpha_i}, C_\alpha) \to 0,$$

as $\alpha_i \to \alpha$. *The converse of proposition also holds.*

We shall define addition and scalar multiplication of fuzzy sets in E^n levelsetwise, that is, for $u, v \in E^n$ and $c \in \mathbb{R}/\{0\}$.

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha$$

and

$$[cu]^\alpha = c[u]^\alpha.$$

Proposition 2.20 *[40]* E^n *is closed under addition and scalar multiplication.*

Since E^n is a space of certain function $u : \mathbb{R}^n \to [0,1]$ an obvious candidate for a metric on E^n is the function space metric

$$d(u,v) = \sup\{|u(x) - v(x)| : x \in \mathbb{R}^n\}.$$

Note that

$$d(cu, cv) = d(u,v), \text{for all } u, v \in E^n \text{ and } c \in \mathbb{R} \setminus \{0\}.$$

Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorff metric

$$d_H(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}$$

where $d(b,A) = \inf\{d(b,a) : a \in A\}$. It is clear that $(E^n), d_H)$ is a complete metric space [54].

The supremum metric D_∞ on E^n is defined by $D_\infty(u,v) = \sup\{d_H([u]^\alpha, [v]^\alpha), \alpha \in I\}$ for all $u, v \in E^n$
The $L_{p,\infty}$ metric d_p on E^n are a class of metrics for $1 \le p < \infty$ defined by :

$$d_p(u,v) = \left(\int_0^1 d_H([u]^\alpha, [v]^\alpha)^p d\alpha\right)^{\frac{1}{p}}$$

Proposition 2.21 *[40]* (E^n, d_p) *is metric space* $1 \le p < \infty$.

Proposition 2.22 *[40]* (E^n, D_∞) *is a complete metric space .*

Proposition 2.23 *[40]* $(E^n(k), d)$ *is a complete metric space for any nonempty compact subset* k *of* $\mathbb{R}^n$.

Proposition 2.24 *[40]* $(E^n(k), d_p)$ *is a complete metric space for any nonempty compact subset* k *of* $\mathbb{R}^n$. *and* $1 \le p < \infty$

Proposition 2.25 *[40] (E^n, D_∞) is not a separable metric space.*

Proposition 2.26 *[40] Let k a nonempty compact subset of $\mathbb{R}^n$.Then $(E^n(k), D_\infty)$ is separable.*

Proposition 2.27 *[40] For $1 \leq p < \infty, (E^n(k), d_p)$ is separable.*

We will characterize compact sets of E^n with respect to various metric topologies. By compact we mean sequentially compact, that is, $U \in (E^n, \nu)$ is said to be compact if any sequence U_k in U has a subsequence converging to a point in U in the metric ν. Criteria for compactness with respect to the D_∞ metric are quite straightforward because (E^n, D_∞) be can regarded as a closed subspace (K^n, D_H) via its sendo-graph. Now, we recalling (prop1.6), a subset H of K^{n+1} is compact if and only if it is closed and uniformly bounded.

Theorem 2.2 *[40] The set $(E^n(k), D_\infty)$ is compact if and only if k is a nonempty compact subset of $\mathbb{R}^n$.*

Theorem 2.3 *[40] A closed set U of (E^n, d_∞) is compact if and only if U is uniformly support-bounded and $U^*$is equi-left-continuous in $\alpha \in I$ uniformly in $x \in S^{n-1}$.*

2.28 Fuzzy set Valued Mapping of Real Variables

2.28.1 Continuity and measurability

Definition 2.29 *We shall say that a fuzzy set valued mapping $F : T \to E^n$, where T is a domain of $\mathbb{R}^k$ is continuous at $t_0 \in T$ if for every $\varepsilon > 0$, there exists a $\delta = \delta(t_0, \varepsilon)$ such that*

$$d_\infty(F(t), F(t_0)) \leq \varepsilon$$

for all $t \in T$ with $\|t - t_0\| \leq \varepsilon$.
Alternatively,we can write in terms of the convergence of sequences,that is as
$\lim_{t_k \to t_0} d_\infty(F(t_k), F(t_0)) = 0$ *for all* $\{t_k\} \subset T$ *with* $t_k \to t_0$.

Example 2.30 *Define $F : (-1, 1) \to E^1$ levelwise as $[F(t)]^0 = [0, 1], [F(t)]^1 = \{1\}$ for all $t \in (-1, 1)$ and*

$$[F(t)]^\alpha = \begin{cases} \{1\}; \ \textit{for } t \in (-1, 0], \\ [1 - (1-\alpha)^{\frac{1}{t}}, 1] \ \textit{for } t \in (0, 1), \end{cases}$$

when $0 < \alpha < 1$.

2.30.1 Upper and Lower semi continuity

Upper and Lower semi continuity of fuzzy set valued mapping $F : T \to E^n$ is defined level set wise uniformly in $\alpha \in I$.

Definition 2.31 *F is upper semi-continuous at $t_0 \in T$ if for every $\varepsilon > 0$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that*

$$d^*_H([F(t)]^\alpha, [F(t_0)]^\alpha) \leq \varepsilon$$

for all $\alpha \in I$ and $t \in T$with $\|t - t_0\| \leq \delta$.

Definition 2.32 *F is lower semi-continuous at $t_0 \in T$ if for every $\varepsilon > 0$,there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that*

$$d^*_H([F(t_0)]^\alpha, [F(t)]^\alpha) \leq \varepsilon$$

for all $\alpha \in I$ and $t \in T$with $\|t - t_0\| \leq \delta$

We can be writ as :

$$[F(t)]^\alpha) \subseteq [F(t_0)]^\alpha + \varepsilon S^n_1$$

and

$$[F(t_0)]^\alpha \subseteq [F(t)]^\alpha) + \varepsilon S^n_1$$

for all $\alpha \in I$with $\|t - t_0\| \leq \delta$. Respectively, where $S^n_1 = \{x \in \mathbb{R}^n : \|x\| < 1\}$.

2.32.1 Measurability of a mapping $F : T \to E^n$

Definition 2.33 *We say that a mapping $F : T \to E^n$is measurable if $\{t \in T; f(T) \in \mathfrak{B}\}$is a Borel subsets of $\mathbb{R}^k$ for all Borel subset $\mathfrak{B}$of the metric space (E^n, d_∞), or equivalently if $\{t \in T; d_\infty(F(t), u) < \varepsilon\}$ is a Borel subsets of $\mathbb{R}^k$ for all $\varepsilon > 0$ and $u \in E^n$.*

Definition 2.34 *We say that $F : T \to E^n$is strongly measurable if the level set mapping $[F(t_.)]^\alpha : T \to K^n_c$are measurable (if prop4.1.3) for all $\alpha \in I$.*

Proposition 2.35 *[40] If $F : T \to E^n$is strongly measurable, then it is measurable.*

Proposition 2.36 *[40] If $F : T \to E^n$is upper or lower semi-continuous at $t_0 \in T$ then $F$is strongly measurable on T.*

Proposition 2.37 *[40] Let $F_i : T \to E^n$ be strongly measurable for $i = 1, 2...$and suppose that $\lim_{i\to\infty} d_\infty(F_i(t), F(t)) = 0$ for almost $t \in T$ Then $F : T \to E^n$is strongly measurable.*

2.37.1 Differentiation

Definition 2.38 *A mapping $F : T \to E^n$is Hukuhara differentiable at $t_0 \in T \subseteq \mathbb{R}^1$ if for some $h_0 > 0$, the Hukuhara differences*

$$F(t_0 + \Delta t)\underline{h}F(t_0); F(t_0)\underline{h}F(t_0 - \Delta t)$$

exist in E^n for all $0 < \Delta t < \underline{h}$ and if there exists an $F'(t_0) \in E^n$ such that

$$\lim_{\Delta t \to 0} d_\infty(\frac{F(t_0 + \Delta t)\underline{h}F(t_0)}{\Delta t}, F'(t_0)) = 0$$

and

$$\lim_{\Delta t \to 0} d_\infty(\frac{F(t_0)\underline{h}F(t_0 - \Delta t)}{\Delta t}, F'(t_0)) = 0$$

Here $F'(t_0)$is called the Hukuhara derivative of F at t_0

Proposition 2.39 *[40] If $F : T \to E^n$ is Hukuhara differentiable at $t_0 \in T \subseteq \mathbb{R}$,then diam$[F(t)]^\alpha$ is non decreasing int at t_0 for each $\alpha \in I$.*

Proposition 2.40 *If $F : T \to E^n$ is Hukuhara (or Bolylev)differentiable at $t_0 \in T \subseteq \mathbb{R}$,then:*

(i) its derivative $F'(t_0))$ is unique;

(ii) it is continuous at t_0.

Proposition 2.41 *[40] If $F, G : T \to E^n$ are Hukuhara (or Bolylev) differentiable at $t_0 \in T \subseteq \mathbb{R}$,then: $F + G$ and cF for all $c \in \mathbb{R}$ Hukuhara (or Bolylev) differentiable at $t_0 \in T$ and*

(i) $(F + G)'(t_0) = F'(t_0) + G'(t_0)$,

(ii) $(cF)'(t_0) = cF'(t_0)$.

Proposition 2.42 *[40] Let $F : T \to E^n$ is Hukuhara (or Bolylev) differentiable at each $t_0 \in T \subseteq \mathbb{R}$, with derivative $F' : T \to E^n$ continuous at each $t_0 \in T$, then for all $t_0, t_1 \in E^n$ with $t_0 < t_1$and*

$$d_\infty(F(t_1), F(t_0)) \leq |t_1 - t_0| \max_{t_0 \leq t \leq t_1} \|F'(t)\|.$$

Proposition 2.43 *$r : T \to \mathbb{R}^n$ is differentiable at $t_0 \in T$ with derivative $r'(t_0)$, then $F : T \to E^n$ defined by $F(t) = \chi_{r(t)}$ is Hukuhara differentiable at t_0 with derivative $F'(t_0) = \chi_{r'(t_0)}$. Moreover, if $F : T \to E^1$ is Hukuhara differentiable at (t_0) and $[F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$ for $\alpha \in I$, then f_α and g_α are differentiable at t_0 for each $\alpha \in I$,and the Hukuhara derivative $F'(t_0)$ has level sets $[F'(t_0)]^\alpha = [f'_\alpha(t_0), g'_\alpha(t_0)]$ for each $\alpha \in I$.*

2.43.1 Integration

Theorem 2.4 *[40] If $F : [0,1] \to E^n$is strongly measurable,then it is integrable over $[a,b] \subseteq [0,1]$ with $a < b$.*

Proposition 2.44 *[40] If $F, G : [0,1] \to E^n$ are integrable ,then $F + G$ and $\lambda F$for any $\lambda \in \mathbb{R}$ are integrable at with*

(i) $\displaystyle\int_0^1 (F(t) + G(t))dt = \int_0^1 F(t)dt + \int_0^1 G(t)dt$

(ii) $\displaystyle\int_0^1 \lambda F(t)dt = \lambda \int_0^1 F(t)dt.$

Proposition 2.45 *[40] If $F, G : [0,1] \to E^n$ are integrable with $F(t) \leq G(t)$ for each $t \in [0,1]$, then $\displaystyle\int_0^1 F(t)dt \leq \int_0^1 G(t)dt$*

Theorem 2.5 *Let $F_i, F : [0,1] \to E^n; 1, 2, ...$ be strongly measurable and uniformly integrally bounded. If $F_i(t) \to F(t)$ in the d_∞metric as $i \to \infty$ for all $t \in [0,1]$,
then $u_i = \int_0^1 (F_i(t))dt \to u = \int_0^1 (F(t))dt$ as $i \to \infty$*

Proposition 2.46 *[40] If $F, G : [0,1] \to E^n$ are integrable then $d_\infty(F(.), G(.)) : [0,1] \to \mathbb{R}^1$ is integrable and*

$$d_\infty\left(\int_0^1 F(t)dt, \int_0^1 G(t)dt\right) \leq \int_0^1 d_\infty(F(t), G(t))dt.$$

Theorem 2.6 *[40] If $F : [0,1] \to E^n$ is continuous,then*

$$d_\infty\left(\int_a^{t'} F(s)ds, \int_a^t F(s)ds\right) \leq |t' - t| \max_{t \leq s \leq t'} \|[F(s)]^0\|,$$

for all $0 \leq a \leq t \leq s \leq t' \leq 1$.

Theorem 2.7 *[40] If $F : [0,1] \to E^n$ is continuous, then for all $t \in (a,1) \subseteq [0,1]$ the integral $G(t) = \displaystyle\int_a^t F(s)ds$ is Hukuhara differentiable with Hukuhara derivative $G'(t) = F(t)$.*

If $F : [0,1] \to E^n$ is Hukuhara differentiable and its Hukuhara derivative F' is integrable over $[0,1]$, then $F(t) = F(t_0) + \displaystyle\int_{t_0}^t F(s)ds$ for all $0 \leq t_0 \leq t \leq 1$.

2.47 Fractional calculus

In this section, we introduce notations, definitions and preliminary Lemmas concerning to fuzzy fractional calculus theory.

2.47.1 Special functions

Definition 2.48 *The Gamma function $\Gamma(x)$ is defined by the Euler formula*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

It's well-know that

(a) $\Gamma(x+1) = x\Gamma(x)$.

(b) $\Gamma(1) = 1$.

(c) $\Gamma(n) = (n-1)!$.

(d) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(e) $\Gamma(n+\frac{1}{2}) = \dfrac{(2n)!}{2^{2n} n!}\sqrt{\pi}, n \in N$

Definition 2.49 *for $\theta \in C$ and $0 < \alpha < 1$, we say Mittag-Leffler exponential function and we denote $E_\alpha(.)$ the following function:*

$$E_\alpha(\theta) = \sum_{k=0}^{\infty} \frac{\theta^k}{\Gamma(1+\alpha k)}$$

we remark that E_1 is usual exponentiel,$E_1(\theta) = exp(\theta)$

$$D^\alpha u + \lambda u = 0; t > 0$$

with the initial condition $u(0) = 1$

2.49.1 Fuzzy fractional calculus

Definition 2.50 *Let $x, y \in E$. If exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y, it is denoted by $z = x \ominus y$.*

We denote by $C(I; E^n)$ the space of all continuous fuzzy functions on I. Also, we denote by $L^1(T, E)$ the space of all fuzzy function $f : I \longrightarrow E$ with are Lebesgue integrable on the bounded interval I.

