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# THÈSE
# DE DOCTORAT

Présentée par:

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Spécialité : Mathématique
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Intitulée

## Problème de Darboux pour des équations différentielles hyperboliques d'ordre fractionnaire

Soutenue le: . . . . . . . . . . . . . . . . .
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# Publications

1. M. Benchohra, J. Henderson and A. Heris, Fractional partial random differential equations with delay, *PanAmer. Math. J.* **27** (1) (2017), 29-40.

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3. M. Benchohra and A. Heris, Random impulsive partial hyperbolic fractional differential equations, *Nonlinear Dyn. Syst. Theory* (to appear).

# Abstract

The objective of this thesis is to present the existence of random solutions for the fractional partial random differential equations in Banach spaces . Some equations present delay which may be fnite, infinite, or state-dependent. Our results will be obtained by means the measure of noncompactness and a random fixed point theorem with stochastic domain.

**Key words:**Random differential equation; left-sided mixed Riemann-Liouville integral; Caputo fractional order derivative; Banach space; Darboux problem; measure of noncompactness .

**AMS Subject Classification :** 26A33.

# Résumé

L'objectif de cette thèse est de présenter, des résultats d'existence des solutions du problème de Darboux pour des équations différentielles hyperboliques d'ordre fractionnaire avec un effet dans un espace de Banach. On a considéré ce problème avec retard fini, infini et dépendant de l'etat. Nos résultats sont basées sur théorème du point fixe et la mesure de non compacité.

**Mots clé:** Equations différentielles; l'intégrale d'ordre fractionnaire au sens de Riemann-Liouville; dérivée de Caputo; espace de Banach; problème de Darboux; mesure de non compacité.

**Classification AMS:** 26A33.

# Contents

# Introduction

Fractional calculus is generalization of ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus, starting from some speculations of G.W. Lebeniz(1967) and L. Euler(1730) and since then, it has continued to be developed up to nowadays. Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particulary true of problems in mechanical vibrations and the related fields of engineering and mathematical physics. we can find numerous applications of differential and integral equation of fractional order in finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering ( [16,38,55,56,60]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [10], Baleanu *et al.* [16], Kilbas *et al.* [44], Lakshmikantham *et al*, the papers of Abbas *et al.*

The theory of impulsive integer order differential equations and inclusions have become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. The study of impulsive fractional differential equations and inclusions was initiated in the 1960's by Milman and Myshkis [52, 53]. At present the foundations of the general theory are already laid, and many of them are investigated in detail in the books of Aubin [13], Berhoun [20] and the references therein. There was an intensive development of the impulse theory, especially in the area of impulsive differential equations and inclusions with fixed moments; see for example [46, 57]. Recently in [2, 9], we have considered some classes of hyperbolic partial differential equations involving the Caputo fractional derivative and impulses at fixed time. The theory of impulsive differential equations and inclusions with variable time is relatively less developed due to the difficulties created by the state-dependent impulses.Some interesting extensions to impulsive differential equations with variable times have been done by Bajo and Liz [15], Abbas *et al.* [1], Belarbi and Benchohra [17], Benchohra *et al.* [18, 19], Frigon and O'Regan [28, 29, 30], Kaul *et al.* [41], Kaul and Liu [42, 43], Lakshmikantham *et al.* [47] and the references cited therein.

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last year; see for instance [33, 34] and the references therein. The literature related to partial functional differential equations with state-dependent delay is limited; see for instance [7, 37]. The literature related to ordinary and partial functional differential equations with delay for which $\rho(t, \cdot) = t$ or $(\rho 1(x, y, \cdot), \rho 2(x, y, \cdot)) = (x, y)$ is very extensive; see for instance [5, 6, 32] and the references therein.

Random differential equations and random integral equations have been studied systematically in Ladde and Lakshmikantham [45] and Bharucha-Reid [21], respectively.They are good models in various branches of science and engineering since random factors and uncertainties have been taken into consideration. Hence, the study of the fractional differential equations with random parameters seem to be a natural one. We refer the reader to the monographs [21, 45, 58], the papers [25, 26, 40] and the references therein.

Initial value problems for fractional differential equations with random parameters have been studied by Lupulescu and Ntouyas [50]. The basic tool in the study of the problems for random fractional differential equations is to treat it as a fractional differential equation in some appropriate Banach space. In [51], authors proved the existence results for a random fractional equation under a Carathéodory condition.

This thesis is devoted to the the existence of random solutions for the fractional partial random differential equations. Some equations present delay which may be finite, infinite, or state-dependent. The tools used include measure of noncompactness and a random fixed point theorem with stochastic domain. This thesis is arranged and organized as follows:

In Chapter 1, we introduce some notations, definitions,lemmas and fixed point theorems which are used throughout this thesis.

In chapter 2, we prove the existence of random solutions for the fractional partial random differential equations.

In Chapter 3, in section two of chapter 3, we study the existence results for the Darboux problem of partial fractional random differential equations with delay. Later, we give similar results to nonlocal initial value problem.
In Section 3.3, we prove the existence of random solutions for the fractional partial random differential equations with infinite delay.

In chapter 4, we shall be concerned to the existence for the fractional partial random differential equations with state-dependent delay.
In Section 4.2, we study the existence results for the Darboux problem of partial fractional random differential equations with finite delay.
In Section 4.3, we study the existence results for the Darboux problem of partial fractional random differential equations with infinite delay.

Finally, in Chapter 5, we discuss the existence of random solutions for the impulsive partial fractional random differential equations.

# Chapter 1

# Preliminary

In this chapter, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this thesis.

## 1.1 Some Notations and Definitions

Let $J := [0, a] \times [0, b],\ a, b > 0$. Denote $L^1(J)$ the space of Bochner-integrable functions $u : J \to E$ with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(x, y)\|_E dydx.$$

$L^\infty(J)$ the Banach space of functions $u : J \to \mathbb{R}$ which are essentially bounded with the norm

$$\|u\|_{L^\infty} = \inf\{c > 0 : \|u(x, y)\| \leq c,\ a.e.\ (x, y) \in J\}.$$

As usual, by $AC(J)$ we denote the space of absolutely continuous functions from $J$ into $E$.

Let $\beta_E$ be the $\sigma$-algebra of Borel subsets of $E$. A mapping $v : \Omega \to E$ is said to be measurable if for any $B \in \beta_E$, one has

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \subset \mathcal{A}.$$

To define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

**Definition 1.1.1.** *A mapping $T : \Omega \times E \to E$ is called jointly measurable if for any $B \in \beta_E$, one has*

$$T^{-1}(B) = \{(w, v) \in \Omega \times E : T(w, v) \in B\} \subset \mathcal{A} \times \beta_E,$$

*where $\mathcal{A} \times \beta_E$ is the direct product of the $\sigma$-algebras $\mathcal{A}$ and $\beta_E$ those defined in $\Omega$ and $E$ respectively.*

**Lemma 1.1.1.** *Let $T : \Omega \times E \to E$ be a mapping such that $T(\cdot, v)$ is measurable for all $v \in E$, and $T(w, \cdot)$ is continuous for all $w \in \Omega$. Then the map $(w, v) \mapsto T(w, v)$ is jointly measurable.*

**Definition 1.1.2.** *A function $f : J \times E \times \Omega \to E$ is called random Carathéeodory if the following conditions are satisfied:*

*(i) The map $(x, y, w) \to f(x, y, u, w)$ is jointly measurable for all $u \in E$, and*

*(ii) The map $u \to f(x, y, u, w)$ is continuous for almost all $(x, y) \in J$ and $w \in \Omega$.*

Let $T : \Omega \times E \to E$ be a mapping. Then $T$ is called a random operator if $T(w, u)$ is measurable in $w$ for all $u \in E$ and it is expressed as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is a random operator on $E$. A random operator $T(w)$ on $E$ is called continuous (resp. compact, totally bounded and completely continuous) if $T(w, u)$ is continuous (resp. compact, totally bounded and completely continuous) in $u$ for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [40].

**Definition 1.1.3.** *[27] Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of $Y$ and $C$ be a mapping from $\Omega$ into $\mathcal{P}(Y)$. A mapping $T : \{(w, y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called random operator with stochastic domain $C$ if $C$ is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w, y) \in D\}$ is measurable. $T$ will be called continuous if every $T(w)$ is continuous. For a random operator $T$, a mapping $y : \Omega \to Y$ is called random (stochastic) fixed point of $T$ if for $P-$almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.*

## 1.2 Some Properties of Partial Fractional Calculus

In this section, we introduce notations, definitions and preliminary Lemmas concerning to partial fractional calculus theory.

Let $\theta = (0,0),\ r_1,\ r_2 > 0$ and $r = (r_1, r_2)$. For $f \in L^1(J)$, the expression

$$(I_\theta^r f)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)\,dt\,ds,$$

is called the left-sided mixed Riemann-Liouville integral of order $r$, where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt;\ \xi > 0.$

In particular,

$$(I_\theta^\theta u)(x,y) = u(x,y),\ (I_\theta^\sigma u)(x,y) = \int_0^x \int_0^y u(s,t)\,dt\,ds;\ \text{for almost all } (x,y) \in J,$$

where $\sigma = (1,1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)$, when $u \in L^1(J)$. Note also that when $u \in \mathcal{C}$, then $(I_\theta^r u) \in \mathcal{C}$, moreover

$$(I_\theta^r u)(x,0) = (I_\theta^r u)(0,y) = 0;\ x \in [0,a],\ y \in [0,b].$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second order partial derivative.

**Definition 1.2.1.** *[61] Let $r \in (0,1]\times(0,1]$ and $u \in L^1(J)$. The Caputo fractional-order derivative of order $r$ of $u$ is defined by the expression*

$${}^cD_\theta^r u(x,y) = (I_\theta^{1-r} D_{xy}^2 u)(x,y).$$

The case $\sigma = (1,1)$ is included and we have

$$({}^cD_\theta^\sigma u)(x,y) = (D_{xy}^2 u)(x,y);\ \text{for almost all } (x,y) \in J.$$

## 1.3 The phase space $\mathcal{B}$

The notation of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato (see [31]). For further applications see for instance the books [35, 39, 48] and their references. For any $(x,y) \in J$ denote

$\mathcal{E}_{(x,y)} := [0, x] \times \{0\} \cup \{0\} \times [0, y]$, furthermore in case $x = a,\ y = b$ we write simply $\mathcal{E}$. Consider the space $(\mathcal{B}, \|(\cdot,\cdot)\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0] \times (-\infty, 0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

$(A_1)$ If $z : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n$ continuous on $J$ and $z_{(x,y)} \in \mathcal{B}$, for all $(x, y) \in \mathcal{E}$, then there are constants $H, K, M > 0$ such that for any $(x, y) \in J$ the following conditions hold:

$(i)$ $z_{(x,y)}$ is in $\mathcal{B}$;

$(ii)$ $\|z(x, y)\| \leq H\|z_{(x,y)}\|_{\mathcal{B}}$,

$(iii)$ $\|z_{(x,y)}\|_{\mathcal{B}} \leq K \sup_{(s,t)\in[0,x]\times[0,y]} \|z(s, t)\| + M \sup_{(s,t)\in\mathcal{E}_{(x,y)}} \|z_{(s,t)}\|_{\mathcal{B}}$,

$(A_2)$ For the function $z(.,.)$ in $(A_1)$, $z_{(x,y)}$ is a $\mathcal{B}$-valued continuous function on $J$.

$(A_3)$ The space $\mathcal{B}$ is complete.

Now, we present some examples of phase spaces [23,24].

**Example 1.3.1.** *Let $\mathcal{B}$ be the set of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0],\ \alpha, \beta \geq 0$, with the seminorm*

$$\|\phi\|_{\mathcal{B}} = \sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s, t)\|.$$

*Then we have $H = K = M = 1$. The quotient space $\widehat{\mathcal{B}} = \mathcal{B}/\|.\|_{\mathcal{B}}$ is isometric to the space $C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n)$ of all continuous functions from $[-\alpha, 0] \times [-\beta, 0]$ into $\mathbb{R}^n$ with the supremum norm, this means that partial differential functional equations with finite delay are included in our axiomatic model.*

**Example 1.3.2.** *Let $C_\gamma$ be the set of all continuous functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ for which a limit $\lim_{\|(s,t)\|\to\infty} e^{\gamma(s+t)}\phi(s, t)$ exists, with the norm*

$$\|\phi\|_{C_\gamma} = \sup_{(s,t)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(s+t)}\|\phi(s, t)\|.$$

*Then we have $H = 1$ and $K = M = \max\{e^{-(a+b)}, 1\}$.*

**Example 1.3.3.** *Let $\alpha, \beta, \gamma \geq 0$ and let*

$$\|\phi\|_{CL_\gamma} = \sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,t)\| + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\gamma(s+t)}\|\phi(s,t)\|dtds.$$

*be the seminorm for the space $CL_\gamma$ of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0]$ measurable on $(-\infty, -\alpha] \times (-\infty, 0] \cup (-\infty, 0] \times (-\infty, -\beta]$, and such that $\|\phi\|_{CL_\gamma} < \infty$. Then*

$$H = 1,\ K = \int_{-\alpha}^{0}\int_{-\beta}^{0} e^{\gamma(s+t)}dtds,\ M = 2.$$

## 1.4 Measure of noncompactness

Let $\mathcal{M}_X$ denote the class of all bounded subsets of a metric space $X$.

**Definition 1.4.1.** *Let $X$ be a complete metric space. A map $\alpha : \mathcal{M}_X \to [0, \infty)$ is called a measure of noncompactness on $X$ if it satisfies the following properties for all $B, B_1, B_2 \in \mathcal{M}_X$.*

*(MNC.1)* $\alpha(B) = 0$ *if and only if $B$ is precompact (Regularity),*

*(MNC.2)* $\alpha(B) = \alpha(\overline{B})$ *(Invariance under closure),*

*(MNC.3)* $\alpha(B_1 \cup B_2) = \alpha(B_1) + \alpha(B_2)$ *(Semi-additivity).*

For more details on measure of noncompactness and its properties see [12].

**Example 1.4.1.** *In every metric space $X$, the map $\phi : \mathcal{M}_X \to [0, \infty)$ with $\phi(B) = 0$ if $B$ is relatively compact and $\phi(B) = 1$ otherwise is a measure of noncompactness, the so-called discrete measure of noncompactness [ [14], Example1, p. 19].*

**Lemma 1.4.1.** *[22] If $Y$ is a bounded subset of Banach space $X$, then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\alpha(Y) \leq 2\alpha(\{y_k\}_{k=1}^{\infty}) + \epsilon.$$

**Lemma 1.4.2.** *[54] $\{u_k\}_{k=1}^{\infty} \subset L^1(J)$ is uniformly integrable, then $\alpha(\{u_k\}_{k=1}^{\infty})$ is measurable and for each $(x, y) \in J$,*

$$\alpha\left(\left\{\int_0^x\int_0^y u_k(s,t)dtds\right\}_{k=1}^{\infty}\right) \leq 2\int_0^x\int_0^y \alpha(\{u_k(s,t)\}_{k=1}^{\infty})dtds.$$

## 1.5 Fixed Point Theorems

Fixed point theory plays an important role in our existence results, therefore we state the following lemma.

**Lemma 1.5.1.** *[49] Let $F$ be a closed and convex subset of a real Banach space, let $G : F \to F$ be a continuous operator and $G(F)$ be bounded. If there exist a constant $k \in [0, 1)$ such that for each bounded subset $B \subset F$,*

$$\alpha(G(B)) \leq k\alpha(B),$$

*then G has a fixed point in $F$.*

# Chapter 2

# Fractional Partial Random Differential Equations

## 2.1 Introduction

We study in this chapter the existence of random solutions for the following fractional partial random differential equations

$$^{c}D_{\theta}^{r}u(x,y,w)=f(x,y,u(x,y,w),w);\ \text{ for a.a. } (x,y)\in J:=[0,a]\times[0,b],\ w\in\Omega, \tag{2.1}$$

with the initial conditions

$$\begin{cases} u(x,0,w)=\varphi(x,w);\ x\in[0,a], \\ u(0,y,w)=\psi(y,w);\ y\in[0,b], \qquad w\in\Omega, \\ \varphi(0,w)=\psi(0,w), \end{cases} \tag{2.2}$$

where $a,b>0,\ \theta=(0,0),\ ^{c}D_{\theta}^{r}$ is the fractional Caputo derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1],\ (\Omega,\mathcal{A})$ is a measurable space, $f:J\times E\times\Omega\to E$ is a given continuous function, $(E,\|\cdot\|_E)$ is a real Banach space, $\varphi:[0,a]\times\Omega\to E,\ \psi:[0,b]\times\Omega\to E$ are given are given functions such that $\varphi(\cdot,w)$ and $\psi(\cdot,w)$ are absolutely continuous functions for all $w\in\Omega$, and $\varphi(x,\cdot)$ and $\psi(y,\cdot)$ are measurable for all $x\in[0,a]$ and $y\in[0,b]$ respectively, and $\mathcal{C}$ is the Banach space of all continuous functions from $J$ into $E$ with the supremum (uniform) norm $\|\cdot\|_{\infty}$.

