*N° d'ordre :*

REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
ACULTE DES SCIENCES EXACTES
SIDI BEL ABBES

# THESE
# DE DOCTORAT

*Présenté par*

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*Spécialité : MATHEMATIQUES*
*Option : EQUATIONS AUX DERIVEES PARTIELLES*

*Intitulé*

## Existence globale et stabilisation des équations aux dérivées partielles

*Soutenu le : 29-11-2016*
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# Contents

## Notations

$\Omega$: Bounded domain in $\mathbb{R}^N$.
$\Gamma$: Topological boundary of $\Omega$.
$x = (x_1, x_1, ..., x_N)$:Generic point of $\mathbb{R}^N$.
$dx = dx_1 dx_1 ... dx_N$: Lebesgue measuring on $\Omega$.
$\nabla u$: Gradient of $u$.
$\Delta u$: Laplacien of $u$.
$f^+, f^-$ : $\max(f,0)$, $\max(-f,0)$.
a.e: Almost everywhere.
$p'$: Conjugate of $p$, i.e $\frac{1}{p} + \frac{1}{p'} = 1$.
$D(\Omega)$: Space of differentiable functions with compact support in $\Omega$.
$D'(\Omega)$: Distribution space.
$C^k(\Omega)$: Space of functions k-times continuously differentiable in $\Omega$.
$C_0(\Omega)$: Space of continuous functions null board in $\Omega$.
$L^p(\Omega)$: Space of functions p-th power integrated on $\Omega$ with measure of $dx$.
$\|f\|_p = \left( \int_\Omega |f(x)|^p \right)^{\frac{1}{p}}$.
$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega), \nabla u \in (L^p(\Omega))^N \right\}$.
$W^{1,p}(\Omega)$: The closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.
$\|u\|_{1,p} = \left( \|u\|_p^p + \|\nabla u\|_p^p \right)^{\frac{1}{p}}$.
$W_0^{1,p}(\Omega)$: The closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.
$W_0^{-1,p'}(\Omega)$: The dual space of $W_0^{1,p}(\Omega)$.
H: Hilbert space.
$H_0^1 = W_0^{1,2}(\Omega)$.
If X is a Banach space
$L^p(0,T;X) = \left\{ f : (0,T) \to X \text{is measurable}; \int_0^T \|f(t)\|_X^p dt < \infty \right\}$.
$L^\infty(0,T;X) = \left\{ f : (0,T) \to X \text{is measurable}; ess - \sup_{t \in (0,T)} \|f(t)\|_X^p \right\}$.
$C^k([0,T];X)$: Space of functions k-times continuously differentiable for $[0,T] \to X$.
$D([0,T];X)$: Space of functions continuously differentiable with compact support in $[0,T]$.
$B_X = \{x \in X; \|x\| \leq 1\}$: unit ball.

# REMERCIEMENTS

Tout le mérite de ce travail revient aux Professeurs, Monsieurs Ali Hakem et Abbès Benaissa en qui je reconnais maîtrise et competence dans ce domaine d'EDP, leur habilité, clairvoyance font d'eux des encadreurs de référence.
Des années de travail laborieux, des discussions fructueuses m'ont été très bénéfiques sur autant de domaines que l'analyse fonctionnelle, les EDP, la stabilisation des systèmes distribués,...ainsi trouve t-il mes expressions de remerciements et de gratitude les plus sincères.

Mes remerciements vont aussi à Messieur Abbès Benaissa de l'honneur qu'il ma fait en président le jury de cette thèse ainsi qu'a Messieurs, K.Belghaba, Abdeli Mama et Mechab Mustapha , pour avoir accépté de faire partie de mon jury. Je les remercie pour l'intérêt qu'ils ont porté à mon travail.

Je tiens, à exprimer mes remerciements les plus chaleureux à tous mes collègues du département de mathématiques et informatique de Université d'Oran

En dernier lieu, je tiens à remercier vivement toute ma famille pour leur soutien constant. Je tiens à exprimer toute ma redevance a mes parents, mes frères et soeurs pour leurs soutiens durant ces années. Je dédie à vous tous ce modeste travail.

Merci à toute personne ayant contibué à l'aboutissement de ce travail.

# Introduction

### Stabilisation of somme class of partial differential equations

This thesis is devoted to the study of global existence , asymptotic behavior in time of solutions to nonlinear evolutions equations and systems of hyperbolic, parabolic type . The decreasing of classical energy plays a crucial role in the study of global existence and in stabilisation of various systems.
In this thesis, the main objective is to give a global existence and stabilisation results. This work consists in four chapter, the first one for wave equations with delay term.
The second one, for wave equations with a dynamic boundary and time varying delay term in the presence of a memory term.
The third one, we deal with bresse system with delay terms and infinite memories.
The last one is devoted to quasilinear parabolic system with viscoelastic term.
The purpose of stabilisation is to attenuate the vibrations by feedlack, it consists to guarantee the decay of the energy of solutions towards 0 in away, more less fast.
More precisely, we are interested to determine the asymptotic behavior of the energy denoted by $E(t)$ and to give an estimate of the decay rate of the energy.
There are several type of stabilisation

1) Strong stabilization: $E(t) \to 0$, as $t \to \infty$.
2) Logarithmic stabilization: $E(t) \leq c(\log(t))^{-\delta}$, $\forall t > 0, (c, \delta > 0)$.
3) Polynomial stabilization: $E(t) \leq ct^{-\delta}$, $\forall t > 0, (c, \delta > 0)$
4) Uniform stabilization: $E(t) \leq ce^{-\delta t}$, $\forall t > 0, (c, \delta > 0)$.

# Chapter 1

# Preliminary

In this chapter we will introduce and state without proofs some important materials needed in the proof of our results,

## 1.1 Banach Spaces-Definition and properties

We first review some basic facts from calculus in the most important class of linear spaces" Banach spaces".

**Definition 1.1.1** . *A Banach space is a complete normed linear space X. Its dual space X' is the linear space of all continuous linear functional* $f : X \to \mathbb{R}$.

**Proposition 1.1.1** *([34]) $X'$ equipped with the norm $\|.\|_{X'}$ defined by*

$$\|f\|_{X'} = \sup\{|f(u)| : \|u\| \leq 1\}, \tag{1.1}$$

*is also a Banach space.*
*We shall denote the value of $f \in X'$ at $u \in X$ by either $f(u)$ or $\langle f, u\rangle_{X',X}$.*

**Remark 1.1.1** *From $X'$we construct the bidual or second dual $X'' = (X')'$. Furthermore, with each $u \in X$ we can define $\varphi(u) \in X''$ by $\varphi(u)(f) = f(u), f \in X'$, this satisfies clearly $\|\varphi(x)\| \leq \|u\|$. Moreover, for each $u \in X$ there is an $f \in X'$ with $f(u) = \|u\|$ and $\|f\| = 1$, so it follows that $\|\varphi(x)\| = \|u\|$.*

**Definition 1.1.2** . *Since $\varphi$ is linear we see that*

$$\varphi : X \to X'',$$

*is a linear isometry of $X$ onto a closed subspace of $X''$, we denote this by*

$$X \hookrightarrow X''.$$

**Definition 1.1.3** . *If $\varphi$ is onto $X''$ we say $X$ is reflexive, $X \cong X''$.*

**Theorem 1.1.1** *([81]). Let X be Banach space. Then, X is reflexive, if and only if,*

$$B_X = \{x \in X : \|x\| \leq 1\},$$

*is compact with the weak topology $\sigma(X, X')$. (See the next subsection for the definition of $\sigma(X, X')$).*

***Definition 1.1.4*** *. Let X be a Banach space, and let $(u_n)_{n\in N}$ be a sequence in X. Then $u_n$ converges strongly to u in X if and only if*

$$lim \|u_n - u\|_X = 0,$$

*and this is denoted by $u_n \to u$, or $\lim\limits_{n\to\infty} u_n = u$.*

**Definition 1.1.5** *The Banach space E is said to be separable if there exists a countable subset D of E which is dense in E, i.e. $\overline{D} = E$.*

**Proposition 1.1.2** *If E is reflexive and if F is a closed vector subspace of E, then F is reflexive.*

**Corollary 1.1.1** *The following two assertions are equivalent: (i) E is reflexive; (ii) $E'$ is reflexive.*

### 1.1.1 The weak and weak star topologies

Let X be a Banach space and $f \in X'$. Denote by

$$\begin{aligned} \varphi_f : X \to\ & \mathbb{R} \\ x \to\ & \varphi_f(x), \end{aligned} \tag{1.2}$$

when $f$ cover $X'$, we obtain a family $(\varphi_f)_{f\in X'}$ of applications to X in $\mathbb{R}$.

**Definition 1.1.6** *The weak topology on X, denoted by $\sigma(X, X')$, is the weakest topology on X for which every $(\varphi_f)_{f\in X'}$ is continuous.*

We will define the third topology on $X'$, the weak star topology, denoted by $\sigma(X', X)$. For all $x \in X$. Denote by

$$\begin{aligned} \varphi_f : X' \to\ & \mathbb{R} \\ f \to\ & \varphi_x(f) = \langle f, x\rangle_{X',X}, \end{aligned} \tag{1.3}$$

when $x$ cover X, we obtain a family $(\varphi_x)_{x\in X'}$ of applications to $X'$ in $\mathbb{R}$.

**Definition 1.1.7** *. The weak star topology on $X'$ is the weakest topology on $X'$ for which every $(\varphi_x)_{x\in X'}$ is continuous.*

**Remark 1.1.2** *([81]) Since $X \subset X''$, it is clear that, the weak star topology $\sigma(X', X)$ is weakest then the topology $\sigma(X', X'')$, and this later is weakest then the strong topology.*

**Definition 1.1.8** *A sequence $(u_n)$ in $X$ is weakly convergent to $x$ if and only if*

$$\underset{n\to\infty}{lim} f(u_n) = f(u),$$

*for every $f \in X'$, and this is denoted by $u_n \rightharpoonup u$*

**Remark 1.1.3** *([81])*

1. *If the weak limit exist, it is unique.*
2. *If $u_n \to u \in X(strongly)$, then $u_n \rightharpoonup u(weakly)$.*
3. *If $dimX < +\infty$, then the weak convergent implies the strong convergent.*

**Proposition 1.1.3** *On the compactness in the three topologies in the Banach space $X$:*

1. *First, the unit ball*
   $$B' \equiv \{x \in X : \|x\| \leq 1\}, \tag{1.4}$$
   *in $X$ is compact if and only if $dim(X) < \infty$.*
2. *Second, the unit ball $B'$ in $X'$( The closed subspace of a product of compact spaces) is weakly compact in $X'$ if and only if $X$ is reflexive.*
3. *Third, $B'$ is always weakly star compact in the weak star topology of $X'$.*

**Proposition 1.1.4** *([81]) Let $(f_n)$ be a sequence in $X'$. We have:*

1. $[f_n \rightharpoonup^* f \text{ in } \sigma(X', X)] \Leftrightarrow [f_n(x) \rightharpoonup^* f(x),\ \forall x \in X]$.
2. *If $f_n \to f(strongly)$, then $f_n \rightharpoonup f$, in $\sigma(X', X'')$,*
   *If $f_n \rightharpoonup f$ in $\sigma(X', X'')$, then $f_n \rightharpoonup^* f$, in $\sigma(X', X)$.*
3. *If $f_n \rightharpoonup^* f$ in $\sigma(X', X)$, then $\|f_n\|$ is bounded and $\|f\| \leq \liminf \|f_n\|$.*
4. *If $f_n \rightharpoonup^* f$ in $\sigma(X', X)$, and $x_n \to x(strongly)$ in $X$, then $f_n(x_n) \to f(x)$.*

### 1.1.2 Hilbert spaces

Now, we give some important results on these spaces here.

**Definition 1.1.9** *A Hilbert space H is a vectorial space supplied with inner product* $\langle u, \upsilon\rangle$ *such that* $\|u\| = \sqrt{\langle u, u\rangle}$ *is the norm which let H complete.*

**Theorem 1.1.2** *(Riesz). If* $(H; \langle ., .\rangle)$ *is a Hilbert space,* $\langle ., .\rangle$ *being a scalar product on H, then* $H' = H$ *in the following sense: to each* $f \in H'$ *there corresponds a unique* $x \in H$ *such that* $f = \langle x, .\rangle$ *and* $\|f\|'_H = \|x\|_H$.

**Remark 1.1.4** *: From this theorem we deduce that* $H'' = H$. *This means that a Hilbert space is reflexive.*

**Theorem 1.1.3** *([81]). Let* $(u_n)_{n\in N}$ *is a bounded sequence in the Hilbert space H, it posses a subsequence which converges in the weak topology of H.*

**Theorem 1.1.4** *([81]). In the Hilbert space, all sequence which converges in the weak topology is bounded.*

**Theorem 1.1.5** *([81]). Let* $(u_n)_{n\in N}$ *be a sequence which converges to u, in the weak topology and* $(\upsilon_n)_{n\in N}$ *is an other sequence which converge weakly to* $\upsilon$, *then*

$$\lim_{n\to\infty} \langle \upsilon_n, u_n\rangle = \langle \upsilon, u\rangle \tag{1.5}$$

**Theorem 1.1.6** *([81]). Let X be a normed space, then the unit ball*

$$B' \equiv \{x \in X : \|x\| \leq 1\}, \tag{1.6}$$

*of* $X'$ *is compact in* $\sigma(X', X)$.

## 1.2 Functional Spaces

### 1.2.1 The $L^p(\Omega)$ spaces

**Definition 1.2.1** *Let* $1 \leq p \leq \infty$, *and let* $\Omega$ *be an open domain in* $\mathbb{R}^n$, $n \in \mathbb{N}$ *Define the standard Lebesgue space* $L^p(\Omega)$, *by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |f(x)|^p dx < \infty \right\}. \tag{1.7}$$

**Notation 1.2.1** *For* $p \in \mathbb{R}$ *and* $1 \leq p < \infty$, *denote by*

If $p = \infty$, we have

$$\begin{cases} L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \text{ is measurable and there exists a constant C such that} \\ |f(x)| \leq C \; a.e \; in \; \Omega\} \end{cases}$$

Also, we denote by

$$\|f\|_\infty = \inf\{C, \; |f(x)| \leq C \; a.e \; in \; \Omega\}. \tag{1.8}$$

**Notation 1.2.2** *For $p \in \mathbb{R}$ and $1 \leq p \leq \infty$, we denote by $q$ the conjugate of $p$ i.e. $\frac{1}{p} + \frac{1}{q} = 1$.*

**Theorem 1.2.1** *([81]) $L^p(\Omega)$ is a Banach space, for all $1 \leq p \leq \infty$*

**Remark 1.2.1** *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx, \tag{1.9}$$

*is a Hilbert space.*

**Theorem 1.2.2** *([81]) For $1 < p < \infty$, $L^p(\Omega)$ is reflexive space.*

### 1.2.2 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

**Theorem 1.2.3** *([34], Holder's inequality) Let $1 \leq p \leq \infty$. Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then, $fg \in L^p(\Omega)$ and*

$$\int_\Omega |fg|dx \leq \|f\|_p\|g\|_q.$$

**Lemma 1.2.1** *([81], Young's inequality) Let $f \in L^p(\mathbb{R})$ and $g \in L^g(\mathbb{R})$ with $1 < p < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$. Then $f * g \in L^r(\mathbb{R})$ and*

$$\|f * g\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}.$$

**Lemma 1.2.2** *([81]) Let $1 \leq p \leq r \leq q$, $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$, and $1 \leq \alpha \leq 1$. Then*

$$\|u\|_{L^r} \leq \|u\|_{L^p}^\alpha \|u\|_{L^q}^{1-\alpha}.$$

**Lemma 1.2.3** *([81]) If $\mu(\Omega) < \infty$, $1 \leq p \leq q \leq \infty$, then $L^q \hookrightarrow L^p$, and*

$$\|u\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p}-\frac{1}{q}}\|u\|_{L^q}.$$

### 1.2.3 The $W^{m,p}(\Omega)$ spaces

**Proposition 1.2.1** *Let $\Omega$ be an open domain in $\mathbb{R}^N$. Then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that*

$$\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx, \text{ for all } \varphi \in D(\Omega),$$

*where $1 \leq p \leq \infty$, and it's well-known that $f$ is unique.*

Now, we will introduce the Sobolev spaces: The Sobolev space $W^{k,p}(\Omega)$ is defined to be the subset of $L^p$ such that function $f$ and its weak derivatives up to some order $k$ have a finite $L^p$ norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega).\ \ \forall \alpha; |\alpha| \leq k\}\ ,$$

With this definition, the Sobolev spaces admit a natural norm,

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}, \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^\infty(\Omega)}\ , \text{ for } p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\|\ .\ \|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the $H^k$ inner product is defined in terms of the $L^2$ inner product:

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)}\ .$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H^m_0(\Omega)$ (respectively $W^{k,p}_0(\Omega)$).
Now, we introduce a space of functions with values in a space $X$ (a separable Hilbert space). The space $L^2(a, b; X)$ is a Hilbert space for the inner product

$$(f, g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X\, dt$$

We note that $L^\infty(a,b;X) = (L^1(a,b;X))'$.
Now, we define the Sobolev spaces with values in a Hilbert space $X$
For $k \in \ N$, $p \in [1,\infty]$, we set:

$$W^{k,p}(a,b;X) = \left\{ v \in L^p(a,b;X); \frac{\partial v}{\partial x_i} \in L^p(a,b;X). \ \ \forall i \leq k \right\} \ ,$$

The Sobolev space $W^{k,p}(a,b;X)$ is a Banach space with the norm

$$\begin{aligned} \|f\|_{W^{k,p}(a,b;X)} &= \left( \sum_{i=0}^{k} \left\| \frac{\partial f}{\partial x_i} \right\|^p_{L^p(a,b;X)} \right)^{1/p}, \ \text{for } p < +\infty \\ \|f\|_{W^{k,\infty}(a,b;X)} &= \sum_{i=0}^{k} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^\infty(a,b;X)}, \qquad \text{for } p = +\infty \end{aligned}$$

The spaces $W^{k,2}(a,b;X)$ form a Hilbert space and it is noted $H^k(0,T;X)$. The $H^k(0,T;X)$ inner product is defined by:

$$(u,v)_{H^k(a,b;X)} = \sum_{i=0}^{k} \int_a^b \left( \frac{\partial u}{\partial x^i}, \frac{\partial v}{\partial x^i} \right)_X dt \ .$$

**Theorem 1.2.4** *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

*where $p^*$ is given by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ (where $p = n, p^* = \infty$). Moreover there exists a constant $C = C(p,n)$ such that*

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)} \forall u \in W^{1,p}(\mathbb{R}^n).$$

**Corollary 1.2.1** *Let $1 \leq p < n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [p, p^*]$$

*with continuous imbedding.*

For the case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [n, +\infty[$$

**Theorem 1.2.5** *Let $p > n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

*with continuous imbedding.*

**Corollary 1.2.2** *Let $\Omega$ a bounded domain in $\mathbb{R}^n$ of $C^1$ class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & 1 \leq p < \infty, \text{ then } W^{1,p}(\Omega) \subset L^{p^*}(\Omega) \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \\ if & p = n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n, \text{ then } W^{1,p}(\Omega) \subset L^\infty(\Omega) \end{array}$$

*with continuous imbedding.*
*Moreover, if $p > n$, we have: $\forall u \in W^{1,p}(\Omega)$,*

$$|u(x) - u(y)| \leq C|x - y|^\alpha \|u\|_{W^{1,p}(\Omega)} \ a.e\ x, y \in \Omega$$

with $\alpha = 1 - \frac{n}{p} > 0$ and $C$ is a constant which depend on $p, n$ and $\Omega$. In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

**Corollary 1.2.3** *Let $\Omega$ a bounded domain in $\mathbb{R}^n$ of $C^1$ class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & p < n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega) \forall q \in [1, p^*[ \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \\ if & p = n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n, \text{ then } W^{1,p}(\Omega) \subset C(\overline{\Omega}) \end{array}$$

*with compact imbedding.*

**Remark 1.2.2** *We remark in particular that*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

*with compact imbedding for $1 \leq p \leq \infty$ and for $p \leq q < p^*$.*

**Corollary 1.2.4**

$$\begin{array}{ll} if & \frac{1}{p} - \frac{m}{n} > 0, \text{ then } W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \text{ where } \frac{1}{q} = \frac{1}{p} - \frac{m}{n}. \\ if & \frac{1}{p} - \frac{m}{n} = 0, \text{ then } W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [p, +\infty[. \\ if & \frac{1}{p} - \frac{m}{n} < 0, \text{ then } W^{m,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \end{array}$$

*with continuous imbedding.*

**Lemma 1.2.4** *(Sobolev-poincarés inequality)*

$$If \quad 2 \leq q \leq \frac{2n}{n-2}, n \geq 3$$

$$q \geq 2, n = 1, 2,$$

*then*

$$\|u\|_q \leq C(q, \Omega)\|\nabla u\|_2,$$

*for all $u \in H_0^1(\Omega)$.*

**Remark 1.2.3** *For all $\varphi \in H^2(\Omega)$, $\Delta\varphi \in L^2(\Omega)$ and for $\Gamma$ sufficiently smooth, we have*

$$\|\varphi(t)\|_{H^2(\Omega)} \leq C\|\Delta\varphi(t)\|_{L^2(\Omega)}.$$

**Proposition 1.2.2** *([81], Green's formula) For all $u \in H^2(\Omega)$, $\upsilon \in H^1(\Omega)$ we have*

$$-\int_\Omega \Delta u \upsilon dx = \int_\Omega \nabla u \nabla \upsilon dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \upsilon d\sigma,$$

*where $\dfrac{\partial u}{\partial \eta}$ is a normal derivation of $u$ at $\Gamma$.*

### 1.2.4 The $L^p(0,T,X)$ spaces

Let X be a Banach space, denote by $L^p(0,T,X)$ the space of measurable functions

**Definition 1.2.2**

$$f : ]0,T[ \rightarrow X \qquad t \rightarrow f(t). \tag{1.10}$$

*such that*

$$\left(\int_0^T \|f(t)\|_X^p dt\right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, for\ 1 \leq p < \infty. \tag{1.11}$$

If $p = \infty$,

$$\|f\|_{L^p(0,T,X)} = \sup_{t\in]0,T[} ess\|f(t)\|_X. \tag{1.12}$$

**Theorem 1.2.6** *([81]) The space $L^p(0,T,X)$ is complete.*

We denote by $D'(0,T,X)$ the space of distributions in $]0,T[$ which take its values in X, and let us define

$$D'(0,T,X) = \pounds(D]0,T[,X),$$

where $\pounds(\phi,\varphi)$ is the space of the linear continuous applications of $\phi$ *to* $\varphi$. Since $u \in D'(0,T,X)$, we define the distribution derivation as

$$\frac{\partial u}{\partial t}(\varphi) = -u\left(\frac{d\varphi}{dt}\right), \quad \forall\varphi \in D(]0,T[),$$

and since $u \in L^p(0,T,X)$, we have

$$u(\varphi) = \int_0^T u(t)\varphi(t)dt, \quad \forall\varphi \in D(]0,T[),$$

We will introduce some basic results on the $L^p(0,T,X)$ space. These results, will be very useful in the other chapters of this thesis

**Lemma 1.2.5** *([81]) Let* $f \in L^p(0,T,X)$ *and* $\dfrac{\partial f}{\partial t} \in L^p(0,T,X)$, $(1 \leq p \leq \infty)$, *then, the function* $f$ *is continuous from* $[0,T]$ *to* $X$*. i.e.* $f \in C^1(0,T,X)$.

**Lemma 1.2.6** *([81]) Let* $\varphi = ]0,T[\times\Omega$ *an open bounded domain in* $\mathbb{R}\times\ \mathbb{R}^n$, *and let* $g_\mu$, $g$ *are two functions in* $L^q(]0,T[,L^q(\Omega))$, $1 < q < \infty$ *such that*

$$\|g_\mu\|_{L^q(]0,T[,L^q(\Omega))} \leq C, \forall \mu \in \mathbb{N} \tag{1.13}$$

*and*

$$g_\mu \to g \quad in \quad \varphi,$$

*then*

$$g_\mu \to g \quad in \quad L^q(\varphi).$$

**Theorem 1.2.7** *([81])* $L^p(0,T,X)$ *equipped with the norm* $\|.\|_{L^q(]0,T[,X)}$, $1 \leq p \leq \infty$ *is a Banach space.*

**Proposition 1.2.3** *([81]) Let* $X$ *be a reflexive Banach space,* $X'$ *it's dual, and* $1 \leq p,q < \infty$, $\frac{1}{p}+\frac{1}{q}=1$. *Then the dual of* $L^p(0,T,X)$ *is identify algebraically and topologically with* $L^q(0,T,X')$.

**Proposition 1.2.4** *([81]) Let* $X$, $Y$ *be to Banach space,*$X \subset Y$ *with continuous embedding, then we have*

$$L^p(0,T,X) \subset L^p(0,T,Y)$$

*with continuous embedding*

The following compactness criterion will be useful for nonlinear evolution problem, especially in the limit of the nonlinear terms.

**Proposition 1.2.5** *([82]) Let* $B_0, B, B_1$ *be Banach spaces with* $B_0 \subset B \subset B_1$, *assume that the embedding* $B_0 \hookrightarrow B$ *is compact and* $B \hookrightarrow B_1$ *are continuous. Let* $1 < p,q < \infty$, *assume further that* $B_0$ *and* $B_1$ *are reflexive.*
*Define*

$$W \equiv \{u \in L^p(0,T,B_0) : u' \in L^q(0,T,B_1)\}. \tag{1.14}$$

*Then, the embedding* $W \hookrightarrow L^p(0,T,B)$ *is compact.*

### 1.2.5 Some Algebraic inequalities

Since our study based on some known algebraic inequalities, we want to recall few of them here

**Lemma 1.2.7** *([81] The Cauchy-Schwartz inequality) Every inner product satisfies the Cauchy-Schwarz inequality*

$$\langle x_1, x_2 \rangle \leq \|x_1\| \|x_2\|. \tag{1.15}$$

*The equality sign holds if and only if $x_1$ and $x_1$ are dependent.*

**Lemma 1.2.8** *([81]Young's inequalities) For all $a, b \in \mathbb{R}^+$, we have*

$$ab \leq \alpha a^2 + \frac{1}{4\alpha} b^2 \tag{1.16}$$

*where $\alpha$ is any positive constant.*

**Lemma 1.2.9** *([81]) For $a, b \geq 0$, the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{1.17}$$

*where, $\frac{1}{p} + \frac{1}{q} = 1$.*

## 1.3 Integral Inequalities

We will recall some fundamental integral inequalities introduced by A. Haraux, V. Komornik and A.Guesmia to estimate the decay rate of the energy.

### 1.3.1 A result of exponential decay

The estimation of the energy decay for some dissipative problems is based on the following lemma:

**Lemma 1.3.1** *([20]) Let $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function and assume that there is a constant $A > 0$ such that*

$$\forall t \geq 0, \qquad \int_t^{+\infty} E(\tau)\, d\tau \leq \frac{1}{A} E(t). \tag{1.18}$$

*Then we have*

$$\forall t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}. \tag{1.19}$$

**Proof of Lemma 1.3.1**.
The inequality (1.19) is verified for $t \leq \frac{1}{A}$, this follows from the fact that $E$ is a decreasing function. We prove that (1.19) is verified for $t \geq \frac{1}{A}$. Introduce the function

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_t^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (1.18) we find that

$$\forall t \geq 0, \quad h'(t) + Ah(t) \leq 0.$$

Let

$$T_0 = \sup\{t,\ h(t) > 0\}. \tag{1.20}$$

For every $t < T_0$, we have

$$\frac{h'(t)}{h(t)} \leq -A,$$

thus

$$h(0) \leq e^{-At} \leq \frac{1}{A} E(0)\, e^{-At}, \quad \text{for} \quad 0 \leq t < T_0. \tag{1.21}$$

Since $h(t) = 0$ if $t \geq T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Let $\varepsilon > 0$. As $E$ is positive and decreasing, we deduce that

$$\forall t \geq \varepsilon, \quad E(t) \leq \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} E(\tau)\, d\tau \leq \frac{1}{\varepsilon} h(t-\varepsilon) \leq \frac{1}{A\varepsilon} E(0)\, e^{\varepsilon t}\, e^{-At}.$$

Choosing $\varepsilon = \frac{1}{A}$, we obtain

$$\forall t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}.$$

The proof of Lemma 1.3.1 is now completed.

### 1.3.2 A result of polynomial decay

**Lemma 1.3.2** *([20]) Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ ( $\mathbb{R}_+ = [0, +\infty)$) be a non-increasing function and assume that there are two constants $q > 0$ and $A > 0$ such that*

$$\forall t \geq 0, \quad \int_t^{+\infty} E^{q+1}(\tau)\, d\tau \leq \frac{1}{A} E^q(0) E(t). \tag{1.22}$$

*Then we have:*

$$\forall t \geq 0, \quad E(t) \leq E(0) \left( \frac{1+q}{1+A\,q\,t} \right)^{1/q}. \tag{1.23}$$

**Remark 1.3.1** It is clear that Lemma 1.3.1, is similar to Lemma 1.3.2 in the case of $q = 0$.

**Proof of Lemma 1.3.2**.
If $E(0)=0$, then $E\equiv 0$ and there is nothing to prove. Otherwise, replacing the function $E$ by the function $\frac{E}{E(0)}$ we may assume that $E(0)=1$.
Introduce the function

$$h:\mathbb{R}_+\longrightarrow\mathbb{R}_+,\qquad h(t)=\int_t^{+\infty}E(\tau)\,d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (1.22) we find that

$$\forall t\geq 0,\quad -h'\geq (Ah)^{1+q}.$$

where

$$T_0=\sup\{t,\ h(t)>0\}.$$

Integrating in $[0,t]$ we obtain that

$$\forall 0\leq t<T_0, h(t)^{-q}-h(0)^{-\sigma}\geq \sigma\omega^{1+q}t,$$

hence

(1.24) $$0\leq t<T_0,\quad h(t)\leq \left(h^{-q}(0)+qA^{1+q}\,t\right)^{-1/q}.$$

Since $h(t)=0$ if $t\geq T_0$, this inequality holds in fact for every $t\in\mathbb{R}_+$. Since

$$h(0)\leq\frac{1}{A}E(0)^{1+q}=\frac{1}{A},$$

by (1.22), the right-hand side of (1.24) is less than or equal to:

(1.25) $$\left(h^{-q}(0)+qA^{1+q}\,t\right)^{-1/q}\leq\frac{1}{A}(1+A\,q\,t)^{-1/q}.$$

From other hand, $E$ being nonnegative and non-increasing, we deduce from the definition of $h$ and the above estimate that:

$$\begin{aligned}\forall s\geq 0,\ E\left(\frac{1}{A}+(q+1)s\right)^{q+1} &\leq \frac{1}{\frac{1}{A}+q+1}\int_s^{\frac{1}{A}+(q+1)s}E(\tau)^{q+1}\,d\tau\\ &\leq \frac{A}{1+Aqs}h(s)\leq\frac{A}{1+Aqs}\frac{1}{A}(1+Aqs)^{-\frac{1}{q}},\end{aligned}$$

hence

$$\forall S\geq 0,\quad E\left(\frac{1}{A}+(q+1)S\right)\leq\frac{1}{(1+A\,q\,S)^{1/q}}.$$

Choosing $t=\frac{1}{A}+(1+q)s$ then the inequality (1.23) follows.Note that letting $q\to 0$ in this theorem we obtain (1.23).

### 1.3.3 New integral inequalities of P. Martinez

The above inequalities are verified only if the energy function is integrable, we will try to resolve this problem by introducing some weighted integral inequalities, so we can estimate the decay rate of the energy when it is slow.

**Lemma 1.3.3** *([20]) Let $E: \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ an increasing $C^1$ function such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{when} \quad t \to +\infty. \tag{1.26}$$

*Assume that there exist $q \geq 0$ and $A > 0$ such that*

$$\int_S^{+\infty} E(t)^{q+1}\phi'(t)\,dt \leq \frac{1}{A}\,E(0)^q E(S), \quad 0 \leq S < +\infty. \tag{1.27}$$

*then we have*

$$\text{if } q > 0, \quad \text{then } E(t) \leq E(0)\left(\frac{1+q}{1+q\,A\,\phi(t)}\right)^{\frac{1}{q}}, \ \ \forall t \geq 0,$$

$$\text{if } q = 0, \quad \text{then } E(t) \leq E(0)\,e^{1-A\,\phi(t)}, \ \ \forall t \geq 0.$$

**Proof of Lemma 1.3.3.**
This Lemma is a generalization of Lemma 1.3.1, Let $f: \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$, (we notice that $\phi^{-1}$ has a meaning by the hypotheses assumed on $\phi$). $f$ is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain $f$ is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain

$$\begin{aligned}
\int_{\phi(S)}^{\phi(T)} f(x)^{q+1}\,dx &= \int_{\phi(S)}^{\phi(T)} E\left(\phi^{-1}(x)\right)^{q+1}\,dx = \int_S^T E(t)^{q+1}\phi'(t)\,dt \\
&\leq \frac{1}{A}\,E(0)^q E(S) \\
&= \frac{1}{A}\,E(0)^q f(\phi(S)), \quad 0 \leq S < T < +\infty.
\end{aligned}$$

Setting $s := \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\forall s \geq 0, \quad \int_s^{+\infty} f(x)^{q+1}\,dx \leq \frac{1}{A}\,E(0)^q f(s).$$

Thanks to Lemma 1.3.1, we deduce the desired results.

### 1.3.4 Generalized inequalities of A. Guesmia

**Lemma 1.3.4** *([20]) Let $E: \mathbb{R}_+ \to \mathbb{R}_+$ differentiable function, $\lambda \in \mathbb{R}_+$ and $\Psi: \mathbb{R}_+ \to \mathbb{R}_+$ convex and increasing function such that $\Psi(0) = 0$. Assume that*

$$\int_s^{+\infty} \Psi(E(t))\,dt \le E(s), \quad \forall s \ge 0.$$

$$E'(t) \le \lambda E(t), \quad \forall t \ge 0.$$

*Then $E$ satisfies the estimate*

$$E(t) \le e^{\tau_0 \lambda T_0} d^{-1}\Big(e^{\lambda(t-h(t))}\Psi\Big(\psi^{-1}\Big(h(t)+\psi(E(0))\Big)\Big)\Big), \quad \forall t \ge 0,$$

*where*

$$\psi(t) = \int_t^1 \frac{1}{\Psi(s)}\,ds, \quad \forall t > 0,$$

$$d(t) = \begin{cases} \Psi(t) & if \quad \lambda = 0, \\ \displaystyle\int_0^t \frac{\Psi(s)}{s}\,ds & if \quad \lambda > 0, \end{cases} \quad \forall t \ge 0,$$

$$h(t) = \begin{cases} K^{-1}(D(t)), & if \quad t > T_0, \\ 0 & if \quad t \in [0, T_0], \end{cases}$$

$$K(t) = D(t) + \frac{\psi^{-1}(t+\psi(E(0)))}{\Psi(\psi^{-1}(t+\psi(E(0))))}\, e^{\lambda t}, \quad \forall t \ge 0,$$

$$D(t) = \int_0^t e^{\lambda s}\,ds, \quad \forall t \ge 0,$$

$$T_0 = D^{-1}\Big(\frac{E(0)}{\Psi(E(0))}\Big), \quad \tau_0 = \begin{cases} 0, & if \quad t > T_0, \\ 1, & if \quad t \in [0, T_0]. \end{cases}$$

**Remark 1.3.2** If $\lambda = 0$ (that is $E$ is non increasing), then we have

$$E(t) \le \psi^{-1}\Big(h(t)+\psi(E(0))\Big), \quad \forall t \ge 0 \tag{1.28}$$

where $\psi(t) = \displaystyle\int_t^1 \frac{1}{\Psi(s)}, ds$ for $t > 0$, $h(t) = 0$ for $0 \le t \le \dfrac{E(0)}{\Psi(E(0))}$ and

$$h^{-1}(t) = t + \frac{\psi^{-1}\Big(t+\psi(E(0))\Big)}{\Psi\Big(\psi^{-1}\Big(t+\psi(E(0))\Big)\Big)}, \quad t > 0.$$

This particular result generalizes the one obtained by Martinez ([20]) in the particular case of
$\Psi(t) = dt^{p+1}$ with $p \ge 0$ and $d > 0$, and improves the one obtained by Eller, Lagnese and Nicaise ([32]).