Definition 2.51 *Let $f : [a,b] \to E, x_0 \in (a,b)$ and $\Phi(x) = \dfrac{1}{\Gamma(1-q)} \displaystyle\int_a^x \frac{f(t)}{(x-t)^q}\, dt$. We say that f is Riemann-Liouville H-differentiable about order $0 \leq q \leq 1$ at x_0, if there exists an element ${}^{RL}D^q_{a^+} f(x_0) \in C^F, 0 \leq q \leq 1$ such that for all $h > 0$*

(1)

$$^{RL}D^q_{a^+}f(x_0) = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0+h) \ominus \Phi(x_0)}{h} = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0-h)}{h}$$

or

(2)

$$^{RL}D^q_{a^+}f(x_0) = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0+h)}{-h} = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0-h) \ominus \Phi(x_0)}{-h}.$$

For sake of simplicity, we say that a fuzzy-valued function f is $[1,q]$-differentiable if it is differentiable as in case (1), and is $[2,q]$-differentiable if it is differentiable as in case (2).

Definition 2.52 *The Riemann-Liouville fractional integral of order q for a function* $f : [a,b] \to E^n$ *is defined by :*

$$^{RL}I^q_{a+}f(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s)ds.$$

When $a = 0$*, we write* $^{RL}I^q f(t)$*.*

Definition 2.53 *For a function* $f : [a,b] \to E^n$ *the Riemann-Liouville derivative of fractional order* $q > 0$ *is defined by*

$$^{RL}D^q_{a+}f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-q-1} f(s)ds,$$

with $n = [q]+1$ *and* $[q]$ *is the integer part of* q*. Obviously for* $q \in (0,1]$*, we have*

$$^{RL}D^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} f(s)ds.$$

Theorem 2.8 *Let* $f(x) \in C([0,a], E^n) \bigcap L^1([0,a], E^n)$*, the fuzzy Riemann-Liouville integral of fuzzy-valued function* f *is defined as following:*

$$\left[^{RL}I^q_{a+}f(t)\right]^\alpha = \left[^{RL}I^q_{a+}f_1^\alpha(t), ^{RL}I^q_{a+}f_2^\alpha(t)\right], \quad 0 \leq \alpha \leq 1,$$

where

$$^{RL}I^q_{a+}f_1^\alpha(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f_1^\alpha(s)ds, \quad 0 \leq \alpha \leq 1,$$

and

$$^{RL}I^q_{a+}f_2^\alpha(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f_2^\alpha(s)ds, \quad 0 \leq \alpha \leq 1.$$

Definition 2.54 *We say that the function f is fuzzy Riemann-Liouville fractional differentiable of order q at* x_0 *if there exist an element* $^{RL}D^q_{a^+}f(x_0)$ *such that for all* $h > 0$ *sufficiently small*

(1)

$$^{RL}D^q_{a^+}f(x_0) = \lim_{h\longrightarrow 0^+}\frac{\Phi(x_0+h)\ominus\Phi(x_0)}{h} = \lim_{h\longrightarrow 0^+}\frac{\Phi(x_0)\ominus\Phi(x_0-h)}{h}$$

or

(2)

$$^{RL}D^q_{a^+}f(x_0) = \lim_{h\longrightarrow 0^+}\frac{\Phi(x_0)\ominus\Phi(x_0+h)}{-h} = \lim_{h\longrightarrow 0^+}\frac{\Phi(x_0-h)\ominus\Phi(x_0)}{-h}.$$

Theorem 2.9 *[26] Let $f(t) \in C([0,a],E^n)\bigcap L^1([0,a],E^n)$ and $t \in [0,a]$, $0 < q \leq 1$, and $0 \leq \alpha \leq 1$. The fuzzy Caputo H-differentiable function of fuzzy-valued is defined as following:*

i) The case $^c[(i)-q]$*:*

$$(^cD^q_{a+}f^\alpha(t) = [(^cD^q_{a+}f^\alpha_1(t),(^cD^q_{a+}f^\alpha_2(t)]\,.$$

ii) The case $^c[(ii)-q]$

$$(^cD^q_{a+}f)(t;\alpha) = [(^cD^q_{a+}f^\alpha_2(t),(^cD^q_{a+}f^\alpha_1(t)]\,,$$

where

$$\begin{aligned} ^cD^q_{a+}f^\alpha_1(t) &= {}^{RL}D^q_{a+}\left[f^\alpha_1(t) - \Sigma^{n-1}_{k=0}\frac{x^k}{k!}f^{\alpha(k)}_1(a)\right](x) \\ &= \frac{1}{\Gamma(n-q)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-q-1}\left[f^\alpha_1(t) - \Sigma^{n-1}_{k=0}\frac{t^k}{k!}f^{\alpha(k)}_1(a)\right]dt, \end{aligned}$$

and

$$\begin{aligned} ^cD^q_{a+}f^\alpha_2(t) &= {}^{RL}D^q_{a+}\left[f^\alpha_2(t) - \Sigma^{n-1}_{k=0}\frac{x^k}{k!}f^{\alpha(k)}_2(a)\right](x) \\ &= \frac{1}{\Gamma(n-q)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-q-1}\left[f^\alpha_2(t) - \Sigma^{n-1}_{k=0}\frac{t^k}{k!}f^{\alpha(k)}_2(a)\right]dt. \end{aligned}$$

Let $x : I \longrightarrow E$ be a fuzzy function, we have

$$[x(t)]^\alpha = [x^\alpha_1(t), x^\alpha_2(t)],\ t \in I,\ \alpha \in [0,1].$$

Let $x \in C([0,a];E^n)\bigcap L^1([0,a];E^n)$, and define the fuzzy fractional primitive of q order for x;

$$I^q_{a+}x(t) = \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}x(s)ds, t \in [0,a],$$

by

$$[I^q x(t)]^\alpha = \frac{1}{\Gamma(q)}\left[\int_0^t (t-s)^{q-1}x^\alpha_1(s)ds, \int_0^t (t-s)^{q-1}x^\alpha_2(s)ds\right],\quad t \in [0,a].$$

Also, the following properties are obvious:

(i) $I^q(cx)(t) = cI^qx(t)$ for each $c \in E^n$

(ii) $I^q(x+y)(t) = I^q x(t) + I^q y(t)$.

Theorem 2.10 *Let $x : [0,a] \longrightarrow E^n$ be a function Caputo fractional differentiable, and we denote*

$$[x(t)]^\alpha = [x_1^\alpha(t), x_2^\alpha(t)],$$

then the boundary function $x_1^\alpha(t)$, $x_2^\alpha(t)$ are Caputo differentiable and

1) Case ${}^c[(i)-q]$, we have

$$[{}^cD^q x(t)]^\alpha = [{}^cD^q x_1^\alpha(t), {}^c D^q x_2^\alpha(t)].$$

2) Case ${}^c[(ii)-q]$, we have

$$[{}^cD^q x(t)]^\alpha = [{}^cD^q x_2^\alpha(t), {}^c D^q x_1^\alpha(t)].$$

2.54.1 Some Properties of Fuzzy partial fractional calculus

Definition 2.55 *Let $q = (q_1, q_2) \in (0,\infty) \times (0,\infty)$ and $u \in L^1(J, \mathbb{E}^n)$. The left-sided mixed Riemann-Liouville integral of order q of u is defined by*

$$(I_0^q u)(x,y) = \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} u(s,t)dtds,$$

where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt,\ \xi > 0$.

In particular,

$$(I_0^0 u)(x,y) = u(x,y),\ (I_0^1 u)(x,y) = \int_0^x \int_0^y u(s,t)dtds;\ \text{for almost all } (x,y) \in J.$$

For instance, $I_0^q u$ exists for all $q_1, q_2 > 0$, when $u \in L^1(J, \mathbb{E}^n)$. Note also that when $u \in C(J, \mathbb{E}^n)$, then $(I_0^q u) \in C(J, \mathbb{E}^n)$, moreover

$$(I_0^q u)(x,0) = (I_0^q u)(0,y) = 0;\ x \in [0,a],\ y \in [0,b].$$

By $1-q$ we mean $(1-q_1, 1-q_2) \in (0,1] \times (0,1]$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second order partial derivative.

Definition 2.56 *Let $q \in (0,1] \times (0,1]$ and $u \in L^1(J, \mathbb{R}^n)$. The Caputo fractional-order derivative of order q of u is defined by the expression $({}^cD_0^q u)(x,y) = (I_0^{1-q} D_{xy}^2 u)(x,y)$ and the mixed fractional Riemann-Liouville derivative of order q of u is defined by the expression $({}^{RL}D_0^q u)(x,y) = (D_{xy}^2 I_0^{1-q} u)(x,y)$.*

The case $q=(1,1)$ is included and we have

$$({}^cD_0^1u)(x,y)=({}^{RL}D_0^1u)(x,y)=(D_{xy}^2u)(x,y),\ \text{for almost all } (x,y)\in J.$$

Let $a_1\in[0,a],\ z^+=(a_1,0)\in J,\ J_z=[a_1,a]\times[0,b]$. For $w\in L^1(J_z,\mathbb{E}^n)$, the expression

$$(I_{z^+}^q w)(x,y)=\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_{a_1}^x\int_0^y(x-s)^{q_1-1}(y-t)^{q_2-1}w(s,t)dtds,$$

is called the left-sided mixed Riemann-Liouville integral of order q of w. The Caputo fractional-order derivative of order q of w is defined by :

$$({}^cD_{z^+}^q w)(x,y)=(I_{z^+}^{1-q}D_{xy}^2w)(x,y)$$

and the mixed fractional Riemann-Liouville derivative of order q of w is defined by :

$$({}^{RL}D_{z^+}^q w)(x,y)=(D_{xy}^2I_{z^+}^{1-q}w)(x,y).$$

Remark 2.57 *(Relation between ${}^{RL}D_0^q$ and ${}^cD_0^q$) Let $u\in L^1(J,\mathbb{R}^n)$ and $\varphi:[0,a]\to\mathbb{E}^n$, $\psi:[0,b]\to\mathbb{E}^n$ be given absolutely continuous functions such that $u(x,0)=\varphi(x);\ x\in[0,a],\ u(0,y)=\psi(y);\ y\in[0,b]$ and $\varphi(0)=\psi(0)$. Then we have*

$$({}^{RL}D_0^qu)(x,y)=\lambda(x,y)+({}^cD_0^qu)(x,y);\ (x,y)\in J,$$

where

$$\begin{aligned}\lambda(x,y) &= \frac{x^{-q_1}}{\Gamma(q_2)\Gamma(1-q_1)}\int_0^y(y-t)^{-q_2}\dot{\psi}(t)dt\\ &+ \frac{y^{-q_2}}{\Gamma(q_1)\Gamma(1-q_2)}\int_0^x(x-t)^{-q_1}\dot{\varphi}(t)dt+\frac{x^{-q_1}y^{-q_2}\varphi(0)}{\Gamma(1-q_1)\Gamma(1-q_2)},\end{aligned}$$

and the dot denotes differentiation.

Let $f,g\in L^1(J,\mathbb{E}^n)$.

Lemma 2.58 *([3, 4]) A function $u\in AC(J,\mathbb{E}^n)$ such that its mixed derivative D_{xy}^2 exists and is integrable on J is a solution of problem*

$$\begin{cases}({}^cD_0^qu)(x,y)=f(x,y);\ (x,y)\in J,\\ u(x,0)=\varphi(x);\ x\in[0,a],\ u(0,y)=\psi(y);\ y\in[0,b],\\ \varphi(0)=\psi(0),\end{cases}$$

if and only if $u(x,y)$ satisfies

$$u(x,y)=\mu(x,y)+(I_0^qf)(x,y);\ (x,y)\in J,$$

where

$$\mu(x,y)=\varphi(x)+\psi(y)-\varphi(0).$$

Lemma 2.59 *([7]) A function $u \in AC(J, \mathbb{R}^n)$ such that the mixed derivative $D^2_{xy}(u-g)$ exists and is integrable on J is a solution of problem*

$$\begin{cases} {}^cD^q_0[u(x,y)-g(x,y)] = f(x,y); \ (x,y) \in J, \\ u(x,0) = \varphi(x); \ x \in [0,a], \ u(0,y) = \psi(y); \ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases}$$

if and only if $u(x,y)$ satisfies

$$u(x,y) = \mu(x,y) + g(x,y) - g(x,0) - g(0,y) + g(0,0) + I^q_0(f)(x,y); \ (x,y) \in J.$$

Let $h \in C([x_k, x_{k+1}] \times [0,b], \mathbb{R}^n)$, $z_k = (x_k, 0)$, $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = a$ and

$$\mu_k(x,y) = u(x,0) + u(x_k^+, y) - u(x_k^+, 0); \quad k = 0, \dots, m.$$

Lemma 2.60 *([5, 6]) A function $u \in AC([x_k, x_{k+1}] \times [0,b], \mathbb{R}^n)$; $k = 0, \dots, m$ whose q-derivative exists on $[x_k, x_{k+1}] \times [0,b]$, $k = 0, \dots, m$ is a solution of the differential equation*

$$({}^cD^q_{z_k} u)(x,y) = h(x,y); \ (x,y) \in [x_k, x_{k+1}] \times [0,b],$$

if and only if $u(x,y)$ satisfies

$$u(x,y) = \mu_k(x,y) + (I^q_{z_k} h)(x,y); \ (x,y) \in [x_k, x_{k+1}] \times [0,b].$$

Let $J' := J \backslash \{(x_1, y), \dots, (x_m, y), \ y \in [0,b]\}$, $I_k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $k = 0, 1, \dots, m$ and denote $\mu(x,y) := \mu_0(x,y)$; $(x,y) \in J$.