## 2.2 Existence Results

**Definition 2.2.1.** *By a random solution of the random problem (2.1)-(2.2) we mean a measurable function $u : \Omega \to AC(J)$ that satisfies the equation (2.1) a.a. on $J \times \Omega$ and the initial conditions (2.2) are satisfied.*

Let $h \in L^1(J, \mathbb{R}^n)$. We need the following lemma:

**Lemma 2.2.1.** *[3, 10] A function $u \in \mathcal{AC}(J, \mathbb{R}^n)$ is a solution of problem*

$$\begin{cases} {}^cD^r_\theta u(x,y) = h(x,y); \text{ for a.a. } (x,y) \in J := [0,a] \times [0,b], \\ u(x,0) = \varphi(x);\ x \in [0,a], \\ u(0,y) = \psi(y);\ y \in [0,b], \\ \varphi(0) = \psi(0). \end{cases}$$

*if and only if $u$ if and only if $u(x,y)$ satisfies*

$$u(x,y) = \mu(x,y) + I^r_\theta h(x,y); \text{ for a.a. } (x,y) \in J,$$

*where*

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

Let us assume that the function $f$ is random Carathéeodory on $J \times E \times \Omega$. From the above Lemma, we have the following Lemma.

**Lemma 2.2.2.** *Let $0 < r_1, r_2 \leq 1$. A function $u \in \Omega \times \mathcal{AC}$ is a solution of the random fractional integral equation*

$$u(x,y,w) = \mu(x,y,w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds, \tag{2.3}$$

*where*

$$\mu(x,y,w) = \varphi(x,w) + \psi(y,w) - \varphi(0,w),$$

*if and only if $u$ is a solution of the random problem (2.1)-(2.2).*

The following hypotheses will be used in the sequel.

$(H_1)$ The functions $w \mapsto \varphi(x,0,w)$ *and* $w \mapsto \psi(0,y,w)$ are measurable and bounded for a.e. $x \in [0,a]$ and $y \in [0,b]$ respectively,

$(H_2)$ The function $f$ is random Carathéeodory on $J \times E \times \Omega$,

$(H_3)$ There exist functions $p_1, p_2 : J \times \Omega \to [0,\infty)$ with $p_i(.,w) \in AC(J,[0,\infty))L^\infty(J,[0,\infty));\ i = 1,2$ such that

$$\|f(x,y,u,w)\|_E \leq p_1(x,y,w) + p_2(x,y,w)\|u\|_E,$$

for all $u \in E$, $w \in \Omega$ and a.e. $(x,y) \in J$,

$(H_4)$ for any bounded $B \subset E$,

$$\alpha(f(x,y,B,w)) \leq p(x,y,w)\alpha(B),\ for\ a.e.\ (x,y) \in J,$$

$(H_5)$ There exists a random function $R : \Omega \to (0,\infty)$ such that

$$\mu^*(w) + \frac{(p_1^*(w) + p_2^*(w)R(w))a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \leq R(w),$$

where

$$\mu^*(w) = \sup_{(x,y)\in J}\|\mu(x,y,w)\|_E,\ p_i^*(w) = \sup ess_{(x,y)\in J}p_i(x,y,w);\ i = 1,2.$$

**Theorem 2.2.1.** *Assume that hypotheses $(H_1)-(H_5)$ hold. If*

$$\ell := \frac{4q^*a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1,$$

*then the problem (2.1)-(2.2) has a random solution defined on $J$.*

**Proof.** From th hypothesis $(H_2), (H_3)$, for each $w \in \Omega$ and almost all $(x,y) \in J$, we have that $f(x,y,u(x,y,w),w)$ is in $L^1$. By using Lemma 2.2.2, the problem (2.1)-(2.2) is equivalent to the integral equation

$$u(x,y,w) = \mu(x,y,w) + \int_0^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(s,t,u(s,t,w),w)dtds;$$

for each $w \in \Omega$ and a.e. $(x,y) \in J$.

Define the operator $N : \Omega \times \mathcal{C} \to \mathcal{C}$ by

$$(N(w)u)(x,y) = \mu(x,y,w) + \int_0^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(s,t,u(s,t,w),w)dtds.$$

Since the functions $\varphi, \psi$ and $f$ are absolutely continuous , then the function $\mu$ and the indefinite integral are absolutely continuous for all $w \in \Omega$ and almost all $(x, y) \in J$. Again, as the map $\mu$ is continuous for all $w \in \Omega$ and the indefinite integral is continuous on $J$, then $N(w)$ defines a mapping $N : \Omega \times \mathcal{C} \to \mathcal{C}$. Hence $u$ is a solution for the problem (2.1)-(2.2) if and only if $u = (N(w))u$. We shall show that the operator $N$ satisfies all conditions of Lemma 1.5.1. The proof will be given in several steps.

**Step 1:** $N(w)$ is a random operator with stochastic domain on $\mathcal{C}$.
Since $f(x, y, u, w)$ is random Carathéodory, the map $w \to f(x, y, u, w)$ is measurable in view of Definition 1.1.1.
Similarly, the product $(x-s)^{r_1-1}(y-t)^{r_2-1} f(s, t, u(s, t, w), w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$w \mapsto \mu(x, y, w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s, t, u(s, t, w), w) dt ds,$$

is measurable. As a result, $N$ is a random operator on $\Omega \times \mathcal{C}$ into $\mathcal{C}$.

Let $W : \Omega \to \mathcal{P}(\mathcal{C})$ be defined by

$$W(w) = \{u \in \mathcal{C} : \|u\|_\infty \leq R(w)\},$$

with $W(w)$ bounded, closed, convex and solid for all $w \in \Omega$. Then $W$ is measurable by Lemma [ [27], Lemma 17]. Let $w \in \Omega$ be fixed, then from $(H_4)$, for any $u \in w(w)$, we get

$$\begin{aligned}
& \|(N(w)u)(x, y)\|_E \\
\leq\ & \|\mu(x, y, w)\|_E + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|f(s, t, u(s, t, w), w)\|_E dt ds \\
\leq\ & \|\mu(x, y, w)\|_E + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_1(s, t, w) dt ds \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_2(s, t, w) \|u(s, t, w)\|_E dt ds \\
\leq\ & \mu^*(w) + \frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dt ds \\
& + \frac{p_2^*(w)R(w)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dt ds \\
\leq\ & \mu^*(w) + \frac{(p_1^*(w) + p_2^*(w)R(w))a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
\leq\ & R(w).
\end{aligned}$$

Therefore, $N$ is a random operator with stochastic domain $W$ and $N(w) : W(w) \to N(w)$. Furthermore, $N(w)$ maps bounded sets into bounded sets in $\mathcal{C}$.

**Step 2:** $N(w)$ is continuous.
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $\mathcal{C}$. Then, for each $(x, y) \in J$ and $w \in \Omega$, we have

$$\|(N(w)u_n)(x,y) - (N(w)u)(x,y)\|_E \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}$$
$$\times \|f(s,t,u_n(s,t,w),w) - f(s,t,u(s,t,w),w)\|_E dtds.$$

Using the Lebesgue Dominated Convergence Theorem, we get

$$\|N(w)u_n - N(w)u\|_\infty \to 0 \text{ as } n \to \infty.$$

As a consequence of Steps 1 and 2, we can conclude that $N(w) : W(w) \to N(w)$ is a continuous random operator with stochastic domain $W$, and $N(w)(W(w))$ is bounded.

**Step 3:** For each bounded subset $B$ of $W(w)$ we have

$$\alpha(N(w)B) \leq \ell\alpha(B).$$

Let $w \in \Omega$ be fixed. From Lemmas 1.4.1 and 1.4.2, for any $B \subset W$ and any

$\epsilon > 0$, there exists a sequence $\{u_n\}_{n=0}^{\infty} \subset B$, such that for all $(x,y) \in J$, we have

$$\begin{aligned}
& \alpha((N(w)B)(x,y)) \\
& = \alpha\left(\left\{\mu(x,y) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u(s,t,w),w)dtds;\ u \in B\right\}\right) \\
\leq\ & 2\alpha\left(\left\{\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_n(s,t;w),w)dtds\right\}_{n=1}^{\infty}\right) + \epsilon \\
\leq\ & 4\int_0^x \int_0^y \alpha\left(\left\{\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_n(s,t,w),w)\right\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq\ & 4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha\left(\{f(s,t,u_n(s,t,w),w)\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq\ & 4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)\alpha\left(\{u_n(s,t,w)\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq\ & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dsdt\right) \alpha\left(\{u_n\}_{n=1}^{\infty}\right) + \epsilon \\
\leq\ & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dtds\right) \alpha(B) + \epsilon \\
\leq\ & \frac{4p_2^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \alpha(B) + \epsilon \\
& = \ell\alpha(B) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\alpha(N(B)) \leq \ell\alpha(B).$$

It follows from Lemma 1.5.1 that for each $w \in \Omega$, $N$ has at least one fixed point in $W$. Since $\bigcap_{w\in\Omega} intW(w) \neq \emptyset$ the hypothesis that a measurable selector of $intW$ exists holds. By Lemma 1.5.1, $N$ has a stochastic fixed point, i.e., the problem (2.1)-(2.2) has at least one random solution on $\mathcal{C}$.

## 2.3 An Example

Let $E = \mathbb{R}$, $\Omega = (-\infty, 0)$ be equipped with the usual $\sigma$-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Given a measurable function $u : \Omega \to AC([0,1] \times [0,1])$, consider the following partial functional random differential equation of the form

$$({}^cD_\theta^r u)(x,y,w) = \frac{w^2 e^{-x-y-3}}{1+w^2+5|u(x,y,w)|};\ \text{a.a. } (x,y)\in J=[0,1]\times[0,1],\ w\in\Omega, \tag{2.4}$$

with the initial conditions

$$\begin{cases} u(x,0,w) = x\sin w;\ x\in[0,1], \\ u(0,y,w) = y^2\cos w;\ y\in[0,1], \end{cases} \quad w\in\Omega, \tag{2.5}$$

where $(r_1, r_2) \in (0,1]\times(0,1]$. Set

$$f(x,y,u(x,y,w),w) = \frac{w^2}{(1+w^2+5|u(x,y,w)|)e^{x+y+3}},\ (x,y)\in[0,1]\times[0,1],\ w\in\Omega.$$

The functions $w \mapsto \varphi(x,0,w) = x\sin w$ *and* $w\mapsto \psi(0,y,w) = y^2\cos w$ are measurable and bounded with

$$|\varphi(x,0,w)|\le 1,\ |\psi(0,y,w)|\le 1,$$

hence, the conditions $(H_1)$ is satisfied.
Clearly, the map $(x,y,w)\mapsto f(x,y,u,w)$ is jointly continuous for all $u\in\mathbb{R}$ and hence jointly measurable for all $u\in\mathbb{R}$. Also the map $u\mapsto f(x,y,u,w)$ is continuous for all $(x,y)\in J$ and $w\in\Omega$. So the function $f$ is Carathéodory on $[0,1]\times[0,1]\times\mathbb{R}\times\Omega$.
For each $u\in\mathbb{R},\ (x,y)\in[0,1]\times[0,1]$ and $w\in\Omega$, we have

$$|f(x,y,u,w)|\le 1+\frac{5}{e^3}|u|.$$

Hence the conditions $(H_3)$ is satisfied with $p_1(x,y,w) = p_1^* = 1$ and $p_2(x,y,w) = p_2^* = \frac{5}{e^3}$.
Also, the conditions $(H_4)$ is satisfied.

We shall show that condition $\ell < 1$ holds with $a=b=1$. For each $(r_1,r_2)\in(0,1]\times(0,1]$ we get

$$\begin{aligned} \ell &= \frac{4p_2^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &= \frac{20}{e^3\Gamma(1+r_1)\Gamma(1+r_2)} \\ &< \frac{20}{e^3} \\ &< 1. \end{aligned}$$

Consequently, Theorem 2.2.1 implies that the problem (2.4)-(2.5) has a random solution defined on $[0,1]\times[0,1]$.

# Chapter 3

# Fractional Partial Random Differential Equations with Delay

## 3.1 Introduction

We study in this chapter existence results for the Darboux problem of partial fractional random differential equations with delay

## 3.2 Fractional Partial Random Differential Equations with finite Delay

### 3.2.1 Introduction

In this section, we discuss the existence of random solutions for the following fractional partial random differential equations with finite delay

$$({}^cD_0^r u)(x,y,w) = f(x,y,u_{(x,y)},w), \text{ if } (x,y) \in J := [0,a] \times [0,b], w \in \Omega, \tag{3.1}$$

$$u(x,y,w) = \phi(x,y,w), \text{ if } (x,y) \in \tilde{J} := [-\alpha,a] \times [-\beta,b] \backslash (0,a] \times (0,b], w \in \Omega, \tag{3.2}$$

$$u(x,0,w) = \varphi(x,w),\ x \in [0,a],\ u(0,y,w) = \psi(y,w),\ y \in [0,b], w \in \Omega, \tag{3.3}$$

where $\alpha,\beta,a,b > 0$, ${}^cD_0^r$ is the standard Caputo's fractional derivative of order $-r = (r_1,r_2) \in (0,1] \times (0,1]$, $(\Omega,\mathcal{A})$ is a measurable space, $f : J \times C \times \Omega \to E$ is a given function, $\phi : \tilde{J} \times \Omega \to E$ is a given continuous function, $\varphi : [0,a] \times \Omega \to E, \psi : [0,b] \times \Omega \to E$ are given absolutely continuous functions with $\varphi(x,w) = \phi(x,0,w),\ \psi(y,w) = \phi(0,y,w)$ for each $x \in [0,a],\ y \in [0,b],\ w \in \Omega$ and $C :=$

$C([-\alpha,0]\times[-\beta,0],E)$ is the space of continuous functions on $[-\alpha,0]\times[-\beta,0]$.

If $u\in C_{(a,b)}=C([-\alpha,a]\times[-\beta,b],E);\ a,b,\alpha,\beta>0$ then for any $(x,y)\in J$ define $u_{(x,y)}$ by

$$u_{(x,y)}(s,t,w)=u(x+s,y+t,w),$$

for $(s,t)\in[-\alpha,0]\times[-\beta,0]$. Here $u_{(x,y)}(\cdot,\cdot,w)$ represents the history of the state from time $x-\alpha$ up to the present time $x$ and from time $y-\beta$ up to the present time $y$.

Next we consider the following nonlocal initial value problem

$$({}^cD_0^r u)(x,y,w)=f(x,y,u_{(x,y)},w),\ \text{if } (x,y)\in J:=[0,a]\times[0,b], w\in\Omega, \tag{3.4}$$

$$u(x,y,w)=\phi(x,y,w),\ \text{if } (x,y)\in\tilde{J}:=[-\alpha,a]\times[-\beta,b]\backslash(0,a]\times(0,b], w\in\Omega, \tag{3.5}$$

$$\begin{cases} u(x,0,w)+Q(u)=\varphi(x,w);\ x\in[0,a], \\ u(0,y,w)+K(u)=\psi(y,w);\ y\in[0,b], \end{cases} \quad w\in\Omega, \tag{3.6}$$

where $f,\phi,\varphi,\psi$ are as in problem (3.1)-(3.3) and $Q,K:C(J,E)\to E$ are given continuous functions.

