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Contents

Introduction

Stabilization of evolution problems

Problems of global existence and stability in time of Partial Differential Equations made object, recently, of many work. In this thesis we were interested in study of the global existence and the stabilization of some evolution equations.

The purpose of stabilization is to attenuate the vibrations by feedback, thus it consists in guaranteeing the decrease of energy of the solutions to 0 in a more or less fast way by a mechanism of dissipation.

More precisely, the problem of stabilization consists in determining the asymptotic behaviour of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero.

This problem has been studied by many authors for various systems. In our study, we obtain severals type of stabilization

1) Strong stabilization: $E(t) \to 0$, as $t \to \infty$.

2) Logarithmic stabilization: $E(t) \leq c(\log(t))^{-\delta}, \forall t > 0, (c, \delta > 0)$.

3) polynomial stabilization: $E(t) \leq ct^{-\delta}, \forall t > 0, (c, \delta > 0)$

4) uniform stabilization: $E(t) \leq ce^{-\delta t}, \forall t > 0, (c, \delta > 0)$.

For wave equation with dissipation of the form $u'' - \Delta_x u + g(u') = 0$, stabilization problems have been investigated by many authors:
When $g : \mathbb{R} \to \mathbb{R}$ is continuous and increasing function such that $g(0) = 0$, global existence of solutions is known for all initial conditions (u_0, u_1) given in $H_0^1(\Omega) \times L^2(\Omega)$. This result is, for instance, a consequence of the general theory of nonlinear semi-groups of contractions generated by a maximal monotone operator (see Brézis **[?]**).

Moreover, if we impose on the control the condition $\forall \lambda \neq 0, g(\lambda) \neq 0$, then strong asymptotic stability of solutions occurs in $H_0^1(\Omega) \times L^2(\Omega)$, i.e.,

$$(u, u') \to (0, 0) \text{ strongly in } H_0^1(\Omega) \times L^2(\Omega),$$

without speed of convergence. These results follows, for instance, from the invariance principle of Lasalle (see for example C. M. Dafermos, A. Haraux **[20]**, , F. Conrad, M. Pierre). If the solution goes to 0 as time goes to ∞, how to get energy decay rates?

Dafermos has written in 1978 "Another advantage of this approach is that it is so simplistic that it requires only quite weak assumptions on the dissipative mechanism. The corresponding drawback is that the deduced information is also weak, never yielding, for example, decay rates of solutions."

Many authors have worked since then on energy decay rates. First results were obtained for linear stabilization, then for polynomial stabilization (see A. Haraux **[11]**, V. Komornik **[14]** M. Nakao **[29]**, and E. Zuazua **[?]**) and then extended to arbitrary growing feedbacks (close to 0). In the same time, geometrical aspects were considered.

By combining the multiplier method with the techniques of micro-local analysis, Lasiecka et al **[9]**, **[16]**, **[17]**-**[19]** have investigated different dissipative systems of partial differential equations (with Dirichlet and Neumann boundary conditions) under general geometrical conditions with nonlinear feedback without any growth restrictions near the origin or at infinity. The computation of decay rates is reduced to solving an appropriate explicitly given ordinary differential equation of monotone type. More precisely, the following explicit decay estimate of the energy is obtained:

$$E(t) \leq h(\frac{t}{t_0} - 1), \quad \forall t \geq t_0, \tag{1}$$

where $t_0 > 0$ and h is the solution of the following differential equation:

$$h'(t) + q(h(t)) = 0, \quad \forall t \geq 0 \quad \text{and} \quad h(0) = E(0) \tag{2}$$

and the function q is determined entirely from the behavior at the origin of the nonlinear feedback by proving that E satisfies

$$(Id - q)^{-1}\Big(E((m+1)t_0)\Big) \leq E(mt_0), \quad \forall m \in \mathbb{N}.$$

In this thesis, the main objective is to give a global existence and stabilization results. This work consists in two chapter, the first, for wave equations with delay term.
the second, for wave equations with a nonlinear delay term.
• In the chapter I, we prove the global existence of its solutions in Sobolev spaces by means of semigroup theory under a condition between the weight of the delay terms in the feedbacks and the weight of the terms without delay. Furthermore, we study the asymptotic behavior of solutions using multiplier method.
• In the chapter II, We prove the global existence of its solutions in Sobolev spaces by means of semigroup theory under a condition between the weight of the delay terms in the feedbacks and the weight of the terms without delay. Furthermore, we study the asymptotic behavior of solutions using multiplier method.

Chapter 1: Global existence and energy decay of solutions to a Bresse system with delay terms.

In this chapter we investigate the existence and decay properties of solutions for the initial boundary value problem of the linear Bresse system of the type

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x+\psi+l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1\varphi_t + \mu_2\varphi_t(x,t-\tau_1) = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x+\psi+l\omega) + \widetilde{\mu}_1\psi_t + \widetilde{\mu}_2\psi_t(x,t-\tau_2) = 0 \\ \rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x+\psi+l\omega) + \widetilde{\widetilde{\mu}}_1\omega_t + \widetilde{\widetilde{\mu}}_2\omega_t(x,t-\tau_3) = 0 \end{cases}$$

and prove the global existence of its solutions in Sobolev spaces by means of semigroup theory under a condition between the weight of the delay terms in the feedbacks and the weight of the terms without delay. Furthermore, we study the asymptotic behavior of solutions using multiplier method.

Chapter 2:Well-posedness and energy decay of solutions to a nonlinear Bresse system with delay terms.

In this chapter we investigate the existence and decay properties of solutions for the initial boundary value problem of the linear Bresse system of the type

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x+\psi+l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1 g_1(\varphi_t(x,t)) & \\ +\mu_2 g_2(\varphi_t(x,t-\tau_1)) = 0 & \text{in }]0,1[\times]0,+\infty[\\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x+\psi+l\omega) + \widetilde{\mu}_1\widetilde{g}_1(\psi_t(x,t)) & \\ +\widetilde{\mu}_2\widetilde{g}_2(\psi_t(x,t-\tau_2)) = 0 & \text{in }]0,1[\times]0,+\infty[\\ \rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x+\psi+l\omega) & \\ +\widetilde{\widetilde{\mu}}_1\widetilde{\widetilde{g}}_1(\omega_t(x,t)) + \widetilde{\widetilde{\mu}}_2\widetilde{\widetilde{g}}_2(\omega_t(x,t-\tau_3)) = 0 & \text{in }]0,1[\times]0,+\infty[\\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 & t \geq 0 \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x) & x \in]0,1[\\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & x \in]0,1[\\ \psi_t(x,t-\tau) = f_0(x,t-\tau) & \text{in }]0,1[\times]0,\tau[\end{cases}$$

and prove the global existence of its solutions in Sobolev spaces by means of semigroup theory under a condition between the weight of the delay terms in the feedbacks and the weight of the terms without delay. Furthermore, we study the asymptotic behavior of solutions using multiplier method.

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1. M. A. Ait Yala , M.Miloudi and M. Mokhtari Enhanced Mathematical Modeling of DOP Plasticizer Migration from PVC into Liquid (Methanol). Columbia International Publishing, American Journal of Materials Science and Technology (2014) Vol. 3 No. 1 pp. 22-32,doi:10.7726/ajmst.2014.1003.

2. A. Benaissa ,M.Miloudi and M. Mokhtari Global existence and energy decay of solutions to a Bresse system with delay terms. Comment.Math.Univ.Carolin. 56,2 (2015) 169-186.

3. A. Benaissa , M.Miloudi and M. Mokhtari Well-posedness and energy decay of solutions to a nonlinear Bresse system with delay terms.Differential Equations and Dynamical Systems(submitted)

Chapter 1

Preliminaries

1.1 Sobolev spaces

We denote by Ω an open domain in $\mathbb{R}^n, n \geq 1$, with a smooth boundary $\Gamma = \partial\Omega$. In general, some regularity of Ω will be assumed. We will suppose that either

$$\Omega \text{ is Lipschitz,}$$

i.e., the boundary Γ is locally the graph of a Lipschitz function, or

$$\Omega \text{ is of class } \mathcal{C}^r, r \geq 1,$$

i.e., the boundary Γ is a manifold of dimension $n \geq 1$ of class $\mathcal{C}^r$. In both cases we assume that Ω is totally on one side of Γ. These definitions mean that locally the domain Ω is below the graph of some function ψ, the boundary Γ is represented by the graph of ψ and its regularity is determined by that of the function ψ. Moreover, it is necessary to note that a domain with a continuous boundary is never on both sides of its boundary at any point of this boundary and that a Lipschitz boundary has almost everywhere a unit normal vector ν.

We will also use the following multi-index notation for partial differential derivatives of a function:

$$\partial_i^k u = \frac{\partial^k u}{\partial x_i^k} \text{ for all } k \in \mathbb{N} \text{ and } i = 1, ..., n,$$
$$D^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} u = \frac{\partial^{\alpha_1 + \ldots + \alpha_n} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}},$$
$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + \ldots + \alpha_n.$$

We denote by $\mathcal{C}(D)$ (respectively $\mathcal{C}^k(D), k \in \mathbb{N}$ or $k = +\infty$) the space of real continuous functions on D (respectively the space of k times continuously differentiable functions on D), where D plays the role of Ω or its closure $\overline{\Omega}$. The space of real $\mathcal{C}^\infty$ functions on Ω with a compact support in Ω is denoted by $\mathcal{C}_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$ as in the distributions theory of Schwartz.The distributions space on Ω is denoted by $\mathcal{D}'(\Omega)$, i.e., the space of continuous linear form over $\mathcal{D}(\Omega)$.

For $1 \leq p \leq \infty$, we call $L^p(\Omega)$ the space of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx\right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

$$\|f\|_{L^\infty(\Omega)} = \sup_\Omega |f(x)| < +\infty \quad \text{for} \quad p = +\infty$$

The space $L^p(\Omega)$ equipped with the norm $f \longrightarrow \|f\|_{L^p}$ is a Banach space: it is reflexive and separable for $1 < p < \infty$ (its dual is $L^{\frac{p}{p-1}}(\Omega)$), separable but not reflexive for $p = 1$ (its dual is $L^\infty(\Omega)$), and not separable, not reflexive for $p = \infty$ (its dual contains strictly $L^1(\Omega)$). In particular the space $L^2(\Omega)$ is a Hilbert space equipped with the scalar product defined by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

We denote by $L^p_{loc}(\Omega)$ the space of functions which are L^p on any bounded sub-domain of Ω.

Similar space can be defined on any open set other than Ω, in particular, on the cylinder set $\Omega\times]a, b[$ or on the set $\Gamma\times]a, b[$, where $a, b \in \mathbb{R}$ and $a < b$.

Let U be a Banach space, $1 < p < +\infty$ and $-\infty \leq a < b \leq +\infty$, then $L^p(a, b; U)$ is the space of L^p functions f from (a, b) into U which is a Banach space for the norm

$$\|f\|_{L^p(a,b;U)} = \left(\int_a^b \|f(x)\|_U^p \, dt\right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

and for the norm

$$\|f\|_{L^\infty(a,b;U)} = \sup_{t\in(a,b)} \|f(x)\|_U < +\infty \quad \text{for} \quad p = +\infty$$

Similarly, for a Banach space $U, k \in \mathbb{N}$ and $-\infty < a < b < +\infty$, we denote by $C([a, b]; U)$ (respectively $C^k([a, b]; U)$) the space of continuous functions (respectively the space of k times continuously differentiable functions) f from $[a, b]$ into U, which are Banach spaces, respectively, for the norms

$$\|f\|_{\mathcal{C}(a,b;U)} = \sup_{t\in(a,b)} \|f(x)\|_U, \quad \|f\|_{\mathcal{C}^k(a,b;U)} = \sum_{i=0}^k \left\|\frac{\partial^i f}{\partial t^i}\right\|_{\mathcal{C}(a,b;U)}$$

1.1.1 Definition of Sobolev Spaces

Now, we will introduce the Sobolev spaces: The Sobolev space $W^{k,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order k have a finite L^p norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega). \ \forall\alpha; |\alpha| \leq k\} \ ,$$

With this definition, the Sobolev spaces admit a natural norm,

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}, \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^\infty(\Omega)}, \text{ for } p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\| \, . \, \|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f,g)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)} \; .$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H^m_0(\Omega)$ (respectively $W^{k,p}_0(\Omega)$).

Now, we introduce a space of functions with values in a space X (a separable Hilbert space).

The space $L^2(a,b;X)$ is a Hilbert space for the inner product

$$(f,g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X \, dt$$

We note that $L^\infty(a,b;X) = (L^1(a,b;X))'$.

Now, we define the Sobolev spaces with values in a Hilbert space X

For $k \in \mathbb{N}$, $p \in [1,\infty]$, we set:

$$W^{k,p}(a,b;X) = \left\{ v \in L^p(a,b;X); \frac{\partial v}{\partial x_i} \in L^p(a,b;X). \;\; \forall i \leq k \right\} ,$$

The Sobolev space $W^{k,p}(a,b;X)$ is a Banach space with the norm

$$\|f\|_{W^{k,p}(a,b;X)} = \left(\sum_{i=0}^{k} \left\| \frac{\partial f}{\partial x_i} \right\|^p_{L^p(a,b;X)} \right)^{1/p}, \text{ for } p < +\infty$$

$$\|f\|_{W^{k,\infty}(a,b;X)} = \sum_{i=0}^{k} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^\infty(a,b;X)}, \qquad \text{for } p = +\infty$$

The spaces $W^{k,2}(a,b;X)$ form a Hilbert space and it is noted $H^k(0,T;X)$. The $H^k(0,T;X)$ inner product is defined by:

$$(u,v)_{H^k(a,b;X)} = \sum_{i=0}^{k} \int_a^b \left(\frac{\partial u}{\partial x^i}, \frac{\partial v}{\partial x^i} \right)_X dt \ .$$

Theorem 1.1 *Let $1 \le p \le n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

where p^* is given by $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$ (where $p = n, p^* = \infty$). Moreover there exists a constant $C = C(p,n)$ such that

$$\|u\|_{L^{p^*}} \le C \|\nabla u\|_{L^p(\mathbb{R}^n)} \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corollaire 1.1 *Let $1 \le p < n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [p, p^*]$$

with continuous imbedding.

For the case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [n, +\infty[$$

Theorem 1.2 *Let $p > n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

with continuous imbedding.

Corollaire 1.2 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \le p \le \infty$. We have*

$$\begin{array}{ll} if & 1 \le p < \infty, \text{ then } W^{1,p}(\Omega) \subset L^{p^*}(\Omega) \text{ where } \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if & p = n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n, \text{ then } W^{1,p}(\Omega) \subset L^\infty(\Omega) \end{array}$$

with continuous imbedding.
Moreover, if $p > n$, we have: $\forall u \in W^{1,p}(\Omega)$,

$$|u(x) - u(y)| \le C|x-y|^\alpha \|u\|_{W^{1,p}(\Omega)} \ a.e \ x, y \in \Omega$$

with $\alpha = 1 - \dfrac{n}{p} > 0$ and C is a constant which depend on p, n and Ω. In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Corollaire 1.3 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{ll} if & p < n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega) \forall q \in [1, p^*[\text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \\ if & p = n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if & p > n, \text{ then } W^{1,p}(\Omega) \subset C(\overline{\Omega}) \end{array}$$

with compact imbedding.

Remark 1.1 *We remark in particular that*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

with compact imbedding for $1 \leq p \leq \infty$ and for $p \leq q < p^*$.

Corollaire 1.4

$$\begin{array}{ll} if & \frac{1}{p} - \frac{m}{n} > 0, \text{ then } W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \text{ where } \frac{1}{q} = \frac{1}{p} - \frac{m}{n}. \\ if & \frac{1}{p} - \frac{m}{n} = 0, \text{ then } W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [p, +\infty[. \\ if & \frac{1}{p} - \frac{m}{n} < 0, \text{ then } W^{m,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \end{array}$$

with continuous imbedding.

1.2 Weak convergence

Let $(E; \|.\|_E)$ a Banach space and E' its dual space, i.e., the Banach space of all continuous linear forms on E endowed with the norm $\|.\|'_E$ defined by

$$\|f\|_{E'} =: \sup_{x \neq 0} \frac{|\langle f, x\rangle|}{\|x\|}$$

; where $\langle f, x\rangle$; denotes the action of f on$x$, i.e.$\langle f, x\rangle := f(x)$. In the same way, we can define the dual space of E' that we denote by E''. (The Banach space E'' is also called the bi-dual space of E.) An element x of E can be seen as a continuous linear form on E' by setting $x(f) := \langle x, f\rangle$, which means that $E \subset E''$:

Definition 1.1 *The Banach space E is said to be reflexive if $E = E''$.*

Definition 1.2 *The Banach space E is said to be separable if there exists a countable subset D of E which is dense in E, i.e. $\overline{D} = E$.*

Theorem 1.3 *(Riesz). If $(H; \langle .,.\rangle)$ is a Hilbert space, $\langle .,.\rangle$ being a scalar product on H, then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $x \in H$ such that $f = \langle x, .\rangle$ and $\|f\|'_H = \|x\|_H$*

Remark : From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.

Proposition 1.1 *If E is reflexive and if F is a closed vector subspace of E, then F is reflexive.*

Corollaire 1.5 *The following two assertions are equivalent: (i) E is reflexive; (ii) E' is reflexive.*

1.2.1 Weak, weak star and strong convergence

Definition 1.3 *(Weak convergence in E). Let $x \in E$ and let $\{x_n\} \subset E$. We say that $\{x_n\}$ weakly converges to x in E, and we write $x_n \rightharpoonup x$ in E, if*

$$\langle f, x_n\rangle \to \langle f, x\rangle$$

for all $f \in E'$.

Definition 1.4 *(weak convergence in E'). Let $f \in E'$ and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly converges to f in E', and we write $f_n \rightharpoonup f$ in E', if*

$$\langle f_n, x\rangle \to \langle f, x\rangle$$

for all $x \in E''$.

Definition 1.5 *(weak star convergence). Let $f \in E'$ and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly star converges to f in E', and we write $f_n \rightharpoonup *f$ in E' if;*

$$\langle f_n, x \rangle \to \langle f, x \rangle$$

for all $x \in E$.

Remark As $E \subset E''$ we have $f_n \rightharpoonup f$ in E' imply $f_n \rightharpoonup *f$ in E'. When E is reflexive, the last definitions are the same, i.e, weak convergence in E' and weak star convergence coincide.

Definition 1.6 *(strong convergence). Let $x \in E$(resp. $f \in E'$) and let $\{x_n\} \subset E$ (resp $\{f_n\} \subset E'$). We say that $\{x_n\}$ (resp. $\{f_n\}$) strongly converges to x (resp. f), and we write $x_n \to x$ in E (resp. $f_n \to f$ in E'), if*

$$\underset{n}{lim} \|x_n - x\|_E = 0; (resp.\ \underset{n}{lim} \|f_n - f\|'_E = 0)$$

Proposition 1.2 *Let $x \in E$, let $\{x_n\} \subset E$, let $f \in E'$ and let $\{f_n\} \subset E'$.*

i. If $x_n \to x$ in E then $x_n \rightharpoonup x$ in E.

ii. If $x_n \rightharpoonup x$ in E then $\{x_n\}$ is bounded.

iii. If $x_n \rightharpoonup x$ in E then $\liminf_{n\to\infty} \|x_n\|_E \geq \|x\|_E$

iv. If $f_n \to f$ in E' then $f_n \rightharpoonup f$ inE' (and so $f_n \overset{*}{\rightharpoonup} f$ in E').

v. If $f_n \rightharpoonup f$ in E' then $\{f_n\}$ is bounded.

vi. If $f_n \rightharpoonup f$ in E' then then $\liminf_{n\to\infty} \|f_n\|'_E \geq \|f\|'_E$

Proposition 1.3 *(finite dimension). If $\dim E < \infty$ then strong, weak and weak star convergence are equivalent.*

1.2.2 Weak and weak star compactness

In finite dimension, i.e, $\dim E < \infty$, we have Bolzano-Weierstrass's theorem (which is a strong compactness theorem).

Theorem 1.4 *(Bolzano-Weierstrass). If $\dim E < \infty$ and if $\{x_n\} \subset E$) is bounded, then there exist $\in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ strongly converges to x.*

The following two theorems are generalizations, in infinite dimension, of Bolzano-Weierstrass's theorem.

Theorem 1.5 *(weak star compactness, Banach-Alaoglu-Bourbaki). Assume that E is separable and consider $\{f_n\} \subset E')$. If $\{x_n\}$ is bounded, then there exist $f \in E'$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ weakly star converges to f in E'.*

Theorem 1.6 *(weak compactness, Kakutani-Eberlein). Assume that E is reflexive and consider $\{x_n\} \subset E)$. If $\{x_n\}$ is bounded, then there exist $x \in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ weakly converges to x in E.*

Weak, weak star convergence and compactness in $L^p(\Omega)$.

Definition 1.7 *(weak convergence in $L^p(\Omega)$ with $1 \le p < \infty$). Let Ω an open subset of $\mathbb{R}^n$ .We say that the sequence $\{f_n\}$ of $L^p(\Omega)$ weakly converges to $f \in L^p(\Omega)$, if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^q; \; (\frac{1}{p} + \frac{1}{q} = 1)$$

Definition 1.8 *(weak star convergence in $L^\infty(\Omega)$). We say that the sequence $\{f_n\} \subset L^\infty(\Omega)$ weakly star converges to $f \in L^\infty(\Omega)$, if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^1(\Omega)$$

Theorem 1.7 *(weak compactness in $L^p(\Omega)$) with $1 < p < \infty$. Given $\{f_n\} \subset L^p(\Omega)$, if $\{f_n\}$ is bounded, then there exist $f \in L^p(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \rightharpoonup f$ in $L^p(\Omega)$.*

Theorem 1.8 *(weak star compactness in $L^\infty(\Omega)$.*
Given $\{f_n\} \subset L^\infty(\Omega)$, if $\{f_n\}$ is bounded, then there exist $f \in L^\infty(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$.

Generalities. In what follows, Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and $1 \le p \le \infty$.
Weak and weak star convergence in Sobolev spaces
For $1 \le p \le \infty$, $W^{1;p}(\Omega)$ is a Banach space. Denote the space of all restrictions to Ω of C^1-differentiable functions from $\mathbb{R}^N$ to $\mathbb{R}$ with compact support in R^N by $C^1(\overline{\Omega})$.

Theorem 1.9 *for every $1 \le p \le \infty$ $C^1(\overline{\Omega}) \subset W^{1;p}(\Omega) \subset L^p(\Omega)$, and, for $1 < p < \infty$, $C^1(\overline{\Omega})$ is dense in $W^{1;p}(\Omega)$.*

Definition 1.9 *(weak convergence in $W^{1;p}(\Omega)$ with $1 \le p < \infty$).)*
We say the $\{f_n\} \subset W^{1;p}(\Omega)$ weakly converges to $f \in W^{1;p}(\Omega)$, and we write $f_n \rightharpoonup f$ in $W^{1;p}(\Omega)$, if $f_n \rightharpoonup f$ in $L^p(\Omega)$ and $\nabla f_n \rightharpoonup \nabla f$ in $L^p(\Omega;\, \mathbb{R}^N)$

Definition 1.10 *(weak convergence in $W^{1;\infty}(\Omega)$*
. We say the $\{f_n\} \subset W^{1;\infty}(\Omega)$ weakly star converges to $f \in W^{1;\infty}(\Omega)$, and we write $f_n \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$, if $f_n \overset{*}{\rightharpoonup} f$ in $L^p(\Omega)$ and $\nabla f_n \overset{*}{\rightharpoonup} \nabla f$ in $L^\infty(\Omega;\, \mathbb{R}^N)$

Theorem 1.10 *(Rellich). Let* $1 \leq p \leq \infty$, $\{f_n\} \subset W^{1;p}(\Omega)$ *and* $f \in W^{1;p}(\Omega)$; *if* $f_n \rightharpoonup f$ *in* $W^{1;p}(\Omega)$ *when* $1 \leq p < \infty$ *(resp.*$f_n \overset{*}{\rightharpoonup} f$ *in* $W^{1;\infty}(\Omega)$*) when* $p = \infty$*) then* $f_n \to f$ *in* $L^p(\Omega)$*), which means that for every* $1 \leq p \leq \infty$*, the weak convergence in* $W^{1;p}(\Omega)$ *imply the strong convergence in* $L^p(\Omega)$*.*

Theorem 1.11 *Let* $1 < p \leq \infty$ *and let* $\{f_n\} \subset W^{1;p}(\Omega)$ *. If* $\{f_n\}$ *is bounded, then there exist* $f \in W^{1;p}(\Omega)$ *and a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *such that* $f_{n_k} \rightharpoonup f$ *in* $W^{1;p}(\Omega)$ *when* $1 < p < \infty$ *(resp.* $f_{n_k} \overset{*}{\rightharpoonup} f$ *in* $W^{1;\infty}(\Omega)$*)*

As a consequence of this theorem we have

Corollaire 1.6 *Let* $1 < p \leq \infty$*and let* $\{f_n\} \subset W^{1;p}(\Omega)$ *. If* $\{f_n\}$ *is bounded, then there exist* $f \in W^{1;p}(\Omega)$ *and a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *such that* $f_{n_k} \to f$ *in* $L^p(\Omega)$ *and* $\nabla f_{n_k} \rightharpoonup \nabla f$ *in* $L^p(\Omega)$ *when* $1 < p < \infty$ *(resp.*$\nabla f_{n_k} \overset{*}{\rightharpoonup} \nabla f$ *in* $L^\infty(\Omega)$*)*

Theorem 1.12 *. If*$N < p \leq \infty$ *and if* $\{f_n\} \subset W^{1;p}(\Omega)$ *is bounded, then there exist* $f \in W^{1;p}(\Omega)$ *and a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *such that* $\{f_{n_k}\}$ *converges uniformly to f, and* $\nabla f_{n_k} \rightharpoonup \nabla f$ *in* $W^{1;p}(\Omega)$ *when* $N < p < \infty$ *(resp.* $\nabla f_{n_k} \overset{*}{\rightharpoonup} \nabla f$ *in* $W^{1;\infty}$*)*

1.3 Fadeo-Galerkin method

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$.

$$(P) \qquad \begin{cases} u''(t) + A(t)u(t) = f(t), \quad t \in [0,T] \\ (x,0) = u_0(x), \ u'(x,0) = u_1(x); \end{cases}$$

where u and f are unknown and given function, respectively, mapping the closed interval $[0,T] \subset \mathbb{R}$ into a real separable Hilbert space H ,$A(t)$ $(0 \leq t \leq T)$ are linear bounded operators in H acting in the energy space $V \subset H$.
Assume that $\langle A(t)u(t), v(t) \rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t;.,.)$ is a bilinear continuous in V.
The problem (P) can be formulated as: Found the solution $u(t)$ such that

$$(\widetilde{P}) \qquad \begin{cases} u \in C([0,T];V), u' \in C([0,T];H) \\ \langle u''(t), v \rangle + a(t; u(t), v) = \langle f, v \rangle \text{ in } D'(]0,T[) \\ u_0 \in V \ , \ u_1 \in H; \end{cases}$$

This problem can be resolved with the approximation process of Fadeo-Galerkin.