**Proof of Lemma 1.3.4**.
Because $E'(t) \leq \lambda E(t)$ imply $E(t) \leq e^{\lambda(t-t_0)}E(t_0)$ for all $t \geq t_0 \geq 0$, then, if $E(t_0) = 0$ for some $t_0 \geq 0$, then $E(t) = 0$ for all $t \geq t_0$, and then there is nothing to prove in this case. So we assume that $E(t) > 0$ for all $t \geq 0$ without loss of generality. Let:

$$L(s) = \int_s^{+\infty} \Psi(E(t))\, dt, \quad \forall s \geq 0.$$

We have, $L(s) \leq E(s)$, for all $s \geq 0$. The function $L$ is positive, decreasing and of class $C^1(\mathbb{R}_+)$ satisfying

$$-L'(s) = \Psi(E(s)) \geq \Psi(L(s)), \quad \forall s \geq 0.$$

The function $\psi$ is decreasing, then

$$\Big(\psi(L(s))\Big)' = \frac{-L'(s)}{\Psi(L(s))} \geq 1, \quad \forall s \geq 0.$$

Integration on $[0,t]$, we obtain

$$\psi(L(t)) \geq t + \psi(E(0)), \quad \forall t \geq 0. \tag{1.29}$$

Since $\Psi$ is convex and $\Psi(0) = 0$, we have

$$\Psi(s) \leq \Psi(1)s, \ \forall s \in [0,1] \quad \textit{and} \quad \Psi(s) \geq \Psi(1)s, \ \forall s \geq 1,$$

then $\lim\limits_{t\to 0} \psi(t) = +\infty$ and $[\psi(E(0)), +\infty[ \subset \mathit{Image}\,(\psi)$. Then (1.29) imply that

$$L(t) \leq \psi^{-1}\Big(t + \psi(E(0))\Big), \quad \forall t \geq 0. \tag{1.30}$$

Now, for $s \geq 0$, let

$$f_s(t) = e^{-\lambda t}\int_s^t e^{\lambda\tau}\, d\tau, \quad \forall t \geq s.$$

The function $f_s$ is increasing on $[s, +\infty[$ and strictly positive on $]s, +\infty[$ such that

$$f_s(s) = 0 \quad \textit{and} \quad f_s'(t) + \lambda f_s(t) = 1, \quad \forall t \geq s \geq 0,$$

and the function $d$ is well defined, positive and increasing such that:

$$d(t) \leq \Psi(t) \quad \textit{and} \quad \lambda t d'(t) = \lambda\Psi(t), \quad \forall t \geq 0,$$

then

$$\begin{aligned}
\partial_\tau\Big(f_s(\tau)d(E(\tau))\Big) &= f_s'(\tau)d(E(\tau)) + f_s(\tau)E'(\tau)d'(E(\tau)) \\
&\leq \Big(1 - \lambda f_s(\tau)\Big)\Psi(E(\tau)) + \lambda f_s(\tau)\Psi(E(\tau)) \\
&= \Psi(E(\tau)), \quad \forall \tau \geq s \geq 0.
\end{aligned}$$

Integrating on $[s,t]$, we obtain

$$L(s) \geq \int_s^t \Psi(E(\tau))\,d\tau \geq f_s(t)d(E(t)), \quad \forall t \geq s \geq 0. \tag{1.31}$$

Since $\underset{t\to+\infty}{lim} d(s) = +\infty$, $d(0)=0$ and $d$ is increasing, then (1.30) and (1.31) imply

$$E(t) \leq d^{-1}\left(\inf_{s\in[0,t[} \frac{\psi^{-1}\Big(s+\psi(E(0))\Big)}{f_s(t)}\right), \quad \forall t > 0. \tag{1.32}$$

Now, let $t > T_0$ and

$$J(s) = \frac{\psi^{-1}\Big(s+\psi(E(0))\Big)}{f_s(t)}, \quad \forall s \in [0,t[.$$

The function $J$ is differentiable and we have

$$J'(s) = f_s^{-2}(t)\Big[e^{-\lambda(t-s)}\psi^{-1}\Big(s+\psi(E(0))\Big) - f_s(t)\Psi\Big(\psi^{-1}\Big(s+\psi(E(0))\Big)\Big)\Big].$$

Then

$$J'(s) = 0 \;\Leftrightarrow\; K(s) = D(t) \quad and \quad J'(s) < 0 \;\Leftrightarrow\; K(s) < D(T).$$

Since $K(0) = \dfrac{E(0)}{\Psi(E(0))}$, $D(0)=0$ and $K$ and $D$ are increasing (because $\psi^{-1}$ is decreasing and $s \mapsto \dfrac{s}{\Psi(s)}$, $s > 0$, is non increasing thanks to the fact that $\Psi$ is convex). Then, for $t > T_0$,

$$\inf_{s\in[0,t[} J(s) = J\Big(K^{-1}(D(t))\Big) = J(h(t)).$$

Since $h$ satisfies $J'(h(t)) = 0$, we conclude from (1.32) our desired estimate for $t > T_0$.
For $t \in [0, T_0]$, we have just to note that $E'(t) \leq \lambda E(t)$ and the fact that $d \leq \Psi$ implies

$$E(t) \leq e^{\lambda t}E(0) \leq e^{\lambda T_0}E(0) \leq e^{\lambda T_0}\Psi^{-1}\Big(e^{\lambda t}\Psi(E(0))\Big) \leq e^{\lambda T_0}d^{-1}\Big(e^{\lambda t}\Psi(E(0))\Big).$$

**Remark 1.3.3** Under the hypotheses of Lemma 1.3.4, we have $\underset{t\to+\infty}{lim} E(t) = 0$. Indeed, we have just to choose $s = \frac{1}{2}t$ in (1.32) instead of $h(t)$ and note that $d^{-1}(0) = 0$, $\underset{t\to+\infty}{lim} \psi^{-1}(t) = 0$ and $\underset{t\to+\infty}{lim} f_{\frac{1}{2}t}(t) > 0$.

**Lemma 1.3.5** *([Guesmia 20]) Let $E : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a differentiable function , $a : \mathbb{R}^+ \longrightarrow \mathbb{R}^{+*}$ and $\lambda : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ two continuous functions .Assume that there exist $r \geq 0$ such that*

$$\int_s^{+\infty} E^{r+1}(t)dt \leq a(s)E(s), \forall s \geq 0 \tag{1.33}$$

$$E'(t) \leq \lambda(t)E(t), \quad \forall t \geq 0 \tag{1.34}$$

*Then E verifies ,for all* $t \geq 0$,

$$E(t) \leq \frac{E(0)}{\omega(0)}\omega(h(t)exp(\widetilde{\lambda}(t) - \widetilde{\lambda}(h(t)))exp(-\int_0^{h(t)} \omega(\tau)d\tau), \ \ if\ r = 0$$

*and*

$$E(t) \leq \omega(h(t))exp(\widetilde{\lambda}(t) - \widetilde{\lambda}(h(t))) \left[ \left( \frac{\omega(0)}{E(0)} \right)^r + r \int_0^{h(t)} \omega(\tau)^{r+1} d\tau \right]^{-1/r} \quad if\ r > 0$$

*where* $\widetilde{\lambda}(t) = \int_0^t \lambda(\tau)d\tau$

**Proof of Lemma 1.3.5**.
If $E(s) = 0$ or $a(s) = 0$ for one $s \geq 0$ ,the first inequality implies $E(t) = 0$ for $t \geq s$,we suppose then that $E(t) > 0$ and $a(t) > 0$ for $t \geq 0$
Put $\omega = \dfrac{1}{a}$ and $\Psi(s) = \int_s^{+\infty} E^{r+1}(t)dt$; we have

$$\Psi(s) \leq \frac{1}{\omega(s)} E(s), \qquad \forall s \geq 0. \tag{1.35}$$

the function $\Psi$ is decreasing ,positive and of class $C^1$ on $\mathbb{R}^+$ and verifies:

$$\Psi'(s) = -E^{r+1}(s) \leq -(\omega(s)\Psi(s))^{r+1}, \quad \forall s \geq 0$$

then

$$\Psi(s) \leq \Psi(0)exp\left(\int_0^s \omega(\tau)d\tau\right) \leq \frac{E(0)}{\omega(0)}exp\left(\int_0^s \omega(\tau)d\tau\right) \qquad \text{if } r = 0 \tag{1.36}$$

$$\Psi(s) \leq \left( (\frac{\omega(0)}{E(0)})^r + \int_0^s (\omega(\tau))^{r+1} d\tau \right)^{-1/r} \qquad \text{if } r > 0 \tag{1.37}$$

Now we put for all $s \geq 0$,

$$f_s(t) = exp(-(r+1)\widetilde{\lambda}(t)) \int_s^t exp((r+1)\widetilde{\lambda}(\tau))d\tau, \quad \forall t \geq s \tag{1.38}$$

where $f_s(s) = 0$ and $f_s'(t) + (r+1)\lambda(t)f_s(t) = 1$ , $\forall t \geq s \geq 0$.
Under the second hypothesis in the lemma, we deduce

$$E^{r+1}(t) \geq \partial_t(f_s(t)E^{r+1}(t)); \forall t \geq s \geq 0 \tag{1.39}$$

hence

$$\Psi(s) \geq \int_s^{g(s)} E^{r+1}(t) \geq f_s(g(s))E^{r+1}(g(s))); \forall s \geq 0 \tag{1.40}$$

where $g: \mathbb{R}^+ \longrightarrow \mathbb{R}^{+*}$ with $I_s(g(s)) = 0$ , $I_s$ is defined by

$$I_s(t) = (\omega(s))^{r+1} \int_s^t exp((r+1)\widetilde{\lambda}(\tau))d\tau$$

Let $t > g(0)$ and $s = h(t)$ with

$$h(t) = \begin{cases} 0, \text{ if } & t \in [0, g(0)] \\ \max g^{-1(t)} \text{ if } & t \in ]g(0), +\infty[ \end{cases}$$

Hence we have $g(s) = t$ and we deduce from (1.40 that, for all $t \geq g(0)$,

$$\Psi(h(t)) \geq f_{h(t)}(t)E^{r+1}(t) = \left( exp(-(r+1)\widetilde{\lambda}(t)) \int_{h(t)}^t exp((r+1)\widetilde{\lambda}(\tau))d\tau \right) E^{r+1}(t)$$

We conclude from (1.36 and (1.37 that ,for all $t > g(0)$,

$$E(t) \leq \frac{E(0)}{\omega(0)} exp(\widetilde{\lambda}(t)) \left( \int_{h(t)}^t exp(\widetilde{\lambda}(\tau))d\tau \right)^{-1} exp\left( -\int_0^{h(t)} \omega(\tau)d\tau \right) \text{ if } r = 0$$

and

$$E(t) \leq exp(\widetilde{\lambda}(t)) \left( \int_{h(t)}^t exp((r+1)\widetilde{\lambda}(\tau))d\tau \right)^{\frac{-1}{r+1}} \times$$
$$\left( (\tfrac{\omega(0)}{E(0)})^r + r\int_0^{h(t)} (\omega(\tau))^{r+1} d\tau \right)^{\frac{-1}{r(r+1)}} \text{ if } r > 0$$

The fact that $I_{h(t)}^t = I_s(g(s)) = 0$ , we obtain the result of the lemma for $t > g(0)$.
If $t \in [0, g(0)]$ the second inequality of the lemma implies that

$$E(t) \leq E(0)exp(\widetilde{\lambda}(t))$$

Since $h(t) = 0$ on $[0, g(0)]$, $E(0)exp(\widetilde{\lambda}(t))$ is identically equal to the left hand side of the results of the lemma.That conclude the proof.

**Lemma 1.3.6** *([Guesmia 20]) Let* $E: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *be a differentiable function* , $a_1, a_2 \in \mathbb{R}^{+*}$ *and* $a_3, \lambda, r, p \in \mathbb{R}^+$ *such that*

$$a_3\lambda(r+1) < 1$$

*and for all* $0 \leq s \leq T < +\infty$ ,

$$\int_s^T E^{r+1}(t)dt \leq a_1(s)E(s) + a_2E^{p+1}(s) + a_3E^{r+1}(T),$$

$$E'(t) \leq \lambda E(t), \quad \forall t \geq 0$$

*Then there exist two positive constants* $\omega$ *and* $c$ *such that ,for all* $t \geq 0$,

$$E(t) \leq ce^{-\omega t}, \ \textit{if}\ r = 0$$

$$E(t) \leq c(1+t)^{-1/r}, \ \textit{if}\ r > 0 \quad \textit{and}\ \lambda = 0$$

$$E(t) \leq c(1+t)^{\frac{-1}{r(r+1)}}, \ \textit{if}\ r > 0 \quad \textit{and}\ \lambda > 0$$

**Proof of Lemma 1.3.6**.
We show that $E$ verifies the inequality (1.33).Applying the lemma (1.3.5),we have

$$\begin{aligned} a_3 E^{r+1}(T) &= a_3 \int_s^T E'^{r+1}(t)dt + a_3 E^{r+1}(s) \\ &\leq a_3(r+1) \int_s^T \lambda E^{r+1}(t)dt + a_3 E^{r+1}(s) \end{aligned}$$

Under (1.33),we obtain:

$$\int_s^{+\infty} E^{r+1}(t)dt \leq b(s)E(s), \qquad \forall s \geq 0 \tag{1.41}$$

where

$$b(s) = \frac{a_1 + a_2 E^p(s) + a_3 E^r(s)}{1 - a_3\lambda(r+1)}, \qquad \forall s \geq 0$$

We consider the function $f_0$ defined in (1.38)and integrating on $[0, s]$ the inequality

$$E^{r+1}(t) \geq \partial_t(f_0(t)E^{r+1}(t)), \quad \forall t \geq 0$$

we obtain under (1.41)

$$b(0)E(0) \geq \int_0^s E^{r+1}(t)dt \geq f_0(s)E^{r+1}(s), \qquad \forall s \geq 0$$

then

$$E(s) \leq \left(\frac{b(0)E(0)}{f_0(s)}\right)^{\frac{1}{r+1}}, \qquad \forall s \geq 0$$

on the other hand, the conditions of the lemma implies that

$$E(s) \leq E(0)exp(\widetilde{\lambda}(s) \quad \forall s \geq 0$$

Hence

$$E(s) \leq \min\left\{E(0)exp(\widetilde{\lambda}(s), \left(\frac{b(0)E(0)}{f_0(s)}\right)^{\frac{1}{r+1}}\right\} = d(s) \quad \forall s \geq 0$$

$d$ is continuous and positive and

$$b(s) \leq \frac{a_1 + a_2(d(s))^p + a_3(d(s))^r}{1 - a_3\lambda(r+1)}, \qquad \forall s \geq 0$$

Hence we can conclude from (1.41) the first inequality (1.33) of the lemma (1.3.5) with

$$a(s) = \frac{a_1 + a_2(d(s))^p + a_3(d(s))^r}{1 - a_3\lambda(r+1)}, \qquad \forall s \geq 0.$$

This completes the proof.

## 1.4 Existence Methods

### 1.4.1 Faedo-Galerkin's approximations

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle ., . \rangle$ and the associated norm $\|.\|$ .

$$(P) \qquad \begin{cases} u''(t) + A(t)u(t) = f(t), & t \in [0,T] \\ (x,0) = u_0(x), \;\; u'(x,0) = u_1(x); \end{cases}$$

where $u$ and $f$ are unknown and given function, respectively, mapping the closed interval $[0,T] \subset \mathbb{R}$ into a real separable Hilbert space $H$ ,$A(t)$ $(0 \leq t \leq T)$ are linear bounded operators in $H$ acting in the energy space $V \subset H$.
Assume that $\langle A(t)u(t), v(t)\rangle = a(t; u(t), v(t))$ , for all $u, v \in V$; where $a(t; ., .)$ is a bilinear continuous in V.
The problem $(P)$ can be formulated as: Found the solution $u(t)$ such that

$$(\widetilde{P}) \qquad \begin{cases} u \in C([0,T]; V), u' \in C([0,T]; H) \\ \langle u''(t), v\rangle + a(t; u(t), v) = \langle f, v\rangle \text{ in } D'(]0,T[) \\ u_0 \in V \;,\;\; u_1 \in H; \end{cases}$$

This problem can be resolved with the approximation process of Fadeo-Galerkin.

Let $V_m$ a sub-space of $V$ with the finite dimension $d_m$, and let $\{w_{jm}\}$ one basis of $V_m$ such that .

1. $V_m \subset V(\dim V_m < \infty), \forall m \in \mathbb{N}$
2. $V_m \to V$ such that, there exist a dense subspace $\vartheta$ in $V$ and for all $\upsilon \in \vartheta$ we can get sequence $\{u_m\}_{m\in\mathbb{N}} \in V_m$ and $u_m \to u$ in V.
3. $V_m \subset V_{m+1}$ and $\overline{\cup_{m\in\mathbb{N}} V_m} = V$.

we define the solution $u_m$ of the approximate problem

$$(P_m)\qquad \begin{cases} u_m(t) = \sum_{j=1}^{d_m} g_j(t) w_{jm} \\ u_m \in C([0,T];V_m), u'_m \in C([0,T];V_m) \quad , u_m \in L^2(0,T;V_m) \\ \langle u''_m(t), w_{jm}\rangle + a(t;u_m(t), w_{jm}) = \langle f, w_{jm}\rangle, \quad 1 \le j \le d_m \\ u_m(0) = \sum_{j=1}^{d_m} \xi_j(t) w_{jm} \ , \ u'_m(0) = \sum_{j=1}^{d_m} \eta_j(t) w_{jm} \end{cases}$$

where

$$\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \longrightarrow u_0 \text{ in V as } m \longrightarrow \infty$$

$$\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \longrightarrow u_1 \text{ in V as } m \longrightarrow \infty$$

By virtue of the theory of ordinary differential equations,the system $(P_m)$ has unique local solution which is extend to a maximal interval $[0, t_m[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$, to obtain one solution defined for all $t > 0$.

### 1.4.2 A priori estimation and convergence

Using the following estimation

$$\|u_m\|^2 + \|u'_m\|^2 \le C \left\{ \|u_m(0)\|^2 + \|u'_m(0)\|^2 + \int_0^T \|f(s)\|^2 ds \right\} \ ; \quad 0 \le t \le T$$

and the Gronwall lemma we deduce that the solution $u_m$ of the approximate problem $(P_m)$ converges to the solution $u$ of the initial problem $(P)$.The uniqueness proves that $u$ is the solution.

### 1.4.3 Gronwall's lemma

**Lemma 1.4.1** *Let $T > 0$, $g \in L^1(0,T)$, $g \ge 0$ a.e and $c_1$, $c_2$ are positives constants.Let $\varphi \in L^1(0,T)$ $\varphi \ge 0$ a.e such that $g\varphi \in L^1(0,T)$ and*

$$\varphi(t) \le c_1 + c_2 \int_0^t g(s)\varphi(s) ds \quad a.e \ in \ (0,T).$$

*then, we have*

$$\varphi(t) \le c_1 exp \left( c_2 \int_0^t g(s) ds \right) \quad a.e \ in \ (0,T).$$

### 1.4.4 Semigroups approach

**Definition 1.4.1** *([83]). Let $X$ be a Banach space. A one parameter family $T(t)$ for $0 \leq t < \infty$ of bounded linear operators from $X$ into $X$ is a semigroup bounded linear operator on $X$ if*

- *$T(0) = I$, ($I$ is the identity operator on $X$).*
- *$T(t+s) = T(t).T(s)$ for every $t, s \geq 0$ (the semigroup property).*

*A semigroup of bounded linear operators, $T(t)$, is uniformly continuous if*

$$\lim_{t \to 0} \|T(t) - I\| = 0.$$

*The linear operator $A$ defined by*

$$D(A) = \left\{ x \in X; \lim_{t \to 0} \frac{T(t)x - x}{t} \quad exists \right\}$$

*and*

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt}|_{t=0} \ \forall x \in D(A)$$

*is the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is the domain of $A$.*

**Theorem 1.4.1** *([83])(Lumer-Phillips). Let A be a linear operator with dense domain D(A) in X*

- *If $A$ is dissipative and there is a $\lambda_0 > 0$ such that the range , $R(\lambda_0 I - A) = X$, then $A$ is the infinitesimal generator of a $C_0$ semigroup of contraction on $X$.*
- *If $A$ is the infinitesimal generator of a $C_0$ semigroup of contractions on $X$ then $R(\lambda_0 I - A) = X \ \forall \ \lambda > 0$ and $A$ is dissipative.*

**Remark 1.4.1** *In the last chapter we will give an example for ilustration*

# Chapter 2

# Global existence and asymptotic behavior for a coupled system of viscoelastic wave equations with a delay term

### 2.0.5 Introduction

In this chapter we consider a coupled system of viscoelastic wave equations with a delay term, firstly global existence of the solutions is proved by Faedo-Galerkin's procedure. Furthermore, we study the asymptotic behavior in using multiplier technique introduced by A. Guessmia [20]. We consider the following problem :

$$
\begin{cases}
u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g_1(t-s)\Delta u(s)ds \\
+\sum_{i=1}^{2} \mu_i u_t(x, t-\tau(i)) + f_1(u, \upsilon) = 0, \\
\upsilon_{tt} - \Delta \upsilon - \Delta \upsilon_{tt} + \int_0^t g_2(t-s)\Delta u(s)ds \\
+\sum_{i=1}^{2} \alpha_i \upsilon_t(x, t-\tau(i)) + f_2(u, \upsilon) = 0, \\
u(x,t) = 0,\ \upsilon(x,t) = 0 \qquad on \quad \Gamma \times (0, +\infty), \\
u(x,0) = u_0(x),\ \upsilon(x,0) = \upsilon_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \\
u_t(x, t-\tau(2)) = \phi_0(x, t-\tau(2)),\ x \in \Omega,\ t \in (0, \tau_2), \\
\upsilon_t(x, t-\tau(2)) = \phi_1(x, t-\tau(2)),\ x \in \Omega,\ t \in (0, \tau_2),\ \tau(1) = 0,\ \tau(2) = \tau_2.
\end{cases} \tag{2.1}
$$

Where $\Omega$ is a bounded domain in $R^n, n \in N^*$, with a smooth boundary $\partial\Omega$,and $g_1, g_2 : R^+ \to R^+$, $\phi_i(.,.) : R^2 \to R$ $i = 1, 2$, are given functions which will be specified later, $\tau_2 > 0$ is a time delay, where $\mu_1, \alpha_1, \alpha_2, \mu_2$ are positive real numbers and the initial data

$(u_0,u_1,f_0),(\upsilon_0,\upsilon_1,f_1)$ belonging to a suitable space. Problems of this type arise in material science and physics.
Recently, the authors of [2] considered the following coupled system of quasilinear viscoelastic equation in canonical form without delay terms

$$(2.2)\quad \begin{cases} |u_t|^\rho u_{tt} - \Delta u - \gamma_1 \Delta u_{tt} + \int_0^t g_1(t-s)\Delta u(s)ds + f_1(x,u) = 0, & in\ \Omega \times (0,+\infty), \\ |\upsilon_t|^\rho \upsilon_{tt} - \Delta \upsilon - \gamma_2 \Delta \upsilon_{tt} + \int_0^t g_2(t-s)\Delta \upsilon(s)ds + f_2(x,u) = 0, & in\ \Omega \times (0,+\infty), \end{cases}$$

where $\Omega$ is a bounded domain in $R^n$ $(n \geq 1)$with a smooth boundary $\partial\Omega, \gamma_1, \gamma_2 \geq 0$ are constants and $\rho$ is a real number such that $0 < \rho < \frac{2}{(n-2)}$ if $n \geq 3$ or $\rho > 0$ if $n = 1, 2$. The functions $u_0, u_1, \upsilon_0$ and $\upsilon_1$ are given initial data. The relaxations functions $g_1$ and $g_2$ are continuous functions and $f_1(u,\upsilon), f_2(u,\upsilon)$ represent the nonlinear terms. The authors proved the energy decay result using the perturbed energy method.
Many authors considered the initial boudary value problem as follows

$$(2.3)\quad \begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + h_1(u_t) = f_1(x,u), & in\ \Omega \times (0,+\infty), \\ \upsilon_{tt} - \Delta \upsilon + \int_0^t g_2(t-s)\Delta \upsilon(s)ds + h_2(\upsilon_t) = f_2(x,u), & in\ \Omega \times (0,+\infty), \end{cases}$$

when the viscoelastic terms $g_i$ $(i = 1, 2.)$ are not taken into account in (2.3) , Agre and Rammaha [4] obtained several results related to local and global existence of a weak solution, they showed that any weak solution blow-up in finite time with negative initial energy. Later Said-Houari [6] extended this blow up result to positive initial energy. Conversly, in the presence of the memory term ($g_i \neq 0$ $(i = 1, 2.)$, there are numerous results related to the asymptotic behavior and blow up of solutions of viscoelastic systems. For example, Liang and Gao [7] studied problem (2.3) with $h_1(u_t) = -\Delta u_t$, $h_2(\upsilon_t) = -\Delta u_t$. They obtained that, under suitable conditions on the functions $g_i, f_i, i = 1, 2$, and certain initial data in the stable set, the decay rate of the energy functions is exponential. On the contrary, for certain initial data in the unstable set, there are solutions with positive initial energy that blow-up in finite time. For $h_1(u_t) = |u_t|^{m-1}u_t$ and $h_2(\upsilon_t) = |\upsilon_t|^{r-1}\upsilon_t$. Hun and Wang [8] estabilished several results related to local existence, global existence and finite time blow-up ( the initial energy $E(0) < 0$). on the other hand, Messaoudi and Tatar [9] considered the following problem

$$(2.4)\quad \begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + f_1(x,u) = 0, & in\ \Omega \times (0,+\infty), \\ \upsilon_{tt} - \Delta \upsilon + \int_0^t g_2(t-s)\Delta \upsilon(s)ds + f_2(x,u) = 0, & in\ \Omega \times (0,+\infty), \end{cases}$$

where the functions $f_1$ *and* $f_2$ satisfy the following assumptions

$$\begin{cases} |f_1(u,v)| \leq d(|u|^{\beta_1} + |v|^{\beta_2}), \\ |f_2(u,v)| \leq d(|u|^{\beta_3} + |v|^{\beta_4}), \end{cases}$$

for some constant $d > 0$ and $\beta_i \geq 0, \beta_i \leq \frac{n}{(n-2)}$, $i = 1,2,3,4$. They obtained that the solution goes to zero with an exponential or polynomial rate, depending on the decay rate of the relaxation functions $g_i, i = 1,2$.
Muhammad I.M [26] considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + f_1(v,u) = 0, \quad in\ \Omega \times (0,+\infty), \\ v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)ds + f_2(v,u) = 0, \quad in\ \Omega \times (0,+\infty), \end{cases} \tag{2.5}$$

and proved the well-posedness and energy decay result for wider class of relaxation functions. In the present paper, we analyze the influence of the viscoelastic terms, damping terms and delay terms on the solutions to (2.1). Under suitable assumptions on the functions $g_i(.)$, $f_i(.,.)(i = 1.2)$, the initial data and the parameters in the equations, we establish several results concerning global existence , asymptotic behavior and boundedness of solutions to (2.1).

### 2.0.6 Preliminary Results

In this subsection, we present some material for the proof of our result. For the relaxation function $g_i$ we assume
$(A_1)$ : The relaxations functions $g_1$ and $g_2$ are of class $C^1$ and satisfy, for $s \geq 0$

$$\begin{cases} g_1(s) \geq 0,\ 1 - \int_0^\infty g_1(s)ds = l_1 > 0, \\ g_2(s) \geq 0,\ 1 - \int_0^\infty g_2(s)ds = l_2 > 0, \end{cases}$$

$$\begin{cases} g_1'(t) \leq -rg_1(t),\ \forall t \geq 0, \\ g_2'(t) \leq -rg_2(t),\ \forall t \geq 0, \end{cases}$$

and $g_2'(s) \leq 0$ and $g'(s) \leq 0$.
We take $f_1$,$f_2$ as in [10]

$$f_1(u,v) = a|u+v|^{p-1}(u+v) + b|u|^{\frac{p-3}{2}}|v|^{\frac{p+1}{2}}u, \tag{2.6}$$

$$f_2(u,v) = a|u+v|^{p-1}(u+v) + b|v|^{\frac{p-3}{2}}|u|^{\frac{p+1}{2}}v. \tag{2.7}$$

With $a, b > 0$ Further, one can easily verify that

$$uf_1(u,v) + vf_2(u,v) = (p+1)F(u,v), \forall (u,v) \in R^2.$$

Where

$$F(u,\upsilon)=\frac{1}{(p+1)}(a|u+\upsilon|^{p+1}+2b|u\upsilon|^{\frac{p+1}{2}}),\quad f_1(u,\upsilon)=\frac{\partial F}{\partial u},\quad f_2(u,\upsilon)=\frac{\partial F}{\partial \upsilon}.$$

$(A_2)$ : there exists $c_0, c_1 > 0$, such that

$$c_0(|u|^{p+1}+|\upsilon|^{p+1})\leq F(u,\upsilon)\leq c_1(|u|^{p+1}+|\upsilon|^{p+1}),\forall(u,\upsilon)\in R^2.$$

and

$$\left|\frac{\partial f_i}{\partial u}(u,\upsilon)\right|+\left|\frac{\partial f_i}{\partial \upsilon}(u,\upsilon)\right|\leq C(|u|^{p-1}+|\upsilon|^{p-1}),\quad i=1,2\quad where\quad 1\leq p<6$$

$(A_3)$ :

(2.8) $$if\ n=1,2;\quad p\geq 3\quad if\ n=3;\quad p=3$$

**Lemma 2.0.2** *(Sobolev-Poincaré inequality). Let* $2\leq m\leq\frac{2n}{n-2}$. *The inequality*

$$\|u\|_m\leq c_s\|\nabla u\|_2\quad for\quad u\in H_0^1(\Omega),$$

*holds with some positive constant* $c_s$.

**Lemma 2.0.3** *([11]). For any* $g\in C^1$ *and* $\phi\in H^1(0,T)$, *we have*

$$\int_0^t\int_\Omega g(t-s)\varphi\varphi_t dxds=-\frac{1}{2}\frac{d}{dt}\left((go\varphi)(t)+\int_0^t g(s)ds\|\varphi\|_2^2\right)-g(t)\|\varphi\|_2^2+(g'o\varphi)(t),$$

*where*

$$(go\varphi)(t)=\int_0^t g(t-s)\int_\Omega|\varphi(s,x)-\varphi(t,x)|^2dxds.$$

**Lemma 2.0.4** *([11]). Suppose that (2.8) holds. Then there exists* $\rho>0$ *such that for any* $(u,\upsilon)\in H_0^1(\Omega)^2$, *we have*

$$\|u+\upsilon\|_{p+1}^{p+1}+2\|u\upsilon\|_{\frac{p+1}{2}}^{\frac{p+1}{2}}\leq\rho(l_1\|\nabla u\|_2^2+l_2\|\nabla\upsilon\|_2^2)^{\frac{p+1}{2}}.$$

**Lemma 2.0.5** . *Guessmia ([20])* $E:R^+\to R^+$ *be a differentiable function* $a_1,a_2\in R^{+*}$ *and* $a_3,\lambda\in R^+$ *such that*

$$a_3\lambda(r+1)<1,$$

*and for all* $0\leq S\leq T<+\infty$,

$$\int_S^T E^{q+1}(t)dt\leq a_1(S)E(S)+a_2E^{q+1}(S)+a_3E^{r+1}(T),$$

$$E'(t) \leq \lambda E(t), \quad \forall t \geq 0,$$

*then there exist two positive constants and c such that, for all* $t \geq 0$

$$E(t) \leq ce^{-\omega t}, if r > 0,$$
$$E(t) \leq c(1+t)^{\frac{-1}{r}}, if r > 0 \quad and \quad \lambda = 0,$$
$$E(t) \leq c(1+t)^{\frac{-1}{r(r+1)}}, if r > 0 \quad and \quad \lambda > 0.$$

***Remark 2.0.1*** . *Avoiding the complexity of the matter , we take* $a = b = 1$ *in* $(2.6) - (2.7)$ .

### 2.0.7 Global existence

In order to prove the existence of solutions of problem (2.1), we introduce the new variables $z_1$, $z_2$ as in [12]

$$z_1(x, k_1, t) = u_t(x, t - \tau_2 k_1), x \in \Omega, k_1 \in (0, 1),$$
$$z_2(x, k_2, t) = u_t(x, t - \tau_2 k_2), x \in \Omega, k_2 \in (0, 1),$$

which implies that

$$\tau_2 z_1'(x, k_1, t) + z_{k_1}(x, k_1, t) = 0 \in \Omega \times (0, 1) \times (0, \infty),$$

$$\tau_2 z_2'(x, k_2, t) + z_{k_2}(x, k_2, t) = 0 \in \Omega \times (0, 1) \times (0, \infty),$$

therefore, problem (2.1) is equivalent to

$$(2.9) \qquad \begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g_1(t-s)\Delta u(s)ds \\ +\mu_1 u_t(x,t) + \mu_2 z_1(x,1,t) + f_1(u,\upsilon) = 0, \\ \upsilon_{tt} - \Delta \upsilon - \Delta \upsilon_{tt} + \int_0^t g_2(t-s)\Delta u(s)ds \\ +\alpha_1 \upsilon_t(x,t) + \alpha_2 z_2(x,1,t) + f_2(u,\upsilon) = 0, \\ \tau_2 z_1'(x,k_1,t) + z_{k_1}(x,k_1,t) = 0 \in \Omega \times (0,1) \times (0,\infty), \\ \tau_2 z_2'(x,k_2,t) + z_{k_2}(x,k_2,t) = 0 \in \Omega \times (0,1) \times (0,\infty), \\ z_1(x,0,t) = u_t(x,t), x \in \Omega, t > 0, \\ z_2(x,0,t) = \upsilon_t(x,t), x \in \Omega, t > 0, \\ z_1(x,k_1,0) = \phi_0(x, -\tau_2 k_1), x \in \Omega, \\ z_2(x,k_2,0) = \phi_1(x, -\tau_2 k_2), x \in \Omega, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \\ \upsilon(x,0) = \upsilon_0(x), \upsilon_t(x,0) = \upsilon_1(x), x \in \Omega, \\ u(x,t) = 0, \upsilon(x,t) = 0, x \in \partial\Omega, t \geq 0. \end{cases}$$

In the following, we will give sufficient conditions for the well-posedness of problem (2.9) by using the Fadeo-Galerkin's method.