### 3.2.2 Existence Results

**Lemma 3.2.1.** *Let $0<r_1,r_2\le 1, \mu(x,y,w)=\varphi(x,w)+\psi(y,w)-\varphi(0,w)$. A function $u\in\Omega\times C_{(a,b)}$ is a solution of the random problem (3.1)-(3.3) if u satisfies condition (3.2) for $(x,y)\in\tilde{J}, w\in\Omega$ and $u$ is a solution of the equation*

$$u(x,y,w)=\mu(x,y,w)+\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(x,y,u_{(s,t)},w)dtds$$

*for $(x,y)\in J, w\in\Omega$*

We will need to introduce the following hypotheses which are be assumed there after:

$(H_1)$ The functions $w\mapsto\varphi(x,0,w)$ *and* $w\mapsto\psi(0,y,w)$ are measurable and bounded for a.e. $x\in[0,a]$ and $y\in[0,b]$ respectively,

$(H_2)$ The function $\Phi$ is measurable for $(x,y)\in\tilde{J}$

$(H_3)$ The function $f$ is random Carathéeodory on $J\times C\times\Omega$,

$(H_4)$ There exist functions $p_1, p_2 : J \times \Omega \to [0, \infty)$ with $p_i(., w) \in AC(J, [0, \infty))L^\infty(J, [0, \infty));\ i = 1, 2$ such that

$$\|f(x, y, u, w)\|_E \le p_1(x, y, w) + p_2(x, y, w)\|u\|_C,$$

for all $u \in C$,$w \in \Omega$ and a.e. $(x, y) \in J$,

$(H_5)$ For any bounded $B \subset E$,

$$\alpha(f(x, y, B, w)) \le P_2(x, y, w)\alpha(B),\ for\ a.e.\ (x, y) \in J,$$

Set

$$\mu^*(w) = \sup_{(x,y)\in J} \|\mu(x, y, w)\|_E,\ p_i^*(w) = \sup ess_{(x,y)\in J} p_i(x, y, w);\ i = 1, 2.$$

**Theorem 3.2.1.** *Assume that hypotheses $(H_1) - (H_5)$ hold. If*

$$\ell := \frac{4p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} < 1,$$

*then the problem (3.1)-(3.3) has a random solution defined on $[-\alpha, a] \times [-\beta, b]$.*

**Proof.** Define the operator $N : \Omega \times C_{(a,b)} \to C_{(a,b)}$ by

$$N(u)(x, y) = \begin{cases} \phi(x, y, w), & (x, y) \in \tilde{J}, w \in \Omega, \\ \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} & \\ \times f(s, t, u_{(s,t)}, w)dtds, & (x, y) \in J, w \in \Omega. \end{cases} \tag{3.7}$$

Since the functions $\varphi, \psi$ and $f$ are absolutely continuous , the function $\mu$ and the indefinite integral are absolutely continuous for all $w \in \Omega$ and almost all $(x, y) \in J$. Again, as the map $\mu$ is continuous for all $w \in \Omega$ and the indefinite integral is continuous on $J$, then $N(w)$ defines a mapping $N : \Omega \times C_{(a,b)} \to C_{(a,b)}$. Hence $u$ is a solution for the problem (3.1)-(3.3) if and only if $u = (N(w))u$. We shall show that the operator $N$ satisfies all conditions of Lemma 1.5.1. The proof will be given in several steps.

**Step 1:** $N(w)$ is a random operator with stochastic domain on $C_{(a,b)}$.
Since $f(x, y, u, w)$ is random Carathéodory, the map $w \to f(x, y, u, w)$ is measurable in view of Definition 1.1.1. Similarly, the product $(x-s)^{r_1-1}(y-t)^{r_2-1} f(s, t, u_{(s,t)}, w)$

of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$w \mapsto \mu(x,y,w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)},w)dtds,$$

is measurable. As a result, $N$ is a random operator on $\Omega \times C_{(a,b)}$ into $C_{(a,b)}$.

Let $W : \Omega \to \mathcal{P}(C_{(a,b)})$ be defined by

$$W(w) = \{u \in C_{(a,b)} : \|u\|_\infty \leq R(w)\},$$

With $R(\cdot)$ is chosen appropriately. From instance, we assume that

$$R(w) \geq \frac{\mu^*(w) + p_1^*(w)\frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}{1 - p_2^*(w)\frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}.$$

with $W(w)$ bounded, closed, convex and solid for all $w \in \Omega$. Then $W$ is measurable by Lemma [ [27], Lemma 17]. Let $w \in \Omega$ be fixed, then from $(H_4)$, for any $u \in w(w)$, we get

$$\begin{aligned}
& \|(N(w)u)(x,y)\|_E \\
\leq\ & \|\mu(x,y,w)\|_E + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|f(s,t,u_{(s,t)},w)\|_E dtds \\
\leq\ & \|\mu(x,y,w)\|_E + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_1(s,t,w)dtds \\
& \quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_2(s,t,w)\|u_{(s,t)}\|_\infty dtds \\
\leq\ & \mu^*(w) + \frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \\
& \quad + \frac{p_2^*(w)R(w)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \\
\leq\ & \mu^*(w) + \frac{(p_1^*(w) + p_2^*(w)R(w))a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
\leq\ & R(w).
\end{aligned}$$

Therefore, $N$ is a random operator with stochastic domain $W$ and $N(w) : W(w) \to N(w)$. Furthermore, $N(w)$ maps bounded sets into bounded sets in $C_{(a,b)}$.

**Step 2:** $N(w)$ is continuous.
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C_{(a,b)}$. Then, for each $(x,y) \in J$ and $w \in \Omega$, we have

$$\|(N(w)u_n)(x,y) - (N(w)u)(x,y)\|_E \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}$$
$$\times \|f(s,t,u_{n(s,t)},w) - f(s,t,u_{(s,t)},w)\|_E dtds.$$

Using the Lebesgue Dominated Convergence Theorem, we get

$$\|N(w)u_n - N(w)u\|_\infty \to 0 \text{ as } n \to \infty.$$

As a consequence of Steps 1 and 2, we can conclude that $N(w) : W(w) \to N(w)$ is a continuous random operator with stochastic domain $W$, and $N(w)(W(w))$ is bounded.

**Step 3:** For each bounded subset $B$ of $W(w)$ we have

$$\alpha(N(w)B) \leq \ell\alpha(B).$$

Let $w \in \Omega$ be fixed. From Lemmas 1.4.1 and 1.4.2, for any $B \subset W$ and any $\epsilon > 0$, there exists a sequence $\{u_n\}_{n=0}^\infty \subset B$, such that for all $(x,y) \in J$, we have

$$\begin{aligned}
&\alpha((N(w)B)(x,y)) \\
&= \alpha\left(\left\{\mu(x,y) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_{(s,t)},w)dtds;\ u \in B\right\}\right) \\
\leq\ & 2\alpha\left(\left\{\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_{n(s,t)},w)dtds\right\}_{n=1}^\infty\right) + \epsilon \\
\leq\ & 4\int_0^x \int_0^y \alpha\left(\left\{\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_{n(s,t)},w)\right\}_{n=1}^\infty\right) dtds + \epsilon \\
\leq\ & 4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha\left(\{f(s,t,u_{n(s,t)},w)\}_{n=1}^\infty\right) dtds + \epsilon \\
\leq\ & 4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)\alpha\left(\{u_{n(s,t)}\}_{n=1}^\infty\right) dtds + \epsilon \\
\leq\ & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dsdt\right) \alpha\left(\{u_n\}_{n=1}^\infty\right) + \epsilon \\
\leq\ & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dtds\right) \alpha(B) + \epsilon \\
\leq\ & \frac{4p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \alpha(B) + \epsilon \\
&= \ell\alpha(B) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\alpha(N(B)) \leq \ell\alpha(B).$$

It follows from Lemma 1.5.1 that for each $w \in \Omega$, $N$ has at least one fixed point in $W$. Since $\bigcap_{w\in\Omega} intW(w) \neq \emptyset$, and the hypothesis that a measurable selector of $intW$ exists holds. By Lemma 1.5.1, $N$ has a stochastic fixed point, i.e., the problem (3.1)-(3.3) has at least one random solution on $C_{(a,b)}$.

Let us assume that the function $f$ is random Carathéeodory on $J \times C \times \Omega$. From the above Lemma, we have.

**Lemma 3.2.2.** *Let $0 < r_1, r_2 \leq 1, \mu(x,y,w) = \varphi(x,w) + \psi(y,w) - \varphi(0,w)$. A function $u \in \Omega \times C_{(a,b)}$ is a solution of the random problem (3.4)-(3.6) if $u$ satisfies condition (3.5) for $(x,y) \in \tilde{J}, w \in \Omega$, and the integral equation*

$$u(x,y,w) = \mu(x,y,w) - Q(u) - H(u) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(x,y,u_{(s,t)},w)dtds$$

*for $(x,y) \in J, w \in \Omega$*

The following hypotheses will be used in the sequel.

$(H_1)$ The functions $w \mapsto \varphi(x,0,w)$ *and* $w \mapsto \psi(0,y,w)$ are measurable and bounded for a.e. $x \in [0,a]$ and $y \in [0,b]$ respectively,

$(H_2)$ The function $\Phi$ is measurable for $(x,y) \in \tilde{J}$

$(H_3)$ The function $f$ is random Carathéeodory on $J \times C \times \Omega$,

$(H_4)$ there exist constants $d^*, \tilde{d} > 0$ such that

$$\|Q(u)\| \leq d^*(1 + \|u\|),$$

and

$$\|Q(u)\| \leq \tilde{d}(1 + \|u\|),$$

for $u \in C(J,E)$,

$(H_5)$ There exist functions $p_1, p_2 : J \times \Omega \to [0,\infty)$ with $p_i(\cdot,w) \in AC(J,[0,\infty))L^\infty(J,[0,\infty));\ i = 1,2$ such that

$$\|f(x,y,u,w)\|_E \leq p_1(x,y,w) + p_2(x,y,w)\|u\|_C,$$

for all $u \in C$, $w \in \Omega$, and a.e. $(x,y) \in J$,

$(H_6)$ There exist functions $p_3, p_4 : J \times \Omega \to [0, \infty)$ with $p_i(\cdot, w) \in AC(J, [0, \infty))L^\infty(J, [0, \infty));\ i = 3, 4$ such that for each $w \in \Omega$ and for any bounded $B \subset C$,

$$\alpha_c(f(x, y, B, w)) \le P_2(x, y, w)\alpha_c(B),\ for\ a.e.\ (x, y) \in J,$$

$$\alpha_c(Q(B)) \le P_3(x, y, w)\alpha_c(B),\ for\ a.e.\ (x, y) \in J,$$

and

$$\alpha_c(H(B)) \le P_4(x, y, w)\alpha_c(B),\ for\ a.e.\ (x, y) \in J,$$

here $\alpha, \alpha_c$ designe respectively the measures of noncompactness on X and C,

$(H_7)$ There exists a random function $R : \Omega \to (0, \infty)$ such that

$$\mu^*(w) + (\tilde{d} + d^*)(1 + R(w)) + \frac{(p_1^*(w) + p_2^*(w)R(w))a^{r_1}b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \le R(w),$$

where

$$\mu^*(w) = \sup_{(x,y)\in J} \|\mu(x, y, w)\|_E,\ p_i^*(w) = \sup ess_{(x,y)\in J} p_i(x, y, w);\ i = 1, 2,$$

**Theorem 3.2.2.** *Assume that hypotheses $(H_1) - (H_7)$ hold. If*

$$\ell := 2(p_3^*(w) + p_4^*(w)) + \frac{4p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} < 1,$$

*then the problem (3.4)-(3.6) has a random solution defined on $[-\alpha, a] \times [-\beta, b]$.*

**Proof.** Define the operator $N : \Omega \times C_{(a,b)} \to C_{(a,b)}$ by

$$(N(w)u)(x, y) = \begin{cases} \phi(x, y, w), & (x, y) \in \tilde{J}, w \in \Omega \\ \mu(x, y, w) - Q(u) - H(u) & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} & \\ \times f(s, t, u_{(s,t)}, w)dtds, & (x, y) \in J, w \in \Omega. \end{cases} \tag{3.8}$$

Since the functions $\varphi, \psi$ and $Q, H$ and $f$ are absolutely continuous , the function $\mu$ and the indefinite integral are absolutely continuous for all $w \in \Omega$ and almost all $(x, y) \in J$. Again, as the map $\mu$ is continuous for all $w \in \Omega$ and the indefinite integral is continuous on $J$, then $N(w)$ defines a mapping $N : \Omega \times C_{(a,b)} \to C_{(a,b)}$. Hence $u$ is a solution for the problem (3.4)-(3.6) if and only if $u = (N(w))u$. We shall show that the operator $N$ satisfies all conditions of Lemma 1.5.1. The proof will be given in several steps.

**Step 1:** $N(w)$ is a random operator with stochastic domain on $C_{(a,b)}$.
Since $f(x, y, u, w)$ is random Carathéodory, the map $w \to f(x, y, u, w)$ is measurable in view of Definition 1.1.1.
Similarly, the product $(x - s)^{r_1-1}(y - t)^{r_2-1}f(s, t, u_{(s,t)}, w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$w \mapsto \mu(x, y, w) - Q(u) - H(u) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s, t, u_{(s,t)}, w) dt ds,$$

is measurable. As a result, $N$ is a random operator on $\Omega \times C_{(a,b)}$ into $C_{(a,b)}$.

Let $W : \Omega \to \mathcal{P}(C_{(a,b)})$ be defined by

$$W(w) = \{u \in C_{(a,b)} : \|u\|_\infty \leq R(w)\},$$

with $W(w)$ bounded, closed, convex and solid for all $w \in \Omega$. Then $W$ is measurable by Lemma [ [27], Lemma 17]. Let $w \in \Omega$ be fixed, then from $(H_4)$, for any $u \in w(w)$, we get

$$\begin{aligned}
& \|(N(w)u)(x,y)\|_E \\
\leq\ & \|\mu(x,y,w)\|_E + \|Q(u)\| + \|H(u)\| \\
& \quad + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|f(s,t,u_{(s,t)},w)\|_E dtds \\
\leq\ & \|\mu(x,y,w)\|_E + \tilde{d}(1+\|u\|) + d^*(1+\|u\|) \\
& \quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_1(s,t,w) dtds \\
& \quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_2(s,t,w)\|u_{(s,t)}\|_\infty dtds \\
\leq\ & \mu^*(w) + (\tilde{d}+d^*)(1+R(w)) + \frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \\
& \quad + \frac{p_2^*(w)R(w)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \\
\leq\ & \mu^*(w) + (\tilde{d}+d^*)(1+R(w)) + \frac{(p_1^*(w)+p_2^*(w)R(w))a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
\leq\ & R(w).
\end{aligned}$$

Therefore, $N$ is a random operator with stochastic domain $W$ and $N(w) : W(w) \to N(w)$. Furthermore, $N(w)$ maps bounded sets into bounded sets in $C_{(a,b)}$.

**Step 2:** $N(w)$ is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C_{(a,b)}$. Then, for each $(x,y) \in J$ and $w \in \Omega$, we have

$$\begin{aligned}
& \|(N(w)u_n)(x,y) - (N(w)u)(x,y)\|_E \leq \|Q(u_n) - Q(u)\| + \|H(u_n) - H(u)\| \\
& + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|f(s,t,u_{n(s,t)},w) - f(s,t,u_{(s,t)},w)\|_E dtds.
\end{aligned}$$

Using the Lebesgue Dominated Convergence Theorem, we get

$$\|N(w)u_n - N(w)u\|_\infty \to 0 \text{ as } n \to \infty.$$

As a consequence of Steps 1 and 2, we can conclude that $N(w) : W(w) \to N(w)$ is a continuous random operator with stochastic domain $W$, and $N(w)(W(w))$ is bounded.