Lemma 2.61 *([5, 6]) Let $h : J \longrightarrow \mathbb{R}^n$ be continuous. A function u whose q-derivative exists on J' is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases} \mu(x,y) + \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} h(s,t) dt ds; \\ \quad \text{if } (x,y) \in [0, x_1] \times [0,b], \\ \mu(x,y) + \sum_{i=1}^k (I_i(u(x_i^-, y)) - I_i(u(x_i^-, 0))) \\ + \frac{1}{\Gamma(q_1)\Gamma(q_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{q_1-1}(y-t)^{q_2-1} h(s,t) dt ds \\ + \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_{x_k}^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} h(s,t) dt ds; \\ \quad \text{if } (x,y) \in (x_k, x_{k+1}] \times [0,b], \ k = 1, \dots, m, \end{cases}$$

if and only if u is a solution of the fractional IVP

$$\begin{cases} {}^cD^q u(x,y) = h(x,y); \quad (x,y) \in J', \\ u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)); \ y \in [0,b]; \quad k = 1, \dots, m. \end{cases}$$

Lemma 2.62 *([2]) Let $h : J \longrightarrow \mathbb{R}^n$ be continuous. A function u whose q-derivative exists on J' is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases} \mu(x,y) + \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} h(s,t)dtds; \\ \quad \text{if } (x,y) \in [0,x_1] \times [0,b], \\ \varphi(x) + I_k(u(x_k,y)) - I_k(u(x_k,0)) \\ + \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_{x_k}^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} h(s,t)dtds; \\ \quad \text{if } (x,y) \in (x_k, x_{k+1}] \times [0,b], \ k = 1, \dots, m, \end{cases}$$

if and only if u is a solution of the fractional IVP

$$\begin{cases} {}^cD^q u(x,y) = h(x,y); \quad (x,y) \in J', \\ u(x_k^+, y) = I_k(u(x_k,y)); \ y \in [0,b], \quad k = 1, \dots, m. \end{cases}$$

Let $f \in C(J, \mathbb{R}^*)$ and $g \in L^1(J, \mathbb{R})$.

Lemma 2.63 *A function $u \in AC(J, \mathbb{R})$ such that the mixed derivative $D^2_{xy}(\frac{u}{f})$ exists and is integrable on J is a solution of problem*

$$\begin{cases} {}^cD_0^q \Big(\frac{u(x,y)}{f(x,y)} \Big) = g(x,y), \ (x,y) \in J, \\ u(x,0) = \varphi(x); \ x \in [0,a], \ u(0,y) = \psi(y); \ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{2.63.1}$$

if and only if $u(x,y)$ satisfies

$$u(x,y) = f(x,y)\Big(\mu_0(x,y) + (I_0^q g)(x,y)\Big); \ (x,y) \in J, \tag{2.63.2}$$

for $(x,y) \in J$, where

$$\mu_0(x,y) = \frac{\varphi(x)}{f(x,0)} + \frac{\psi(y)}{f(0,y)} - \frac{\varphi(0)}{f(0,0)}.$$

Proof: Let $u(x,y)$ be a solution of problem (2.63.1). Then, taking into account the definition of the derivative ${}^cD_0^q$, we have

$$I_0^{1-q}(D^2_{xy}) \frac{u(x,y)}{f(x,y)} = g(x,y).$$

Hence, we obtain

$$I_0^q(I_0^{1-q} D^2_{xy}) \frac{u(x,y)}{f(x,y)} = (I_0^q g)(x,y),$$

then

$$I_0^1(D^2_{xy}) \frac{u(x,y)}{f(x,y)} = (I_0^q g)(x,y).$$

Since

$$I_0^1(D_{xy}^2)\frac{u(x,y)}{f(x,y)} = \frac{u(x,y)}{f(x,y)} - \frac{u(x,0)}{f(x,0)} - \frac{u(0,y)}{f(0,y)} + \frac{u(0,0)}{f(0,0)},$$

we have

$$u(x,y) = f(x,y)\Big(\mu_0(x,y) + (I_0^q g)(x,y)\Big).$$

Now let $u(x,y)$ satisfies (2.63.2). It is clear that $u(x,y)$ satisfy

$${}^cD_0^q\Big(\frac{u(x,y)}{f(x,y)}\Big) = g(x,y), \text{ on } J.$$

Lemma 2.63 is proved.

Let $f \in C([x_k,x_{k+1}]\times[0,b],\mathbb{R}^*)$, $g \in L^1([x_k,x_{k+1}]\times[0,b],\mathbb{R})$, $z_k=(x_k,0)$, and

$$\mu_{0,k}(x,y) = \frac{u(x,0)}{f(x,0)} + \frac{u(x_k^+,y)}{f(x_k^+,y)} - \frac{u(x_k^+,0)}{f(x_k^+,0)}; \quad k=0,\dots,m.$$

Lemma 2.64 *([8]) A function $u \in AC([x_k,x_{k+1}]\times[0,b],\mathbb{R})$, $k=0,\dots,m$ such that the mixed derivative $D_{xy}^2(\frac{u}{f})$ exists and is integrable on $[x_k,x_{k+1}]\times[0,b]$, $k=0,\dots,m$ is a solution of the differential equation*

$${}^cD_{z_k}^q\Big(\frac{u}{f}\Big)(x,y) = g(x,y); \ (x,y)\in[x_k,x_{k+1}]\times[0,b],$$

if and only if $u(x,y)$ satisfies

$$u(x,y) = f(x,y)\Big(\mu_{0,k}(x,y) + (I_{z_k}^q g)(x,y)\Big); \ (x,y)\in[x_k,x_{k+1}]\times[0,b].$$

Let $\mu' := \mu_{0,0}$.

Lemma 2.65 *([8]) Let $f: J \longrightarrow \mathbb{R}^*$, $g: J \longrightarrow \mathbb{R}$ be continuous. A function u such that the mixed derivative $D_{xy}^2(\frac{u}{f})$ exists and is integrable on J' is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases} f(x,y)[\mu'(x,y) + \dfrac{1}{\Gamma(q_1)\Gamma(q_2)}\displaystyle\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}g(s,t)dtds]; \\ \text{if } (x,y)\in[0,x_1]\times[0,b], \\ f(x,y)[\mu'(x,y) + \displaystyle\sum_{i=1}^k\Big(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y)} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0)}\Big) \\ +\dfrac{1}{\Gamma(q_1)\Gamma(q_2)}\displaystyle\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{q_1-1}(y-t)^{q_2-1}g(s,t)dtds \\ +\dfrac{1}{\Gamma(q_1)\Gamma(q_2)}\displaystyle\int_{x_k}^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}g(s,t)dtds]; \\ \text{if } (x,y)\in(x_k,x_{k+1}]\times[0,b], \ k=1,\dots,m, \end{cases}$$

if and only if u is a solution of the fractional IVP

$$\begin{cases} {}^cD^q(\frac{u}{f})(x,y) = g(x,y); \quad (x,y) \in J', \\ u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)); \ y \in [0,b]; \quad k = 1, \dots, m. \end{cases}$$

2.66 Fixed point theorems

In this section we give some fixed point theorems that will be used in the sequel.

Definition 2.67 *A space Z is called an absolute retract (written $Z \in AR$) if Z is metrizable and for any metrizable space W and any embedding $h : Z \longrightarrow W$ the set $h(Z)$ is a retract of W.*

Theorem 2.11 *[45]. Let $X \in AR$ and $F : X \longrightarrow X$ a continuous and completely continuous map, then F has a fixed point.*

By $\overline{U}$ and ∂U we denote the closure of U and the boundary of U respectively. Let us start by stating a well known result, the Nonlinear Alternative.

Theorem 2.12 *(Nonlinear alternative of Leray Schauder type) ([45]) Let X be a Banach space and C a nonempty convex subset of X. Let U a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \longrightarrow C$ continuous and compact operator.*
Then either

(a) T has fixed points. Or

(b) There exist $u \in \partial U$ and $\lambda \in [0,1]$ with $u = \lambda T(u)$.

Lemma 2.68 *([45]) Let X be a Banach space and C a nonempty convex subset of X. Let U a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \longrightarrow \mathcal{P}(C)$ an upper semicontinuous and compact multivalued operator.*
Then either

(a) T has fixed points. Or

(b) There exist $u \in \partial U$ and $\lambda \in [0,1]$ with $u \in \lambda T(u)$.

Chapter 3

Fuzzy H-integral equations of fractional order

3.1 Introduction

Dubois and Prade [43, 44] introduced the concept of integration of fuzzy function. Alternative approaches were later suggested by Goetschel and Voxman [48], Kaleva [77], Nanda [62] and others. While Goetssch and Voxman preferred a Riemann integral type approach, Kaleva chose to define integral of fuzzy function by using the Lebesgue-type concept of integration. For more information about integration of fuzzy functions and fuzzy integral equations, for instance, see [43, 48, 77, 62, 76, 80, 74].

In this chapter we proposed the fuzzy H-integral equations of Riemann-Liouville fractional order generalized H-integral this equation takes the form

$$y(t) = f(t) + \frac{1}{\Gamma(1-q)} \int_0^t \frac{g(s, y(s))}{(t-s)^q}\, ds, \tag{3.1.1}$$

where $f : [0, T] \to E^n$ and $g : [0, T] \times E^n \to E^n$,

Our approche using the method of successive approximation is the main tool in our analysis. This result can be considered as a contribution to the literature. [24][1]

3.2 Existence and uniqueness results

In this section, we will study Eq(3.1.1) assuming that the following assumptions are satisfied, Let L and T be positive numbers:

(a_1) $f : [0, T] \to E^n$ is continuous and bounded.

(a_2) $g : [0, T] \times E^n \to E^n$ is continuous and satisfies the Lipschitz condition, i.e,

$$D\left(g(t, y_i(t)), g(t, y_j(t))\right) \leq L\ D\left(y_i(t), y_j(t)\right),\ t \in [0, T],$$

[1] M. Benchohra, A. Boukenkoul, An Existence and uniqueness theorem for fuzzy H-integral equations of fractional order, *Malaya J. Mat.* 2(2)(2014) 151-159.

where $y_i : [0,T] \to E^n$, $i = 1,\ 2$ and $y_j : [0,T] \to E^n$,$j = 1,\ 2, i \neq j$

(a_3) $g(t, \hat{0})$ is bounded on $[0,T]$.

Now, we are in a position to state and prove our main result in paper

Theorem 3.1 Let the assumptions $(a_1) - (a_3)$ be satisfied. If

$$T < \left(\frac{\Gamma(2-q)}{L}\right)^{\frac{1}{1-q}},$$

then Eq(3.1.1) has a unique solution y on $[0,T]$ defined as the following:

(1) In the case ${}^{RL}(1;q)$ differentiability, the successive iterations

$$\begin{aligned} y_0(t) &= f(t) \\ y_{n+1}(t) &= f(t) + {}^{RL}I_0{}^q g(t, y_n(t)), \quad n = 0, 1, 2, \ldots \end{aligned} \tag{3.2.2}$$

(2) In the case ${}^{RL}(2;q)$ differentiability, the successive iterations

$$\begin{aligned} \hat{y}_0(t) &= f(t) \\ \hat{y}_{n+1}(t) &= f(t) \ominus {}^{RL}I_0{}^q g(t, \hat{y}_n(t)), \quad n = 0, 1, 2, \ldots \end{aligned} \tag{3.2.3}$$

are uniformly convergent to y on $[0,T]$.

Proof:

(1) Case (1): If f is ${}^{RL}(1;q)$ differentiable
First we prove that y_n are bounded on $[0,T]$. We have $y_0 = f(t)$ is bounded, then (a_1). Assume that y_{n-1} is bounded. From (3.2.2) we have

$$\begin{aligned} D(y_n(t), \hat{0}) &= D\left(f(t) +^{RL} I_0^q g(t, y_{n-1}(t)),\ \hat{0}\right) \\ &\leq D\left(f(t),\ \hat{0}\right) + D\left({}^{RL}I_0^q g(t, y_{n-1}(t)),\ \hat{0}\right) \\ &\leq D\left(f(t),\ \hat{0}\right) + \frac{1}{\Gamma(1-q)} \int_0^t D\left(\frac{g(s, y_{n-1}(s))}{(t-s)^q}\ ,\ \hat{0}\right) ds \\ &\leq D\left(f(t),\ \hat{0}\right) + \frac{1}{\Gamma(1-q)} \sup_{0 \leq t \leq T} D(g(t, y_{n-1}(t)),\ \hat{0}) \int_0^t \frac{ds}{(t-s)^q}. \end{aligned}$$

But

$$\begin{aligned} D(g(t, y_{n-1}(t)),\ \hat{0}) &\leq D(g(t, y_{n-1}(t)),\ g(t,\ \hat{0})) + D(g(t,\ \hat{0}),\ \hat{0}) \\ &\leq L\ D(y_{n-1}(t),\ \hat{0}) + D(g(t,\ \hat{0}),\ \hat{0}). \end{aligned}$$

So

$$\begin{aligned} D(y_n(t), \hat{0}) &\leq D\left(f(t),\ \hat{0}\right) + \frac{T^{1-q}}{\Gamma(2-q)} \sup_{0 \leq t \leq T} [L\ D(y_{n-1}(t),\ \hat{0}) + D(g(t,\ \hat{0}),\ \hat{0})] \\ &\leq D\left(f(t),\ \hat{0}\right) + \sup_{0 \leq t \leq T} D(y_{n-1}(t),\ \hat{0}) + \frac{T^{1-q}}{\Gamma(2-q)} \sup_{0 \leq t \leq T} D(g(t,\ \hat{0}),\ \hat{0}). \end{aligned}$$