1.3.1 General method

Let V_m a sub-space of V with the finite dimension d_m, and let $\{w_{jm}\}$ one basis of V_m. we define the solution u_m of the approximate problem

$$(P_m) \qquad \begin{cases} u_m(t) = \displaystyle\sum_{j=1}^{d_m} g_j(t) w_{jm} \\ u_m \in C([0,T];V_m), u'_m \in C([0,T];V_m) \quad , u_m \in L^2(0,T;V_m) \\ \langle u''_m(t), w_{jm}\rangle + a(t; u_m(t), w_{jm}) = \langle f, w_{jm}\rangle, \quad 1 \le j \le d_m \\ u_m(0) = \displaystyle\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \ , \ u'_m(0) = \displaystyle\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \end{cases}$$

where

$$\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \longrightarrow u_0 \text{ in V as } m \longrightarrow \infty$$

$$\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \longrightarrow u_1 \text{ in V as } m \longrightarrow \infty$$

By virtue of the theory of ordinary differential equations,the system (P_m) has unique local solution which is extend to a maximal interval $[0, t_m[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$, to obtain one solution defined for all $t > 0$.

1.3.2 A priori estimation and convergence

Using the following estimation

$$\|u_m\|^2 + \|u'_m\|^2 \le C(\|u_m(0)\|^2 + \|u'_m(0)\|^2 + \int_0^T |f(s)|^2 ds) \ ; \quad 0 \le t \le T$$

and the Gronwall lemma we deduce that the solution u_m of the approximate problem (P_m) converges to the solution u of the initial problem (P).The uniqueness proves that u is the solution.

1.3.3 Gronwall lemma

Lemma 1.1 *Let $T > 0$, $g \in L^1(0,T)$, $g \ge 0$ a.e and c_1, c_2 are positives constants.Let $\varphi \in L^1(0,T)$ $\varphi \ge 0$ a.e such that $g\varphi \in L^1(0,T)$ and*

$$\varphi(t) \le c_1 + c_2 \int_0^t g(s)\varphi(s)ds \quad a.e \ in \ (0,T).$$

then, we have

$$\varphi(t) \leq c_1 exp\ (c_2 \int_0^t g(s)ds) \quad a.e\ in\ (0,T).$$

1.4 Convex analysis

1.4.1 Fenchel conjugate functions

Let V be a topological vector space and let V' be its dual space with bilinear duality form $\langle .,. \rangle_{V,V'}$.

Definition 1.11 *(Conjugate function)*
Let $F : V \longrightarrow \overline{\rm I\!R}$ be an extend real valued function. The function $F^* : V' \longrightarrow \overline{\rm I\!R}$ defined by

$$F^*(f) = \sup_{u \in V}(\langle f, u\rangle_{V,V'} - F(u)), \quad \forall f \in V'$$

is said to by Fenchel (convex) conjugate or conjugate function of F.
The mapping $F \longrightarrow F^*$ is called the Legendre -Fenchel transformation.

Proposition 1.4 *Let $F : V \longrightarrow \overline{\rm I\!R}$ be a given extend real valued function, the following statements are true*

- *i.$F^*(f) + F(u) \geq \langle f, u\rangle_{V,V'}, \quad \forall f \in V',\ \forall u \in V$*
- *ii.Let f be in the dual $V'$of V and $\lambda \in$ ${\rm I\!R}$,the conjugate of affine function $u \longrightarrow (\langle f, u\rangle_{V,V'} - \lambda$ is less than F if and only if*

$$F^*(f) \leq \lambda$$

- *iii.If F is identically equal to $+\infty$ then F^* is identically equal to $-\infty$ .Moreover ,if F is proper ,then the relation:$F^*(f) = \sup_{u \in V}(\langle f, u\rangle_{V,V'} - F(u))$ may be restricted to the points u in the effective domain of F $(dom(F))$.*
- *iv.The function F^* is always in $\Gamma(V')$ (since F^* is the point-wise supremum of a family of affine continuous functions of v').Therefore, F^* is always o lower semi-continuous convex function on V'. Moreover, if F^*takes the value $-\infty$ then F^* is identically equal to $-\infty$.*

Proposition 1.5 *(i)Let F and G be tow given extend real valued functions of V into $\overline{\rm I\!R}$, the following properties hold:*

1. *$F^*(0) = -\inf_{u \in V} F(u)$.*
2. *If F is less than G then G^* is less than F^*.*

3. *If* $G(u) = F(\alpha u),\ \forall u \in V,$ *with* $\alpha \neq 0$ *then* $G^*(f) = F^*(f/\alpha)\ ,\forall f \in V'$.
4. $(\alpha F)^*(f) = \alpha F^*(f/\alpha),\ \forall \in V',\ \ \forall \alpha > 0.$
5. $(F+\beta)^* = F^* - \beta,\ \ \forall \beta \in \mathbb{R}.$

(ii)Given a family $(F_i)_{i\in J}$ *of functions from* V *into* $\overline{\mathbb{R}}$*, we have*

$$(\inf_{i\in J} F_i)^* = \sup_{i\in J} F_i^*$$

$$\sup_{i\in J} F_i^* \leq \inf_{i\in J} (F_i)^*$$

(iii)For every $a \in V$ *we denote by* F_a *the translated function (i.e,*$F_a(u) = F(u-a), \forall u \in V$*).Then* ${F_a}^*(f) = F^*(f) + \langle f, u\rangle_{V,V'},\ \ \ \forall f \in V'$.

Theorem 1.13 *(Fenchel duality)Let* V *be a locally convex Hausdorff topological vector space over* $\mathbb{R}$ *with its dual* V'*. Let* F *and* G *be two power convex functions of* V *into* $\overline{\mathbb{R}}$ *.Assume that there exists* $u_0 \in dom(F) \cap dom(G)$ *such that* F *is continuous in* u_0 *.Then*

$$\inf_{u\in V}(F(u) + G(u)) = \sup_{f\in V'}(-F^*(-f) - G^*(f)).$$

Proof:From Fenchel inequality, we have for any function H

$$H^*(f) + H(u) \geq (\langle f, u\rangle_{V,V'},\ \ \forall u \in V\ ,\ \forall f \in V'$$

consequently, we have that

$$\inf_{u\in V}(F(u) + G(u)) \geq \sup_{f\in V'}(-F^*(-f) - G^*(f)).$$

(this fact is usually referred to as weak duality).
Denote $p := \inf_{u\in V}(F(u) + G(u))$, $q := \sup_{f\in V'}(-F^*(-f) - G^*(f))$ and $C := epiF$. To complete the proof, we show that $p \leq q$.
If $p = -\infty$ there is nothing to prove.Suppose now that $p \neq -\infty$.
It is clear that the interior of C : $intC$ is not empty (because F is continuous in u_0).
We introduce now the following sets:

$$A := intC,$$

$$B := \{(\lambda, u) \in V \times \mathbb{R} : \lambda \leq p - G(u)\}$$

The set A and B are convex (since F and G are convex) and disjoint (according to the definition of p), therefore, (because of Hahn-Banach's first geometric form) there exist a non zero continuous linear function $f \in V'$ and $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$H = \{(\lambda, u) \in V \times \mathbb{R} : (\langle f, u\rangle_{V,V'} + \alpha\lambda = \beta\}$$

and

$$(\langle f,u\rangle_{V,V'}+\alpha\lambda\geq\beta,\forall(u,\lambda)\in C,$$
$$(\langle f,u\rangle_{V,V'}+\alpha\lambda\leq\beta,\forall(u,\lambda)\in B,$$

By taking $u=u_0$ in the first part of the last inequality and by passing to the limit on $(\lambda\longrightarrow+\infty)$ we can deduce that $\alpha\geq 0$.
Prove now that $\alpha\neq 0$; for this we proceed by contradiction. Assume that $\alpha=0$, then according to the last inequalities, we arrive at

$$\langle f,u\rangle_{V,V'}\geq\beta,\forall u\in dom(F),\text{ and },\langle f,u\rangle_{V,V'}\leq\beta,\forall u\in dom(G).$$

In particular $\langle f,u_0\rangle_{V,V'}=\beta$ (since $u_0\in dom(F)\cap dom(G)$) and then $\langle f,u-u_0\rangle_{V,V'}\geq 0$ for all u in $dom(F)$ .Consequently, $f=0$ since $dom(F)$ is neighborhood of u_0. We thus have $\alpha>0$.
According to

$$\langle f,u\rangle_{V,V'}+\alpha\lambda\geq\beta,\forall(u,\lambda)\in C,$$
$$\langle f,u\rangle_{V,V'}+\alpha\lambda\leq\beta,\forall(u,\lambda)\in B,$$

and dividing by $\alpha>0$, we obtain easily that

$$F^*(-f_\alpha)\leq-\beta_\alpha,$$

$$G^*(f_\alpha)\leq\beta_\alpha-p$$

and then $f_\alpha=f/\alpha$ and $\beta_\alpha=\beta/\alpha$.
Therefore ,$p\leq q$.This complete the proof.

Examples

1.Let C be a non-empty subset of topological vector space V and χ_C be its indicator function. Then the conjugate function ${\chi_C}^*$ is defined by

$${\chi_C}^*(f)=\sup_{u\in C}\langle f,u\rangle_{V,V'}$$

and is called the support function of C. Moreover, if C is a closed and convex set, χ_C is closed and convex, and by the conjugacy theorem the conjugate of its support function is its indicator function.
2.Let $(V,\|.\|)$ be a Banach space, $(V',\|.\|_*)$ its dual, $\Psi_\alpha:t\in\mathbb{R}\longrightarrow|t|^\alpha/\alpha$ and $F_\alpha:V\longrightarrow\mathbb{R}$ such that $F_\alpha(u)=\Psi_\alpha(\|u\|)$, where $1<\alpha<\infty$. Then

$$\begin{aligned}F_\alpha^*(f)&=\sup_{u\in V}(\langle f,u\rangle_{V,V'}-F_\alpha(u))\\&=\sup_{\lambda\geq 0}\left(\|f\|_*\lambda-\frac{\lambda^\alpha}{\alpha}\right)\end{aligned}$$

Hence (by analyzing the function $r(\lambda):=\theta\lambda-\lambda^\alpha/\alpha$ where $\theta:=\|f\|_*$ and $\lambda\in[0,+\infty[$, $F_\alpha^*(f)=\|f\|_*^{\alpha^*}/\alpha^*$ where $1/\alpha+1/\alpha^*=1$.Consequently

$$F_\alpha^*(f)=\Psi_{\alpha^*}(\|f\|_*)$$

3.We finish with an interesting example for the boundary valued problems in a lemma form.

Lemma 1.2 *Let $(V, \|.\|)$ be a Banach space, $(V', \|.\|_*)$ its dual and C be a non-empty closed and convex subset or V. Consider the convex and lower semi-continuous real-valued function F on V given by*

$$F(v) := \langle f, v\rangle_{V,V'} + \chi_C(v-u) \quad \forall v \in V$$

where $u \in V$ and $f \in V'$ are given elements.
then the conjugate of F is

$$F^*(g) = \langle g-f, u\rangle_{V,V'} + \chi_{C^*}(g-f) \quad \forall g \in V'$$

where $C^* = \{g \in V' : \ \langle g, v\rangle_{V,V'} = 0 \ \ \forall v \in C\}$ (which is said to be the polar set of C)

Proof.Let $g \in V'$, we have

$$\begin{aligned} F^*(g) &= \sup_{v\in V}(\langle g, v\rangle_{V,V'} - \langle f, v\rangle_{V,V'} - \chi_C(v-u)) \\ &= \sup_{w\in C}\langle g-f, w+u\rangle_{V,V'} \\ &= \langle g-f, u\rangle_{V,V'} + \sup_{w\in C}\langle g-f, w\rangle_{V,V'} \end{aligned}$$

This completes the proof (since $\sup_{w\in C}\langle g-f, w\rangle_{V,V'} = \chi^*_C(g-f) = \chi_{C^*}(g-f)$).

1.4.2 Legendre transformation

In mathematics, the Legendre transformation or Legendre transform, named after Adrien-Marie Legendre, is an operation that transforms one real-valued function of a real variable into another. Specifically, the Legendre transform of a convex function F is the function F^* defined by

$$F^*(p) = \sup(px - F(x))$$

where "sup" represents the supremum. If F is differentiable, then $F^*(p)$ can be interpreted as the negative of the y-intercept of the tangent line to the graph of F that has slope p. In particular, the value of x that attains the maximum has the property :$F'(x) = p$

That is, the derivative of the function F becomes the argument to the function F^*. In particular, if F is convex (or concave up), then F^* satisfies the functional equation

$$F^*(F'(x)) = xF'(x) - F(x)$$

The Legendre transform is its own inverse. Like the familiar Fourier transform, the Legendre transform takes a function $F(x)$ and produces a function of a different variable p. However, while the Fourier transform consists of an integration with a kernel, the Legendre transform uses maximization as the transformation procedure. The transform is especially well behaved if $F(x)$ is a convex function. The Legendre transformation is an application of the duality relationship between points and lines. The functional relationship specified by $F(x)$ can be represented equally well as a set of (x, y) points, or as a set of tangent lines specified by their slope and intercept values. The Legendre transformation can be generalized to the Legendre-Fenchel transformation. It is commonly used in thermodynamics and in the Hamiltonian formulation of classical mechanics.

1.4.3 Jensen inequality

Let (Ω, A, μ) be a measure space, such that $\mu(\Omega) = 1$. If g is a real-valued function that is μ-integrable, and if is a convex function on the real line, then:

$$\varphi\left(\int_\Omega g\ d\mu\right) \leq \int_\Omega \varphi \circ g\ d\mu$$

In real analysis, we may require an estimate on $\varphi\left(\int_a^b g(x)\ dx\right)$ where a, b are real numbers, and g is a non-negative real-valued function that is Lebesgue-integrable. In this case, the Lebesgue measure of $[a, b]$ don't need to be unity. However, by integration by substitution, the interval can be rescaled so that it has measure unity. Then Jensen's inequality can be applied to get

$$\varphi\left(\int_a^b g(x)\ dx\right) \leq \frac{1}{b-a}\int_a^b \varphi((b-a)g(x))\ dx$$

1.5 Aubin -Lions lemma

The Aubin-Lions lemma is a result in the theory of Sobolev spaces of Banach space-valued functions. More precisely, it is a compactness criterion that is very useful in the study of nonlinear evolutionary partial differential equations. The result is named after the French mathematicians Thierry Aubin and Jacques-Louis Lions. We complete the preliminaries by the useful inequalities of Gagliardo-Nirenberg and Sobolev-Poincaré.

Lemma 1.3 *LetX_0,X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Assume that$X_0$ is compactly embedded in X and that X is continuously embedded in X_1; assume also that X_0 and X_1 are reflexive spaces. For $1 < p, q < +\infty$, let*

$$W = \{u \in L^p([0,T]; X_0)/ \quad \dot{u} \in L^q([0,T]; X_1)\}$$

Then the embedding of W into $L^p([0,T]; X)$ is also compact.

Lemma 1.4 (Gagliardo-Nirenberg) *Let $1 \leq r < q \leq +\infty$ and $p \leq q$. Then, the inequality*

$$\|u\|_{W^{m,q}} \leq C\|u\|^{\theta}_{W^{m,p}}\|u\|^{1-\theta}_r \quad \textit{for} \quad u \in W^{m,p} \bigcap L^r$$

holds with some $C > 0$ *and*

$$\theta = \left(\frac{k}{n} + \frac{1}{r} - \frac{1}{q}\right)\left(\frac{m}{n} + \frac{1}{r} - \frac{1}{p}\right)^{-1}$$

provided that $0 < \theta \leq 1$ *(we assume* $0 < \theta < 1$ *if* $q = +\infty$*).*

Lemma 1.5 (Sobolev-Poincaré inequality) *Let q be a number with* $2 \leq q < +\infty$ $(n = 1, 2)$ *or* $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$, *then there is a constant* $c_* = c(\Omega, q)$ *such that*

$$\|u\|_q \leq c_* \|\nabla u\|_2 \quad \text{for} \quad u \in H_0^1(\Omega).$$

1.6 Integrale Inequalities.

We will recall some fundamental integral inequalities introduced by A. Haraux ,V. Komornik and A.Guesmia to estimate the decay rate of the energy.

1.6.1 Case of exponential decay

The estimation of the energy decay for some dissipative problems is based on the following lemma:

Lemma 1.6 *Let* $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *be a non-increasing function and assume that there is a constant* $A > 0$ *such that*

$$\forall t \geq 0, \qquad \int_t^{+\infty} E(\tau)\, d\tau \leq \frac{1}{A} E(t). \tag{1.1}$$

Then we have

$$\forall t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}. \tag{1.2}$$

Proof of Lemma 1.6.
The inequality (1.2) is verified for $t \leq \frac{1}{A}$, this follows from the fact that E is a decreasing function. We prove that (1.2) is verified for $t \geq \frac{1}{A}$. Introduce the function

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_t^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (1.1) we find that

$$\forall t \geq 0, \quad h'(t) + Ah(t) \leq 0.$$

Let

$$T_0 = \sup\{t,\ h(t) > 0\}. \tag{1.3}$$

For every $t < T_0$, we have

$$\frac{h'(t)}{h(t)} \leq -A,$$

thus

$$h(0) \leq e^{-At} \leq \frac{1}{A} E(0)\, e^{-At}, \quad \text{for} \quad 0 \leq t < T_0. \tag{1.4}$$

Since $h(t) = 0$ if $t \geq T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Let $\varepsilon > 0$. As E is positive and decreasing, we deduce that

$$\forall t \geq \varepsilon, \quad E(t) \leq \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} E(\tau)\, d\tau \leq \frac{1}{\varepsilon} h(t-\varepsilon) \leq \frac{1}{A\varepsilon} E(0)\, e^{\varepsilon t}\, e^{-At}.$$

Choosing $\varepsilon = \frac{1}{A}$, we obtain

$$\forall t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}.$$

The proof of Lemma 1.6 is now completed.

1.6.2 Case of polynomial decay

Lemma 1.7 ([14]) *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ = [0, +\infty)$) be a non-increasing function and assume that there are two constants $q > 0$ and $A > 0$ such that*

$$\forall t \geq 0, \qquad \int_t^{+\infty} E^{q+1}(\tau)\, d\tau \leq \frac{1}{A} E^q(0) E(t). \tag{1.5}$$

Then we have:

$$\forall t \geq 0, \quad E(t) \leq E(0) \left(\frac{1+q}{1+A\,q\,t} \right)^{1/q}. \tag{1.6}$$

Remark 1.2 It is clear that Lemma 1.6, is similar to Lemma 1.7 in the case of $q = 0$.

Proof of Lemma 1.7.

If $E(0) = 0$, then $E \equiv 0$ and there is nothing to prove. Otherwise, replacing the function E by the function $\dfrac{E}{E(0)}$ we may assume that $E(0) = 1$.

Introduce the function

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_t^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (1.5) we find that

$$\forall t \geq 0, \quad -h' \geq (Ah)^{1+q}.$$

where

$$T_0 = \sup\{t,\ h(t) > 0\}.$$

Integrating in $[0,t]$ we obtain that

$$\forall 0 \le t < T_0, h(t)^{-q} - h(0)^{-\sigma} \ge \sigma\omega^{1+q}t,$$

hence

$$0 \le t < T_0, \quad h(t) \le \left(h^{-q}(0) + qA^{1+q}\, t\right)^{-1/q}. \tag{1.7}$$

Since $h(t) = 0$ if $t \ge T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Since

$$h(0) \le \frac{1}{A}E(0)^{1+q} = \frac{1}{A},$$

by (1.5), the right-hand side of (1.7) is less than or equal to:

$$\left(h^{-q}(0) + qA^{1+q}\, t\right)^{-1/q} \le \frac{1}{A}(1 + A\, q\, t)^{-1/q}, \tag{1.8}$$

From other hand, E being nonnegative and non-increasing, we deduce from the definition of h and the above estimate that:

$$\begin{aligned} \forall s \ge 0,\ E\left(\frac{1}{A} + (q+1)s\right)^{q+1} &\le \frac{1}{\frac{1}{A} + q + 1}\int_s^{\frac{1}{A}+(q+1)s} E(\tau)^{q+1}\, d\tau \\ &\le \frac{A}{1+Aqs}h(s) \le \frac{A}{1+Aqs}\frac{1}{A}(1+Aqs)^{-\frac{1}{q}}, \end{aligned}$$

hence

$$\forall S \ge 0, \quad E\left(\frac{1}{A} + (q+1)S\right) \le \frac{1}{(1 + A\, q\, S)^{1/q}}.$$

Choosing $t = \dfrac{1}{A} + (1+q)s$ then the inequality (1.6) follows.Note that letting $q \to 0$ in this theorem we obtain (1.6).

1.7 New integral inequalities of P. Martinez

The above inequalities are verified only if the energy function is integrable, we will try to resolve this problem by introducing some weighted integral inequalities, so we can estimate the decay rate of the energy when it is slow. 1.6.

Lemma 1.8 ([?]) *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^1 function such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{when} \quad t \to +\infty. \tag{1.9}$$

Assume that there exist $q \ge 0$ and $A > 0$ such that

$$\int_S^{+\infty} E(t)^{q+1}\phi'(t)\, dt \le \frac{1}{A}\, E(0)^q E(S), \quad 0 \le S < +\infty. \tag{1.10}$$

then we have

$$\text{if } q > 0, \quad \text{then } E(t) \le E(0)\left(\frac{1+q}{1+q\,A\,\phi(t)}\right)^{\frac{1}{q}}, \;\; \forall t \ge 0,$$

$$\text{if } q = 0, \quad \text{then } E(t) \le E(0)\, e^{1-A\,\phi(t)}, \;\; \forall t \ge 0.$$

Proof of Lemma 1.8.
This Lemma is a generalization of Lemma 1.6, Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$, (we notice that ϕ^{-1} has a meaning by the hypotheses assumed on ϕ). f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain

$$\begin{aligned}
\int_{\phi(S)}^{\phi(T)} f(x)^{q+1}\,dx &= \int_{\phi(S)}^{\phi(T)} E\left(\phi^{-1}(x)\right)^{q+1}\,dx = \int_S^T E(t)^{q+1}\phi'(t)\,dt \\
&\le \frac{1}{A}\,E(0)^q E(S) \\
&= \frac{1}{A}\,E(0)^q f(\phi(S)), \quad 0 \le S < T < +\infty.
\end{aligned}$$

Setting $s := \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\forall s \ge 0, \quad \int_s^{+\infty} f(x)^{q+1}\,dx \le \frac{1}{A}\;E(0)^q f(s).$$

Thanks to Lemma 1.6, we deduce the desired results.

1.8 Generalized inequalities of A. Guesmia

Lemma 1.9 (Guesmia [12]) *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ differentiable function, $\lambda \in \mathbb{R}_+$ and $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ convex and increasing function such that $\Psi(0) = 0$. Assume that*

$$\textstyle\int_s^{+\infty} \Psi(E(t))\,dt \le E(s), \quad \forall s \ge 0.$$

$$E'(t) \le \lambda E(t), \quad \forall t \ge 0.$$

Then E satisfies the estimate

$$E(t) \le e^{\tau_0 \lambda T_0} d^{-1}\Big(e^{\lambda(t-h(t))}\Psi\Big(\psi^{-1}\Big(h(t) + \psi(E(0))\Big)\Big)\Big), \quad \forall t \ge 0,$$

where

$$\psi(t) = \int_t^1 \frac{1}{\Psi(s)}\,ds, \quad \forall t > 0,$$

$$d(t) = \begin{cases} \Psi(t) & if \quad \lambda = 0, \\ \displaystyle\int_0^t \frac{\Psi(s)}{s}\, ds & if \quad \lambda > 0, \end{cases} \quad \forall t \geq 0,$$

$$h(t) = \begin{cases} K^{-1}(D(t)), & if \quad t > T_0, \\ 0 & if \quad t \in [0, T_0], \end{cases}$$

$$K(t) = D(t) + \frac{\psi^{-1}\big(t + \psi(E(0))\big)}{\Psi\big(\psi^{-1}\big(t + \psi(E(0))\big)\big)}\, e^{\lambda t}, \quad \forall t \geq 0,$$

$$D(t) = \int_0^t e^{\lambda s}\, ds, \quad \forall t \geq 0,$$

$$T_0 = D^{-1}\Big(\frac{E(0)}{\Psi(E(0))}\Big), \quad \tau_0 = \begin{cases} 0, & if \quad t > T_0, \\ 1, & if \quad t \in [0, T_0]. \end{cases}$$

Remark 1.3 If $\lambda = 0$ (that is E is non increasing), then we have

$$E(t) \leq \psi^{-1}\Big(h(t) + \psi(E(0))\Big), \quad \forall t \geq 0 \tag{1.11}$$

where $\psi(t) = \displaystyle\int_t^1 \frac{1}{\Psi(s)}, ds$ for $t > 0$, $h(t) = 0$ for $0 \leq t \leq \dfrac{E(0)}{\Psi(E(0))}$ and

$$h^{-1}(t) = t + \frac{\psi^{-1}\Big(t + \psi(E(0))\Big)}{\Psi\Big(\psi^{-1}\Big(t + \psi(E(0))\Big)\Big)}, \quad t > 0.$$

This particular result generalizes the one obtained by Martinez [25] in the particular case of
$\Psi(t) = dt^{p+1}$ with $p \geq 0$ and $d > 0$, and improves the one obtained by Eller, Lagnese and Nicaise [25].