**Step 3:** For each bounded subset $B$ of $W(w)$ we have

$$\alpha_c(N(w)B) \leq \ell \alpha_c(B).$$

Let $w \in \Omega$ be fixed. From Lemmas 1.4.1 and 1.4.2, for any $B \subset W$ and any $\epsilon > 0$, there exists a sequence $\{u_n\}_{n=0}^{\infty} \subset B$, such that for all $(x, y) \in J$, we have

$$\begin{aligned}
& \alpha_c((N(w)B)(x,y)) \\
= \; & \alpha\Bigg(\Bigg\{\mu(x,y) - Q(u) - H(u) \\
& \quad + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_{(s,t)},w)dtds;\ u \in B\Bigg\}\Bigg) \\
\leq \; & 2\alpha\left\{-Q(u_n) - H(u_n) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_{n(s,t)},w)dtds\right\}_{n=1}^{\infty} + \epsilon \\
\leq \; & 2\alpha\left\{Q(u_n)\right\} + 2\alpha\left\{H(u_n)\right\} \\
& \quad +4 \int_0^x \int_0^y \alpha\left(\left\{\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_{n(s,t)},w)\right\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq \; & 2p_3(s,t,w)\alpha\left(\{u_n\}_{n=1}^{\infty}\right) + 2p_4(s,t,w)\alpha\left(\{u_n\}_{n=1}^{\infty}\right) \\
& \quad +4 \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha\left(\{f(s,t,u_{n(s,t)},w)\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq \; & 2p_3(s,t,w)\alpha_c(B) + 2p_4(s,t,w)\alpha_c(B) \\
& +4 \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)\alpha\left(\{u_{n(s,t)}\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq \; & 2(p_3^*(w) + p_4^*(w))\alpha_c(B) \\
& + \left(4 \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dsdt\right) \alpha\left(\{u_n\}_{n=1}^{\infty}\right) + \epsilon \\
\leq \; & 2(p_3^*(w) + p_4^*(w))\alpha_c(B) \\
& + \left(4 \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dtds\right) \alpha_c(B) + \epsilon \\
\leq \; & \left(2(p_3^*(w) + p_4^*(w)) + \frac{4p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right)\alpha_c(B) + \epsilon \\
= \; & \ell\alpha_c(B) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\alpha_c(N(B)) \leq \ell\alpha_c(B).$$

It follows from Lemma 1.5.1 that for each $w \in \Omega$, $N$ has at least one fixed point in $W$. Since $\bigcap_{w\in\Omega} intW(w) \neq \emptyset$, and the hypothesis that a measurable selector of $intW$ exists holds. By Lemma 1.5.1, $N$ has a stochastic fixed point, i.e., the problem (3.4)-(3.6) has at least one random solution on $C_{(a,b)}$.

### 3.2.3 An Example

Let $E = \mathbb{R}$, $\Omega = (-\infty, 0)$ be equipped with the usual $\sigma$-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Consider the following partial functional random differential equation of the form

$$({}^cD_\theta^r u)(x,y,w) = \frac{w^2 e^{-x-y-3}}{1+w^2+5|u(x,y,w)|}; \text{ a.a. } (x,y) \in J = [0,1]\times[0,1], \; w \in \Omega, \tag{3.9}$$

$$u(x,y,w) = x\sin w + y^2\cos w, \; (x,y) \in [-1,1]\times[-2,1]\backslash(0,1]\times(0,1], \tag{3.10}$$

$$u(x,0,w) = x\sin w; \; x \in [0,1], \; u(0,y,w) = y^2\cos w; \; y \in [0,1], \; w \in \Omega. \tag{3.11}$$

where $(r_1, r_2) \in (0,1]\times(0,1]$. Set

$$f(x,y,u(x,y,w),w) = \frac{w^2}{(1+w^2+5|u(x-1,y-2,w)|)e^{x+y+3}}, \; (x,y) \in J, \; w \in \Omega.$$

The functions $w \mapsto \varphi(x,0,w) = x\sin w$ , $w \mapsto \psi(0,y,w) = y^2\cos w$
and $w \mapsto \Phi(x,y,w) = x\sin w + y^2\cos w$ are measurable and bounded with

$$|\varphi(x,0,w)| \le 1, \; |\psi(0,y,w)| \le 1,$$

hence, the conditions $(H_1)$ is satisfied.
Clearly, the map $(x,y,w) \mapsto f(x,y,u,w)$ is jointly continuous for all $u \in \mathbb{R}$ and hence jointly measurable for all $u \in \mathbb{R}$. Also the map $u \mapsto f(x,y,u,w)$ is continuous for all $(x,y) \in J$ and $w \in \Omega$. So the function $f$ is Carathéodory on $[0,1]\times[0,1]\times\mathbb{R}\times\Omega$.
For each $u \in \mathbb{R}$, $(x,y) \in [0,1]\times[0,1]$ and $w \in \Omega$, we have

$$|f(x,y,u,w)| \le 1 + \frac{5}{e^3}|u|.$$

Hence the conditions $(H_4)$ is satisfied with $p_1(x,y,w) = p_1^* = 1$ and $p_2(x,y,w) = p_2^* = \frac{5}{e^3}$.
Also, the conditions $(H_5)$ is satisfied.

We shall show that condition $\ell < 1$ holds with $a = b = 1$. For each $(r_1, r_2) \in (0,1]\times(0,1]$ we get

$$\begin{aligned} \ell &= \frac{4p_2^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &= \frac{20}{e^3\Gamma(1+r_1)\Gamma(1+r_2)} \\ &< \frac{20}{e^3} \\ &< 1. \end{aligned}$$

Consequently, Theorem 3.2.1 implies that the problem $(3.9)-(3.11)$ has a random solution defined on $[-1,1]\times[-2,1]$.

## 3.3 Fractional Partial Random Differential Equations with infinite Delay

### 3.3.1 Introduction

We study in this section the existence of random solutions for the following fractional partial random differential equations with infinite delay

$$({}^cD_0^r u)(x,y,w) = f(x,y,u_{(x,y)},w), \text{ if } (x,y)\in J := [0,a]\times[0,b], w\in\Omega, \quad (3.12)$$

$$u(x,y,w) = \phi(x,y,w), \text{ if } (x,y)\in \tilde{J} := (-\infty,a]\times(-\infty,b]\backslash(0,a]\times(0,b], w\in\Omega, \quad (3.13)$$

$$u(x,0,w) = \varphi(x,w),\ x\in[0,a],\ u(0,y,w) = \psi(y,w),\ y\in[0,b], w\in\Omega, \quad (3.14)$$

where $a,b>0$, ${}^cD_0^r$ is the standard Caputo's fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $(\Omega,\mathcal{A})$ is a measurable space, $f: J\times\mathcal{B}\times\Omega\to E$ is a given function, $\phi:\tilde{J}\times\Omega\to E$ is a given continuous function, $\varphi:[0,a]\times\Omega\to E, \psi:[0,b]\times\Omega\to E$ are given absolutely continuous functions with $\varphi(x,w)=\phi(x,0,w),\ \psi(y,w)=\phi(0,y,w)$ for each $x\in[0,a],\ y\in[0,b],\ w\in\Omega$ and $\mathcal{B}$ is called a phase space that will be specified later. We denote by $u_{(x,y)}$ the element of $\mathcal{B}$ defined by

$$u_{(x,y)}(s,t,w) = u(x+s,y+t,w);\ (s,t)\in(-\infty,0]\times(-\infty,0],$$

here $u_{(x,y)}(\cdot,\cdot,w)$ represents the history of the state from time $-\infty$ up to the present time $x$ and from time $-\infty$ up to the present time $y$.

### 3.3.2 Existence Results

Let us assume that the function $f$ is random Carathéeodory on $J\times\mathcal{B}\times\Omega$. From the above Lemma, we have the following Lemma.

Let the space

$$\Delta = \{u:(-\infty,a]\times(-\infty,b]\to E:\ u_{(x,y)}\in\mathcal{B} \text{ for } (x,y)\in\mathcal{E} \text{ and } u|_J \text{ is continuous}\}.$$

**Lemma 3.3.1.** *Let* $0 < r_1, r_2 \leq 1, \mu(x,y,w) = \varphi(x,w) + \psi(y,w) - \varphi(0,w)$. *A function* $u \in \Omega \times \Delta$ *is a solution of the random problem (3.12)-(3.14) if* $u$ *satisfies condition (3.13) for* $(x,y) \in \tilde{J}, w \in \Omega$ *and* $u$ *is a solution of the equation*

$$u(x,y,w) = \mu(x,y,w) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(x,y,u_{(s,t)},w)dtds$$

*for* $(x,y) \in J, w \in \Omega$

The following hypotheses will be used in the sequel.

$(H_1)$ The functions $w \mapsto \varphi(x,0,w)$ *and* $w \mapsto \psi(0,y,w)$ are measurable and bounded for a.e. $x \in [0,a]$ and $y \in [0,b]$ respectively,

$(H_2)$ The function $\Phi$ is measurable for $(x,y) \in \tilde{J}$

$(H_3)$ The function $f$ is random Carathéeodory on $J \times \mathcal{B} \times \Omega$,

$(H_4)$ There exist functions $p_1, p_2 : J \times \Omega \to [0,\infty)$ with $p_i(.,w) \in AC(J,[0,\infty))L^\infty(J,[0,\infty))$; $i = 1,2$ such that

$$\|f(x,y,u,w)\|_E \leq p_1(x,y,w) + p_2(x,y,w)\|u\|_{\mathcal{B}},$$

for all $u \in \mathcal{B}$, $w \in \Omega$, and a.e. $(x,y) \in J$,

$(H_5)$ For any bounded $B \subset E$,

$$\alpha(f(x,y,B,w)) \leq P_2(x,y,w)\alpha(B),\ for\ a.e.\ (x,y) \in J,$$

where

$$p_i^*(w) = \sup ess_{(x,y)\in J} p_i(x,y,w);\ i = 1,2,$$

**Theorem 3.3.1.** *Assume that hypotheses* $(H_1) - (H_6)$ *hold. If*

$$\ell := \frac{4p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1,$$

*then the problem (3.12)-(3.14) has a random solution defined on* $(-\infty,a] \times (-\infty,b]$.

**Proof.**

Define the operator $N : \Omega \times \Delta \to \Delta$ by

$$(N(w)u)(x,y) = \begin{cases} \phi(x,y,w), & (x,y) \in \tilde{J}, w \in \Omega \\ \mu(x,y,w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} & \\ \times \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} & \\ \times f(s,t,u_{(s,t)},w)dtds, & (x,y) \in J, w \in \Omega. \end{cases} \tag{3.15}$$

Since the functions $\varphi, \psi$ and $f$ are absolutely continuous , then the function $\mu$ and the indefinite integral are absolutely continuous for all $w \in \Omega$ and almost all $(x,y) \in J$. Again, as the map $\mu$ is continuous for all $w \in \Omega$ and the indefinite integral is continuous on $J$, then $N(w)$ defines a mapping $N : \Omega \times \Delta \to \Delta$. Hence $u$ is a solution for the problem (3.12)-(3.14) if and only if $u = (N(w))u$.

Let $v(\cdot,\cdot,\cdot) : (-\infty, a] \times (-\infty, b] \times \Omega \to E$ be a function defined by,

$$v(x,y,w) = \begin{cases} \phi(x,y,w), & (x,y) \in \tilde{J}', w \in \Omega, \\ \mu(x,y,w), & (x,y) \in J, w \in \Omega. \end{cases}$$

Then $v_{(x,y)} = \phi$ for all $(x,y) \in \mathcal{E}$. For each I continuous on $J$ with $I(x,y,w) = 0$ for each $(x,y) \in \mathcal{E}$ we denote by $\overline{I}$ the function defined by

$$\overline{I}(x,y,w) = \begin{cases} 0, & (x,y) \in \tilde{J}', w \in \Omega, \\ I(x,y,w) & (x,y) \in J, w \in \Omega. \end{cases}$$

If $u(\cdot,\cdot,\cdot)$ satisfies the integral equation,

$$u(x,y,w) = \mu(x,y,w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)},w)dtds,$$

we can decompose $u(\cdot,\cdot,\cdot)$ as $u(x,y,w) = \overline{I}(x,y,w) + v(x,y,w)$; $(x,y) \in J$, which implies $u_{(x,y)} = \overline{I}_{(x,y)} + v_{(x,y)}$, for every $(x,y) \in J$, and the function $I(\cdot,\cdot,\cdot)$ satisfies

$$I(x,y,w) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\overline{I}_{(s,t)} + v_{(s,t)},w)dtds.$$

Set

$$C_0 = \{I \in C(J,E) : \ I(x,y) = 0 \ \text{ for } (x,y) \in \mathcal{E}\},$$

and let $\|.\|_{(a,b)}$ be the seminorm in $C_0$ defined by

$$\|I\|_{(a,b)} = \sup_{(x,y)\in\mathcal{E}} \|I_{(x,y)}\|_{\mathcal{B}} + \sup_{(x,y)\in J} \|I(x,y)\| = \sup_{(x,y)\in J} \|I(x,y)\|, \ I \in C_0.$$

$C_0$ is a Banach space with norm $\|.\|_{(a,b)}$. Let the operator $P : \Omega \times \ C_0 \to C_0$ be defined by

$$(P(w)I)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\overline{I}_{(s,t)} + v_{(s,t)}, w)dtds, \tag{3.16}$$

for each $(x,y) \in J$. Then the operator $N$ has a fixed point is equivalent to $P$ has a fixed point, and so we turn to proving that $P$ has a fixed point.We shall show that the operator $P$ satisfies all conditions of Lemma 1.5.1. The proof will be given in several steps.

**Step 1:** $P(w)$ is a random operator with stochastic domain on $C_0$.
Since $f(x,y,u,w)$ is random Carathéodory, the map $w \to f(x,y,u,w)$ is measurable in view of Definition 1.1.1. Similarly, the product $(x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)}, w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$w \mapsto \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\overline{I}_{(s,t)} + v_{(s,t)}, w)dtds,$$

is measurable. As a result, $P$ is a random operator on $\Omega \times C_0$ into $C_0$.

Let $W : \Omega \to \mathcal{P}(C_0)$ be defined by

$$W(w) = \{I \in C_0 : \|I\|_(a,b) \leq R(w)\},$$

With $R(\cdot)$ is chosen appropriately. From instance, we assume that

$$R(w) \geq \frac{((k\|\phi(0,0)\| + M\|\phi\|)p_2^*(w) + p_1^*(w))\frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}{1 - Kp_2^*(w)\frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}.$$

with $W(w)$ bounded, closed, convex and solid for all $w \in \Omega$. Then $W$ is measurable by Lemma [ [27], Lemma 17]. Let $w \in \Omega$ be fixed, then from $(H_4)$, for any $u \in w(w)$, we get

$$\begin{aligned}
&\|(P(w)I)(x,y)\| \\
\leq\ & \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|f(s,t,\overline{I}_{(s,t)}+v_{(s,t)},w)\| dtds \\
\leq\ & \frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \\
&+ \frac{p_2^*(w)R^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \\
\leq\ & \frac{(p_1^*(w)+p_2^*(w)R^*(w))a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
\leq\ & R(w).
\end{aligned}$$

where

$$\begin{aligned}
\|\overline{I}_{(s,t)}+v_{(s,t)}\|_{\mathcal{B}} &\leq \|\overline{I}_{(s,t)}\|_{\mathcal{B}} + \|v_{(s,t)}\|_{\mathcal{B}} \\
&\leq KR(w) + K\|\phi(0,0)\| + M\|\phi\|_{\mathcal{B}} := R^*(w).
\end{aligned}$$

Therefore, $P$ is a random operator with stochastic domain $W$ and $P(w): W(w) \to W(w)$. Furthermore, $P(w)$ maps bounded sets into bounded sets in $C_0$.

**Step 2:** $P(w)$ is continuous.
Let $\{I_n\}$ be a sequence such that $I_n \to u$ in $C_0$. Then, for each $(x,y) \in J$ and $w \in \Omega$, we have

$$\begin{aligned}
&\|(P(w)I_n)(x,y) - (P(w)I)(x,y)\|_E \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
&\times \|f(s,t,\overline{I}_{n(s,t)}+v_{n(s,t)},w) - f(s,t,\overline{I}_{(s,t)}+v_{(s,t)},w)\|_F dtds.
\end{aligned}$$

Using the Lebesgue Dominated Convergence Theorem, we get

$$\|P(w)I_n - P(w)I\|_\infty \to 0 \text{ as } n \to \infty.$$

As a consequence of Steps 1 and 2, we can conclude that $P(w): W(w) \to W(w)$ is a continuous random operator with stochastic domain $W$, and $P(w)(W(w))$ is bounded.