This proves that y_n is bounded. Therefore, $\{y_n\}$ is a sequence of bounded functions on $[0,T]$. Second we prove that y_n are continuous on $[0,T]$. For $0 \le t \le \tau \le T$, we have

$$\begin{aligned}
D(y_n(t), y_n(\tau)) &\le D(f(t), f(\tau)) + \frac{1}{\Gamma(1-q)}\, D\left(\int_0^t \frac{g(s, y_{n-1}(s))}{(t-s)^q}\, ds, \int_0^\tau \frac{g(s, y_{n-1}(s))}{(\tau-s)^q}\, ds\right) \\
&\le D\left(f(t), f(\tau)\right) + \frac{1}{\Gamma(1-q)} D\left(\int_0^t \frac{g(s, y_{n-1}(s))}{(t-s)^q}\, ds, \int_0^t \frac{g(s, y_{n-1}(s))}{(\tau-s)^q}\, ds\right) \\
&\quad + \frac{1}{\Gamma(1-q)} D\left(\int_t^\tau \frac{g(s, y_{n-1}(s))}{(\tau-s)^q}\, ds,\ \hat{0}\right) \\
&\le D\left(f(t), f(\tau)\right) + \frac{1}{\Gamma(1-q)} \int_0^t D\left(\frac{g(s, y_{n-1}(s))}{(t-s)^q}, \frac{g(s, y_{n-1}(s))}{(\tau-s)^q}\right) ds \\
&\quad + \frac{1}{\Gamma(1-q)} \int_t^\tau D\left(\frac{g(s, y_{n-1}(s))}{(\tau-s)^q},\ \hat{0}\right) ds \\
&\le D\left(f(t), f(\tau)\right) + \frac{1}{\Gamma(1-q)} \sup_{0\le t\le T} D(g(t, y_{n-1}(t)),\ \hat{0}) \\
&\quad \int_0^t |(t-s)^{-q} - (\tau-s)^{-q}|\, ds \\
&\quad + \frac{1}{\Gamma(1-q)} \sup_{0\le t\le T} D(g(t, y_{n-1}(t)),\ \hat{0}) \int_t^\tau \frac{ds}{(\tau-s)^q}\, ds \\
&\le D\left(f(t), f(\tau)\right) + \frac{1}{\Gamma(-q)} [|t-\tau|^(1-q) - |t^1 - q - \tau^1 - q|] \\
&\quad \sup_{0\le t\le T} D(g(t, y_{n-1}(t)),\ \hat{0}) \\
&\quad + \frac{1}{\Gamma(2-q)}\ |t-\tau|^\alpha \sup_{0\le t\le T} D(g(t, y_{n-1}(t)),\ \hat{0})
\end{aligned}$$

$$\begin{aligned}
&\le D\left(f(t), f(\tau)\right) + \frac{1}{\Gamma(2-q)} [2\ |t-\tau|^{(1-q)} - |t^{(1-q)} - \tau^{(1-q)}|] \\
&\quad \sup_{0\le t\le T} D(g(t, y_{n-1}(t)),\ \hat{0}) \\
&\le D\left(f(t), f(\tau)\right) + \frac{1}{\Gamma(2-q)} [2\ |t-\tau|^(1-q) - |t^(1-q) - \tau^(1-q)|] \\
&\quad \sup_{0\le t\le T} [L\ D(g(y_{n-1}(t)),\ \hat{0}) + D(g(t, \hat{0}),\ \hat{0})].
\end{aligned}$$

The last inequality, by symmetry, is valid for all $t,\ \tau \in [0,T]$ regardless whether or not $t \le \tau$. Thus, $D(y_n(t), y_n(\tau)) \to 0$ as $t \to \tau$. Therefore, the sequence $\{y_n\}$ is continuous on $[0,T]$. For $n \geq 1$, we have

$$\begin{aligned}
D(y_{n+1}(t), y_n(t)) &= \frac{1}{\Gamma(1-q)} \, D\left(\int_0^t \frac{g(s,y_n(s))}{(t-s)^q}\,ds, \int_0^t \frac{g(s,y_{n-1}(s))}{(t-s)^q}\,ds\right) \\
&\leq \frac{1}{\Gamma(1-q)} \int_0^t D\left(\frac{g(s,y_n(s))}{(t-s)^q}, \int_0^t \frac{g(s,y_{n-1}(s))}{(t-s)^q}\right)\,ds \\
&\leq \frac{1}{\Gamma(1-q)} \int_0^t D\left(g(s,y_n(s)), g(s,y_{n-1}(s))\right) \frac{ds}{(t-s)^q} \\
&\leq \frac{1}{\Gamma(1-q)} \sup_{0\leq t\leq T} D(g(t,y_n(t)), g(t,y_{n-1}(t))) \int_0^t \frac{ds}{(t-s)^q} \\
&\leq \frac{L\,T^{(1-q)}}{\Gamma(2-q)} \sup_{0\leq t\leq T} D(y_n(t), y_{n-1}(t)) \\
&\leq \left(\frac{L\,T^{(1-q)}}{\Gamma(2-q)}\right)^2 \sup_{0\leq t\leq T} D(y_{n-1}(t), y_{n-2}(t)) \\
&\;\;\vdots \\
&\leq \left(\frac{L\,T^{(1-q)}}{\Gamma(2-q)}\right)^n \sup_{0\leq t\leq T} D(y_1(t), y_0(t)). \qquad (3.2.4)
\end{aligned}$$

But

$$\begin{aligned}
D(y_1(t), y_0(t)) &= \frac{1}{\Gamma((1-q))} \, D\left(\int_0^t \frac{g(s,f(s))}{(t-s)^q}\,ds, \hat{0}\right) \\
&\leq \frac{1}{\Gamma((1-q))} \int_0^t D\left(\frac{g(s,f(s))}{(t-s)^q}, \hat{0}\right)\,ds \\
&\leq \frac{1}{\Gamma((1-q))} \sup_{0\leq t\leq T} D(g(t,f(t)),\hat{0}) \int_0^t \frac{ds}{(t-s)^q}.
\end{aligned}$$

Thus

$$\sup_{0\leq t\leq T} D(y_1(t), y_0(t)) \leq \frac{T^{(1-q)}}{\Gamma(2-q)}[LM+N] := R,$$

where

$$M = \sup_{0\leq t\leq T} D(f(t),\hat{0}) \;\text{ and }\; N = \sup_{0\leq t\leq T} D(g(t,\hat{0}),\hat{0}).$$

Therefore (3.2.4) takes the form

$$D(y_{n+1}(t), y_n(t)) \leq R\,\left(\frac{L\,T^{(1-q)}}{\Gamma(2-q)}\right)^n. \qquad (3.2.5)$$

Next, we show that for each $t \in [0,T]$ the sequence $\{y_n(t)\}$ is a Cauchy sequence in E^n. Let $m_1,\ m_2$ be such that $m_2 > m_1$ and $t \in [0,T]$. Then, by using (3.2.5), we have

$$\begin{aligned} D(y_{m_1}(t), y_{m_2}(t)) &\leq D(y_{m_2}(t), y_{m_2-1}(t)) + D(y_{m_2-1}(t), y_{m_2-2}(t)) \\ &\quad + \ldots + D(y_{m_1+1}(t), y_{m_1}(t)) \\ &\leq R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_2-1} + R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_2-2} \\ &\quad + \ldots + R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_1} \\ &= R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_2-1}\left[1 + \frac{\Gamma(2-q)}{L\,T^{1-q}} + \left(\frac{\Gamma(2-q)}{L\,T^{1-q}}\right)^2 \right. \\ &\qquad \left. + \ldots + \left(\frac{\Gamma(2-q)}{L\,T^{1-q}}\right)^{m_2-m_1-1}\right] \\ &= R\left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^{m_2-1}\left[\frac{1-\left(\frac{\Gamma(2-q)}{L\,T^{1-q}}\right)^{m_2-m_1}}{1-\frac{\Gamma(2-q)}{L\,T^{1-q}}}\right]. \end{aligned}$$

The right hand side of the last inequality tends to zero as $m_1,\ m_2 \to \infty$. This implies that $\{y_n(t)\}$ is a Cauchy sequence. Consequently, the sequence $\{y_n(t)\}$ is convergent, by the completeness of the metric space (E^n, D). If we denote $y(t) = \lim\limits_{n\to\infty} y_n(t)$, then $y(t)$ satisfies (3.1.1). It is continuous and bounded on $[0,T]$. To prove the uniqueness, let $x(t)$ be a continuous solution of (3.1.1) on $[0,T]$. Then

$$x(t) = f(t) + {}^{RL}I^q g(t, x(t)), \quad t \geq 0.$$

Now, for $n \geq 1$, we have

$$\begin{aligned} D(x(t), y_n(t)) &= D\left({}^{RL}I^{1-q} g(t, x(t)), {}^{RL}I^{1-q} g(t, y_n(t))\right) \\ &\leq \frac{1}{\Gamma(1-q)} \int_0^t D\left(\frac{g(s, x(s))}{(t-s)^q}, \int_0^t \frac{g(s, y_n(s))}{(t-s)^q}\right) ds \\ &\leq \frac{1}{\Gamma(1-q)} \int_0^t /8 + D\left(g(s, x(s)), g(s, y_n(s))\right) \frac{ds}{(t-s)^q} \\ &\leq \frac{1}{\Gamma(1-q)} \sup_{0\leq t\leq T} D(g(t, x(t)), g(t, y_n(t))) \int_0^t \frac{ds}{(t-s)^q} \\ &\leq \frac{L\,T^{1-q}}{\Gamma(2-q)} \sup_{0\leq t\leq T} D(x(t), y_n(t)) \\ &\ \ \vdots \\ &\leq \left(\frac{L\,T^{1-q}}{\Gamma(2-q)}\right)^n \sup_{0\leq t\leq T} D(x(t), y_0(t)). \end{aligned}$$

Since $\frac{L\,T^{1-q}}{\Gamma(2-q)} < 1$

$$\lim_{n\to\infty} y_n(t) = x(t) = y(t), \quad t \in [0,T].$$

This completes the proof.

(2) Case (2): If f is ${}^{RL}(2;q)$ differentiable, with the same argument as above, we can prove that the solution is (3.2.3) with

$$\lim_{n\to\infty} \hat{y}_n(t) = \hat{x}(t) = \hat{y}(t), \quad t \in [0, T].$$

Chapter 4

Fuzzy Differential Equations Involving Caputo Derivative

4.1 Introduction

Recently, Agarwal et al. [12] have proposed a concept of solution for fractional differential equations with uncertainty. They have considered the Riemann-Liouville differentiability concept based on the Hukuhara differentiability to solve uncertain fractional differential equations. One cane find some recently monographs and research in the field of fractional differential equations and theie solutions in Belmekki et al [28], Jumari [51], Kosmatov [55] , Latshmikantham and Vatsala [57], Nieto [63, 64] and Shuqin [72]. Moreover, we suggest the concept of fractional derivatives under Caputo's differentiability by applying Hukuhara difference which is named as Caputo's H-differentiability.Similar to the deterministic cases, construction of Caputo's derivatives are based on the definitions of Riemann-Liouville derivatives in fuzzy cases.
Our aim in this chapter is to study the existence of the solution for fuzzy Caputo fractional differential equations with initial condition. The Banach contraction principle and a fixed point theorem for absolute retract spaces are used to investigate the existence of fuzzy solutions for the following fractional differential equation with initial condition:

$$^{c}D^{q}x(t) = f(t, x(t)), \quad t \in I := [0, a], \quad 0 < q < 1, \tag{4.1.1}$$

$$x(0) = x_0, \tag{4.1.2}$$

where $f : I \times E^n \to E^n$ is a given function satisfying suitable conditions, $x_0 \in E^n$, and E^n is the set of fuzzy numbers.

4.2 The Main Results

[1]

[1]M. Benchohra, **A. Boukenkoul**, Fuzzy Fractional Differential Equations Involving Caputo Derivative. Submitted.

In this section, our first result is based on Banach fixed point theorem, we discuss the existence and uniqueness of solutions for the problem (4.1.1)–(4.1.2).

Definition 4.3 *A mapping $x : I \to E^n$ is a solution to problem (4.1.1)–(4.1.2) if it is levelwise continuous and satisfies the integral equation:*

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds. \tag{4.3.3}$$

Theorem 4.1 *Assume that*

(C1) *The function $f : I \times E^n \longrightarrow E^n$ is continous.*

(C2) *there exists a constant $k > 0$ such that*

$$d(f(t,x); f(t,y)) \leq kd(x;y), \quad \text{for each } t \in J \text{ and } x, y \in E^n.$$

If $\dfrac{ka^q}{\Gamma(q+1)} < 1$, then the problem (4.1.1)–(4.1.2) has a unique solution on $[0,a]$.

Proof. We proceed as follows (see th2 [89]) .

Case $^c[i-q]$ H-differentiability: The problem (4.1.1)–(4.1.2) is equivalent to the following fractional differential system

$$^cD^q x_1^\alpha(t) = f_\alpha(t, x_1^\alpha(t), x_2^\alpha(t)), \quad 0 < q < 1, \quad x_1^\alpha(0) = x_1(0), \tag{4.3.4}$$

$$^cD^q x_2^\alpha(t) = g_\alpha(t, x_1^\alpha(t), x_2^\alpha(t)), \quad 0 < q < 1, \quad x_2^\alpha(0) = x_2(0). \tag{4.3.5}$$

We define the operator $N : C([0,a]; E^n) \longrightarrow C([0,a]; E^n)$ by

$$Nx_1^\alpha(t) = x_1(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_\alpha(s, x_1^\alpha(s), x_2^\alpha(t)) ds. \tag{4.3.6}$$

Let $x_1, y_1 \in A$ and $t \in [0,a]$. Then we have

$$\begin{aligned} d(Nx_1^\alpha(t), Ny_1^\alpha(t)) &= d\left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_\alpha(s, x_1^\alpha(s), x_2^\alpha(t)) ds; \right. \\ &\quad \left. \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_\alpha(s, y_1^\alpha(s), y_2^\alpha(t)) ds \right) \\ &\leq \frac{k}{\Gamma(q)} d(x_1^\alpha; y_1^\alpha) \int_0^t (t-s)^{q-1} ds \\ &\leq \frac{ka^q}{\Gamma(q+1)} d(x_1^\alpha; y_1^\alpha). \end{aligned}$$

Thus

$$d(Nx_1^\alpha(t), Ny_1^\alpha(t)) \leq \frac{ka^q}{\Gamma(q+1)} d(x_1^\alpha; y_1^\alpha).$$

Consequently, N is a contraction, as a consequence of Banach fixed point theorem, we deduce that N has a unique fixed point which is a solution of the problem (6.3.3)

We transform now the problem (4.3.5) into fixed point problem, we consider the operator N by

$$Nx_2^{\alpha}(t) = x_2(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g_{\alpha}(s, x_1^{\alpha}(t), x_2^{\alpha}(s))ds. \tag{4.3.7}$$

The fixed point of the operator N is a solution of the problem (4.3.5).
We conclude the existence of a unique solution for the problem (4.1.1)–(4.1.2).

Case $^c[ii-q]$ **H-differentiability**: The problem (4.1.1)–(4.1.2) is equivalent to the following fractional differential system

$$^cD^q x_1^{\alpha}(t) = g_{\alpha}(t, x_1^{\alpha}(t), x_2^{\alpha}(t)), \quad 0 < q < 1, \quad x_1^{\alpha}(0) = x_1(0), \tag{4.3.8}$$

$$^cD^q x_2^{\alpha}(t) = f_{\alpha}(t, x_1^{\alpha}(t), x_2^{\alpha}(t)), \quad 0 < q < 1, \quad x_2^{\alpha}(0) = x_2(0). \tag{4.3.9}$$

Transform the problem (4.3.8) into fixed point problem, so we consider the operator

$$Nx_1^{\alpha}(t) = x_1(0) \ominus \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g_{\alpha}(s, x_1^{\alpha}(s), x_2^{\alpha}(t))ds. \tag{4.3.10}$$

By the same technic, we can prove that there exists a unique solution for the problem (4.3.8)–(4.3.9).