**Theorem 2.0.2** . *Suppose that $\mu_2 < \mu_1$, $\alpha_2 < \alpha_1$, $(A_1)-(A_3)$ holds. Assume that $((u_0,u_1),(\upsilon_0,\upsilon_1)) \in (H_0^1(\Omega))^2$ and $(\phi_0,\phi_1) \in (L^2(\Omega\times(0,1))^2$. Then there exists a unique solution $((u,z_1),(\upsilon,z_2))$ of (2.9) satisfying*

$$u(t),\upsilon(t) \in C([-\tau_2,\infty);H_0^1(\Omega)) \cap C^1([-\tau_2,\infty);L^2(\Omega)),$$

$$u_t(t),\upsilon_t(t) \in L^2([-\tau_2,\infty);H_0^1(\Omega)) \cap L^2([-\tau_2,\infty)\times\Omega).$$

*Proof* We use the standard Faedo-Galerkin's method to construct approximate solution. Let $\{w_j\}_{j=1}^{\infty}$ is a basis of $D(A) = H^2(\Omega)\bigcap H_0^1(\Omega)$. It is known that $\{w_j\}_{j=1}^{\infty}$ forms an orthonormal basis for $L^2(\Omega)$ as well as for $H_0^1(\Omega)$. Moreover, the linear span of $\{w_j\}_{j=1}^{\infty}$ is dense in $L^q(\Omega)$ for any $1 \le q < \infty$. $W_n$ is the linear span of $\{w_1.....w_n\}$, we define also for $1 \le j \le n$, the sequence $\varphi_j(x,k)$ as follows $\varphi_j(x,0) = w_j(x)$. Then we may extend $\varphi_j(x,0)$ by $\varphi_j(x,k)$ over $L^2(\Omega\times[0,1])$ and denote $V_n$ to be the space generated by
$\{\varphi_1,....\ \varphi_n\}$, $n = 1,2,3$.

$$u^n(t) = \sum_{j=1}^{n} u_{k,j}(t)w_j \quad z_1^n(t) = \sum_{j=1}^{n} z_{k_1,j}(t)\varphi_j(x,k),$$

$$\upsilon^n(t) = \sum_{j=1}^{n} \upsilon_{k,j}(t)w_j, \quad z_2^n(t) = \sum_{j=1}^{n} z_{k_2,j}(t)\varphi_j(x,k),$$

where $((u^n(t),z_1^n(t)),(\upsilon^n(t),z_2^n(t)))$ are the solutions of the following approximate problem corresponding to (2.9) then $((u^n(t),z_1^n(t)),(\upsilon^n(t),z_2^n(t)))$ verify the following system of ODEs:

$$\text{(2.10)}\quad \begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + \displaystyle\int_0^t g_1(t-s)\Delta u(s)ds \\ +\mu_1 u_t(x,t) + \mu_2 z_1(x,1,t) + f_1(u,\upsilon) = 0, \\ \upsilon_{tt} - \Delta \upsilon - \Delta \upsilon_{tt} + \displaystyle\int_0^t g_2(t-s)\Delta u(s)ds \\ +\alpha_1 \upsilon_t(x,t) + \alpha_2 z_2(x,1,t) + f_2(u,\upsilon) = 0, \\ \tau_2 z_1'(x,k_1,t) + z_{k_1}(x,k_1,t) = 0 \in \Omega\times(0,1)\times(0,\infty), \\ \tau_2 z_2'(x,k_2,t) + z_{k_2}(x,k_2,t) = 0 \in \Omega\times(0,1)\times(0,\infty), \\ z_1(x,0,t) = u_t(x,t), x\in\Omega, t>0, \\ z_2(x,0,t) = \upsilon_t(x,t), x\in\Omega, t>0, \\ z_1(x,k_1,0) = \phi_0(x,-\tau_2 k_1), x\in\Omega, \\ z_2(x,k_2,0) = \phi_1(x,-\tau_2 k_2), x\in\Omega, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x\in\Omega, \\ \upsilon(x,0) = \upsilon_0(x), \upsilon_t(x,0) = \upsilon_1(x), x\in\Omega, \\ u(x,t) = 0, \upsilon(x,t) = 0, x\in\partial\Omega, t\ge 0. \end{cases}$$

$$\text{(2.11)}\quad \begin{aligned} &\langle u_{tt}^n(t), w_j\rangle + \langle \nabla u^n(t), \nabla w_j\rangle + \langle \nabla u_{tt}^n(t), \nabla w_j\rangle + \left\langle \int_0^t g_1(t-s)\nabla u^n(t)ds, \nabla w_j \right\rangle \\ &+ \langle \mu_1 u_t^n(x,t), w_j\rangle + \langle \mu_2 z_1^n(x,1,t), w_j\rangle + \langle f_1(u^n(t),\upsilon^n(t)), w_j\rangle = 0, \end{aligned}$$

$$\langle \upsilon_{tt}^n(t), w_j\rangle + \langle \nabla \upsilon^n(t), \nabla w_j\rangle + \langle \nabla \upsilon_{tt}^n(t), \nabla w_j\rangle + \left\langle \int_0^t g_2(t-s)\nabla \upsilon^n(t)ds, \nabla w_j\right\rangle$$
$$+\langle \alpha_1 \upsilon_t^n(x,t), w_j\rangle + \langle \alpha_2 z_2^n(x,1,t), w_j\rangle + \langle f_2(u^n(t), \upsilon^n(t)), w_j\rangle = 0, \tag{2.12}$$

for $j = 1.....n$. More specifically

$$u^n(0) = \sum_{j=1}^n u_{k,j}(0)w_j, \upsilon^n(0) = \sum_{j=1}^n \upsilon_{k,j}(0)w_j, \tag{2.13}$$

$$u_t^n(0) = \sum_{j=1}^n u'_{k,j}(0)w_j, \upsilon_t^n(0) = \sum_{j=1}^n \upsilon'_{k,j}(0)w_j, \tag{2.14}$$

where

$$u^n(0) = \langle u^0, w_j\rangle, \upsilon^n(0) = \langle \upsilon^0, w_j\rangle, \upsilon_t^n(0) = \langle \upsilon^1, w_j\rangle, \upsilon_t^n(0) = \langle \upsilon^1, w_j\rangle,$$

$j = 1, ...., n$. Obviously, $u^n(0) \to u^0$, $\upsilon^n(0) \to \upsilon^0$ strongly in $H_0^1(\Omega)$, $u_t^n(0) \to u^1$, $\upsilon_t^n(0) \to \upsilon^1$ strongly in $L^2(\Omega)$ as $n \to \infty$.

$$(\tau_2 z_{1t}^n(x, k_1, t) + z_{1k}^n(x, k_1, t))\varphi_i = 0, \tag{2.15}$$

$$(\tau_2 z_{2t}^n(x, k_2, t) + z_{2k}^n(x, k_2, t))\varphi_i = 0, \tag{2.16}$$

$$z_1^n(0) = z_1^n \to f_0, z_2^n(0) = z_2^n \to f_1 \ in \ L^2(\Omega \times (0,1)). \tag{2.17}$$

Let $\xi_1$, $\xi_2$ be positive constants such that

$$\tau_2\mu_2 < \xi_1 < \tau_2(2\mu_1 - \mu_2), \tag{2.18}$$

$$\tau_2\alpha_2 < \xi_1 < \tau(2\alpha_1 - \alpha_2). \tag{2.19}$$

**Step one.**

**Energy estimates.**

Multiplying equation (2.11) by $u'_{k,j}(t)$ and the equation (2.12) by $\upsilon'_{k,j}(t)$, and summing with respect to j, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|u_t^n\|_2^2+\frac{1}{2}\|\upsilon_t^n\|_2^2+\frac{1}{2}\|\nabla u^n(t)\|_2^2+\frac{1}{2}\|\nabla \upsilon^n(t)\|_2^2+\frac{1}{2}\|\nabla u_t^n(t)\|_2^2\right]\\
&+\frac{d}{dt}\left[\frac{1}{2}\|\nabla \upsilon_t^n(t)\|_2^2+\int_\Omega F(u^n,\upsilon^n)dx\right]+\frac{\mu_1}{2}\|u_t^n\|_2^2+\frac{\alpha_1}{2}\|\upsilon_t^n\|_2^2\\
&+\mu_2\int_\Omega z_1(x,1,t)u_t^n(x,t)dx+\alpha_2\int_\Omega z_2(x,1,t)\upsilon_t^n(x,t)dx\\
&-\int_0^t g_1(t-s)\int_\Omega \nabla u^n(s)\nabla u_t^n(s)dxds-\int_0^t g_2(t-s)\int_\Omega \nabla \upsilon^n(s)\nabla \upsilon_t^n(s)dxds,
\end{aligned}
\tag{2.20}
$$

Using a lemma 2.0.3 and integrating (2.20) over (0,t), we get

$$
\begin{aligned}
&\frac{1}{2}\|u_t^n\|_2^2+\frac{1}{2}\|\upsilon_t^n\|_2^2+\frac{1}{2}\|\nabla u_t^n\|_2^2+\frac{1}{2}\|\nabla \upsilon_t^n\|_2^2+\int_\Omega F(u^n,\upsilon^n)dx\\
&\frac{1}{2}\left(1-\int_0^t g_1(s)ds\right)\|\nabla u^n(t)\|_2^2+\frac{1}{2}\left(1-\int_0^t g_2(s)ds\right)\|\nabla \upsilon^n(t)\|_2^2\\
&+\frac{1}{2}(g_1 o\nabla u^n)(t)+\frac{1}{2}(g_2 o\nabla \upsilon^n)(t)+\mu_1\int_0^t\|u_t^n(s)\|_2^2ds+\alpha_1\int_0^t\|\upsilon_t^n(s)\|_2^2ds\\
&+\mu_2\int_0^t\int_\Omega z_1^n(x,1,s)u_t^n dxds+\alpha_2\int_0^t\int_\Omega z_2^n(x,1,s)\upsilon_t^n dxds\\
&+\frac{1}{2}\int_0^t g_1(s)\|\nabla u_t^n\|_2^2ds+\frac{1}{2}\int_0^t g_2(s)\|\nabla \upsilon_t^n\|_2^2ds-\frac{1}{2}\int_0^t(g_1' o\nabla u^n)(s)ds\\
&-\frac{1}{2}\int_0^t(g_2' o\nabla \upsilon^n)(s)ds,
\end{aligned}
\tag{2.21}
$$

we multiply the equation (2.15) by $\frac{\xi_1}{\tau_2}z_{k_1,j}(t)$ and the equation (2.16) by $\frac{\xi_2}{\tau_2}z_{k_2,j}(t)$, summing with respect to j and integrating the result over $\Omega\times(0,1)$ to obtain

$$
\begin{aligned}
\xi_1\int_\Omega\int_0^1(z_1^n)'z_1^n(x,k_1,t)dk_1dx &= \frac{-\xi_1}{2\tau_2}\int_\Omega\int_0^1\frac{\partial}{\partial k_1}(z_1^n)^2(x,k_1,t)dk_1dx,\\
&= \frac{-\xi_1}{2\tau_2}\int_\Omega((z_1^n)^2(x,1,t)-(z_1^n)^2(x,0,t))dx,
\end{aligned}
\tag{2.22}
$$

then

$$
\frac{\xi_1}{2}\frac{d}{dt}\int_\Omega\int_0^1(z_1^n)^2(x,k_1,t)dk_1dx=-\frac{\xi_1}{2\tau_2}\int_\Omega(z_1^n)^2(x,k_1,t)dx+\frac{\xi_1}{2\tau}\|u_t^n\|_2^2,
\tag{2.23}
$$

in the same manner

$$
\frac{\xi_2}{2}\frac{d}{dt}\int_\Omega\int_0^1(z_2^n)^2(x,k_2,t)dk_2dx=-\frac{\xi_2}{2\tau_2}\int_\Omega(z_2^n)^2(x,k_2,t)dx+\frac{\xi_2}{2\tau}\|\upsilon_t^n\|_2^2.
\tag{2.24}
$$

Summing (2.21), (2.23) and (2.24), we get

$$
\begin{aligned}
&E^n(t)+\left(\mu_1-\frac{\xi_1}{2\tau_2}\right)\int_0^t \|u_t^n\|_2^2 ds+\left(\alpha_1-\frac{\xi_2}{2\tau_2}\right)\int_0^t \|\upsilon_t^n\|_2^2 ds\\
&+\frac{\xi_1}{2}\int_0^t\int_\Omega (z_1^n)^2(x,1,s)dxds+\frac{\xi_2}{2}\int_0^t\int_\Omega (z_2^n)^2(x,1,s)dxds\\
&+\mu_2\int_\Omega\int_0^t z_2^n(x,1,s)\upsilon_t^n(x,t)dxds+\alpha_2\int_\Omega\int_0^t z_2^n(x,1,s)\upsilon_t^n(x,t)dxds.
\end{aligned}
\tag{2.25}
$$

Using Young and Gauchy-Shwartz inequalities, we obtain

$$
\begin{aligned}
&E^n(t)+\left(\mu_1-\tfrac{\xi_1}{2\tau_2}-\tfrac{\mu_2}{2}\right)\int_0^t \|u_t^n\|_2^2 ds+\left(\alpha_1-\tfrac{\xi_2}{2\tau_2}-\tfrac{\alpha_2}{2}\right)\int_0^t \|\upsilon_t^n\|_2^2 ds\\
&+\left(\tfrac{\xi_1}{2\tau_2}-\tfrac{\mu_2}{2}\right)\int_0^t\int_\Omega (z_1^n)^2(x,1,s)dxds+\left(\tfrac{\xi_2}{2\tau_2}-\tfrac{\alpha_2}{2}\right)\int_0^t\int_\Omega (z_2^n)^2(x,1,s)dxds\\
&=E^n(0).
\end{aligned}
\tag{2.26}
$$

Where E(t) is the energy of the solution defined by the following formula

$$
\begin{aligned}
&E(t)=\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|\upsilon_t(t)\|_2^2+\frac{1}{2}\|\nabla u_t(t)\|_2^2+\frac{1}{2}\|\nabla \upsilon_t(t)\|_2^2\\
&+\frac{1}{2}\left(1-\int_0^t g_1(s)ds\right)\|\nabla u(t)\|_2^2+\frac{1}{2}\left(1-\int_0^2 g_2(s)ds\right)\|\nabla \upsilon(t)\|_2^2\\
&+\frac{\xi_1}{2}\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx+\frac{\xi_2}{2}\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx\\
&+\int_\Omega F(u,\upsilon)dx+\frac{1}{2}(g_1 o\nabla u)(t)+\frac{1}{2}(g_2 o\nabla \upsilon)(t).
\end{aligned}
\tag{2.27}
$$

we shall prove that the problem $(2.11)-(2.17)$ admits a local solution in $[0,t_m)$, $0<t_m<T$, for an arbitrary $T>0$. The extension of the solution to the whole interval $[0,T]$ is a consequence of the estimates below.

**Step two.**

**First estimate.** We Multiply the equation (2.11) by $w_j=-\Delta u_t^n$, and equation (2.12) by $w_j=-\Delta \upsilon_t^n$ and summing with respect to n from 1 to n, respectively, using a lemma 2.0.3, we get

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|\nabla u_t^n\|_2^2+\frac{1}{2}\|\Delta u_t^n\|_2^2+\left(1-\int_0^t g_1(s)ds+\right)\|\Delta u^n\|_2^2+(g_1 o\Delta u^n)(t)\right]\\
&+\frac{1}{2}g_1(t)\|\Delta u^n\|_2^2-\frac{1}{2}(g_1' o\Delta u^n)(t)+\frac{\mu_1}{2}\|\nabla u_t^n\|_2^2\\
&+\mu_2\int_\Omega z_1^n(x,1,t)\Delta u_t^n(t)dx=\int_\Omega f_1(u^n(t),\upsilon^n(t))\Delta u^n(t)dx,
\end{aligned}
\tag{2.28}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|\nabla \upsilon_t^n\|_2^2+\frac{1}{2}\|\Delta \upsilon_t^n\|_2^2+\left(1-\int_0^t g_2(s)ds+\right)\|\Delta \upsilon^n\|_2^2+(g_2 o\Delta \upsilon^n)(t)\right]\\
&+\frac{1}{2}g_2(t)\|\Delta \upsilon^n\|_2^2-\frac{1}{2}(g_2' o\Delta \upsilon^n)(t)+\frac{\alpha_1}{2}\|\nabla \upsilon_t^n\|_2^2\\
&+\alpha_2\int_\Omega z_2^n(x,1,t)\Delta \upsilon_t^n(t)dx=\int_\Omega f_2(u^n(t),\upsilon^n(t))\Delta \upsilon^n(t)dx.
\end{aligned}\tag{2.29}
$$

Using Young's inequalities, summing (2.28)-(2.29), and integrating over $(0,t)$, we get

$$
\begin{aligned}
&\frac{1}{2}[\|\nabla u_t^n\|_2^2+\|\nabla \upsilon_t^n\|_2^2+\|\Delta u_t^n\|_2^2+\|\Delta \upsilon_t^n\|_2^2+\left(1-\int_0^t g_1(s)ds\right)\|\Delta u^n\|_2^2\\
&+\left(1-\int_0^t g_2(s)ds\right)\|\Delta \upsilon^n\|_2^2+\frac{1}{2}((g_1 o\Delta u^n)(t)+\frac{1}{2}(g_2 o\Delta \upsilon^n)(t)\\
&+\frac{1}{2}\int_0^t g_1(s)\|\Delta u^n(s)\|_2^2 ds+\frac{1}{2}\int_0^t g_2(s)\|\Delta \upsilon^n(s)\|_2^2 ds-\frac{1}{2}\int_0^t (g_1' o\Delta \upsilon^n)(s)ds\\
&-\frac{1}{2}\int_0^t (g_2' o\Delta \upsilon^n)(s)ds+\frac{\mu_1}{2}\int_0^t \|\nabla u_t^n(s)\|_2^2 ds+\frac{\alpha_1}{2}\int_0^t \|\nabla \upsilon_t^n(s)\|_2^2 ds\\
&+\mu_2\int_0^t\int_\Omega (z_1^n)^2(x,1,s)dsdx+\mu_2\int_0^t \|\Delta u_t^n(s)\|_2^2 ds\\
&+\alpha_2\int_0^t\int_\Omega (z_2^n)^2(x,1,s)dsdx+\alpha_2\int_0^t \|\Delta \upsilon_t^n(s)\|_2^2 dxds\\
&\le \frac{1}{2}\|\nabla u^{1n}\|_2^2+\frac{1}{2}\|\nabla \upsilon^{1n}\|_2^2+\frac{1}{2}\|\Delta u^{0n}\|_2^2+\frac{1}{2}\|\Delta \upsilon^{0n}\|_2^2\\
&+\int_\Omega\begin{pmatrix} f_1(u^n,\upsilon^n)\Delta u^n-f_1(u^0,\upsilon^0)\Delta u^0\\ f_2(u^n,\upsilon^n)\Delta \upsilon^n-f_2(u^n,\upsilon^n)\Delta \upsilon^n\end{pmatrix}dx\\
&-\int_0^t\int_\Omega\begin{pmatrix} \frac{\partial}{\partial u}f_1(u^n,\upsilon^n)u_t^n\Delta u^n+\frac{\partial}{\partial \upsilon}f_1(u^n,\upsilon^n)\upsilon_t^n\Delta u^n\\ +\frac{\partial}{\partial \upsilon}f_2(u^n,\upsilon^n)u_t^n\Delta \upsilon^n+\frac{\partial}{\partial \upsilon}f_2(u^n,\upsilon^n)\upsilon_t^n\Delta \upsilon^n\end{pmatrix}dxds,
\end{aligned}\tag{2.30}
$$

where

$$
c_0=\frac{1}{2}\|\nabla u^{1n}\|_2^2+\frac{1}{2}\|\nabla \upsilon^{1n}\|_2^2+\frac{1}{2}\|\Delta u^{0n}\|_2^2+\frac{1}{2}\|\Delta \upsilon^{0n}\|_2^2,
$$

is a positive constant, we just need to estimate the right hand terms of (2.30). Applying Holder's inequality, Sobolev embedding theorem inequality

$$
\begin{aligned}
\left|\int_\Omega f_1(u^n(s),\upsilon^n(s))\Delta u^n(s)dxds\right| &\le \int_\Omega \left(|u^n|^p + |\upsilon^n|^p + |u^n|^{\frac{p-1}{2}}|\upsilon^n|^{\frac{p+1}{2}}\right)|\Delta u^n|dx,\\
&\le C\left(\|u^n\|_{2p}^p + \|\upsilon^n\|_{2p}^p\right)\|\Delta u^n\|_2,\\
&+ C\left(\|u^n\|_{\frac{p-1}{2}}^{3(p-1)}\|u^n\|_{\frac{p+1}{2}}^{\frac{3(p+1)}{2}}\right)\|\Delta u^n\|_2,\\
&\le C\left(\|\nabla u^n\|_2^p + \|\nabla \upsilon^n\|_2^p\right)\Delta u^n\|_2\\
&+ C\left(\|\nabla u^n\|_2^{\frac{p-1}{2}}\|\nabla \upsilon^n\|_2^{\frac{p+1}{2}}\right)\|\Delta u^n\|_2,\\
&\le C\left(\|\Delta u^n\|_2^2 + \|\nabla u^n\|_2^{2p}\right)\\
&+ C\left(\|\nabla \upsilon^n\|_2^{2p} + \|\nabla u^n\|_2^{p-1}\|\nabla u^n\|_2^{p+1}\right),\\
&\le C\|\Delta u^n\|_2^2 + c.
\end{aligned}
\tag{2.31}
$$

Likewise, we obtain

$$
\left|\int_\Omega f_1(u^n,\upsilon^n)\Delta u^n dx\right| \le C\|\Delta \upsilon^n\|_2^2 + c. \tag{2.32}
$$

Now we estimate $I := \int_\Omega \frac{\partial}{\partial u} f_1(u^n,\upsilon^n)u_t^n \Delta u^n dx,$

then, by $(A_2)$ and Young's inequality we get

$$
\begin{aligned}
|I| &\le c\int_\Omega (|u^n|^{p-1} + |\upsilon^n|^{p-1})|u_t^n||\Delta u^n|dx,\\
&\le c(\|u_t^n\| + \|u^n\|_{2p}^{p-1}\|u_t^n\|_{2p} + \|\upsilon_n\|_{2p}^{p-1}\|u_t^n\|_{2p})\|\Delta u^n\|_2.
\end{aligned}
\tag{2.33}
$$

We get

$$
\begin{aligned}
|I| &\le c(\|\nabla u_t^n\|_2^{p-1} + \|\nabla \upsilon_t^n\|_2^{p-1})\|\nabla u_t^n\|_2\|\Delta u^n\|_2,\\
&\le c\|\nabla u_t^n\|_2\|\Delta u^n\|_2,\\
&\le c\|\Delta u^n\|_2^2 + c\|\nabla u_t^n\|_2^2.
\end{aligned}
\tag{2.34}
$$

Let

$$
y^n(t) = \|\nabla u_t^n(t)\|_2^2 + \|\nabla \upsilon_t^n(t)\|_2^2 + \|\Delta \upsilon^n(t)\|_2^2 + \|\Delta u^n(t)\|_2^2. \tag{2.35}
$$

Then, we infer from (2.31) – (2.34) that

$$
y^n(t) + \|\nabla u_t^n\|_2^2 \le C_0 + C\int_0^t (y^n(s))ds. \tag{2.36}
$$

Using the Gronwall type inequality, we can get

$$
y^n(t) \le C. \tag{2.37}
$$

Hence from (2.27) and (2.37), we obtain

$$
\begin{aligned}
&\|u_t^n(t)\|_2^2 + \|\upsilon_t^n(t)\|_2^2 + \|\nabla u^n(t)\|_2^2 + \|\nabla \upsilon^n(t)\|_2^2 + \|\nabla u_t^n(t)\|_2^2 + \|\nabla \upsilon_t^n(t)\|_2^2\\
&+ \int_0^1\int_\Omega z_1^n(x,1,s)dxds + \int_0^1\int_\Omega z_2^n(x,1,s)dxds + (g_1 o \nabla u^n)(t) + (g_2 o \nabla \upsilon^n)(t)\\
&+ \int_0^1\int_\Omega z_1^n(x,k_1,s)dxds + \int_0^1\int_\Omega z_2^n(x,k_2,s)dxds + \int_\Omega F(u,\upsilon)dx \le L_1,
\end{aligned}
\tag{2.38}
$$

where $L_1$ is a positive constant depending on the parameter $E(0)$.
**Step three.**
**Second estimate.**
In same manner we Multiply the equation (2.11) by $w_j = -\Delta u_{tt}^n$, (2.12) by $w_j = -\Delta \upsilon_{tt}^n$ and summing with respect to j from 1 to n, respectively

$$\begin{aligned} &\frac{1}{2}\|\nabla u_{tt}^n\|_2^2 + \int_\Omega \Delta u_{tt}^n \Delta u^n dx + \|\Delta u_{tt}^n\|_2^2 + \frac{\mu_1}{2}\frac{d}{dt}\|\nabla u_t^n\|_2^2 \\ = &-\int_\Omega \Delta u(t)\Delta u_{tt}^n dx + \int_0^t \int_\Omega g_1(t-s)u^n(\tau)\Delta u_{tt}^n(t)dxd\tau \\ - &\mu_2 \int_\Omega z_1^n(x,1,t)\Delta u_{tt}^n dx + \int_\Omega f_1(u^n,\upsilon^n)\Delta u_{tt}^n dx, \end{aligned} \tag{2.39}$$

$$\begin{aligned} &\frac{1}{2}\|\nabla \upsilon_{tt}^n\|_2^2 + \int_\Omega \Delta \upsilon_{tt}^n \Delta \upsilon^n dx + \|\Delta \upsilon_{tt}^n\|_2^2 + \frac{\alpha_1}{2}\frac{d}{dt}\|\nabla \upsilon_t^n\|_2^2 \\ = &-\int_\Omega \Delta u(t)\Delta \upsilon_{tt}^n dx + \int_0^t \int_\Omega g_2(t-s)\upsilon^n(\tau)\Delta \upsilon_{tt}^n(t)dxd\tau \\ - &\alpha_2 \int_\Omega z_2^n(x,1,t)\Delta \upsilon_{tt}^n dx + \int_\Omega f_2(u^n,\upsilon^n)\Delta \upsilon_{tt}^n dx. \end{aligned} \tag{2.40}$$

Summing (2.39)-(2.40) we obtain

$$\begin{aligned} &\frac{1}{2}\|\nabla u_{tt}^n\|_2^2 + \int_\Omega \Delta u_{tt}^n \Delta u^n dx + \|\Delta u_{tt}^n\|_2^2 + \frac{\mu_1}{2}\frac{d}{dt}\|\nabla u_t^n\|_2^2 \\ &+\frac{1}{2}\|\nabla \upsilon_{tt}^n\|_2^2 + \int_\Omega \Delta \upsilon_{tt}^n \Delta \upsilon^n dx + \|\Delta \upsilon_{tt}^n\|_2^2 + \frac{\alpha_1}{2}\frac{d}{dt}\|\nabla \upsilon_t^n\|_2^2 \\ = &-\int_\Omega \Delta u(t)\Delta u_{tt}^n dx + \int_0^t \int_\Omega g_1(t-s)u^n(\tau)\Delta u_{tt}^n(t)dxd\tau \\ - &\int_\Omega \Delta u(t)\Delta \upsilon_{tt}^n dx + \int_0^t \int_\Omega g_2(t-s)\upsilon^n(\tau)\Delta \upsilon_{tt}^n(t)dxd\tau \\ - &\mu_2 \int_\Omega z_1^n(x,1,t)\Delta u_{tt}^n dx - \alpha_2 \int_\Omega z_2^n(x,1,t)\Delta \upsilon_{tt}^n dx \\ + &\int_\Omega f_1(u^n,\upsilon^n)\Delta u_{tt}^n + f_2(u^n,\upsilon^n)\Delta \upsilon_{tt}^n dx. \end{aligned} \tag{2.41}$$

Exploiting Holder, Young's inequalities, and lemma 2.0.3, for $\epsilon > 0$, $c > 0$ from the first estimate we have

$$\left| -\int_\Omega \Delta u(t)\Delta u_{tt}^n dx \right| \le \epsilon c \|\Delta u_{tt}^n\|_2^2 + \frac{c}{4\epsilon}\|\Delta u^n\|_2^2, \tag{2.42}$$

$$\left| \int_0^t g_1(t-s)\int_\Omega u^n(\tau)\Delta u_{tt}^n(t)dxd\tau \right| \le \epsilon \|\Delta u_{tt}^n\|_2^2 + \frac{(1-l_1)g_1(0)}{4\epsilon}\int_0^t \|\nabla u^n(s)\|_2^2 ds, \tag{2.43}$$

$$\left| -\int_\Omega z_1^n(x,1,t)\Delta u_{tt}^n dx \right| \le \frac{\epsilon}{\mu_2}\|\Delta u_{tt}^n\|_2^2 + \frac{\mu_2 c_s^2}{4\epsilon}\int_\Omega (z_1^n)^2(x,1,s)dx, \tag{2.44}$$

$$\left|-\int_\Omega \Delta \upsilon(t)\Delta \upsilon_{tt}^n dx\right| \leq \epsilon c\|\Delta \upsilon_{tt}^n\|_2^2 + \frac{c}{4\epsilon}\|\Delta \upsilon^n\|_2^2, \tag{2.45}$$

$$\left|\int_0^t g_2(t-s)\int_\Omega \upsilon^n(\tau)\Delta \upsilon_{tt}^n(t)dxd\tau\right| \leq \epsilon\|\Delta \upsilon_{tt}^n\|_2^2 + \frac{(1-l_2)g_2(0)}{4\epsilon}\int_0^t \|\nabla \upsilon^n(s)\|_2^2 ds, \tag{2.46}$$

$$\left|-\int_\Omega z_2^n(x,1,t)\Delta \upsilon_{tt}^n dx\right| \leq \frac{\epsilon}{\alpha_2}\|\Delta \upsilon_{tt}^n\|_2^2 + \frac{\alpha_2 c_s^2}{4\epsilon}\int_\Omega (z_2^n)^2(x,1,s)dx, \tag{2.47}$$

$$\begin{aligned}
\left|\int_\Omega f_1(u^n(s),\upsilon^n(s))\Delta u_{tt}^n(s)dxds\right| &\leq \int_\Omega (|u^n|^p + |\upsilon^n|^p) \\
&+ \left(|u^n|^{\frac{p-1}{2}}|\upsilon^n|^{\frac{p+1}{2}}\right)|\Delta u_{tt}^n|dx, \\
&\leq C\left(\|u^n\|_{2p}^p + \|\upsilon^n\|_{2p}^p\right)|\Delta u_{tt}^n| \\
&+ \left(\|u^n\|_{\frac{p-1}{2}}^{3(p-1)}\|u^n\|_{\frac{p+1}{2}}^{\frac{3(p+1)}{2}}\right)\|\Delta u_{tt}^n\|_2, \\
&\leq C\left(\|\nabla u^n\|_2^p + \|\nabla \upsilon^n\|_2^p\right)\|\Delta u_{tt}^n\|_2 \\
&+ \left(\|\nabla u^n\|_2^{\frac{p-1}{2}}\|\nabla \upsilon^n\|_2^{\frac{p+1}{2}}\right)\|\Delta u_{tt}^n\|_2, \\
&\leq C\left(\|\Delta u_{tt}^n\|_2^2 + \|\nabla u^n\|_2^{2p}\right)\|\Delta u_{tt}^n\|_2 \\
&+ \left(\|\nabla \upsilon^n\|_2^{2p} + \|\nabla u^n\|_2^{p-1}\|\nabla u^n\|_2^{p+1}\right)\|\Delta u_{tt}^n\|_2, \\
&\leq C\|\Delta u_{tt}^n\|_2^2 + c.
\end{aligned} \tag{2.48}$$

Likewise, we obtain

$$\left|\int_\Omega f_2(u^n,\upsilon^n)\Delta u^n dx\right| \leq C\|\Delta \upsilon_{tt}^n\|_2^2 + c. \tag{2.49}$$

Substituting these estimates (2.42)-(2.49) into (2.41), then integrating the obtained inequality over (0,t) and using (2.38), we deduce that

$$\begin{aligned}
&\int_0^t \|\nabla u_{tt}^n(s)\|_2^2 ds + \left(1-\epsilon(c+1+\frac{1}{\mu_2}\right)\int_0^t \|\Delta u_{tt}^n(s)\|_2^2 ds + \frac{\mu_1}{2}\|u_t^n\|_2^2 \\
&+\int_0^t \|\nabla \upsilon_{tt}^n(s)\|_2^2 ds + \left(1-\epsilon(c+1+\frac{1}{\alpha_2}\right)\int_0^t \|\Delta \upsilon_{tt}^n(s)\|_2^2 ds + \frac{\alpha_1}{2}\|\upsilon_t^n\|_2^2 \\
&\leq c\epsilon T + \frac{\mu_2 c_s^2}{4\epsilon}L_1 + (1+(1-l)g_1(0)T)T + c_3 \\
&+c\epsilon T + \frac{\alpha_2 c_s^2}{4\epsilon}L_1 + (1+(1-l)g_2(0)T)T + c_4.
\end{aligned} \tag{2.50}$$

Where $c_3, c_4$ are positive constants depending only on $\|u^1\|_2^2$ and $\|\upsilon^1\|_2^2$ . Choosing $\epsilon > 0$ small enough in (2.50) we obtain the second estimate

$$\int_0^t \|\nabla u_{tt}^n(s)\|_2^2 ds + \int_0^t \|\nabla \upsilon_{tt}^n(s)\|_2^2 ds + \int_0^t \|\Delta u_{tt}^n(s)\|_2^2 ds + \int_0^t \|\Delta \upsilon_{tt}^n(s)\|_2^2 ds \leq L_2, \tag{2.51}$$

where $L_2$ is a positive constant independent of $n \in N$ and $t \in [0, T)$.
We observe that estimates (2.38) and (2.50) imply that there exists a subsequence $(u^n, z_1^n), (\upsilon^n, z_2^n)$ such that

$$u^n \to u \text{ weakly star in } L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \tag{2.52}$$

$$\upsilon^n \to \upsilon \text{ weakly star in } L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \tag{2.53}$$

$$u_t^n \to u_t \text{ weakly star in } L^\infty(0, T; H_0^1(\Omega)), \tag{2.54}$$

$$\upsilon_t^n \to \upsilon_t \text{ weakly star in } L^\infty(0, T; H_0^1(\Omega)), \tag{2.55}$$

$$u_{tt}^n \to u_{tt} \text{ weakly star in } L^\infty(0, T; L^2(\Omega)), \tag{2.56}$$

$$\upsilon_{tt}^n \to \upsilon_{tt} \text{ weakly star in } L^\infty(0, T; L^2(\Omega)), \tag{2.57}$$

$$z_1^n \to z_1 \text{ weakly star in } L^\infty(0, T; L^2(\Omega) \times (0, 1)), \tag{2.58}$$

$$z_2^n \to z_2 \text{ weakly star in } L^\infty(0, T; L^2(\Omega) \times (0, 1)). \tag{2.59}$$

$$z_1^n(x, 1, t) \to \psi_1 \text{ weakly star in } L^2(\Omega \times (0, T)) \tag{2.60}$$

$$z_2^n(x, 1, t) \to \psi_2 \text{ weakly star in } L^2(\Omega \times (0, T)) \tag{2.61}$$

Further, by Aubin's lemma [54], it follows from (2.28) and (2.50) that there exists a subsequence $(u^n(t), \upsilon^n(t))$ still represented by the same notation, such that

$$u^n \to u \text{ strongly in } L^2(0, T; L^2(\Omega)), \tag{2.62}$$

$$\upsilon^n \to \upsilon \text{ strongly in } L^2(0, T; L^2(\Omega)), \tag{2.63}$$

Then

$$u^n \to u \text{ and } \upsilon^n \to \upsilon \text{ a.e in } (0, T) \times \Omega, \tag{2.64}$$

and

$$u_t^n \to u_t \text{ and } \upsilon_t^n \to \upsilon_t \text{ a.e in } (0, T) \times \Omega, \tag{2.65}$$

**Analysis of nonlinear term**

$$
\begin{aligned}
\|f_1(u_i,\upsilon_i)\|_{L^2(\Omega\times(0,T))} &= \int_0^T\int_\Omega (|u_i(s)|^p+|\upsilon_i(s)|^p+|u_i(s)|^{\frac{p-1}{2}}|\upsilon_i(s)|^{\frac{p+1}{2}})dsdx,\\
&\le c_s^p\int_0^T\|\nabla u_i(s)\|^p ds + c_s^p\int_0^T\|\nabla \upsilon_i(s)\|^p ds,\\
&+ c_s^{\frac{p-1}{2}}\int_0^T\|\nabla u_i(s)\|^{\frac{p-1}{2}}ds + c_s^{\frac{p+1}{2}}\int_0^T\|\nabla \upsilon_i(s)\|^{\frac{p+1}{2}}ds,\\
&\le 2c_s^pTL_1^P + c_s^{\frac{p-1}{2}}TL_1^{\frac{p-1}{2}}TL_1^{\frac{p-1}{2}} + c_s^{\frac{p+1}{2}}TL_1^{\frac{p+1}{2}}TL_1^{\frac{p+1}{2}},\\
&= C.
\end{aligned}
\tag{2.66}
$$

In the same way for $f_2(u_i,\upsilon_i)$

$$\|f_2(u_i,\upsilon_i)\|_{L^2(\Omega\times(0,T))}\le C. \tag{2.67}$$

From the (2.66) and (2.67) we deduce that

$$
\begin{aligned}
f_1(u_i,\upsilon_i)&\to f_1(u,\upsilon)\ weakly\ in\ L^2(0,T;L^2(\Omega)),\\
f_1(u_i,\upsilon_i)&\to f_1(u,\upsilon)\ weakly\ in\ L^2(0,T;L^2(\Omega)),
\end{aligned}
\tag{2.68}
$$

For suitable functions $u,\upsilon\ \in L^\infty(0,T;H_0^1(\Omega)),\ z_1,z_2\in\ L^\infty(0,T;L^2(\Omega\times(0.1)),\ \psi_1,\psi_2\in L^2(\Omega\times(0,T))$. We have to show that $((u,z_1),(\upsilon,z_2))$ is a solution of (2.1).
Using the embedding

$$L^\infty(0,T;H_0^1(\Omega))\hookrightarrow L^2(0,T;H_0^1(\Omega)),$$

$$H^1((0,T)\times\Omega)\hookrightarrow L^2((0,T)\times\Omega).$$

From (2.54)-(2.55) we have that $u_t^n,\upsilon_t^n$ are bounded in

$$L^\infty((0,T);H_0^1(\Omega))\hookrightarrow L^2((0,T);H_0^1(\Omega)),$$

then $u_{tt}^n,\upsilon_{tt}^n$ are bounded in

$$L^\infty((0,T);L^2(\Omega))\hookrightarrow L^2((0,T);L^2(\Omega)).$$

Consequently, $u_t^n,\upsilon_t^n$ are bounded in

$$H^1((\Omega)\times(0,T)),$$

using Aubin-Lions theorem [54], we can extract a subsequence $(u^\xi)$ of $(u^n)$ and $(\upsilon^\xi)$ of $(\upsilon^n)$ such that

$$u_t^\xi\to u_t\ strongly\ in\ L^2(\Omega\times(0,T)), \tag{2.69}$$

$$\upsilon_t^\xi\to \upsilon_t\ strongly\ in\ L^2(\Omega\times(0,T)), \tag{2.70}$$

therefore

$$u_t^\xi \to u_t \textit{ strongly and a.e. in } (\Omega \times (0,T)), \tag{2.71}$$

$$\upsilon_t^\xi \to \upsilon_t \textit{ strongly and a.e. in } (\Omega \times (0,T)), \tag{2.72}$$

similarly

$$z_1^\xi \to z_1 \textit{ strongly in } L^2(0,T;L^2(\Omega \times (0,1)), \tag{2.73}$$

$$z_2^\xi \to z_2 \textit{ strongly in } L^2(0,T;L^2(\Omega \times (0,1)), \tag{2.74}$$