Proof of Lemma 1.9 Because $E'(t) \leq \lambda E(t)$ imply $E(t) \leq e^{\lambda(t-t_0)} E(t_0)$ for all $t \geq t_0 \geq 0$, then, if $E(t_0) = 0$ for some $t_0 \geq 0$, then $E(t) = 0$ for all $t \geq t_0$, and then there is nothing to prove in this case. So we assume that $E(t) > 0$ for all $t \geq 0$ without loss of generality. Let:

$$L(s) = \int_s^{+\infty} \Psi(E(t))\, dt, \quad \forall s \geq 0.$$

We have, $L(s) \leq E(s)$, for all $s \geq 0$. The function L is positive, decreasing and of class $C^1(\mathbb{R}_+)$ satisfying

$$-L'(s) = \Psi(E(s)) \geq \Psi(L(s)), \quad \forall s \geq 0.$$

The function ψ is decreasing, then

$$\Big(\psi(L(s))\Big)' = \frac{-L'(s)}{\Psi(L(s))} \geq 1, \quad \forall s \geq 0.$$

Integration on $[0,t]$, we obtain

$$\psi(L(t)) \geq t + \psi(E(0)), \quad \forall t \geq 0. \tag{1.12}$$

Since Ψ is convex and $\Psi(0) = 0$, we have

$$\Psi(s) \leq \Psi(1)s,\ \forall s \in [0,1] \quad and \quad \Psi(s) \geq \Psi(1)s,\ \forall s \geq 1,$$

then $\underset{t\to 0}{lim}\, \psi(t) = +\infty$ and $[\psi(E(0)), +\infty[\subset Image\,(\psi)$. Then (1.12) imply that

$$L(t) \leq \psi^{-1}\Big(t + \psi(E(0))\Big), \quad \forall t \geq 0. \tag{1.13}$$

Now, for $s \geq 0$, let

$$f_s(t) = e^{-\lambda t}\int_s^t e^{\lambda\tau}\, d\tau, \quad \forall t \geq s.$$

The function f_s is increasing on $[s, +\infty[$ and strictly positive on $]s, +\infty[$ such that

$$f_s(s) = 0 \quad and \quad f_s'(t) + \lambda f_s(t) = 1, \quad \forall t \geq s \geq 0,$$

and the function d is well defined, positive and increasing such that:

$$d(t) \leq \Psi(t) \quad and \quad \lambda t d'(t) = \lambda\Psi(t), \quad \forall t \geq 0,$$

then

$$\begin{aligned} \partial_\tau\Big(f_s(\tau)d(E(\tau))\Big) &= f_s'(\tau)d(E(\tau)) + f_s(\tau)E'(\tau)d'(E(\tau)) \\ &\leq \Big(1 - \lambda f_s(\tau)\Big)\Psi(E(\tau)) + \lambda f_s(\tau)\Psi(E(\tau)) \\ &= \Psi(E(\tau)), \quad \forall \tau \geq s \geq 0. \end{aligned}$$

Integrating on $[s,t]$, we obtain

$$L(s) \geq \int_s^t \Psi(E(\tau))\, d\tau \geq f_s(t)d(E(t)), \quad \forall t \geq s \geq 0. \tag{1.14}$$

Since $\underset{t\to +\infty}{lim}\, d(s) = +\infty$, $d(0) = 0$ and d is increasing, then (1.13) and (1.14) imply

$$E(t) \leq d^{-1}\left(\inf_{s\in[0,t[} \frac{\psi^{-1}\Big(s + \psi(E(0))\Big)}{f_s(t)}\right), \quad \forall t > 0. \tag{1.15}$$

Now, let $t > T_0$ and

$$J(s) = \frac{\psi^{-1}\Big(s + \psi(E(0))\Big)}{f_s(t)}, \quad \forall s \in [0, t[.$$

The function J is differentiable and we have

$$J'(s) = f_s^{-2}(t)\Big[e^{-\lambda(t-s)}\psi^{-1}\Big(s+\psi(E(0))\Big) - f_s(t)\Psi\Big(\psi^{-1}\Big(s+\psi(E(0))\Big)\Big)\Big].$$

Then

$$J'(s) = 0 \;\Leftrightarrow\; K(s) = D(t) \quad and \quad J'(s) < 0 \;\Leftrightarrow\; K(s) < D(T).$$

Since $K(0) = \dfrac{E(0)}{\Psi(E(0))}$, $D(0) = 0$ and K and D are increasing (because ψ^{-1} is decreasing and $s \mapsto \dfrac{s}{\Psi(s)}$, $s > 0$, is non increasing thanks to the fact that Ψ is convex). Then, for $t > T_0$,

$$\inf_{s\in[0,t[} J(s) = J\Big(K^{-1}(D(t))\Big) = J(h(t)).$$

Since h satisfies $J'(h(t)) = 0$, we conclude from (1.15) our desired estimate for $t > T_0$.

For $t \in [0, T_0]$, we have just to note that $E'(t) \leq \lambda E(t)$ and the fact that $d \leq \Psi$ implies

$$E(t) \leq e^{\lambda t}E(0) \leq e^{\lambda T_0}E(0) \leq e^{\lambda T_0}\Psi^{-1}\Big(e^{\lambda t}\Psi(E(0))\Big) \leq e^{\lambda T_0}d^{-1}\Big(e^{\lambda t}\Psi(E(0))\Big).$$

Remark 1.4 Under the hypotheses of Lemma 1.9, we have $\lim\limits_{t\to+\infty} E(t) = 0$. Indeed, we have just to choose $s = \frac{1}{2}t$ in (1.15) instead of $h(t)$ and note that $d^{-1}(0) = 0$, $\lim\limits_{t\to+\infty} \psi^{-1}(t) = 0$ and $\lim\limits_{t\to+\infty} f_{\frac{1}{2}t}(t) > 0$.

Lemma 1.10 (Guesmia [12]) *Let $E : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a differentiable function , $a : \mathbb{R}^+ \longrightarrow \mathbb{R}^{+*}$ and $\lambda : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ two continuous functions .Assume that there exist $r \geq 0$ such that*

$$\int_s^{+\infty} E^{r+1}(t)dt \leq a(s)E(s), \forall s \geq 0 \tag{1.16}$$

$$E'(t) \leq \lambda(t)E(t), \qquad \forall t \geq 0 \tag{1.17}$$

Then E verifies ,for all $t \geq 0$,

$$E(t) \leq \frac{E(0)}{\omega(0)}\omega(h(t)exp(\widetilde{\lambda}(t) - \widetilde{\lambda}(h(t)))exp(-\int_0^{h(t)} \omega(\tau)d\tau), \;\; if\; r = 0$$

and

$$E(t) \leq \omega(h(t))exp(\widetilde{\lambda}(t) - \widetilde{\lambda}(h(t)))\left[\left(\frac{\omega(0)}{E(0)}\right)^r + r\int_0^{h(t)} \omega(\tau)^{r+1}d\tau\right]^{-1/r} \quad if\; r > 0$$

where $\widetilde{\lambda}(t) = \displaystyle\int_0^t \lambda(\tau)d\tau$

Proof of Lemma 1.10 If $E(s) = 0$ or $a(s) = 0$ for one $s \geq 0$,the first inequality implies $E(t) = 0$ for $t \geq s$,we suppose then that $E(t) > 0$ and $a(t) > 0$ for $t \geq 0$
Put $\omega = \frac{1}{a}$ and $\Psi(s) = \int_s^{+\infty} E^{r+1}(t)dt$; we have

$$\Psi(s) \leq \frac{1}{\omega(s)} E(s), \qquad \forall s \geq 0. \tag{1.18}$$

the function Ψ is decreasing ,positive and of class C^1 on $\mathbb{R}^+$ and verifies:

$$\Psi'(s) = -E^{r+1}(s) \leq -(\omega(s)\Psi(s))^{r+1}, \quad \forall s \geq 0$$

then

$$\Psi(s) \leq \Psi(0) exp\left(\int_0^s \omega(\tau)d\tau\right) \leq \frac{E(0)}{\omega(0)} exp\left(\int_0^s \omega(\tau)d\tau\right) \qquad \text{if } r = 0 \tag{1.19}$$

$$\Psi(s) \leq \left((\frac{\omega(0)}{E(0)})^r + \int_0^s (\omega(\tau))^{r+1} d\tau \right)^{-1/r} \qquad \text{if } r > 0 \tag{1.20}$$

Now we put for all $s \geq 0$,

$$f_s(t) = exp(-(r+1)\widetilde{\lambda}(t)) \int_s^t exp((r+1)\widetilde{\lambda}(\tau))d\tau, \quad \forall t \geq s \tag{1.21}$$

where $f_s(s) = 0$ and $f_s'(t) + (r+1)\lambda(t) f_s(t) = 1$, $\forall t \geq s \geq 0$.
Under the second hypothesis in the lemma, we deduce

$$E^{r+1}(t) \geq \partial_t (f_s(t) E^{r+1}(t)); \forall t \geq s \geq 0 \tag{1.22}$$

hence

$$\Psi(s) \geq \int_s^{g(s)} E^{r+1}(t) \geq f_s(g(s)) E^{r+1}(g(s))); \forall s \geq 0 \tag{1.23}$$

where $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^{+*}$ with $I_s(g(s)) = 0$, I_s is defined by

$$I_s(t) = (\omega(s))^{r+1} \int_s^t exp((r+1)\widetilde{\lambda}(\tau))d\tau$$

Let $t > g(0)$ and $s = h(t)$ with

$$h(t) = \begin{cases} 0, \text{ if } & t \in [0, g(0)] \\ \max g^{-1(t)} \text{ if } & t \in]g(0), +\infty[\end{cases}$$

Hence we have $g(s) = t$ and we deduce from (1.23 that, for all $t \geq g(0)$,

$$\Psi(h(t)) \geq f_{h(t)}(t) E^{r+1}(t) = \left(exp(-(r+1)\widetilde{\lambda}(t)) \int_{h(t)}^t exp((r+1)\widetilde{\lambda}(\tau))d\tau \right) E^{r+1}(t)$$

We conclude from (1.19 and (1.20 that ,for all $t > g(0)$,

$$E(t) \leq \frac{E(0)}{\omega(0)} exp(\widetilde{\lambda}(t)) \left(\int_{h(t)}^{t} exp(\widetilde{\lambda}(\tau))d\tau \right)^{-1} exp\left(- \int_{0}^{h(t)} \omega(\tau)d\tau \right) \text{ if } r = 0$$

and

$$E(t) \leq exp(\widetilde{\lambda}(t)) \left(\int_{h(t)}^{t} exp((r+1)\widetilde{\lambda}(\tau))d\tau \right)^{\frac{-1}{r+1}} \times$$
$$\left((\tfrac{\omega(0)}{E(0)})^{r} + r \int_{0}^{h(t)} (\omega(\tau))^{r+1} d\tau \right)^{\frac{-1}{r(r+1)}} \quad \text{if } r > 0$$

The fact that $I_{h(t)}^{t} = I_s(g(s)) = 0$, we obtain the result of the lemma for $t > g(0)$.
If $t \in [0, g(0)]$ the second inequality of the lemma implies that

$$E(t) \leq E(0) exp(\widetilde{\lambda}(t))$$

Since $h(t) = 0$ on $[0, g(0)]$, $E(0)exp(\widetilde{\lambda}(t))$ is identically equal to the left hand side of the results of the lemma.That conclude the proof.

Lemma 1.11 (Guesmia [12]) *Let* $E : \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$ *be a differentiable function ,* $a_1, a_2 \in \mathbb{R}^{+*}$ *and* $a_3, \lambda, r, p \in \mathbb{R}^{+}$ *such that*

$$a_3\lambda(r+1) < 1$$

and for all $0 \leq s \leq T < +\infty$ *,*

$$\int_{s}^{T} E^{r+1}(t)dt \leq a_1(s)E(s) + a_2E^{p+1}(s) + a_3E^{r+1}(T),$$

$$E'(t) \leq \lambda E(t), \qquad \forall t \geq 0$$

Then there exist two positive constants ω *and* c *such that ,for all* $t \geq 0$*,*

$$E(t) \leq ce^{-\omega t}, \text{ if } r = 0$$

$$E(t) \leq c(1+t)^{-1/r}, \text{ if } r > 0 \quad \text{and } \lambda = 0$$

$$E(t) \leq c(1+t)^{\frac{-1}{r(r+1)}}, \text{ if } r > 0 \quad \text{and } \lambda > 0$$

Proof of Lemma 1.11:
We show that E verifies the inequality (1.16).Applying the lemma (1.10),we have

$$a_3E^{r+1}(T) = a_3 \int_{s}^{T} E'^{r+1}(t)dt + a_3E^{r+1}(s)$$
$$\leq a_3(r+1) \int_{s}^{T} \lambda E^{r+1}(t)dt + a_3E^{r+1}(s)$$

Under (1.16),we obtain:

$$\int_s^{+\infty} E^{r+1}(t)dt \leq b(s)E(s), \quad \forall s \geq 0 \tag{1.24}$$

where

$$b(s) = \frac{a_1 + a_2 E^p(s) + a_3 E^r(s)}{1 - a_3\lambda(r+1)}, \quad \forall s \geq 0$$

We consider the function f_0 defined in (1.21)and integrating on $[0, s]$ the inequality

$$E^{r+1}(t) \geq \partial_t(f_0(t)E^{r+1}(t)), \quad \forall t \geq 0$$

we obtain under (1.24)

$$b(0)E(0) \geq \int_0^s E^{r+1}(t)dt \geq f_0(s)E^{r+1}(s), \quad \forall s \geq 0$$

then

$$E(s) \leq \left(\frac{b(0)E(0)}{f_0(s)}\right)^{\frac{1}{r+1}}, \quad \forall s \geq 0$$

on the other hand, the conditions of the lemma implies that

$$E(s) \leq E(0)exp(\widetilde{\lambda}(s) \quad \forall s \geq 0$$

Hence

$$E(s) \leq \min\left\{E(0)exp(\widetilde{\lambda}(s), \left(\frac{b(0)E(0)}{f_0(s)}\right)^{\frac{1}{r+1}}\right\} = d(s) \quad \forall s \geq 0$$

d is continuous and positive and

$$b(s) \leq \frac{a_1 + a_2(d(s))^p + a_3(d(s))^r}{1 - a_3\lambda(r+1)}, \quad \forall s \geq 0$$

Hence we can conclude from (1.24) the first inequality (1.16) of the lemma (1.10) with

$$a(s) = \frac{a_1 + a_2(d(s))^p + a_3(d(s))^r}{1 - a_3\lambda(r+1)}, \quad \forall s \geq 0$$

This completes the proof.

Chapter 2

Global existence and energy decay of solutions to a Bresse system with delay terms.

2.1 Introduction

In this thesis we investigate the existence and decay properties of solutions for the initial boundary value problem of the linear Bresse system of the type

$$(P)\quad \begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1\varphi_t + \mu_2\varphi_t(x, t-\tau_1) = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + \widetilde{\mu}_1\psi_t + \widetilde{\mu}_2\psi_t(x, t-\tau_2) = 0 \\ \rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x + \psi + l\omega) + \widetilde{\widetilde{\mu}}_1\omega_t + \widetilde{\widetilde{\mu}}_2\omega_t(x, t-\tau_3) = 0 \end{cases}$$

where $(x,t) \in (0,L) \times (0,+\infty), \tau_i > 0\ (i = 1,2,3)$ is a time delay, $\mu_1, \mu_2, \widetilde{\mu}_1, \widetilde{\mu}_2, \widetilde{\widetilde{\mu}}_1, \widetilde{\widetilde{\mu}}_2$ are positive real numbers. This system is subject to the Dirichlet boundary conditions

$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = \omega(0,t) = \omega(L,t) = 0, \quad t > 0$$

and to the initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \\ \psi_t(x,0) = \psi_1(x), \quad \omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), \quad x \in (0,L) \\ \varphi_t(x, t-\tau_1) = f_0(x, t-\tau_1), \quad \text{in } (0,L) \times [0,\tau_1] \\ \psi_t(x, t-\tau_2) = \tilde{f}_0(x, t-\tau_2), \quad \text{in } (0,L) \times [0,\tau_2] \\ \omega_t(x, t-\tau_3) = \tilde{\tilde{f}}_0(x, t-\tau_3), \quad \text{in } (0,L) \times [0,\tau_3] \end{cases}$$

where the initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, f_0, \tilde{f}_0, \tilde{\tilde{f}}_0)$ belong to a suitable Sobolev space. By ω, ψ and φ we are denoting the longitudinal, vertical and shear angle displacements.

The original Bresse system is given by the following equations (see **[8]**) :

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN + F_1, \\ \rho_2\psi_{tt} = M_x - Q + F_2, \\ \rho_1\omega_{tt} = N_x - lQ + F_3, \end{cases}$$

where we use N, Q and M to denote the axial force, the shear force and the bending moment respectively. These forces are stress-strain relations for elastic behavior and given by

$$N = Eh(\omega_x - l\varphi), \quad Q = Gh(\varphi_x + \psi + l\omega), \quad \text{and } M = EI\psi_x,$$

where G, E, I and h are positive constants. Finally, by the terms F_i we are denoting external forces.

The Bresse system without delay (i.e $\mu_2 = \widetilde{\mu_2} = \widetilde{\widetilde{\mu_2}} = 0$), is more general than the well-known Timoshenko system where the longitudinal displacement ω is not considered $l = 0$. There are a number of publications concerning the stabilization of Timoshenko system with different kinds of damping (see **[15]**, **[26]**, **[27]** and **[33]**). Raposo et al. **[34]** proved the exponential decay of the solution for the following linear system of Timoshenko-type beam equations with linear frictional dissipative terms:

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1\varphi_t = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + \widetilde{\mu_1}\psi_t = 0. \end{cases}$$

Messaoudi and Mustafa **[26]** (see also **[33]**) considered the stabilization for the following Timoshenko system with nonlinear internal feedbacks:

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + g_1(\psi_t) = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + g_2(\psi_t) = 0. \end{cases}$$

Recently, Park and Kang **[33]** considered the stabilization of the Timoshenko system with weakly nonlinear internal feedbacks.
In **[23]**, Liu and Rao considered a thermoelastic Bresse system that consists of three wave equations and two heat equations coupled in certain way.

The two wave equations for the longitudinal displacement and the shear angle displacement are effectively globally damped by the dissipation from the two heat equations. The wave equation about the vertical displacement is subject to a weak thermal damping and indirectly damped through the coupling. They establish exponential energy decay rate when the vertical and the longitudinal waves have the same speed of propagation. Otherwise, a polynomial-type decay is established.

Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see **[36]**). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. In recent years, the PDEs with time delay effects have become an active area of research and arise in many pratical problems (see for example **[1]**, **[37]**). The presence of delay may be a source of instability. For example, it was proved in **[10]** that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay.

To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see **[31]** and **[39]**). For instance, in **[31]** the authors studied the wave equation with a linear internal damping term with constant delay and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they found a sequence of delays for which the solution will be instable if $\mu_2 \geq \mu_1$.

The main approach used in **[31]**, is an observability inequality obtained with a Carleman estimate. The same results were showed if both the damping and the delay acting in the boundary domain. We also recall the result by Xu, Yung and Li **[39]**, where the authors proved the same result as in **[31]** for the one space dimension by adopting the spectral analysis approach. Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (P) for linear damping and delay terms.

To obtain global solutions to the problem (P), we use the argument combining the semigroup theory (see **[31]** and **[8]**) with the energy estimate method. To prove decay estimates, we use a multiplier method.

2.2 Preliminaries and main results

First assume the following hypotheses:
(H1)

$$|\mu_2| < \mu_1, \ |\widetilde{\mu_2}| < \widetilde{\mu_1}, |\widetilde{\widetilde{\mu_2}}| < \widetilde{\widetilde{\mu_1}}. \tag{2.1}$$

We first state some lemmas which will be needed later.

Lemma 2.1 (Sobolev-Poincaré's inequality) *Let q be a number with $2 \leq q < +\infty$. Then there is a constant $c_* = c_*((0,1), q)$ such that*

$$\|\psi\|_q \leq c_* \|\psi_x\|_2 \quad \text{for} \quad \psi \in H_0^1((0,1)).$$

Lemma 2.2 ([13], [15]) *Let $\mathcal{E} : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and assume that there are two constants $\sigma > -1$ and $\omega > 0$ such that*

$$\int_S^{+\infty} \mathcal{E}^{1+\sigma}(t)dt \leq \frac{1}{\omega}\mathcal{E}^{\sigma}(0)\mathcal{E}(S). \quad 0 \leq S < +\infty, \tag{2.2}$$

then we have

$$\mathcal{E}(t) = 0 \quad \forall t \geq \frac{\mathcal{E}(0)^{\sigma}}{\omega|\sigma|} \qquad \forall t \geq 0, \quad \text{if} \quad -1 < \sigma < 0, \tag{2.3}$$

$$\mathcal{E}(t) \le \mathcal{E}(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}} \quad \forall t \ge 0, \quad if \quad \sigma > 0 \tag{2.4}$$

$$\mathcal{E}(t) \le \mathcal{E}(0)e^{1-\omega t} \qquad \forall t \ge 0, \quad if \quad \sigma = 0. \tag{2.5}$$

We introduce, as in **[31]**, the new variables

$$\begin{array}{lll} z_1(x,\rho,t) = \phi_t(x, t-\tau_1\rho), & x\in(0,L),\ \rho\in(0,1), & t>0,\\ z_2(x,\rho,t) = \psi_t(x, t-\tau_2\rho), & x\in(0,L),\ \rho\in(0,1), & t>0,\\ z_3(x,\rho,t) = \omega_t(x, t-\tau_3\rho), & x\in(0,L),\ \rho\in(0,1), & t>0. \end{array} \tag{2.6}$$

Then, we have

$$\tau_i z_{it}(x,\rho,t) + z_{i\rho}(x,\rho,t) = 0, \quad \text{in } (0,L)\times(0,1)\times(0,+\infty) \text{ for } i=1,2,3. \tag{2.7}$$

Therefore, problem (P) takes the form:

$$\left\{\begin{array}{c} \rho_1\varphi_{tt}(x,t) - Gh(\varphi_x+\psi+l\omega)_x(x,t) - lEh(\omega_x - l\varphi)(x,t)\\ +\mu_1\varphi_t(x,t) + \mu_2 z_1(x,1,t) = 0,\\ \tau_1 z_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) = 0,\\ \rho_2\psi_{tt}(x,t) - EI\psi_{xx}(x,t) + Gh(\varphi_x+\psi+l\omega)(x,t)\\ +\widetilde{\mu}_1\psi_t(x,t) + \widetilde{\mu}_2 z_2(x,1,t) = 0,\\ \tau_2 z_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) = 0,\\ \rho_1\omega_{tt}(x,t) - Eh(\omega_x - l\varphi)_x(x,t) + lGh(\varphi_x+\psi+l\omega)(x,t)\\ +\widetilde{\widetilde{\mu}}_1\omega_t(x,t) + \widetilde{\widetilde{\mu}}_2 z_3(x,1,t) = 0,\\ \tau_3 z_{3t}(x,\rho,t) + z_{3\rho}(x,\rho,t) = 0. \end{array}\right. \tag{2.8}$$

The above system subjected to the following initial and boundary conditions

$$\left\{\begin{array}{cc} \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = \omega(0,t) = \omega(L,t) & t>0\\ z_1(x,0,t) = \varphi_t(x,t), z_2(x,0,t) = \psi_t(x,t), z_3(x,0,t) = \omega_t(x,t), & x\in(0,L),\ t>0\\ \varphi(x,0) = \varphi_0, \varphi_t(x,0) = \varphi_1, \psi(x,0) = \psi_0, \psi_t(x,0) = \psi_1, \omega(x,0) = \omega_0, \omega_t(x,0) = \omega_1, & x\in(0,L),\\ z_1(x,1,t) = f_1(x,t-\tau_1), \quad \text{in } (0,L)\times(0,\tau_1) & \\ z_2(x,1,t) = f_2(x,t-\tau_2), \quad \text{in } (0,L)\times(0,\tau_2) & \\ z_3(x,1,t) = f_3(x,t-\tau_3) \quad \text{in } (0,L)\times(0,\tau_3). & \end{array}\right. \tag{2.9}$$

Let ξ_1, ξ_2 and ξ_3 be positive constants such that

$$\left\{\begin{array}{l} \tau_1|\mu_2| < \xi_1 < \tau_1(2\mu_1 - |\mu_2|),\\ \tau_2|\widetilde{\mu}_2| < \xi_2 < \tau_2(2\widetilde{\mu}_1 - |\widetilde{\mu}_2|),\\ \tau_3|\widetilde{\widetilde{\mu}}_2| < \xi_3 < \tau_3(2\widetilde{\widetilde{\mu}}_1 - |\widetilde{\widetilde{\mu}}_2|), \end{array}\right. \tag{2.10}$$

thanks to hypothesis $(H1)$. We define the energy associated to the solution of the problem (2.8) by the following formula:

$$\mathcal{E}(t) = \frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_1}{2}\|\omega_t\|_2^2 + \frac{EI}{2}\|\psi_x\|_2^2 + \frac{Gh}{2}\|\varphi_x + \psi + l\omega\|_2^2 + \frac{Eh}{2}\|\omega_x - l\varphi\|_2^2 + \sum_{i=1}^{3} \frac{\xi_i}{2} \int_0^1 \|z_i(x,\rho,t)\|_2^2 \, d\rho. \tag{2.11}$$

We have the following theorem.

Theorem 2.1 *Let* $(\varphi_0, \varphi_1, f_1(.,-.\tau_1), \psi_0, \psi_1, f_2(.,-.\tau_2), \omega_0, \omega_1, f_3(.,-.\tau_3)) \in (H_0^1(0,L) \times L^2(0,L) \times L^2((0,L) \times (0,1)))^3$.
Assume that the hypotheses $(H1)$ *holds. Then problem* (P) *admits a unique solution*

$$\begin{gathered} \varphi \in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)), \\ \psi \in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)) \\ \omega \in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)), \\ z_1, z_2, z_3 \in C([0,+\infty); L^2((0,L) \times (0,1))). \end{gathered}$$

In addition, we have the following decay estimate:

$$\mathcal{E}(t) \leq c\mathcal{E}(0)e^{-\omega t}, \qquad \forall\, t \geq 0, \tag{2.12}$$

while c *and* ω *are positive constants, independent of the initial data.*

We finish this section by giving an explicit upper bound for the derivative of the energy.