Our second result in this section is based on a fixed point theorem for absolute retract spaces. To prove our results, we introduce following conditions

(H1) The function $f : I \times E^n \to E^n$ is levelwise continuous.

(H2) There exists nondecreasing continuous function $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ and $p \in L^1(I, \mathbb{R}_+)$ such that

$$d(f(t,x), \hat{0}) \le p(t)\psi(d(x, \hat{0})), \quad \text{for all } t \in I, \ x \in E^n.$$

(H3) There exists $M > 0$, with

$$M - \left(\frac{p(t)\psi(M)}{\Gamma(q+1)} \right) \ge 0.$$

(H4) For all $t \in I$ the set

$$\left\{ x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))ds, \ x \in A \right\}$$

is totally bounded subset of E^n, where

$$A = \left\{ x \in C([0,a], E^n) : \ d(x(t), \hat{0}) \le M, \ t \in I \right\}.$$

Theorem 4.2 *Let the conditions* (H1)–(H4) *be satisfied, then the problem (4.1.1)–(4.1.2) has at least one solution.*

Proof. Transform the problem into a fixed point problem. It is clear that the solutions of the problem (4.1.1)–(4.1.2) are fixed points of the operator

$$N : C([0,a],E^n) \longrightarrow C([0,a],E^n)$$

defined by

$$Nx(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))ds.$$

We make

$$A \cong B \equiv \{\bar{J}_x \in C([0,a],E^n) : \ x \in C([0,a],E^n) \text{ and } d(x(t),\hat{0}) \le M, \ t \in [0,a]\}.$$

We see that B is a subset convex of Banach space $C([0,a],E^n)$, so in particular B is an absolute retract. We proceed as follows

Case $^c[i-q]$ **H-differentiability**: The problem (4.1.1)–(4.1.2) is equivalent to the following fractional differential system

$$^cD^q x_1^\alpha(t) = f_\alpha(t, x_1^\alpha(t), x_2^\alpha(t)), \quad 0<q<1, \quad x_1^\alpha(0) = x_1(0), \tag{4.3.11}$$

$$^cD^q x_2^\alpha(t) = g_\alpha(t, x_1^\alpha(t), x_2^\alpha(t)), \quad 0<q<1, \quad x_2^\alpha(0) = x_2(0). \tag{4.3.12}$$

We define the operator N by

$$Nx_1^\alpha(t) = x_1(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_\alpha(s, x_1^\alpha(t), x_2^\alpha(s))ds. \tag{4.3.13}$$

Clearly the fixed points of the operator $N : C([0,a],E^n) \longrightarrow C([0,a],E^n)$ are solutions of the problem (4.3.11), we shall show that $N : C([0,a],E^n) \longrightarrow C([0,a],E^n)$ is continuous and completely continuous. The proof will be given in several steps.

Step 1. $N(A) \subset A$.

Let $x \in A$ and $t \in [0,a]$. (H2) implies that

$$\begin{aligned}
d(Nx_1^\alpha(t),\hat{0}) &= d\left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x_1^\alpha(s),x_2^\alpha(t))ds, \hat{0}\right) \\
&\le d\left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x_1^\alpha(s),x_2^\alpha(t))ds, \hat{0}\right) \\
&\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} d(f(s,x_1^\alpha(s),x_2^\alpha(t)),\hat{0})ds \\
&\le \frac{p(t)\psi(M)}{\Gamma(q+1)} = M.
\end{aligned}$$

Step 2. N is continuous.

Let the sequence $[x_n(t)]^\alpha = [x^\alpha_{n,1}(t), x^\alpha_{n,2}(t)]$ of A such that $x^\alpha_{n,1} \longrightarrow x^\alpha_1$ element of A in $C([0,a], E^n)$, then

$$\begin{array}{rcl} d(Nx^\alpha_{n,1}(t), Nx^\alpha_1(t)) & = & d\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x^\alpha_{n,1}(s), x^\alpha_{n,2}(t))ds;\right. \\ & & \left.\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x^\alpha_1(s), x^\alpha_2(t))ds\right) \\ & \leq & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} d(f(s, x^\alpha_{n,1}(s), x^\alpha_{n,2}(t)); f(s, x^\alpha_1(s), x^\alpha_2(t)))ds. \end{array}$$

Let $\rho_n(s) = d(f(s, x^\alpha_{n,1}(s), x^\alpha_{n,2}(t)); f(s, x^\alpha_1(s), x^\alpha_2(t)))$, since f is continuous, we have

$$\rho_n(s) \longrightarrow 0, \quad \text{as } \ n \longrightarrow +\infty, \ s \in J.$$

From (H1), (H2) and (H3), we have,

$$\begin{array}{rcl} \rho_n(s) & \leq & d(f(s, x^\alpha_{n,1}(s), x^\alpha_{n,2}(t)); \hat{0}) + d(f(s, x^\alpha_{n,1}(s), x^\alpha_{n,2}(t)); \hat{0}) \\ & \leq & p(s)\left[\psi(d(x^\alpha_{n,1}(s); \hat{0})) + \psi(d(x^\alpha_1(s); \hat{0}))\right] \\ & \leq & 2p(s)\psi(M). \end{array}$$

As a result

$$\lim_{n\longrightarrow+\infty} \int_0^a \rho_n(s)ds = \int_0^a \lim_{n\longrightarrow+\infty} \rho_n(s)ds = 0.$$

Then

$$d(Nx^\alpha_{n,1}(t), Nx^\alpha_1(t)) \longrightarrow 0.$$

Thus N is continuous.

Step 3. The operator N is equicontinuous.

Let $l_1, l_2 \in [0,a]$, $l_1 < l_2$ and $x \in A$. Then

$$\begin{aligned}
d(Nx_1^\alpha(l_2), Nx_1^\alpha(l_1)) &= d\left(\frac{1}{\Gamma(q)}\int_0^{l_2}(l_2-s)^{q-1}f(s,x_1^\alpha(s),x_2^\alpha(t))ds;\right.\\
&\quad \left.\frac{1}{\Gamma(q)}\int_0^{l_1}(l_1-s)^{q-1}f(s,x_1^\alpha(s),x_2^\alpha(t))ds\right)\\
&\leq \frac{1}{\Gamma(q)}d\left(\int_0^{l_1}(l_2-s)^{q-1}f(s,x_1^\alpha(s),x_2^\alpha(t))ds\right.\\
&\quad +\int_{l_1}^{l_2}(l_2-s)^{q-1}f(s,x_1^\alpha(s),x_2^\alpha(t))ds;\\
&\quad \left.\int_0^{l_1}(l_1-s)^{1-q}f(s,x_1^\alpha(s),x_2^\alpha(t))ds\right)\\
&\leq \frac{1}{\Gamma(q)}d\left(\int_0^{l_1}\left[(l_2-s)^{q-1}-(l_1-s)^{q-1}\right]f(s,x_1^\alpha(s),x_2^\alpha(t))ds;\hat{0}\right)\\
&\quad +\frac{1}{\Gamma(q)}d\left(\int_{l_1}^{l_2}(l_2-s)^{q-1}f(s,x_1^\alpha(s),x_2^\alpha(t))ds;\hat{0}\right)\\
&\leq \frac{p(t)\psi(M)}{\Gamma(q)}\left(\int_0^{l_1}\left[(l_2-s)^{q-1}-(l_1-s)^{q-1}\right]ds\right.\\
&\quad \left.+\int_{l_1}^{l_2}(l_2-s)^{q-1}ds\right)\\
&\leq \frac{p(t)\psi(M)}{\Gamma(q)}\left[2(l_2-l_1)^q+(l_1^q-l_2^q)\right].
\end{aligned}$$

Then

$$d(Nx_1^\alpha(l_2), Nx_1^\alpha(l_1)) \longrightarrow 0, \quad \text{as } l_1 \longrightarrow l_2.$$

Consider now the problem (4.3.5), we define the operator by

$$Nx_2^\alpha(t) = x_2(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g_\alpha(s,x_1^\alpha(t),x_2^\alpha(s))ds. \tag{4.3.14}$$

Using the same reasoning as that used for the problem (4.3.11), we can conclude that there exists at least one solution.

Case $^c[ii-q]$ **H-differentiability**: The problem (4.1.1)–(4.1.2) is equivalent to the following fractional differential system

$$^cD^qx_1^\alpha(t) = g_\alpha(t,x_1^\alpha(t),x_2^\alpha(t)), \quad 0<q<1, \quad x_1^\alpha(0)=x_1(0), \tag{4.3.15}$$

$$^cD^qx_2^\alpha(t) = f_\alpha(t,x_1^\alpha(t),x_2^\alpha(t)), \quad 0<q<1, \quad x_2^\alpha(0)=x_2(0). \tag{4.3.16}$$

Transform the problem (4.3.15) into fixed point problem, so we consider the operator

$$Nx_1^\alpha(t) = x_1(0) \ominus \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g_\alpha(s,x_1^\alpha(s),x_2^\alpha(t))ds. \tag{4.3.17}$$

By the same technic, we can prove that there exists at least one solution for the problem (4.3.15)–(4.3.16).

We conclude the existence at least a solution for the problem (4.1.1)–(4.1.2).

4.4 An Example

Consider the crisp differential equation

$$^{c}D^{q}x(t) = -x(t), \quad t \in I := [0, a], \quad 0 < q < 1, \tag{4.4.18}$$

$$x_0 = (1, |2|, |3|, 4) \in E \tag{4.4.19}$$

where x_0 is a fuzzy trapezoidal number that is $[x_0]^\alpha = [1+\alpha, 4-\alpha]$ we obtain the system

$$^{c}D^{q}x_1^\alpha(t) = -x_2^\alpha(t)), \quad x_1^\alpha(0) = 1 - \alpha, \tag{4.4.20}$$

$$^{c}D^{q}x_2^\alpha(t) = -x_1^\alpha(t), \quad x_2^\alpha(0) = 4 - \alpha. \tag{4.4.21}$$

We can denote this system by

$$^{c}D^{q}W(t) = AW(t), \quad W_0 = C$$

where

$$W(t) = \begin{bmatrix} x_1^\alpha(t) \\ x_2^\alpha(t) \end{bmatrix}, A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1+\alpha \\ 4-\alpha \end{bmatrix}$$

Using the same method that in (see [70]),we obtain the solution of system.It is given by

$$W(t) = E_{q,q}(At^q)C$$

where

$$E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha(k+1))}.$$

is the classical Mittag-Leffler function.
After calculus we have

$$x_1^\alpha(t) = \sum_{n=0}^{\infty} \frac{t^{(2n+1)q-1}}{\Gamma(q(2n+1))}(1+\alpha) - \sum_{n=0}^{\infty} \frac{t^{(2n+2)q-1}}{\Gamma(q(2n+2))}(3-\alpha)$$

$$x_2^\alpha(t) = \sum_{n=0}^{\infty} \frac{t^{(2n+1)q-1}}{\Gamma(q(2n+1))}(3-\alpha) - \sum_{n=0}^{\infty} \frac{t^{(2n+2)q-1}}{\Gamma(q(2n+2))}(1+\alpha)$$

It easy to ensure that $[x_1^\alpha(t), x_2^\alpha(t)]$ and $[^{c}D^{q}x_1^\alpha(t), ^{c}D^{q}x_2^\alpha(t)]$ are valid level sets and using Stacking Theorem, pile up the level $[x_1^\alpha(t), x_2^\alpha(t)]$ to a fuzzy solution $x(t)$.

Chapter 5

Darboux problem for fuzzy implicit fractional differential equations

5.1 Introduction

Fractional calculus is a generalization of differentiation and integration to an arbitrary order. First work, devoted exclusively to the subject of fractional calculus, is the book by Oldham and Spanier [87]. A rigorous study of fractional calculus can be found in [70].

The fractional differential equations has recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, Physics, economy and science. We can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [22, 82, 83, 84, 88, 73]. In recent years, there has been a significant development in fractional differential equations. One can see the monographs of Abbas *et al.* [1], Kilbas *et al.* [85], Lakshmikantham *et al.* [86], and the references therein. Agarwal *et al.* [12] proposed the concept of solution for fractional differential equation with uncertainty, and they have considered the Riemman-Liouville's differentiability to solve fuzzy fractional differential equation which is a combination of Hukuhara difference and Riemman-Liouville's derivative. In this chapter, we investigate the solution of Caputo's fuzzy fractional differential equations. This chapter is organized as follow. In Section 2 we recall some basic knowledge of fuzzy calculus and fractional calculus. In Section 3 several basic concepts and properties of fuzzy fractional calculus are presented. We use the fixed point approach. For an application of the above cited approach to fuzzy differential equations see [89].

Our aim in this chapter is to study the existence of solution for fuzzy partial hyperbolic differential equations with nonlocal condition involving Caputo derivative. The Banach contraction principle and a fixed point theorem for absolute retract spaces are used to investigate the existence of fuzzy solutions for the following fractional differential equation with nonlocal condition:

$$({}^cD_0^q u)(x,y) = f(x,y,u(x,y),{}^c D_0^q u)(x,y)), \text{ if } (x,y) \in J := [0,a] \times [0,b], \tag{5.1.1}$$

$$\begin{cases} u(x,0) = \varphi(x), & x \in [0,a], \\ u(0,y) = \psi(y), & y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{5.1.2}$$

where $a, b > 0$, ${}^cD_0^q$ is the Caputo's fractional derivative of order $q = (q_1, q_2) \in (0,1] \times (0,1]$, $f : J \times E^n \times E^n \to E^n$ is a given continuous function, $\varphi : [0,a] \to E^n, \psi : [0,b] \to E^n$ are given absolutely continuous functions with $\varphi(x) = \phi(x,0),\ \psi(y) = \phi(0,y)$ for each $x \in [0,a],\ y \in [0,b]$ and $\phi(0,0) = \psi(0) = \varphi(0)$.

We present two results for the problem (5.1.1)–(5.1.2), the first one is based on Banach's contraction principle and the second one on the theorem for absolute retract spaces.

5.2 The Main Results

[1]

Let us start by defining what we mean by a solution of the problem (5.1.1)–(5.1.2).