Now, we will pass to the limit in (2.11)-(2.12). Taking $n = \xi, \forall w_j \in W_n$, $\forall \varphi_j \in V_n$ in (2.11)-(2.12) and fixed $j < \xi$,

$$\begin{aligned} &\int_\Omega u_{tt}^\xi(t) w_j dx + \int_\Omega \nabla u^\xi(t) \nabla w_j dx + \int_\Omega \nabla u_{tt}^\xi(t) \nabla w_j dx \\ &+ \int_\Omega \int_0^t g_1(t-s) \nabla u^\xi(t) ds \nabla w_j dx dt + \int_\Omega \mu_1 u_t^\xi(t) w_j dx \\ &+ \int_\Omega \mu_2 z_1^\xi(x,1,t) w_j dx = \int_\Omega f_1(u^\xi(t) \upsilon^\xi(t) w_j dx \end{aligned} \tag{2.75}$$

$$\begin{aligned} &\int_\Omega \upsilon_{tt}^\xi(t) w_j dx + \int_\Omega \nabla \upsilon^\xi(t) \nabla w_j dx + \int_\Omega \nabla \upsilon_{tt}^\xi(t) \nabla w_j dx \\ &+ \int_\Omega \int_0^t g_2(t-s) \nabla \upsilon^\xi(t) ds \nabla w_j dx dt + \int_\Omega \alpha_1 \upsilon_t^\xi(t) w_j dx \\ &+ \int_\Omega \alpha_2 z_2^\xi(x,1,t) w_j dx = \int_\Omega f_2(u^\xi(t) \upsilon^\xi(t) w_j dx \end{aligned} \tag{2.76}$$

by using the property of continuous of the operator in the distributions space and due to (2.52)-(2.61) we have

$$\int_\Omega u_{tt}^\xi(t) w_j dx \rightharpoonup^* \int_\Omega u_{tt}(t) w_j dx \textit{ in } D'(0,T), \tag{2.77}$$

$$\int_\Omega \upsilon_{tt}^\xi(t) w_j dx \rightharpoonup^* \int_\Omega \upsilon_{tt}(t) w_j dx \textit{ in } D'(0,T), \tag{2.78}$$

$$\int_\Omega \nabla u^\xi(t) \nabla w_j dx \rightharpoonup^* \int_\Omega \nabla u(t) \nabla w_j dx \textit{ in } L^\infty(0,T), \tag{2.79}$$

$$\int_\Omega \nabla \upsilon^\xi(t), \nabla w_j dx \rightharpoonup^* \int_\Omega \nabla u(t) \nabla w_j dx \textit{ in } L^\infty(0,T), \tag{2.80}$$

$$\int_\Omega \nabla u^\xi_{tt}(t)\nabla w_j dx \rightharpoonup^* \int_\Omega \nabla u_{tt}(t)\nabla w_j dx \ in\ L^\infty(0,T), \tag{2.81}$$

$$\int_\Omega \nabla \upsilon^\xi_{tt}(t)\nabla w_j dx \rightharpoonup^* \int_\Omega \nabla u_{tt}(t)\nabla w_j dx \ in\ L^\infty(0,T), \tag{2.82}$$

$$\begin{aligned} &\int_\Omega \int_0^t g_1(t-s)\nabla u^\xi(t) ds \nabla w_j dx dt \\ &\rightharpoonup^* \int_\Omega \int_\Omega \int_0^t g_1(t-s)\nabla u(t) ds \nabla w_j dx dt \ in\ L^\infty(0,T), \end{aligned} \tag{2.83}$$

$$\begin{aligned} &\int_\Omega \int_0^t g_2(t-s)\nabla \upsilon^\xi(t) ds \nabla w_j dx dt \\ &\rightharpoonup^* \int_\Omega \int_\Omega \int_0^t g_2(t-s)\nabla \upsilon(t) ds \nabla w_j dx dt \ in\ L^\infty(0,T), \end{aligned} \tag{2.84}$$

$$\int_\Omega f_1(u^\xi(t)\upsilon^\xi(t) w_j dx \rightharpoonup^* \int_\Omega f_1(u^\xi(t)\upsilon^\xi(t) w_j dx \ in\ L^\infty(0,T), \tag{2.85}$$

$$\int_\Omega f_2(u^\xi(t)\upsilon^\xi(t) w_j dx \rightharpoonup^* \int_\Omega f_2(u^\xi(t)\upsilon^\xi(t) w_j dx \ in\ L^\infty(0,T), \tag{2.86}$$

$$\int_\Omega \mu_1 u^\xi_t(t), w_j dx \rightharpoonup^* \int_\Omega \mu_1 u_t(t) w_j dx \ in\ L^\infty(0,T), \tag{2.87}$$

$$\int_\Omega \mu_2 z^\xi_1(x,1,t) w_j dx \rightharpoonup^* \int_\Omega \mu_2 z_1(x,1,t) w_j dx \ in\ L^\infty(0,T), \tag{2.88}$$

$$\int_\Omega \alpha_1 \upsilon^\xi_t(t) w_j dx \rightharpoonup^* \int_\Omega \alpha_1 \upsilon_t(t) w_j dx \ in\ L^\infty(0,T), \tag{2.89}$$

$$\int_\Omega \alpha_2 z^\xi_2(x,1,t) w_j dx \rightharpoonup^* \int_\Omega \alpha_2 z_2(x,1,t) w_j dx \ in\ L^\infty(0,T), \tag{2.90}$$

as $\xi \to \infty$ the convergence (2.77)-(2.90) permits us to deduce that

$$\begin{aligned}
&\int_\Omega u^\xi_{tt}(t)w_j dx + \int_\Omega \nabla u^\xi(t)\nabla w_j dx + \int_\Omega \nabla u^\xi_{tt}(t)\nabla w_j dx\\
&+\int_\Omega\int_0^t g_1(t-s)\nabla u^\xi(t)ds\nabla w_j dxdt + \int_\Omega \mu_1 u^\xi_t(t)w_j dx\\
&+\int_\Omega \mu_2 z^\xi_1(x,1,t)w_j dx = \int_\Omega f_1(u^\xi(t),\upsilon^\xi(t)w_j dx,\\
&\to \int_\Omega u_{tt}(t)w_j dx + \int_\Omega \nabla u(t)\nabla w_j dx + \int_\Omega \nabla u_{tt}(t)\nabla w_j dx\\
&+\int_\Omega\int_0^t g_1(t-s)\nabla u(t)ds\nabla w_j dxdt + \int_\Omega \mu_1 u_t(t)w_j dx\\
&+\int_\Omega \mu_2 z_1(x,1,t)w_j dx = \int_\Omega f_1(u(t),\upsilon(t)w_j dx,
\end{aligned} \tag{2.91}$$

$$\begin{aligned}
&\int_\Omega \upsilon^\xi_{tt}(t)w_j dx + \int_\Omega \nabla \upsilon^\xi(t)\nabla w_j dx + \int_\Omega \nabla \upsilon^\xi_{tt}(t)\nabla w_j dx\\
&+\int_\Omega\int_0^t g_2(t-s)\nabla \upsilon^\xi(t)ds, \nabla w_j dxdt + \int_\Omega \alpha_1 \upsilon^\xi_t(t)w_j dx\\
&+\int_\Omega \alpha_2 z^\xi_2(x,1,t)w_j dx = \int_\Omega f_2(u^\xi(t),\upsilon^\xi(t)w_j dx,\\
&\to \int_\Omega \upsilon_{tt}(t)w_j dx + \int_\Omega \nabla \upsilon(t)\nabla w_j dx + \int_\Omega \nabla \upsilon_{tt}(t)\nabla w_j dx\\
&+\int_\Omega\int_0^t g_2(t-s)\nabla \upsilon(t)ds\nabla w_j dxdt + \int_\Omega \alpha_1 \upsilon_t(t)w_j dx\\
&+\int_\Omega \alpha_2 z_2(x,1,t)w_j dx = \int_\Omega f_2(u(t),\upsilon(t))
\end{aligned} \tag{2.92}$$

using (2.15)-(2.16) and exploiting the convergence (2.58), (2.59) we deduce

$$\int_0^T\int_0^1\int_\Omega (\tau_2\frac{\partial}{\partial t}z^\xi_1 + \frac{\partial}{\partial \rho}z^\xi_1)\varphi_j\ dx\ d\rho\ dt \to \int_0^T\int_0^1\int_\Omega (\tau_2\frac{\partial}{\partial t}z_1 + \frac{\partial}{\partial \rho}z_1)\varphi_j\ dx\ d\rho\ dt \tag{2.93}$$

$$\int_0^T\int_0^1\int_\Omega (\tau_2\frac{\partial}{\partial t}z^\xi_2 + \frac{\partial}{\partial \rho}z^\xi_2)\varphi_j\ dx\ d\rho\ dt \to \int_0^T\int_0^1\int_\Omega (\tau_2\frac{\partial}{\partial t}z_2 + \frac{\partial}{\partial \rho}z_2)\varphi_j\ dx\ d\rho\ dt \tag{2.94}$$

as $\xi \to +\infty$. Hence, this completes our proof of existence result of system (2.11)-(2.17). ■

**Remark 2.0.2** *By virtue of the theory of ordinary differential equations, the system* (2.9) *has local solution which is extended to a maximal interval* $[0, T_k[$ *with (* $0 < T_k \leq +\infty$).

Now we will prove that the solution obtained above is global and bounded in time, for this purpose, we define

$$I(t) = \xi_1 \int_\Omega \int_0^1 z_1^2(x,k_1,t)dk_1dx + \xi_2 \int_\Omega \int_0^1 z_2^2(x,k_2,t)dk_2dx + (p+1)\int_\Omega F(u,\upsilon)dx \\ + \left(1 - \int_0^t g_1(s)ds\right)\|\nabla u(t)\|_2^2 + \left(1 - \int_0^t g_2(s)ds\right)\|\nabla \upsilon(t)\|_2^2, \tag{2.95}$$

$$J(t) = \frac{\xi_1}{2} \int_\Omega \int_0^1 z_1^2(x,k_1,t)dk_1dx + \frac{\xi_2}{2} \int_\Omega \int_0^1 z_2^2(x,k_2,t)dk_2dx + \int_\Omega F(u,\upsilon)dx \\ +\frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g_2(s)ds\right)\|\nabla \upsilon(t)\|_2^2, \tag{2.96}$$

we observe that

$$E(t) = \frac{1}{2}(\|u_t(t)\|_2^2 + \|\upsilon_t(t)\|_2^2 + \|\nabla u_t(t)\|_2^2 + \|\nabla \upsilon_t(t)\|_2^2) + J(t). \tag{2.97}$$

**Lemma 2.0.6** . *Let $((u,z_1),(\upsilon,z_2))$, be the solution of problem (2.9). Assume further that $I(0) > 0$ and*

$$\alpha = \rho\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}} < 1. \tag{2.98}$$

*Then $I(t) > 0$ $\forall t$ moreover the solution of problem (2.9) is global and bounded, where $\rho$ is a positive constant appeared in lemma 2.0.4*

*Proof* Since $I(0) > 0$, then there exists (by continuity of $u(t)$) $T^* < T$ such that

$$I(t) \geq 0, \tag{2.99}$$

for all $t \in [0, T^*]$. From (2.95) (2.96) gives that

$$\begin{aligned} J(t) &\geq \frac{p-1}{2(p+1)}\left[l_1\|\nabla u\|_2^2 + l_2\|\nabla \upsilon\|_2^2 + \zeta_1 \int_0^1 \int_\Omega z_1^2(x,k_1,t)dk_1dx\right] \\ &+ \frac{p-1}{2(p+1)}\left[\zeta_2 \int_0^1 \int_\Omega z_2^2(x,k_2,t)dk_2dx\right] + \frac{1}{p+1}I(t) \\ &\geq \frac{p-1}{2(p+1)}\left[\|\nabla u\|_2^2 + \|\nabla \upsilon\|_2^2\right]. \end{aligned} \tag{2.100}$$

Thus by (2.100),(2.97) we deduce

$$\begin{aligned} \|\nabla u\|_2^2 + \|\nabla \upsilon\|_2^2 &\leq \frac{2(p+1)}{p-1}J(t) \leq \frac{2(p+1)}{(p-1)}E(t), \\ &\leq \frac{2(p+1)}{(p-1)}E(0),\ \forall t \in [0,T^*]. \end{aligned} \tag{2.101}$$

Employing lemma 2.0.4,$(A_1)$, we obtain

$$
\begin{aligned}
(p+1)\int_\Omega F(u(t_0,\upsilon(t_0))dx &\leq \rho(l_1\|\nabla u(t_0)\|_2^2+l_2\|\nabla\upsilon(t_0)\|_2^2)^{\frac{p+1}{2}},\\
&\leq \rho\left(\tfrac{2(p+1)}{p-1}\right)^{\frac{p-1}{2}}(\beta_1\|\nabla u(t_0)\|_2^2+\|\nabla\upsilon(t_0)\|_2^2),\\
&= \alpha(\beta_1\|\nabla u(t_0)\|_2^2+\|\nabla\upsilon(t_0)\|_2^2),\\
&< (\beta_1\|\nabla u(t_0)\|_2^2+\|\nabla\upsilon(t_0).
\end{aligned}
\tag{2.102}
$$

Hence, we conclude from (2.102) that $I(t)>0$ on $[0,t_0]$ which contradicts thus $I(t)>0$ on $[0,T]$
which completes the proof.

■

**Lemma 2.0.7** . *Let $((u,z_1),(\upsilon,z_2))$ be a solution of the problem (2.9). Then the energy functional defined by (2.97) satisfies*

$$
\begin{aligned}
&E'(t) = -\left(\mu_1-\frac{\xi_1}{2\tau_2}-\frac{\mu_2}{2}\right)\|u_t\|_2^2\\
&-\left(\alpha_1-\frac{\xi_2}{2\tau_2}-\frac{\alpha_2}{2}\right)\|\upsilon_t\|_2^2-\left(\frac{\xi_1}{2\tau_2}-\frac{\mu_2}{2}\right)\int_\Omega (z_1^n)^2(x,1,t)dx\\
&+\left(\frac{\xi_2}{2\tau_2}-\frac{\alpha_2}{2}\right)\int_\Omega (z_2^n)^2(x,1,t)dx \leq 0.
\end{aligned}
\tag{2.103}
$$

*Proof* after deriving the equation (2.26) we get the desired result. ■

**Remark 2.0.3** . *Due to the conditions (2.18),(2.19) we have* $\left(\mu_1-\frac{\xi_1}{2\tau_2}-\frac{\mu_2}{2}\right)>0$, $\left(\alpha_1-\frac{\xi_2}{2\tau_2}-\frac{\alpha_2}{2}\right)>0$, $\left(\frac{\xi_1}{2\tau_2}-\frac{\mu_2}{2}\right)>0$, $\left(\frac{\xi_2}{2\tau_2}-\frac{\alpha_2}{2}\right)>0$.

### 2.0.8 Asymptotic behavior

In this section we use the multiplier method introduced by A. Guessmia [20], we get the following result

**Theorem 2.0.3** . *Suppose that $\mu_2<\mu_1$, $\alpha_2<\alpha_1$, $(A_1)$, (2.98) holds. With satisfy $I(0)>0$ Assume that $((u_0,u_1),(\upsilon_0,\upsilon_1))\in(H_0^1(\Omega))^2$ and $(\phi_0,\phi_1)\in(L^2(\Omega\times(0,1)))^2$. Then the solution of problem (3.1) is global and bounded, Furthermore, we have the following decay property*

$$E(t)\leq cE(0)e^{-\omega t} \quad ,t\geq 0,$$

*where c, $\omega$ , are positive constants, independent of the initial data*

**Remark 2.0.4** *we denote by $c_i$ various constants which may be different at different occurences.*

**proof**. First, we prove $T=\infty$, it is sufficient to show that $l_1\|\nabla u\|_2^2+l_2\|\nabla \upsilon\|_2^2$ is bounded independently of t. we have from (2.97)

$$\begin{aligned} E(0)\geq E(t) &= \frac{1}{2}\|u'(t)\|_2^2+\frac{1}{2}\|\upsilon'(t)\|_2^2+J(t),\\ &\geq \frac{1}{2}\|u'(t)\|_2^2+\frac{1}{2}\|\upsilon'(t)\|_2^2\\ &+ \left(\frac{p-1}{2(p+1)}\right)(l_l\|\nabla u\|_2^2+l_l\|\nabla \upsilon\|_2^2),\\ &\geq (l_l\|\nabla u\|_2^2+l_l\|\nabla \upsilon\|_2^2). \end{aligned}$$

We multiply the first equation of (2.9) by $E^q u$, the second equation of (2.9) by $E^q\upsilon$, integrating over $[S,T]\times\Omega$ we get

$$\begin{aligned} 0 &= \int_S^T E^q\int_\Omega [uu''-|\nabla u|^2-\Delta u_{tt}u-\Delta \upsilon_{tt}\upsilon+\mu_1 u'u\\ &+\mu_2 z_1(x,1,t)u+\upsilon''\upsilon+\alpha_1\upsilon'\upsilon+\alpha_2 z_2(x,1,t)\upsilon+(uf_1(u,\upsilon)+\upsilon f_2(u,\upsilon))]dxdt\\ &-\int_0^t g_1(t-s)\Delta\upsilon(s)\upsilon(t)dsdxdt-\int_0^t g_2(t-s)\Delta\upsilon(s)\upsilon(t)dsdxdt \end{aligned} \tag{2.104}$$

$$\begin{aligned} 0 &= \left[E^q\int_\Omega uu'dx\right]_S^T-\int_S^T(qE'E^{q-1})\int_\Omega uu'dxdt-2\int_S^T E^q\int_\Omega u'^2dxdt\\ &+\int_S^T E^q\int_\Omega(u'^2+|\nabla u|^2)dxdt+\left[E^q\int_\Omega \upsilon\upsilon'dx\right]_S^T-\int_S^T(qE'E^{q-1})\int_\Omega \upsilon\upsilon'dxdt\\ &+2\int_S^T E^q\int_\Omega \upsilon'^2dxdt+\int_S^T E^q\int_\Omega(\upsilon'^2+|\nabla\upsilon|^2)dxdt+\int_S^T E^q u'\Delta u_t ds\\ &+\mu_1\int_S^T E^q\int_\Omega uu'dxdt+\mu_2\int_S^T E^q\int_\Omega uz_1(x,1,t)dxdt\\ &+\mu_3\int_S^T E^q\int_\Omega \upsilon\upsilon'dxdt+\mu_4\int_S^T E^q\int_\Omega \upsilon z_2(x,1,t)dxdt\\ &-\int_S^T E^q\int_\Omega(p+1)F(u,\upsilon)dxdt-\int_S^T E'E^{q-1}u\Delta u_t dt\\ &-\int_S^T E^q\int_\Omega\int_0^t g_1(t-s)\Delta\upsilon(s)\upsilon(t)dsdxdt\\ &-\int_S^T E^q\int_\Omega\int_0^t g_2(t-s)\Delta\upsilon(s)\upsilon(t)dsdxdt+[E^q u\Delta u_t]_S^T\\ &+[E^q\upsilon\Delta\upsilon_t]_S^T-\int_S^T E'E^{q-1}\upsilon\Delta\upsilon_t dt+\int_S^T E^q\upsilon'\Delta\upsilon_t dt. \end{aligned} \tag{2.105}$$

Similarly, we multiply the third equation in (2.9) by $E^q e^{-2\tau k_1} z_1 E^q e^{-2\tau k_2} z_2$ we get

$$
\begin{aligned}
0 &= \int_S^T E^q \int_\Omega \int_0^1 e^{-2\tau k_1} z_1(\tau z_1' + z_{k_1}) dx dt, \\
&= \left[ E^q \int_\Omega \int_0^1 \tau \int_0^1 e^{-2\tau k_1} z_1^2 dx dk_1 \right]_S^T - \tau \int_S^T (qE'E^{q-1}) \int_\Omega \int_0^1 e^{-2\tau k_1} z_1^2 dx dk_1 dt \\
&+ \int_S^T E^q \int_\Omega \int_0^1 \left( \frac{\partial}{\partial k_1}(e^{-2\tau k_1}) + 2\tau e^{-2\tau k_1} z_1^2 \right) dx dk_1 dt, \qquad (2.106)\\
&= \left[ E^q \int_\Omega \int_0^1 \tau e^{-2\tau k_1} z_1^2 dx dk_1 \right]_S^T - \tau \int_S^T (qE'E^{q-1}) \int_\Omega \int_0^1 e^{-2\tau k_1} z_1^2 dx dk_1 dt \\
&+ \int_S^T E^q \int_\Omega (e^{-2\tau} z_1^2(x,1,t) - z_1^2(x,0,t)) dx dt + 2\tau \int_S^T E^q \int_\Omega \int_0^1 e^{-2\tau k_1} z_1^2 dx dk_1 dt.
\end{aligned}
$$

In the same manner for the forth equation in (2.9)

$$
\begin{aligned}
&\int_S^T E^q \int_\Omega \int_0^1 e^{-2\tau k_2} z_2(\tau z_2' + z_{k_2}) dx dt \\
&= \left[ E^q \int_\Omega \int_0^1 \tau e^{-2\tau k_2} z_2^2 dx dk_2 \right]_S^T - \tau \int_S^T (qE'E^{q-1}) \int_\Omega \int_0^1 e^{-2\tau k_2} z_2^2 dx dk_2 dt \qquad (2.107)\\
&+ \int_S^T E^q \int_\Omega (e^{-2\tau} z_2^2(x,1,t) - z_2^2(x,0,t)) dx dt + 2\tau \int_S^T E^q \int_\Omega \int_0^1 e^{-2\tau k_2} z_2^2 dx dk_2 dt.
\end{aligned}
$$

Taking their sum, we obtain

$$
\begin{aligned}
&A \int_S^T E^{q+1} dt \le - \left[ E^q \int_\Omega u u' dx \right]_S^T + \int_S^T (qE'E^{q-1}) \int_\Omega u u' dx dt \\
&+2 \int_S^T E^q \int_\Omega u'^2 dx dt - \mu_1 \int_S^T E^q \int_\Omega u u' dx dt - \mu_2 \int_S^T E^q \int_\Omega u z_1(x,1,t) dx dt \\
&+p \int_S^T E^q \int_\Omega F(u,\upsilon) dx dt - \left[ E^q \int_\Omega \int_0^1 \tau e^{-2\tau k_1} z_1^2 dx dk_1 \right]_S^T \\
&\tau \int_S^T (qE'E^{q-1}) \int_\Omega \int_0^1 e^{-2\tau k_2} z_2^2 dx dk_1 dt + \int_S^T E^q \int_\Omega (e^{-2\tau} z_1^2(x,1,t) - z_1^2(x,0,t)) dx dt \\
&- \left[ E^q \int_\Omega \upsilon \upsilon' dx \right]_S^T + \int_S^T (qE'E^{q-1}) \int_\Omega \upsilon \upsilon' dx dt + 2 \int_S^T E^q \int_\Omega \upsilon'^2 dx dt \qquad (2.108)\\
&-\mu_3 \int_S^T E^q \int_\Omega \upsilon \upsilon' dx dt - \mu_4 \int_S^T E^q \int_\Omega u z_2(x,1,t) dx dt \\
&- \left[ E^q \int_\Omega \int_0^1 \tau e^{-2\tau k_2} z_2^2 dx dk_2 \right]_S^T \tau \int_S^T (qE'E^{q-1}) \int_\Omega \int_0^1 e^{-2\tau k_2} z_2^2 dx dk_2 dt \\
&+ \int_S^T E^q \int_\Omega (e^{-2\tau} z_2^2(x,1,t) - z_2^2(x,0,t)) dx dt \\
&+ \int_S^T E^q \int_\Omega \int_0^t g_1(t-s) \Delta \upsilon(s) \upsilon(t) ds dx dt + \int_S^T E^q \int_\Omega \int_0^t g_2(t-s) \Delta \upsilon(s) \upsilon(t) ds dx dt
\end{aligned}
$$

Where $A = 2\min\{1, \frac{2\tau e^{-2\tau}}{\zeta_1}, \frac{2\tau e^{-2\tau}}{\zeta_2}\}$. Using the Cauchy-Schwartz and poincaré inequalities and the energy identity, we get

$$\begin{aligned}\left|E^q\int_\Omega uu'dx\right| &\leq \epsilon_1 c_*\|\nabla u\|_2^2+\frac{1}{4\epsilon_1}\|u'\|_2^2,\\ &\leq cE(t)^{q+1},\end{aligned} \tag{2.109}$$

$$\begin{aligned}&\left|qE'E^{q-1}\int_\Omega uu'dx\right|\\ \leq\;& cE^q(t)|E'(t)|\\ \leq\;& cE^q(t)|-\left(\mu_1-\tfrac{\xi_1}{2\tau_2}-\tfrac{\mu_2}{2}\right)\|u'\|_2^2-\left(\alpha_1-\tfrac{\xi_2}{2\tau_2}-\tfrac{\alpha_2}{2}\right)\|\upsilon'\|_2^2\\ -\;& \left(\tfrac{\xi_1}{2\tau_2}-\tfrac{\mu_2}{2}\right)\int_\Omega z_1^2(x,1,t)dx-\left(\frac{\xi_2}{2\tau_2}-\frac{\alpha_2}{2}\right)\int_\Omega z_2^2(x,1,t)dx|\\ \leq\;& E^q(t)\left[\left(\mu_1-\tfrac{\xi_1}{2\tau_2}-\tfrac{\mu_2}{2}\right)\|u'\|_2^2+\left(\alpha_1-\tfrac{\xi_2}{2\tau_2}-\tfrac{\alpha_2}{2}\right)\|\upsilon'\|_2^2\right]\\ +\;& E^q(t)\left[\left(\tfrac{\xi_1}{2\tau_2}-\tfrac{\mu_2}{2}\right)\int_\Omega z_1^2(x,1,t)dx+\left(\frac{\xi_2}{2\tau_2}-\frac{\alpha_2}{2}\right)\int_\Omega z_2^2(x,1,t)dx\right]\\ \leq\;& E^q(t)(-E'(t)),\end{aligned} \tag{2.110}$$

$$\begin{aligned}&\left|qE'E^{q-1}\int_\Omega\int_0^1 e^{-2\tau_2 k_1}z_1^2dxdk_1\right|\\ \leq\;& cE^q(t)|E'(t)|\\ \leq\;& cE^q(t)|-\left(\mu_1-\tfrac{\xi_1}{2\tau_2}-\tfrac{\mu_2}{2}\right)\|u'\|_2^2-\left(\alpha_1-\tfrac{\xi_2}{2\tau_2}-\tfrac{\alpha_2}{2}\right)\|\upsilon'\|_2^2\\ -\;& \left(\tfrac{\xi_1}{2\tau_2}-\tfrac{\mu_2}{2}\right)\int_\Omega z_1^2(x,1,t)dx-\left(\frac{\xi_2}{2\tau_2}-\frac{\alpha_2}{2}\right)\int_\Omega z_2^2(x,1,t)dx|\\ \leq\;& E^q(t)\left[\left(\mu_1-\tfrac{\xi_1}{2\tau_2}-\tfrac{\mu_2}{2}\right)\|u'\|_2^2+\left(\alpha_1-\tfrac{\xi_2}{2\tau_2}-\tfrac{\alpha_2}{2}\right)\|\upsilon'\|_2^2\right]\\ +\;& E^q(t)\left[\left(\tfrac{\xi_1}{2\tau_2}-\tfrac{\mu_2}{2}\right)\textstyle\int_\Omega z_1^2(x,1,t)dx+\left(\tfrac{\xi_2}{2\tau_2}-\tfrac{\alpha_2}{2}\right)\int_\Omega z_2^2(x,1,t)dx\right]\\ \leq\;& E^q(t)(-E'(t)),\end{aligned} \tag{2.111}$$

$$\begin{aligned}\int_S^T E^q\int_\Omega uz_1(x,1,t)dxdt &\leq \int_S^T E^q(\epsilon_1 c_*^2\|\nabla u\|_2^2+\frac{1}{4\epsilon_1}\|z_1(x,1,t)\|_2^2)dt\\ &\leq c\int_S^T E^q(-E'+cE(t))dt,\end{aligned} \tag{2.112}$$

$$\begin{aligned}\int_S^T E^q\int_\Omega u'^2dxdt &\leq \int_S^T E^q(-E')dt\\ &\leq cE^{q+1}(S),\end{aligned} \tag{2.113}$$

$$\begin{aligned}\left[E^q\int_\Omega u'udx\right]_s^T &= E^q(S)\int_\Omega u'(S)u(S)dx\\ &- E^q(T)\int_\Omega u'(T)u(T)dx\\ &\leq CE^{q+1}(S),\end{aligned} \tag{2.114}$$

$$-\left[E^q\int_\Omega\int_0^1 \tau e^{-2\tau k_1} z_1^2 dxdk_1\right]_S^T \leq E^q(S)\int_\Omega\int_0^1 e^{-2\tau k_1}|z_1(x,k_1,S)|^2 dxdk_1 - E^q(T)\int_\Omega\int_0^1 e^{-2\tau k_1}|z_1(x,k_1,T)|^2 dxdk_1 \leq cE^{q+1}(S), \tag{2.115}$$

and

$$\int_S^T (qE'E^{q-1})\int_\Omega\int_0^1 e^{-2\tau k_1}|z_1|^2 dxdk_1dt \leq c\int_S^T qE'E^q dt \leq cE^{q+1}(S), \tag{2.116}$$

and

$$\int_S^T E^q\int_\Omega e^{-2\tau}|z_1(x,1,t)|^2 dxdt \leq c\int_S^T E^q(-E')dt \leq cE^{q+1}(S), \tag{2.117}$$

and

$$\int_S^T E^q\int_\Omega e^{-2\tau}|z_1(x,0,t)|^2 dxdt = \int_S^T\int_\Omega E^q e^{-2\tau}|u'(x,t)|^2 dxdt \leq c\int_S^T E^q(-E')dt \leq cE^{q+1}(S), \tag{2.118}$$

using the lemma 2.0.4, to get

$$\left|p\int_\Omega F(u,\upsilon)dx\right| \leq p\rho(l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla\upsilon(t)\|_2^2)^{\frac{p+1}{2}} \leq p\rho\left(\frac{2(p+1)}{p-1}\right)^{\frac{p-1}{2}}(l_1\|\nabla u(t)\|_2^2 + l_2\|\nabla\upsilon(t)\|_2^2) \leq c\epsilon_2 E(t), \tag{2.119}$$

$$\begin{aligned}
\int_\Omega \int_0^t & g_1(t-s)\nabla u(t)(\nabla u(s)-\nabla u(t))dsdx \\
= & \frac{1}{2}\left[\int_0^t g_1(t-s)(\|\nabla u(s)\|_2^2 + \|\nabla u(t)\|_2^2)ds - \int_0^t g_1(t-s)(\|\nabla u(t)-\nabla u(s)\|_2^2)ds\right] \\
- & \int_\Omega\int_0^t g_1(s)|\nabla u(t)|^2 dsdx, \\
= & -\frac{1}{2}\int_\Omega\int_0^t g_1(s)|\nabla u(s)|^2 dsdx + \frac{1}{2}\int_0^t g_1(t-s)ds\|\nabla u(s)\|_2^2 ds - \frac{1}{2}(g_1 o\nabla u)(t).
\end{aligned} \tag{2.120}$$

This yields

$$
\begin{aligned}
&-\int_\Omega \int_0^t g_1(t-s)\Delta u(s)u(t)dsdx = \int_\Omega \int_0^t g_1(t-s)\nabla u(s)\nabla u(t)dsdx \\
&= \frac{1}{2}\left[\int_0^t g_1(t-s)\|\nabla u(t)\|_2^2 ds + \int_0^t g_1(t-s)\|\nabla u(s)\|_2^2 ds - (g_1 o \nabla u)(t)\right].
\end{aligned}
\tag{2.121}
$$

Similarly

$$
\begin{aligned}
&-\int_\Omega \int_0^t g_2(t-s)\Delta \upsilon(s)\upsilon(t)dsdx = \int_\Omega \int_0^t g_2(t-s)\nabla \upsilon(s)\nabla \upsilon(t)dsdx \\
&= \frac{1}{2}\left[\int_0^t g_2(t-s)\|\nabla \upsilon(t)\|_2^2 ds + \int_0^t g_2(t-s)\|\nabla \upsilon(s)\|_2^2 ds - (g_2 o \nabla \upsilon)(t)\right].
\end{aligned}
\tag{2.122}
$$

Employing Young's inequality for convolution $\|\varphi * \phi\| \leq \|\varphi\|\|\phi\|$, and using the potential well method's we easily find

$$
\begin{aligned}
\int_0^t \int_S^T g_1(t-s)\|\nabla u(s)\|_2^2 dsdt, &\leq \int_0^\infty g_1(t)dt \int_S^T \|\nabla u(t)\|_2^2 dt, \\
&\leq (1-l_1)\int_S^T \|\nabla u(t)\|_2^2 dt, \\
&\leq (1-l_1)\int_S^T E(t)dt,
\end{aligned}
\tag{2.123}
$$

in the same manner

$$
\begin{aligned}
\int_0^t \int_S^T g_2(s)\|\nabla u(t)\|_2^2 dsdt &\leq \int_0^\infty \int_S^T g_2(s)ds\|\nabla u(t)\|_2^2 dt \\
&\leq (1-l_2)\int_S^T E(t)dt,
\end{aligned}
\tag{2.124}
$$

exploiting (2.123) to obtain

$$
\begin{aligned}
\int_S^T (g_1 o \nabla u)(t)dt &= \int_0^t \int_S^T g_1(t-s)\|\nabla u(s) - \nabla u(t)\| dsdt, \\
&\leq \int_0^t g_1(t)dt \int_S^T \|\nabla u(t)\| dt, \\
&\leq \int_0^\infty g_1(t)dt \int_S^T l_1 \|\nabla u(t)\| dt, \\
&\leq (1-l_1)\int_S^T E(t)dt.
\end{aligned}
\tag{2.125}
$$

Likewise we obtain

$$
\int_S^T (g_2 o \nabla u)(t)dt \leq (1-l_2)\int_S^T E(t)dt
\tag{2.126}
$$

$$
\begin{aligned}
& \int_S^T \left| -\int_\Omega \int_0^t g_1(t-s)\Delta \upsilon(s)\upsilon(t) dt ds dx \right| \\
= & \left| \frac{1}{2} \int_S^T \left[ \int_0^t g_1(t-s) \|\nabla \upsilon(s)\|_2^2 ds - (g_1 o \nabla \upsilon)(t) \right] \right| dt \\
+ & \int_S^T \left| \frac{1}{2} \left[ \int_0^t g_1(t) \|\nabla \upsilon(t)\|_2^2 ds \right] \right| dt \\
\le & \int_S^T \epsilon \frac{1}{2} \left| \int_0^t g_1(t) \|\nabla \upsilon(t)\|_2^2 ds \right| dt + \epsilon \frac{1}{2} \int_S^T |(g_1 o \nabla \upsilon)(t)| \, dt \\
+ & \epsilon \int_S^T \frac{1}{2} \left| \int_0^t g_1(t) \|\nabla \upsilon(t)\|_2^2 ds \right| dt \\
\le & \epsilon \int_S^T (\frac{1}{2} - l_1) E(t) dt,
\end{aligned}
\tag{2.127}
$$

for some $\epsilon$ sufficiently small we take $\epsilon(\frac{1}{2} - l_1) > 0$
in the same way it gives

$$
\int_S^T \left| -\int_\Omega \int_0^t g_2(t-s)\Delta \upsilon(s)\upsilon(t) ds dx \right| dt \le \epsilon(\frac{1}{2} - l_1) \int_S^T E(t) dt,
\tag{2.128}
$$

then

$$
\int_S^T E^q \left| -\int_\Omega \int_0^t g_1(t-s)\Delta \upsilon(s)\upsilon(t) ds dx dt \right| \le \epsilon(\frac{1}{2} - l_1) \int_S^T E^{q+1}(t) dt,
\tag{2.129}
$$

$$
\int_S^T E^q \left| -\int_\Omega \int_0^t g_2(t-s)\Delta \upsilon(s)\upsilon(t) ds dx dt \right| \le \epsilon(\frac{1}{2} - l_2) \int_S^T E^{q+1}(t) dt,
\tag{2.130}
$$

for the second and third equation we use the same technique.
Combining all above inequalities, and choosing $\epsilon_2$ small enough, we deduce from (2.109)-(2.131) that

$$
\int_S^T E^{q+1} dt \le c(E^{q+1}(S) + E^{q+1}(T)).
\tag{2.131}
$$

Where c is a positive constant independent of E(0). From the last inequality, and the conditions of lemma 1.3.6 are satisfied, then

$$
E(t) \le cE(0)e^{-\omega t} \quad t \ge 0.
\tag{2.132}
$$

■

# Chapter 3

# Asymptotic behavior for a weak viscoelastic wave equations with a dynamic boundary and time varying delay term

## 3.1 Introduction

In this paper, we consider the weak viscoelastic wave equations with dynamic boundary conditions related to the Kelvin Voigt damping and delay term acting on the boundary in a bounded domain. Under appropriate conditions on $\mu_1$ and $\mu_2$, we prove the asymptotic behavior by making use an appropriate Lyapunov functional.