**Step 3:** For each bounded subset $B$ of $W(w)$ we have

$$\alpha(P(w)B) \leq \ell\alpha(B).$$

Let $w \in \Omega$ be fixed. From Lemmas 1.4.1 and 1.4.2, for any $B \subset W$ and any $\epsilon > 0$, there exists a sequence $\{I_n\}_{n=0}^{\infty} \subset B$, such that for all $(x,y) \in J$, we have

$$\begin{aligned}
& \alpha(P(w)B)(x,y)) \\
& = \alpha\left(\left\{\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,\overline{I}_{(s,t)} + v_{(s,t)}, w)dtds;\ I \in B\right\}\right) \\
\leq\ & 2\alpha\left(\left\{\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,\overline{I}_{n(s,t)} + v_{n(s,t)}, w)dtds\right\}_{n=1}^{\infty}\right) + \epsilon \\
\leq\ & 4\int_0^x \int_0^y \alpha\left(\left\{\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,\overline{I}_{n(s,t)} + v_{n(s,t)}, w)\right\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq\ & 4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha\left(\{f(s,t,\overline{I}_{n(s,t)} + v_{n(s,t)}, w)\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq\ & 4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)\alpha\left(\{\overline{I}_{n(s,t)} + v_{n(s,t)}\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq\ & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dsdt\right) \alpha\left(\{u_{n(s,t)}\}_{n=1}^{\infty}\right) + \epsilon \\
\leq\ & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dsdt\right) \alpha\left(\{u_n\}_{n=1}^{\infty}\right) + \epsilon \\
\leq\ & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dsdt\right) \alpha\left(\{I_n\}_{n=1}^{\infty}\right) + \epsilon \\
\leq\ & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dtds\right) \alpha(B) + \epsilon \\
\leq\ & \frac{4p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\alpha(B) + \epsilon \\
& = \ell\alpha(B) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\alpha(P(B)) \leq \ell\alpha(B).$$

It follows from Lemma 1.5.1 that for each $w \in \Omega$, $P$ has at least one fixed point in $W$. Since $\bigcap_{w\in\Omega} intW(w) \neq \emptyset$ the hypothesis that a measurable selector of $intW$ exists holds. By Lemma 1.5.1, $N$ has a stochastic fixed point, i.e., the problem (3.12)-(3.14) has at least one random solution.

### 3.3.3 An Example

Let $E = \mathbb{R},\ \Omega = (-\infty, 0)$ be equipped with the usual $\sigma$-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Consider the following partial functional random differential equation of the form

$$({}^cD^r_\theta u)(x,y,w) = \frac{ce^{x+y-\gamma(x+y)}\|u_{(x,y)}\|}{(e^{x+y}+e^{-x-y})(1+w^2+\|u_{(x,y)})\|};\ \text{ a.a. } (x,y)\in J,\ w\in\Omega, \tag{3.17}$$

$$u(x,y,w) = x\sin w + y^2\cos w,\ (x,y)\in(-\infty,1]\times(-\infty,1]\backslash(0,1]\times(0,1],\ w\in\Omega, \tag{3.18}$$

$$u(x,0,w) = x\sin w;\ x\in[0,1],\ u(0,y,w) = y^2\cos w;\ y\in[0,1],\ w\in\Omega. \tag{3.19}$$

where $J = [0,1]\times[0,1] c = \frac{8}{\Gamma(r_1+1)\Gamma(r_2+1)}$ and $\gamma$ a positive real constant.
Let

$$\mathcal{B}_\gamma = \{u\in C((-\infty,0]\times(-\infty,0],\mathbb{R}) : \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u(\theta,\eta) \text{ exists in } \mathbb{R}\}.$$

The norm of $\mathcal{B}_\gamma$ is given by

$$\|u\|_\gamma = \sup_{(\theta,\eta)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(\theta+\eta)}|u(\theta,\eta)|.$$

Let

$$E := [0,1]\times\{0\}\cup\{0\}\times[0,1],$$

and $u : (-\infty,1]\times(-\infty,1]\to\mathbb{R}$ such that $u_{(x,y)}\in\mathcal{B}_\gamma$ for $(x,y)\in E$, then

$$\lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u_{(x,y)}(\theta,\eta) = \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta-x+\eta-y)}u(\theta,\eta)$$

$$= e^{-\gamma(x+y)}\lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u(\theta,\eta) < \infty.$$

Hence $u_{(x,y)}\in\mathcal{B}_\gamma$. Finally we prove that

$$\|u_{(x,y)}\|_\gamma = K\sup\{|u(s,t)| : (s,t)\in[0,x]\times[0,y]\} + M\sup\{\|u_{(s,t)}\|_\gamma : (s,t)\in E_{(x,y)}\},$$

where $K = M = 1$ and $H = 1$.
If $x+\theta\le 0,\ y+\eta\le 0$ we get

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t)\in(-\infty,0]\times(-\infty,0]\},$$

and if $x+\theta\ge 0,\ y+\eta\ge 0$ then we have

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t)\in[0,x]\times[0,y]\}.$$

Thus for all $(x+\theta, y+\eta) \in [0,1]\times[0,1]$, we get

$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t)\in(-\infty,0]\times(-\infty,0]\}+\sup\{|u(s,t)| : (s,t)\in[0,x]\times[0,y]\}.$

Then

$$\|u_{(x,y)}\|_\gamma = \sup\{\|u_{(s,t)}\|_\gamma : (s,t)\in E\}+\sup\{|u(s,t)| : (s,t)\in[0,x]\times[0,y]\}.$$

$(\mathcal{B}_\gamma, \|.\|_\gamma)$ is a Banach space. We conclude that $\mathcal{B}_\gamma$ is a phase space. Set

$$f(x,y,u_{(x,y)},w) = \frac{ce^{x+y-\gamma(x+y)}\|u_{(x,y)}\|}{(e^{x+y}+e^{-x-y})(1+w^2+\|u_{(x,y)})\|},\ (x,y)\in[0,1]\times[0,1].$$

The functions $w\mapsto\varphi(x,0,w) = x\sin w$ , $w\mapsto\psi(0,y,w) = y^2\cos w$
and $w\mapsto\Phi(x,y,w) = x\sin w + y^2\cos w$ are measurable and bounded with

$$|\varphi(x,0,w)|\le 1,\ |\psi(0,y,w)|\le 1,$$

hence, the conditions $(H_1)$ is satisfied.
Clearly, the map $(x,y,w)\mapsto f(x,y,u,w)$ is jointly continuous for all $u\in\mathcal{B}_\gamma$ and hence jointly measurable for all $u\in\mathcal{B}_\gamma$. Also the map $u\mapsto f(x,y,u,w)$ is continuous for all $(x,y)\in J$ and $w\in\Omega$. So the function $f$ is Carathéodory on $[0,1]\times[0,1]\times\mathcal{B}_\gamma\times\Omega$.
For each $u\in\mathcal{B}_\gamma,\ (x,y)\in[0,1]\times[0,1]$ and $w\in\Omega$, we have

$$|f(x,y,u_{(x,y)})| \le 1+\frac{1}{c}\|u\|_B.$$

Hence the conditions $(H_4)$ is satisfied with $p_1(x,y,w) = p_1^* = 1$ and $p_2(x,y,w) = p_2^* = \frac{1}{c}$.
Also, the conditions $(H_5)$ is satisfied.

We shall show that condition $\ell < 1$ holds with $a = b = 1$. For each $(r_1,r_2)\in(0,1]\times(0,1]$ we get

$$\begin{aligned}\ell &= \frac{4p_2^*a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\\ &= \frac{4}{c\Gamma(1+r_1)\Gamma(1+r_2)}\\ &< \frac{1}{2}\\ &< 1.\end{aligned}$$

Consequently, Theorem 3.3.1 implies that the problem $(3.17)-(3.19)$ has a random solution defined on $(-\infty,1]\times(-\infty,1]$.

# Chapter 4

# Fractional Partial Random Differential Equations with State-Dependent Delay

## 4.1 Introduction

In this chapter, we shall be concerned to the existence for the following fractional partial random differential equations with state-dependent delay

## 4.2 Fractional Partial Random Differential Equations with finite Delay

### 4.2.1 Introduction

In this section, we shall be concerned with the existence of solutions for the following fractional partial random differential equations:

$$({}^cD_0^r u)(x,y,w) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)},w),\rho_2(x,y,u_{(x,y)},w))},w), \text{ if } J := [0,a]\times[0,b], w\in\Omega, \tag{4.1}$$

$$u(x,y,w) = \phi(x,y,w), \text{ if } (x,y)\in\tilde{J} := [-\alpha,a]\times[-\beta,b]\backslash(0,a]\times(0,b], w\in\Omega, \tag{4.2}$$

$$u(x,0,w) = \varphi(x,w),\ x\in[0,a],\ u(0,y,w) = \psi(y,w),\ y\in[0,b], w\in\Omega, \tag{4.3}$$

where $\alpha,\beta,a,b>0$, ${}^cD_0^r$ is the standard Caputo's fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $(\Omega,\mathcal{A})$ is a measurable space, $f: J\times C\times\Omega\to$

$E,\ \rho_1, \rho_2 : J \times C \times \Omega \to E$ are given functions, $\phi : \tilde{J} \times \Omega \to E$ is a given continuous function, $\varphi : [0,a] \times \Omega \to E, \psi : [0,b] \times \Omega \to E$ are given absolutely continuous functions with $\varphi(x,w) = \phi(x,0,w),\ \psi(y,w) = \phi(0,y,w)$ for each $x \in [0,a],\ y \in [0,b],\ w \in \Omega, (E, \|\cdot\|_E)$ a separable Banach space, and $C := C([-\alpha,0] \times [-\beta,0], E)$ is the space of continuous functions on $[-\alpha,0] \times [-\beta,0]$.

If $u \in C([-\alpha,a] \times [-\beta,b], E);\ a,b,\alpha,\beta > 0$ then for any $(x,y) \in J$ define $u_{(x,y)}$ by

$$u_{(x,y)}(s,t,w) = u(x+s, y+t, w),\ \text{for } (s,t) \in [-\alpha,0] \times [-\beta,0].$$

Here $u_{(x,y)}(\cdot,\cdot,w)$ represents the history of the state $u$.

### 4.2.2 Existence Results

Let us assume that the function $f$ is random Carathéeodory on $J \times C \times \Omega$. From the above Lemma, we have the following Lemma.

**Lemma 4.2.1.** *Let $0 < r_1, r_2 \leq 1, \mu(x,y,w) = \varphi(x,w) + \psi(y,w) - \varphi(0,w)$. A function $u \in \Omega \times C_{(a,b)}$ is a solution of the random problem (4.1)-(4.3) if u satisfies condition (4.2) for $(x,y) \in \tilde{J}, w \in \Omega$ and $u$ is a solution of the equation*

$$u(x,y,w) = \mu(x,y,w) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}, w)dtds$$

*for $(x,y) \in J, w \in \Omega$*

Set $\mathcal{R} := \mathcal{R}_{(\rho_1^-,\rho_2^-)}$

$$= \{(\rho_1(s,t,u,w), \rho_2(s,t,u,w)) : (s,t) \in J, u_{(s,t)}(\cdot,\cdot,w) \in C, w \in \Omega,\ \rho_i(s,t,u,w) \leq 0;\ i = 1,2\}.$$

We always assume that $\rho_i : J \times C \times \Omega \to E;\ i = 1,2$ are continuous and the function $(s,t) \longmapsto u_{(s,t)}$ is continuous from $\mathcal{R}$ into $C$.

Let us introduce the following hypotheses which are assumed after.

$(H_1)$ The functions $w \mapsto \varphi(x,0,w)$ *and* $w \mapsto \psi(0,y,w)$ are measurable and bounded for a.e. $x \in [0,a]$ and $y \in [0,b]$ respectively,

$(H_2)$ The function $\Phi$ is measurable for $(x,y) \in \tilde{J}$

$(H_3)$ The function $f$ is random Carathéeodory on $J \times C \times \Omega$,

$(H_4)$ There exist functions $p_1, p_2 : J \times \Omega \to [0, \infty)$ with $p_i(\cdot, w) \in AC(J, [0, \infty))L^\infty(J, [0, \infty));\ i = 1, 2$ such that

$$\|f(x, y, u, w)\|_E \leq p_1(x, y, w) + p_2(x, y, w)\|u\|_C,$$

for all $u \in C$, $w \in \Omega$ and a.e. $(x, y) \in J$,

$(H_5)$ For any bounded $B \subset E$,

$$\alpha(f(x, y, B, w)) \leq P_2(x, y, w)\alpha(B),\ for\ a.e.\ (x, y) \in J,$$

$(H_6)$ There exists a random function $R : \Omega \to (0, \infty)$ such that

$$\mu^*(w) + \frac{(p_1^*(w) + p_2^*(w)R(w))a^{r_1}b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \leq R(w),$$

where

$$\mu^*(w) = \sup_{(x,y)\in J} \|\mu(x, y, w)\|_E,\ p_i^*(w) = \sup ess_{(x,y)\in J} p_i(x, y, w);\ i = 1, 2,$$

**Theorem 4.2.1.** *Assume that hypotheses $(H_1) - (H_6)$ hold. If*

$$\ell := \frac{4p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} < 1,$$

*then the problem (4.1)-(4.3) has a random solution defined on $[-\alpha, a] \times [-\beta, b]$.*

**Proof.** Define the operator $N : \Omega \times C_{(a,b)} \to C_{(a,b)}$ by

$$(N(w)u)(x, y) = \begin{cases} \phi(x, y, w), & (x, y) \in \tilde{J}, w \in \Omega \\ \mu(x, y, w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} & \\ \times f(s, t, u_{(\rho_1(s,t,u_{(s,t)},w), \rho_2(s,t,u_{(s,t)},w)), w})dtds, & (x, y) \in J, w \in \Omega. \end{cases} \tag{4.4}$$

Since the functions $\varphi, \psi$ and $f$ are absolutely continuous, the function $\mu$ and the indefinite integral are absolutely continuous for all $w \in \Omega$ and almost all $(x, y) \in J$. Again, as the map $\mu$ is continuous for all $w \in \Omega$ and the indefinite integral is continuous on $J$, then $N(w)$ defines a mapping $N : \Omega \times C_{(a,b)} \to C_{(a,b)}$. Hence $u$ is a solution for the problem (4.1)-(4.3) if and only if $u = (N(w))u$. We shall show that the operator $N$ satisfies all conditions of Lemma 1.5.1. The proof will be given in several steps.

**Step 1:** $N(w)$ is a random operator with stochastic domain on $C_{(a,b)}$.
Since $f(x,y,u,w)$ is random Carathéodory, the map $w \to f(x,y,u,w)$ is measurable in view of Definition 1.1.1. Similarly, the function $(s,t) \mapsto (x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u_{(s,t)},w)$ is measure as the product of a continuous and a measurable function. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$w \mapsto \mu(x,y,w)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w)dtds,$$

is measurable. As a result, $N$ is a random operator on $\Omega \times C_{(a,b)}$ into $C_{(a,b)}$.

Let $W:\Omega \to \mathcal{P}(C_{(a,b)})$ be defined by

$$W(w)=\{u\in C_{(a,b)}:\|u\|_\infty \le R(w)\},$$

with $R(\cdot)$ is chosen appropriately. From instance, we assume that

$$R(w)\ge \frac{\mu^*+p_1^*(w)\frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}{1-p_2^*(w)\frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}.$$

Clearly, $W(w)$ is bounded, closed, convex and solid for all $w\in\Omega$. Then $W$ is measurable by Lemma 17 of [27]. Let $w\in\Omega$ be fixed, then from $(H_4)$, for any $u\in w(w)$, we get

$$\begin{aligned}
&\|(N(w)u)(x,y)\|_E\\
\le\ &\|\mu(x,y,w)\|_E+\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w)\|_E dtds\\
\le\ &\|\mu(x,y,w)\|_E+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}p_1(s,t,w)dtds\\
+\ &\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}p_2(s,t,w)\|u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\|_E dtds\\
\le\ &\mu^*(w)+\frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}dtds\\
+\ &\frac{p_2^*(w)R(w)}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}dtds\\
\le\ &\mu^*(w)+\frac{(p_1^*(w)+p_2^*(w)R(w))a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\\
\le\ &R(w).
\end{aligned}$$

Therefore, $N$ is a random operator with stochastic domain $W$ and $N(w) : W(w) \to N(w)$. Furthermore, $N(w)$ maps bounded sets into bounded sets in $C_{(a,b)}$.