Definition 5.3 *A function $u \in C(J)$ such that $u(x,y)$, $D_{(0,x)}^{q_1} u(x,y)$, $D_{(0,y)}^{q_2} u(x,y)$, $D_{(0,0)}^{q} u(x,y)$ are continuous for $(x,y) \in J$ and $I_{(0,0)}^{1-q} u(x,y) \in AC(J)$ is said to be a solution of (5.1.1)–(5.1.2) if u satisfies equation (5.1.1) and conditions (5.1.2) on J.*

For the existence of solutions for the problem (5.1.1)-(5.1.2) we need the following lemma.

Lemma 5.4 *[2] Let a function $f(x,y,u,z) : J \times E^n \times E^n \to E^n$ be continuous. Then problem (5.1.1)–(5.1.2) is equivalent to the problem of the solution of the equation*

$$g(x,y) = f(x,y,\mu(x,y) + I_0^q g(x,y), g(x,y)),$$

and if $g \in C(J)$ is the solution of this equation, then $u(x,y) = \mu(x,y) + I_0^q g(x,y)$, where

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

Further, we present conditions for the existence and uniqueness of a solution of problem (5.1.1)-(5.1.2).

Theorem 5.1 *Assume that the following hypotheses hold:*

(H_1) *$f : J \times E^n \times E^n \to E^n$ is a continuous function;*

(H_2) *For any $u, v, z, w \in E^n$ and $(x,y) \in J$, there exists $k > 0$, $0 < l < 1$ such that*

$$d(f(x,y,u,z); f(x,y,v,w)) \leq kd(u;v) + ld(z;w)$$

[1]M. Benchohra, **A. Boukenkoul**, On fuzzy fractional order derivatives and Darboux problem for implicit differential equations. Submitted.

If

$$\frac{ka^{q_1}b^{q_2}}{(1-l)\Gamma(q_1+1)\Gamma(q_2+1)} < 1, \tag{5.4.3}$$

then there exists a unique solution for IVP (5.1.1)-(5.1.2) on J.

Proof: Transform the problem (5.1.1)-(5.1.2) into a fixed point problem.

Consider the operator $N : C(J) \longrightarrow C(J)$ defined by,

$$N(u)(x,y) = \mu(x,y) + I^q_{(0,0)}g(x,y),$$

where $g \in C(J)$ such that

$$g(x,y) = f(x,y,u(x,y),g(x,y)).$$

By Lemma 5.4, the problem of finding the solutions of the IVP (5.1.1)-(5.1.2) is reduced to finding the solutions of the operator equation $N(u) = u$.

By Theorem [89], we proceed as follows

Case ${}^c[i-q]$ **H-differentiability**: The problem (5.1.1)-(5.1.2) is equivalent to the following fractional differential system

$$\begin{cases} ({}^cD^q_0 u^\alpha_1)(x,y) = f_\alpha(x,y,u^\alpha_1(x,y),u^\alpha_2(x,y)), & \text{if } (x,y) \in J; \\ u^\alpha_1(x,0) = \varphi^\alpha_1(x), \;\; u^\alpha_1(0,y) = \psi^\alpha_1(y), & \text{if } \; x \in [0,a], \;\; y \in [0,b], \end{cases} \tag{5.4.4}$$

and

$$\begin{cases} ({}^cD^q_0 u^\alpha_2)(x,y) = g_\alpha(x,y,u^\alpha_1(x,y),u^\alpha_2(x,y)), & \text{if } (x,y) \in J; \\ u^\alpha_2(x,0) = \varphi^\alpha_2(x), \;\; u^\alpha_2(0,y) = \psi^\alpha_2(y), & \text{if } \; x \in [0,a], \;\; y \in [0,b], \end{cases} \tag{5.4.5}$$

We define the operator N by

$$N(v^\alpha_1)(x,y) = \mu^\alpha_1(x,y) + I^q_{(0,0)}g(x,y)$$

with

$\mu^\alpha_1(x,y) = \varphi^\alpha_1(x,y) + \psi^\alpha_1(x,y) - \varphi_1(0)$.

Let $v, w \in C(J)$. Then, for $(x,y) \in J$, we have

$$\begin{aligned} d(N(v^\alpha_1)(x,y); N(w^\alpha_1)(x,y)) \;\; \leq \;\; & \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} \\ & \times d(g(s,t),h(s,t))dtds \end{aligned} \tag{5.4.6}$$

where $g, h \in C(J)$ such that

$$g(x,y) = f(x,y,v(x,y),g(x,y))$$

and

$$h(x,y) = f(x,y,v(x,y),h(x,y))$$

By (H_2), we get

$$d(g(x,y),h(x,y)) \leq kd(v(x,y);w(x,y)) + ld(g(x,y),h(x,y)).$$

Then

$$\begin{aligned} d(g(x,y),h(x,y)) &\leq \frac{k}{1-l} d(v(x,y);w(x,y)) \\ &\leq \frac{k}{1-l} d(v;w). \end{aligned}$$

Thus (5.4.6) implies that

$$d(N(v_1^\alpha)(x,y);N(w_1^\alpha)(x,y)) \leq$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y d(g(x,y),h(x,y))|(x-s)^{q_1-1}||(y-t)^{q_2-1}|dtds \\ &\leq \frac{k}{(1-l)\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} d(v_{1(s,t)}^\alpha; w_{1(s,t)}^\alpha) dtds \\ &\leq \frac{k}{(1-l)\Gamma(q_1)\Gamma(q_2)} d_C(v_1^\alpha;w_1^\alpha) \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} dtds \\ &\leq \frac{ka^{q_1}b^{q_2}}{(1-l)\Gamma(q_1+1)\Gamma(q_2+1)} d(v_1^\alpha;w_1^\alpha). \end{aligned}$$

Consequently,

$$d(N(v_1^\alpha);N(w_1^\alpha)) \leq \frac{ka^{q_1}b^{q_2}}{(1-l)\Gamma(q_1+1)\Gamma(q_2+1)} d(v_1^\alpha;w_1^\alpha).$$

By (6.3.3), N is a contraction, and hence N has a unique fixed point by Banach's contraction principle.

We transform now the problem (6.3.5) into fixed point problem, we consider the operator N such that:

$$N(v_2^\alpha)(x,y) = \mu_2^\alpha(x,y) + I_{(0,0)}^q g(x,y)$$

with

$\mu_2^\alpha(x,y) = \varphi_2^\alpha(x,y) + \psi_2^\alpha(x,y) - \varphi_2(0).$

The fixed point of the operator N is a solution of the problem (6.3.5).
We conclude the existence of a unique solution for the problem (5.1.1)–(5.1.2).

Case ${}^c[ii{-}q]$ **H-differentiability**: The problem (5.1.1)–(5.1.2) is equivalent to the following fractional differential system

$$\begin{cases} ({}^cD_0^q u_1^\alpha)(x,y) = g_\alpha(x,y,u_1^\alpha(x,y),u_2^\alpha(x,y)), & \text{if } (x,y) \in J; \\ u_1^\alpha(x,0) = \varphi_1^\alpha(x), \ u_1^\alpha(0,y) = \psi_1^\alpha(y), & \text{if } \ x \in [0,a], \ y \in [0,b], \end{cases} \tag{5.4.7}$$

and

$$\begin{cases} ({}^cD_0^q u_2^\alpha)(x,y) = f_\alpha(x,y,u_1^\alpha(x,y),u_2^\alpha(x,y)), & \text{if } (x,y)\in J; \\ u_2^\alpha(x,0) = \varphi_2^\alpha(x),\ u_2^\alpha(0,y) = \psi_2^\alpha(y), & \text{if } \ x\in[0,a],\ y\in[0,b], \end{cases} \tag{5.4.8}$$

Transform the problem (6.3.8) into fixed point problem, so we consider the operator:

$$N(u_1^\alpha)(x,y) = \mu_1^\alpha(x,y) \ominus I_{(0,0)}^q g_\alpha(s,t)$$

with
$\mu_1^\alpha(x,y) = \varphi_1^\alpha(x,y) + \psi_1^\alpha(x,y)$.

By the same technic, we can prove that there exists at least one solution for the problem (5.1.1)-(5.1.2).

Our second result in this section is based on a fixed point theorem for absolute retract spaces.

Theorem 5.2 *Assume that* (H_1) *and the following hypothesis holds:*

(H_3) *There exist* $A,B,C\in C(J,\mathbb{R}_+)$ *such that*

$$d(f(x,y,u,v);\hat{0}) \leq A(x,y) + B(x,y)d(u;\hat{0}) + C(x,y)d(v;\hat{0}),$$

(H_4) *For all* $(x,y)\in J$ *the set*

$$\left\{\mu(x,y) + I_{(0,0)}^q g(x,y),\ u\in\Theta\right\}$$

is totally bounded subset of E^n*, where*

$$\Theta = \left\{u\in C(J,E^n):\ d(u(x,y),\hat{0}) \leq M,\ (x,y)\in J\right\},$$

with

$g(x,y) = f(x,y,u(x,y),g(x,y))$ *and* $\mu(x,y) = \varphi(x,y) + \psi(x,y) - \varphi(0)$.

Then the IVP (5.1.1)-(5.1.2) has at least one solution on J.

Proof: It is clear that the solutions of the problem (5.1.1)–(5.1.2) are fixed points of the operator

$$N: C(J,E^n) \longrightarrow C(J,E^n)$$

defined by:

$$N(u)(x,y) = \mu(x,y) + I_{(0,0)}^q g(x,y)$$

We make

$$\Theta \cong \tilde{B} \equiv \{\bar{J}_u \in C(J,E^n):\ u\in C(J,E^n) \text{ and } d(u(x,y),\hat{0}) \leq M,\ (x,y)\in J\}.$$

We see that $\tilde{B}$ is a subset convex of Banach space $C(J, E^n)$, so in particular $\tilde{B}$ is an absolute retract.

Case $^c[i-q]$ **H-differentiability**: The problem (5.1.1)–(5.1.2) is equivalent to the following fractional differential system

$$\begin{cases} ({}^cD_0^q u_1^\alpha)(x,y) = g_\alpha(x,y,u_1^\alpha(x,y),u_2^\alpha(x,y)), & \text{if } (x,y)\in J;\\ u_1^\alpha(x,0)=\varphi_1^\alpha(x),\ u_1^\alpha(0,y)=\psi_1^\alpha(y), & \text{if } \ x\in[0,a],\ y\in[0,b], \end{cases} \tag{5.4.9}$$

and

$$\begin{cases} ({}^cD_0^q u_2^\alpha)(x,y) = f_\alpha(x,y,u_1^\alpha(x,y),u_2^\alpha(x,y)), & \text{if } (x,y)\in J;\\ u_2^\alpha(x,0)=\varphi_2^\alpha(x),\ u_2^\alpha(0,y)=\psi_2^\alpha(y), & \text{if } \ x\in[0,a],\ y\in[0,b], \end{cases} \tag{5.4.10}$$

Transform the problem (6.3.12) into a fixed point problem. Consider the operator $N : C(J,E^n)\to C(J,E^n)$ defined by,

$$N(u_1^\alpha)(x,y) = \mu_1^\alpha(x,y) + I^q_{(0,0)} f_\alpha(x,y)$$

with

$\mu_1^\alpha(x,y) = \varphi_1^\alpha(x,y)+\psi_1^\alpha(x,y)-\varphi_1(0)$. Clearly the fixed points of the operator $N : C(J,E^n)\longrightarrow C(J,E^n)$ are solutions of the problem (6.1.2), we shall show that $N : C(J,E^n)\longrightarrow C(J,E^n)$ is continuous and completely continuous. The proof will be given in several steps.

Step 1. $N(\Theta)\subset\Theta$.

Let $u\in\Theta$ and $(x,y)\in J$, there exists exists a positive constant ℓ such that, for each $u\in B_\ell=\{u\in C(J,E^n) : d(u(x,y),\hat{0})\le M,\ (x,y)\in J\}$. we have $d(N(u^\alpha)(x,y),\hat{0})\le\ell$. By $((H_3))$ we have

$$\begin{aligned} d(g(x,t),\hat{0}) &\le A(x,y)+B(x,y)d(u;\hat{0})+C(x,y)d(g;\hat{0})\\ &\le \|A\|_\infty+\|B\|_\infty d(u;\hat{0})+\|C\|_\infty d(g;\hat{0}), \end{aligned}$$

Then

$$d(g(x,t),\hat{0})\le\frac{\|A\|_\infty+\|B\|_\infty M}{1-\|C\|_\infty}$$

Thus

$$\begin{aligned} d(Nu_1^\alpha((x,y)),\hat{0}) &\le d(\mu_1^\alpha(x,y),\hat{0})\\ &+ d\left(\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}\times f_\alpha(s,t)dtds,\hat{0}\right)\\ &\le M+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\left(\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}d(f_\alpha(s,t),\hat{0})dtds\right)\\ &\le M+\frac{\|A\|_\infty+M\|B\|_\infty}{(1-\|C\|_\infty)\Gamma(q_1)\Gamma(q_2)}a^{q_1}b^{q_2} := \ell. \end{aligned}$$

Step 2: *N is continuous.*

Let the sequence $[U_n(x;y)]^\alpha = [U^\alpha_{n,1}(x;y), U^\alpha_{n,2}(x;y)]$ of Θ such that $U^\alpha_{n,1} \longrightarrow U^\alpha_1$ element of Θ in $C(J,E^n)$ and $U^\alpha_{n,2} \longrightarrow U^\alpha_2$ element of Θ in $C(J,E^n)$. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J,E^n)$. Let $\eta > 0$ be such that $d(u_n;\widehat{0}) \leq \eta$. Then

$$d(N(u^\alpha_{n,1})(x,y); N(u^\alpha_1)(x,y)) \leq$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y |x-s|^{q_1-1}|y-t|^{q_2-1} d(f(s,t,u^\alpha_{n,1}); f(s,t,u^\alpha_1))dtds \\
&\leq \frac{\|A\|_\infty + M\|B\|_\infty}{(1-\|C\|_\infty)q_1q_2\Gamma(q_1)\Gamma(q_2)} a^{q_1}b^{q_2} d_\infty(f(.,.,u^\alpha_{n,1}(.,.)); f(.,.,u^\alpha_1(.,.)))
\end{aligned}$$

Since f is a continuous function, we have

$$\begin{aligned}
d_\infty(N(u^\alpha_{n,1}); N(u^\alpha_1)) &\leq \frac{\|A\|_\infty + M\|B\|_\infty}{\Gamma(q_1+1)\Gamma(q_2+1)} a^{q_1}b^{q_2} d_\infty(f(.,.,u^\alpha_{n,1})(.,.); f(.,.,u^\alpha_1(.,.)) \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus N is a continuous.