Lemma 2.3 *Let* $(\varphi, \psi, \omega, z_1, z_2, z_3)$ *be a solution of the problem (2.8). Then, the energy functional defined by (2.11) satisfies*

$$\begin{aligned} \mathcal{E}'(t) \leq & -\left(\mu_1 - \frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\|\varphi_t\|_2^2 \\ & -\left(\widetilde{\mu_1} - \frac{\xi_2}{2\tau_2} - \frac{|\widetilde{\mu_2}|}{2}\right)\|\psi_t\|_2^2 \\ & -\left(\widetilde{\widetilde{\mu_1}} - \frac{\xi_3}{2\tau_3} - \frac{|\widetilde{\widetilde{\mu_2}}|}{2}\right)\|\omega_t\|_2^2 \\ & -\left(\frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\|z_1(x,1,t)\|_2^2 \\ & -\left(\frac{\xi_2}{2\tau_2} - \frac{|\widetilde{\mu_2}|}{2}\right)\|z_2(x,1,t)\|_2^2 \\ & -\left(\frac{\xi_3}{2\tau_3} - \frac{|\widetilde{\widetilde{\mu_2}}|}{2}\right)\|z_3(x,1,t)\|_2^2. \end{aligned} \tag{2.13}$$

Proof. Multiplying the first equation in (2.8) by φ_t, the third equation by ψ_t, the fifth equation by ω_t, integrating over $(0,L)$ and using integration by parts, we get

$$\frac{1}{2}\rho_1\frac{d}{dt}\|\varphi_t\|_2^2 - Gh\int_0^L(\varphi_x+\psi+l\omega)_x\varphi_t dx - lEh\int_0^L(\omega_x-l\varphi)\varphi_t dx$$
$$+\mu_1\|\varphi_t\|_2^2+\mu_2\int_0^L z_1(x,1,t)\varphi_t dx = 0$$

$$\frac{1}{2}\rho_2\frac{d}{dt}\|\psi_t\|_2^2+\frac{EI}{2}\|\psi_x\|_2^2+Gh\int_0^L(\varphi_x+\psi+l\omega)\psi_t dx$$
$$+\widetilde{\mu_1}\|\psi_t\|_2^2+\widetilde{\mu_2}\int_0^L z_1(x,1,t)\psi_t = 0$$

$$\frac{1}{2}\rho_1\frac{d}{dt}\|\omega_t\|_2^2 - Eh\int_0^L(\omega_x-l\varphi)_x\omega_t dx + lGh\int_0^L(\varphi_x+\psi+l\omega)\omega_t dx$$
$$+\widetilde{\widetilde{\mu_1}}\|\omega_t\|_2^2+\widetilde{\widetilde{\mu_2}}\int_0^L z_3(x,1,t)\omega_t dx = 0.$$

Then

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\rho_1}{2}\|\varphi_t\|_2^2+\frac{\rho_2}{2}\|\psi_t\|_2^2+\frac{\rho_1}{2}\|\omega_t\|_2^2+\frac{EI}{2}\|\psi_x\|_2^2+\frac{Gh}{2}\|\varphi_x+\psi+l\omega\|_2^2+\frac{Eh}{2}\|\omega_x-l\varphi\|_2^2\right)\\
&+\quad \mu_1\|\varphi_t\|_2^2+\widetilde{\mu_1}\|\psi_t\|_2^2+\widetilde{\widetilde{\mu_1}}\|\omega_t\|_2^2+\mu_2\int_0^L z_1(x,1,t)\varphi_t dx+\widetilde{\mu_2}\int_0^L z_2(x,1,t)\psi_t dx\\
&+\quad \widetilde{\widetilde{\mu_2}}\int_0^L z_3(x,1,t)\omega_t dx = 0
\end{aligned}\tag{2.14}$$

Multiplying the equation in (2.7) by $\xi_i z_i$ and integrating over $(0,L)\times(0,1)$, to obtain:

$$\begin{aligned}
\frac{\xi_i}{2}\frac{d}{dt}\int_0^L\int_0^1 z_i^2(x,\rho,t)d\rho dx &= -\frac{\xi_i}{\tau_1}\int_0^L\int_0^1 z_i z_{i\rho} d\rho dx\\
&= \frac{\xi_i}{2\tau_i}\int_0^L\left(z_i^2(x,0,t)-z_i^2(x,1,t)\right)dx\\
&= \frac{\xi_i}{2\tau_i}\left[\|z_i^2(x,0,t)\|_2^2-\|z_i(x,1,t)\|_2^2\right],
\end{aligned}\tag{2.15}$$

where $z_1(x,0,t)=\varphi_t(x,t),\ z_2(x,0,t)=\psi_t(x,t)$ and $z_3(x,0,t)=\omega_t(x,t)$. From (2.11), (2.14), (2.15) and using Young inequality we get

$$\begin{aligned}
\mathcal{E}'(t) = &-\left(\mu_1-\frac{\xi_1}{2\tau_1}\right)\|\varphi_t\|_2^2-\left(\widetilde{\mu_1}-\frac{\xi_2}{2\tau_2}\right)\|\psi_t\|_2^2-\left(\widetilde{\widetilde{\mu_1}}-\frac{\xi_3}{2\tau_3}\right)\|\omega_t\|_2^2\\
&-\sum_{i=1}^{3}\frac{\xi_i}{2\tau_i}\|z_i(x,1,t)\|_2^2-\mu_2\int_0^L z_1(x,1,t)\varphi_t\,dx\\
&-\widetilde{\mu_2}\int_0^L z_1(x,1,t)\psi_t\,dx-\widetilde{\widetilde{\mu_2}}\int_0^L z_3(x,1,t)\omega_t\,dx.
\end{aligned}\tag{2.16}$$

Due to Young's inequality, we have

$$\begin{aligned}
\int_0^L z_1(x,1,t)\varphi_t(x,t)\,dx &\le \frac{1}{2}\|\varphi_t(x,t)\|_2^2 + \frac{1}{2}\|z_1(x,1,t)\|_2^2 \\
\int_0^L z_2(x,1,t)\varphi_t(x,t)\,dx &\le \frac{1}{2}\|\psi_t(x,t)\|_2^2 + \frac{1}{2}\|z_2(x,1,t)\|_2^2 \\
\int_0^L z_3(x,1,t)\omega_t(x,t)\,dx &\le \frac{1}{2}\|\omega_t(x,t)\|_2^2 + \frac{1}{2}\|z_3(x,1,t)\|_2^2.
\end{aligned} \tag{2.17}$$

Inserting (2.17) into (2.16), we obtain

$$\begin{aligned}
\mathcal{E}'(t) \le &-\left(\mu_1 - \frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\|\varphi_t\|_2^2 \\
&-\left(\widetilde{\mu}_1 - \frac{\xi_2}{2\tau_2} - \frac{|\widetilde{\mu}_2|}{2}\right)\|\psi_t\|_2^2 \\
&-\left(\widetilde{\widetilde{\mu}}_1 - \frac{\xi_3}{2\tau_3} - \frac{|\widetilde{\widetilde{\mu}}_2|}{2}\right)\|\omega_t\|_2^2 \\
&-\left(\frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\|z_1(x,1,t)\|_2^2 \\
&-\left(\frac{\xi_2}{2\tau_2} - \frac{|\widetilde{\mu}_2|}{2}\right)\|z_2(x,1,t)\|_2^2 \\
&-\left(\frac{\xi_3}{2\tau_3} - \frac{|\widetilde{\widetilde{\mu}}_2|}{2}\right)\|z_3(x,1,t)\|_2^2.
\end{aligned}$$

This completes the proof of the lemma.

□

2.3 Global existence

In this section we will give well-posedness results for problem (2.8) and (2.9) using semigroup theory. Let us introduce the semigroup representation of the Bresse system (2.8) and (2.9). Let $U = (\varphi, \varphi_t, z_1, \psi, \psi_t, z_2, \omega, \omega_t, z_3)^T$ and rewrite (2.8) and (2.9) as

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (\varphi_0, \varphi_1, f_1(.,-.\tau_1), \psi_0, \psi_1, f_2(.,-.\tau_2), \omega_0, \omega_1, f_3(.,-.\tau_3)), \end{cases} \tag{2.18}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix}\varphi\\ u\\ z_1\\ \psi\\ v\\ z_2\\ \omega\\ \tilde{\omega}\\ z_3\end{pmatrix}=\begin{pmatrix} u\\ \frac{Gh}{\rho_1}(\varphi_x+\psi+l\omega)_x+\frac{lEh}{\rho_1}(\omega_x-l\varphi)-\frac{\mu_1}{\rho_1}u-\frac{\mu_2}{\rho_1}z_1(.,1)\\ -\frac{1}{\tau_1}z_{1\rho}\\ v\\ \frac{EI}{\rho_2}\psi_{xx}-\frac{Gh}{\rho_2}(\varphi_x+\psi+l\omega)-\frac{\tilde{\mu}_1}{\rho_2}v-\frac{\tilde{\mu}_2}{\rho_2}z_2(.,1)\\ -\frac{1}{\tau_2}z_{2\rho}\\ \tilde{\omega}\\ \frac{Eh}{\rho_1}(\omega_x-l\varphi)_x-\frac{lGh}{\rho_1}(\varphi_x+\psi+l\omega)-\frac{\tilde{\tilde{\mu}}_1}{\rho_1}\tilde{\omega}-\frac{\tilde{\tilde{\mu}}_2}{\rho_1}z_3(.,1)\\ -\frac{1}{\tau_3}z_{3\rho}\end{pmatrix}$$

with domain

$$\begin{aligned} D(\mathcal{A})=\{&(\varphi,u,z_1,\psi,v,z_2,\omega,\tilde{\omega},z_3)^T \text{ in } H: u=z_1(.,0),\\ &v=z_2(.,0),\tilde{\omega}=z_3(.,0), \text{ in } (0,L)\}, \end{aligned} \tag{2.19}$$

where

$$H=(H^2(0,L)\cap H_0^1(0,L)\times H_0^1(0,L)\times L^2(0,L,H^1(0,1)))^3.$$

Now, the energy space $\mathcal{H}$ is defined as

$$\mathcal{H}=H_0^1(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,1)).$$

For $U=(\varphi,u,z_1,\psi,v,z_2,\omega,\tilde{\omega},z_3)^T,\overline{U}=(\overline{\varphi},\overline{u},\overline{z}_1,\overline{\psi},\overline{v},\overline{z}_2,\overline{\omega},\overline{\tilde{\omega}},\overline{z}_3)^T$ and for ξ_i positive constants satisfying (2.10), we define the following inner product in $\mathcal{H}$

$$\begin{aligned}\langle U,\overline{U}\rangle_{\mathcal{H}}=&\int_0^L\Big(\rho_1u\overline{u}+\rho_2v\overline{v}+\rho_1\tilde{\omega}\overline{\tilde{\omega}}+EI\psi_x\overline{\psi}_x\\ &+Gh(\varphi_x+\psi+l\omega)(\overline{\varphi}_x+\overline{\psi}+l\overline{\omega})\\ &+Eh(\omega_x-l\varphi)(\overline{\omega}_x-l\overline{\varphi})+\sum_{i=1}^{3}\xi_i\int_0^1 z_i\overline{z}_i\,d\rho\Big)\,dx.\end{aligned}$$

We show that the operator $\mathcal{A}$ generates a C_0- semigroup in $\mathcal{H}$. In this step, we prove that the operator $\mathcal{A}$ is dissipative. Let $U=(\varphi,u,z_1,\psi,v,z_2,\omega,\tilde{\omega},z_3)^T$. Using (2.18), (2.13) and the fact that

$$\mathcal{E}(t)=\frac{1}{2}\|U\|_{\mathcal{H}}^2, \tag{2.20}$$

we get

$$\begin{aligned}\langle\mathcal{A}U,U\rangle_{\mathcal{H}}=&-\mu_1\int_0^L u^2dx-\tilde{\mu}_1\int_0^L v^2dx-\tilde{\tilde{\mu}}_1\int_0^L\tilde{\omega}^2dx\\ &-\mu_2\int_0^L z_1(x,1)udx-\tilde{\mu}_2\int_0^L z_2(x,1)vdx-\tilde{\tilde{\mu}}_2\int_0^L z_3(x,1)\tilde{\omega}\,dx\\ &-\sum_{i=1}^{3}\frac{\xi_i}{\tau_i}\int_0^L\int_0^1 z_i(x,\rho)z_{i\rho}(x,\rho)\,d\rho\,dx.\\ &\leq 0.\end{aligned} \tag{2.21}$$

Consequently, the operator $\mathcal{A}$ is dissipative. Now, we will prove that the operator $\lambda I-\mathcal{A}$ is surjective for $\lambda > 0$. For this purpose, let $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathcal{H}$, we seek $U = (\varphi, u, z_1, \psi, v, z_2, \omega, \tilde{\omega}, z_3)^T \in D(\mathcal{A})$ solution of the following system of equations

$$\begin{cases} \lambda\varphi - u = f_1, \\ \lambda u - \dfrac{Gh}{\rho_1}(\varphi_x - \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) + \dfrac{\mu_1}{\rho_1}u + \dfrac{\mu_2}{\rho_1}z_1(.,1) = f_2, \\ \lambda z_1 + \dfrac{1}{\tau_1}z_{1\rho} = f_3, \\ \lambda\psi - v = f_4, \\ \lambda v - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) + \dfrac{\tilde{\mu}_1}{\rho_2}v + \dfrac{\tilde{\mu}_2}{\rho_2}z_2(.,1) = f_5, \\ \lambda z_2 + \dfrac{1}{\tau_2}z_{2\rho} = f_6, \\ \lambda\omega - \tilde{\omega} = f_7, \\ \lambda\tilde{\omega} - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) + \dfrac{\tilde{\tilde{\mu}}_1}{\rho_1}\tilde{\omega} + \dfrac{\tilde{\tilde{\mu}}_2}{\rho_1}z_3(.,1) = f_8, \\ \lambda z_3 + \dfrac{1}{\tau_3}z_{3\rho} = f_9. \end{cases} \tag{2.22}$$

Suppose that we have found φ, ψ and ω. Therefore, the first, the fourth and the seventh equation in (2.22) give

$$\begin{cases} u = \lambda\varphi - f_1, \\ v = \lambda\psi - f_4, \\ \tilde{\omega} = \lambda\omega - f_7. \end{cases} \tag{2.23}$$

It is clear that $u \in H_0^1(0, L), v \in H_0^1(0, L)$ and $\omega \in H_0^1(0, L)$. Furthermore, by (2.22) we can find $z_i (i = 1, 2, 3)$ as

$$z_1(x, 0) = u(x), z_2(x, 0) = v(x), z_3(x, 0) = \tilde{\omega}(x), \quad \text{for } x \in (0, L). \tag{2.24}$$

Following the same approach as in **[31]**, we obtain, by using equations for z_i in (2.22),

$$\begin{aligned} z_1(x, \rho) &= u(x)e^{-\lambda\tau_1\rho} + \tau_1 e^{-\lambda\tau_1\rho}\int_0^\rho f_3(x, s)e^{\lambda\tau_1 s}\, ds, \\ z_2(x, \rho) &= v(x)e^{-\lambda\tau_2\rho} + \tau_2 e^{-\lambda\tau_2\rho}\int_0^\rho f_6(x, s)e^{\lambda\tau_2 s}\, ds, \\ z_3(x, \rho) &= \tilde{\omega}(x)e^{-\lambda\tau_3\rho} + \tau_3 e^{-\lambda\tau_3\rho}\int_0^\rho f_9(x, s)e^{\lambda\tau_3 s}\, ds. \end{aligned}$$

From (2.23), we obtain

$$\begin{cases} z_1(x, \rho) = \lambda\varphi(x)e^{-\lambda\tau_1\rho} - f_1 e^{-\lambda\tau_1\rho} + \tau_1 e^{-\lambda\tau_1\rho}\displaystyle\int_0^\rho f_3(x, s)e^{\lambda\tau_1 s}\, ds, \\ z_2(x, \rho) = \lambda\psi(x)e^{-\lambda\tau_2\rho} - f_4 e^{-\lambda\tau_2\rho} + \tau_2 e^{-\lambda\tau_2\rho}\displaystyle\int_0^\rho f_6(x, s)e^{\lambda\tau_2 s}\, ds, \\ z_3(x, \rho) = \lambda\omega(x)e^{-\lambda\tau_3\rho} - f_7 e^{-\lambda\tau_3\rho} + \tau_3 e^{-\lambda\tau_3\rho}\displaystyle\int_0^\rho f_9(x, s)e^{\lambda\tau_3 s}\, ds. \end{cases} \tag{2.25}$$

By using (2.22) and (2.23) the functions φ, ψ and ω satisfying the following system

$$\left\{\begin{aligned} &\lambda^2\varphi - \frac{Gh}{\rho_1}(\varphi_x - \psi + l\omega)_x - \frac{lEh}{\rho_1}(\omega_x - l\varphi) + \frac{\mu_1}{\rho_1}u + \frac{\mu_2}{\rho_1}z_1(.,1) = f_2 + \lambda f_1, \\ &\lambda^2\psi - \frac{EI}{\rho_2}\psi_{xx} + \frac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) + \frac{\tilde{\mu}_1}{\rho_2}v + \frac{\tilde{\mu}_2}{\rho_2}z_2(.,1) = f_5 + \lambda f_4, \\ &\lambda^2\omega - \frac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \frac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) + \frac{\tilde{\tilde{\mu}}_1}{\rho_1}\tilde{\omega} + \frac{\tilde{\tilde{\mu}}_2}{\rho_1}z_3(.,1) = f_8 + \lambda f_7. \end{aligned}\right. \tag{2.26}$$

Solving system (2.26) is equivalent to finding $(\varphi, \psi, \omega) \in (H^2 \cap H_0^1(0, L))^3$ such that

$$\left\{\begin{aligned} &\int_0^L (\rho_1\lambda^2\varphi w + Gh(\varphi_x - \psi + l\omega)w_x - lEh(\omega_x - l\varphi)w + \mu_1 uw + \mu_2 z_1(.,1)w)\,dx \\ &\qquad\qquad = \int_0^L \rho_1(f_2 + \lambda f_1)w\,dx, \\ &\int_0^L (\rho_2\lambda^2\psi\chi + EI\psi_x\chi_x + Gh(\varphi_x + \psi + l\omega)\chi + \tilde{\mu}_1 v\chi + \tilde{\mu}_2 z_2(.,1)\chi)\,dx \\ &\qquad\qquad = \int_0^L \rho_2(f_5 + \lambda f_4)\chi\,dx, \\ &\int_0^L (\rho_1\lambda^2\omega\zeta + Eh(\omega_x - l\varphi)\zeta_x + lGh(\varphi_x + \psi + l\omega)\zeta + \tilde{\tilde{\mu}}_1\tilde{\omega}\zeta + \tilde{\tilde{\mu}}_2 z_3(.,1)\zeta)\,dx \\ &\qquad\qquad = \int_0^L \rho_1(f_8 + \lambda f_7)\zeta\,dx \end{aligned}\right. \tag{2.27}$$

for all $(w, \chi, \zeta) \in H_0^1(0, L) \times H_0^1(0, L) \times H_0^1(0, L)$. From (2.25), we have

$$\left\{\begin{aligned} z_1(x,1) &= \lambda\varphi(x)e^{-\lambda\tau_1} - f_1 e^{-\lambda\tau_1} + \tau_1 e^{-\lambda\tau_1}\int_0^1 f_3(x,s)e^{\lambda\tau_1 s}\,ds, \\ z_2(x,1) &= \lambda\psi(x)e^{-\lambda\tau_2} - f_4 e^{-\lambda\tau_2} + \tau_2 e^{-\lambda\tau_2}\int_0^1 f_6(x,s)e^{\lambda\tau_2 s}\,ds, \\ z_3(x,1) &= \lambda\omega(x)e^{-\lambda\tau_3} - f_7 e^{-\lambda\tau_3} + \tau_3 e^{-\lambda\tau_3}\int_0^1 f_9(x,s)e^{\lambda\tau_3 s}\,ds. \end{aligned}\right.$$

Consequently, problem (2.27) is equivalent to the problem

$$a((\varphi, \psi, \omega), (w, \chi, \zeta)) = L(w, \chi, \zeta) \tag{2.28}$$

where the bilinear form $a : [H_0^1(0, L) \times H_0^1(0, L) \times H_0^1(0, L)]^2 \to \mathbb{R}$ and the linear form $L : H_0^1(0, L) \times H_0^1(0, L) \times H_0^1(0, L) \to \mathbb{R}$ are defined by

$$\begin{aligned} a((\varphi, \psi, \omega), (w, \chi, \zeta)) &= \int_0^L (\rho_1\lambda^2\varphi w + Gh(\varphi_x - \psi + l\omega)(w_x + \chi + l\zeta))\,dx \\ &+ \int_0^L (\rho_2\lambda^2\psi\chi + EI\psi_x\chi_x)\,dx + \int_0^L (\rho_1\lambda^2\omega\zeta + Eh(\omega_x - l\varphi)(\zeta_x - lw)\,dx \\ + \int_0^L \lambda\varphi(\mu_1 + \mu_2 e^{-\lambda\tau_1})w\,dx &+ \int_0^L \lambda\varphi(\widetilde{\mu_1} + \widetilde{\mu_2}e^{-\lambda\tau_2})w\,dx + \int_0^L \lambda\varphi(\widetilde{\widetilde{\mu}}_1 + \widetilde{\widetilde{\mu}}_2 e^{-\lambda\tau_3})w\,dx \end{aligned}$$

and

$$L(w,\chi,\zeta)=\int_0^L(\mu_1f_1-\mu_2M_1)w\,dx+\int_0^L(\widetilde{\mu}_1f_4-\widetilde{\mu}_2M_2)\chi\,dx+\int_0^L(\widetilde{\widetilde{\mu}}_1f_7-\widetilde{\widetilde{\mu}}_2M_3)\zeta\,dx$$
$$+\int_0^L\rho_1(f_2+\lambda f_1)w\,dx+\int_0^L\rho_2(f_5+\lambda f_4)\chi\,dx+\int_0^L\rho_1(f_8+\lambda f_7)\zeta\,dx.$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all

$$(w,\chi,\zeta)\in H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L).$$

problem (2.28) admits a unique solution

$$(\varphi,\psi,\omega)\in H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L).$$

Applying the classical elliptic regularity, it follows from (2.27) that

$$(\varphi,\psi,\omega)\in H^2(0,L)\times H^2(0,L)\times H^2(0,L).$$

Therefore, the operator $\lambda I-A$ is surjective for any $\lambda>0$. Consequently, the existence result of Theorem 2.1 follows from the Hille Yosida theorem. □

2.4 Asymptotic Behavior

First we state and prove a lemma that will be needed to establish the asymptotic behavior.

Lemma 2.4 *There exists a positive constant C such that the following inequality holds for every $(\varphi,\psi,\omega)\in(H_0^1(0,L))^3$*

$$\int_0^L(|\varphi_x|^2+|\psi_x|^2+|\omega_x|^2)\,dx\le C\int_0^L(EI|\psi_x|^2+Gh|\varphi_x+\psi+l\omega|^2+Eh|\omega_x-l\varphi|^2)\,dx \le \mathcal{E}(t). \tag{2.29}$$

Proof. We will argue by contradiction. Indeed, let us suppose that (2.29) is not true. So, we can find a sequence $\{(\varphi_\nu,\psi_\nu,\omega_\nu)\}_{\nu\in\mathbb{N}}$ in $(H_0^1(0,L))^3$ satisfying

$$\int_0^L(EI|\psi_{\nu x}|^2+Gh|\varphi_{\nu x}+\psi+l\omega_\nu|^2+Eh|\omega_{\nu x}-l\varphi_\nu|^2)\,dx\le\frac{1}{\nu} \tag{2.30}$$

and

$$\int_0^L(|\varphi_{\nu x}|^2+|\psi_{\nu x}|^2+|\omega_{\nu x}|^2)\,dx=1. \tag{2.31}$$

From (2.31), the sequence $\{(\varphi_\nu,\psi_\nu,\omega_\nu)\}_{\nu\in\mathbb{N}}$ is bounded in $(H_0^1(0,L))^3$. Since the embedding $H_0^1(0,L)\hookrightarrow L^2(0,L)$ is compact, then the sequence $\{(\varphi_\nu,\psi_\nu,\omega_\nu)\}_{\nu\in\mathbb{N}}$ converge strongly in $(L^2(0,L))^3$.
From (2.30)

$$\psi_{\nu x}\to 0 \text{ strongly in } L^2(0,L). \tag{2.32}$$

Using Poincaré's inequality we can conclude that

$$\psi_{\nu}\to 0 \text{ strongly in } L^2(0,L). \tag{2.33}$$

Now, setting $\varphi_\nu\to\varphi$ and $\omega_\nu\to\omega$ strongly in $L^2(0,L)$.
From (2.30), we have

$$\varphi_{\nu x}+\psi_\nu+l\omega_\nu\to 0 \text{ strongly in } L^2(0,L). \tag{2.34}$$

Then

$$\varphi_{\nu x}+\psi_\nu+l\omega_\nu=\varphi_{\nu x}+\psi_\nu+l(\omega_\nu-\omega)+l\omega\to 0 \text{ strongly in } L^2(0,L). \tag{2.35}$$

which implies that

$$\varphi_{\nu x}\to -l\omega \text{ strongly in } L^2(0,L). \tag{2.36}$$

Then, $\{\varphi_\nu\}_n$ is a Cauchy sequence in $H^1(0,L)$. Therefore $\{\varphi_\nu\}_n$ converge to a function φ_1 in $H^1(0,L)$. Consequently $\{\varphi_\nu\}_n$ converge to φ_1 in $L^2(0,L)$. Thus by the uniqueness of the limit $\varphi_1=\varphi$. Moreover $\varphi\in H_0^1(0,L)$.
From (2.36) we deduce that

$$\varphi_x+l\omega=0 \text{ a.e } x\in(0,L). \tag{2.37}$$

Similarly, we have

$$\omega_x-l\varphi=0 \text{ a.e } x\in(0,L) \tag{2.38}$$

and $\omega\in H_0^1(0,L)$.
(2.37) and (2.38) provides us $\varphi=\omega=0$, contradicting (2.31).