**Step 2:** $N(w)$ is continuous.
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C_{(a,b)}$. Then, for each $(x, y) \in J$ and $w \in \Omega$, we have

$$\|(N(w)u_n)(x,y) - (N(w)u)(x,y)\|_E \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}$$
$$\times \|f(s,t,u_{n(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}, w) - f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}, w)\|_E dtds.$$

Using the Lebesgue Dominated Convergence Theorem, we get

$$\|N(w)u_n - N(w)u\|_\infty \to 0 \text{ as } n \to \infty.$$

As a consequence of Steps 1 and 2, we can conclude that $N(w) : W(w) \to N(w)$ is a continuous random operator with stochastic domain $W$, and $N(w)(W(w))$ is bounded.

**Step 3:** For each bounded subset $B$ of $W(w)$ we have

$$\alpha(N(w)B) \leq \ell\alpha(B).$$

Let $w \in \Omega$ be fixed. From Lemmas 1.4.1 and 1.4.2, for any $B \subset W$ and any

$\epsilon > 0$, there exists a sequence $\{u_n\}_{n=0}^{\infty} \subset B$, such that for all $(x,y) \in J$, we have

$$\begin{aligned}
& \alpha((N(w)B)(x,y)) \\
= & \ \alpha\left(\left\{\mu(x,y) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\right.\right. \\
& \left.\left. \qquad \times f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w)dtds;\ u \in B\right\}\right) \\
\leq & \ 2\alpha\left(\left\{\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_{n(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w)dtds\right\}_{n=1}^{\infty}\right) + \epsilon \\
\leq & \ 4\int_0^x \int_0^y \alpha\left(\left\{\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_{n(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w)\right\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq & \ 4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha\left(\{f(s,t,u_{n(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w)\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq & \ 4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)\alpha\left(\{u_{n(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dsdt\right) \alpha\left(\{u_n\}_{n=1}^{\infty}\right) + \epsilon \\
\leq & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dtds\right) \alpha(B) + \epsilon \\
\leq & \ \frac{4p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\alpha(B) + \epsilon \\
& = \ell\alpha(B) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\alpha(N(B)) \leq \ell\alpha(B).$$

It follows from Lemma 1.5.1 that for each $w \in \Omega$, $N$ has at least one fixed point in $W$. Since $\bigcap_{w\in\Omega} intW(w) \neq \emptyset$, and a measurable selector of $intW$ exists, Lemma 1.5.1 implies that $N$ has a stochastic fixed point, i.e., the problem (4.1)-(4.3) has at least one random solution on $C_{(a,b)}$.

### 4.2.3 An Example

Let $E = \mathbb{R}$, $\Omega = (-\infty, 0)$ be equipped with the usual $\sigma$-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Consider the following partial functional random differential equation of the form

$$({}^cD_0^r u)(x,y) = \frac{|u(x-\sigma_1(u(x,y,w)),y-\sigma_2(u(x,y,w)),w)|+2}{e^{x+y+4}(1+w^2+5|u(x-\sigma_1(u(x,y,w)),y-\sigma_2(u(x,y,w)),w)|)},\ \text{if } (x,y)\in J, \tag{4.5}$$

$$u(x,y,w) = x\sin w + y^2\cos w,\ (x,y)\in[-1,1]\times[-2,1]\backslash(0,1]\times(0,1],\ w\in\Omega, \tag{4.6}$$

$$u(x,0,w) = x\sin w;\ x\in[0,1],\ u(0,y,w) = y^2\cos w;\ y\in[0,1],\ w\in\Omega, \tag{4.7}$$

where $\sigma_1 \in C(\mathbb{R},[0,1]),\ \sigma_2\in C(\mathbb{R},[0,2])$. Set

$$\rho_1(x,y,\varphi,w) = x-\sigma_1(\varphi(0,0,w)),$$

$$\rho_2(x,y,\varphi,w) = y-\sigma_2(\varphi(0,0,w)),$$

where $(x,y)\in J=[0,1]\times[0,1], \varphi(\cdot,\cdot,w)\in C([-1,0]\times[-2,0],\mathbb{R}), w\in\Omega,$

$$f(x,y,\varphi,w) = \frac{|\varphi|+2}{(e^{x+y+4})(1+w^2+5|\varphi|)},\ (x,y)\in J,\ \varphi\in C([-1,0]\times[-2,0],\mathbb{R}).$$

The functions $w\mapsto \varphi(x,0,w) = x\sin w$ , $w\mapsto\psi(0,y,w) = y^2\cos w$ and $w\mapsto\Phi(x,y,w) = x\sin w + y^2\cos w$ are measurable and bounded with

$$|\varphi(x,0,w)|\le 1,\ |\psi(0,y,w)|\le 1,$$

hence, the conditions $(H_1)$ is satisfied.
Clearly, the map $(x,y,w)\mapsto f(x,y,u,w)$ is jointly continuous for all $u\in\mathbb{R}$ and hence jointly measurable for all $u\in\mathbb{R}$. Also the map $u\mapsto f(x,y,u,w)$ is continuous for all $(x,y)\in J$ and $w\in\Omega$. So the function $f$ is Carathéodory on $[0,1]\times[0,1]\times\mathbb{R}\times\Omega$.
For each $\varphi\in C([-1,0]\times[-2,0],\mathbb{R}),\ (x,y)\in[0,1]\times[0,1]$ and $w\in\Omega$, we have

$$|f(x,y,u,w)|\le 1+\frac{5}{e^3}|u|.$$

Hence the conditions $(H_4)$ is satisfied with

$$p_1(x,y,w) = p_1^* = 1,\quad p_2(x,y,w) = p_2^* = \frac{5}{e^3}.$$

Also, the conditions $(H_5)$ is satisfied.

We shall show that condition $\ell < 1$ holds with $a = b = 1$. For each $(r_1, r_2) \in (0,1] \times (0,1]$ we get

$$\begin{aligned}
\ell &= \frac{4p_2^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
&= \frac{20}{e^3\Gamma(1+r_1)\Gamma(1+r_2)} \\
&< \frac{20}{e^3} \\
&< 1.
\end{aligned}$$

Consequently, Theorem 4.2.1 implies that the problem $(4.5) - (4.7)$ has a random solution defined on $[-1,1] \times [-2,1]$.

## 4.3 Fractional Partial Random Differential Equations with infinite Delay

### 4.3.1 Introduction

In this section, we shall be concerned with the existence of solutions for the following fractional partial random differential equations:

$$({}^cD_0^r u)(x,y,w) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)},w),\rho_2(x,y,u_{(x,y)},w))}), \text{ if } (x,y) \in J, w \in \Omega, \quad (4.8)$$

$$u(x,y,w) = \phi(x,y,w), \text{ if } (x,y) \in \tilde{J} := (-\infty,a] \times (-\infty,b] \backslash (0,a] \times (0,b], w \in \Omega, \quad (4.9)$$

$$u(x,0,w) = \varphi(x,w),\ x \in [0,a],\ u(0,y,w) = \psi(y,w),\ y \in [0,b], w \in \Omega, \quad (4.10)$$

where $\alpha, \beta, a, b > 0$, $J := [0,a] \times [0,b]$ ${}^cD_0^r$ is the standard Caputo's fractional derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $(\Omega, \mathcal{A})$ is a measurable space, $f : J \times \mathcal{B} \times \Omega \to E,\ \rho_1, \rho_2 : J \times \mathcal{B} \times \Omega \to E$ are given functions, $\phi : \tilde{J} \times \Omega \to E$ is a given continuous function, $\varphi : [0,a] \times \Omega \to E, \psi : [0,b] \times \Omega \to E$ are given absolutely continuous functions with $\varphi(x,w) = \phi(x,0,w),\ \psi(y,w) = \phi(0,y,w)$ for each $x \in [0,a],\ y \in [0,b],\ w \in \Omega$and $\mathcal{B}$ is called a phase space that will be specified later. We denote by $u_{(x,y)}$ the element of $\mathcal{B}$ defined by

$$u_{(x,y)}(s,t,w) = u(x+s, y+t, w);\ (s,t) \in (-\infty,0] \times (-\infty,0],$$

here $u_{(x,y)}(\cdot,\cdot,w)$ represents the history of the state from time $-\infty$ up to the present time $x$ and from time $-\infty$ up to the present time $y$.

### 4.3.2 Existence Results

Let us assume that the function $f$ is random Carathéeodory on $J \times \mathcal{B} \times \Omega$.

Let the space

$\Delta = \{u : (-\infty, a]\times(-\infty, b] \to E : \ u_{(x,y)} \in \mathcal{B}$ for $(x, y) \in \mathcal{E}$ and $u|_J$ is continuous$\}$.

From the above Lemma, we have.

**Lemma 4.3.1.** *Let $0 < r_1, r_2 \leq 1, \mu(x, y, w) = \varphi(x, w) + \psi(y, w) - \varphi(0, w)$. A function $u \in \Omega \times \Delta$ is a solution of the random problem (4.8)-(4.10) if $u$ satisfies condition (4.9) for $(x, y) \in \tilde{J}, w \in \Omega$, and the integral equation*

$$u(x, y, w) = \mu(x, y, w) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))}, w)dtds$$

*for $(x, y) \in J, w \in \Omega$*

Set $\mathcal{R}' := \mathcal{R}'_{(\rho_1^-,\rho_2^-)}$

$= \{(\rho_1(s, t, u, w), \rho_2(s, t, u, w)) : (s, t, u, w) \in J\times\mathcal{B}\times\Omega, \ \rho_i(s, t, u, w) \leq 0; \ i = 1, 2\}$.

We always assume that $\rho_i : J \times \mathcal{B} \times \Omega \to E; \ i = 1, 2$ are continuous and the function $(s, t) \longmapsto u_{(s,t)}$ is continuous from $\mathcal{R}$ into $\mathcal{B}$.

We will need to introduce the following hypothesis:

$(C_\phi)$ There exists a continuous bounded function $L : \mathcal{R}'_{(\rho_1^-,\rho_2^-)} \to (0, \infty)$ such that

$$\|\phi_{(s,t)}\|_{\mathcal{B}} \leq L(s, t)\|\phi\|_{\mathcal{B}}, \text{ for any}(s, t) \in \mathcal{R}'.$$

In the sequel we will make use of the following generalization of a consequence of the phase space axioms ( [36], Lemma 2.1).

**Lemma 4.3.2.** *If $u \in \Omega$, then*

$$\|u_{(s,t)}\|_{\mathcal{B}} = (M + L')\|\phi\|_{\mathcal{B}} + K \sup_{(\theta,\eta)\in[0,\max\{0,s\}]\times[0,\max\{0,t\}]} \|u(\theta, \eta)\|,$$

*where*

$$L' = \sup_{(s,t)\in\mathcal{R}'} L(s, t).$$

The following hypotheses will be used in the sequel.

$(H_1)$ The functions $w \mapsto \varphi(x,0,w)$ *and* $w \mapsto \psi(0,y,w)$ are measurable and bounded for a.e. $x \in [0,a]$ and $y \in [0,b]$ respectively,

$(H_2)$ The function $\Phi$ is measurable for $(x,y) \in \tilde{J}$

$(H_3)$ The function $f$ is random Carathéeodory on $J \times \mathcal{B} \times \Omega$,

$(H_4)$ There exist functions $p_1, p_2 : J \times \Omega \to [0,\infty)$ with $p_i(\cdot,w) \in AC(J,[0,\infty))L^\infty(J,[0,\infty));\ i = 1,2$ such that

$$\|f(x,y,u,w)\|_E \leq p_1(x,y,w) + p_2(x,y,w)\|u\|_{\mathcal{B}},$$

for all $u \in \mathcal{B}$, $w \in \Omega$ and a.e. $(x,y) \in J$,

$(H_5)$ For any bounded $B \subset E$,

$$\alpha(f(x,y,B,w)) \leq P_2(x,y,w)\alpha(B),\ for\ a.e.\ (x,y) \in J,$$

where

$$p_i^*(w) = \sup ess_{(x,y)\in J} p_i(x,y,w);\ i = 1,2,$$

**Theorem 4.3.1.** *Assume that hypotheses $(H_1) - (H_5)$ and $(C_\phi)$ hold. If*

$$\ell := \frac{4p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1,$$

*then the problem (4.8)-(4.10) has a random solution defined on $(-\infty,a] \times (-\infty,b]$.*

$$(N(w)u)(x,y) = \begin{cases} \phi(x,y,w), & (x,y) \in \tilde{J}, w \in \Omega \\ \mu(x,y,w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} & \\ \times f(s,t,u_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))},w)dtds, & (x,y) \in J, w \in \Omega. \end{cases} \tag{4.11}$$

Since the functions $\varphi, \psi$ and $f$ are absolutely continuous , then the function $\mu$ and the indefinite integral are absolutely continuous for all $w \in \Omega$ and almost all $(x,y) \in J$. Again, as the map $\mu$ is continuous for all $w \in \Omega$ and the indefinite integral is continuous on $J$, then $N(w)$ defines a mapping $N : \Omega \times \Delta \to \Delta$. Hence $u$ is a solution for the problem (4.8)-(4.10) if and only if $u = (N(w))u$.

Let $v(\cdot,\cdot,\cdot) : (-\infty,a] \times (-\infty,b] \times \Omega \to E$ be a function defined by,

$$v(x,y,w) = \begin{cases} \phi(x,y,w), & (x,y) \in \tilde{J}', w \in \Omega, \\ \mu(x,y,w), & (x,y) \in J, w \in \Omega. \end{cases}$$

Then $v_{(x,y)} = \phi$ for all $(x,y) \in \mathcal{E}$. For each I continuous on $J$ with $I(x,y,w) = 0$ for each $(x,y) \in \mathcal{E}$ we denote by $\overline{I}$ the function defined by

$$\overline{I}(x,y,w) = \begin{cases} 0, & (x,y) \in \tilde{J}', w \in \Omega, \\ I(x,y,w) & (x,y) \in J, w \in \Omega. \end{cases}$$

If $u(\cdot,\cdot,\cdot)$ satisfies the integral equation,

$$\begin{aligned} u(x,y,w) &= \mu(x,y,w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ &\times f(s,t,u_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))}, w)dtds, \end{aligned}$$

we can decompose $u(\cdot,\cdot,\cdot)$ as $u(x,y,w) = \overline{I}(x,y,w) + v(x,y,w);\ (x,y) \in J$, which implies $u_{(x,y)} = \overline{I}_{(x,y)} + v_{(x,y)}$, for every $(x,y) \in J$, and the function $I(\cdot,\cdot,\cdot)$ satisfies

$$I(x,y,w) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}$$

$$\times f(s,t,\overline{I}_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))} + v_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))}, w)dtds.$$

Set

$$C_0 = \{I \in C(J,E) : \ I(x,y) = 0 \ \text{ for } (x,y) \in \mathcal{E}\},$$

and let $\|.\|_{(a,b)}$ be the seminorm in $C_0$ defined by

$$\|I\|_{(a,b)} = \sup_{(x,y)\in\mathcal{E}} \|I_{(x,y)}\|_{\mathcal{B}} + \sup_{(x,y)\in J} \|I(x,y)\| = \sup_{(x,y)\in J} \|I(x,y)\|,\ I \in C_0.$$

$C_0$ is a Banach space with norm $\|.\|_{(a,b)}$. Let the operator $P : \Omega \times\ C_0 \to C_0$ be defined by

$$(Pw)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}$$

$$\times f(s,t,\overline{I}_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))} + v_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))}, w)dtds. \tag{4.12}$$

for each $(x,y) \in J$. Then the operator $N$ has a fixed point is equivalent to $P$ has a fixed point, and so we turn to proving that $P$ has a fixed point.We shall show that the operator $P$ satisfies all conditions of Lemma 1.5.1. The proof will be given in several steps.