$$(3.1)\quad \begin{cases} u_{tt} - \Delta u - \delta \Delta u_t - \alpha(t) \displaystyle\int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u, & in \quad \Omega \times (0,+\infty), \\ u = 0, & on \quad \Gamma_0 \times (0,+\infty), \\ u_{tt} = -a \left[ \dfrac{\partial u}{\partial \upsilon}(x,t) + \delta \dfrac{\partial u_t}{\partial \upsilon}(x,t) + \alpha(t) \displaystyle\int_0^t g(t-s)\Delta u(s) \dfrac{\partial u}{\partial \upsilon}(x,s)ds \right] & \\ -a\left[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau(t))\right], & on \quad \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau(t)) = f_0(x,t-\tau(t)), & on \quad \Gamma_1 \times (0,+\infty). \end{cases}$$

where $u = u(x,t)$ , $t \geq 0$ , $x \in \Omega$, $\Delta$ denotes the Laplacian operator with respect to the $x$ variable, $\Omega$ is a regular and bounded domain of $R^N$, $(N \geq 1)$, $\partial\Omega = \Gamma_1 \cup \Gamma_0$, $\Gamma_1 \cap \Gamma_0 = \emptyset$ and $\frac{\partial}{\partial \nu}$denotes the unit outer normal derivative, $\mu_1$ and $\mu_2$ are positive constants. Moreover, $\tau(t) > 0$ represents the time varying delay term and $u_0$, $u_1$, $f_0$ are given functions belonging to suitable spaces that will be precised later. This type of problems arises (for example) in modeling of longitudinal vibrations in a homogeneous bar on which there are viscous effects. The term $\Delta u_t$, indicates that the stress is proportional not only to the strain, but also to

the strain rate. See [28].
This type of problem without delay (i.e $\mu_i = 0$), has been considered by many authors during the past decades and many results have been obtained (see [28], [30], [32], [33], [46]).
The main difficulty of the problem considered is related to the non ordinary boundary conditions defined on $\Gamma_1$. Very little attention has been paid to this type of boundary conditions. We mention a few particular results in the one dimensional without delay term for a linear damping (m=1)and $g = 0$ ([36-48], [33]). From the mathematical point of view, these problems do not neglect acceleration terms on the boundary. Such types of boundary conditions are usually called dynamic boundary conditions. They are not only important from the theoretical point of view but also arise in several physical applications. For instance in one space dimension, problem (3.1) can modelize the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end. The dynamic boundary conditions represent the Newton's law for the attached mass, (see [30, 28, 32] for more details). Which arise when we consider the transverse motion of a flexible membrane whose boundary may be affected by the vibrations only in a region. Also some of them as in problem (3.1) appear when we assume that is an exterior domain of $R^3$ in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [28] for more details). Among the early results dealing with the dynamic boundary conditions are those of Grobbelaar-Van Dalsen [34, 35] in which the author has made contributions to this field and in [39] the authors have studied the following problem :

$$\begin{cases} u_{tt} - \Delta u + \delta \Delta u_t = |u|^{p-1}u, & in \quad \Omega \times (0, +\infty), \\ u = 0, & on \quad \Gamma_0 \times (0, +\infty), \\ u_{tt} = -a \left[ \dfrac{\partial u}{\partial \upsilon}(x,t) + \delta \dfrac{\partial u_t}{\partial \upsilon}(x,t) + \alpha |u_t|^{m-1} u_t(x,t) \right], & on \quad \Gamma_1 \times (0, +\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \end{cases}$$

and they have obtained several results concerning local existence which extended to the global existence by using stable sets, the authors have obtained also the energy decay and the blow up of the solutions for initial energy positive.
In absence of delay ($\mu_2 = 0$), the problem of existence and energy decay have been extensively studied by several authors (see [28], [30], [32], [34] ) and many energy estimates have been derived for arbitrary growing feedbacks (polynomial, exponential or logarithmic decay).
very recently the authors in [55] studied the following problem:

$$\begin{cases} u_{tt} - \Delta u + b(x) + f(u) = 0, & in \quad \Omega \times (0, +\infty), \\ u(x,t) = 0, & on \quad \Gamma_0 \times (0, +\infty), \\ \dfrac{\partial u}{\partial \nu} + g(u_t(x,t)) = 0, & on \quad \Gamma_1 \times (0, +\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \end{cases}$$

they proved the existence, uniqueness and uniform stability of strong and weak solutions of the nonlinear wave equation in bounded domains with nonlinear damped boundary conditions with restrictions on function $f(u)$ ; $g(u_t)$ and $b(x)$ . They proved the existence by means of the Glerkin method and obtain the asymptotic behavior by using perturbed energy method and combining some ideas of Kmornik and Zuazua (see [80]).
It is widely known that delay effects, which arise in many practical problems, source of some instabilities, in this way Datko and Nicaise [72,73] showed that a small delay in a boundary control turns a well-behave hyperbolic system into a wild one which in turn, becomes a source of instability, where they proved that the energy is exponentially stable under the condition

$$\mu_2 < \mu_1. \tag{3.2}$$

Recently, inspired by the works of Al and Nicaise [37] , Sthéphan Gherbi and B. Said-Houari [39] considered the following problem in bounded domain:

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t = 0, & in \quad \Omega \times (0, +\infty), \\ u = 0, & on \quad \Gamma_0 \times (0, +\infty), \\ u_{tt} = -a \left[ \dfrac{\partial u}{\partial \upsilon}(x,t) + \alpha \dfrac{\partial u_t}{\partial \upsilon}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) \right], & on \quad \Gamma_1 \times (0, +\infty), \\ \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x, t-\tau) = f_0(x, t-\tau), & on \quad \Gamma_1 \times (0, +\infty), \end{cases}$$

and obtained several results concerning global existence and exponential decay rates for various signs of $\mu_1$,$\mu_2$.
The case of time varying delay in the wave equation has been studied recently by Nicaise, Valein and Fridman , ([37] in one-space dimension and in the linear case in problem (3.1) and proved an exponential stability result under the condition

$$\mu_2 < \sqrt{1-d}\mu_1,$$

where the constant d satisfies

$$\tau'(t) \le d < 1, \quad \forall t > 0.$$

In( [12]) Nicaise, Pignotti and Valein extended the above result to higher-space dimension and established an exponential decay.
Very recently Z-Y Zhang, J- Huang, Z-H. Liu, and M. Sun [54] , have sudied a more general model than the above one

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t h(t-s)ds + a u_t(x, t-\tau(t)) = 0, & in \quad \Omega \times (0, +\infty), \\ u(x,t) = 0, & on \quad \Gamma_0 \times (0, +\infty), \\ \\ \frac{\partial u}{\partial \nu} + g(u_t(x,t)) = 0, & on \quad \Gamma_1 \times (0, +\infty), \\ \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x, t-\tau(t)) = f_0(x, t-\tau(t)), & on \quad \Gamma_1 \times (0, +\infty). \end{cases}$$

Since it contains nonlinear term in the boundary. They investigated a nonlinear viscoelastic equation with interior time-varying delay and nonlinear dissipative boundary feedback. Under suitable assumptions on the relaxation function and time-varying delay effect together with nonlinear dissipative boundary feedback, they proved the global existence of weak solutions and asymptotic behavior of the energy by using the Faedo-Galerkin method and the perturbed energy method.
Motivated by the previous works, it is interesting to investigate the rate of decay of solutions by using an appropriate Lyapunov functional Precisely, we show that the decay rate of energy function is exponential depending on both functions $\sigma(t)$ and $\alpha(t)$ that will be precised later.

## 3.2 Preliminary Results

In this section, we present some material for the proof of our result. For the relaxation function g, $\alpha$ and $\sigma$ we assume
$(A_0)$ $g, \alpha : R_+ \to R_+$ are nonincreasing differentiable functions satisfying

$$(3.3)\; g(0) > 0,\; l_0 = \int_0^\infty g(s)ds < \infty,\; \alpha(t) > 0, \quad 1 - \alpha(t)\int_0^t g(s)ds = l > 0 \; for \; t > 0,$$

there exists a nonincreasing differentiable function $\sigma : R^+ \to R^+$ satisfying

$$g'(t) \le -\sigma(t)g(t), \quad \sigma(t) > 0, \qquad for\; t > 0,\; \lim_{t\to\infty} \frac{-\alpha'(t)}{\sigma(t)\alpha(t)} = 0.$$

$(A_1)$ $\tau$ is a function such that

$$(3.4)\qquad \tau \in W^{2,\infty}([0,T]), \quad \forall T > 0,$$

$$(3.5)\qquad 0 < \tau_0 \le \tau(t) \le \tau_1, \quad \forall t > 0,$$

$$(3.6)\qquad \tau'(t) \le d < 1, \quad \forall t > 0,$$

where $\tau_0$ and $\tau_1$ are two positive constants.
$(A_2)$

$$(3.7)\qquad \mu_2 < \sqrt{1-d}\,\mu_1.$$

We choose $\xi$ such that

$$(3.8)\qquad \frac{\mu_2}{\sqrt{1-d}} < \xi < 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}.$$

We denote

$$V = \left\{v \in H_0^1(\Omega) : v = 0 \quad on \quad \Gamma_0\right\} = H_{\Gamma_0}^1(\Omega),$$

we denote $\langle .,. \rangle$ the scalar product in $L^2(\Omega)$ i.e. $\langle u, v\rangle = \int_\Omega u(x,t)v(x,t)dx$. Also we mean by $\|.\|_q$ the $L^q(\Omega)$ norm for $1 \le q \le \infty$, and by $\|.\|_{q.\Gamma_1}$ the $L^q(\Gamma_1)$ norm.

Let T¿0 be a real number and X a Banach space endowed with norm $\|.\|_X$. $L^p(0,T;X)$, $1 \leq p < \infty$ denotes the space of functions $f$ which are $L^p$ over (0,T) with values in X, which are measurable and $f \in L^p(0,T;X)$. This space is a Banach space endowed with the norm

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f\|_X^p dt\right)^{\frac{1}{p}}.$$

$L^\infty(0,T;X)$ denotes the space of functions $f$ :$]0,T[\to X$ which are measurable and $f \in L^\infty(0,T)$. This space is a Banach space endowed with the norm :

$$\|f\|_{L^\infty(0,T;X)} = ess \sup_{0<t<T} \|f\|_X.$$

We recall that if X and Y are two Banach spaces such that $X \hookrightarrow Y$( continuous embedding), then

$$L^p(0,T;X) \hookrightarrow L^p(0,T;Y), \quad 1 \leq p \leq \infty.$$

We will also use the embedding

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^p(\Omega), \quad 2 \leq p \leq \bar{p} \quad \text{where} \quad \bar{p} = \begin{cases} \frac{2N}{N-2} & \text{if} \quad N \geq 3, \\ +\infty & \text{if} \quad N = 1,2, \end{cases}$$

and also

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^p(\Gamma_1), \quad 2 \leq q \leq \bar{q} \quad \text{where} \quad \bar{q} = \begin{cases} \frac{2(N-1)}{N-2} & \text{if} \quad N \geq 3, \\ +\infty & \text{if} \quad N = 1,2. \end{cases}$$

We denote $V = H^1_{\Gamma_0}(\Omega) \cap L^2(\Gamma_1)$.
Now we give some estimates related to the convolution operator. By direct calculations, as in [18-19] we find

$$\begin{aligned} \sigma(t)(g*u,u_t) &= -\frac{\sigma(t)}{2}g(t)\|u(t)\|_2^2 - \frac{d}{dt}\left[\frac{\sigma(t)}{2}(g\ o\ u)(t) - \frac{\sigma(t)}{2}\left(\int_0^\infty g(s)ds\right)\|u(t)\|_2^2\right] \\ &+\tfrac{\sigma(t)}{2}(g'\ o\ u)(t) + \tfrac{\sigma'(t)}{2}(g\ o\ u)(t) - \tfrac{\sigma'(t)}{2}\int_0^\infty g(s)ds\|u(t)\|_2^2, \end{aligned} \tag{3.9}$$

where

$$(g*u)(t) = \int_0^\infty g(t-s)u(s)ds,\ g\ o\ u = \int_0^\infty g(t-s)u(s)ds\|u(t)-u(s)\|_2^2 ds, \tag{3.10}$$

and

$$(g*u,u) \leq 2\left(\int_0^t g(s)ds\right)\|u(t)\|_2^2 + \frac{1}{4}(g\ o\ u)(t). \tag{3.11}$$

Let us consider the new variable $z$ as in [12] ,

$$z(x,k,t)=u_t(x,t-\tau(t)k),\ \ x\in\Gamma_1, k\in(0,1),$$

which implies that

$$\tau(t)z_t(x,k,t)+(1-\tau'(t)k)z_k(x,k,t)=0, \quad in \quad \Gamma_1\times(0,1)\times(0,\infty).$$

Therefore, problem (3.1) is equivalent to:

$$(3.12)\begin{cases} u_{tt}-\Delta u-\delta\Delta u_t+\alpha(t)\displaystyle\int_0^t g(t-s)\Delta u(s)ds=|u|^{p-2}u, \quad in \quad \Omega\times(0,\infty),\\ u_{tt}=-a\left[\dfrac{\partial u}{\partial \upsilon}(x,t)+\delta\dfrac{\partial u_t}{\partial \upsilon}(x,t)+\alpha(t)\displaystyle\int_0^t g(t-s)\Delta u(s)\dfrac{\partial u}{\partial \upsilon}(x,s)ds\right]\\ +\left[\mu_1 u_t(x,t)+\mu_2 z_k(x,1,t)\right], \qquad on \quad \Gamma_1\times(0,+\infty),\\ \tau(t)z_t(x,k,t)+z_k(x,k,t)=0, \qquad in \quad \Gamma_1\times(0,1)\times(0,\infty),\\ z(x,k,0)=f_0(x,-\tau k), \qquad x\in\Gamma_1,\\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), \qquad x\in\Omega,\\ u(x,t)=0, \qquad x\in\Gamma_0, \quad t\geq 0. \end{cases}$$

**Remark**. For seeking of simplicity, we take $a=1$ in (3.12) .
Now inspired by [43,44], we define the modified energy functional related with problem (3.12) by

$$(3.13)\quad \begin{aligned} E(t) &=\frac{1}{2}\left(1-\alpha(t)\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2+\frac{\xi(t)\tau(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,s))dkd\gamma\\ &+\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|u_t(t)\|_{2.\Gamma_1}^2-\frac{1}{p}\|u(t)\|_p^p+\alpha(t)(g\ o\ \nabla u)(t). \end{aligned}$$

**Lemma 3.2.1** . *Let $2\leq p\leq \bar{q}$ and $(u,z)$ be a solution of the problem (3.12). Then the energy functional defined by (3.12) satisfies*

$$(3.14)\quad \begin{aligned} E'(t) &\leq -\left(\frac{\xi(1-\tau'(t))}{2}-\frac{\mu_2\sqrt{1-d}}{2}\right)\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma-\delta\|\nabla u_t(t)\|_2^2\\ &-\left(\mu_1-\frac{\xi}{2}-\frac{\mu_2}{2\sqrt{1-d}}\right)\|u_t(t)\|_{2.\Gamma_1}^2+\frac{\alpha(t)}{2}(g'\ o\ \nabla u)(t)\\ &-\frac{\alpha'(t)}{2}\int_0^t g(s)ds\|\nabla u(t)\|_2^2-\frac{\alpha(t)}{2}g(t)\|\nabla u(t)\|_2^2. \end{aligned}$$

*Proof* By multiplying the first and second equation in (3.12) by $u_t(t)$, and integrating the first equation over $\Omega$ and the second equation over $\Gamma_1$, using the Green's formula, we get

$$
\begin{aligned}
(3.15)\qquad & \frac{d}{dt}\left[\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|u_t(t)\|_{2.\Gamma_1}^2+\frac{1}{2}\|\nabla u(t)\|_2^2-\frac{1}{p}\|u(t)\|_p^p\right]\\
+ & \ \mu_1\int_{\Gamma_1}\|u_t(t)\|_{2.\Gamma_1}^2 d\gamma+\int_{\Gamma_1}\mu_2 z(\gamma,1,t)u_t(t)d\gamma\\
+ & \ \alpha(t)(g' o\nabla u)(t)-\alpha'(t)\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2\\
- & \ \frac{\alpha(t)}{2}g(t)\|\nabla u(t)\|_2^2+\frac{\alpha'(t)}{2}(g\ o\ \nabla u)(t)+\delta\|\nabla u_t(t)\|_2^2=0.
\end{aligned}
$$

We multiply the third equation in (3.12) by $\xi(t)z$ and integrate over $\Gamma_1\times(0,1)$ to obtain

$$
\begin{aligned}
(3.16)\qquad & \xi(t)\tau(t)\int_{\Gamma_1}\int_0^1 z_t z(\gamma,k,t)dkd\gamma\\
= & \ -\frac{\xi(t)}{2}\int_{\Gamma_1}\int_0^1(1-\tau'(t)k)\frac{\partial}{\partial k}z^2(\gamma,k,t)dkd\gamma.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
(3.17)\qquad & \frac{d}{dt}\left(\frac{\xi(t)\tau(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma\right)\\
= & \ -\frac{\xi(t)}{2}\int_0^1\int_{\Gamma_1}\frac{\partial}{\partial k}((1-\tau'(t)k)z^2(\gamma,k,t))dkd\gamma\\
+ & \ \frac{\xi'(t)\tau(t)}{2}\int_0^1\int_{\Gamma_1}z^2(\gamma,k,t)dkd\gamma\\
= & \ \frac{\xi(t)}{2}\int_{\Gamma_1}(z^2(\gamma,0,t)-z^2(\gamma,1,t))d\gamma+\frac{\xi(t)\tau'(t)}{2}\int_{\Gamma_1}z^2(\gamma,1,t)d\gamma\\
+ & \ \frac{\xi'(t)\tau(t)}{2}\int_0^1\int_{\Gamma_1}z^2(\gamma,k,t)dkd\gamma\\
\leq & \ \frac{\xi(t)}{2}\int_{\Gamma_1}(z^2(\gamma,0,t)-z^2(\gamma,1,t))d\gamma+\frac{\xi(t)\tau'(t)}{2}\int_{\Gamma_1}z^2(\gamma,1,t)d\gamma.
\end{aligned}
$$

From (3.15) , (3.17) and Young's inequality, we get

$$
\begin{aligned}
(3.18)\qquad E'(t) \leq & \ -\left(\mu_1-\frac{\xi(t)}{2}\right)\|u_t(t)\|_{2.\gamma_1}^2-\left(\frac{\xi(t)(1-\tau'(t))}{2}\right)\int_{\Gamma_1}z^2(\gamma,k,t)d\gamma\\
- & \ \mu_2\int_{\Gamma_1}z(\gamma,1,t)u_t(\gamma,t)d\gamma+\frac{\alpha(t)}{2}(g'\ o\ \nabla u)(t)-\delta\|\nabla u_t(t)\|_2^2\\
- & \ \frac{\alpha'(t)}{2}\int_0^t g(s)ds\|\nabla u(t)\|_2^2-\frac{\alpha(t)}{2}g(t)\|\nabla u(t)\|_2^2.
\end{aligned}
$$

Due to Young's inequality, we have

$$
(3.19)\qquad \mu_2\int_{\Gamma_1}z(\gamma,1,t)u_t(\gamma,t)d\gamma\leq\frac{\mu_2}{2\sqrt{1-d}}\|u_t(t)\|_{2.\Gamma_1}^2+\frac{\mu_2\sqrt{1-d}}{2}\int_{\Gamma_1}z^2(\gamma,1,t)d\gamma.
$$

Inserting (3.19) into (3.18), we obtain

$$
\begin{aligned}
E'(t) \leq & -\left(\frac{\xi(1-\tau'(t))}{2} - \frac{\mu_2\sqrt{1-d}}{2}\right)\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma - \delta\|\nabla u_t(t)\|_2^2 \\
& -\left(\mu_1 - \frac{\xi}{2} - \frac{\mu_2}{2\sqrt{1-d}}\right)\|u_t(t)\|_{2.\Gamma_1}^2 + \frac{\alpha(t)}{2}(g' \ o \ \nabla u)(t) \\
& -\frac{\alpha'(t)}{2}\int_0^t g(s)ds\|\nabla u(t)\|_2^2 - \frac{\alpha(t)}{2}g(t)\|\nabla u(t)\|_2^2.
\end{aligned}
\tag{3.20}
$$

This completes the proof. ■

**Remark**. Since $-\frac{\alpha'(t)}{2}\int_0^\infty g(s)ds\|\nabla u(t)\|^2 > 0$. E(t) may not be non-increasing.

**Remark**. The following result to problem (3.12) can be established by combining arguments of ([40],[52]).

**Theorem 3.2.1** *Let $2 \leq p \leq \bar{q}$ and then given $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$, $f_0 \in L^2(\Gamma_1 \times (0,1))$. Suppose that $(A_0)-(A_2)$ hold. Then the problem (3.12) admits a unique weak solution satisfying*

$$
u \in L^\infty((0,T); H^1_{\Gamma_0}(\Omega)), \quad u_t \in L^\infty((0,T); H^1_{\Gamma_0}(\Omega)) \cap L^\infty((0,T); L^2(\Gamma_1)),
$$
$$
u_{tt} \in L^\infty((0,T); L^2(\Omega)) \cap L^\infty((0,T); L^2(\Gamma_1)).
$$

## 3.3 Asymptotic Behavior

In this section, we establish the asymptotic behavior for the solutions. We define the following perturbed function:

$$
L(t) = ME(t) + \epsilon\alpha(t)\psi(t) + \epsilon\alpha(t)I(t) + \epsilon\frac{\delta\alpha(t)}{2}\|\nabla u\|_2^2, \tag{3.21}
$$

where

$$
\psi(t) = \int_\Omega uu_t dx + \int_{\Gamma_1} uu_t d\gamma, \tag{3.22}
$$

and

$$
I(t) = \xi(t)\int_{\Gamma_1}\int_0^1 e^{-k\tau(t)}z^2(\gamma,k,t)dkd\gamma. \tag{3.23}
$$

We need also the following lemma

**Lemma 3.3.1** *Let (u,z) be a solution of problem (3.12), then there exists two positive constants $\lambda_1$,$\lambda_2$ such that*

$$
\lambda_1 E(t) \leq L(t) \leq \lambda_2 E(t), \quad t \geq 0, \tag{3.24}
$$

*for M sufficiently large .*

*Proof* Thank's to the Cauchy Schwarz and Young's inequalities, and using the fact that $\|u\|_{2.\Gamma_1} \leq B\|\nabla u\|_2$, we have

$$|\psi(t)| \leq \frac{1}{\omega}\|u_t\|_2^2 + \frac{1}{4\omega}\|u_t\|_{2.\Gamma}^2 + \omega\|\nabla u\|_2^2 + \omega B^2\|\nabla u\|_2^2, \tag{3.25}$$

it follows from (3.23) that $\forall c > 0$

$$\begin{aligned} |I(t)| &= \left|\xi(t)\int_{\Gamma_1}\int_0^1 e^{-k\tau(t)} z^2(\gamma, k, s))dkd\gamma\right| \\ &\leq c\xi(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma, k, s))dkd\gamma. \end{aligned} \tag{3.26}$$

Hence, combining (3.25),(3.26), and using the fact that $\alpha(t) < \alpha(0)$. This yields

$$\begin{aligned} |L(t) - ME(t)| &= \epsilon\alpha(t)\psi(t) + \xi(t)\int_{\Gamma_1}\int_0^1 e^{-k\tau(t)} z^2(\gamma, k, t)dkd\gamma \\ &\leq \frac{\epsilon}{\omega}\|u_t\|_2^2 + \frac{\epsilon}{4\omega}\|u_t\|_{2.\Gamma_1}^2 + (\epsilon\omega + \epsilon B^2)\|\nabla u\|_2^2 \\ &+ c\xi(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma, k, t)dkd\gamma + \epsilon\frac{\delta\alpha(t)}{2}\|\nabla u\|_2^2. \end{aligned} \tag{3.27}$$

Where $c_1 = \frac{\epsilon}{\omega}$, $c_2 = \frac{\epsilon}{4\omega}$, $c_3 = (\epsilon\omega + \epsilon B^2)$, $c_4 = c$, then we can write

$$|L(t) - ME(t)| \leq c_5 E(t), \tag{3.28}$$

where $c_5 = max(c_1, c_2, c_3, c_4)$. Thus, from the definition of E(t) and selecting M sufficiently large,

$$\lambda_1 E(t) \leq L(t) \leq \lambda_2 E(t). \tag{3.29}$$

Where $\lambda_1 = (M - \epsilon c_5)$, $\lambda_2 = (M + \epsilon c_5)$. This completes the proof.

■

**Lemma 3.3.2** *The functional defined in (3.23) satisfies*

$$\frac{d}{dt}I(t) \leq \frac{\xi(t)}{2\tau_0}\|u_t\|_{2.\Gamma_1}^2 - \xi(t)\left(\frac{1-d}{2\tau_1}\right)\int_{\Gamma_1}\int_0^1 z^2(\gamma, 1, t)d\gamma - \frac{\tau'(t)\eta_1}{2\tau_1}\int_{\Gamma_1}\int_0^1 z^2(\gamma, k, t)dkd\gamma.$$

*Where $\eta_1$, $\eta_2$ , $\tau_0$, $\tau_1$ and $d$ are a positive constants and $\xi(t)$ are positive and bounded functions such that $\xi_0 = \sup_{t\geq 0}\xi(t)$, $\xi_1 = \inf_{t\geq 0}\xi(t)$,*

*Proof* Taking derivative of (3.23) produces

$$
\begin{aligned}
\frac{d}{dt}I(t) &= \frac{d}{dt}\left(\xi(t)e^{-k\tau(t)}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t))dkd\gamma\right)\\
&= \left[\xi'(t)e^{-\tau(t)k}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t))dkd\gamma - \xi(t)ke^{-\tau(t)k}\tau'(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t))dkd\gamma\right]\\
&+ \frac{1}{\tau(t)}e^{-\tau(t)k}\tau(t)\int_{\Gamma_1}\int_0^1 \frac{d}{dt}z^2(\gamma,k,t))dkd\gamma\\
&= \left[\xi'(t)e^{-\tau(t)k}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t))dkd\gamma - \xi(t)ke^{-\tau(t)k}\tau'(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t))dkd\gamma\right]\\
&+ \frac{1}{\tau(t)}e^{-\tau(t)k}\int_{\Gamma_1}\int_0^1 \frac{\partial}{\partial k}(1-\tau'(t)k)z^2(\gamma,k,t))dkd\gamma\\
&\le \left[\xi'(t)e^{-\tau(t)k}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma - \xi(t)ke^{-\tau(t)k}\tau'(t)\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t))dkd\gamma\right]\\
&+ \frac{1}{\tau(t)}\left[\xi(t)\int_{\Gamma_1}[z^2(\gamma,0,t))d\gamma - z^2(\gamma,1,t)d\gamma] + \xi(t)\tau'(t)\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma\right]\\
&\le \frac{\xi(t)}{2\tau_0}\|u_t\|^2_{2.\Gamma_1} - \xi(t)\left(\frac{1-d}{2\tau_1}\right)\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma - \tau'(t)\eta_1\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma.
\end{aligned}
\tag{3.30}
$$

■

**Lemma 3.3.3** . *The functional $\psi(t)$ defined in (3.22) satisfies*

$$
\begin{aligned}
\frac{d}{dt}\psi(t) &\le \|u_t\|_2^2 + \|u_t\|^2_{2.\Gamma_1} - (1-2n-\eta_1)\|\nabla u\|_2^2 + \|u\|_p^p\\
&+ \frac{\alpha(t)}{4}(g\ o\ \nabla u)(t) + \frac{c}{4\eta}\int_{\Gamma_1}|u_t(\gamma,t))|^2 d\gamma\\
&+ \frac{c}{4\eta}\int_{\Gamma_1}|z(\gamma,1,t))|^2 d\gamma,
\end{aligned}
\tag{3.31}
$$

*where* $n = \left(1 - \frac{2\alpha(t)}{\lambda}\int_0^t g(s)ds\right) > 0,\quad \eta_1 = 2\epsilon\eta c_s^2 B^2 > 0$ *and* $(1-2n-\eta_1) > 0.$

*Proof* Taking derivative of $\psi$ and using the problem (3.12) , we have

$$
\begin{aligned}
\frac{d}{dt}\psi(t) &\le \|u_t\|_2^2 + \|u_t\|^2_{2.\Gamma_1} - \|\nabla u\|_2^2 + \|u\|_p^p + \alpha(t)(g*\nabla u.\nabla u)\\
&- \mu_1\int_{\Gamma_1} u_t u d\gamma - \mu_2\int_{\Gamma_1} z(\gamma,1,t)ud\gamma.
\end{aligned}
\tag{3.32}
$$

Young's inequality produces $\forall\ \ \epsilon > 0$ and put $|\sigma(t)| \le c$

$$
\left|\int_{\Gamma_1} u_t(\gamma,t)u(\gamma,t)d\gamma\right| \le \eta c_s^2 B^2\epsilon\|\nabla u\|_2^2 + \frac{c}{4\eta}\int_{\Gamma_1}|u_t|^2 d\gamma
\tag{3.33}
$$

$$\left|\int_{\Gamma_1} z(\gamma,1,t))u(\gamma,t)d\gamma\right| \leq \eta c_s^2 B^2 \epsilon \|\nabla u\|_2^2 + \frac{c}{4\eta}\int_{\Gamma_1} |z(\gamma,1,t))|^2 d\gamma, \tag{3.34}$$

$$\alpha(t)(g * \nabla u.\nabla u) \leq \frac{2\alpha(t)}{\lambda}\int_0^t g(s)ds\|\nabla u\|_2^2 + \frac{\alpha(t)}{4}(g\ o\ \nabla u)(t), \tag{3.35}$$

inserting (3.33)-(3.35) in (3.32) gives

$$\begin{aligned} \frac{d}{dt}\psi(t) \leq\ & \|u_t\|_2^2 + \|u_t\|_{2.\Gamma_1}^2 - \left[1 - \frac{2\alpha(t)}{\lambda}\int_0^t g(s)ds - 2\epsilon\eta c_s^2 B^2\right]\|\nabla u\|_2^2 + \|u\|_p^p \\ & + \frac{\alpha(t)}{4}(g\ o\ \nabla u)(t) + \frac{c}{4\eta}\int_{\Gamma_1} |u_t(\gamma,t)|^2 d\gamma + \frac{c}{4\eta}\int_{\Gamma_1} |z(\gamma,1,t)|^2 d\gamma, \end{aligned} \tag{3.36}$$

then

$$\begin{aligned} \frac{d}{dt}\psi(t) &\leq \|u_t\|_2^2 + \|u_t\|_{2.\Gamma_1}^2 - (1 - 2n - \eta_1)\|\nabla u\|_2^2 + \|u\|_p^p \\ &+ \frac{\alpha(t)}{4}(g\ o\ \nabla u)(t) + \frac{c}{4\eta}\int_{\Gamma_1} |u_t(\gamma,t))|^2 d\gamma \\ &+ \frac{c}{4\eta}\int_{\Gamma_1} |z(\gamma,1,t))|^2 d\gamma, \end{aligned} \tag{3.37}$$

where $n = \left(1 - \frac{2\alpha(t)}{\lambda}\int_0^t g(s)ds\right) > 0$, $\eta_1 = 2\epsilon\eta c_s^2 B^2 > 0$ and $(1 - 2n - \eta_1) > 0$, which completes the proof.

■

**Lemma 3.3.4** *Let L(t) the functional defined in (3.21), then L(t) satisfies*

$$\frac{d}{dt}L(t) \leq -\alpha(t)C_1 E(t) + C_2\alpha(t)(g\ o\ \nabla u)(t),\ \forall t \geq 0. \tag{3.38}$$

*Proof* We take the derivative of (3.21), we get

$$\begin{aligned} \frac{d}{dt}L(t) &= ME'(t) + \epsilon\alpha(t)\psi'(t) + \epsilon\alpha^{'}(t)\psi(t) + \epsilon\alpha'(t)I(t) + \epsilon\alpha(t)I'(t) \\ &+ \epsilon\frac{\delta\alpha^{'}(t)}{2}\|\nabla u\|_2^2 + \epsilon\delta\alpha(t)\int_\Omega \nabla u\nabla u_t dx, \end{aligned} \tag{3.39}$$

making use of the inequalities

$$\alpha^{'}(t)\left|\int_\Omega u u_t dx\right| \leq \alpha^{'}(t)\frac{c_s^2}{\alpha_1}\|\nabla u\|_2^2 + \alpha^{'}(t)\alpha_1^2\|u_t\|_2^2, \tag{3.40}$$

and

$$\alpha^{'}(t)\left|\int_{\Gamma_1} u u_t d\gamma\right| \leq \alpha^{'}(t)\frac{c_s^2 B^2}{\alpha_1}\|\nabla u\|_2^2 + \alpha^{'}(t)\alpha_1^2\|u_t\|_{2.\Gamma_1}^2, \tag{3.41}$$

using lemma 3.1.1 , and (3.12) so $L'(t)$ gives the form:

$$
\begin{aligned}
L'(t) &= -Ma_1\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma - Ma_2\|u_t\|^2_{2.\Gamma_1} + \frac{M\alpha(t)}{2}(g' \ o\ \nabla u)(t)\\
&- \frac{M\alpha'(t)}{2}\int_0^t g(s)ds\|\nabla u\|_2^2 - \frac{M\alpha(t)}{2}g(t)\|\nabla u\|_2^2 - M\delta\|\nabla u_t\|_2^2\\
&+ \epsilon\alpha(t)\|u_t\|_2^2 + \epsilon\alpha(t)\|u_t\|^2_{2.\Gamma_1} - \epsilon\alpha(t)(1-2n-\eta_1)\|\nabla u\|_2^2\\
&+ \epsilon\alpha(t)\|u\|_p^p + \epsilon\frac{\alpha(t)^2}{4}(g\ o\ \nabla u)(t) + \epsilon\frac{\alpha(t)}{4\eta}\|u_t\|^2_{2.\Gamma_1}\\
&+ \epsilon\frac{\alpha(t)}{4\eta}\|z(\gamma,1,t)\|^2_{2.\Gamma_1} + \epsilon\frac{\alpha'(t)c_s^2}{\alpha_1}\|\nabla u\|_2^2 + \epsilon\alpha'(t)\alpha_1^2\|u_t\|_2^2\\
&+ \epsilon\frac{\alpha'(t)c_s^2B^2}{\alpha_1}\|\nabla u\|_2^2 + \epsilon\alpha'(t)\alpha_1^2\|u_t\|^2_{2.\Gamma_1} + \epsilon\frac{\alpha(t)\xi(t)}{2\tau_0}\|u_t\|^2_{2.\Gamma_1}\\
&+ \epsilon\alpha'(t)\xi(t)\int_{\Gamma_1}\int_0^1 e^{-k\tau(t)}z^2(\gamma,k,t)dkd\gamma + \epsilon\frac{\delta\alpha'(t)}{2}\|\nabla u\|_2^2\\
&- \epsilon\tau(t)\xi(t)\alpha(t)\frac{\tau'(t)\eta_1}{2\tau_1\xi_0}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma\\
&- \epsilon\alpha(t)\xi(t)\left(\frac{1-d}{2\tau_1}\right)\int_{\Gamma_1} z^2(\gamma,1,t)d\gamma,
\end{aligned}
\tag{3.42}
$$

using the fact that $\alpha(t) < \alpha(0)$ , we conclude

$$
\begin{aligned}
L'(t) &= -\alpha(t)\epsilon\left((1-2n-\eta_1) - (c_s^2(1+B^2) - \tfrac{\delta}{2})\tfrac{\alpha'(t)}{\alpha(t)}\right)\|\nabla u\|_2^2\\
&+ \epsilon\alpha(t)\left(1+\alpha_1^2\tfrac{\alpha'(t)}{\alpha(t)} + \tfrac{1}{4\eta} + \tfrac{\xi(t)}{2\tau_0}\right)\|u_t\|^2_{2.\Gamma_1} + \epsilon\alpha(t)\left(1+\alpha_1^2\tfrac{\alpha'(t)}{\alpha(t)}\right)\|u_t\|_2^2\\
&+ \epsilon\alpha(t)\|u\|_p^p - \delta M\|\nabla u_t\|_2^2 + \epsilon\tfrac{\alpha(t)^2}{4}(g\ o\ \nabla u)(t) + \epsilon\tfrac{\alpha(t)}{4\eta}\|z(\gamma,1,t)\|^2_{2.\Gamma_1}\\
&- \alpha(t)\left(\tfrac{Ma_2\sigma(t)}{\alpha(0)} - \epsilon\tfrac{\xi(1-d)}{2\tau_1\alpha(0)}\right)\int_{\Gamma_1} z^2(\gamma,1,t))d\gamma\\
&- \alpha(t)\left(\tfrac{Ma_1\sigma(t)}{\alpha(0)} - \epsilon\tfrac{\xi\alpha_2}{\tau_0}\right)\|u_t\|^2_{2.\Gamma_1}.\\
&- \epsilon\tau(t)\xi(t)\alpha(t)\tfrac{\tau'(t)\eta_1}{2\tau_1\xi_0}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma.
\end{aligned}
\tag{3.43}
$$

Consequently, using the definition of the energy (3.13), for any positive constant M, we obtain:

$$
\begin{aligned}
L'(t) &= -\alpha(t)\epsilon\left((1-2n-\eta_1)-(c_s^2(1+B^2)-\tfrac{\delta}{2})\tfrac{\alpha'(t)}{\alpha(t)}\right)\|\nabla u\|_2^2\\
&- \epsilon\alpha(t)\left(\tfrac{M}{2}-1\right)\|u\|_p^p-\epsilon\alpha(t)\left(\tfrac{M}{2}-\alpha_1^2\left(1+\alpha_1^2\tfrac{\alpha'(t)}{\alpha(t)}\right)\right)\|u_t\|_2^2\\
&- \epsilon\alpha(t)\left(\tfrac{M}{2}-\left(1+\alpha_1^2\tfrac{\alpha'(t)}{\alpha(t)}+\tfrac{1}{4\eta}+\tfrac{\xi(t)}{2\tau_0}\right)\right)\|u_t\|_{2.\Gamma_1}^2+\tfrac{\alpha(t)M}{2}\|u_t\|_2^2\\
&+ \epsilon\tfrac{\alpha(t)M}{2}\|u_t\|_{2.\Gamma_1}^2-\epsilon\tfrac{M\alpha(t)^2}{4}(g\ o\ \nabla u)(t)+\epsilon\tfrac{M\alpha(t)^2}{2}(g\ o\ \nabla u)(t)\\
&- M\delta\|\nabla u_t\|_2^2+\epsilon\tfrac{\alpha(t)}{4\eta}\|u_t\|_{2.\Gamma_1}^2+\epsilon\tfrac{\alpha(t)^2}{4\eta}\|z(\gamma,1,t)\|_{2.\Gamma_1}^2\\
&- \alpha(t)\left(\tfrac{Ma_2\sigma(t)}{\alpha(0)}-\epsilon\tfrac{\xi\alpha_2}{\tau_0}\right)\|u_t\|_{2.\Gamma_1}^2\\
&- \alpha(t)\left(\tfrac{Ma_1\sigma(t)}{\alpha(0)}-\epsilon\tfrac{\xi(1-d)}{2\tau_1\alpha(0)}\right)\int_{\Gamma_1}z^2(\gamma,1,t)d\gamma\\
&- \epsilon\alpha(t)\tau(t)\xi(t)\tfrac{\tau'(t)\eta_1}{2\tau_1\xi_0}\int_{\Gamma_1}\int_0^1 z^2(\gamma,k,t)dkd\gamma.
\end{aligned}
\tag{3.44}
$$

First, we fix $n-\eta_1>0$ such that $1-2n-\eta_1>0$ and then take $M>0$ such that $\left(\frac{M}{2}-1\right)>0$, since $\lim\limits_{t\to\infty}\dfrac{\alpha'(t)}{\alpha(t)}=0$, we can choose $t_0>0$ sufficiently large so that
$\left(\frac{M}{2}-\alpha_1^2\left(1+\frac{\alpha'(t)}{\alpha(t)}\right)\right)>0$, $\left((1-2n-\eta_1)-(c_s^2(1+B^2)-\frac{\delta}{2})\frac{\alpha'(t)}{\alpha(t)}\right)>0$, $\left(\frac{Ma_2\sigma(t)}{\alpha(0)}-\epsilon\frac{\bar{\xi}\alpha_2}{\tau_0}\right)>0$. $\frac{\tau'(t)\eta_1}{2\tau_1\xi_0}>0$, $\left(\frac{Ma_1\sigma(t)}{\alpha(0)}-\epsilon\frac{\bar{\xi}(1-d)}{2\tau_1\alpha(0)}\right)>0$, $\left(\frac{M}{2}-\left(1+\alpha_1^2\frac{\alpha'(t)}{\alpha(t)}+\frac{1}{4\eta}+\frac{\xi(t)}{2\tau_0}\right)\right)>0$.
By using the poincaré and trace inequalities

$$\|u_t\|_2^2\le C\|\nabla u_t\|_2^2,$$

and

$$\|u_t\|_{2.\Gamma_1}^2\le C\|\nabla u_t\|_2^2.$$

Then (3.44) takes the form:

$$\frac{d}{dt}L(t)\le -M\alpha(t)c\epsilon E(t)-(M\delta-\epsilon M\alpha(0)C)\|\nabla u_t\|_2^2+\epsilon\frac{\alpha(0)M}{2}\alpha(t)(g\ o\ \nabla u)(t), \tag{3.45}$$

then, choosing $\epsilon$ small enough such that $(M\delta-\epsilon M\alpha(0)C)>0$
we obtain

$$\frac{d}{dt}L(t)\le -M\alpha(t)c\epsilon E(t)+\epsilon\frac{\alpha(0)M}{2}\alpha(t)(g\ o\ \nabla u)(t), \tag{3.46}$$

setting $\theta=\frac{M\epsilon}{\lambda_2}$, $C_1=c\theta$, $C_2=\epsilon\frac{\alpha(0)M}{2}$ and

$$\frac{d}{dt}L(t)\le -\alpha(t)C_1E(t)+C_2\alpha(t)(g\ o\ \nabla u)(t),\ \forall t\ge 0. \tag{3.47}$$

The proof is completed.