□

From now on, we denote by c various positive constants which may be different at different occurrences.
Multiplying the first equation in (2.8) by $\mathcal{E}^q\varphi$, the third equation by $\mathcal{E}^q\psi$ and the fifth equation by $\mathcal{E}^q\omega$. We obtain

$$0=\int_S^T\mathcal{E}^q\int_0^L\varphi\left(\rho_1\varphi_{tt}-Gh(\varphi_x+\psi+l\omega)_x-lEh(\omega_x-l\varphi)+\mu_1\varphi_t+\mu_2 z_1(x,1,t)\right)dxdt$$

$$\begin{aligned}0 \;=\; & \left[\mathcal{E}^q\rho_1\int_0^L\varphi\varphi_t dx\right]_S^T-\int_S^T\rho_1 q\mathcal{E}'\mathcal{E}^{q-1}\int_0^L\varphi\varphi_t dxdt-\rho_1\int_S^T\mathcal{E}^q\|\varphi_t\|_2^2dt\\ &-\int_S^T\mathcal{E}^q\int_0^L\varphi_x Gh(\varphi_x+\psi+l\omega)dxdt-\int_S^T\mathcal{E}^q\int_0^L\varphi(lEh)(\omega_x-l\varphi)dxdt\\ &+\mu_1\int_S^T\mathcal{E}^q\int_0^L\varphi_t\varphi dxdt+\mu_2\int_S^T\mathcal{E}^q\int_0^L\varphi z_1(x,1,t)dxdt\end{aligned}$$

$$0 = \int_S^T \mathcal{E}^q \int_0^L \psi\left(\rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + \widetilde{\mu_1}\psi_t + \widetilde{\mu_2}z_2(x,1,t)\right)dxdt$$

$$\begin{aligned}0 \;=\; & \left[\mathcal{E}^q\rho_2\int_0^L \psi\psi_t dx\right]_S^T - \int_S^T \rho_2 q\mathcal{E}'\mathcal{E}^{q-1}\int_0^L \psi\psi_t dxdt - \rho_2\int_S^T \mathcal{E}^q\|\psi_t\|_2^2 dt \\ & + \int_S^T \mathcal{E}^q EI\|\psi_x\|_2^2 dt + \int_S^T \mathcal{E}^q\int_0^L \psi\psi Gh(\varphi_x + \psi + l\omega))dxdt \\ & + \widetilde{\mu_1}\int_S^T \mathcal{E}^q\int_0^L \psi\psi_t dxdt + \widetilde{\mu_2}\int_S^T \mathcal{E}^q\int_0^L \psi z_2(x,1,t)dxdt\end{aligned}$$

$$0 = \int_S^T \mathcal{E}^q \int_0^L \omega\left(\rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x + \psi + l\omega) + \widetilde{\widetilde{\mu_1}}\omega_t + \widetilde{\widetilde{\mu_2}}z_3(x,1,t)\right)dxdt$$

$$\begin{aligned}0 \;=\; & \left[\mathcal{E}^q\rho_1\int_0^L \omega\omega_t dx\right]_S^T - \int_S^T \rho_1 q\mathcal{E}'\mathcal{E}^{q-1}\int_0^L \omega\omega_t dxdt \\ & -\rho_1\int_S^T \mathcal{E}^q\|\omega_t\|_2^2 dt + \int_S^T \mathcal{E}^q\int_0^L Eh\omega_x(\omega_x - l\varphi)dxdt + \int_S^T \mathcal{E}^q\int_0^L \omega(lGh)(\varphi_x + \psi + l\omega)dxdt \\ & + \widetilde{\widetilde{\mu_1}}\int_S^T \mathcal{E}^q\int_0^L \omega_t\omega dxdt + \widetilde{\widetilde{\mu_2}}\int_S^T \mathcal{E}^q\int_0^L \omega z_3(x,1,t)dxdt\end{aligned}$$

Taking their sum, we obtain

$$\begin{aligned}0 \;=\; & \left[\mathcal{E}^q\rho_1\int_0^L \varphi\varphi_t dx\right]_S^T + \left[\mathcal{E}^q\rho_2\int_0^L \psi\psi_t dx\right]_S^T + \left[\mathcal{E}^q\rho_1\int_0^L \omega\omega_t dx\right] \\ & - \int_S^T \rho_1 q\mathcal{E}'\mathcal{E}^{q-1}\int_0^L (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t + \rho_1\omega\omega_t)\,dxdt \\ & -2\rho_1\int_S^T \mathcal{E}^q\|\varphi_t\|_2^2 dt - 2\rho_2\int_S^T \mathcal{E}^q\|\psi_t\|_2^2 dt - 2\rho_1\int_S^T \mathcal{E}^q\|\omega_t\|_2^2 dt \\ & + \int_S^T \mathcal{E}^q\left(\rho_1\|\varphi_t\|_2^2 + \rho_2\|\psi_t\|_2^2 + \rho_1\|\omega_t\|_2^2 + Gh\|\varphi_x + \psi + l\omega\|_2^2 + EI\|\psi_t\|_2^2 + Eh\|\omega_x - l\psi\|_2^2\right) \\ & + \mu_1\int_S^T \mathcal{E}^q\int_0^L \varphi_t\varphi dxdt + \mu_2\int_S^T \mathcal{E}^q\int_0^L \varphi z_1(x,1,t)dxdt \\ & + \widetilde{\mu_1}\int_S^T \mathcal{E}^q\int_0^L \psi\psi_t dxdt + \widetilde{\mu_2}\int_S^T \mathcal{E}^q\int_0^L \psi z_2(x,1,t)dxdt \\ & + \widetilde{\widetilde{\mu_1}}\int_S^T \mathcal{E}^q\int_0^L \omega_t\omega dxdt + \widetilde{\widetilde{\mu_2}}\int_S^T \mathcal{E}^q\int_0^L \omega z_3(x,1,t)dxdt\end{aligned} \tag{2.39}$$

Similarly, we multiply the equation of (2.7) by $\mathcal{E}^q\xi_i e^{-2\tau_i\rho}z_i(x,\rho,t)$ and get

$$0 \;=\; \int_S^T \mathcal{E}^q\int_0^L\int_0^1 e^{-2\tau_i\rho}\xi_i z_i(\tau_i z_{it} + z_{i\rho})\,d\rho dxdt$$

$$= \left[\frac{1}{2}\xi_i\tau_i\mathcal{E}^q\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx\right]_S^T - \frac{\tau_i\xi_i}{2}\int_S^T q\mathcal{E}^{q-1}\mathcal{E}'\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dxdt$$
$$+\int_S^T \mathcal{E}^q\xi_i\int_0^L\int_0^1 \frac{e^{-2\tau_i\rho}}{2}\frac{d}{d\rho}(z_i^2)\,d\rho dxdt \tag{2.40}$$

$$0 = \left[\frac{1}{2}\xi_i\tau_i\mathcal{E}^q\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx\right]_S^T - \frac{\tau_i\xi_i}{2}\int_S^T q\mathcal{E}^{q-1}E'\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dxdt$$
$$+\frac{\xi_i}{2}\int_S^T \mathcal{E}^q\int_0^L\int_0^1 \left[\frac{d}{d\rho}\left(e^{-2\tau_i\rho}z_i^2\right) + 2\tau_i e^{-2\tau_i\rho}z_i^2\right]\,d\rho dxdt$$

$$0 = \left[\frac{1}{2}\xi_i\tau_i\mathcal{E}^q\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx\right]_S^T - \frac{\tau_i\xi_i}{2}\int_S^T q\mathcal{E}^{q-1}\mathcal{E}'\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dxdt$$
$$+\frac{\xi_i}{2}\int_S^T \mathcal{E}^q\int_0^L \left[e^{-2\tau_i}z_i^2(x,1,t) - z_i^2(x,0,t)\right]dxdt$$
$$+\xi_i\tau_i\int_S^T\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dxdt$$

Recalling the definition of $\mathcal{E}$ and from (2.39), (2.40), we get

$$\begin{aligned}
A\int_S^T \mathcal{E}^{q+1}dt \quad \leq \quad & -\left[\rho_1\mathcal{E}^q\int_0^L \varphi\varphi_t\,dx\right]_S^T-\left[\rho_2\mathcal{E}^q\int_0^L \psi\psi_t dx\right]_S^T-\left[\rho_1\mathcal{E}^q\int_0^L \omega\omega_t\,dx\right]_S^T\\
&+\int_S^T q\mathcal{E}'\mathcal{E}^{q-1}\int_0^L(\rho_1\varphi\varphi_t+\rho_2\psi\psi_t+\rho_1\omega\omega_t)\,dxdt\\
&+2\int_S^T \mathcal{E}^q\left(\rho_1\|\varphi_t\|_2^2+\rho_2\|\psi_t\|_2^2+\rho_1\|\omega_t\|_2^2\right)\,dt\\
&-\mu_1\int_S^T \mathcal{E}^q\int_0^L \varphi_t\varphi dxdt-\mu_2\int_S^T\mathcal{E}^q\int_0^L \varphi z_1(x,1,t)\,dxdt \qquad (2.41)\\
&-\widetilde{\mu}_1\int_S^T \mathcal{E}^q\int_0^L \psi\psi_t dxdt-\widetilde{\mu}_2\int_S^T\mathcal{E}^q\int_0^L \psi z_2(x,1,t)\,dxdt\\
&-\widetilde{\widetilde{\mu}}_1\int_S^T \mathcal{E}^q\int_0^L \omega_t\omega dxdt-\widetilde{\widetilde{\mu}}_2\int_S^T\mathcal{E}^q\int_0^L \omega z_3(x,1,t)\,dxdt\\
&-\sum_{i=1}^3\left[\frac{1}{2}\xi_i\tau_i\mathcal{E}^q\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx\right]_S^T\\
&+\sum_{i=1}^3\frac{\tau_i\xi_i}{2}\int_S^T q\mathcal{E}^{q-1}E'\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dxdt\\
&-\sum_{i=1}^3\frac{\xi_i}{2}\int_S^T \mathcal{E}^q e^{-2\tau_i}\int_0^L z_i^2(x,1,t)\,dxdt\\
&+\sum_{i=1}^3\frac{\xi_i}{2}\int_S^T \mathcal{E}^q\|z_i(x,0,t)\|_2^2\,dt
\end{aligned}$$

where $A=2\min\{1,2\tau_1e^{-2\tau_1},2\tau_2e^{-2\tau_2},2\tau_3e^{-2\tau_3}\}$. Using the Young and Sobolev-Poincaré inequalities and Lemma 2.4, we find that

$$\begin{aligned}
-\left[\mathcal{E}^q\int_0^L \varphi\varphi_t\,dx\right]_S^T \quad &= \quad \mathcal{E}^q(S)(S)\int_0^L \varphi(S)\varphi_t(S)\,dx-\mathcal{E}^q(T)\int_0^L\varphi(T)\varphi_t(T)\,dx\\
&\leq \quad C\mathcal{E}^{q+1}(S)
\end{aligned}$$

$$\left|\int_S^T(q\mathcal{E}'\mathcal{E}^{q-1})\int_0^L(\rho_1\varphi\varphi_t+\rho_2\psi\psi_t+\rho_1\omega\omega_t)\,dxdt\right|\leq c\int_S^T(-\mathcal{E}')\mathcal{E}^q\,dt\leq c\mathcal{E}^{q+1}(S)$$

$$\left|\frac{1}{2}\xi_i\tau_i\mathcal{E}^q\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,dxd\rho\right|\leq c\mathcal{E}(S)^{q+1}\quad \forall t\geq S$$

$$\int_S^T\mathcal{E}^q\int_0^L u_t^2\,dxdt \quad\leq\quad c\int_S^T\mathcal{E}^q(-\mathcal{E}')\,dt$$

$$\leq \; c\mathcal{E}^{q+1}(S)$$

$$\begin{aligned}\int_S^T \mathcal{E}^q \xi_i \int_0^L e^{-2\tau_i} z_i^2(x,1,t)\,dxdt &\leq c\int_S^T \mathcal{E}^q(-\mathcal{E}')\,dt \\ &\leq c\mathcal{E}^{q+1}(S)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\int_S^T \mathcal{E}^q \xi_i \int_0^L z_i^2(x,0,t))\,dxdt &= \frac{1}{2}\int_S^T \mathcal{E}^q \xi_i \int_0^L \varphi'^2\,dxdt \\ &\leq c\mathcal{E}^{q+1}(S),\end{aligned}$$

$$\left|\frac{\tau_i\xi_i}{2}\int_S^T q\mathcal{E}^{q-1}\mathcal{E}'\int_0^L\int_0^1 e^{-2\tau_i\rho} z_i^2 dxd\rho\,dt\right| \leq c\int_S^T (-\mathcal{E}')\mathcal{E}^q\,dt \leq c\mathcal{E}^{q+1}(S)$$

$$\begin{aligned}\left|\int_S^T \mathcal{E}^q \int_0^L \varphi\varphi_t\,dxdt\right| &\leq \varepsilon\int_S^T \mathcal{E}^q\int_0^L \varphi^2\,dxdt + c(\varepsilon)\int_S^T \mathcal{E}^q\int_0^L \varphi_t^2\,dxdt \\ &\leq \varepsilon c\int_0^L \mathcal{E}^{q+1}\,dt + c(\varepsilon)\int_S^T \mathcal{E}^q\int_0^L \varphi_t^2\,dxdt \\ &\leq \varepsilon c\int_0^L \mathcal{E}^{q+1}\,dt + c(\varepsilon)\int_S^T \mathcal{E}^q(-\mathcal{E}')\,dt \\ &\leq \varepsilon c\int_0^L \mathcal{E}^{q+1}\,dt + c(\varepsilon)\mathcal{E}(S)^{q+1} \qquad (2.42)\end{aligned}$$

and

$$\begin{aligned}\left|\int_S^T \mathcal{E}^q\int_0^L \varphi z_1(x,1,t)\,dxdt\right| &\leq \varepsilon_1\int_S^T \mathcal{E}^q\int_0^L \varphi^2\,dxdt + c(\varepsilon_1)\int_S^T \mathcal{E}^q\int_0^L z_1(x,1,t)^2\,dxdt \\ &\leq \varepsilon_1 c\int_S^T \mathcal{E}^{q+1}\,dt + c(\varepsilon_1)\int_S^T \mathcal{E}^q\int_0^L z_1(x,1,t)^2 \\ &\leq \varepsilon_1 c\int_S^T \mathcal{E}^{q+1}\,dt + c(\varepsilon_1)\int_S^T \mathcal{E}^q(-\mathcal{E}')\,dt \\ &\leq \varepsilon_1 c\int_S^T \mathcal{E}^{q+1}\,dt + c(\varepsilon_1)\mathcal{E}^{q+1}(S). \qquad (2.43)\end{aligned}$$

$$\left|\int_S^T \mathcal{E}^q\int_0^L \psi\psi_t\,dxdt\right| \leq \varepsilon' c\int_S^T \mathcal{E}^{q+1}\,dt + c(\varepsilon')\mathcal{E}(S)^{q+1} \qquad (2.44)$$

$$\left|\int_S^T \mathcal{E}^q\int_0^L \psi z_2(x,1,t)\,dxdt\right| \leq \varepsilon_1' c\int_S^T \mathcal{E}^{q+1}\,dt + c(\varepsilon_1')\mathcal{E}(S)^{q+1} \qquad (2.45)$$

$$\left|\int_S^T \mathcal{E}^q \int_0^L \omega\omega_t \, dxdt\right| \leq \varepsilon'' c \int_S^T \mathcal{E}^{q+1} \, dt + c(\varepsilon'')\mathcal{E}(S)^{q+1} \tag{2.46}$$

$$\left|\int_S^T \mathcal{E}^q \int_0^L \omega z_3(x,1,t) \, dxdt\right| \leq \varepsilon_1'' c \int_S^T \mathcal{E}^{q+1} \, dt + c(\varepsilon_1'')\mathcal{E}(S)^{q+1} \tag{2.47}$$

Choosing $\varepsilon, \varepsilon_1, \varepsilon', \varepsilon_1', \varepsilon''$ and ε_1'' small enough, we deduce from (2.41),(2.42), (2.43), (2.44),(2.45), (2.45), (2.46) and (2.47) that

$$\int_S^T \mathcal{E}^{q+1} \, dt \leq c\mathcal{E}^{q+1}(S),$$

where c is a positive constant independent of $E(0)$. We choose $q = 0$. Hence, we deduce from Lemma 2.2 that

$$\mathcal{E}(t) \leq c\mathcal{E}(0)e^{-\omega t}, \qquad t \geq 0.$$

This ends the proof of Theorem 2.1.

Chapter 3

Well-posedness and energy decay of solutions to a nonlinear Bresse system with delay terms.

3.1 Introduction

We consider the Bresse system in bounded domain with a delay term in the nonlinear

$$(P)\begin{cases}\rho_1\varphi_{tt}-Gh(\varphi_x+\psi+l\omega)_x-lEh(\omega_x-l\varphi)+\mu_1g_1(\varphi_t(x,t))+\mu_2g_2(\varphi_t(x,t-\tau_1))=0 & \text{in }]0,1[\times]0,+\infty[\\ \rho_2\psi_{tt}-EI\psi_{xx}+Gh(\varphi_x+\psi+l\omega)+\widetilde{\mu_1}\widetilde{g}_1(\psi_t(x,t))+\widetilde{\mu_2}\widetilde{g}_2(\psi_t(x,t-\tau_2))=0 & \text{in }]0,1[\times]0,+\infty[\\ \rho_1\omega_{tt}-Eh(\omega_x-l\varphi)_x+lGh(\varphi_x+\psi+l\omega)+\widetilde{\widetilde{\mu_1}}\widetilde{\widetilde{g}}_1(\omega_t(x,t))+\widetilde{\widetilde{\mu_2}}\widetilde{\widetilde{g}}_2(\omega_t(x,t-\tau_3))=0 & \text{in }]0,1[\times]0,+\infty[\\ \varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=0 & t\geq 0\\ \psi(x,0)=\psi_0(x),\quad \psi_t(x,0)=\psi_1(x) & x\in]0,1[\\ \varphi(x,0)=\varphi_0(x),\quad \varphi_t(x,0)=\varphi_1(x) & x\in]0,1[\\ \psi_t(x,t-\tau)=f_0(x,t-\tau) & \text{in }]0,1[\times]0,\tau[\end{cases}$$

3.2 Preliminaries and main results

First assume the following hypotheses:
(H1) $g_1 : \mathbb{R}\to\mathbb{R}, (resp, \widetilde{g}_1 : \mathbb{R}\to\mathbb{R},\ \ \widetilde{\widetilde{g}}_1 : \mathbb{R}\to\mathbb{R})$ is a non-decreasing function of the class $C(\mathbb{R})$ such that there exist $\epsilon_1,\ c_1,\ c_2, \widetilde{\epsilon}_1, \widetilde{c}_1,\ \widetilde{c}_2, \widetilde{\widetilde{\epsilon}}_1, \widetilde{\widetilde{c}}_1,\ \widetilde{\widetilde{c}}_2>0$ and a convex and increasing function $H : \mathbb{R}_+\to\mathbb{R}_+$ of the class $C^1(\mathbb{R}_+)\cap C^2(]0,\infty[)$ satisfying $H(0)=0$, and H linear on $[0,\epsilon']$ or ($H'(0)=0$ and $H''>0$ on $]0,\epsilon']$), such that

$$\begin{array}{ll} c_1|s|\leq|g_1(s)|\leq c_2|s| & \text{if } |s|\geq\epsilon'\\ \widetilde{c}_1|s|\leq|\widetilde{g}_1(s)|\leq \widetilde{c}_2|s| & \text{if } |s|\geq\widetilde{\epsilon}'\\ \widetilde{\widetilde{c}}_1|s|\leq|\widetilde{\widetilde{g}}_1(s)|\leq \widetilde{\widetilde{c}}_2|s| & \text{if } |s|\geq\widetilde{\widetilde{\epsilon}}'. \end{array} \tag{3.1}$$

$$\begin{array}{ll} s^2+g_1^2(s)\leq H^{-1}(sg_1(s)) & \text{if } |s|\leq\epsilon'\\ s^2+\widetilde{g}_1^2(s)\leq H^{-1}(s\widetilde{g}_1(s)) & \text{if } |s|\leq\widetilde{\epsilon}'\\ s^2+\widetilde{\widetilde{g}}_1^2(s)\leq H^{-1}(s\widetilde{\widetilde{g}}_1(s)) & \text{if } |s|\leq\widetilde{\widetilde{\epsilon}}'. \end{array} \tag{3.2}$$

$g_2 : \mathbb{R} \to \mathbb{R}$ (resp $\widetilde{g}_2 : \mathbb{R} \to \mathbb{R}$, $\widetilde{\widetilde{g}}_2 : \mathbb{R} \to \mathbb{R}$) is an odd non-decreasing function of the class $C^1(\mathbb{R})$ such that there exist $c_3, \alpha_1, \alpha_2, \widetilde{c}_3, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\widetilde{c}}_3, \widetilde{\widetilde{\alpha}}_1, \widetilde{\widetilde{\alpha}}_2 > 0$

$$\begin{array}{l} |g_2'(s)| \leq c_3 \\ |\widetilde{g}_2'(s)| \leq \widetilde{c}_3 \\ |\widetilde{\widetilde{g}}_2'(s)| \leq \widetilde{\widetilde{c}}_3. \end{array} \tag{3.3}$$

$$\begin{array}{l} \alpha_1 \ s g_2(s) \leq G_2(s) \leq \alpha_2 \ s g_1(s) \\ \widetilde{\alpha}_1 \ s \widetilde{g}_2(s) \leq \widetilde{G}_2(s) \leq \widetilde{\alpha}_2 \ s \widetilde{g}_1(s) \\ \widetilde{\widetilde{\alpha}}_1 \ s \widetilde{\widetilde{g}}_2(s) \leq \widetilde{\widetilde{G}}_2(s) \leq \widetilde{\widetilde{\alpha}}_2 \ s \widetilde{\widetilde{g}}_1(s). \end{array} \tag{3.4}$$

where

$$\begin{array}{l} G_2(s) = \displaystyle\int_0^s g_2(r)\,dr \\ \widetilde{G}_2(s) = \displaystyle\int_0^s \widetilde{g}_2(r)\,dr \\ \widetilde{\widetilde{G}}_2(s) = \displaystyle\int_0^s \widetilde{\widetilde{g}}_2(r)\,dr. \end{array}$$

and

$$\begin{array}{l} \alpha_2 \mu_2 < \alpha_1 \mu_1 \\ \widetilde{\alpha}_2 \widetilde{\mu}_2 < \widetilde{\alpha}_1 \widetilde{\mu}_1, \\ \widetilde{\widetilde{\alpha}}_2 \widetilde{\widetilde{\mu}}_2 < \widetilde{\widetilde{\alpha}}_1 \widetilde{\widetilde{\mu}}_1. \end{array} \tag{3.5}$$

We first state some Lemmas which will be needed later.

Lemma 3.1 (Sobolev-Poincaré's inequality) *Let q be a number with $2 \leq q < +\infty$ $(n = 1, 2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$. Then there is a constant $c_* = c_*((0,1), q)$ such that*

$$\|\psi\|_q \leq c_* \|\nabla \psi\|_2 \quad \textit{for} \quad \psi \in H_0^1((0,1)).$$

We introduce as in **[31]** the new variable

$$\begin{array}{rcll} z_1(x,\rho,t) & = & \phi_t(x, t-\tau_1\rho), & x \in (0,1),\ \rho \in (0,1), \quad t > 0, \\ z_2(x,\rho,t) & = & \psi_t(x, t-\tau_2\rho), & x \in (0,1),\ \rho \in (0,1), \quad t > 0, \\ z_3(x,\rho,t) & = & \omega_t(x, t-\tau_3\rho), & x \in (0,1),\ \rho \in (0,1), \quad t > 0. \end{array} \tag{3.6}$$

Then, we have

$$\tau_i z_{it}(x,\rho,t) + z_{i\rho}(x,\rho,t) = 0, \quad \text{in } (0,L) \times (0,1) \times (0,+\infty) \text{ for } i = 1,2,3. \tag{3.7}$$

Therefore, problem (P) is equivalent to:

$$
\begin{cases}
\rho_1\varphi_{tt} - Gh(\varphi_x+\psi+l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1 g_1(\varphi_t(x,t)) & \\
\qquad\qquad +\mu_2 g_2(z_1(x,1,t)) = 0 & \text{in }]0,1[\times]0,+\infty[,\\
\tau_1 z_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) = 0 & \text{in }]0,1[\times]0,1[\times]0,+\infty[,\\
\rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x+\psi+l\omega) + \widetilde{\mu_1}\widetilde{g_1}(\psi_t(x,t)) & \\
\qquad\qquad +\widetilde{\mu_2}\widetilde{g_2}(z_2(x,1,t)) = 0 & \text{in }]0,1[\times]0,+\infty[,\\
\tau_2 z_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) = 0 & \text{in }]0,1[\times]0,1[\times]0,+\infty[,\\
\rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x+\psi+l\omega) & \\
\qquad\qquad +\widetilde{\widetilde{\mu_1}}\widetilde{\widetilde{g_1}}(\omega_t(x,t)) + \widetilde{\widetilde{\mu_2}}\widetilde{\widetilde{g_2}}(z_2(x,1,t)) = 0 & \text{in }]0,1[\times]0,+\infty[,\\
\tau_3 z_{3t}(x,\rho,t) + z_{3\rho}(x,\rho,t) = 0 & \text{in }]0,1[\times]0,1[\times]0,+\infty[,\\
\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 & t \geq 0,\\
\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x) & x \in]0,1[,\\
\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & x \in]0,1[,\\
\omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x) & x \in]0,1[,\\
\varphi_t(x,t-\tau_1) = f_1(x,t-\tau_1) & \text{in }]0,1[\times]0,\tau[,\\
\psi_t(x,t-\tau_2) = f_2(x,t-\tau_2) & \text{in }]0,1[\times]0,\tau[,\\
\omega_t(x,t-\tau_3) = f_3(x,t-\tau_3) & \text{in }]0,1[\times]0,\tau[,
\end{cases}
\tag{3.8}
$$

Let ξ_1 be a positive constant such that

$$\tau_1\frac{\mu_2(1-\alpha_1)}{\alpha_1} < \xi_1 < \tau_1\frac{\mu_1 - \alpha_2\mu_2}{\alpha_2}. \tag{3.9}$$

Let ξ_2 be a positive constant such that

$$\tau_2\frac{\widetilde{\mu_2}(1-\widetilde{\alpha_1})}{\widetilde{\alpha_1}} < \xi_2 < \tau_2\frac{\widetilde{\mu_1} - \widetilde{\alpha_2}\widetilde{\mu_2}}{\widetilde{\alpha_2}}. \tag{3.10}$$

Let ξ_3 be a positive constant such that

$$\tau_3\frac{\widetilde{\widetilde{\mu_2}}(1-\widetilde{\widetilde{\alpha_1}})}{\widetilde{\widetilde{\alpha_1}}} < \xi_3 < \tau_3\frac{\widetilde{\widetilde{\mu_1}} - \widetilde{\widetilde{\alpha_2}}\widetilde{\widetilde{\mu_2}}}{\widetilde{\widetilde{\alpha_2}}}. \tag{3.11}$$

We define the energy associated to the solution of the problem (3.8) by the following formula:

$$
\begin{aligned}
E(t) = \frac{1}{2}\Big(&\frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_1}{2}\|\omega_t\|_2^2 + \frac{EI}{2}\|\psi_x\|_2^2 + \frac{Gh}{2}\|\varphi_x+\psi+l\omega\|_2^2 + \frac{Eh}{2}\|\omega_x - l\varphi\|_2^2\Big)\\
&+\xi_1\int_0^1\int_0^1 G_2(z_1(x,\rho,t))\,d\rho\,dx + \xi_2\int_0^1\int_0^1 \widetilde{G}_2(z_2(x,\rho,t))\,d\rho\,dx.\\
&+\xi_3\int_0^1\int_0^1 \widetilde{\widetilde{G}}_2(z_3(x,\rho,t))\,d\rho\,dx.
\end{aligned}
\tag{3.12}
$$

We have the following theorem.