Step 3: *The operator N is equicontinuous.*

Let $(x_1,y_1),(x_2,y_2) \in (0,a] \times (0,b]$, $x_1 < x_2$, $y_1 < y_2$, $u \in B_{\eta^*} = \{u \in C(J,E^n) : d(u;\hat{0}) \leq \eta^*\}$, be a bounded set of $C_{(a,b)}$ and let $u \in B_{\eta^*}$. Then

$$d(N(u^\alpha_1)(x_2,y_2); N(u^\alpha_1)(x_1,y_1)) \leq d(\mu(x_1,y_1); \mu(x_2,y_2))$$

$$\begin{aligned}
&+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^{x_1}\int_0^{y_1}[(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}-(x_1-s)^{q_1-1}(y_1-t)^{q_2-1}]\\
&\times d(f(s,t);\widehat{o})dtds+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}\\
&d\left(f(s,t);\widehat{o}\right)dtds\\
&+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}d(f(s,t);\widehat{o})dtds\\
&+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}d(f(s,t);\widehat{o})dtds\\
\leq\ & d(\mu(x_1,y_1);\mu(x_2,y_2))+\frac{\|A\|_\infty+\|B\|_\infty\eta^*}{(1-\|C\|_\infty)\Gamma(q_1)\Gamma(q_2)}\\
&\times\int_0^{x_1}\int_0^{y_1}[(x_1-s)^{q_1-1}(y_1-t)^{q_2-1}-(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}]dtds\\
&+\frac{\|A\|_\infty+\|B\|_\infty\eta^*}{(1-\|C\|_\infty)\Gamma(q_1)\Gamma(q_2)}\int_{y_1}^{y_2}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}dtds\\
&+\frac{\|A\|_\infty+\|B\|_\infty\eta^*}{(1-\|C\|_\infty)\Gamma(q_1)\Gamma(q_2)}\int_{y_1}^{y_2}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}dtds\\
&+\frac{\|A\|_\infty+\|B\|_\infty\eta^*}{(1-\|C\|_\infty)\Gamma(q_1)\Gamma(q_2)}\int_0^{y_1}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}dtds\\
\leq\ & d(\mu(x_1,y_1);\mu(x_2,y_2))\\
&+\frac{\|A\|_\infty+\|B\|_\infty\eta^*}{(1-\|C\|_\infty)\Gamma(q_1)\Gamma(q_2)}[2y_2^{q_2}(x_2-x_1)^{q_1}+2x_2^{q_1}(y_2-y_1)^{q_2}\\
&+x_1^{q_1}y_1^{q_2}-x_2^{q_1}y_2^{q_2}-2(x_2-x_1)^{q_1}(y_2-y_1)^{q_2}].
\end{aligned}$$

As $x_1\to x_2$, $y_1\to y_2$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $x_1<x_2<0$, $y_1<y_2<0$ and $x_1\leq 0\leq x_2$, $y_1\leq 0\leq y_2$ is obvious. As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous.

Case $^c[ii-q]$ **H-differentiability**: The problem (5.1.1)–(5.1.2) is equivalent to the following fractional differential system

$$\begin{cases} ({}^cD_0^q u_1^\alpha)(x,y)=g_\alpha(x,y,u_1(x,y)^\alpha,u_2^\alpha(x,y)), & \text{if } (x,y)\in J;\\ u_1^\alpha(x,0)=\varphi_1^\alpha(x),\ u_1^\alpha(0,y)=\psi_1^\alpha(y), & \text{if } \ x\in[0,a],\ y\in[0,b],\end{cases} \tag{5.4.11}$$

and

$$\begin{cases} ({}^cD_0^q u_2^\alpha)(x,y)=f_\alpha(x,y,u_{1(x,y)}^\alpha,u_2^\alpha(x,y)), & \text{if } (x,y)\in J;\\ u_2^\alpha(x,0)=\varphi_2^\alpha(x),\ u_2^\alpha(0,y)=\psi_2(y), & \text{if } \ x\in[0,a],\ y\in[0,b],\end{cases} \tag{5.4.12}$$

Transform the problem (5.4.11) into fixed point problem, so we consider the operator

$$N(u_1^\alpha)(x,y)=\mu_1^\alpha(x,y)\ominus\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y(x-s)^{q_1-1}(y-t)^{q_2-1}\times g_\alpha(s,t,u_1^\alpha(s,t),u_2^\alpha(s,t))dtds$$

By the same technic, we can prove that there exists at least one solution for the problem (5.4.11) and finally the IVP(5.1.1)-(5.1.2) has a last on solution.

5.5 An Example

As an application of our results we consider the following fuzzy partial hyperbolic functional differential equations of the form

$$({}^cD_0^q u)(x,y) = \frac{1}{(5e^{x+y+2})(1+d(u(x,y),\hat{0})+d({}^cD_0^q u(x,y),\hat{0}))}, \quad \text{if } (x,y) \in [0,1]\times[0,1], \tag{5.5.13}$$

$$u(x,0)=x,\ x\in[0,1],\ u(0,y)=y^2,\ y\in[0,1], \tag{5.5.14}$$

Set

$$f(x,y,u(x,y),{}^cD_0^q u(x,y)) = \frac{1}{(5e^{x+y+2})(1+d(u(x,y),\hat{0})+d({}^cD_0^q u(x,y),\hat{0}))},\ (x,y)\in[0,1]\times[0,1].$$

For each $u,\overline{u},v,\overline{v}\in E^1$ and $(x,y)\in[0,1]\times[0,1]$ we have

$$d(f(x,y,u,v);f(x,y,\overline{u},\overline{v})) \le \frac{1}{5e^2}(d(u;\overline{u})+d(v;\overline{v})).$$

Hence condition (H_2) is satisfied with $k=l=\dfrac{1}{5e^2}$. We shall show that condition (5.4.3) holds with $a=b=1$. Indeed

$$\frac{ka^{q_1}b^{q_2}}{(1-l)\Gamma(q_1+1)\Gamma(q_2+1)} = \frac{1}{(5e^2-1)\Gamma(q_1+1)\Gamma(q_2+1)} < 1,$$

which is satisfied for each $(q_1,q_2)\in(0,1]\times(0,1]$. Consequently Theorem 6.1 implies that problem (6.4.20)-(6.4.21) has a unique solution defined on $[0,1]\times[0,1]$.

Chapter 6

Fuzzy Fractional Partial Hyperbolic Differential Equations

6.1 Introduction

The existence of solution for fuzzy partial hyperbolic differential equations with nonlocal condition involving Caputo derivative will be studied in this chapter. The Banach contraction principle and a fixed point theorem for absolute retract spaces are used to investigate the existence of fuzzy solutions for the following fractional differential equation with nonlocal condition:

$$({}^cD_0^q u)(x,y) = f(x,y,u(x,y)), \text{ if } (x,y) \in J := [0,a] \times [0,b], \tag{6.1.1}$$

$$u(x,0) = \varphi(x),\ x \in [0,a],\ u(0,y) = \psi(y),\ y \in [0,b], \tag{6.1.2}$$

where $a, b > 0$, ${}^cD_0^r$ is the Caputo's fractional derivative of order $q = (q_1, q_2) \in (0,1] \times (0,1]$, $f : J \times E^n \to E^n$ is a given continuous function, $\varphi : [0,a] \to E^n, \psi : [0,b] \to E^n$ are given absolutely continuous functions with $\varphi(x) = \phi(x,0),\ \psi(y) = \phi(0,y)$ for each $x \in [0,a],\ y \in [0,b]$ and $\phi(0,0) = \psi(0) = \varphi(0)$.

6.2 The Main Results

[1]

In this section, our first result is based on Banach fixed point theorem, we discuss the existence and uniqueness of solutions for the problem (6.1.1)–(6.1.2).

Definition 6.3 *A function $u \in C_{(a,b)} := C([0,a] \times [0,b], E^n)$ with its mixed derivative D^2_{xy} exists and is integrable on Jis said to be a solution of (6.1.1)-(6.1.2) if u satisfies equations (6.1.1) and the condition (6.1.2) on J.*

[1] M. Benchohra, **A. Boukenkoul**, Fuzzy Partial Hyperbolic Differential Equations. Accepted. Journal of Fuzzy Mathematics.

Further, we present conditions for the existence and uniqueness of a solution of problem (6.1.1)-(6.1.2).

Theorem 6.1 *Assume that the following hypotheses hold:*

(H_1) $f : J \times E^n \to E^n$ *is a continuous function;*

(H_2) *For any* $u,\ v \in E^n$ *and* $(x,y) \in J$, *there exists* $k > 0$ *such that*

$$d(f(x,y,u); f(x,y,v)) \leq k d_C(u;v).$$

If

$$\frac{k a^{q_1} b^{q_2}}{\Gamma(q_1+1)\Gamma(q_2+1)} < 1, \tag{6.3.3}$$

then there exists a unique solution for IVP (6.1.1)-(6.1.2) on $[0,a] \times [0,b]$.

Proof: We proceed as follows:(see Th. 2 [89])

Case $^c[i-q]$ **H-differentiability**: The problem (6.1.1)-(6.1.2) is equivalent to the following fractional differential system

$$({}^cD_0^q u_1^\alpha)(x,y) = f_\alpha(x,y,u_1^\alpha(x,y),u_2^\alpha(x,y)),\ \text{if } (x,y) \in J, \tag{6.3.4}$$

$$u_1^\alpha(x,0) = \varphi_1^\alpha(x),\ x \in [0,a],\ u_1^\alpha(0,y) = \psi_1^\alpha(y),\ y \in [0,b], \tag{6.3.5}$$

and

$$({}^cD_0^q u_2^\alpha)(x,y) = g_\alpha(x,y,u_1^\alpha(x,y),u_2^\alpha(x,y)),\ \text{if } (x,y) \in J, \tag{6.3.6}$$

$$u_2^\alpha(x,0) = \varphi_2^\alpha(x),\ x \in [0,a],\ u_2^\alpha(0,y) = \psi_2^\alpha(y),\ y \in [0,b], \tag{6.3.7}$$

Transform the problem (6.3.4)-(6.3.5) into a fixed point problem. Consider the operator $N : C_{(a,b)} \to C_{(a,b)}$ defined by,

$$N(u_1^\alpha)(x,y) = \mu_1^\alpha(x,y) + \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} \times f_\alpha(s,t,u^\alpha_{1(s,t)})dtds,$$

with

$\mu_1^\alpha(x,y) = \varphi_1^\alpha(x,y) + \psi_1^\alpha(x,y).$

Let $v,w \in C_{(a,b)}$. Then, for $(x,y) \in [0,a] \times [0,b]$,

$$d(N(v_1^\alpha)(x,y); N(w_1^\alpha)(x,y)) \leq$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y d(f(s,t,v^\alpha_{1(s,t)}); f(s,t,w^\alpha_{1(s,t)}))|(x-s)^{q_1-1}||(y-t)^{q_2-1}|dtds\\
&\leq \frac{k}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} d_C(v^\alpha_{1(s,t)}; w^\alpha_{1(s,t)})dtds\\
&\leq \frac{k}{\Gamma(q_1)\Gamma(q_2)} d_J(v_1^\alpha; w_1^\alpha)\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}dtds\\
&\leq \frac{k x^{q_1} y^{q_2}}{\Gamma(q_1+1)\Gamma(q_2+1)} d_J(v_1^\alpha; w_1^\alpha).
\end{aligned}$$

Consequently,

$$d_J(N(v_1^\alpha); N(w_1^\alpha)) \le \frac{ka^{q_1}b^{q_2}}{\Gamma(q_1+1)\Gamma(q_2+1)} d_J(v_1^\alpha; w_1^\alpha).$$

By (6.3.3), N is a contraction, and hence N has a unique fixed point by Banach's contraction principle.

We transform now the problem (6.3.6)–(6.3.7) into fixed point problem, we consider the operator N such that:

$$N(u_2^\alpha)(x,y) = \mu_2^\alpha(x,y) + \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} \times f_\alpha(s,t,u_2^\alpha(s,t))dtds$$

with

$\mu_2^\alpha(x,y) = \varphi_2^\alpha(x,y) + \psi_2^\alpha(x,y).$

The fixed point of the operator N is a solution of the problem (6.3.6)–(6.3.7).

We conclude the existence of a unique solution for the problem (6.1.1)–(6.1.2).

Case $^c[ii{-}q]$ **H-differentiability**: The problem (6.1.1)–(6.1.2) is equivalent to the following fractional differential system

$$({}^cD_0^q u_1^\alpha)(x,y) = g_\alpha(x,y,u_1^\alpha(x,y),u_2^\alpha(x,y)), \text{ if } (x,y) \in J, \tag{6.3.8}$$

$$u_1^\alpha(x,0) = \varphi_1^\alpha(x),\ x \in [0,a],\ u_1^\alpha(0,y) = \psi_1^\alpha(y),\ y \in [0,b], \tag{6.3.9}$$

and

$$({}^cD_0^q u_2^\alpha)(x,y) = f_\alpha(x,y,u_1^\alpha(x,y),u_2^\alpha(x,y)), \text{ if } (x,y) \in J, \tag{6.3.10}$$

$$u_2^\alpha(x,0) = \varphi_2^\alpha(x),\ x \in [0,a],\ u_2^\alpha(0,y) = \psi_2^\alpha(y),\ y \in [0,b], \tag{6.3.11}$$

Transform the problem (6.3.8)–(6.3.9) into fixed point problem, so we consider the operator:

$$N(u_1^\alpha)(x,y) = \mu_1^\alpha(x,y) \ominus \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} \times g_\alpha(s,t,u_1^\alpha(s,t))dtds$$

with

$\mu_1^\alpha(x,y) = \varphi_1^\alpha(x,y) + \psi_1^\alpha(x,y).$

By the same technic, we can prove that there exists at least one solution for the problem (6.1.1)-(6.1.2).

Our second result in this section is based on a fixed point theorem for absolute retract spaces.