**Step 1:** $P(w)$ is a random operator with stochastic domain on $C_0$.
Since $f(x,y,u,w)$ is random Carathéodory, the map $w \to f(x,y,u,w)$ is measurable in view of Definition 1.1.1.

Similarly, the product $(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))},w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$w \mapsto \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}$$

$$\times f(s,t,\overline{I}_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))}+v_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))},w)dtds,$$

is measurable. As a result, $P$ is a random operator on $\Omega\times C_0$ into $C_0$.

Let $W:\Omega\to\mathcal{P}(C_0)$ be defined by

$$W(w)=\{I\in C_0:\|I\|_(a,b)\leq R(w)\},$$

With $R(\cdot)$ is chosen appropriately. From instance, we assume that

$$R(w)\geq\frac{((k\|\phi(0,0)\|+(M+L')\|\phi\|_{\mathcal{B}})p_2^*(w)+p_1^*(w))\frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}{1-Kp_2^*(w)\frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}.$$

with $W(w)$ bounded, closed, convex and solid for all $w\in\Omega$. Then $W$ is measurable by Lemma [ [27], Lemma 17]. Let $w\in\Omega$ be fixed, then from $(H_4)$, for any $u\in w(w)$, we get

$$\begin{aligned}
&\|(P(w)I)(x,y)\|\leq\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\\
&\times\|f(s,t,\overline{I}_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))}+v_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))},w)\|dtds\\
\leq\;&\frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}dtds\\
&\quad+\frac{p_2^*(w)R^*(w)}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}dtds\\
\leq\;&\frac{(p_1^*(w)+p_2^*(w)R^*(w))a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\\
\leq\;&R(w).
\end{aligned}$$

where

$$\begin{aligned}
\|\overline{I}_{(s,t)}+v_{(s,t)}\|_{\mathcal{B}}&\leq\|\overline{I}_{(s,t)}\|_{\mathcal{B}}+\|v_{(s,t)}\|_{\mathcal{B}}\\
&\leq KR(w)+K\|\phi(0,0)\|+(M+L')\|\phi\|_{\mathcal{B}}:=R^*(w).
\end{aligned}$$

Therefore, $P$ is a random operator with stochastic domain $W$ and $P(w) : W(w) \to W(w)$. Furthermore, $P(w)$ maps bounded sets into bounded sets in $C_0$.

**Step 2:** $P(w)$ is continuous.
Let $\{I_n\}$ be a sequence such that $I_n \to I$ in $C_0$. Then, for each $(x,y) \in J$ and $w \in \Omega$, we have

$$\begin{aligned}
& \|(P(w)I_n)(x,y) - (P(w)I)(x,y)\|_E \\
\leq \; & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
& \times \|f(s,t,\overline{I}_{n(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))} + v_{n(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))}, w) \\
& - f(s,t,\overline{I}_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))} + v_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))}, w)\|_E dtds \\
\leq \; & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
& \times \|f(s,t,\overline{I}_{n(s,t)} + v_{n(s,t)}, w) - f(s,t,\overline{I}_{(s,t)} + v_{(s,t)}, w)\|_E dtds.
\end{aligned}$$

Using the Lebesgue Dominated Convergence Theorem, we get

$$\|P(w)I_n - P(w)I\|_\infty \to 0 \text{ as } n \to \infty.$$

As a consequence of Steps 1 and 2, we can conclude that $P(w) : W(w) \to W(w)$ is a continuous random operator with stochastic domain $W$, and $P(w)(W(w))$ is bounded.

**Step 3:** For each bounded subset $B$ of $W(w)$ we have

$$\alpha(P(w)B) \leq \ell\alpha(B).$$

Let $w \in \Omega$ be fixed. From Lemmas 1.4.1 and 1.4.2, for any $B \subset W$ and any

$\epsilon > 0$, there exists a sequence $\{I_n\}_{n=0}^{\infty} \subset B$, such that for all $(x,y) \in J$, we have

$$
\begin{aligned}
& \alpha(P(w)B)(x,y)) \\
= \; & \alpha\Bigg(\Bigg\{ \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \\
& \times f(s,t,\overline{I}_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))} + v_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))}, w)dtds;\; I \in B\Bigg\}\Bigg) \\
\leq \; & 2\alpha\Bigg(\Bigg\{ \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \\
& \times f(s,t,\overline{I}_{n(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))} + v_{n(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w))}, w)dtds\Bigg\}_{n=1}^{\infty}\Bigg) + \epsilon \\
\leq \; & 4\int_0^x \int_0^y \alpha\left(\left\{ \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,\overline{I}_{n(s,t)} + v_{n(s,t)}, w)\right\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq \; & 4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha\left(\{f(s,t,\overline{I}_{n(s,t)} + v_{n(s,t)}, w)\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq \; & 4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)\alpha\left(\{\overline{I}_{n(s,t)} + v_{n(s,t)}\}_{n=1}^{\infty}\right) dtds + \epsilon \\
\leq \; & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dsdt\right) \alpha\left(\{u_{n(s,t)}\}_{n=1}^{\infty}\right) + \epsilon \\
\leq \; & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dsdt\right) \alpha\left(\{u_n\}_{n=1}^{\infty}\right) + \epsilon \\
\leq \; & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dsdt\right) \alpha\left(\{I_n\}_{n=1}^{\infty}\right) + \epsilon \\
\leq \; & \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dtds\right) \alpha(B) + \epsilon \\
\leq \; & \frac{4p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\alpha(B) + \epsilon \\
& = \ell\alpha(B) + \epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, then

$$\alpha(P(B)) \leq \ell\alpha(B).$$

It follows from Lemma 1.5.1 that for each $w \in \Omega$, $P$ has at least one fixed point in $W$. Since $\bigcap_{w\in\Omega} intW(w) \neq \emptyset$ the hypothesis that a measurable selector of $intW$ exists holds. By Lemma 1.5.1, $N$ has a stochastic fixed point, i.e., the problem (4.8)-(4.10) has at least one random solution on.

### 4.3.3 An Example

Let $E = \mathbb{R}$, $\Omega = (-\infty, 0)$ be equipped with the usual $\sigma$-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Consider the following partial functional random differential equation of the form

$$({}^cD_0^r u)(x,y) = \frac{ce^{x+y-\gamma(x+y)}}{e^{x+y}+e^{-x-y}}$$

$$\times \frac{|u(x-\sigma_1(u(x,y)), y-\sigma_2(u(x,y)))|}{1+w^2+|u(x-\sigma_1(u(x,y)), y-\sigma_2(u(x,y)))|}; \text{ a.a.}(x,y) \in [0,1]\times[0,1],\ w \in \Omega, \tag{4.13}$$

$$u(x,y,w) = x\sin w + y^2\cos w,\ (x,y) \in (-\infty,1]\times(-\infty,1]\backslash(0,1]\times(0,1],\ w \in \Omega, \tag{4.14}$$

$$u(x,0,w) = x\sin w;\ x \in [0,1],\ u(0,y,w) = y^2\cos w;\ y \in [0,1],\ w \in \Omega. \tag{4.15}$$

where $c = \frac{8}{\Gamma(r_1+1)\Gamma(r_2+1)}$ and $\gamma$ a positive real constant.
Let

$$\mathcal{B}_\gamma = \{u \in C((-\infty,0]\times(-\infty,0],\mathbb{R}) : \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u(\theta,\eta) \text{ exists in } \mathbb{R}\}.$$

The norm of $\mathcal{B}_\gamma$ is given by

$$\|u\|_\gamma = \sup_{(\theta,\eta)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(\theta+\eta)}|u(\theta,\eta)|.$$

Let

$$\mathcal{E} := [0,1]\times\{0\}\cup\{0\}\times[0,1],$$

and $u : (-\infty,1]\times(-\infty,1] \to \mathbb{R}$ such that $u_{(x,y)} \in \mathcal{B}_\gamma$ for $(x,y) \in \mathcal{E}$, then

$$\lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u_{(x,y)}(\theta,\eta) = \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta-x+\eta-y)}u(\theta,\eta)$$

$$= e^{-\gamma(x+y)} \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u(\theta,\eta) < \infty.$$

Hence $u_{(x,y)} \in \mathcal{B}_\gamma$. Finally we prove that

$$\|u_{(x,y)}\|_\gamma = K\sup\{|u(s,t)| : (s,t) \in [0,x]\times[0,y]\} + M\sup\{\|u_{(s,t)}\|_\gamma : (s,t) \in \mathcal{E}_{(x,y)}\},$$

where $K = M = 1$ and $H = 1$.
If $x+\theta \leq 0,\ y+\eta \leq 0$ we get

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t) \in (-\infty,0]\times(-\infty,0]\},$$

and if $x+\theta \geq 0,\ y+\eta \geq 0$ then we have

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t) \in [0,x] \times [0,y]\}.$$

Thus for all $(x+\theta, y+\eta) \in [0,1] \times [0,1]$, we get

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t) \in (-\infty,0]\times(-\infty,0]\}+\sup\{|u(s,t)| : (s,t) \in [0,x]\times[0,y]\}.$$

Then

$$\|u_{(x,y)}\|_\gamma = \sup\{\|u_{(s,t)}\|_\gamma : (s,t) \in \mathcal{E}\} + \sup\{|u(s,t)| : (s,t) \in [0,x] \times [0,y]\}.$$

$(\mathcal{B}_\gamma, \|.\|_\gamma)$ is a Banach space. We conclude that $\mathcal{B}_\gamma$ is a phase space.Set

$$\rho_1(x,y,\varphi,w) = x - \sigma_1(\varphi(0,0,w)),$$

$$\rho_2(x,y,\varphi,w) = y - \sigma_2(\varphi(0,0,w)),$$

where $(x,y) \in J, \varphi(\cdot,\cdot,w) \in \mathcal{B}_\gamma, w \in \Omega$,

$$f(x,y,\varphi,w) = \frac{ce^{x+y-\gamma(x+y)}|\varphi|}{(e^{x+y}+e^{-x-y})(1+w^2+|\varphi|)},\ (x,y) \in [0,1]\times[0,1],\ \varphi \in \mathcal{B}_\gamma,\ w \in \Omega.$$

The functions $w \mapsto \varphi(x,0,w) = x \sin w$ , $w \mapsto \psi(0,y,w) = y^2 \cos w$
and $w \mapsto \Phi(x,y,w) = x \sin w + y^2 \cos w$ are measurable and bounded with

$$|\varphi(x,0,w)| \leq 1,\ |\psi(0,y,w)| \leq 1,$$

hence, the conditions $(H_1)$ is satisfied.
Clearly, the map $(x,y,w) \mapsto f(x,y,u,w)$ is jointly continuous for all $u \in \mathcal{B}_\gamma$ and hence jointly measurable for all $u \in \mathcal{B}_\gamma$. Also the map $u \mapsto f(x,y,u,w)$ is continuous for all $(x,y) \in J$ and $w \in \Omega$. So the function $f$ is Carathéodory on $[0,1] \times [0,1] \times \mathcal{B}_\gamma \times \Omega$.
For each $u \in \mathcal{B}_\gamma,\ (x,y) \in [0,1] \times [0,1]$ and $w \in \Omega$, we have

$$|f(x,y,u_{(x,y)})| \ \leq \ 1 + \frac{1}{c}\|u\|_B.$$

Hence the conditions $(H_4)$ is satisfied with $p_1(x,y,w) = p_1^* = 1$ and $p_2(x,y,w) = p_2^* = \frac{1}{c}$.
Also, the conditions $(H_5)$ is satisfied.

We shall show that condition $\ell < 1$ holds with $a = b = 1$. For each $(r_1, r_2) \in (0, 1] \times (0, 1]$ we get

$$
\begin{aligned}
\ell &= \frac{4p_2^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
&= \frac{4}{c\Gamma(1+r_1)\Gamma(1+r_2)} \\
&< \frac{1}{2} \\
&< 1.
\end{aligned}
$$

Consequently, Theorem $4.3.1$ implies that the problem $(4.13) - (4.15)$ has a random solution defined on $(-\infty, 1] \times (-\infty, 1]$.

# Chapter 5

# Random Impulsive Partial Hyperbolic Fractional Differential Equations

## 5.1 Introduction

In this chapter, we discuss the existence of random solutions for the following impulsive partial fractional random differential equations:

$$\begin{cases} {}^cD^r_{x_k}u(x,y,w) = f(x,y,u(x,y,w),w); \text{ if } (x,y) \in J_k,\ k = 0,\dots,m, w \in \Omega, \\ u(x_k^+,y,w) = u(x_k^-,y,w) + I_k(u(x_k^-,y,w)); \text{ if } y \in [0,b],\ k = 1,\dots,m, w \in \Omega, \\ u(x,0,w) = \varphi(x,w);\ x \in [0,a], w \in \Omega, \\ u(0,y,w) = \psi(y,w);\ y \in [0,b], w \in \Omega, \\ \varphi(0,w) = \psi(0,w), \end{cases} \tag{5.1}$$

where $J_0 = [0,x_1] \times [0,b],\ J_k := (x_k,x_{k+1}] \times [0,b];\ k = 1,\dots,m,\ a,b > 0,\ \theta_k = (x_k,0);\ k = 0,\dots,m,\ {}^cD^r_{x_k}$ is the fractional Caputo derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1],\ 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = a, (\Omega,\mathcal{A})$ is a measurable space, $f : J \times E \times \Omega \to E;\ I_k : E \to E;\ k = 1,\dots,m$ are given continuous functions, $\varphi : [0,a] \times \Omega \to E$ and $\psi : [0,b] \times \Omega \to E$ are given absolutely continuous functions. Here $u(x_k^+,y,w)$ and $u(x_k^-,y,w)$ denote the right and left limits of $u(x,y,w)$ at $x = x_k$, respectively.

## 5.2 Existence Results

Let the space

$$\begin{aligned} PC = \big\{ & u : J \to E : u \in C(J_k);\ k = 0, 1, \dots, m, \text{ and there} \\ & \text{exist } u(x_k^-, y) \text{ and } u(x_k^+, y);\ k = 1, \dots, m, \\ & \text{with } u(x_k^-, y) = u(x_k, y) \text{ for each } y \in [0, b] \big\}. \end{aligned}$$

We need the following auxiliary lemma.

**Lemma 5.2.1.** *[8] Let $0 < r_1, r_2 \le 1,\ \mu(x, y) = \varphi(x) + \psi(y) - \varphi(0)$ and let $f : J \times E \to E$ be continuous. A function $u \in PC(J)$ is a solution of the fractional integral equation*

$$u(x, y) = \begin{cases} \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} f(s, t, u(s, t)) dt ds; \\ \textit{if } (x, y) \in [0, x_1] \times [0, b], \\ \mu(x, y) + \sum_{i=1}^k (I_i(u(x_i^-, y)) - I_i(u(x_i^-, 0))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1 - 1}(y - t)^{r_2 - 1} f(s, t, u(s, t)) dt ds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} f(s, t, u(s, t)) dt ds; \\ \textit{if } (x, y) \in (x_k, x_{k+1}] \times [0, b],\ k = 1, \dots, m, \end{cases}$$

*if and only if $u$ is a solution of the problem*

$$\begin{cases} {}^cD_{x_k}^r u(x, y) = f(x, y, u(x, y)); & \textit{if } (x, y) \in J_k,\ k = 0, \dots, m, \\ u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)); & \textit{if } y \in [0, b],\ k = 1, \dots, m, \\ u(x, 0) = \varphi(x);\ x \in [0, a], \\ u(0, y) = \psi(y);\ y \in [0, b], \\ \varphi(0) = \psi(0). \end{cases}$$

Consider the space

$$\begin{aligned} PC = PC(J \times \Omega) = \big\{ & u : J \times \Omega \to E : u(\cdot, \cdot, w) \text{ is continuous on } J_k; \\ & k = 0, 1, \dots, m, \text{ and there exist } u(x_k^-, y, w) \\ & \text{and } u(x_k^+, y, w);\ k = 1, \dots, m, \text{ with} \\ & u(x_k^-, y, w) = u(x_k, y, w) \text{ for each } y \in [0, b], w \in \Omega \big\}. \end{aligned}$$

This set is a Banach space with the norm

$$\|u\|_{PC} = \sup_{(x,y) \in J} \|u(x, y, w)\|_E.$$

we have the following lemma.