Theorem 3.1 *Let* $(\varphi_0,\varphi_1),(\psi_0,\psi_1),(\omega_0,\omega_1) \in (H^2(0,1)\cap H_0^1(0,1))\times H_0^1(0,1)$, $f_0 \in H_0^1((0,1);H^1(0,1))$ *satisfy the compatibility condition*

$$f_1(.,0) = \varphi_1, \quad f_2(.,0) = \psi_1, \quad f_3(.,0) = \omega_1.$$

Assume that the hypothese (H1) hold. Then the problem (P) admits a unique weak solution

$$\psi,\varphi,\omega \in L^\infty_{loc}((-\tau,\infty);H^2(0,1)\cap H^1_0(0,1)), \quad \psi_t,\varphi_t,\omega_t \in L^\infty_{loc}((-\tau,\infty);H^1_0(0,1)),$$
$$\psi_{tt},\varphi_{tt},\omega_{tt} \in L^\infty_{loc}((-\tau,\infty);L^2(0,1))$$

and, for some constants ω_1,ω_2 and ω_3,ϵ_0 we obtain the following decay property:

$$E(t) \le \omega_1 H_1^{-1}(\omega_2 t+\omega_3), \quad \forall t>0, \tag{3.13}$$

where

$$H_1(t)=\int_t^1 \frac{1}{H_2(s)}\,ds \tag{3.14}$$

and

$$H_2(t)=\begin{cases} t & \text{if } H \text{ is linear on } [0,\epsilon'],\\ tH'(\epsilon_0 t) & \text{if } H'(0)=0 \text{ and } H''>0 \text{ on }]0,\epsilon'].\end{cases}$$

Remark 3.1 1. *By the mean value Theorem for integrals and the monotonicity of g_2, $\widetilde{g}_2$, $\widetilde{\widetilde{g}}_2$, we find that*

$$G_2(s)=\int_0^s g_2(r)\,dr \le sg_2(s),$$
$$\widetilde{G}_2(s)=\int_0^s \widetilde{g}_2(r)\,dr \le s\widetilde{g}_2(s),$$
$$\widetilde{\widetilde{G}}_2(s)=\int_0^s \widetilde{\widetilde{g}}_2(r)\,dr \le s\widetilde{\widetilde{g}}_2(s).$$

Then, $\alpha_1\le\alpha_2\le 1, \widetilde{\alpha}_1\le\widetilde{\alpha}_2\le 1, \widetilde{\widetilde{\alpha}}_1\le\widetilde{\widetilde{\alpha}}_2\le 1.$

2. *We need the condition (3.3) only to prove global existence, so if we study the energy decay, we can replace the linear growth order of the function $g_2(s),\widetilde{g}_2(s),\widetilde{\widetilde{g}}_2(s)$ for large $|s|$ by nonlinear polynomial growth.*

Proof of Theorem 3.1. We finish this section by giving an explicit upper bound for the derivative of the energy.

Lemma 3.2 *Let $(\varphi,\psi,\omega,z_1,z_2,z_3)$ be a solution of the problem (3.8). Then, the energy functional defined by (3.12) satisfies*

$$\begin{aligned} E'(t) \le &-\left(\mu_1-\frac{\xi_1\alpha_2}{\tau_1}-\mu_2\alpha_2\right)\int_0^1 \varphi_t g_1(\varphi_t)-\left(\frac{\xi_1}{\tau_1}\alpha_1-\mu_2(1-\alpha_1)\right)\int_0^1 z_1(x,1,t)g_2(z_1(x,1,t))dx\\ &-\left(\widetilde{\mu_1}-\frac{\xi_2\widetilde{\alpha_2}}{\tau_2}-\widetilde{\mu_2}\widetilde{\alpha_2}\right)\int_0^1 \psi_t\widetilde{g_1}(\psi_t)-\left(\frac{\xi_2}{\tau_2}\widetilde{\alpha_1}-\widetilde{\mu_2}(1-\widetilde{\alpha_1})\right)\int_0^1 z_2(x,1,t)\widetilde{g_2}(z_2(x,1,t))dx\\ &-\left(\widetilde{\widetilde{\mu_1}}-\frac{\xi_3\widetilde{\widetilde{\alpha_2}}}{\tau_3}-\widetilde{\widetilde{\mu_2}}\widetilde{\widetilde{\alpha_2}}\right)\int_0^1 \omega_t\widetilde{\widetilde{g_1}}(\omega_t)dx-\left(\frac{\xi_3}{\tau_3}\widetilde{\widetilde{\alpha_1}}-\widetilde{\widetilde{\mu_2}}(1-\widetilde{\widetilde{\alpha_1}})\right)\int_0^1 z_3(x,1,t)\widetilde{\widetilde{g_2}}(z_3(x,1,t))dx\\ &\le 0 \end{aligned} \tag{3.15}$$

Proof.

$$\frac{1}{2}\left(\frac{\rho_1}{2}\|\varphi_t\|_2^2+\frac{\rho_2}{2}\|\psi_t\|_2^2+\frac{\rho_1}{2}\|\omega_t\|_2^2+\frac{EI}{2}\|\psi_x\|_2^2+\frac{Gh}{2}\|\varphi_x+\psi+l\omega\|_2^2+\frac{Eh}{2}\|\omega_x-l\varphi\|_2^2\right)= \\ -\mu_1\int_0^1 \varphi_t g_1(\varphi_t)\ dx-\mu_2\int_0^1 \varphi_t(x,t)g_2(z_1(x,1,t))\ dx \\ -\widetilde{\mu_1}\int_0^1 \psi_t\widetilde{g_1}(\psi_t)\ dx-\widetilde{\mu_2}\int_0^1 \psi_t(x,t)\widetilde{g_2}(z_2(x,1,t))\ dx \\ -\widetilde{\widetilde{\mu_1}}\int_0^1 \omega_t\widetilde{\widetilde{g_1}}(\omega_t)\ dx-\widetilde{\widetilde{\mu_2}}\int_0^1 \omega_t(x,t)\widetilde{\widetilde{g_2}}(z_3(x,1,t))\ dx. \tag{3.16}$$

We multiply the second equation in (3.8) by $\xi_1 g_2(z_1(x,\rho,t))$, the Fourth equation in (3.8) by $\xi_2\widetilde{g_2}(z_2(x,\rho,t))$ and the Sixth equation in (3.8) by $\xi_3\widetilde{\widetilde{g_2}}(z_3(x,\rho,t))$ and integrate the result over $(0,1)\times(0,1)$, to obtain:

$$\begin{aligned}\xi_1\int_0^1\int_0^1 z_1' g_2(z(x,\rho,t))\,d\rho\,dx &= -\frac{\xi_1}{\tau_1}\int_0^1\int_0^1\frac{\partial}{\partial\rho}G_2(z(x,\rho,t))\,d\rho\,dx \\ &= -\frac{\xi_1}{\tau_1}\int_0^1 (G_2(z(x,1,t))-G_2(z(x,0,t)))\,dx.\end{aligned} \tag{3.17}$$

Then

$$\xi_1\frac{d}{dt}\int_0^1\int_0^1 G_2(z_1(x,\rho,t))\,d\rho\,dx=-\frac{\xi_1}{\tau_1}\int_0^1 G_2(z_1(x,1,t))\,dx+\frac{\xi_1}{\tau_1}\int_0^1 G_2(\varphi_t)\,dx. \tag{3.18}$$

$$\xi_2\frac{d}{dt}\int_0^1\int_0^1 \widetilde{G_2}(z_2(x,\rho,t))\,d\rho\,dx=-\frac{\xi_2}{\tau_2}\int_0^1 \widetilde{G_2}(z_2(x,1,t))\,dx+\frac{\xi_2}{\tau_2}\int_0^1 \widetilde{G_2}(\psi_t)\,dx. \tag{3.19}$$

$$\xi_3\frac{d}{dt}\int_0^1\int_0^1 \widetilde{\widetilde{G_2}}(z_3(x,\rho,t))\,d\rho\,dx=-\frac{\xi_3}{\tau_3}\int_0^1 \widetilde{\widetilde{G_2}}(z_3(x,1,t))\,dx+\frac{\xi_3}{\tau_3}\int_0^1 \widetilde{\widetilde{G_2}}(\omega_t)\,dx. \tag{3.20}$$

From (3.16), (3.18),(3.19),(3.20) and using Young inequality we get

$$\begin{aligned}E'(t) &= -\left(\mu_1-\frac{\xi_1\alpha_2}{\tau_1}\right)\int_0^1\varphi_t g_1(\varphi_t)\,dx-\frac{\xi_1}{\tau_1}\int_0^1 G_2(z_1(x,1,t))\,dx-\mu_2\int_0^1\varphi_t(t)g_2(z_1(x,1,t))dx \\ &\quad -\left(\widetilde{\mu_1}-\frac{\xi_2\widetilde{\alpha_2}}{\tau_2}\right)\int_0^1\psi_t\widetilde{g_1}(\psi_t)dx-\frac{\xi_2}{\tau_2}\int_0^1\widetilde{G_2}(z_2(x,1,t)),dx-\widetilde{\mu_2}\int_0^1\psi_t(t)\widetilde{g_2}(z_2(x,1,t))dx \\ &\quad -\left(\widetilde{\widetilde{\mu_1}}-\frac{\xi_3\widetilde{\widetilde{\alpha_2}}}{\tau_3}\right)\int_0^1\omega_t\widetilde{\widetilde{g_1}}(\omega_t),dx-\frac{\xi_3}{\tau_3}\int_0^1\widetilde{\widetilde{G_2}}(z_3(x,1,t))dx-\widetilde{\widetilde{\mu_2}}\int_0^1\omega_t(t)\widetilde{\widetilde{g_2}}(z_3(x,1,t))dx.\end{aligned} \tag{3.21}$$

Let us denote $G_2^*, (\widetilde{G}_2^*, \widetilde{\widetilde{G}}_2^*)$ to be the conjugate function of the convex function $G_2, (\widetilde{G}_2, \widetilde{\widetilde{G}}_2)$, i.e.,
$G_2^*(s) = \sup_{t\in\mathbb{R}^+}(st - G_2(t)), (\widetilde{G}_2^*(s) = \sup_{t\in\mathbb{R}^+}(st - \widetilde{G}_2(t)), \widetilde{\widetilde{G}}_2^*(s) = \sup_{t\in\mathbb{R}^+}(st - \widetilde{\widetilde{G}}_2(t)))$.
Then $G_2^*, (\widetilde{G}_2^*, \widetilde{\widetilde{G}}_2^*)$ is the Legendre transform of $G_2, (\widetilde{G}_2, \widetilde{\widetilde{G}}_2)$, which is given by (see Arnold **[3]**, p. 61-62, and Lasiecka **[18]**)

$$\begin{array}{l} G_2^*(s) = s(G_2')^{-1}(s) - G_2[(G_2')^{-1}(s)], \quad \forall s \geq 0, \\ \widetilde{G}_2^*(s) = s(\widetilde{G}_2')^{-1}(s) - \widetilde{G}_2[(\widetilde{G}_2')^{-1}(s)], \quad \forall s \geq 0, \\ \widetilde{\widetilde{G}}_2^*(s) = s(\widetilde{\widetilde{G}}_2')^{-1}(s) - \widetilde{\widetilde{G}}_2[(\widetilde{\widetilde{G}}_2')^{-1}(s)], \quad \forall s \geq 0. \end{array} \tag{3.22}$$

and satisfies the following inequality

$$\begin{array}{l} st \leq G_2^*(s) + G_2(t), \quad \forall s, t \geq 0, \\ st \leq \widetilde{G}_2^*(s) + \widetilde{G}_2(t), \quad \forall s, t \geq 0, \\ st \leq \widetilde{\widetilde{G}}_2^*(s) + \widetilde{\widetilde{G}}_2(t), \quad \forall s, t \geq 0. \end{array} \tag{3.23}$$

Then, from the definition of $G_2, (\widetilde{G}_2, \widetilde{\widetilde{G}}_2)$, we get

$$\begin{array}{l} G_2^*(s) = s g_2^{-1}(s) - G_2(g_2^{-1}(s)), \\ \widetilde{G}_2^*(s) = s \widetilde{g}_2^{-1}(s) - \widetilde{G}_2(\widetilde{g}_2^{-1}(s)), \\ \widetilde{\widetilde{G}}_2^*(s) = s \widetilde{\widetilde{g}}_2^{-1}(s) - \widetilde{\widetilde{G}}_2(\widetilde{\widetilde{g}}_2^{-1}(s)). \end{array}$$

Hence

$$\begin{array}{rcl} G_2^*(g_2(z_1(x,1,t))) & = & z_1(x,1,t) g_2(z_1(x,1,t)) - G_2(z_1(x,1,t)) \\ & \leq & (1-\alpha_1) z_1(x,1,t) g_2(z_1(x,1,t)). \\ \widetilde{G}_2^*(\widetilde{g}_2(z_2(x,1,t))) & = & z_2(x,1,t) \widetilde{g}_2(z_2(x,1,t)) - \widetilde{G}_2(z_2(x,1,t)) \\ & \leq & (1-\widetilde{\alpha}_1) z_2(x,1,t) \widetilde{g}_2(z_2(x,1,t)). \\ \widetilde{\widetilde{G}}_2^*(\widetilde{\widetilde{g}}_2(z_3(x,1,t))) & = & z_3(x,1,t) \widetilde{\widetilde{g}}_2(z_3(x,1,t)) - \widetilde{\widetilde{G}}_2(z_3(x,1,t)) \\ & \leq & (1-\widetilde{\widetilde{\alpha}}_1) z_3(x,1,t) \widetilde{\widetilde{g}}_2(z_3(x,1,t)). \end{array} \tag{3.24}$$

Making use of (3.21), (3.23) and (3.24), we have

$$
\begin{aligned}
E'(t) \leq & -\left(\mu_1 - \frac{\xi_1\alpha_2}{\tau_1}\right)\int_0^1 \varphi_t g_1(\varphi_t)\ dx - \frac{\xi_1}{\tau_1}\int_0^1 G_2(z_1(x,1,t))dx \\
& +\mu_2\int_0^1 \left(G_2(\varphi_t) + G_2^*(g_2(z_1(x,1,t)))\right) dx \\
& -\left(\widetilde{\mu_1} - \frac{\xi_2\widetilde{\alpha_2}}{\tau_2}\right)\int_0^1 \psi_t\widetilde{g_1}(\psi_t)dx - \frac{\xi_2}{\tau_2}\int_0^1 \widetilde{G_2}(z_2(x,1,t))dx \\
& +\widetilde{\mu_2}\int_0^1 \left(\widetilde{G_2}(\psi_t) + \widetilde{G_2}^*(\widetilde{g_2}(z_2(x,1,t)))\right) dx \\
& -\left(\widetilde{\widetilde{\mu_1}} - \frac{\xi_3\widetilde{\widetilde{\alpha_2}}}{\tau_3}\right)\int_0^1 \omega_t\widetilde{\widetilde{g_1}}(\omega_t)dx - \frac{\xi_3}{\tau_3}\int_0^1 \widetilde{\widetilde{G_2}}(z_3(x,1,t))dx \\
& +\widetilde{\widetilde{\mu_2}}\int_0^1 \left(\widetilde{\widetilde{G_2}}(\omega_t) + \widetilde{\widetilde{G_2}}^*(\widetilde{\widetilde{g_2}}(z_3(x,1,t)))\right) dx \\
\leq & -\left(\mu_1 - \frac{\xi_1\alpha_2}{\tau_1} - \mu_2\alpha_2\right)\int_0^1 \varphi_t g_1(\varphi_t)dx - \frac{\xi_1}{\tau_1}\int_0^1 G_2(z_1(x,1,t))dx \\
& +\mu_2\int_0^1 G_2^*(g_2(z_1(x,1,t)))dx \\
& -\left(\widetilde{\mu_1} - \frac{\xi_2\widetilde{\alpha_2}}{\tau_2} - \widetilde{\mu_2}\widetilde{\alpha_2}\right)\int_0^1 \psi_t\widetilde{g_1}(\psi_t)dx - \frac{\xi_2}{\tau_2}\int_0^1 \widetilde{G_2}(z_2(x,1,t))dx \\
& +\widetilde{\mu_2}\int_0^1 \widetilde{G_2}^*(\widetilde{g_2}(z_2(x,1,t)))dx \\
& -\left(\widetilde{\widetilde{\mu_1}} - \frac{\xi_3\widetilde{\widetilde{\alpha_2}}}{\tau_3} - \widetilde{\widetilde{\mu_2}}\widetilde{\widetilde{\alpha_2}}\right)\int_0^1 \omega_t\widetilde{\widetilde{g_1}}(\omega_t)dx - \frac{\xi_3}{\tau_3}\int_0^1 \widetilde{\widetilde{G_2}}(z_3(x,1,t))dx \\
& +\widetilde{\widetilde{\mu_2}}\int_0^1 \widetilde{\widetilde{G_2}}^*(\widetilde{\widetilde{g_2}}(z_3(x,1,t)))dx.
\end{aligned} \tag{3.25}
$$

Using (3.4) and (3.9), we obtain

$$
\begin{aligned}
E'(t) \leq & -\left(\mu_1 - \frac{\xi_1\alpha_2}{\tau_1} - \mu_2\alpha_2\right)\int_0^1 \varphi_t g_1(\varphi_t)dx \\
& -\left(\frac{\xi_1}{\tau_1}\alpha_1 - \mu_2(1-\alpha_1)\right)\int_0^1 z_1(x,1,t)g_2(z_1(x,1,t))dx \\
& -\left(\widetilde{\mu_1} - \frac{\xi_2\widetilde{\alpha_2}}{\tau_2} - \widetilde{\mu_2}\widetilde{\alpha_2}\right)\int_0^1 \psi_t\widetilde{g_1}(\psi_t)dx \\
& -\left(\frac{\xi_2}{\tau_2}\widetilde{\alpha_1} - \widetilde{\mu_2}(1-\widetilde{\alpha_1})\right)\int_0^1 z_2(x,1,t)\widetilde{g_2}(z_2(x,1,t))dx \\
& -\left(\widetilde{\widetilde{\mu_1}} - \frac{\xi_3\widetilde{\widetilde{\alpha_2}}}{\tau_3} - \widetilde{\widetilde{\mu_2}}\widetilde{\widetilde{\alpha_2}}\right)\int_0^1 \omega_t\widetilde{\widetilde{g_1}}(\omega_t)dx \\
& -\left(\frac{\xi_3}{\tau_3}\widetilde{\alpha_1} - \widetilde{\mu_2}(1-\widetilde{\widetilde{\alpha_1}})\right)\int_0^1 z_3(x,1,t)\widetilde{\widetilde{g_2}}(z_3(x,1,t))dx \\
\leq & \ 0.
\end{aligned}
$$

3.3 Global Existence

We are now ready to prove Theorem 3.1 in the next two sections.

Throughout this section we assume $\varphi_0, \psi_0 \in H^2 \cap H_0^1(0,1), \varphi_1, \psi_1 \in H_0^1(0,1)$ and $f_0 \in H_0^1((0,1); H^1(0,1))$.

We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \dots, w_k\}$ where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2 \cap H_0^1$.

Now, we define for $1 \leq j \leq k$ the sequence $\phi_j(x, \rho)$ as follows:

$$\phi_j(x, 0) = w_j.$$

Then, we may extend $\phi_j(x, 0)$ by $\phi_j(x, \rho)$ over $L^2((0,1) \times (0,1))$ and denote Z_k the space generated by $\{\phi_1, \phi_2, \dots, \phi_k\}$.

We construct approximate solutions $(\varphi_k, \psi_k, \omega_k, z_{1k}, z_{2k}, z_{3k}), k = 1, 2, 3, \dots$, in the form

$$\begin{aligned}
\varphi_k(t) &= \sum_{j=1}^{k} g_{jk} w_j, \quad z_{1k}(t) = \sum_{j=1}^{k} h_{jk} \phi_j, \\
\psi_k(t) &= \sum_{j=1}^{k} \widetilde{g}_{jk} w_j, \quad z_{2k}(t) = \sum_{j=1}^{k} \widetilde{h}_{jk} \phi_j, \\
\omega_k(t) &= \sum_{j=1}^{k} \widetilde{\widetilde{g}}_{jk} w_j, \quad z_{3k}(t) = \sum_{j=1}^{k} \widetilde{\widetilde{h}}_{jk} \phi_j.
\end{aligned}$$

where $g_{jk}, \widetilde{g}_{jk}, \widetilde{\widetilde{g}}_{jk}, h_{jk}, \widetilde{h}_{jk}$ and $\widetilde{\widetilde{h}}_{jk}$, $j = 1, 2, \dots, k$, are determined by the following ordinary differential equations:

$$\begin{cases}
\rho_1(\varphi_k''(t), w_j) + Gh(\varphi_{kx}(t), w_{jx}) + Gh(\psi + l\omega)(t), w_j) + lEh(\omega + l\varphi)(t), w_j) \\
+\mu_1(g_1(\varphi_k'), w_j) + \mu_2(g_2(z_1k(.,1)), w_j) = 0 \quad 1 \leq j \leq k, \\
z_{1k}(x, 0, t) = \varphi_k'(x, t)
\end{cases} \tag{3.26}$$

$$\varphi_k(0) = \varphi_{0k} = \sum_{j=1}^{k} (\varphi_0, w_j) w_j \to \varphi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{3.27}$$

$$\varphi_k'(0) = \varphi_{1k} = \sum_{j=1}^{k} (\varphi_1, w_j) w_j \to \varphi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{3.28}$$

$$\begin{cases}
\rho_2(\psi_k''(t), w_j) + El(\psi_{kx}(t), w_{jx}) + Gh((\varphi_{kx} + \psi + l\omega)(t), w_j), \\
+\widetilde{\mu}_1(\widetilde{g}_1(\psi_k'), w_j) + \widetilde{\mu}_2(\widetilde{g}_2(z_{2k}(.,1)), w_j) = 0 \quad 1 \leq j \leq k, \\
z_{2k}(x, 0, t) = \psi_k'(x, t)
\end{cases} \tag{3.29}$$

$$\psi_k(0) = \psi_{0k} = \sum_{j=1}^{k} (\psi_0, w_j) w_j \to \psi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{3.30}$$

$$\psi_k'(0) = \psi_{1k} = \sum_{j=1}^{k} (\psi_1, w_j) w_j \to \psi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{3.31}$$

$$\begin{cases} \rho_1(\omega_k''(t), w_j) + Eh(\omega_{kx}(t), w_{jx}) + lEh(\varphi_{kx})(t), w_j) + lGh(\varphi_{kx} + \psi + l\omega)(t) w_j), \\ +\tilde{\tilde{\mu}}_1(\tilde{\tilde{g}}_1(\omega_k'), w_j) + \tilde{\tilde{\mu}}_2(\tilde{\tilde{g}}_2(z_3k(.,1)), w_j) = 0 \quad 1 \le j \le k, \\ z_{3k}(x,0,t) = \omega_k'(x,t) \end{cases} \tag{3.32}$$

$$\omega_k(0) = \omega_{0k} = \sum_{j=1}^{k} (\omega_0, w_j) w_j \to \omega_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{3.33}$$

$$\omega_k'(0) = \omega_{1k} = \sum_{j=1}^{k} (\omega_1, w_j) w_j \to \omega_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{3.34}$$

and

$$(\tau z_1 k t + z_1 k \rho, \phi_j) = 0, \quad 1 \le j \le k, \tag{3.35}$$

$$z_1 k(\rho, 0) = z_{01k} = \sum_{j=1}^{k} (f_1, \phi_j)\phi_j \to f_1 \text{ in } H_0^1((0,1); H^1(0,1)) \text{ as } k \to +\infty. \tag{3.36}$$

$$(\tau z_2 k t + z_2 k \rho, \phi_j) = 0, \quad 1 \le j \le k, \tag{3.37}$$

$$z_{2k}(\rho, 0) = z_{02k} = \sum_{j=1}^{k} (f_2, \phi_j)\phi_j \to f_2 \text{ in } H_0^1((0,1); H^1(0,1)) \text{ as } k \to +\infty. \tag{3.38}$$

$$(\tau z_{3kt} + z_3 k \rho, \phi_j) = 0, \quad 1 \le j \le k, \tag{3.39}$$

$$z_{3k}(\rho, 0) = z_{03k} = \sum_{j=1}^{k} (f_3, \phi_j)\phi_j \to f_3 \text{ in } H_0^1((0,1); H^1(0,1)) \text{ as } k \to +\infty. \tag{3.40}$$

By virtue of the theory of ordinary differential equations, the system (3.26)-(3.40) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \le +\infty$) by Zorn lemma since the nonlinear terms in (3.26, 3.29,3.32) are locally Lipschitz continuous. Note that $(\varphi_k(t), \psi_k(t), \omega_k(t))$ is from the class C^2.

In the next step we obtain a priori estimates for the solution, such that it can be extended outside $[0, T_k[$ to obtain one solution defined for all $t > 0$.

We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for $(\varphi_k, \psi_k, \omega_k, z_{1k}, z_{2k}, z_{3k})$.