■

**Theorem 3.3.1** *There exist two positive constants $C_0$, $\theta$ and $t_1$ such that*

$$E(t) \leq C_0 e^{-\theta \int_{t_1}^{t} \alpha(s)\sigma(s)ds} \tag{3.48}$$

*Proof* Multiplying (3.47) by $\sigma(t)$ and using the lemma 3.1.1. We get

$$\begin{aligned} \sigma(t)\tfrac{d}{dt}L(t) &\leq -C_1\alpha(t)\sigma(t)E(t) + C_2\alpha(t)\sigma(t)(g\ o\ \nabla u)(t) \\ &\leq -C_1\alpha(t)\sigma(t)E(t) - C_2\alpha(t)\sigma(t)(g'\ o\ \nabla u)(t) \\ &\leq -C_1\alpha(t)\sigma(t)E(t) + C_2\left(-2\tfrac{d}{dt}E(t) - \alpha'(t)\int_0^t g(s)ds\|\nabla u\|_2^2\right). \end{aligned} \tag{3.49}$$

Since $\sigma$ is nonincreasing, from the definition of E(t) and assumption $(A_0)$, we have

$$\frac{d}{dt}\left(\sigma(t)L(t) + 2C_2E(t)\right) \leq -\alpha(t)\sigma(t)\left(C_1 + \frac{2C_2 l_0 \alpha'(t)}{\lambda l \alpha(t)\sigma(t)}\right)E(t)\ for\ t > t_0,$$

as we have $\lim\limits_{t\to\infty} \dfrac{2C_2 l_0 \alpha'(t)}{\lambda l \alpha(t)\sigma(t)} = 0$, we can choose $t_1 > t_0$ such that $C_3 = C_1 + \frac{2C_2 l_0 \alpha'(t)}{\lambda l \alpha(t)\sigma(t)} > 0$ for $t > t_1$.
Now let $\chi(t) = \sigma(t)L(t) + 2C_2E(t)$. Then we can verify that

$$\theta_1 E(t) \leq \chi(t) \leq \theta_2 E(t). \tag{3.50}$$

Where $\theta_1$, $\theta_2$ are two positive constants, thus we arrive at

$$\frac{d}{dt}\chi(t) \leq -C_4\alpha(t)\sigma(t)\chi(t)\ for\ t > t_1.$$

Integrating the previous differential inequality between $t_1$ and t gives the following estimate for the function $\chi$

$$\chi(t) \leq \chi(t_1)e^{-C_4\int_{t_1}^{t}\alpha(s)\sigma(s)ds},\ \forall t \geq t_1.$$

Consequently, by using (3.50), we conclude

$$E(t) \leq \hat{C}e^{-C_4\int_{t_1}^{t}\alpha(s)\sigma(s)ds},\ \forall t \geq t_1.$$

This completes the proof.

■

**Remark 3.3.1** *We illustrate the energy decay rate given by Theorem 2 through the following examples which are introduced in [43,44].*

1. *If $g(t) = ae^{-b(1+t)^{\nu}}$, $\alpha(t) = \frac{1}{1+t}$ for $a, b > 0$ and $0 < \nu \leq 1$, then $\sigma(t) = b\nu(1+t)^{\nu-1}$ satisfies $(A_0)$. Thus (3.48) gives the estimate*

$$E(t) \leq C_0 e^{-\theta(1+t)^{\nu-1}}.$$

2. *If* $g(t) = ae^{-b\ln^{\nu}(1+t)}$, $\alpha(t) = \frac{1}{\ln(1+t)}$ *for* $a, b > 0$ *and* $1 < \nu$, *then* $\sigma(t) = \frac{b\nu\ln^{\nu-1}(1+t)}{(1+t)}$ *satisfies* $(A_0)$. *Thus (3.48) gives the estimate*

$$E(t) \leq C_0 e^{-\theta\ln^{\nu}(1+t)}.$$

3. *If* $g(t) = e^{-at}$ , $\alpha(t) = \frac{b}{(1+t)}$ *for* $a, b > 0$ *then* $\sigma(t) \equiv a$ *satisfies* $(A_0)$. *Thus (48) gives the estimate*

$$E(t) \leq C_0(1+t)^{-\theta ab}.$$

4. *If* $g(t) = e^{-at}$, $\alpha(t) \equiv b$. *Note that in these case (3.48) reduces to one of [13].*

# Chapter 4

# Global existence and energy decay of solutions to a bresse system with delay terms and infinite memories

## 4.1 Introduction

In this chapter We consider the Bresse system in bounded domain with delay terms in the internal feedbacks and infinite memories acting in the three equations of the system. First, we prove the global existence of its solutions in Sobolev spaces by means of semigroup theory. Furthermore, the asymptotic stability is given by using an appropriate Lyapunov functional.

$$
\left\{
\begin{array}{l}
\rho_1\varphi_{tt} - k_1(\varphi_x + \psi + l\omega)_x - lk_3(\omega_x - l\varphi) + \mu_1\varphi_t + \mu_2\varphi_t(x, t - \tau_1) \\
+ \displaystyle\int_0^\infty g_1(s)\varphi_{xx}(x, t - s)ds = 0, \\
\\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + l\omega) + \widetilde{\mu_1}\psi_t + \widetilde{\mu_2}\psi_t(x, t - \tau_2) \\
+ \displaystyle\int_0^\infty g_2(s)\psi_{xx}(x, t - s)ds = 0, \\
\\
\rho_1\omega_{tt} - k_3(\omega_x - l\varphi)_x + lk_1(\varphi_x + \psi + l\omega) + \widetilde{\widetilde{\mu_1}}\omega_t + \widetilde{\widetilde{\mu_2}}\omega_t(x, t - \tau_3) \\
+ \displaystyle\int_0^\infty g_3(s)\omega_{xx}(x, t - s)ds = 0,
\end{array}
\right.
\tag{4.1}
$$

where $(x, t) \in (0, L) \times (0, +\infty), \tau_i > 0$ $(i = 1, 2, 3)$ is a time delay, $\mu_1, \mu_2, \widetilde{\mu_1}, \widetilde{\mu_2}, \widetilde{\widetilde{\mu_1}}, \widetilde{\widetilde{\mu_2}}$ are positive real numbers. This system is subject to the Dirichlet boundary conditions

$$\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = \omega(0, t) = \omega(L, t) = 0, \quad t > 0$$

and to the initial conditions

$$\begin{cases} \varphi(x,-t)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \quad \psi(x,-t)=\psi_0(x), \quad x\in(0,L) \\ \psi_t(x,0)=\psi_1(x), \quad \omega(x,-t)=\omega_0(x), \quad \omega_t(x,0)=\omega_1(x), \qquad x\in(0,L) \\ \varphi_t(x,t-\tau_1)=f_0(x,t-\tau_1), \quad \text{in } (0,L)\times[0,\tau_1] \\ \psi_t(x,t-\tau_2)=\tilde{f}_0(x,t-\tau_2), \quad \text{in } (0,L)\times[0,\tau_2] \\ \omega_t(x,t-\tau_3)=\tilde{\tilde{f}}_0(x,t-\tau_3), \quad \text{in } (0,L)\times[0,\tau_3]. \end{cases}$$

The initial data $(\varphi_0,\varphi_1,\psi_0,\psi_1,\omega_0,\omega_1,f_0,\tilde{f}_0,\tilde{\tilde{f}}_0)$ belong to a suitable Sobolev space. By $\omega,\psi$ and $\varphi$ we are denoting the longitudinal, vertical and shear angle displacements. The original Bresse system is given by the following equations (see [**63**]) :

$$\begin{cases} \rho_1\varphi_{tt}=Q_x+lN+F_1, \\ \rho_2\psi_{tt}=M_x-Q+F_2, \\ \rho_1\omega_{tt}=N_x-lQ+F_3, \end{cases}$$

where we use $N,Q$ and $M$ to denote the axial force, the shear force and the bending moment respectively. These forces are stress-strain relations for elastic behavior and given by

$$N=Eh(\omega_x-l\varphi), \quad Q=Gh(\varphi_x+\psi+l\omega), \quad \text{and } M=EI\psi_x,$$

where $G,E,I$ and $h$ are positive constants. Finally, by the terms $F_i$ we are denoting external forces.
The Bresse system without delay (i.e $\mu_2=\widetilde{\mu}_2=\widetilde{\widetilde{\mu}}_2=0$), is more general than the well-known Timoshenko system where the longitudinal displacement $\omega$ is not considered $l=0$. There are a number of publications concerning the stabilization of Timoshenko system with different kinds of damping (see [**64**], [**65**], [**66**] and [**67**]). Raposo et al. [**68**] proved the exponential decay of the solution for the following linear system of Timoshenko-type beam equations with linear frictional dissipative terms:

$$\begin{cases} \rho_1\varphi_{tt}-Gh(\varphi_x+\psi+l\omega)_x-lEh(\omega_x-l\varphi)+\mu_1\varphi_t=0 \\ \\ \rho_2\psi_{tt}-EI\psi_{xx}+Gh(\varphi_x+\psi+l\omega)+\widetilde{\mu}_1\psi_t=0. \end{cases}$$

Messaoudi and Mustafa [**65**] (see also [**67**]) considered the stabilization for the following Timoshenko system with nonlinear internal feedbacks:

$$\begin{cases} \rho_1\varphi_{tt}-Gh(\varphi_x+\psi+l\omega)_x-lEh(\omega_x-l\varphi)+g_1(\psi_t)=0 \\ \rho_2\psi_{tt}-EI\psi_{xx}+Gh(\varphi_x+\psi+l\omega)+g_2(\psi_t)=0. \end{cases}$$

Recently, Park and Kang [**67**] considered the stabilization of the Timoshenko system with weakly nonlinear internal feedbacks.
In [**69**], Liu and Rao considered a thermoelastic Bresse system that consists of three wave equations and two heat equations coupled in certain way. The two wave equations for the longitudinal displacement and the shear angle displacement are effectively globally damped by

the dissipation from the two heat equations. The wave equation about the vertical displacement is subject to a weak thermal damping and indirectly damped through the coupling. They establish exponential energy decay rate when the vertical and the longitudinal waves have the same speed of propagation. Otherwise, a polynomial-type decay is established.
Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see [**70**]). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. In recent years, the PDEs with time delay effects have become an active area of research and arise in many practical problems (see for example [**71**], [**72**]). The presence of delay may be a source of instability. For example, it was proved in [**73**] that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [**74**] and [**75**]). For instance, in [**74**] the authors studied the wave equation with a linear internal damping term with constant delay and determined suitable relations between $\mu_1$ and $\mu_2$, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they found a sequence of delays for which the solution will be instable if $\mu_2 \geq \mu_1$. The main approach used in [**74**], is an observability inequality obtained with a Carleman estimate. The same results were showed if both the damping and the delay acting in the boundary domain. We also recall the result by Xu, Yung and Li [**75**], where the authors proved the same result as in [**74**] for the one space dimension by adopting the spectral analysis approach.
Motivated by the previous works it is interesting to give more general decay result to (1.1), by combining the idea of ( [79],[80]). Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (1.1) for linear damping , delay terms, and infinite memories . To obtain global solutions to the problem (1.1), we use the argument combining the semigroup theory (see [**74**] and [**76**]) with the energy estimate method. To prove decay estimates, we use a Lyapunov functional methods.

## 4.2 Preliminary Results

First assume the following hypotheses:

(H1)

$$|\mu_2| < \mu_1, \quad |\widetilde{\mu_2}| < \widetilde{\mu_1}, \quad |\widetilde{\widetilde{\mu_2}}| < \widetilde{\widetilde{\mu_1}}. \tag{4.2}$$

(H2) $g_i : R_+ \to R_+$ are differentiable non-increasing function and integrable on $R_+$ such that there exists a non-increasing differentiable function $\zeta : R^+ \to R^+$ satisfying

$$g_i'(t) \leq -\zeta(t) g_i(t),$$

and there exists a positive constant $k_0$ satisfying, for any $(\varphi, \psi, \omega) \in (H_0^1(]0,L[))^3$,

$$\begin{aligned} k_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + \omega_x^2\right) dx \ & \leq \int_0^L (k_2\psi_x^2 + k_1 \left(\varphi_x + \psi + l\omega\right)^2 + k_3(\omega_x - l\omega)^2) \, dx \\ & - \int_0^L \left(\int_0^{+\infty} g_1(s)ds\right) \varphi_x^2 dx + \int_0^L \left(\int_0^{+\infty} g_2(s)ds\right) \psi_x^2 dx \\ & + \int_0^L \left(\int_0^{+\infty} g_3(s)ds\omega_x^2\right) dx. \end{aligned}$$

By contradiction arguments, it is easy to see that there exists a positive constant $\widetilde{k_0}$ such that, for $(\varphi, \psi, \omega) \in (H_0^1(]0,L[))^3$,

$$\widetilde{k_0} \int_0^L \left(\varphi_x^2 + \psi_x^2 + \omega_x^2\right) dx \leq \int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + l\omega)^2 + k_3(w_x - l\varphi)^2\right) dx. \tag{4.3}$$

The above inequality will be proved later in lemma 4.3.1. Moreover if

$$g_i^0 := \int_0^{+\infty} g_i(s)ds < \widetilde{k_0}, i = 1, 2, 3, \tag{4.4}$$

then (4.3) is satisfied with

$$k_0 = \widetilde{k_0} - \max\left\{g_1^0, g_2^0, g_3^0\right\}.$$

On the other hand, thanks to Poincare's inequality, there exists a positive constant $\widetilde{k_0}$ such that, for $(\varphi, \psi, \omega) \in (H_0^1(]0,L[))^3$,

$$\int_0^L (k_2\psi_x^2 + k_1 \left(\varphi_x + \psi + l\omega\right)^2 + k_3(\omega_x - l\varphi)^2) \, dx \leq \widetilde{k_0} \int_0^L \left(\varphi_x^2 + \psi_x^2 + \omega_x^2\right) dx. \tag{4.5}$$

## 4.3 Well-posedness

In order to prove the well-posedness result, we have to make the following operations: We introduce, as in **[74]**, the new variables

$$\begin{aligned} z_1(x,\rho,t) &= \phi_t(x, t-\tau_1\rho), \quad x \in (0,L),\ \rho \in (0,1), \quad t > 0, \\ z_2(x,\rho,t) &= \psi_t(x, t-\tau_2\rho), \quad x \in (0,L),\ \rho \in (0,1), \quad t > 0, \\ z_3(x,\rho,t) &= \omega_t(x, t-\tau_3\rho), \quad x \in (0,L),\ \rho \in (0,1), \quad t > 0. \end{aligned} \tag{4.6}$$

Also as in [79], the new variables

$$\begin{cases} \eta_1(x,t,s) = \varphi(x,t) - \varphi(x,t-s) \quad in \quad ]0,L[\times R_+ \times R_+, \\ \eta_2(x,t,s) = \psi(x,t) - \psi(x,t-s) \quad in \quad ]0,L[\times R_+ \times R_+, \\ \eta_3(x,t,s) = \omega(x,t) - \omega(x,t-s) \quad in \quad ]0,L[\times R_+ \times R_+. \end{cases}$$

These functionals satisfy

$$\begin{cases} \partial_t\eta_1 + \partial_s\eta_1 - \varphi_t = 0 & in \quad ]0, L[\times R_+ \times R_+, \\ \partial_t\eta_2 + \partial_s\eta_2 - \psi_t = 0 & in \quad ]0, L[\times R_+ \times R_+, \\ \partial_t\eta_3 + \partial_s\eta_3 - \omega_t = 0 & in \quad ]0, L[\times R_+ \times R_+, \\ \eta_i(0, t, s) = \eta_i(L, t, s) = 0, & in \quad R_+ \times R_+, \\ \eta_i(x, t, 0) = 0, & in \quad ]0, L[\times R_+, i = 1, 2, 3. \end{cases}$$

In order to convert our problem to a system of first-order ordinary differential equations, we note the following:

$$\eta_i^0(x, s) = \eta_i(x, 0, s), \;\; i = 1, 2, 3. \tag{4.7}$$

Then, we have

$$\tau_i z_{it}(x, \rho, t) + z_{i\rho}(x, \rho, t) = 0, \quad \text{in } (0, L) \times (0, 1) \times (0, +\infty) \text{ for } i = 1, 2, 3. \tag{4.8}$$

Therefore, problem (4.1) takes the form:

$$\begin{cases} \rho_1\varphi_{tt}(x, t) - k_1(\varphi_x + \psi + l\omega)_x(x, t) - lk_3(\omega_x - l\varphi)(x, t) + \mu_1\varphi_t(x, t) \\ +\mu_2 z_1(x, 1, t) + \displaystyle\int_0^\infty g_1(s)\partial_{xx}\eta_1 ds = 0, \\ \tau_1 z_{1t}(x, \rho, t) + z_{1\rho}(x, \rho, t) = 0, \\ \rho_2\psi_{tt}(x, t) - k_2\psi_{xx}(x, t) + k_1(\varphi_x + \psi + l\omega)(x, t) + \widetilde{\mu_1}\psi_t(x, t) \\ +\widetilde{\mu_2} z_2(x, 1, t) + \displaystyle\int_0^\infty g_2(s)\partial_{xx}\eta_2 ds = 0, \\ \tau_2 z_{2t}(x, \rho, t) + z_{2\rho}(x, \rho, t) = 0, \\ \rho_1\omega_{tt}(x, t) - k_3(\omega_x - l\varphi)_x(x, t) + lk_1(\varphi_x + \psi + l\omega)(x, t) + \widetilde{\widetilde{\mu_1}}\omega_t(x, t) \\ +\widetilde{\widetilde{\mu_2}} z_3(x, 1, t) + \displaystyle\int_0^\infty g_3(s)\partial_{xx}\eta_3 ds = 0, \\ \tau_3 z_{3t}(x, \rho, t) + z_{3\rho}(x, \rho, t) = 0. \end{cases} \tag{4.9}$$

The above system subjected to the following initial and boundary conditions

$$\begin{cases} \varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = \omega(0, t) = \omega(L, t), \quad t > 0, \\ z_1(x, 0, t) = \varphi_t(x, t), z_2(x, 0, t) = \psi_t(x, t), & x \in (0, L), t > 0, \\ z_3(x, 0, t) = \omega_t(x, t), x \in (0, L), & x \in (0, L), t > 0, \\ \varphi(x, 0) = \varphi_0, \varphi_t(x, 0) = \varphi_1, \psi(x, 0) = \psi_0, \psi_t(x, 0) = \psi_1 \\ \omega(x, 0) = \omega_0\omega_t(x, 0) = \omega_1, & x \in (0, L), \\ z_1(x, 1, t) = f_1(x, t - \tau_1), & \text{in } (0, L) \times (0, \tau_1) \\ z_2(x, 1, t) = f_2(x, t - \tau_2), & \text{in } (0, L) \times (0, \tau_2), \\ z_3(x, 1, t) = f_3(x, t - \tau_3) & \text{in } (0, L) \times (0, \tau_3) \\ \eta_1(x, t, s) = \eta_1(L, t, s) = 0, & x \in (0, L), t > 0, in \; R_+ \times R_+, \\ \eta_2(x, t, s) = \eta_2(L, t, s) = 0, & x \in (0, L), t > 0, in \; R_+ \times R_+, \\ \eta_3(x, t, s) = \eta_3(L, t, s) = 0, & x \in (0, L), t > 0, in \; R_+ \times R_+, \\ \eta_1(x, t, 0) = 0, & x \in (0, L), t > 0, \; in \; R_+ \times R_+, \\ \eta_2(x, t, 0) = 0, & x \in (0, L), t > 0, \; in \; R_+ \times R_+, \\ \eta_3(x, t, 0) = 0, & x \in (0, L), t > 0, \; in \; R_+ \times R_+. \end{cases} \tag{4.10}$$

Let $\xi_1, \xi_2$ and $\xi_3$ be positive constants such that

$$\left\{ \begin{array}{l} \tau_1|\mu_2| < \xi_1 < \tau_1(2\mu_1 - |\mu_2|), \\ \tau_2|\widetilde{\mu_2}| < \xi_2 < \tau_2(2\widetilde{\mu_1} - |\widetilde{\mu_2}|), \\ \tau_3|\widetilde{\widetilde{\mu_2}}| < \xi_3 < \tau_3(2\widetilde{\widetilde{\mu_1}} - |\widetilde{\widetilde{\mu_2}}|), \end{array} \right. \tag{4.11}$$

thanks to hypothesis $(H1)$. We define the energy associated to the solution of the problem (4.9)-(4.10) by the following formula:

$$\begin{aligned} E(t) &= \frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_1}{2}\|\omega_t\|_2^2 + \frac{k_3}{2}\|\psi_x\|_2^2 \\ &+ \frac{k_3}{2}\|\varphi_x + \psi + l\omega\|_2^2 + \frac{k_3}{2}\|\omega_x - l\varphi\|_2^2 + \sum_{i=1}^{3}\frac{\xi_i}{2}\int_0^1 \|z_i(x,\rho,t)\|_2^2 d\rho \\ &- \int_0^L (g_1^0\varphi_x^2 + g_2^0\psi_x^2 + g_3^0\omega_x^2)dx + \|\eta_1\|_{H_1^*}^2 + \|\eta_2\|_{H_2^*}^2 + \|\eta_3\|_{H_3^*}^2 \end{aligned} \tag{4.12}$$

where

$$H_i^* = \left\{ v : R_+ \to H_0^1(]0,L[), \int_0^L \int_0^\infty g_i(s)v_x^2(s)dsdx < +\infty \right\}.$$

We have the following theorem.

**Theorem 4.3.1** *Assume that the hypotheses $(H1) - (H2)$ hold.*
*Let $(\varphi_0, \varphi_1, f_1(.,-.\tau_1), \psi_0, \psi_1, f_2(.,-.\tau_2), \omega_0, \omega_1, f_3(.,-.\tau_3), \eta_0^1, \eta_0^2, \eta_0^3) \in (H_0^1(0,L) \times L^2(0,L) \times L^2((0,L)\times(0,1)))^3$. Then problem $(4.9) - (4.10)$ admits a unique solution*

$$\left\{ \begin{array}{l} \varphi \in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)), \\ \psi \in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)), \\ \omega \in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)), \\ z_1, z_2, z_3 \in C([0,+\infty); L^2((0,L)\times(0,1))), \\ \eta_1, \eta_2, \eta_3 \in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)). \end{array} \right.$$

We finish this section by giving an explicit upper bound for the derivative of the energy.

**Lemma 4.3.1** *Let $(\varphi, \psi, \omega, z_1, z_2, z_3, \eta_1, \eta_2, \eta_3)$ be a solution of the problem (4.9)-(4.10). Then, the energy functional defined by (4.12) satisfies*

$$\begin{aligned} E'(t) &\le -\left(\mu_1 - \frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\|\varphi_t\|_2^2 - \left(\widetilde{\mu_1} - \frac{\xi_2}{2\tau_2} - \frac{|\widetilde{\mu_2}|}{2}\right)\|\psi_t\|_2^2 \\ &- \left(\widetilde{\widetilde{\mu_1}} - \frac{\xi_3}{2\tau_3} - \frac{|\widetilde{\widetilde{\mu_2}}|}{2}\right)\|\omega_t\|_2^2 - \left(\frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\|z_1(x,1,t)\|_2^2 \\ &- \left(\frac{\xi_2}{2\tau_2} - \frac{|\widetilde{\mu_2}|}{2}\right)\|z_2(x,1,t)\|_2^2 - \left(\frac{\xi_3}{2\tau_3} - \frac{|\widetilde{\widetilde{\mu_2}}|}{2}\right)\|z_3(x,1,t)\|_2^2 \\ &+ \frac{1}{2}\int_0^L\int_0^\infty g_1'(s)(\partial_x\eta_1)^2dsdx + \frac{1}{2}\int_0^L\int_0^\infty g_2'(s)(\partial_x\eta_2)^2dsdx \\ &+ \frac{1}{2}\int_0^L\int_0^\infty g_3'(s)(\partial_x\eta_3)^2dsdx. \end{aligned} \tag{4.13}$$

*Proof* Multiplying the first equation in (4.9) by $\varphi_t$, the third equation by $\psi_t$, the five equation by $\omega_t$, integrating over $(0, L)$ and using integration by parts, we get

$$\begin{aligned}
&\frac{1}{2}\rho_1\frac{d}{dt}\|\varphi_t\|_2^2 - k_1\int_0^L(\varphi_x+\psi+l\omega)_x\varphi_t dx - lk_3\int_0^L(\omega_x-l\varphi)\varphi_t dx + \mu_1\|\varphi_t\|_2^2 \\
&+\mu_2\int_0^L z_1(x,1,t)\varphi_t dx + \int_0^\infty g_1(s)\partial_{xx}\eta_1\varphi_t dsdx = 0 \\
&\frac{1}{2}\rho_2\frac{d}{dt}\|\psi_t\|_2^2 + \frac{k_2}{2}\|\psi_x\|_2^2 + k_1\int_0^L(\varphi_x+\psi+l\omega)\psi_t dx + \widetilde{\mu_1}\|\psi_t\|_2^2 \\
&+\widetilde{\mu_2}\int_0^L z_2(x,1,t)\psi_t dx + \int_0^\infty g_2(s)\partial_{xx}\eta_2\psi_t dsdx = 0 \\
&\frac{1}{2}\rho_1\frac{d}{dt}\|\omega_t\|_2^2 - k_3\int_0^L(\omega_x-l\varphi)_x\omega_t dx + lk_1\int_0^L(\varphi_x+\psi+l\omega)\omega_t dx + \widetilde{\widetilde{\mu_1}}\|\omega_t\|_2^2 \\
&+\widetilde{\widetilde{\mu_2}}\int_0^L z_3(x,1,t)\omega_t dx + \int_0^\infty g_3(s)\partial_{xx}\eta_3\omega_t dsdx = 0.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_1}{2}\|\omega_t\|_2^2 + \frac{k_1}{2}\|\psi_x\|_2^2 + \frac{k_2}{2}\|\varphi_x+\psi+l\omega\|_2^2 + \frac{k_3}{2}\|\omega_x-l\varphi\|_2^2\right) \\
&+\mu_1\|\varphi_t\|_2^2 + \widetilde{\mu_1}\|\psi_t\|_2^2 + \widetilde{\widetilde{\mu_1}}\|\omega_t\|_2^2 + \widetilde{\mu_2}\int_0^L z_1(x,1,t)\psi_t dx + \mu_2\int_0^L z_2(x,1,t)\varphi_t dx \\
&+\widetilde{\widetilde{\mu_2}}\int_0^L z_3(x,1,t)\omega_t dx + \int_0^\infty g_1(s)(\partial_x\eta_1)^2 ds + \int_0^\infty g_2(s)(\partial_x\eta_2)^2 ds \\
&+\int_0^\infty g_3(s)(\partial_x\eta_3)^2 ds + \int_0^\infty g_1(s)\partial_s(\partial_x\eta_1)^2 ds + \int_0^\infty g_2(s)\partial_s(\partial_x\eta_2)^2 ds \\
&+\int_0^\infty g_3(s)\partial_s(\partial_x\eta_3)^2 ds = 0.
\end{aligned} \tag{4.14}$$

Multiplying the second equation in (4.9) by $\xi_i z_i$ and integrating over $(0, L)\times(0, 1)$, to obtain:

$$\begin{aligned}
\frac{\xi_i}{2}\frac{d}{dt}\int_0^L\int_0^1 z_i^2(x,\rho,t)d\rho dx &= -\frac{\xi_i}{\tau_1}\int_0^L\int_0^1 z_i z_{i\rho} d\rho dx \\
&= \frac{\xi_i}{2\tau_i}\int_0^L\left(z_i^2(x,0,t) - z_i^2(x,1,t)\right)dx \\
&= \frac{\xi_i}{2\tau_i}\left[\|z_i^2(x,0,t)\|_2^2 - \|z_i(x,1,t)\|_2^2\right],
\end{aligned} \tag{4.15}$$

where $z_1(x, 0, t) = \varphi_t(x, t)$, $z_2(x, 0, t) = \psi_t(x, t)$ and $z_3(x, 0, t) = \omega_t(x, t)$. From (4.14), (4.15), integrating by parts, and using Young inequality we get

$$
\begin{aligned}
E'(t) \ &= -\left(\mu_1 - \frac{\xi_1}{2\tau_1}\right)\|\varphi_t\|_2^2 - \left(\widetilde{\mu_1} - \frac{\xi_2}{2\tau_2}\right)\|\psi_t\|_2^2 - \left(\widetilde{\widetilde{\mu}}_1 - \frac{\xi_3}{2\tau_3}\right)\|\omega_t\|_2^2 \\
&- \sum_{i=1}^{3} \frac{\xi_i}{2\tau_i}\|z_i(x,1,t)\|_2^2 - \mu_2 \int_0^L z_1(x,1,t)\varphi_t \, dx - \widetilde{\mu_2}\int_0^L z_1(x,1,t)\psi_t \, dx \\
&- \widetilde{\widetilde{\mu}}_2 \int_0^L z_3(x,1,t)\omega_t \, dx + \frac{1}{2}\int_0^L \int_0^\infty g_1'(s)(\partial_x\eta_1)^2 ds dx \\
&+ \frac{1}{2}\int_0^L \int_0^\infty g_2'(s)(\partial_x\eta_2)^2 ds dx + \frac{1}{2}\int_0^L \int_0^\infty g_3'(s)(\partial_x\eta_3)^2 ds dx.
\end{aligned} \tag{4.16}
$$

Due to Young's inequality, we have

$$
\begin{aligned}
\int_0^L z_1(x,1,t)\varphi_t(x,t)\, dx &\le \frac{1}{2}\|\varphi_t(x,t)\|_2^2 + \frac{1}{2}\|z_1(x,1,t)\|_2^2 \\
\int_0^L z_2(x,1,t)\varphi_t(x,t)\, dx &\le \frac{1}{2}\|\psi_t(x,t)\|_2^2 + \frac{1}{2}\|z_2(x,1,t)\|_2^2 \\
\int_0^L z_3(x,1,t)\omega_t(x,t)\, dx &\le \frac{1}{2}\|\omega_t(x,t)\|_2^2 + \frac{1}{2}\|z_3(x,1,t)\|_2^2.
\end{aligned} \tag{4.17}
$$

Inserting (4.17) into (4.16), we obtain

$$
\begin{aligned}
E'(t) \ &\le -\left(\mu_1 - \frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\|\varphi_t\|_2^2 - \left(\widetilde{\mu_1} - \frac{\xi_2}{2\tau_2} - \frac{|\widetilde{\mu_2}|}{2}\right)\|\psi_t\|_2^2 \\
&- \left(\widetilde{\widetilde{\mu}}_1 - \frac{\xi_3}{2\tau_3} - \frac{|\widetilde{\widetilde{\mu}}_2|}{2}\right)\|\omega_t\|_2^2 - \left(\frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\|z_1(x,1,t)\|_2^2 \\
&- \left(\frac{\xi_2}{2\tau_2} - \frac{|\widetilde{\mu_2}|}{2}\right)\|z_2(x,1,t)\|_2^2 - \left(\frac{\xi_3}{2\tau_3} - \frac{|\widetilde{\widetilde{\mu}}_2|}{2}\right)\|z_3(x,1,t)\|_2^2 \\
&+ \frac{1}{2}\int_0^L \int_0^\infty g_1'(s)(\partial_x\eta_1)^2 ds dx + \frac{1}{2}\int_0^L \int_0^\infty g_2'(s)(\partial_x\eta_2)^2 ds dx \\
&+ \frac{1}{2}\int_0^L \int_0^\infty g_3'(s)(\partial_x\eta_3)^2 ds dx.
\end{aligned}
$$

This completes the proof of the lemma. ■

Now, we will give well-posedness results for problem (4.9)-(4.10) by using semigroup theory. Let us introduce the semigroup representation of the Bresse system (4.9)-(4.10). Let $U = (\varphi, \psi, \omega, \varphi_t, \psi_t, \omega_t, z_1, z_2, z_3, \eta_1, \eta_2, \eta_3)^T$ and rewrite (4.9)-(4.10) as

$$
\begin{cases} U' = AU, \\ U(x,0) = U^0(x). \end{cases} \tag{4.18}
$$

$$
U^0(x) = (\varphi_0, \psi_0, \omega_0, \varphi_1, \psi_1, \omega_1, f_1(., -.\tau_1), f_2(., -.\tau_2), f_3(., -.\tau_3), \eta_1^0, \eta_2^0, \eta_3^0),
$$

where the operator A is defined by

$$A\begin{pmatrix}\varphi\\ \psi\\ \omega\\ \varphi_t\\ \psi_t\\ \omega_t\\ z_1\\ z_2\\ z_3\\ \eta_1\\ \eta_2\\ \eta_3\end{pmatrix}=\begin{pmatrix}\varphi_t\\ \psi_t\\ \omega_t\\ \frac{1}{\rho_1}(k_1-\int_0^\infty g_1(s)ds)\varphi_{xx}-\frac{l^2k_3}{\rho_1}\varphi+\frac{k_1}{\rho_1}\psi_x+\frac{l}{\rho_1}(k_1+k_3)\omega_x\\ +\frac{1}{\rho_1}\int_0^\infty g_1(s)\partial_{xx}\eta_1 ds\\ -\frac{\mu_1}{\rho_1}\varphi_t-\frac{\mu_2}{\rho_1}z_1(.,1)\\ \frac{-k_1}{\rho_2}\varphi_x+\frac{1}{\rho_2}(k_2-\int_0^\infty g_2(s)ds)\psi_{xx}-\frac{k_1}{\rho_2}\psi-\frac{-lk_1}{\rho_2}\omega\\ +\frac{1}{\rho_2}\int_0^\infty g_2(s)ds)\partial_{xx}\eta_2 ds\\ -\frac{\widetilde{\mu}_1}{\rho_1}\psi_t-\frac{\widetilde{\mu}_2}{\rho_1}z_2(.,1)\\ \frac{-l}{\rho_1}(k_1+k_3)\varphi_x-\frac{lk_1}{\rho_1}\psi\\ +\frac{1}{\rho_1}(k_3-\int_0^\infty g_3(s)ds)\omega_{xx}-\frac{l^2k_1}{\rho_1}\omega+\frac{1}{\rho_1}\int_0^\infty g_3(s)\partial_{xx}\eta_3 ds\\ -\frac{\widetilde{\widetilde{\mu}}_1}{\rho_1}\omega_t-\frac{\widetilde{\widetilde{\mu}}_2}{\rho_1}z_3(.,1)\\ \frac{-1}{\tau}z_{1\rho}\\ \frac{-1}{\tau}z_{2\rho}\\ \frac{-1}{\tau}z_{3\rho}\\ \varphi_t-\partial_s\eta_1\\ \psi_t-\partial_s\eta_2\\ \omega_t-\partial_s\eta_3\end{pmatrix}$$

with domain

$$H=\left\{(H^2(]0,L[)\cap(H_0^1(]0,L[)))^3\times(H_0^1(]0,L[))^3\times(L^2(0,L;H^1(0,1))^3\times H_1^*\times H_2^*\times H_3^*\right\}. \tag{4.19}$$