Theorem 6.2 *Assume that* (H_1) *and the following hypothesis holds:*

(H_3) *There exist* $A, B \in C(J, \mathbb{R}_+)$ *such that*

$$d(f(x,y,u);\hat{0}) \le A(x,y) + B(x,y)d_C(u;\hat{0}),$$

(H_4) *For all* $(x,y)\in J$ *the set*

$$\left\{\mu(x,y)+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y(x-s)^{q_1-1}(y-t)^{q_2-1}\times f_\alpha(s,t,u_1^\alpha(s,t))dtds,\ u\in\Theta\right\}$$

is totally bounded subset of E^n*, where*

$$\Theta=\left\{u\in C([0,a]\times[0,b],E^n):\ d(u(x,y),\hat{0})\le M,\ (x,y)\in J\right\},$$

with

$\mu(x,y)=\varphi(x,y)+\psi(x,y).$

Then the IVP (6.1.1)-(6.1.2) has at least one solution on J*.*

Proof: It is clear that the solutions of the problem (6.1.1)–(6.1.2) are fixed points of the operator

$$N:C([0,a]\times[0,b],E^n)\longrightarrow C([0,a]\times[0,b],E^n)$$

defined by:

$$N(u)(x,y)=\mu(x,y)+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y(x-s)^{q_1-1}(y-t)^{q_2-1}\times f(s,t,u(s,t))dtds.$$

We make

$$\Theta\cong\tilde{B}\equiv\{\bar{J}_u\in C([0,a]\times[0,b],E^n):\ u\in C([0,a]\times[0,b],E^n)\text{ and }d(u(x,y),\hat{0})\le M,\ (x,y)\in J\}.$$

We see that $\tilde{B}$ is a subset convex of Banach space $C([0,a]\times[0,b],E^n)$, so in particular $\tilde{B}$ is an absolute retract.

Case $^c[i-q]$ **H-differentiability**:The problem (6.1.1)–(6.1.2) is equivalent to the following fractional differential system

$$({}^cD_0^qu_1^\alpha)(x,y)=f_\alpha(x,y,u_1^\alpha(x,y),u_2^\alpha(x,y)),\ \text{if }(x,y)\in J, \tag{6.3.12}$$

$$u_1^\alpha(x,0)=\varphi_1^\alpha(x),\ x\in[0,a],\ u_1^\alpha(0,y)=\psi_1^\alpha(y),\ y\in[0,b], \tag{6.3.13}$$

and

$$({}^cD_0^qu_2^\alpha)(x,y)=g_\alpha(x,y,u_1^\alpha(x,y),u_2^\alpha(x,y)),\ \text{if }(x,y)\in J, \tag{6.3.14}$$

$$u_2^\alpha(x,0)=\varphi_2^\alpha(x),\ x\in[0,a],\ u_2^\alpha(0,y)=\psi_2^\alpha(y),\ y\in[0,b], \tag{6.3.15}$$

Transform the problem (6.3.12)-(6.3.13) into a fixed point problem. Consider the operator $N:C_{(a,b)}\to C_{(a,b)}$ defined by,

$$N(u_1^\alpha)(x,y)=\mu_1^\alpha(x,y)+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y(x-s)^{q_1-1}(y-t)^{q_2-1}\times f_\alpha(s,t,u_1^\alpha(s,t))dtds$$

Clearly the fixed points of the operator $N:C([0,a]\times[0,b],E^n)\longrightarrow C([0,a]\times[0,b],E^n)$ are solutions of the problem (6.1.2), we shall show that $N:C([0,a]\times[0,b],E^n)\longrightarrow C([0,a]\times[0,b],E^n)$ is continuous and completely continuous. The proof will be given in several steps.

Step 1. $N(\Theta) \subset \Theta$.

Let $u \in \Theta$ and $(x,y) \in [0,a] \times [0,b]$, there exists exists a positive constant ℓ such that, for each $u \in B_\ell = \{u \in C([0,a] \times [0,b], E^n) : d(u(x,y), \hat{0}) \leq \ell, \ (x,y) \in J\}$. we have $d(N(u^\alpha)(x,y), \hat{0}) \leq \ell$. By $((H_3))$ we have

$$N(u_1^\alpha)(x,y) = \mu_1^\alpha(x,y) + \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} \times f_\alpha(s,t,u_1^\alpha(s,t))dtds$$

$$\begin{aligned} d(Nu_1^\alpha((x,y)), \hat{0}) &\leq d(\mu_1^\alpha(x,y), \hat{0}) \\ &+ d\left(\tfrac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} \times f_\alpha(s,t,u_1^\alpha(s,t))dtds, \hat{0}\right) \\ &\leq M + \tfrac{1}{\Gamma(q_1)\Gamma(q_2)} \left(\int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} d(f_\alpha(s,t,u_1^\alpha(s,t)), \hat{0})dtds\right) \\ &\leq M + \frac{\|A\|_\infty + M\|B\|_\infty}{\Gamma(q_1)\Gamma(q_2)} a^{q_1} b^{q_2} = \ell. \end{aligned}$$

Step 2: *N is continuous.*

Let the sequence $[U_n(x;y)]^\alpha = [U_{n,1}^\alpha(x;y), U_{n,2}^\alpha(x;y)]$ of Θ such that $U_{n,1}^\alpha \longrightarrow U_1^\alpha$ element of Θ in $C([0,a] \times [0,b], E^n)$ and $U_{n,2}^\alpha \longrightarrow U_2^\alpha$ element of Θ in $C([0,a] \times [0,b], E^n)$. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C_{(a,b)}$. Let $\eta > 0$ be such that $d(u_n; \widehat{0}) \leq \eta$. Then

$$d(N(u_{n,1}^\alpha)(x,y); N(u_1^\alpha)(x,y)) \leq$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y |x-s|^{q_1-1}|y-t|^{q_2-1} d(f(s,t,u_{n,1}^\alpha(s,t)); f(s,t,u_1^\alpha(s,t)))dtds \\ &\leq \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^a \int_0^b |x-s|^{q_1-1}|y-t|^{q_2-1} \sup_{(s,t)\in J} d(f(s,t,u_{n,1}^\alpha(s,t)); f(s,t,u_1^\alpha(s,t)))dtds \\ &\leq \frac{a^{q_1} b^{q_2}}{q_1 q_2 \Gamma(q_1)\Gamma(q_2)} d_\infty(f(.,.,u_{n,1}^\alpha(.,.)); f(.,.,u_1^\alpha(.,.))) \end{aligned}$$

Since f is a continuous function, we have

$$\begin{aligned} d_\infty(N(u_{n,1}^\alpha); N(u_1^\alpha)) &\leq \frac{a^{q_1} b^{q_2}}{\Gamma(q_1+1)\Gamma(q_2+1)} d_\infty(f(.,.,u_{n,1}^\alpha)(.,.); f(.,.,u_1^\alpha(.,.)) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus N is a continuous.

Step 3: *The operator N is equicontinuous.*

Let $(x_1,y_1), (x_2,y_2) \in (0,a] \times (0,b]$, $x_1 < x_2$, $y_1 < y_2$, $u \in B_{\eta^*} = \{u \in C_{(a,b)} : d(u; \hat{0}) \leq \eta^*\}$, be a bounded set of $C_{(a,b)}$ and let $u \in B_{\eta^*}$. Then

$$d(N(u_1^\alpha)(x_2,y_2); N(u_1^\alpha)(x_1,y_1)) = d(\mu(x_1,y_1); \mu(x_2,y_2))$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^{x_1}\int_0^{y_1}[(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}-(x_1-s)^{q_1-1}(y_1-t)^{q_2-1}]\\
&\times d(f(s,t,u_1^\alpha(s,t));\widehat{o})dtds+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}\\
&d\left(f(s,t,u_1^\alpha(s,t));\widehat{o}\right)dtds\\
&+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}d(f(s,t,u_1^\alpha(s,t));\widehat{o})dtds\\
&+\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}d(f(s,t,u_1^\alpha(s,t));\widehat{o})dtds\\
\leq\ & d(\mu(x_1,y_1);\mu(x_2,y_2))+\frac{\|p\|_\infty+\|q\|_\infty\eta^*}{\Gamma(q_1)\Gamma(q_2)}\\
&\times\int_0^{x_1}\int_0^{y_1}[(x_1-s)^{q_1-1}(y_1-t)^{q_2-1}-(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}]dtds\\
&+\frac{\|p\|_\infty+\|q\|_\infty\eta^*}{\Gamma(q_1)\Gamma(q_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}dtds\\
&+\frac{\|p\|_\infty+\|q\|_\infty\eta^*}{\Gamma(q_1)\Gamma(q_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}dtds\\
&+\frac{\|p\|_\infty+\|q\|_\infty\eta^*}{\Gamma(q_1)\Gamma(q_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{q_1-1}(y_2-t)^{q_2-1}dtds\\
\leq\ & d(\mu(x_1,y_1);\mu(x_2,y_2))\\
&+\frac{\|p\|_\infty+\|q\|_\infty\eta^*}{\Gamma(q_1+1)\Gamma(q_2+1)}[2y_2^{q_2}(x_2-x_1)^{q_1}+2x_2^{q_1}(y_2-y_1)^{q_2}\\
&+x_1^{q_1}y_1^{q_2}-x_2^{q_1}y_2^{q_2}-2(x_2-x_1)^{q_1}(y_2-y_1)^{q_2}].
\end{aligned}
$$

As $x_1 \to x_2$, $y_1 \to y_2$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $x_1 < x_2 < 0$, $y_1 < y_2 < 0$ and $x_1 \leq 0 \leq x_2$, $y_1 \leq 0 \leq y_2$ is obvious. As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous.
We can show previously that the operator N is a compact, and it has at least a fixed point.

Case ${}^c[ii-q]$ **H-differentiability**: The problem (6.1.1)–(6.1.2) is equivalent to the following fractional differential system

$$({}^cD_0^q u_1^\alpha)(x,y) = g_\alpha(x,y,u_1(x,y)^\alpha, u_2^\alpha(x,y)),\ \text{if } (x,y)\in J, \tag{6.3.16}$$

$$u_1^\alpha(x,0)=\varphi_1^\alpha(x),\ x\in[0,a],\ u_1^\alpha(0,y)=\psi_1^\alpha(y),\ y\in[0,b], \tag{6.3.17}$$

and

$$({}^cD_0^q u_2^\alpha)(x,y) = f_\alpha(x,y,u^\alpha_{1(x,y)}, u_2^\alpha(x,y)),\ \text{if } (x,y)\in J, \tag{6.3.18}$$

$$u_2^\alpha(x,0)=\varphi_2^\alpha(x),\ x\in[0,a],\ u_2^\alpha(0,y)=\psi_2(y),\ y\in[0,b], \tag{6.3.19}$$

Transform the problem (6.3.16)–(6.3.17) into fixed point problem, so we consider the operator

$$N(u_1^\alpha)(x,y) = \mu_1^\alpha(x,y) \ominus \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} \times g_\alpha(s,t,u_1^\alpha(s,t),u_2^\alpha(s,t))dtds$$

By the same technic, we can prove that there exists at least one solution for the problem (6.3.16)–(6.3.17) and finally the IVP(6.1.1)-(6.1.2) has a last on solution.

6.4 An Example

As an application of our results we consider the following partial hyperbolic functional differential equations of the form

$$({}^cD_0^q u)(x,y) = \frac{1}{(4e^{x+y+2})(1+d(u(x,y),\hat{0}))}, \quad \text{if } (x,y) \in [0,1]\times[0,1], \tag{6.4.20}$$

$$u(x,0) = x,\ x \in [0,1],\ u(0,y) = y^2,\ y \in [0,1], \tag{6.4.21}$$

Set

$$f(x,y,u(x,y)) = \frac{1}{(4e^{x+y+2})(1+d(u(x,y),\hat{0}))},\ (x,y) \in [0,1]\times[0,1].$$

For each $u, \overline{u} \in E^1$ and $(x,y) \in [0,1]\times[0,1]$ we have

$$d(f(x,y,u); f(x,y,\overline{u})) \leq \frac{1}{4e^2} d(u;\overline{u}).$$

Hence condition (H_2) is satisfied with $k = \dfrac{1}{4e^2}$. We shall show that condition (6.3.3) holds with $a = b = 1$. Indeed

$$\frac{ka^{q_1}b^{q_2}}{\Gamma(q_1+1)\Gamma(q_2+1)} = \frac{1}{4e^2\Gamma(q_1+1)\Gamma(q_2+1)} < 1,$$

which is satisfied for each $(q_1,q_2) \in (0,1]\times(0,1]$. Consequently Theorem 6.1 implies that problem (6.4.20)-(6.4.21) has a unique solution defined on $[0,1]\times[0,1]$.

Conclusion

The fuzzy theory has become almost a fashion in the 90's. Many researchers in various scientific fields use the theory formulated by Professor Lotfi Zadeh University of Berkley. This theory is very attractive because it is based on the reasoning intutif and takes into account the subjectivity and vagueness, but it is not a vague theory! This is a rigorous mathematical theory, suitable for the treatment of all that is subjective or uncertain. Many researchers are working and making significant contributions in various fields of science (medical diagnosis, robotics, mathematics,...). Currently fuzzy logic is regarded as a basic tool in Japan and the main directions of rechrche are the combination of fuzzy logic, genetic algorithms and neural system.

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ملخص

في هذه الأطروحة تم إعطاء نظرة شاملة على النظرية الغامضة (الضبابية) المنطق الغامض و حساب التفاضل و التكامل الكسري و بعض نظريات النقط الثابتة.

مساهماتنا تمثلت في مسائل التفاضلية غامضة ذات قيمة ابتدائية و أخرى بإدخال مشتقات كسرية بمفهوم كابيتو و كذا معادلة تفاضلية جزئية قطعية ضبابية بشروط غير محلية.

و يستند النهج المتبع على طريقة التقريبات المتعاقبة و كذا الجمع بين نظرية النقطة الثابتة و نظرية الفضاءات المقلصة بصفة مطلقة.

Resumé :

Dans cette thèse, on a donné un aperçut global sur la théorie du floue, sur la logique floue ainsi sur le calcul différentiel et intégral fractionnaire flou.

Nos contributions portent sur des divers problèmes différentielles floues d’ordre fractionnaire floues, les approches utilisées sont : La méthode des approximations successives ainsi le théorème du point fixe dans des espaces absolument rétractés.

Abstract :

In this thesis, i was given an overall saw on fuzzy theory, on fuzzy logic, and on the fuzzy fractional integral and differential calculus.

Our contributions in this thesis focus on various fuzzy differential equations of fractional order. Our approaches are based on the successive approximation method and   the fixed point theorem in absolute retract space.