The first estimate:

Since the sequences $\varphi_{0k}, \varphi_{1k}, \psi_{0k}, \psi_{1k}, \omega_{0k}, \omega_{1k}$ and z_{0k} converge, then standard calculations, using (3.26)-(3.40), similar to those used to derive (3.41), yield C independent of k such that

$$\begin{array}{c} E_k(t) + a_1 \int_0^t \int_0^1 \varphi_k' g_1(\varphi_k) dxds + a_2 \int_0^t \int_0^1 z_{1k}(x,1,t) g_2(z_{1k}(x,1,t)) dsdx \\ +b_1 \int_0^t \int_0^1 \psi_k' \widetilde{g_1}(\psi_k') dxds + b_2 \int_0^t \int_0^1 z_{2k}(x,1,t) \widetilde{g_2}(z_{2k}(x,1,t)) dxds \\ +c_1 \int_0^t \int_0^1 \omega_k' \widetilde{\widetilde{g_1}}(\omega_k') dxds + c_2 \int_0^t \int_0^1 z_{3k}(x,1,t) \widetilde{\widetilde{g_2}}(z_{3k}(x,1,t)) dxds \\ \leq E_k(0) \leq C \end{array} \tag{3.41}$$

where

$$\begin{array}{c} E_k(t) = \frac{1}{2}\left(\frac{\rho_1}{2}\|\varphi_k'\|_2^2 + \frac{\rho_2}{2}\|\psi_k'\|_2^2 + \frac{\rho_1}{2}\|\omega_k'\|_2^2 + \frac{EI}{2}\|\psi_x\|_2^2 + \frac{Gh}{2}\|\varphi_x + \psi + l\omega\|_2^2 + \frac{Eh}{2}\|\omega_x - l\varphi\|_2^2 \right) \\ +\xi_1 \int_0^1 \int_0^1 G_2(z_{1k}(x,\rho,t))\, d\rho\, dx + \xi_2 \int_0^1 \int_0^1 \widetilde{G}_2(z_{2k}(x,\rho,t))\, d\rho\, dx \\ +\xi_3 \int_0^1 \int_0^1 \widetilde{\widetilde{G}}_2(z_{3k}(x,\rho,t))\, d\rho\, dx. \end{array}$$

$$\begin{array}{ll} a_1 = \left(\mu_1 - \frac{\xi_1 \alpha_2}{\tau_1} - \mu_2 \alpha_2 \right), & a_2 = \left(\frac{\xi_1}{\tau_1} \alpha_1 - \mu_2 (1 - \alpha_1) \right), \\ b_1 = \left(\widetilde{\mu_1} - \frac{\xi_2 \widetilde{\alpha_2}}{\tau_2} - \widetilde{\mu_2}\widetilde{\alpha_2} \right), & b_2 = \left(\frac{\xi_2}{\tau_2} \widetilde{\alpha_1} - \widetilde{\mu_2}(1 - \widetilde{\alpha_1}) \right), \\ c_1 = \left(\widetilde{\widetilde{\mu_1}} - \frac{\xi_3 \widetilde{\widetilde{\alpha_2}}}{\tau_3} - \widetilde{\widetilde{\mu_2}}\widetilde{\widetilde{\alpha_2}} \right), & c_2 = \left(\frac{\xi_3}{\tau_3} \widetilde{\widetilde{\alpha_1}} - \widetilde{\widetilde{\mu_2}}(1 - \widetilde{\widetilde{\alpha_1}}) \right) \end{array} \tag{3.42}$$

for some C independent of k. These estimates imply that the solution $(\varphi_k, \psi_k, \omega_k, z_{1k}, z_{2k}, z_{3k})$ exists globally in $[0, +\infty[$.
Estimate (3.41) yields

$$\varphi_k, \psi_k, \omega_k \text{ are bounded in } L^\infty_{loc}(0, \infty; H^1_0(0,1)) \tag{3.43}$$

$$\varphi_k', \psi_k', \omega_k' \text{ are bounded in } L^\infty_{loc}(0, \infty; L^2(0,1)) \tag{3.44}$$

$$\varphi_k' g_1(\varphi_k'), \psi_k' \widetilde{g_1}(\psi_k'), \omega_k' \widetilde{\widetilde{g_1}}(\omega_k') \text{ is bounded in } L^1((0,1) \times (0,T)) \tag{3.45}$$

$$G_2(z_{1k}(x,\rho,t)), \widetilde{G}_2(z_{2k}(x,\rho,t)), \widetilde{\widetilde{G}}_2(z_{3k}(x,\rho,t)) \text{ is bounded in } L^\infty_{loc}(0, \infty; L^1((0,1) \times (0,1))) \tag{3.46}$$

and

$$z_{1k}(x,1,t)g_2(z_k(x,1,t)), \quad z_{2k}(x,1,t)\widetilde{g}_2(z_{2k}(x,1,t)),$$
$$z_{3k}(x,1,t)\widetilde{\widetilde{g}}_2(z_{3k}(x,1,t)) \text{ is bounded in } L^1((0,1)\times(0,T)) \tag{3.47}$$

The second estimate:

First, we estimate $\varphi_k''(0)$, $\psi_k''(0)$ and $\omega_k''(0)$. Testing (3.26) by $g_{jk}''(t)$, (3.29) by $\widetilde{g}_{jk}''(t)$, (3.32) by $\widetilde{\widetilde{g}}_{jk}''(t)$ and choosing $t=0$ we obtain

$$\rho_1\|\varphi_k''(0)\|_2 \le Gh\left(\|\varphi_{0kx}\|_2 + \|\psi_{0k}\|_2 + l\|\omega_{0k}\|_2\right)$$
$$+lEh\left(\|\omega_{0k}\|_2 + l\|\varphi_{0k}\|_2\right) + \mu_1\|g_1(\varphi_{1k})\|_2 + \mu_2\|g_2(z_{10k})\|_2$$

$$\rho_2\|\psi_k''(0)\|_2 \le El\|\psi_{0kxx}\|_2 + Gh\left(\|\varphi_{0k}\|_2 + \|\psi_{0k}\|_2 + l\|\omega_{0k}\|_2\right)$$
$$+\widetilde{\mu}_1\|\widetilde{g}_1(\psi_{1k})\|_2 + \widetilde{\mu}_2\|\widetilde{g}_2(z_{20k})\|_2$$

and

$$\rho_1\|\omega_k''(0)\|_2 \le Eh(\|\omega_{0kxx}\|_2 + lEh\|\varphi_{0k}\|_2 + lGh(\|\varphi_{0k}\|_2 + \|\psi_{0k}\|_2 + l\|\omega_{0k}\|_2)$$
$$+\widetilde{\widetilde{\mu}}_1\|\widetilde{\widetilde{g}}_1(\omega_{1k})\|_2 + \widetilde{\widetilde{\mu}}_2\|\widetilde{\widetilde{g}}_2(z_{03k})\|_2$$

Since $g_1(\varphi_{1k}), g_2(z_{10k}),\quad \widetilde{g}_1(\psi_{1k}), \widetilde{g}_2(z_{20k}),\quad \widetilde{\widetilde{g}}_1(\omega_{1k}), \widetilde{\widetilde{g}}_2(z_{30k})$ are bounded in $L^2(0,1)$ by (**H1**), (3.27), (3.28) and (3.36) yield

$$\|\varphi_k''(0)\|_2 \le C.$$

(3.30), (3.31) and (3.38) yield

$$\|\psi_k''(0)\|_2 \le C.$$

(3.33), (3.34) and (3.40) yield

$$\|\omega_k''(0)\|_2 \le C.$$

Differentiating (3.26), (3.29) and (3.32) with respect to t, we get

$$\rho_1\varphi_k'''(t) + Gh\varphi_{kxx}'(t) + Gh\psi_{kx}'(t) + lGh\omega_k'(t)$$
$$+lEh\omega_k'(t) + l^2Eh\varphi_k'(t) + \mu_1\varphi_k''(g_1'(\varphi_k')$$
$$+\mu_2 z_{1k}'(x,1,t)(g_2'(z_{1k}(x,1,t)) = 0 \tag{3.48}$$

$$\rho_2\psi_k'''(t) + El\psi_{kxx}'(t) + Gh\varphi_{kx}'(t) + Gh\psi_k'(t) + lGh\omega_k'(t)$$
$$+\widetilde{\mu}_1\psi_k''\widetilde{g'}_1(\psi_k') + \widetilde{\mu}_2 z_{2k}'(x,1,t)\widetilde{g'}_2(z_{2k}(x,1,t)) = 0 \tag{3.49}$$

and

$$\begin{aligned}\rho_1\omega_k'''(t)+Eh\omega_{kxx}'(t)+lEh\varphi_{kx}'(t)+lGh\varphi_{kx}'+lGh\psi_k'(t)+l^2Gh\omega_k'(t)\\ +\widetilde{\widetilde{\mu}}_1\omega_k''(t)\widetilde{\widetilde{g}}'_1(\omega_k')+\widetilde{\widetilde{\mu}}_2 z_{3k}'(x,1,t)\widetilde{\widetilde{g}}'_2(z_{3k}(x,1,t))=0\end{aligned}\tag{3.50}$$

Multiplying (3.48) by $g_{jk}'(t)$, (3.49) by $\widetilde{g}_{jk}'(t)$, and (3.50) by $\widetilde{\widetilde{g}}'_{jk}(t)$ summing over j from 1 to k, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi_k''(t)\|_2^2+Gh\|\varphi_{kx}'(t)\|_2^2\right)+Gh\int_0^1(\psi_{kx}'(t)+l\omega_k'(t))\varphi_{jk}''(t)dx\\ +lEh\int_0^1(\omega_k'(t)+l\varphi_k'(t)\varphi_{jk}''(t)dx+\mu_1\int_0^1\varphi_k''^2(t)g_1'(\varphi_k')dx\\ +\mu_2\int_0^1\varphi_k''(t)z_{1k}'(x,1,t)(g_2'(z_{1k}(x,1,t))dx=0.\end{aligned}\tag{3.51}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi_k''(t)\|_2^2+El\|\psi_{kx}'(t)\|_2^2\right)+Gh\int_0^1\left((\varphi_{kx}'(t)+\psi_k'(t)+l\omega_k'(t)\right)\psi_{jk}''(t)dx\\ +\widetilde{\mu}_1\int_0^1\psi_k''^2(t)\widetilde{g}'_1(\psi_k')dx+\widetilde{\mu}_2\int_0^1\psi_k''(t)z_{2k}'(x,1,t)\widetilde{g}'_2(z_{2k}(x,1,t))dx=0.\end{aligned}\tag{3.52}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\omega_k''(t)\|_2^2+Eh\|\omega_{kx}'(t)\|_2^2\right)+lEh\int_0^1\omega_k''(t)\varphi_{kx}'(t)dx\\ +Gh\int_0^1\omega_k''(t)\left(l\varphi_{kx}'+l\psi_k'(t)+l\omega_k'(t)\right)+\widetilde{\widetilde{\mu}}_1\int_0^1\omega_k''^2(t)\widetilde{\widetilde{g}}'_1(\omega_k')dx\\ +\widetilde{\widetilde{\mu}}_2\int_0^1\omega_k''(t)z_{3k}'(x,1,t)\widetilde{\widetilde{g}}'_2(z_{3k}(x,1,t))dx=0\end{aligned}\tag{3.53}$$

Differentiating (3.35), (3.37) and (3.39) with respect to t, we get

$$(\tau z_{1k}''(t)+\frac{\partial}{\partial\rho}z_{1k}',\phi_j)=0.\tag{3.54}$$

$$(\tau z_{2k}''(t)+\frac{\partial}{\partial\rho}z_{2k}',\phi_j)=0.\tag{3.55}$$

and

$$(\tau z_{3k}''(t)+\frac{\partial}{\partial\rho}z_{3k}',\phi_j)=0.\tag{3.56}$$

Multiplying (3.54), (3.55) and (3.56) by $h_{jk}'(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\tau\frac{d}{dt}\|z_{1k}'(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_{1k}'(t)\|_2^2=0.\tag{3.57}$$

$$\frac{1}{2}\tau\frac{d}{dt}\|z_{2k}'(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_{2k}'(t)\|_2^2=0.\tag{3.58}$$

and

$$\frac{1}{2}\tau\frac{d}{dt}\|z_{3k}'(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_{3k}'(t)\|_2^2=0.\tag{3.59}$$

Taking the sum of (3.51), (3.52),(3.53), (3.57),(3.58) and (3.59), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi_k''(t)\|_2^2+\rho_2\|\psi_k''(t)\|_2^2+\rho_1\|\omega_k''(t)\|_2^2+Gh\|\varphi_{kx}'(t)\|_2^2+El\|\psi_{kx}'(t)\|_2^2+Eh\|\omega_{kx}'(t)\|_2^2\right)\\
&+\frac{1}{2}\frac{d}{dt}lEh\left(\|\omega_k'(t)+l\varphi_k'(t)\|_2^2+Gh\|\varphi_{kx}'(t)+\psi_k'(t)+l\omega_k'(t)\|_2^2\right)\\
&+\frac{1}{2}\tau\frac{d}{dt}\left(\|z_{1k}'(x,\rho,t)\|_{L^2((0,1)\times(0,1))}^2+\|z_{2k}'(x,\rho,t)\|_{L^2((0,1)\times(0,1))}^2+\|z_{3k}'(x,\rho,t)\|_{L^2((0,1)\times(0,1))}^2\right)\\
&+\mu_1\int_0^1\varphi_k''^2(t)g_1'(\varphi_k')dx+\widetilde{\mu}_1\int_0^1\psi_k''^2(t)\widetilde{g}'_1(\psi_k')dx+\widetilde{\widetilde{\mu}}_1\int_0^1\omega_k''^2(t)\widetilde{\widetilde{g}}'_1(\omega_k')dx\\
&+\frac{1}{2}\int_0^1\left(|z_{1k}'(x,1,t)|^2+|z_{2k}'(x,1,t)|^2+|z_{3k}'(x,1,t)|^2\right)dx\\
&=-\mu_2\int_0^1\varphi_k''(t)z_{1k}'(x,1,t)g_2'(z_{1k}(x,1,t))dx-\widetilde{\mu}_2\int_0^1\psi_k''(t)z_{2k}'(x,1,t)\widetilde{g}'_2(z_{2k}(x,1,t))dx\\
&-\widetilde{\widetilde{\mu}}_2\int_0^1\omega_k''(t)z_{3k}'(x,1,t)\widetilde{\widetilde{g}}'_2(z_{3k}(x,1,t))dx+\frac{1}{2}\left(\|\varphi_k''(t)\|_2^2+\|\psi_k''(t)\|_2^2+\|\omega_k''(t)\|_2^2\right)
\end{aligned}
$$

Using (3.3), Cauchy-Schwarz and Young's inequalities, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi_k''(t)\|_2^2+\rho_2\|\psi_k''(t)\|_2^2+\rho_1\|\omega_k''(t)\|_2^2+Gh\|\varphi_{kx}'(t)\|_2^2+El\|\psi_{kx}'(t)\|_2^2+Eh\|\omega_{kx}'(t)\|_2^2\right)\\
&+\frac{1}{2}lEh\frac{d}{dt}\left(\|\omega_k'(t)+l\varphi_k'(t)\|_2^2+Gh\|\varphi_{kx}'(t)+\psi_k'(t)+l\omega_k'(t)\|_2^2\right)\\
&+\frac{1}{2}\tau\frac{d}{dt}\left(\|z_{1k}'(x,\rho,t)\|_{L^2((0,1)\times(0,1))}^2+\|z_{2k}'(x,\rho,t)\|_{L^2((0,1)\times(0,1))}^2+\|z_{3k}'(x,\rho,t)\|_{L^2((0,1)\times(0,1))}^2\right)\\
&+\mu_1\int_0^1\varphi_k''^2(t)g_1'(\varphi_k')dx+\widetilde{\mu}_1\int_0^1\psi_k''^2(t)\widetilde{g}'_1(\psi_k')dx+\widetilde{\widetilde{\mu}}_1\int_0^1\omega_k''^2(t)\widetilde{\widetilde{g}}'_1(\omega_k')dx\\
&+c\int_0^1\left(|z_{1k}'(x,1,t)|^2+|z_{2k}'(x,1,t)|^2+|z_{3k}'(x,1,t)|^2\right)dx\\
&\leq c'\left(\|\varphi_k''(t)\|_2^2+\|\psi_k''(t)\|_2^2+\|\omega_k''(t)\|_2^2\right).
\end{aligned}
$$

Integrating the last inequality over $(0,t)$ and using Gronwall's Lemma, we obtain

$$\begin{aligned}
&\rho_1\|\varphi_k''(t)\|_2^2+\rho_2\|\psi_k''(t)\|_2^2+\rho_1\|\omega_k''(t)\|_2^2+Gh\|\varphi_{kx}'(t)\|_2^2+El\|\psi_{kx}'(t)\|_2^2+Eh\|\omega_{kx}'(t)\|_2^2\\
&\quad+lEh\left(\|\omega_k'(t)+l\varphi_k'(t)\|_2^2+Gh\|\varphi_{kx}'(t)+\psi_k'(t)+l\omega_k'(t)\|_2^2\right)\\
&\quad+\tau\left(\|z_{1k}'(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z_{2k}'(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z_{3k}'(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\right)\\
&\quad\le e^{cT}\left(\rho_1\|\varphi_k''(0)\|_2^2+\rho_2\|\psi_k''(0)\|_2^2+\rho_1\|\omega_k''(0)\|_2^2+Gh\|\varphi_{kx}'(0)\|_2^2+El\|\psi_{kx}'(0)\|_2^2+Eh\|\omega_{kx}'(0)\|_2^2\right)\\
&\quad+e^{cT}lEh\left(\|\omega_k'(0)+l\varphi_k'(0)\|_2^2+Gh\|\varphi_{kx}'(0)+\psi_k'(0)+l\omega_k'(0)\|_2^2\right)\\
&\quad+e^{cT}\tau\left(\|z_{1k}'(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}+\|z_{2k}'(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}+\|z_{3k}'(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}\right)
\end{aligned}$$

.

for all $t\in\mathbb{R}_+$, therefore, we conclude that

$$\varphi_k'',\psi_k'',\omega_k'' \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2) \tag{3.60}$$

$$\varphi_k',\psi_k',\omega_k' \text{ is bounded in } L^\infty_{loc}(0,\infty;H_0^1) \tag{3.61}$$

$$z_{1k}',z_{2k}',z_{3k}' \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2((0,1)\times(0,1))) \tag{3.62}$$

The third estimate. Replacing w_j by $-w_{jxx}$ in (3.26), (3.29) and (3.32), multiplying the result by $g_{jk}'(t)$, $\widetilde{g}_{jk}'(t)$ and $\widetilde{\widetilde{g}}_{jk}'(t)$, summing over j from 1 to k, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi_{kx}'(t)\|_2^2+Gh\|\varphi_{kxx}(t)\|_2^2\right)+Gh\int_0^1(\psi_x(t)+l\omega(t))\varphi_{kxx}'(t)dx\\
&\quad+lEh\int_0^1(\omega(t)+l\varphi(t)\varphi_{kx}'(t)dx+\mu_1\int_0^1\varphi_{kx}'^2(t)g_1'(\varphi_k')dx\\
&\quad+\mu_2\int_0^1\varphi_{kx}'(t)z_{1kx}(x,1,t)(g_2'(z_{1k}(x,1,t))dx=0.
\end{aligned}\tag{3.63}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi_{kx}'(t)\|_2^2+El\|\psi_{kxx}(t)\|_2^2\right)++Gh\int_0^1\left(\varphi_x+\psi+l\omega\right)\psi_{kxx}'(t)dx\\
&+\widetilde{\mu}_1\int_0^1\psi_{kx}'^2(t)\widetilde{g}'_1(\psi_k')dx+\widetilde{\mu}_2\int_0^1\psi_{kx}'(t)z_{2kx}(x,1,t)\widetilde{g}'_2(z_{2k}(x,1,t))dx=0.
\end{aligned}\tag{3.64}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\omega_{kx}'(t)\|_2^2+Eh\|\omega_{kxx}(t)\|_2^2\right)+lEh\int_0^1\omega_{kx}'(t)\varphi_{kxx}(t)dx\\
&\quad+Gh\int_0^1\left(\varphi_x+\psi+l\omega\right)\psi_{kxx}'(t)dx+\widetilde{\widetilde{\mu}}_1\int_0^1\omega_{kx}'^2(t)\widetilde{\widetilde{g}}'_1(\omega_k')dx\\
&\quad+\widetilde{\widetilde{\mu}}_2\int_0^1\omega_{kx}'(t)z_{3kx}(x,1,t)\widetilde{\widetilde{g}}'_2(z_{3k}(x,1,t))dx=0
\end{aligned}\tag{3.65}$$

Replacing ϕ_j by $-\phi_{jxx}$ in (3.35), (3.37), and (3.39) multiplying the resulting equation by $h_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\tau\frac{d}{dt}\|z_{1kx}(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_{1kx}(t)\|_2^2=0. \tag{3.66}$$

$$\frac{1}{2}\tau\frac{d}{dt}\|z_{2kx}(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_{2kx}(t)\|_2^2=0. \tag{3.67}$$

and

$$\frac{1}{2}\tau\frac{d}{dt}\|z_{3kx}(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_{3kx}(t)\|_2^2=0. \tag{3.68}$$

From (3.63), (3.64), (3.65), (3.66), (3.67) and (3.68), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'_{kx}(t)\|_2^2+\rho_1\|\omega'_{kx}(t)\|_2^2+Gh\|\varphi_{kxx}(t)\|_2^2+El\|\psi_{kxx}(t)\|_2^2+Eh\|\omega_{kxx}(t)\|_2^2\right)\\
&+\frac{1}{2}lEh\frac{d}{dt}\left(\|\omega_{kx}(t)+l\varphi_{kx}(t)\|_2^2+Gh\|\varphi_{kxx}(t)+\psi_{kx}(t)+l\omega_{kx}(t)\|_2^2\right)\\
&+\frac{1}{2}\tau\frac{d}{dt}\left(\|z_{1kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z_{2kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z_{3kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\right)\\
&+\mu_1\int_0^1|\varphi'_{kx}(t)|^2g'_1(\varphi'_k)dx+\widetilde{\mu}_1\int_0^1|\psi'_{kx}(t)|^2\widetilde{g}'_1(\psi'_k)dx+\widetilde{\widetilde{\mu}}_1\int_0^1|\omega'_{kx}(t)|^2\widetilde{\widetilde{g}}'_1(\omega'_k)dx\\
&+\frac{1}{2}\int_0^1\left(|z_{1kx}(x,1,t)|^2+|z_{2kx}(x,1,t)|^2+|z_{3kx}(x,1,t)|^2\right)dx\\
&=-\mu_2\int_0^1\varphi'_{kx}(t)z_{1kx}(x,1,t)g'_2(z_{1k}(x,1,t))dx-\widetilde{\mu}_2\int_0^1\psi'_{kx}(t)z_{2kx}(x,1,t)\widetilde{g}'_2(z_{2k}(x,1,t))dx\\
&-\widetilde{\widetilde{\mu}}_2\int_0^1\omega'_{kx}(t)z_{3kx}(x,1,t)\widetilde{\widetilde{g}}'_2(z_{3k}(x,1,t))dx+\frac{1}{2}\left(\|\varphi'_{kx}(t)\|_2^2+\|\psi'_{kx}(t)\|_2^2+\|\omega'_{kx}(t)\|_2^2\right).
\end{aligned}$$

Using (3.3), Cauchy-Schwartz and Young's inequalities, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'_{kx}(t)\|_2^2+\rho_1\|\omega'_{kx}(t)\|_2^2+Gh\|\varphi_{kxx}(t)\|_2^2+El\|\psi_{kxx}(t)\|_2^2+Eh\|\omega_{kxx}(t)\|_2^2\right)\\
&+\frac{1}{2}lEh\frac{d}{dt}\left(\|\omega_{kx}(t)+l\varphi_{kx}(t)\|_2^2+Gh\|\varphi_{kxx}(t)+\psi_{kx}(t)+l\omega_{kx}(t)\|_2^2\right)\\
&+\frac{1}{2}\tau\frac{d}{dt}\left(\|z_{1kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z_{2kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z_{3kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\right)\\
&+\mu_1\int_0^1|\varphi'_{kx}(t)|^2g'_1(\varphi'_k)dx+\widetilde{\mu}_1\int_0^1|\psi'_{kx}(t)|^2\widetilde{g}'_1(\psi'_k)dx+\widetilde{\widetilde{\mu}}_1\int_0^1|\omega'_{kx}(t)|^2\widetilde{\widetilde{g}}'_1(\omega'_k)dx\\
&+c\int_0^1\left(|z_{1kx}(x,1,t)|^2+|z_{2kx}(x,1,t)|^2+|z_{3kx}(x,1,t)|^2\right)dx\\
&\leq c'\left(\|\varphi'_{kx}(t)\|_2^2+\|\psi'_{kx}(t)\|_2^2+\|\omega'_{kx}(t)\|_2^2\right).
\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's Lemma, we have

$$\begin{aligned}
&\left(\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'_{kx}(t)\|_2^2+\rho_1\|\omega'_{kx}(t)\|_2^2+Gh\|\varphi_{kxx}(t)\|_2^2+El\|\psi_{kxx}(t)\|_2^2+Eh\|\omega_{kxx}(t)\|_2^2\right)\\
&\quad +lEh\left(\|\omega_{kx}(t)+l\varphi_{kx}(t)\|_2^2+Gh\|\varphi_{kxx}(t)+\psi_{kx}(t)+l\omega_{kx}(t)\|_2^2\right)\\
&\quad +\tau\left(\|z_{1kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z_{2kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z_{3kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\right)\\
&\quad \le \left(\rho_1\|\varphi'_{kx}(0)\|_2^2+\rho_2\|\psi'_{kx}(0)\|_2^2+\rho_1\|\omega'_{kx}(0)\|_2^2+Gh\|\varphi_{kxx}(0)\|_2^2+El\|\psi_{kxx}(0)\|_2^2+Eh\|\omega_{kxx}(0)\|_2^2\right)\\
&\quad +lEh\left(\|\omega_{kx}(0)+l\varphi_{kx}(0)\|_2^2+Gh\|\varphi_{kxx}(0)+\psi_{kx}(0)+l\omega_{kx}(0)\|_2^2\right)\\
&\quad +\tau\left(\|z_{1kx}(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}+\|z_{2kx}(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}+\|z_{3kx}(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}\right)\\
&.
\end{aligned}$$

for all $t\in \mathbb{R}_+$, therefore, we conclude that

$$\varphi_k,\psi_k,\omega_k \text{ are bounded in } L^\infty_{loc}(0,\infty;H^2\cap H^1_0(0,1)), \tag{3.69}$$

$$z_{1k},z_{2k},z_{3k} \text{ is bounded in } L^\infty_{loc}(0,\infty;H^1_0(0,1;L^2(0,1))). \tag{3.70}$$

Applying Dunford-Petti's theorem we conclude from (3.43), (3.44), (3.45), (3.47), (3.60), (3.61), (3.62), (3.69) and (3.70), after replacing the sequences φ_k,ψ_k and z_k with a subsequence if needed, that

$$\begin{cases}\varphi_k\to\varphi \text{ weak-star in } L^\infty_{loc}(0,\infty;H^2\cap H^1_0(0,1))\\ \psi_k\to\psi \text{ weak-star in } L^\infty_{loc}(0,\infty;H^2\cap H^1_0(0,1)),\\ \omega_k\to\omega \text{ weak-star in } L^\infty_{loc}(0,\infty;H^2\cap H^1_0(0,1))\end{cases} \tag{3.71}$$

$$\begin{cases}\varphi'_k\to\varphi' \text{ weak-star in } L^\infty_{loc}(0,\infty;H^1_0(0,1))\\ \psi'_k\to\psi' \text{ weak-star in } L^\infty_{loc}(0,\infty;H^1_0(0,1)),\\ \omega'_k\to\omega' \text{ weak-star in } L^\infty_{loc}(0,\infty;H^1_0(0,1))\end{cases}$$

$$\begin{cases}\varphi''_k\to\varphi'' \text{ weak-star in } L^\infty_{loc}(0,\infty;L^2(0,1))\\ \psi''_k\to\psi'' \text{ weak-star in } L^\infty_{loc}(0,\infty;L^2(0,1)),\\ \omega''_k\to\omega'' \text{ weak-star in } L^\infty_{loc}(0,\infty;L^2(0,1))\end{cases} \tag{3.72}$$

$$\begin{aligned}
&g_1(\varphi'_k)\to\chi \text{ weak-star in } L^2((0,1)\times(0,T)),\\
&\widetilde{g}_1(\psi'_k)\to\widetilde{\chi} \text{ weak-star in } L^2((0,1)\times(0,T)),\\
&\widetilde{\widetilde{g}}_1(\omega'_k)\to\widetilde{\widetilde{\chi}} \text{ weak-star in } L^2((0,1)\times(0,T)),\\
&z_{1k}\to z_1 \text{ weak-star in } L^\infty_{loc}(0,\infty;H^1_0((0,1);L^2(0,1))),
\end{aligned}$$

$$z_{2k} \to z_2 \text{ weak-star in } L^{\infty}_{loc}(0,\infty;H_0^1((0,1);L^2(0,1)),$$

$$z_{3k} \to z_3 \text{ weak-star in } L^{\infty}_{loc}(0,\infty;H_0^1((0,1);L^2(0,1)),$$

$$\begin{cases} z'_{1k} \to z'_1 \text{ weak-star in } L^{\infty}_{loc}(0,\infty;L^2((0,1)\times(0,1))), \\ z'_{2k} \to z'_2 \text{ weak-star in } L^{\infty}_{loc}(0,\infty;L^2((0,1)\times(0,1))), \\ z'_{3k} \to z'_3 \text{ weak-star in } L^{\infty}_{loc}(0,\infty;L^2((0,1)\times(0,1))), \end{cases} \tag{3.73}$$

$$g_2(z_k(x,1,t)) \to \varphi \text{ weak-star in } L^2((0,1)\times(0,T))$$

$$\widetilde{g}_2(z_k(x,1,t)) \to \psi \text{ weak-star in } L^2((0,1)\times(0,T))$$

$$\widetilde{\widetilde{g}}_2(z_k(x,1,t)) \to \omega \text{ weak-star in } L^2((0,1)\times(0,T))$$

for suitable functions $\varphi,\psi,\omega \in L^{\infty}(0,T;H^2\cap H_0^1(0,1)), z_1,z_2,z_3 \in L^{\infty}(0,T;L^2((0,1)\times(0,1))),$
$\chi,\widetilde{\chi},\widetilde{\widetilde{\chi}} \in L^2((0,1)\times(0,T)), \varphi,\psi,\omega \in L^2((0,1)\times(0,T))$ for all $T \geq 0$. We have to show that $(\varphi,\psi,\omega,z_1,z_2,z_3)$ is a solution of (3.8).