The domain D(A) of A is defined by

$$D(A)=\{U\in H; AU\in H, \eta_i(x,t,0)=0, i=1,2,3\}. \tag{4.20}$$

Now, under hypothesis (H1), the sets $H_i^*$ and H are Hilbert spaces equipped, respectively, with the inner products that generate the norms

$$\|\eta_i\|_{H_i^*}^2=\int_0^L\int_0^{+\infty}g_i(s)(\partial_x\eta_i)^2dsdx,$$

$$\|U\|_H^2=\int_0^L(\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1\omega_t^2+k_2\psi_x^2+k_1(\varphi_x+\psi+l\omega)^2+k_3(\omega_x-l\varphi)^2)dx$$
$$+\int_0^L\sum_{i=1}^3\xi_i\int_0^1 z_i^2\,d\rho-\int_0^L(g_1^0\varphi_x^2+g_2^0\psi_x^2+g_3^0\omega_x^2)\,dx+\|\eta_1\|_{H_1^*}^2+\|\eta_2\|_{H_2^*}^2+\|\eta_3\|_{H_3^*}^2.$$

We show that the operator A generates a $C_0$- semigroup in $H$. In this step, we prove that the operator A is dissipative. Let $U=(\varphi,\psi,\omega,u,v,\tilde{\omega},z_1,z_2,z_3,\eta_1,\eta_2,\eta_3)^T$. Using (4.12) and the fact that

$$E(t)=\frac{1}{2}\|U\|_H^2, \tag{4.21}$$

we get

$$
(4.22)\qquad
\begin{aligned}
\langle AU, U\rangle_H \quad &= -\mu_1 \int_0^L u^2\,dx - \widetilde{\mu}_1 \int_0^L v^2\,dx - \widetilde{\widetilde{\mu}}_1 \int_0^L \widetilde{\omega}^2\,dx \\
&-\mu_2 \int_0^L z_1(x,1)u\,dx - \widetilde{\mu}_2 \int_0^L z_2(x,1)v\,dx - \widetilde{\widetilde{\mu}}_2 \int_0^L z_3(x,1)\widetilde{\omega}\,dx \\
&-\sum_{i=1}^{3} \frac{\xi_i}{\tau_i} \int_0^L \int_0^1 z_i(x,\rho) z_{i\rho}(x,\rho)\,d\rho\,dx - \frac{1}{2}\int_0^L g_1(s)\int_0^\infty \partial_s(\partial_x\eta_1)^2 dsdx \\
&-\frac{1}{2}\int_0^L g_2(s)\int_0^\infty \partial_s(\partial_x\eta_2)^2 dsdx - \frac{1}{2}\int_0^L g_3(s)\int_0^\infty \partial_s(\partial_x\eta_3)^2 dsdx, \\
&\le 0,
\end{aligned}
$$

which, by using the integration by parts and the boundary conditions in (4.10), yields

$$
(4.23)\qquad
\begin{aligned}
\langle AU, U\rangle_H \quad &= -\mu_1 \int_0^L u^2\,dx - \widetilde{\mu}_1 \int_0^L v^2\,dx - \widetilde{\widetilde{\mu}}_1 \int_0^L \widetilde{\omega}^2\,dx \\
&-\mu_2 \int_0^L z_1(x,1)u\,dx - \widetilde{\mu}_2 \int_0^L z_2(x,1)v\,dx - \widetilde{\widetilde{\mu}}_2 \int_0^L z_3(x,1)\widetilde{\omega}\,dx \\
&-\sum_{i=1}^{3} \frac{\xi_i}{\tau_i} \int_0^L \int_0^1 z_i(x,\rho) z_{i\rho}(x,\rho)\,d\rho\,dx + \frac{1}{2}\int_0^L \int_0^\infty (g_1'(s)\partial_x\eta_1)^2 dsdx \\
&+\frac{1}{2}\int_0^L \int_0^\infty (g_2'(s)\partial_x\eta_2)^2 dsdx + \frac{1}{2}\int_0^L \int_0^\infty (g_3'(s)\partial_x\eta_3)^2 dsdx,
\end{aligned}
$$

and then, because, for any $i = 1,2,3$, the kernel $g_i$ is non-increasing,

$$
(4.24)\qquad \langle AU, U\rangle \le 0.
$$

Consequently, the operator A is dissipative. Now, we will prove that the operator $\lambda I - A$ is surjective for $\lambda > 0$. For this purpose, let
$(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12})^T \in H$, we seek
$U = (v_1, v_2, v_3, v_4, v_5, v_6, z_1, z_2, z_3, v_7, v_8, v_9)^T \in D(A)$ solution of the following system of equations

$$
(4.25)\qquad \begin{cases}
\lambda \upsilon_4 + \frac{\mu_1}{\rho_1}\upsilon_4 + \frac{\mu_2}{\rho_1} z_1(.,1) + \frac{1}{\rho_1}(k_1 - g_0^1)\partial_{xx}\upsilon_1 - \frac{l}{\rho_1}(k_1\partial_x\upsilon_3 - lk_3\upsilon_1) \\
-\frac{l}{\rho_1}\left\{k_1\partial_x\upsilon_3 + k_2\partial_x\upsilon_3 - g_0^1\upsilon_7\right\} = f_4, \\
\\
-\frac{1}{\rho_2}\left\{(k_2 - g_0^2)\partial_{xx}\upsilon_2 - k_1\partial_x\upsilon_1 + \frac{g_0^3\rho_2}{\rho_1}\partial_{xx}\upsilon_9\right\} + \frac{lk_1}{\rho_2}\upsilon_3 + \lambda\upsilon_5 \\
+\frac{\widetilde{\mu_1}}{\rho_1}\upsilon_5 + \frac{\widetilde{\mu_2}}{\rho_1} z_2(.,1) = f_6, \\
\\
\frac{1}{\rho_1}\left\{(k_1 + k_2)\partial_x\upsilon_1 - (k_3 - g_0^3)\partial_{xx}\upsilon_3 - g_0^3\partial_{xx}\upsilon_9\right\} + \frac{lk_1}{\rho_1}\upsilon_2 + \frac{l^2k_1}{l_1}\upsilon_3 \\
+\frac{\widetilde{\widetilde{\mu_1}}}{\rho_1}\upsilon_6 + \lambda\upsilon_8 + \frac{\widetilde{\widetilde{\mu_2}}}{\rho_1} z_3(.,1) = f_5, \\
\\
\lambda z_1 + \frac{1}{\tau_1} z_{1\rho} = f_7, \\
\lambda z_2 + \frac{1}{\tau_2} z_{2\rho} = f_8, \\
\lambda z_3 + \frac{1}{\tau_3} z_{3\rho} = f_9, \\
\lambda\upsilon_1 - \upsilon_5 = f_1, \\
\lambda\upsilon_2 - \upsilon_6 = f_2, \\
\lambda\upsilon_3 - \upsilon_4 = f_3, \\
\\
-\upsilon_4 + \lambda\upsilon_7 + \partial_s\upsilon_7 = f_{10}, \\
-\upsilon_5 + \lambda\upsilon_8 + \partial_s\upsilon_8 = f_{11}, \\
-\upsilon_6 + \lambda\upsilon_9 + \partial_s\upsilon_9 = f_{12}.
\end{cases}
$$

Suppose that we have found $\upsilon_1, \upsilon_2$ and $\upsilon_3$. Therefore, the seventh, the eighth and the ninth equation in (4.25) give

$$
(4.26)\qquad \begin{cases}
\upsilon_5 = \lambda\upsilon_1 - f_1, \\
\upsilon_6 = \lambda\upsilon_2 - f_2, \\
\upsilon_4 = \lambda\upsilon_3 - f_3.
\end{cases}
$$

Then it is clear that $\upsilon_1 \in H_0^1(0,L), \upsilon_2 \in H_0^1(0,L)$ and $\upsilon_3 \in H_0^1(0,L)$. Furthermore, by (4.26) we can find $z_i (i = 1,2,3)$ as

$$
(4.27)\qquad z_1(x,0) = \upsilon_5(x), z_2(x,0) = \upsilon_6(x), z_3(x,0) = \upsilon_7(x), \qquad \text{for } x \in (0,L).
$$

Following the same approach as in **[74]**, we obtain, by using equations for $z_i$ in (4.27),

$$
(4.28)\qquad \begin{cases}
z_1(x,\rho) = \upsilon_5(x)e^{-\lambda\tau_1\rho} + \tau_1 e^{-\lambda\tau_1\rho}\displaystyle\int_0^\rho f_7(x,s)e^{\lambda\tau_1 s}\,ds, \\
\\
z_2(x,\rho) = \upsilon_6(x)e^{-\lambda\tau_2\rho} + \tau_2 e^{-\lambda\tau_2\rho}\displaystyle\int_0^\rho f_8(x,s)e^{\lambda\tau_2 s}\,ds, \\
\\
z_3(x,\rho) = \upsilon_7(x)e^{-\lambda\tau_3\rho} + \tau_3 e^{-\lambda\tau_3\rho}\displaystyle\int_0^\rho f_9(x,s)e^{\lambda\tau_3 s}\,ds,
\end{cases}
$$

From (4.28), we obtain

$$
\begin{cases}
z_1(x,\rho) = \lambda \upsilon_1(x) e^{-\lambda\tau_1\rho} - f_1 e^{-\lambda\tau_1\rho} + \tau_1 e^{-\lambda\tau_1\rho} \displaystyle\int_0^\rho f_7(x,s) e^{\lambda\tau_1 s}\, ds, \\[2ex]
z_2(x,\rho) = \lambda \upsilon_2(x) e^{-\lambda\tau_2\rho} - f_2 e^{-\lambda\tau_2\rho} + \tau_2 e^{-\lambda\tau_2\rho} \displaystyle\int_0^\rho f_8(x,s) e^{\lambda\tau_2 s}\, ds, \\[2ex]
z_3(x,\rho) = \lambda \upsilon_3(x) e^{-\lambda\tau_3\rho} - f_3 e^{-\lambda\tau_3\rho} + \tau_3 e^{-\lambda\tau_3\rho} \displaystyle\int_0^\rho f_9(x,s) e^{\lambda\tau_3 s}\, ds,
\end{cases}
\tag{4.29}
$$

$$
\begin{cases}
\upsilon_7 = \left(\displaystyle\int_0^s (\upsilon_4 + f_{10}) e^{\tau} d\tau\right) e^{-s}, \\[2ex]
\upsilon_8 = \left(\displaystyle\int_0^s (\upsilon_5 + f_{11}) e^{\tau} d\tau\right) e^{-s}, \\[2ex]
\upsilon_9 = \left(\displaystyle\int_0^s (\upsilon_6 + f_{12}) e^{\tau} d\tau\right) e^{-s}.
\end{cases}
\tag{4.30}
$$

By using (4.25) and (4.30) the functions $\upsilon_1, \upsilon_2$ and $\upsilon_3$ satisfying the following system

$$
\begin{cases}
\lambda^2 \upsilon_3 + \frac{1}{\rho_1}\left(k_1 + \frac{\mu_1\lambda}{\rho_1}\upsilon_3 - g_0^1\right)\partial_{xx}\upsilon_1 - \frac{l}{\rho_1}\left(k_1\partial_x\upsilon_3 - lk_3\upsilon_1\right) - \frac{l}{\rho_1}\left(k_1\partial_x\upsilon_3 + k_2\partial_x\upsilon_3\right) \\
+\frac{\mu_2}{\rho_1} z_1(.,1) = \left(\frac{\mu_1\lambda}{\rho_1} + 1\right) f_3 + \left(\displaystyle\int_0^s (\lambda\upsilon_3 - f_3 + f_{10}) e^{\tau} d\tau\right) e^{-s}, \\[2ex]
\lambda^2\left(\frac{lk_1}{\rho_1} + \frac{\widetilde{\widetilde{\mu_1}}\lambda}{\rho_2}\right)\upsilon_2 + \frac{l^2k_1}{\rho_1}\upsilon_3 + \frac{1}{\rho_1}\left(k_1 + k_2\partial_x\upsilon_1 - \frac{1}{\rho_1}(k_3 - g_0^3\right)\partial_x\upsilon_3 \\
-\frac{\lambda g_0^3}{\rho_1}\left(\displaystyle\int_0^s (\lambda\upsilon_1 - f_1 + f_{12}) e^{\tau} d\tau\right) e^{-s} + \lambda\left(\displaystyle\int_0^s (\lambda\upsilon_2 - f_2 + f_{11}) e^{\tau} d\tau\right) e^{-s}\lambda \\
+\frac{\widetilde{\widetilde{\mu_2}}}{\rho_1} z_3(.,1) = f_5 + \frac{\widetilde{\widetilde{\mu_1}}}{\rho_1} f_2, \\[2ex]
\left(\lambda^2 + \frac{\widetilde{\mu_1}\lambda}{\rho_1}\right)\upsilon_1 + \frac{lk_1}{\rho_1}\upsilon_3 + \frac{k_1}{\rho_1}\partial_x\upsilon_1 + \frac{lk_1}{\rho_2}\upsilon_3 - \frac{1}{\rho_1}\left(k_2 - g_0^2\right)\partial_{xx}\upsilon_2 + \frac{\widetilde{\mu_2}}{\rho_1} z_2(.,1) \\
-\frac{g_0^3}{\rho_1}\partial_{xx}\left(\displaystyle\int_0^s (\lambda\upsilon_2 - f_2 + f_{12}) e^{\tau} d\tau\right) e^{-s} = \left(\lambda + \frac{\widetilde{\mu_1}}{\rho_1}\right) f_1 + f_6.
\end{cases}
\tag{4.31}
$$

Solving system (4.31) is equivalent to finding $(\upsilon_1, \upsilon_2, \upsilon_3) \in (H^2 \cap H_0^1(0,L))^3$ such that

$$
(4.32) \quad \begin{cases}
\displaystyle\int_0^L \left\{ \lambda^2 \upsilon_3 + \frac{1}{\rho_1}\left(k_1 + \frac{\mu_1 \lambda}{\rho_1}\upsilon_3 - g_0^1\right)\partial_{xx}\upsilon_1 \right\}\phi_1 dx \\
\displaystyle -\int_0^L \left\{ \frac{l}{\rho_1}\left(k_1 \partial_x \upsilon_3 - l k_3 \upsilon_1\right) - \frac{l}{\rho_1}\left(k_1 \partial_x \upsilon_3 + k_2 \partial_x \upsilon_3\right) + \frac{\mu_2}{\rho_1} z_1(.,1) \right\}\phi_1 dx \\
\displaystyle = \int_0^L \left\{ \left(\frac{\mu_1 \lambda}{\rho_1} + 1\right) f_3 + \left(\int_0^s (\lambda \upsilon_3 - f_3 + f_{10}) e^{\tau} d\tau\right) e^{-s} \right\}\phi_1 dx, \\[2ex]
\displaystyle\int_0^L \left\{ \lambda^2 \left(\frac{l k_1}{\rho_1} + \frac{\widetilde{\widetilde{\mu}\lambda}}{\rho_2}\right)\upsilon_2 + \frac{1}{\rho_1}\left(k_1 + k_2 \partial_x \upsilon_1 - \frac{1}{\rho_1}(k_3 - g_0^3\right)\partial_x \upsilon_3 \right\}\phi_2 dx \\
\displaystyle + \int_0^L \left\{ \frac{l^2 k_1}{\rho_1}\upsilon_3 + \frac{\tilde{\tilde{\mu}}_2}{\rho_1} z_3(.,1) \right\}\phi_2 dx \\
\displaystyle = \int_0^L \left\{ \frac{\lambda g_0^3}{\rho_1}\left(\int_0^s (\lambda \upsilon_1 - f_1 + f_{12}) e^{\tau} d\tau\right) e^{-s} + f_5 + \frac{\tilde{\tilde{\mu}}_1}{\rho_1} f_2 \right\}\phi_2 dx \\
\displaystyle + \int_0^L \left\{ \lambda \left(\int_0^s (\lambda \upsilon_2 - f_2 + f_{11}) e^{\tau} d\tau\right) e^{-s}\lambda \right\}\phi_2 dx, \\[2ex]
\displaystyle\int_0^L \left\{ \left(\lambda^2 + \frac{\widetilde{\mu_1}\lambda}{\rho_1}\right)\upsilon_1 + \frac{l k_1}{\rho_1}\upsilon_3 + \frac{k_1}{\rho_1}\partial_x \upsilon_1 - \frac{1}{\rho_1}\left(k_2 - g_0^2\right)\partial_{xx}\upsilon_2 \right\}\phi_3 dx \\
\displaystyle + \int_0^L \left\{ \frac{\widetilde{\mu_2}}{\rho_1} z_2(.,1) + \frac{l k_1}{\rho_2}\upsilon_3 \right\}\phi_3 dx \\
\displaystyle = \int_0^L \left\{ \frac{g_0^3}{\rho_1}\partial_{xx}\left(\int_0^s (\lambda \upsilon_2 - f_2 + f_{12}) e^{\tau} d\tau\right) e^{-s} + \left(\lambda + \frac{\widetilde{\mu_1}}{\rho_1}\right) f_1 + f_6 \right\}\phi_3 dx.
\end{cases}
$$

Consequently, problem (4.32) is equivalent to the problem

$$
(4.33) \qquad a((\upsilon_1, \upsilon_2, \upsilon_3), (\phi_1, \phi_2, \phi_3)) = L(\phi_1, \phi_2, \phi_3)
$$

where the bilinear form $a : [H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)]^2 \to R$ and the linear form $L : H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L) \to R$ are defined by

$$
\begin{aligned}
& a((\upsilon_1, \upsilon_2, \upsilon_3, \phi_1, \phi_2, \phi_3) \\
& = \int_0^L \left\{ \lambda^2 \upsilon_3 + \frac{1}{\rho_1} \left( k_1 + \frac{\mu_1 \lambda}{\rho_1} \upsilon_3 - g_0^1 \right) \partial_{xx} \upsilon_1 \right\} \phi_1 dx \\
& - \int_0^L \left\{ \frac{l}{\rho_1} \left( k_1 \partial_x \upsilon_3 - l k_3 \upsilon_1 \right) \right\} \phi_1 dx \\
& - \int_0^L \left\{ \frac{l}{\rho_1} \left( k_1 \partial_x \upsilon_3 + k_2 \partial_x \upsilon_3 \right) + \frac{\mu_2}{\rho_1} z_1(.,1) \right\} \phi_1 dx \\
& + \int_0^L \left\{ \lambda^2 \left( \frac{l k_1}{\rho_1} + \frac{\widetilde{\widetilde{\mu}} \lambda}{\rho_2} \right) \upsilon_2 + \frac{l^2 k_1}{\rho_1} \upsilon_3 \right\} \phi_2 dx \\
& + \int_0^L \left\{ \frac{1}{\rho_1} \left( k_1 + k_2 \partial_x \upsilon_1 - \frac{1}{\rho_1} (k_3 - g_0^3 \right) \partial_x \upsilon_3 + \frac{\widetilde{\widetilde{\mu_2}}}{\rho_1} z_3(.,1) \right\} \phi_2 dx \\
& + \int_0^L \left\{ \left( \lambda^2 + \frac{\widetilde{\mu_1} \lambda}{\rho_1} \right) \upsilon_1 + \frac{l k_1}{\rho_1} \upsilon_3 + \frac{k_1}{\rho_1} \partial_x \upsilon_1 + \frac{l k_1}{\rho_2} \upsilon_3 \right\} \phi_3 dx \\
& - \int_0^L \left\{ \frac{1}{\rho_1} \left( k_2 - g_0^2 \right) \partial_{xx} \upsilon_2 + \frac{\widetilde{\mu_2}}{\rho_1} z_2(.,1) \right\} \phi_3 dx,
\end{aligned} \tag{4.34}
$$

$$
\begin{aligned}
& L(\phi_1, \phi_2, \phi_3) \\
& = \int_0^L \left\{ \left( \frac{\mu_1 \lambda}{\rho_1} + 1 \right) f_3 + \left( \int_0^s (\lambda \upsilon_3 - f_3 + f_{10}) e^{\tau} d\tau \right) e^{-s} \right\} \phi_1 dx \\
& + \int_0^L \left\{ \frac{\lambda g_0^3}{\rho_1} \left( \int_0^s (\lambda \upsilon_1 - f_1 + f_{12}) e^{\tau} d\tau \right) e^{-s} + f_5 + \frac{\widetilde{\widetilde{\mu_1}}}{\rho_1} f_2 \right\} \phi_2 dx \\
& + \int_0^L \left\{ \lambda \left( \int_0^s (\lambda \upsilon_2 - f_2 + f_{11}) e^{\tau} d\tau \right) e^{-s} \lambda \right\} \phi_2 dx \\
& + \int_0^L \left\{ \frac{g_0^3}{\rho_1} \partial_{xx} \left( \int_0^s (\lambda \upsilon_2 - f_2 + f_{12}) e^{\tau} d\tau \right) e^{-s} \right\} \phi_3 dx, \\
& + \int_0^L \left\{ \left( \lambda + \frac{\widetilde{\mu_1}}{\rho_1} \right) f_1 + f_6 \right\} \phi_3 dx.
\end{aligned} \tag{4.35}
$$

It is easy to verify that $a$ is continuous and coercive, and $L$ is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(\phi_1, \phi_2, \phi_3) \in H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$ problem (4.9)-(4.10) admits a unique solution $(\upsilon_1, \upsilon_2, \upsilon_3) \in H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$. Applying the classical elliptic regularity, it follows from (4.33) that $(\upsilon_1, \upsilon_1, \upsilon_3) \in H^2(0,L) \times H^2(0,L) \times H^2(0,L)$. Therefore, the operator $\lambda I - A$ is surjective for any $\lambda > 0$. Consequently, the existence result of Theorem 3.1 follows from the Lumer-Phillips and Hille-Yosida theorems.

## 4.4 Asymptotic Stability

In this section we prove the asymptotic stability result by constructing a suitable Lyapunov functionnal. Now, let us introduce the following functional

$$I_1(t) = -\rho_1 \int_0^L \varphi_t \int_0^{+\infty} g_1(s)\eta_1 ds dx, \tag{4.36}$$

$$I_2(t) = -\rho_2 \int_0^L \psi_t \int_0^{+\infty} g_2(s)\eta_2 ds dx, \tag{4.37}$$

$$I_3(t) = -\rho_1 \int_0^L \omega_t \int_0^{+\infty} g_3(s)\eta_3 ds dx, \tag{4.38}$$

$$I_4(t) = \int_0^L \left(\rho_1 \varphi\varphi_t + \rho_2 \psi\psi_t + \rho_1 \omega\omega_t\right) dx, \tag{4.39}$$

$$I_5(t) = \int_0^L \int_0^1 \sum_{i=1}^{3} e^{-2\tau_i \rho} z_i^2(x,t,\rho)\, d\rho\, dx, \tag{4.40}$$

$$I_0(t) = I_1(t) + I_2(t) + I_3(t). \tag{4.41}$$

Then the following result holds.

**Lemma 4.4.1** *There exists a positive constant $C$ such that the following inequality holds for every $(\varphi, \psi, \omega) \in (H_0^1(0,L))^3$*

$$\begin{aligned}\int_0^L (|\varphi_x|^2 + |\psi_x|^2 + |\omega_x|^2)\, dx \quad &\le C \int_0^L (k_2|\psi_x|^2 + k_1|\varphi_x + \psi + l\omega|^2) dx \\ &+ k_3|\omega_x - l\varphi|^2\, dx\end{aligned} \tag{4.42}$$

*Proof* We will argue by contradiction. Indeed, let us suppose that is not true. So, we can find a sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu \in N}$ in $(H_0^1(0,L))^3$ satisfying

$$\int_0^L (k_2|\psi_{\nu x}|^2 + k_1|\varphi_{\nu x} + \psi + l\omega_\nu|^2 + k_3|\omega_{\nu x} - l\varphi_\nu|^2)\, dx \le \frac{1}{\nu} \tag{4.43}$$

and

$$\int_0^L (|\varphi_{\nu x}|^2 + |\psi_{\nu x}|^2 + |\omega_{\nu x}|^2)\, dx = 1. \tag{4.44}$$

From (4.43), the sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu \in N}$ is bounded in $(H_0^1(0,L))^3$. Since the embedding $H_0^1(0,L) \hookrightarrow L^2(0,L)$ is compact, then the sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu \in N}$ converge strongly in $(L^2(0,L))^3$.
From (4.43)

$$\psi_{\nu x} \to 0 \text{ strongly in } L^2(0,L). \tag{4.45}$$

Using Poincaré's inequality we can conclude that

$$\psi_\nu \to 0 \text{ strongly in } L^2(0,L). \tag{4.46}$$

Now, setting $\varphi_\nu \to \varphi$ and $\omega_\nu \to \omega$ strongly in $L^2(0,L)$.
From (4.44), we have

$$\varphi_{\nu x} + \psi_\nu + l\omega_\nu \to 0 \text{ strongly in } L^2(0,L). \tag{4.47}$$

Then

$$\varphi_{\nu x} + \psi_\nu + l\omega_\nu = \varphi_{\nu x} + \psi_\nu + l(\omega_\nu - \omega) + l\omega \to 0 \text{ strongly in } L^2(0,L). \tag{4.48}$$

which implies that

$$\varphi_{\nu x} \to -l\omega \text{ strongly in } L^2(0,L). \tag{4.49}$$

Then, $\{\varphi_\nu\}_n$ is a Cauchy sequence in $H^1(0,L)$. Therefore $\{\varphi_\nu\}_n$ converge to a function $\varphi_1$ in $H^1(0,L)$. Consequently $\{\varphi_\nu\}_n$ converge to $\varphi_1$ in $L^2(0,L)$. Thus by the uniqueness of the limit $\varphi_1 = \varphi$. Moreover $\varphi \in H_0^1(0,L)$.
From (4.49) we deduce that

$$\varphi_x + l\omega = 0 \text{ a.e } x \in (0,L). \tag{4.50}$$

Similarly, we have

$$\omega_x - l\varphi = 0 \text{ a.e } x \in (0,L), \tag{4.51}$$

and $\omega \in H_0^1(0,L)$. The limits (4.50) and (4.51) provides us $\varphi = \omega = 0$, contradicting (4.43).
The proof is hence complete ■

**Lemma 4.4.2** *the functional defined in (4.41) satisfies for any $\delta > 0$*

$$\begin{aligned}
I_0'(t) \quad &\leq -\rho_1(g_1^0 - \delta(1+\mu_2)) \int_0^L \varphi_t^2 dx + \widetilde{\mu_1}\delta \int_0^L \psi_t^2 dx \\
&+ \widetilde{\widetilde{\mu_1}}\delta \int_0^L \omega_t^2 dx + c_\delta \int_0^L \left\{\psi_x + (\varphi_x + \psi + l\omega)^2 + (\omega_x - l\varphi)^2\right\} dx \\
&+ c_\delta \int_0^L \int_0^\infty g_1(s)(\partial_x\eta_1)^2 ds dx - c_\delta \int_0^L \int_0^\infty g_1'(s)(\partial_x\eta_1)^2 ds dx \\
&+ c_\delta \int_0^L \int_0^\infty g_2(s)(\partial_x\eta_2)^2 ds dx - c_\delta \int_0^L \int_0^\infty g_2'(s)(\partial_x\eta_2)^2 ds dx \\
&+ c_\delta \int_0^L \int_0^\infty g_3(s)(\partial_x\eta_1)^2 ds dx - c_\delta \int_0^L \int_0^\infty g_3'(s)(\partial_x\eta_3)^2 ds dx \\
&+ c_\delta \int_0^L \left\{z_1^2(x,1,t) + z_2^2(x,1,t) + z_3^2(x,1,t)\right\} dx
\end{aligned} \tag{4.52}$$

*Proof* Differentiating (4.41) with respect to $t$ and using the third equation in (4.9)-(4.10), integrating by parts and using the fact that

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty g_1(s)\eta_1 ds \quad &= \frac{d}{dt}\int_0^\infty g_1(t-s)\left(\varphi(t) - \varphi(s)\right) ds \\
&= \int_0^\infty g_1'(t-s)\left(\varphi(t) - \varphi(s)\right) ds + \left(\int_0^\infty g_1(t-s) ds\right)\varphi_t \\
&= \int_0^\infty g_1'(s)\eta_1 ds + g_1^0\varphi_t,
\end{aligned} \tag{4.53}$$

in the same way for

$$\frac{d}{dt}\int_0^\infty g_2(s)\eta_2 ds = \int_0^\infty g_2'(s)\eta_2 ds + g_2^0\psi_t, \tag{4.54}$$

and

$$\frac{d}{dt}\int_0^\infty g_1(s)\eta_3 ds = \int_0^\infty g_3'(s)\eta_3 ds + g_3^0\omega_t. \tag{4.55}$$

we conclude that

$$\begin{aligned} I_0'(t) &= -\rho_1 g_1^0 \int_0^L \varphi_t^2 dx - \rho_1 \int_0^L \varphi_t \int_0^\infty g_1'(s)\eta_1 dsdx \\ &+k_1 \int_0^L (\varphi_x + \psi + l\omega) \int_0^\infty g_1(s)\partial_x\eta_1 dsdx \\ &-k_3 \int_0^L (\omega_x - l\varphi) \int_0^\infty g_1(s)\eta_1 dsdx \\ &- \int_0^L \varphi_x \left(\int_0^\infty g_1(s)\partial_x\eta_1 ds\right) \\ &+ \int_0^L \left(\int_0^\infty g_1(s)\partial_x\eta_1 ds\right)^2 dx + \mu_1 \int_0^L \varphi_t(x,t) \int_0^\infty g_1(s)\eta_1 dsdx \\ &+\mu_2 \int_0^L z_1(x,1,t) \int_0^\infty g_1(s)\eta_1 dsdx + \widetilde{\mu}_1 \int_0^L \varphi_t(x,t) \int_0^\infty g_2(s)\eta_2 dsdx \\ &+\widetilde{\mu}_2 \int_0^L z_2(x,1,t) \int_0^\infty g_2(s)\eta_2 dsdx + \widetilde{\widetilde{\mu}}_1 \int_0^L \varphi_t(x,t) \int_0^\infty g_3(s)\eta_3 dsdx \\ &+\widetilde{\widetilde{\mu}}_2 \int_0^L z_3(x,1,t) \int_0^\infty g_3(s)\eta_3 dsdx. \end{aligned} \tag{4.56}$$

Using Young's, Poincaré's and Holder's inequalities for the last six terms of the above equality, using the second and third equations of (4.9), we find

$$\begin{aligned} I_0'(t) &\leq -\rho_1(g_1^0 - \delta(1+\mu_2)) \int_0^L \varphi_t^2 dx + \widetilde{\mu}_1\delta \int_0^L \psi_t^2 dx \\ &+\widetilde{\widetilde{\mu}}_1\delta \int_0^L \omega_t^2 dx + c_\delta \int_0^L \left\{\psi_x + (\varphi_x + \psi + l\omega)^2 + (\omega_x - l\varphi)^2\right\} dx \\ &+c_\delta \int_0^L \int_0^\infty g_1(s)(\partial_x\eta_1)^2 dsdx - c_\delta \int_0^L \int_0^\infty g_1'(s)(\partial_x\eta_1)^2 dsdx \\ &+c_\delta \int_0^L \int_0^\infty g_2(s)(\partial_x\eta_2)^2 dsdx - c_\delta \int_0^L \int_0^\infty g_2'(s)(\partial_x\eta_2)^2 dsdx \\ &+c_\delta \int_0^L \int_0^\infty g_3(s)(\partial_x\eta_1)^2 dsdx - c_\delta \int_0^L \int_0^\infty g_3'(s)(\partial_x\eta_3)^2 dsdx \\ &+c_\delta \int_0^L \left\{z_1^2(x,1,t) + z_2^2(x,1,t) + z_3^2(x,1,t)\right\} dx \end{aligned} \tag{4.57}$$

The proof is hence complete.