**Lemma 5.2.2.** *Let* $0 < r_1, r_2 \leq 1,\ \mu(x,y,w) = \varphi(x,w) + \psi(y,w) - \varphi(0,w)$. *A function* $u \in PC$ *is a solution of the random fractional integral equation*

$$u(x,y,w) = \begin{cases} \mu(x,y,w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds; \\ \text{if } (x,y) \in [0,x_1] \times [0,b],\ w \in \Omega, \\ \mu(x,y,w) + \sum_{i=1}^k (I_i(u(x_i^-,y,w)) - I_i(u(x_i^-,0,w))) \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds; \\ \text{if } (x,y) \in (x_k,x_{k+1}] \times [0,b],\ k = 1,\dots,m,\ w \in \Omega, \end{cases} \tag{5.2}$$

*if and only if* $u$ *is a solution of the random problem (5.1).*

The following hypotheses will be used in the sequel.

$(H_1)$ The functions $w \mapsto \varphi(x,0,w)$ *and* $w \mapsto \psi(0,y,w)$ are measurable and bounded for almost each $x \in [0,a]$ and $y \in [0,b]$ respectively,

$(H_2)$ The function $f$ is random Carathéeodory on $J \times E \times \Omega$,

$(H_3)$ There exist functions $p_1, p_2, p_3 : J \times \Omega \to [0,\infty)$ with $p_i(\cdot,w) \in L^\infty(J,[0,\infty));\ i = 1,2,3$ such that

$$\|f(x,y,u,w)\|_E \leq p_1(x,y,w) + p_2(x,y,w)\|u\|_E,$$

and

$$\|I_k(u)\|_E \leq p_3(x,y,w)\|u\|_E,$$

for all $u \in E$, $w \in \Omega$ and almost each $(x,y) \in J$,

$(H_4)$ For any bounded $B \subset E$,

$$\alpha(f(x,y,B,w)) \leq p_2(x,y,w)\alpha(B), \quad \text{for almost each } (x,y) \in J,$$

and

$$\alpha(I_k(B)) \leq p_3(x,y,w)\alpha(B), \text{ for almost each } (x,y) \in J.$$

Set

$$\mu^*(w) = \sup_{(x,y)\in J} \|\mu(x,y,w)\|_E,$$

$$p_i^*(w) = \sup ess_{(x,y)\in J} p_i(x,y,w);\ i = 1,2,3.$$

**Remark 5.2.1.** *Conditions* $(H_3)$ *and* $(H_4)$ *are equivalent [12].*

**Theorem 5.2.1.** *Assume that hypotheses* $(H_1)-(H_3)$ *hold. If*

$$\ell := 2mp_3^*(w) + \frac{4(m+1)p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1,$$

*then the problem (5.1) has a random solution defined on* $J$*.*

**Proof.** By Lemma 5.2.2, the problem (5.1) is equivalent to the integral equation

$$u(x,y,w) = \begin{cases} \mu(x,y,w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t,w),w)dtds; \\ \text{ if } (x,y)\in[0,x_1]\times[0,b],\ w\in\Omega, \\ \mu(x,y,w) + \sum_{i=1}^k (I_i(u(x_i^-,y,w)) - I_i(u(x_i^-,0,w))) \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t,w),w)dtds \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t,w),w)dtds; \\ \text{ if } (x,y)\in(x_k,x_{k+1}]\times[0,b],\ k=1,\dots,m,\ w\in\Omega, \end{cases}$$

for each $w\in\Omega$ and almost each $(x,y)\in J$.

Define the operator $N : PC \to PC$ by

$$\begin{aligned}(Nu)(x,y) &= \mu(x,y,w) + \sum_{i=1}^k (I_i(u(x_i^-,y,w)) - I_i(u(x_i^-,0,w))) \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t,w),w)dtds \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t,w),w)dtds.\end{aligned}$$

Since the functions $\varphi,\psi$ and $I_k$ and $f$ are absolutely continuous, the function $\mu$ and the indefinite integral are absolutely continuous for all $w\in\Omega$ and almost all $(x,y)\in J$. Again, as the maps $\mu$ and $I_k$ are continuous for all $w\in\Omega$ and the indefinite integral is continuous on $J$, then $N(w)$ defines a mapping $N : PC \to PC$. Hence $u$ is a solution for the problem (5.1) if and only if $u = Nu$. We shall show that the operator $N$ satisfies all conditions of Lemma 1.5.1. The proof will be given in several steps.

**Step 1:** $N$ is a random operator with stochastic domain on $PC$.
Since $f(x,y,u,w)$ is random Carathéodory, the map $w \to f(x,y,u,w)$ is measurable in view of Definition 1.1.1.

Similarly, the product $(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t,w),w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions and $I_k$ is measurable. Therefore, the map

$$\begin{aligned} w \mapsto \mu(x,y,w) &+ \sum_{i=1}^{k}(I_i(u(x_i^-,y,w)) - I_i(u(x_i^-,0,w))) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t,w),w)dtds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t,w),w)dtds \end{aligned}$$

is measurable. As a result, $N$ is a random operator from $PC$ into $PC$.

Let $W:\Omega\to\mathcal{P}(PC)$ be defined by

$$W(w) = \{u\in PC : \|u\|_{PC} \le R(w)\},$$

with $R(\cdot)$ being chosen appropriately. For instance, we assume that

$$R(w) \ge \frac{\mu^* + \frac{(m+1)p_1^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}{1 - 2mp_3^*(w) - (m+1)p_2^*(w)\frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}.$$

The set $W(w)$ is bounded, closed, convex and solid for all $w\in\Omega$. Then $W$ is measurable (Lemma 17 ( [27]). Let $w\in\Omega$ be fixed, then from $(H_4)$, for any

$u \in w(w)$, we get

$$
\begin{aligned}
& \|(Nu)(x,y)\|_E \\
\leq\ & \|\mu(x,y,w)\|_E + \sum_{i=1}^{k} \|I_i(u(x_i^-,y,w))\| + \|I_i(u(x_i^-,0,w))\| \\
+\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1-1}(y-t)^{r_2-1} \|f(s,t,u(s,t,w),w)\|_E dtds \\
+\ & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|f(s,t,u(s,t,w),w)\|_E dtds, \\
\leq\ & \|\mu(x,y,w)\|_E + \sum_{i=1}^{k} (p_3(x,y,w)\|u\| + (p_3(x_i,0,w))\|u\|) \\
+\ & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \left( \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1-1}(y-t)^{r_2-1} p_1(s,t,w) dtds \right. \\
+\ & \left. \int_{x_{i-1}}^{x_i} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_2(s,t,w) \|u(s,t,w)\|_E dtds \right) \\
+\ & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_1(s,t,w) dtds \\
+\ & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_2(s,t,w) \|u(s,t,w)\|_E dtds \\
\leq\ & \mu^*(w) + 2mp_3^*(w)R(w) \\
+\ & \sum_{i=1}^{k} \left( \frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_{i-1}}^{x_i} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \right. \\
+\ & \left. \frac{p_2^*(w)R(w)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_{i-1}}^{x_i} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \right) \\
+\ & \frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \\
+\ & \frac{p_2^*(w)R(w)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \\
\leq\ & \mu^*(w) + 2mp_3^*(w)R(w) + \frac{(p_1^*(w) + p_2^*(w)R(w))(m+1)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
\leq\ & R(w).
\end{aligned}
$$

Therefore, $N$ is a random operator with stochastic domain $W$ and $N : W(w) \to$

$W(w)$. Furthermore, $N$ maps bounded sets into bounded sets in $PC$.
**Step 2:** $N$ is continuous.
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC$. Then, for each $(x, y) \in J$ and $w \in \Omega$, we have

$$\begin{aligned}
& \|(Nu_n)(x,y) - (N(w)u)(x,y)\|_E \\
\leq & \sum_{i=1}^{k}(\|I_i(u_n(x_i^-, y, w)) - I_i(u(x_i^-, y, w))\| + \|I_i(u_n(x_i^-, 0, w)) - I_i(u(x_i^-, 0, w))\|) \\
+ & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1-1}(y-t)^{r_2-1} \\
\times & \|f(s,t,u_n(s,t,w),w) - f(s,t,u(s,t,w),w)\|_E dtds \\
+ & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x - s)^{r_1-1}(y-t)^{r_2-1} \\
\times & \|f(s,t,u_n(s,t,w),w) - f(s,t,u(s,t,w),w)\|_E dtds.
\end{aligned}$$

Using the Lebesgue dominated convergence theorem, we get

$$\|Nu_n - Nu\|_\infty \to 0 \text{ as } n \to \infty.$$

As a consequence of Steps 1 and 2, we can conclude that $N : W(w) \to W(w)$ is a continuous random operator with stochastic domain $W$, and $N(W(w))$ is bounded.
**Step 3:** For each bounded subset $B$ of $W(w)$ we have

$$\alpha(NB) \leq \ell\alpha(B).$$

Let $w \in \Omega$ be fixed. From Lemmas 1.4.1 and 1.4.2, for any $B \subset W$ and any $\epsilon > 0$, there exists a sequence $\{u_n\}_{n=0}^{\infty} \subset B$, such that for all $(x, y) \in J$, we have

$$
\begin{aligned}
&\alpha((NB)(x,y))\\
&= \alpha\left\{\mu(x,y,w) + \sum_{i=1}^{k}(I_i(u(x_i^-,y,w)) - I_i(u(x_i^-,0,w)))\right.\\
&+ \sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u(s,t,w),w)dtds\\
&\left.+ \int_{x_k}^{x}\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u(s,t,w),w)dtds;\ u\in B\right\}\\
\leq\ & \alpha\left\{\sum_{i=1}^{k}(I_i(u_n(x_i^-,y,w)) - I_i(u_n(x_i^-,0,w)))\right.\\
&+ \sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_n(s,t,w),w)dtds\\
&\left.+ \int_{x_k}^{x}\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_n(s,t,w),w)dtds\right\}_{n=1}^{\infty} + \epsilon\\
\leq\ & \alpha\left\{\sum_{i=1}^{k}(I_i(u_n(x_i^-,y,w)) - I_i(u_n(x_i^-,0,w)))\right\}_{n=1}^{\infty}\\
&+2\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha\{f(s,t,u_n(s,t,w),w)\}_{n=1}^{\infty}dtds\\
&+2\int_{x_k}^{x}\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha\{f(s,t,u_n(s,t,w),w)\}_{n=1}^{\infty}dtds + \epsilon\\
\leq\ & 2mp_3(x,y,w)\alpha\left(\{u_n(s,t,w)\}_{n=1}^{\infty}\right)\\
&+4\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)\alpha\left(\{u_n(s,t,w)\}_{n=1}^{\infty}\right)dtds\\
&+4\int_{x_k}^{x}\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)\alpha\left(\{u_n(s,t,w)\}_{n=1}^{\infty}\right)dtds + \epsilon\\
\leq\ & 2mp_3(x,y,w)\alpha\left(\{u_n\}_{n=1}^{\infty}\right)\\
&+ \left(4\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)\right)\alpha\left(\{u_n\}_{n=1}^{\infty}\right)dtds\\
&+ \left(4\int_{x_k}^{x}\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dsdt\right)\alpha\left(\{u_n\}_{n=1}^{\infty}\right) + \epsilon
\end{aligned}
$$

$$\begin{aligned}
&\leq \quad 2mp_3(x,y,w)\alpha(B)\\
&\quad + \left(4\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}p_2(s,t,w)dtds\right)\alpha(B)\\
&\quad + \left(4\int_{x_k}^{x}\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}p_2(s,t,w)dtds\right)\alpha(B)+\epsilon\\
&\leq \quad \left(2mp_3^*(w)+\frac{4(m+1)p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right)\alpha(B)+\epsilon\\
&\quad = \ell\alpha(B)+\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we haven

$$\alpha(N(B)) \leq \ell\alpha(B).$$

It follows from Lemma 1.5.1 that for each $w \in \Omega$, $N$ has at least one fixed point in $W$. Since $\bigcap_{w\in\Omega} intW(w) \neq \emptyset$, there exists a measurable selector of $intW$, thus $N$ has a stochastic fixed point, i.e., the problem (5.1) has at least one random solution.

## 5.3 An Example

Let $E = \mathbb{R}$, $\Omega = (-\infty, 0)$ be equipped with the usual $\sigma$-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Given a measurable function $u : \Omega \to AC([0,1]\times[0,1])$, consider the following impulsive partial fractional random differential equations of the form

$$\begin{cases} {}^cD^r_{x_k}u(x,y,w) = \frac{w^2e^{-x-y-3}}{1+w^2+5|u(x,y,w)|}; \ \text{if } (x,y)\in J_k, \ k=0,\dots,m,\\ u(x_k^+,y,w) = u(x_k^-,y,w)+\frac{w^2}{(1+w^2+10|u(x,y,w)|)e^{x+y+10}}; \ \text{if } y\in[0,1], \ k=1,\dots,m, \end{cases} \tag{5.3}$$

where $J = [0,1]\times[0,1]$, $w\in\Omega$, $(r_1,r_2)\in(0,1]\times(0,1]$ with the initial conditions

$$\begin{cases} u(x,0,w) = x\sin w; \ x\in[0,1],\\ u(0,y,w) = y^2\cos w; \ y\in[0,1]. \end{cases} \quad w\in\Omega, \tag{5.4}$$

Set

$$f(x,y,u(x,y,w),w) = \frac{w^2}{(1+w^2+5|u(x,y,w)|)e^{x+y+10}}, \ (x,y)\in[0,1]\times[0,1], \ w\in\Omega,$$

and

$$I_k(u(x_k^-, y, w)) = \frac{w^2}{(1 + w^2 + 10|u(x, y, w)|)e^{x+y+10}},\ y \in [0, 1],\ k = 1, \dots, m,\ w \in \Omega.$$

The functions $w \mapsto \varphi(x, 0, w) = x \sin w$ and $w \mapsto \psi(0, y, w) = y^2 \cos w$ are measurable and bounded with

$$|\varphi(x, 0, w)| \leq 1,\ |\psi(0, y, w)| \leq 1,$$

hence, the condition $(H_1)$ is satisfied.
Clearly, the map $(x, y, w) \mapsto f(x, y, u, w)$ is jointly continuous for all $u \in \mathbb{R}$ and hence jointly measurable for all $u \in \mathbb{R}$. Also the map $u \mapsto f(x, y, u, w)$ is continuous for all $(x, y) \in J$ and $w \in \Omega$. So the function $f$ is Carathéodory on $[0, 1] \times [0, 1] \times \mathbb{R} \times \Omega$.
For each $u \in \mathbb{R},\ (x, y) \in [0, 1] \times [0, 1]$ and $w \in \Omega$, we have

$$|f(x, y, u, w)| \leq 1 + \frac{5}{e^{10}}|u|,$$

and

$$|I_k(u)| \leq \frac{10}{e^{10}}|u|.$$

Hence the condition $(H_4)$ is satisfied with

$$p_1(x, y, w) = p_1^*(w) = 1,\ p_2(x, y, w) = p_2^*(w) = \frac{5}{e^{10}},\ p_3(x, y, w) = p_3^*(w) = \frac{10}{e^{10}}.$$

We shall show that condition $\ell < 1$ holds with $a = b = 1$. Indeed, if we assume, for instance, that the number of impulses $m = 3$, then we have

$$\begin{aligned} \ell &= 2mp_3^*(w) + \frac{4(m+1)p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &= \frac{60}{e^{10}} + \frac{80}{e^{10}\Gamma(1+r_1)\Gamma(1+r_2)} \\ &< 1, \end{aligned}$$

which is satisfied for each $(r_1, r_2) \in (0, 1] \times (0, 1]$. Consequently, Theorem 5.2.1 implies that the problem (5.3)-(5.4) has a random solution defined on $[0, 1] \times [0, 1]$.

# Conclusion and Perspective

In this thesis, we have considered the problem of existence results of existence of random solutions for the fractional partial random differential equations in Banach spaces . Some equations present delay which may be finite, infinite, or state-dependent. Our results will be obtained by means the measure of noncompactness and a random fixed point theorem with stochastic domain.

We project to look for similar problems in the case when the impulses are variable, that is they depend on the state variable.

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