■

**Lemma 4.4.3** *The functional defined in (4.39) satisfies for any* $\epsilon > 0$

$$
\begin{aligned}
I_4'(t) \quad &\leq \int_0^L \left\{ (\rho_1+\epsilon)\varphi_t^2 + (\rho_2+\epsilon)\psi_t^2 + (\rho_1+\epsilon)\omega_t^2 \right\} dx \\
&-c_1 \int_0^L \left\{ \psi_x^2 + (\varphi_x+\psi+l\omega)^2 + (\omega_x - l\varphi)^2 \right\} dx \\
&+c_\epsilon \int_0^L \int_0^\infty \left\{ g_1(s)(\partial_x\eta_1)^2 + g_2(s)(\partial_x\eta_2)^2 + g_3(s)(\partial_x\eta_3)^2 \right\} dx \\
&+c_\epsilon \int_0^L \left\{ z_1^2(x,1,t) + z_2^2(x,1,t) + z_3^2(x,1,t) \right\} dx
\end{aligned} \tag{4.58}
$$

*Proof* Differentiating $I_4(t)$ with respect to $t$, we see that

$$
\begin{aligned}
I_4'(t) \quad &= \int_0^L (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2)dx - k_1 \int_0^L (\varphi_x+\psi+l\omega)^2 dx \\
&+g_1^0 \int_0^L \varphi_x^2 dx - (k_2 - g_2^0) \int_0^L \psi_x^2 dx + g_3^0 \int_0^L \omega_x^2 dx \\
&- \int_0^L \varphi_x \int_0^{+\infty} g_1(s)\partial_x\eta_1 ds dx - \int_0^L \psi_x \int_0^{+\infty} g_2(s)\partial_x\eta_2 ds dx \\
&- \int_0^L \omega_x \int_0^{+\infty} g_3(s)\partial_x\eta_3 ds dx - \mu_1 \int_0^L \varphi_t\varphi dx - \widetilde{\mu_1} \int_0^L \psi_t\psi dx \\
&-\widetilde{\widetilde{\mu_1}} \int_0^L \omega_t\omega dx - \mu_2 \int_0^L z_1(x,1,t)\varphi dx - \widetilde{\mu_2} \int_0^L z_2(x,1,t)\psi dx \\
&-\widetilde{\widetilde{\mu_2}} \int_0^L z_3(x,1,t)\omega dx - k_3 \int_0^L (\omega_x - l\varphi)^2 dx
\end{aligned} \tag{4.59}
$$

Using Young's and Poincare's inequalities, we get for any $\epsilon > 0$

$$
\begin{aligned}
&- \int_0^L \varphi_x \int_0^\infty g_1(s)\partial_x\eta_1 ds dx - \int_0^L \psi_x \int_0^\infty g_2(s)\partial_x\eta_2 ds dx \\
&- \int_0^L \omega_x \int_0^\infty g_3(s)\partial_x\eta_3 ds dx dx \\
&\leq \epsilon \int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2)dx + c \int_0^L \int_0^\infty \sum_{i=1}^3 (g_i(s)(\partial_x\eta_i)^2 ds dx,
\end{aligned} \tag{4.60}
$$

$$
\begin{aligned}
&-\mu_1 \int_0^L \varphi_t\varphi dx - \widetilde{\mu_1} \int_0^L \psi_t\psi dx - \widetilde{\widetilde{\mu_1}} \int_0^L \omega_t\omega dx \\
&\leq \epsilon \textstyle\int_0^L (\varphi_t^2 + \psi_t^2 + \omega_t^2)dx + c_\epsilon \displaystyle\int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2)dx,
\end{aligned} \tag{4.61}
$$

$$
\begin{aligned}
&-\mu_2 \int_0^L z_1(x,1,t)\varphi dx - \widetilde{\mu_2} \int_0^L z_2(x,1,t)\psi dx - \widetilde{\widetilde{\mu_2}} \int_0^L z_3(x,1,t)\omega dx \\
&\leq \int_0^L c_\epsilon \left\{ z_1^2(x,1,t) + z_2^2(x,1,t) + z_3^2(x,1,t) \right\} dx + \epsilon \int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2)dx.
\end{aligned} \tag{4.62}
$$

Inserting (4.60)-(4.62) into (4.59), we find

$$
\begin{aligned}
I_4'(t) \leq & \int_0^L \left\{(\rho_1+\epsilon)\varphi_t^2 + (\rho_2+\epsilon)\psi_t^2 + (\rho_1+\epsilon)\omega_t^2\right\} dx \\
& -(k_0-2\epsilon)\int_0^L (\varphi_x^2+\psi_x^2+\omega_x^2)dx \\
& +c_\epsilon \int_0^L \int_0^\infty \left\{g_1(s)(\partial_x\eta_1)^2 + g_2(s)(\partial_x\eta_2)^2 + g_3(s)(\partial_x\eta_3)^2\right\} dx \\
& +c_\epsilon \int_0^L \left\{z_1^2(x,1,t) + z_2^2(x,1,t) + z_3^2(x,1,t)\right\} dx.
\end{aligned} \tag{4.63}
$$

■

**Lemma 4.4.4** *Then the functional defined in (4.40) satisfies*

$$
\begin{aligned}
\frac{d}{dt} I_5(t) \leq & -2I_6(t) - \frac{e^{-2\tau_1}}{\tau_1}\int_0^1 z_1^2(x,1,t))dx + \frac{1}{\tau_1}\int_0^1 \varphi_t^2(x,t)\,dx \\
& -2I_7(t) - \frac{e^{-2\tau_2}}{\tau_2}\int_0^1 z_2^2(x,1,t))dx + \frac{1}{\tau_2}\int_0^1 \psi_t^2(x,t)\,dx \\
& -2I_8(t) - \frac{e^{-2\tau_3}}{\tau_3}\int_0^1 z_3^2(x,1,t))dx + \frac{1}{\tau_3}\int_0^1 \omega_t^2(x,t)\,dx.
\end{aligned} \tag{4.64}
$$

*Proof* Differentiating (4.40) with respect to $t$ and using the third equation in (4.10), we have

$$\begin{aligned}
\frac{d}{dt}I_5(t) &= \frac{d}{dt}\int_0^1\int_0^1 e^{-2\tau_1\rho}z_1^2(x,\rho,t)\,d\rho\ +\frac{d}{dt}\int_0^1\int_0^1 e^{-2\tau_2\rho}z_2^2(x,\rho,t)\,d\rho \\
&\quad +\frac{d}{dt}\int_0^1\int_0^1 e^{-2\tau_3\rho}z_3^2(x,\rho,t)\,d\rho\ dx \\
&= -\frac{2}{\tau_1}\int_0^1\int_0^1 e^{-2\tau_1\rho}z_{1t}(x,\rho,t)z_1(x,\rho,t))\,d\rho\ dx \\
&\quad -\frac{2}{\tau_2}\int_0^1\int_0^1 e^{-2\tau_2\rho}z_{2t}(x,\rho,t)z_2(x,\rho,t))\,d\rho\ dx \\
&\quad -\tfrac{2}{\tau_2}\int_0^1\int_0^1 e^{-2\tau_3\rho}z_{3t}(x,\rho,t)z_3(x,\rho,t))\,d\rho\ dx \\
&= -\frac{1}{\tau_1}\int_0^1\int_0^1 e^{-2\tau_1\rho}\frac{d}{d\rho}(z_1(x,\rho,t)))^2\,d\rho\ dx \\
&\quad -\tfrac{1}{\tau_2}\int_0^1\int_0^1 e^{-2\tau_2\rho}\frac{d}{d\rho}(z_2(x,\rho,t)))^2\,d\rho\ dx \\
&\quad -\frac{1}{\tau_3}\int_0^1\int_0^1 e^{-2\tau_3\rho}\frac{d}{d\rho}(z_3(x,\rho,t)))^2\,d\rho\ dx \\
&= -\frac{1}{\tau_1}\int_0^1\int_0^1 \Big[\frac{d}{d\rho}\Big(e^{-2\tau_1\rho}z_1^2(x,\rho,t))\Big)+2\tau_1 e^{-2\tau_1\rho}z_1^2(x,\rho,t))\Big]\,d\rho\ dx \\
&\quad -\frac{1}{\tau_1}\int_0^1\int_0^1 \Big[\frac{d}{d\rho}\Big(e^{-2\tau_2\rho}z_2^2(x,\rho,t))\Big)+2\tau e^{-2\tau_2\rho}z_2^2(x,\rho,t))\Big]\,d\rho\ dx \\
&\quad -\frac{1}{\tau_3}\int_0^1\int_0^1 \Big[\frac{d}{d\rho}\Big(e^{-2\tau_3\rho}z_3^2(x,\rho,t))\Big)+2\tau_3 e^{-2\tau_3\rho}z_3^2(x,\rho,t))\Big]\,d\rho\ dx \\
&= -\frac{1}{\tau_1}\int_0^1 \Big[e^{-2\tau_1}z_1^2(x,1,t))-\varphi_t^2(x,t))\Big]\,dx-2\int_0^1\int_0^1 e^{-2\tau_1\rho}z_1^2(x,\rho,t))\,d\rho\ dx \\
&\quad -\frac{1}{\tau_2}\int_0^1 \Big[e^{-2\tau_2}z_2^2(x,1,t))-\psi_t^2(x,t))\Big]\,dx-2\int_0^1\int_0^1 e^{-2\tau_2\rho}z_2^2(x,\rho,t))\,d\rho\ dx \\
&\quad -\frac{1}{\tau_3}\int_0^1 \Big[e^{-2\tau_3}z_3^2(x,1,t))-\omega_t^2(x,t))\Big]\,dx-2\int_0^1\int_0^1 e^{-2\tau_3\rho}z_3^2(x,\rho,t))\,d\rho\ dx \\
&\le -2\int_0^1\int_0^1 e^{-2\tau_1\rho}z_1^2(x,\rho,t))\,d\rho\ dx-\frac{1}{\tau_1}\int_0^1 e^{-2\tau_1}z_1^2(x,1,t))dx \\
&\quad -2\int_0^1\int_0^1 e^{-2\tau_2\rho}z_2^2(x,\rho,t))\,d\rho\ dx-\frac{1}{\tau_2}\int_0^1 e^{-2\tau_2}z_2^2(x,1,t))dx \\
&\quad -2\int_0^1\int_0^1 e^{-2\tau_3\rho}z_3^2(x,\rho,t))\,d\rho\ dx-\frac{1}{\tau_3}\int_0^1 e^{-2\tau_3}z_3^2(x,1,t))dx \\
&\quad +\tfrac{1}{\tau_1}\int_0^1 \varphi_t^2(x,t))dx\,dx+\frac{1}{\tau_2}\int_0^1 \psi_t^2(x,t))dx\,dx+\frac{1}{\tau_3}\int_0^1 \omega_t^2(x,t)dx\,dx \\
&\le -2I_6(t)-\tfrac{e^{-2\tau_1}}{\tau_1}\int_0^1 z_1^2(x,1,t)dx+\frac{1}{\tau_1}\int_0^1 \varphi_t^2(x,t)\,dx \\
&\quad -2I_7(t)-\tfrac{e^{-2\tau_2}}{\tau_2}\int_0^1 z_2^2(x,1,t))dx+\frac{1}{\tau_2}\int_0^1 \psi_t^2(x,t)\,dx \\
&\quad -2I_8(t)-\tfrac{e^{-2\tau_3}}{\tau_3}\int_0^1 z_3^3(x,1,t)dx+\frac{1}{\tau_3}\int_0^1 \omega_t^2(x,t)\,dx.
\end{aligned}$$

Such that

$$I_6(t) = \int_0^L \int_0^1 e^{-2\tau_1\rho} z_1^2(x,\rho,t)dxd\rho, \tag{4.65}$$

$$I_7(t) = \int_0^L \int_0^1 e^{-2\tau_2\rho} z_2^2(x,\rho,t)dxd\rho, \tag{4.66}$$

$$I_8(t) = \int_0^L \int_0^1 e^{-2\tau_3\rho} z_3^2(x,\rho,t)dxd\rho. \tag{4.67}$$

This ends the proof of Lemma 4.3 .4

■

Now, let $N_1, N_2 > 0$ and

$$L(t) = N_1 E(t) + N_2(I_1 + I_2 + I_3) + I_4 + I_5, \tag{4.68}$$

where $E$ is the energy functional associated to (4.9) and defined in (4.12). Note that $E$ is non-increasing according to (4.16),

$$\begin{aligned}
E'(t) \leq & -\left(\mu_1 - \frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\|\varphi_t\|_2^2 - \left(\widetilde{\mu}_1 - \frac{\xi_2}{2\tau_2} - \frac{|\widetilde{\mu_2}|}{2}\right)\|\psi_t\|_2^2 \\
& - \left(\widetilde{\widetilde{\mu}}_1 - \frac{\xi_3}{2\tau_3} - \frac{|\widetilde{\widetilde{\mu}}_2|}{2}\right)\|\omega_t\|_2^2 - \left(\frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\|z_1(x,1,t)\|_2^2 \\
& - \left(\frac{\xi_2}{2\tau_2} - \frac{|\widetilde{\mu_2}|}{2}\right)\|z_2(x,1,t)\|_2^2 - \left(\frac{\xi_3}{2\tau_3} - \frac{|\widetilde{\widetilde{\mu}}_2|}{2}\right)\|z_3(x,1,t)\|_2^2 \\
& + \frac{1}{2}\int_0^L \int_0^\infty g_1'(s)(\partial_x \eta_1)^2 dsdx + \frac{1}{2}\int_0^L \int_0^\infty g_2'(s)(\partial_x \eta_2)^2 dsdx \\
& + \frac{1}{2}\int_0^L \int_0^\infty g_3'(s)(\partial_x \eta_3)^2 dsdx. \\
& \leq 0
\end{aligned}$$

Using (4.52),(4.58) and (4.64) with (4.16), we get

$$
\begin{aligned}
L'(t) & \leq -N_2(c_1-c_\delta)\int_0^L \left\{\psi_x+(\varphi_x+\psi+l\omega)^2+(\omega_x-l\varphi)^2\right\}dx \\
& -\left\{N_2\rho_1(g_1^0-\delta(1+\mu_1))-N_2c-\frac{1}{\tau_1}+N_1\left(\mu_1-\frac{\xi_1}{2\tau_1}-\frac{|\mu_1|}{2}\right)\right\}\int_0^L \varphi_t dx \\
& -\left\{N_1\left(\widetilde{\mu_1}-\frac{\xi_2}{2\tau_2}-\frac{|\widetilde{\mu_2}|}{2}\right)-\frac{1}{\tau_2}-N_2\widetilde{\widetilde{\mu_2}}\right\}\int_0^L \psi_t dx \\
& -\left\{N_1\left(\frac{\xi_1}{2\tau_1}-\frac{|\mu_2|}{2}\right)-(c_\delta+c_\epsilon)N_2+2\frac{e^{-2\tau_1}}{\tau_1}\right\}\|z_1(x,1,t)\|_2^2 \\
& -\left\{N_1\left(\frac{\xi_2}{2\tau_2}-\frac{|\widetilde{\mu_2}|}{2}\right)-(c_\delta+c_\epsilon)N_2+2\frac{e^{-2\tau_2}}{\tau_1}\right\}\|z_2(x,1,t)\|_2^2 \\
& -\left\{N_1\left(\frac{\xi_3}{2\tau_3}-\frac{|\widetilde{\widetilde{\mu_2}}|}{2}\right)-(c_\delta+c_\epsilon)N_2+2\frac{e^{-2\tau_3}}{\tau_1}\right\}\|z_3(x,1,t)\|_2^2 \\
& +N_2(c_\epsilon+c_\delta)\textstyle\int_0^L\displaystyle\sum_{i=1}^3\int_0^\infty g_i(s)(\partial_x\eta_i)^2 ds dx \\
& +\left(\frac{N_1}{2}-c_\delta N_2\right)\int_0^L\int_0^\infty\sum_{i=1}^3 (g_i'(s)\partial_x\eta_i)^2 ds dx \\
& -2\int_0^L\int_0^1\sum_{i=1}^3 e^{-2\tau_i\rho}z_i^2(x,\rho,t)d\rho dx.
\end{aligned}
\tag{4.69}
$$

We choose $N_1$ large enough so that

$$
\beta_1 = N_2(c_1-c_\delta)\{N_2\rho_1(g_1^0-\delta(1+\mu_1))-N_2c-\frac{1}{\tau_1}+N_1\left(\mu_1-\frac{\xi_1}{2\tau_1}-\frac{|\mu_1|}{2}\right)>0,
$$

$$
\beta_2 = \left\{N_1\left(\widetilde{\mu_1}-\frac{\xi_2}{2\tau_2}-\frac{|\widetilde{\mu_2}|}{2}\right)-\frac{1}{\tau_2}-N_2\widetilde{\widetilde{\mu_2}}\right\}>0,
$$

$$
\beta_3 = \left\{N_1\left(\widetilde{\mu_1}-\frac{\xi_2}{2\tau_2}-\frac{|\widetilde{\mu_2}|}{2}\right)-\frac{1}{\tau_2}-N_2\widetilde{\widetilde{\mu_2}}\right\}>0.
$$

such that $\min\{\beta_1,\beta_2,\beta_3\}>0$. (Note that $g_i^0>0$ because $g_i$ is continuous non-negative and $g_i(0)>0$ ) and we find, for some positive constants $c_4$,

$$
\begin{aligned}
L'(t) & \leq -c_4E(t)+N_2(c_\epsilon+c_\delta)\int_0^L\sum_{i=1}^3\int_0^\infty g_i(s)(\partial_x\eta_i)^2 ds dx \\
& +\left(\frac{N_1}{2}-c_\delta N_2\right)\int_0^L\int_0^\infty\sum_{i=1}^3 (g_i'(s)\partial_x\eta_i)^2 ds dx
\end{aligned}
\tag{4.70}
$$

On the other hand, by (4.70) and definition of $E(t)$ and $I_i$, there exists a positive constant $N_4$(not depending on $N_1$) such that

$$(N_1 - N_4)E(t) \leq L(t) \leq ((N_1 + N_4)E(t). \tag{4.71}$$

Thus , choosing $N_1 > N_3$ and using the fact that , $g_i' \leq 0$

$$L'(t) \leq -c_4 E(t) + N_2(c_\epsilon + c_\delta) \int_0^L \sum_{i=1}^{3} \int_0^\infty g_i(s)(\partial_x \eta_i)^2 ds dx. \tag{4.72}$$

**Lemma 4.4.5** [79] *For any $i = 1, 2, 3$, there exist positive $\alpha_i$, the following inequalities hold:*

$$\int_0^L \int_0^\infty g_i(s)(\partial_x \eta_i)^2 \leq -\alpha_i E'(t) \ if \ (H2) \ holds \tag{4.73}$$

$$\zeta(t)L'(t) \leq -\eta_1 \zeta(t)E(t) - 2\eta_2 E'(t), \quad \forall \ t \geq t_0. \tag{4.74}$$

Define $\chi(t) = \zeta(t)L(t) + 2\eta_2 E(t)$, which is equivalent to $E(t)$and $\zeta'(t) \leq 0 \ \forall t \geq 0$, we obtain

$$\begin{aligned} \chi'(t) &\leq \zeta'(t)L(t) - \eta_{14}\zeta(t)E(t) \\ &\leq -\alpha\zeta(t)E(t), \quad \forall \ t \geq t_0. \end{aligned} \tag{4.75}$$

Integrating the last inequality over $(t_0, t)$, we conclude that

$$\chi(t) \leq \chi(0) e^{-\alpha \int_{t_0}^t \zeta(s) ds}. \tag{4.76}$$

Then, the equivalent relation between $\chi(t)$ and $E(t)$ yields

$$E(t) \leq K e^{-\alpha \int_{t_0}^t \zeta(s) ds}. \tag{4.77}$$

This completes the proof.

# Chapter 5

# A general decay result in a quasilinear parabolic system with viscoelastic term

## 5.1 Introduction

In this chapter, we consider

$$A(t)|u_t|^{m-2}u_t - Lu + \int_0^t g(t-s)Lu(s)ds = 0, \tag{5.1}$$

in a bounded domain $\Omega$, $A(t)$ is a bounded and positive definite matrix, $\Omega \in R^n (n \geq 1)$, initial data $(u_0, u_1)$ are given functions belonging to suitable spaces and $g$ a continuously differentiable decaying function .

And $Lu = -div(M\nabla u) = -\sum_{i,j=1}^{N}\left(a_{i,j}(x)\frac{\partial}{\partial x_i}\right)$. The matrix $M = (a_{i,j}(x))$, where $a_{i,j} \in C^1(\overline{\Omega})$, is symmetric and there exists a constant $a_0 > 0$ such that for all $x \in \overline{\Omega}$ and $\zeta = (\zeta_1, \zeta_2, ....., \zeta_N) \in R^N$, we have $\sum_{i,j=1}^{N} a_{i,j}(x)\zeta_j\zeta_i \geq a_0|\zeta|^2$ This system is subjected to the following boundary conditions

$$u(x,t) = 0,\ x \in \partial\Omega,\ t \geq 0, \tag{5.2}$$

and initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega. \tag{5.3}$$

To motivate our work, let us recall some results regarding quasilinear parabolic system. This type of equation arises from a variety of mathematical models in engineering and physical sciences. For example, in the study of a heat conduction in materials with memory, the classical Fourier's law is replaced by the following form (cf.[84]):

$$q = -d\nabla u - \int_{-\infty}^{t} k(x,t)u(x,s)ds, \tag{5.4}$$

where $u$ is the temperature, $d$ is the diffusion coefficient and the integral term represents the memory effect in the material. The study of this type of equations has drawn a considerable attention and many results have been obtained see ([84], [85], [88], [89]). First from a mathematical point of view one would expect that the integral term should be dominated by the leading term in the equation, for example, Messaoudi and Tellab [86] studied the following system

$$A(t)|u_t|^{m-2}u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0, \tag{5.5}$$

with the same conditions in (5.2)-(5.3) and they obtained energy decay result although the memory term makes more complex. Berrimi and Messaoudi. [89] showed that if A satisfies $((A(t)v, v) \geq c_0|v|^2 \quad \forall t \in R^+, v \in R^n)$, then the solutions with small initial energy decay exponentially for $m = 2$ and polynomially if $m > 2$.
Very recently for a framework of blow-up in finite time Gongwei Liu and Hua Chen [85] studied the following system

$$A(t)|u_t|^{m-2}u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u, \tag{5.6}$$

with the same conditions in (5.2)-(5.3) and they have obtained blow-up result for both initial energy positive and negative under suitable conditions on $g$ and $p$.
Motivated by the previous works, in this chapter we investigate problem (5.1) in which we generalize the results obtained in [86], supposing new conditions with which the stability is assured, by using the lemma of Martinez.
Our work is organized as follows. In section 2, we present the preliminaries and some lemma. In section 3 decay property is derived. Our results improves the one in Messaoudi and Tellab [86]

## 5.2 Preliminary Results

In this section, we present some material in the proof of our main result. For the relaxation function $g$ we assume
$(A_0)$ $g : R^+ \to R^+$ is a bounded $C^1$ function satisfying

$$g(0) > 0, \qquad 1 - \int_0^{\infty} g(s)ds = l < 1.$$

$(A_1)$ There exists a nonincreasing differentiable function $\xi : R^+ \to R^+$ satisfying

$$g'(t) \leq -\xi(t)g(t),\ t \geq 0; \qquad \int_0^{+\infty} \xi(s)ds = +\infty.$$

$(A_2)$ We also assume that

$$2 \leq m \leq \frac{2n}{n-2} \quad if \quad n \geq 3; \quad m \geq 2, \quad if \quad n = 1, 2.$$

$(A_3)$ The matrix $M = (a_{i,j}(x))$, where $a_{i,j} \in C^1(\overline{\Omega})$, is symmetric and there exists a constant $a_0 > 0$ such that for all $x \in \overline{\Omega}$ and $\zeta = (\zeta_1, \zeta_2, ....., \zeta_N) \in R^N$, we have$\sum_{i,j=1}^{N} a_{i,j}(x)\zeta_j\zeta_i \geq a_0|\zeta|^2$ where $Lu = -div(M\nabla u) = -\sum_{i,j=1}^{N}\left(a_{i,j}(x)\frac{\partial}{\partial x_i}\right)$

$$a(u(t), \upsilon(t)) = \sum_{i,j=1}^{N} a_{i,j}(x)\frac{\partial u(t)}{\partial x_j}\frac{\partial \upsilon(t)}{\partial x_i}dx = \int_\Omega M\nabla u(t).\nabla\upsilon(t)dx,$$

$$a_1 = \max\left(\sum_{i=1}^{N} \|a_{i,j}\|_\infty^2\right).$$

Where $m \geq 2$ and $\Omega$ is a bounded open subset of $R^n(n \geq 1)$. The values of $u$ are taken in $R^n$ and $A \in C(R^+)$ is a bounded square matrix satisfying

$$c_0|\upsilon|^2 \leq (A(t)\upsilon, \upsilon) \leq c_1|\upsilon|^2, \ \forall t \in R^+, \ \upsilon \in R^n, \tag{5.7}$$

**Remark 5.2.1** *The same as in [85] There are many functions satisfying $(A_1)$ and $(A_2)$. Examples of such functions are*

$$g_1(t) = ae^{-b(t+1)^\alpha}, \qquad 0 < \alpha \leq 1 \qquad and \qquad g_2(t) = a1+t)^\epsilon, \epsilon < -1.$$

We will also be using the embedding

$$H_0^1(\Omega) \hookrightarrow L^p(\Omega), \quad H_0^1(\Omega) \hookrightarrow L^m(\Omega),$$

and poincaré's inequality. The same embedding constant $c_s$ will be used later.

**Lemma 5.2.1** *([85]). Let $E : R_+ \to R_+$ be a nonincreasing function and $\psi : R_+ \to R_+$, be a $C^2$ increasing function, $\psi(0) = 0$ and $\lim_{t\to\infty}\psi(t) = +\infty$. Assume that there exists $c > 0$ for which*

$$\int_S^T E(t)\psi'(t)dt \leq cE(S), \ \forall S \geq 0, \tag{5.8}$$

*then*

$$E(t) \leq \alpha E(0)e^{-\left(\int_0^t \xi(s)ds\right)}, \ \forall t \geq 0, \tag{5.9}$$

*where $\alpha$, $\omega$ are positive constants.*

**Lemma 5.2.2** *(Sobolev-Poincaré inequality). Let $2 \leq m \leq \frac{2n}{n-2}$. The inequality*

$$\|u\|_m \leq c_s\|\nabla u\|_2 \quad for \quad u \in H_0^1(\Omega),$$

*holds with some positive constant $c_s$.*

**Lemma 5.2.3** [85]. *For $u(.,t) \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left( \int_0^t g(t-s)(u(t)-u(s))ds \right)^2 dx \leq (1-l)c_s^2(gou)(t), \tag{5.10}$$

*where $c_s^2$ is the poincaré constant and $l$ is given in $(A_1)$,*
*and*

$$(gou)(t) = \int_0^t g(t-s) \int_\Omega a(u(t)-u(s), u(t)-u(s))dxds.$$

## 5.3 Asymptotic behavior

In this section, we consider the energy decay of solutions associated to the system (5.1)−(5.3). Similarly as in [92] we give a definition of a weak solution of the system (5.1) − (5.3).

**Definition 5.3.1** *A weak solution of* (5.1) − (5.3) *is a function* $u \in C([0,T);[H_0^1(\Omega)]^n) \cap C^1((0,T);[L^m(\Omega)]^n)$, *which satisfies*

$$\begin{array}{rl} & \int_0^t \int_\Omega A(s)|u_s(s)|^{m-2}u_s(x,s)\phi(x,s)dsdx + \int_0^t \int_\Omega Lu(x,s)\phi(x,s)dsdx \\ + & \int_0^s \int_0^t \int_\Omega g(t-\nu)\phi(x,s)Lu(x,\nu)d\nu dxds = 0, \end{array}.$$

*for all $t \in [0,T]$ and $\phi \in C([0,T);[H_0^1(\Omega)]^n)$.*

Now we define the "modified" energy equation related with problem (5.1) − (5.3) by

$$E(t) = \frac{1}{2}(gou)(t) + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) a(u(t),u(t)). \tag{5.11}$$

**Lemma 5.3.1** *Let $u$ be the solution of* (5.1) − (5.3) *then, the energy equation satisfies*

$$E'(t) \leq \frac{1}{2}(g'ou)(t) - \frac{1}{2}g(t)a(u(t),u(t)) - \int_\Omega A(t)|u_t(t)|^m dx. \tag{5.12}$$

*Proof* By multiplying (1) by $u_t(t)$, and integrating over $\Omega$ we get

$$\int_\Omega A(t)|u_t(t)|^m dx - \frac{1}{2}\frac{d}{dt}a(u(t),u(t)) + \int_\Omega \int_0^t g(t-s)M\nabla u(s)\nabla u_t(t)dsdx = 0. \tag{5.13}$$

Note that

$$a(u(t),u_t(t)) = \frac{1}{2}\frac{d}{dt}a(u(t),u(t)), \tag{5.14}$$

we can easily obtain

$$\begin{aligned}
&\int_0^t g(t-s)\int_\Omega M\nabla u(s)\nabla u_t(t)dxds \\
= \quad & \sum_{i,j=1}^N \int_0^t \int_\Omega g(t-s)a_{i,j}(x)\frac{\partial u(s)}{\partial x_j}\frac{\partial u_t(t)}{\partial x_i}dxds \\
= \quad & \sum_{i,j=1}^N \int_0^t \int_\Omega g(t-s)a_{i,j}(x)\frac{\partial u(t)}{\partial x_i}\frac{\partial u_t(t)}{\partial x_i}dxds \\
- \quad & \sum_{i,j=1}^N \int_0^t \int_\Omega g(t-s)a_{i,j}(x)\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u_t(s)}{\partial x_j}\right)\frac{\partial u_t(t)}{\partial x_i}dxds \\
= \quad & \frac{1}{2}\int_0^t g(t-s)\left(\frac{d}{dt}a(u(t),u(t))ds\right)-\frac{1}{2}\int_0^t g(t-s)\left(\frac{d}{dt}a(u(t)-u(s),u(t)-u(s))ds\right) \\
= \quad & \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t-s)a(u(t),u(t)\right)ds-\frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t-s)a(u(t)-u(s),u(t)-u(s))\right)ds \\
- \quad & \frac{1}{2}g(t)a(u(t),u(t))+\frac{1}{2}\int_0^t g'(t-s)a(u(t)-u(s),u(t)-u(s))ds \\
= \quad & -\frac{1}{2}\frac{d}{dt}(g\circ u)(t)+\frac{1}{2}(g'\circ u)(t)+\frac{1}{2}\frac{d}{dt}\left[a(u(t),u(t))\int_0^t g(s)ds\right]-\frac{1}{2}g(t)a(u(t),u(t)),
\end{aligned} \tag{5.15}$$

where

$$(g\circ u)(t)=\int_0^t g(t-s)a(u(t)-u(s),u(t)-u(s))ds, \tag{5.16}$$

from (5.13),(5. 14) and (5.15) we obtain

$$E'(t)\leq -\frac{1}{2}(g'ou)(t)-\frac{1}{2}g(t)a(u(t),u(t))-\int_\Omega A(t)|u_t(t)|^m dx. \tag{5.17}$$

■

**Theorem 5.3.1** . *Let $(u_0,u_1)\in(H_0^1(\Omega))^2$ be given. Suppose that $(A_0)-(A_3)$, (5.7), hold. Then there exist two positive constants $w$ and $K$, depending on the initial data and $c_0$ for which the solution of* $(5.1)-(5.3)$ *satisfies*

$$E(t)\leq Ke^{-w\int_0^t \xi(s)ds}$$

*Proof* From now and on, we denote by $c_i$ various positive constants which may be different at different occurrences. We multiply the equation (5.1) by $\xi(t)u$, integrate over $\Omega\times(S,T)$, and use the boundary conditions to get

$$\begin{aligned}
&\int_a^T\int_\Omega \xi(t)A(t)|u_t(t)|^{m-2}u_t(t)u(t)dxdt-\int_S^T\xi(t)a(u(t),u(t))dtdx \\
+ \quad & \int_S^T\xi(t).\int_\Omega M\nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx=0.
\end{aligned} \tag{5.18}$$

We then estimate

$$
\begin{aligned}
&-\int_\Omega \xi(t).M\nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx \\
= \quad &\int_\Omega \xi(t).M\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\nabla u(t)dsdx-\int_0^t g(s)ds.\xi(t)a(u(t),u(t)),
\end{aligned} \tag{5.19}
$$

by substituting (5.19) in (5.18) and adding the following term in (5.18)

$$
\frac{1}{2}\int_S^T \xi(t)(gou)(t)-\frac{1}{2}\int_S^T \xi(t)(gou)(t), \tag{5.20}
$$

(5.18) becomes

$$
\begin{aligned}
\int_S^T \xi(t)E(t)dt = &-\int_S^T\int_\Omega \xi(t)A(t)|u_t(t)|^{m-2}u_t(t)u(t)dxdt+\int_S^T \xi(t)(gou)(t)dt \\
&-\int_S^T\int_\Omega \xi(t)M.\int_0^t g(t-s)(\nabla u(t)-\nabla u(s)).\nabla u(t)dsdxdt.
\end{aligned} \tag{5.21}
$$

By using equation of lemma 2.3 and the equation (5.7) ,(5.12) the boundedness of A(t), and condition $A_3$, we arrive at

$$
\begin{aligned}
\int_\Omega A(t)|u_t(t)|^{m-2}u_t(t)u(t)dx &\leq \beta\int_\Omega |u(t)|^m dx + c_\beta\int_\Omega |u_t(t)|^m dx \\
&\leq \beta c_s^m\|\nabla u(t)\|^m + c_\beta\int_\Omega |u_t(t)|^m dx \\
&\leq \beta c_s^m\left(\tfrac{2E(0)}{l}\right)^{\frac{m-2}{2}}E(t)-\left(\tfrac{c_\beta}{c_0}\right)E'(t), \quad \forall\beta>0,
\end{aligned} \tag{5.22}
$$

then

$$
\begin{aligned}
\int_s^T\int_\Omega A(t)|u_t(t)|^{m-2}u_t(t)u(t)dxds &\leq \left(\beta c_s^m\left(\tfrac{2E(0)}{l}\right)^{\frac{m-2}{2}}\right)\int_s^T E(t)\xi(t)dt \\
&- \left(\tfrac{c_\beta}{c_0}\right)\int_s^T E'(t)\xi(t)dt, \quad \forall\beta>0,
\end{aligned} \tag{5.23}
$$

we have also

$$
\begin{aligned}
&\int_\Omega \int_0^t Mg(t-s)(\nabla u(t) - \nabla u(s))\nabla u(t)dsdx \\
= \quad & \sum_{i,j=1}^{N} \int_0^t g(t-s) \int_\Omega a_{ij}(x) \frac{\partial u(t)}{\partial x_j} \left( \frac{\partial u(t)}{\partial x_i} - \frac{\partial u(s)}{\partial x_i} \right) dxds \\
\leq \quad & \mu \sum_{i,j=1}^{N} \int_\Omega \left( \int_0^t a_{ij}(x) \frac{\partial u(t)}{\partial x_j} ds \right)^2 dx \\
+ \quad & \frac{1}{\mu} \sum_{i,j=1}^{N} \int_\Omega \left( \int_0^t g(t-s) \left( \frac{\partial u(s)}{\partial x_i} - \frac{\partial u(t)}{\partial x_i} \right) ds \right)^2 dx \\
\leq \quad & \frac{\mu}{a_0} \left( \max \sum_{i,j=1}^{N} \|a_{ij}\|_\infty^2 \right) a(u(t), u(t)) \\
+ \quad & \frac{N}{4a_0\mu}(1-l)(gou)(t) \\
\leq \quad & \frac{\mu}{a_0} \left( \sum_{i,j=1}^{N} \|a_{ij}\|_\infty^2 \right) E(t) + \frac{N}{4a_0\mu}(1-l)(g \circ u)(t),
\end{aligned} \tag{5.24}
$$

and using the fact that

$$
|\xi(t)(gou)(t)| \quad = \xi(t)(gou)(t) \quad \leq \beta(gou)(t) \quad \leq \beta(-E'(t)), \quad . \tag{5.25}
$$

we obtain

$$
\int_s^T \xi(t)(gou)(t)dt \leq \int_s^T \beta(-E'(t))dt. \tag{5.26}
$$

By combining (5.19) – (5.25) we easily deduce the following

$$
\begin{aligned}
&\left\{ 1 - \beta c_s^m \left( \tfrac{2E(0)}{l} \right)^{\frac{m-2}{2}} - \left[ (\alpha+1) \left( \max_{i \leq i \leq N} \textstyle\sum_{i,j=1}^{N} \|a_{ij}\|_\infty^2 \right) + \tfrac{N}{4a_0\mu}(1-l) \right] \right\} \textstyle\int_S^T \xi(t)E(t)dt \\
&\leq \left( \tfrac{c_\beta}{c_0} \xi(0) + e^\beta \right) E(S).
\end{aligned} \tag{5.27}
$$

Where $\quad \xi_0 = sup_{t \geq 0} \xi(t)$,

after some manipulation we get

$$
\int_S^T \xi(t)E(t)dt \leq CE(S), \quad \forall S \geq 0. \tag{5.28}
$$

Such that

$$
C = \frac{\left( \frac{c_\beta}{c_0} \xi(0) + e^\beta \right)}{\left\{ 1 - \beta c_s^m \left( \frac{2E(0)}{l} \right)^{\frac{m-2}{2}} - \left[ (\alpha+1) \left( \max_{i \leq i \leq N} \sum_{i,j=1}^{N} \|a_{ij}\|_\infty^2 \right) + \frac{N}{4a_0\mu}(1-l) \right] \right\}}.
$$

Choosing $\beta$, $\delta_2$, $\epsilon$ small enough and by hypothesis we have $l < 1$. By letting T go to infinity, one can easily see that $(A_1)$ satisfied with $\psi(t) = \int_0^t \xi(s)$ ds. Which completes the proof.

■

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**Abstract.**

The first goal of thisthesisis to provedirectly the exponentialdecay rate of the perturbedenergy of a coupledhyperbolicequationwith dissipative terms and nonlinearones,furthemore, we use the so-calledFaedo-Galerkinsmethods to prove the existence of the solutions .Our second goal isstudy theAsymptoticbehavior for a weakviscoelasticwaveequationswith a dynamicboundary and time varyingdelay, and by applying the energymethodsweobtain the exponentialform of the energywhen t goes to infinity. Our thirdaimisfocused on studing the thewellknowen Bresse system and undersuitable conditions on someparameteres on the equations ,weobtain the stability by exploiting the perturbedLyapunouvfunctionals, alsowestudy the existence by semigroup formulation . Our goal in the last chapterisbased on studying aparabolic system withelepticoperator , by applying the multiplicatormethods of martinez which leads to prove the stability.
**Key words:**Faedo-Galerkinsmethod,energymethod, energydecay .

**Résumé.**

Le premier but de cette thèse est de prouver directement le taux de la décroissance de l'énergie de l'énergie perturbé d'un système hyperbolique couplé avec des termes de dissipations et de sources non linéaires , de plus on utilise la méthode de Faedo-Galerkin pour démontrer l'existence des solutions , notre deuxième but est d'étudier le comportement asymptotique pour un système des ondes avec un dynamique sur la frontière et terme de retard variable, en appliquant la méthode de l'énergie on obtient la forme d' intégrale de l'énergie quand t tend vers l'infini, notre troisième but est abordé le système de Bresse , sous certains conditions sur les paramètres de l'équation, on obtient la stabilité en exploitant la méthode de Lyapunouv, on étudie aussi l'existence par la méthode de semi groupe. Notre but dans le dernier chapitre est basé sur l'étude d'un système parabolique avec un opérateur elliptique, on applique la méthode de multiplicateur de Martinez qui mène à démontrer la stabilité

**Mots clès:** Méthode de Faedo-Galerkin, méthode de l'énergie, décroissance de l'énergie

**ملخص**

في هذه الرسالة نهتم بدراسة بعض سلوكيات لبعض النماذج الرياضية هدفنا الأول هو البرهنة المباشرة لتناقص الطاقة لمعادلات هيباربوليسية مزدوجة مع حدود غير خطية و حدود تشتتية، نستعمل تقنيةفايدوغلركين لإثبات وجود الحلول، هدفنا الثاني هو دراسة السلوك لمعادلة الموجة مع وجود تحركات على الحافة في ضل وجود حد التأخر المتغير بتطبيقتقنية الطاقة نتحصل على الشكل التكاملي للطاقة عندما يؤل الوقت إلي ما لانهاية، هدفنا الثالث وهو التطرق لمعادلات برس بوضع بعض الشروط على ثوابت المعادلات نتحصل على الاستقرار بتطبيق تقنية ليابونوف وندرس أيضا وجود الحلول بتطبيق تقنية سومي غروب، هدفنا الأخير هو دراسة معادلة برابولية مع معامل البتيكي بتطبيق طريقة الضرب لمارتينيز تحصل علي الاستقرار

**كلمات مفتاحية:** تقنية فايدوغلركين، تقنية الطاقة ، تناقص الطاقة