From (3.43) and (3.44) we have φ'_k,ψ'_k and ω'_k is bounded in $L^{\infty}(0,T;H_0^1(0,1))$. Then φ'_k,ψ'_k and ω'_k is bounded in $L^2(0,T;H_0^1)$. Since φ''_k,ψ''_k and ω''_k is bounded in $L^{\infty}(0,T;L^2(0,1))$, then φ''_k,ψ''_k and ω''_k is bounded in $L^2(0,T;L^2(0,1))$. Consequently φ'_k,ψ'_k and ω'_k is bounded in $H^1(Q)$,where $Q=(0,1)\times(0,T)$.

Since the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact, using Aubin-Lions theorem [22] we can extract a subsequence (φ_ν) of (φ_k), (ψ_ν) of (ψ_k) and (ω_ν) of (ω_k)such that

$$\varphi'_\nu \to \varphi' \text{ strongly in } L^2(Q).$$

$$\psi'_\nu \to \psi' \text{ strongly in } L^2(Q).$$

$$\omega'_\nu \to \omega' \text{ strongly in } L^2(Q).$$

Therefore

$$\begin{cases} \varphi'_\nu \to \psi' \text{ strongly and a.e on } Q. \\ \psi'_\nu \to \psi' \text{ strongly and a.e on } Q. \\ \omega'_\nu \to \psi' \text{ strongly and a.e on } Q., \end{cases} \tag{3.74}$$

Similarly we obtain

$$\begin{cases} z'_{1\nu} \to z'_1 \text{ strongly and a.e on } Q. \\ z'_{2\nu} \to z'_2 \text{ strongly and a.e on } Q. \\ z'_{3\nu} \to z'_3 \text{ strongly and a.e on } Q., \end{cases} \tag{3.75}$$

Lemma 3.3 *For each $T>0$, $g_1(\varphi'), g_2(z_1(x,1,t)) \in L^1(Q)$, $\widetilde{g}_1(\psi'), \widetilde{g}_2(z_2(x,1,t)) \in L^1(Q)$, $\widetilde{\widetilde{g}}_1(\omega'), \widetilde{\widetilde{g}}_2(z_3(x,1,t)) \in L^1(Q)$ and*

$\|g_1(\varphi')\|_{L^1(Q)}, \|g_2(z_1(x,1,t))\|_{L^1(Q)} \leq K_1,$

$\|\widetilde{g}_1(\psi')\|_{L^1(Q)}, \|\widetilde{g}_2(z_2(x,1,t))\|_{L^1(Q)} \leq K_2,$

$\|\widetilde{\widetilde{g}}_1(\omega')\|_{L^1(Q)}, \|\widetilde{\widetilde{g}}_2(z_3(x,1,t))\|_{L^1(Q)} \leq K_3,$

where K_1*,* K_2 *and* K_3 *is a constant independent of* t*.*

Proof. By (H1) and (3.3) we have

$$\begin{aligned}
g_1(\varphi_k'(x,t)) &\to g_1(\varphi'(x,t)) \text{ a.e. in } Q,\\
\widetilde{g}_1(\psi_k'(x,t)) &\to \widetilde{g}_1(\psi'(x,t)) \text{ a.e. in } Q,\\
\widetilde{\widetilde{g}}_1(\omega_k'(x,t)) &\to \widetilde{\widetilde{g}}_1(\omega'(x,t)) \text{ a.e. in } Q,\\
0 \leq g_1(\varphi_k'(x,t))\varphi_k'(x,t) &\to g_1(\varphi'(x,t))\varphi'(x,t) \text{ a.e. in } Q\\
0 \leq \widetilde{g}_1(\psi_k'(x,t))\psi_k'(x,t) &\to \widetilde{g}_1(\psi'(x,t))\psi'(x,t) \text{ a.e. in } Q\\
0 \leq \widetilde{\widetilde{g}}_1(\omega_k'(x,t))\omega_k'(x,t) &\to \widetilde{\widetilde{g}}_1(\omega'(x,t))\omega'(x,t) \text{ a.e. in } Q
\end{aligned}$$

Hence, by (3.45) and Fatou's lemma we have

$$\int_0^T \int_0^1 \varphi'(x,t) g_1(\varphi'(x,t))\, dx\, dt \leq K \text{ for } T > 0. \tag{3.76}$$

$$\int_0^T \int_0^1 \psi'(x,t) g_1(\psi'(x,t))\, dx\, dt \leq K' \text{ for } T > 0. \tag{3.77}$$

$$\int_0^T \int_0^1 \omega'(x,t) g_1(\omega'(x,t))\, dx\, dt \leq K" \text{ for } T > 0. \tag{3.78}$$

By Cauchy-Schwarz inequality and using (3.76), we have

$$\begin{aligned}
\int_0^T \int_0^1 |g_1(\varphi'(x,t))|\, dx\, dt &\leq c|Q|^{\frac{1}{2}} \left(\int_0^T \int_0^1 \varphi' g_1(\varphi')\, dx\, dt \right)^{\frac{1}{2}}\\
&\leq c|Q|^{\frac{1}{2}} K^{\frac{1}{2}} \equiv K_1
\end{aligned}$$

$$\begin{aligned}
\int_0^T \int_0^1 |\widetilde{g}_1(\psi'(x,t))|\, dx\, dt &\leq c|Q|^{\frac{1}{2}} \left(\int_0^T \int_0^1 \psi' \widetilde{g}_1(\psi')\, dx\, dt \right)^{\frac{1}{2}}\\
&\leq c|Q|^{\frac{1}{2}} K'^{\frac{1}{2}} \equiv K_2
\end{aligned}$$

$$\begin{aligned}
\int_0^T \int_0^1 |\widetilde{\widetilde{g}}_1(\omega'(x,t))|\, dx\, dt &\leq c|Q|^{\frac{1}{2}} \left(\int_0^T \int_0^1 \omega' \widetilde{\widetilde{g}}_1(\omega')\, dx\, dt \right)^{\frac{1}{2}}\\
&\leq c|Q|^{\frac{1}{2}} K"^{\frac{1}{2}} \equiv K_3
\end{aligned}$$

□

Lemma 3.4 *: $g_1(\varphi'_k) \to g_1(\varphi')$, $\widetilde{g}_1(\psi'_k) \to \widetilde{g}_1(\psi')$ and $\widetilde{\widetilde{g}}_1(\omega'_k) \to \widetilde{\widetilde{g}}_1(\omega')$ in $L^1((0,1)\times(0,T))$,*

$g_2(z_{1k}) \to g_2(z_1)$, $\widetilde{g}_2(z_{2k}) \to \widetilde{g}_2(z_2)$, and $\widetilde{\widetilde{g}}_2(z_{3k}) \to \widetilde{\widetilde{g}}_2(z_3)$ in $L^1((0,1)\times(0,T))$.

Proof: Let $E \subset (0,1)\times[0,T]$, $E' \subset (0,1)\times[0,T]$ and $E'' \subset (0,1)\times[0,T]$ and set

$$E_1 = \left\{ (x,t) \in E;\ g_1(\varphi'_k(x,t)) \le \frac{1}{\sqrt{|E|}} \right\},\quad E_2 = E \setminus E_1,$$

$$E'_1 = \left\{ (x,t) \in E';\ \widetilde{g}_1(\psi'_k(x,t)) \le \frac{1}{\sqrt{|E'|}} \right\},\quad E'_2 = E' \setminus E'_1,$$

$$E''_1 = \left\{ (x,t) \in E'';\ \widetilde{\widetilde{g}}_1(\omega'_k(x,t)) \le \frac{1}{\sqrt{|E''|}} \right\},\quad E''_2 = E'' \setminus E''_1,$$

where $|E|$ is the measure of E. If $M(r) := \inf\{|s|;\ s \in \mathbb{R} \text{ and } |g_1(s)| \ge r\}$,

$|E'|$ is the measure of E'. If $M'(r) := \inf\{|s|;\ s \in \mathbb{R} \text{ and } |\widetilde{g}_1(s)| \ge r\}$,

$|E''|$ is the measure of E''. If $M''(r) := \inf\{|s|;\ s \in \mathbb{R} \text{ and } |\widetilde{\widetilde{g}}_1(s)| \ge r\}$,

$$\int_E |g_1(\varphi'_k)|\,dxdt \le \sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}} \right) \right)^{-1} \int_{E_2} |\varphi'_k g_1(\varphi'_k)|\,dxdt.$$

$$\int_{E'} |\widetilde{g}_1(\psi'_k)|\,dxdt \le \sqrt{|E'|} + \left(M'\left(\frac{1}{\sqrt{|E'|}} \right) \right)^{-1} \int_{E'_2} |\psi'_k \widetilde{g}_1(\psi'_k)|\,dxdt.$$

$$\int_{E''} |\widetilde{\widetilde{g}}_1(\omega'_k)|\,dxdt \le \sqrt{|E''|} + \left(M''\left(\frac{1}{\sqrt{|E''|}} \right) \right)^{-1} \int_{E''_2} |\omega'_k \widetilde{\widetilde{g}}_1(\omega'_k)|\,dxdt.$$

Applying (3.45) we deduce that $\sup_k \int_E |g_1(\varphi'_k)|\,dxdt \to 0$
as $|E| \to 0$, $\sup_k \int_{E'} |\widetilde{g}_1(\psi'_k)|\,dxdt \to 0$ as $|E'| \to 0$, and $\sup_k \int_{E''} |\widetilde{\widetilde{g}}_1(\omega'_k)|\,dxdt \to 0$ as $|E''| \to 0$
From Vitali's convergence theorem we deduce that $g_1(\varphi'_k) \to g_1(\varphi')$, $\widetilde{g}_1(\psi'_k) \to \widetilde{g}_1(\psi')$ and $\widetilde{\widetilde{g}}_1(\omega'_k) \to \widetilde{\widetilde{g}}_1(\omega')$ in $L^1((0,1)\times(0,T))$, hence

$$g_1(\varphi'_k) \to g_1(\varphi') \text{ weak star in } L^2(Q).$$

$$\widetilde{g}_1(\psi'_k) \to \widetilde{g}_1(\psi') \text{ weak star in } L^2(Q).$$

$$\widetilde{\widetilde{g}}_1(\omega'_k) \to \widetilde{\widetilde{g}}_1(\omega') \text{ weak star in } L^2(Q).$$

Similarly, we have

$$g_2(z'_{1k}) \to g_2(z'_1) \text{ weak star in } L^2(Q),$$

$$\widetilde{g}_2(z'_{2k}) \to \widetilde{g}_2(z'_2) \text{ weak star in } L^2(Q),$$

$$\widetilde{\widetilde{g}}_2(z'_{3k}) \to \widetilde{\widetilde{g}}_2(z'_3) \text{ weak star in } L^2(Q),$$

and this imply that

$$\int_0^T \int_0^1 g_1(\varphi'_k)v\,dx\,dt \to \int_0^T \int_0^1 g_1(\varphi')v\,dx\,dt \text{ for all } v \in L^2(0,T;H_0^1) \tag{3.79}$$

$$\int_0^T \int_0^1 \widetilde{g}_1(\psi'_k)v\,dx\,dt \to \int_0^T \int_0^1 \widetilde{g}_1(\psi')v\,dx\,dt \text{ for all } v \in L^2(0,T;H_0^1) \tag{3.80}$$

$$\int_0^T \int_0^1 \widetilde{\widetilde{g}}_1(\omega'_k)v\,dx\,dt \to \int_0^T \int_0^1 \widetilde{\widetilde{g}}_1(\omega')v\,dx\,dt \text{ for all } v \in L^2(0,T;H_0^1) \tag{3.81}$$

$$\int_0^T \int_0^1 g_2(z_{1k})v\,dx\,dt \to \int_0^T \int_0^1 g_2(z_1)v\,dx\,dt \text{ for all } v \in L^2(0,T;H_0^1) \tag{3.82}$$

$$\int_0^T \int_0^1 \widetilde{g}_2(z_{2k})v\,dx\,dt \to \int_0^T \int_0^1 \widetilde{g}_2(z_2)v\,dx\,dt \text{ for all } v \in L^2(0,T;H_0^1) \tag{3.83}$$

$$\int_0^T \int_0^1 \widetilde{\widetilde{g}}_2(z_{3k})v\,dx\,dt \to \int_0^T \int_0^1 \widetilde{\widetilde{g}}_2(z_3)v\,dx\,dt \text{ for all } v \in L^2(0,T;H_0^1) \tag{3.84}$$

as $k \to +\infty$. It follows at once from (3.71), (3.72), (3.79), (3.80), (3.81), (3.82), (3.83),(3.84) and (3.73) that for each fixed $u, v, w \in L^2(0,T;H_0^1(0,1))$ and $w_1, w_2, w_3 \in L^2(0,T;H_0^1((0,1)\times(0,1)))$

$$\begin{aligned}&\int_0^T \int_0^1 (\rho_1\varphi''_k + Gh(\varphi_{kxx} + Gh(\psi_k + l\omega_k) + lEh(\omega_k + l\varphi_k) + \mu_1 g_1(\varphi'_k) + \mu_2 g_2(z_1k))u\,dx\,dt \\ &\to \int_0^T \int_0^1 (\rho_1\varphi'' + Gh(\varphi_{xx} + Gh(\psi + l\omega) + lEh(\omega + l\varphi) + \mu_1 g_1(\varphi') + \mu_2 g_2(z_1))u\,dx\,dt\end{aligned}$$

$$\begin{aligned}&\int_0^T \int_0^1 (\rho_2\psi''_k + El\psi_{kxx} + Gh(\varphi_{kx} + \psi_k + l\omega_k + \widetilde{\mu}_1\widetilde{g}_1(\psi'_k) + \widetilde{\mu}_2\widetilde{g}_2(z_{2k}))v\,dx\,dt \\ &\to \int_0^T \int_0^1 (\rho_2\psi'' + El\psi_{xx} + Gh(\varphi_x + \psi + l\omega + \widetilde{\mu}_1\widetilde{g}_1(\psi') + \widetilde{\mu}_2\widetilde{g}_2(z_2))v\,dx\,dt\end{aligned}$$

$$\begin{aligned}&\int_0^T \int_0^1 (\rho_1\omega''_k + Eh\omega_{kxx} + lEh\varphi_{kx} + lGh(\varphi_{kx} + \psi_k + l\omega_k) + \widetilde{\widetilde{\mu}}_1\widetilde{\widetilde{g}}_1(\omega'_k) + \widetilde{\widetilde{\mu}}_2\widetilde{\widetilde{g}}_2(z_3k))w\,dx\,dt \\ &\to \int_0^T \int_0^1 (\rho_1\omega'' + Eh\omega_{xx} + lEh\varphi_x + lGh(\varphi_x + \psi + l\omega) + \widetilde{\widetilde{\mu}}_1\widetilde{\widetilde{g}}_1(\omega') + \widetilde{\widetilde{\mu}}_2\widetilde{\widetilde{g}}_2(z_3))w\,dx\,dt\end{aligned}$$

$$\int_0^T\int_0^1\int_0^1(\tau z'_{1k}+\frac{\partial}{\partial\rho}z_{1k})w_1\,dx\,d\rho\,dt \to \int_0^T\int_0^1\int_0^1(\tau z'_1+\frac{\partial}{\partial\rho}z_1)w_1\,dx\,d\rho\,dt$$
$$\int_0^T\int_0^1\int_0^1(\tau z'_{2k}+\frac{\partial}{\partial\rho}z_{2k})w_2\,dx\,d\rho\,dt \to \int_0^T\int_0^1\int_0^1(\tau z'_2+\frac{\partial}{\partial\rho}z_2)w_2\,dx\,d\rho\,dt$$
$$\int_0^T\int_0^1\int_0^1(\tau z'_{3k}+\frac{\partial}{\partial\rho}z_{3k})w_3\,dx\,d\rho\,dt \to \int_0^T\int_0^1\int_0^1(\tau z'_3+\frac{\partial}{\partial\rho}z_3)w_3\,dx\,d\rho\,dt$$

as $k \to +\infty$. Hence

$$\int_0^T\int_0^1(\rho_1\varphi''+Gh(\varphi_{xx}+Gh(\psi+l\omega)+lEh(\omega+l\varphi)+\mu_1g_1(\varphi')+\mu_2g_2(z_1))u\,dx\,dt=0$$

$$\int_0^T\int_0^1(\rho_2\psi''+El\psi_{xx}+Gh(\varphi_x+\psi+l\omega+\widetilde{\mu}_1\widetilde{g}_1(\psi')+\widetilde{\mu}_2\widetilde{g}_2(z_2))v\,dx\,dt=0$$

$$\int_0^T\int_0^1(\rho_1\omega''+Eh\omega_{xx}+lEh\varphi_x+lGh(\varphi_x+\psi+l\omega)+\widetilde{\widetilde{\mu}}_1\widetilde{\widetilde{g}}_1(\omega)+\widetilde{\widetilde{\mu}}_2\widetilde{\widetilde{g}}_2(z_3))w\,dx\,dt=0$$

$$\int_0^T\int_0^1\int_0^1(\tau u'+\frac{\partial}{\partial\rho}z)w\,dx\,d\rho\,dt=0, \qquad w_1,w_2,w_3\in L^2(0,T;H_0^1((0,1)\times(0,1))).$$

Thus the problem (P) admits a global weak solution (φ,ψ,ω).

Uniqueness. Let $(\varphi_1,\psi_1,\omega_1,z_1,z_2,z_3)$ and $(\widetilde{\varphi}_1,\widetilde{\psi}_1,\widetilde{\omega}_1,\widetilde{z}_1,\widetilde{z}_2,\widetilde{z}_3)$ be two solutions of problem (3.8). Then $(\widetilde{\widetilde{\varphi}},\widetilde{\widetilde{\psi}},\widetilde{\widetilde{\omega}},\widetilde{\widetilde{z}}_1,\widetilde{\widetilde{z}}_2,\widetilde{\widetilde{z}}_3)=(\varphi_1,\psi_1,\omega_1,z_1,z_2,z_3)$ -$(\widetilde{\varphi}_1,\widetilde{\psi}_1,\widetilde{\omega}_1,\widetilde{z}_1,\widetilde{z}_2,\widetilde{z}_3)$ verifies

$$\begin{cases}
\rho_1\widetilde{\widetilde{\varphi}}_{tt}-Gh(\widetilde{\widetilde{\varphi}}_x+\widetilde{\widetilde{\psi}}+l\widetilde{\widetilde{\omega}})_x-lEh(\widetilde{\widetilde{\omega}}_x-l\widetilde{\widetilde{\varphi}})+\mu_1g_1(\widetilde{\varphi}'(x,t))-\mu_1g_1(\varphi'(x,t)) & \\
\qquad +\mu_2g_2(\widetilde{z}_1(x,1,t)-\mu_2g_2(z_1(x,1,t))=0 & \text{in }]0,1[\times]0,+\infty[,\\
\tau_1\widetilde{\widetilde{z}}_{1t}(x,\rho,t)+\widetilde{\widetilde{z}}_{1\rho}(x,\rho,t)=0 & \text{in }]0,1[\times]0,1[\times]0,+\infty[,\\
\rho_2\widetilde{\widetilde{\psi}}_{tt}-EI\widetilde{\widetilde{\psi}}_{xx}+Gh(\widetilde{\widetilde{\varphi}}_x+\widetilde{\widetilde{\psi}}+l\widetilde{\widetilde{\omega}})+\widetilde{\mu_1}\widetilde{g}_1(\widetilde{\psi}'(x,t))-\widetilde{\mu_1}\widetilde{g}_1(\psi'(x,t)) & \\
\qquad +\widetilde{\mu_2}\widetilde{g}_2(\widetilde{z}_2(x,1,t)-\widetilde{\mu_2}\widetilde{g}_2(z_2(x,1,t))=0 & \text{in }]0,1[\times]0,+\infty[,\\
\tau_2\widetilde{\widetilde{z}}_{2t}(x,\rho,t)+\widetilde{\widetilde{z}}_{2\rho}(x,\rho,t)=0 & \text{in }]0,1[\times]0,1[\times]0,+\infty[,\\
\rho_1\widetilde{\widetilde{\omega}}_{tt}-Eh(\widetilde{\widetilde{\omega}}_x-l\widetilde{\widetilde{\varphi}})_x+lGh(\widetilde{\widetilde{\varphi}}_x+\widetilde{\widetilde{\psi}}+l\widetilde{\widetilde{\omega}})+\widetilde{\widetilde{\mu_1}}\widetilde{\widetilde{g}}_1(\widetilde{\omega}'(x,t))-\widetilde{\widetilde{\mu_1}}\widetilde{\widetilde{g}}_1(\omega'(x,t)) & \\
\qquad +\widetilde{\widetilde{\mu_2}}\widetilde{\widetilde{g}}_2(\widetilde{z}_3(x,1,t))-\widetilde{\widetilde{\mu_2}}\widetilde{\widetilde{g}}_2(z_3(x,1,t))=0 & \text{in }]0,1[\times]0,+\infty[\\
\tau_3\widetilde{\widetilde{z}}_{3t}(x,\rho,t)+\widetilde{\widetilde{z}}_{3\rho}(x,\rho,t)=0 & \text{in }]0,1[\times]0,1[\times]0,+\infty[\\
\widetilde{\widetilde{\varphi}}(0,t)=\widetilde{\widetilde{\varphi}}(1,t)=\widetilde{\widetilde{\psi}}(0,t)=\widetilde{\widetilde{\psi}}(1,t)=0 & t\geq 0,\\
\widetilde{\widetilde{\psi}}(x,0,t)=\widetilde{\psi}'(x,t)-\psi'(x,t) & x\in]0,1[\times]0,+\infty[\\
\widetilde{\widetilde{\varphi}}(x,0,t)=\widetilde{\varphi}'(x,t)-\varphi'(x,t) & x\in]0,1[\times]0,+\infty[\\
\widetilde{\widetilde{\omega}}(x,0,t)=\widetilde{\omega}'(x,t)-\omega'(x,t) & x\in]0,1[\times]0,+\infty[\\
\widetilde{\widetilde{\varphi}}(x,0)=\widetilde{\widetilde{\varphi}}'(x,0)=\widetilde{\widetilde{\psi}}(x,0)=\widetilde{\widetilde{\psi}}'(x,0)=\widetilde{\widetilde{\omega}}(x,0)=\widetilde{\widetilde{\omega}}'(x,0)=0 & x\in]0,1[\\
\widetilde{\widetilde{z}}_1(x,\rho,0)=\widetilde{\widetilde{z}}_2(x,\rho,0)=\widetilde{\widetilde{z}}_3(x,\rho,0)=0 & x\in]0,1[\times]0,1[
\end{cases}\tag{3.85}$$

Multiplying the first equation (3.85) by $\widetilde{\widetilde{\varphi}}'$ the third equation by $\widetilde{\widetilde{\psi}}'$ and the five equation by $\widetilde{\widetilde{\omega}}'$ integrating over $(0,1)\times(0,1)$, we get

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\widetilde{\widetilde{\varphi}}_t\|_2^2\right)+Gh\int_0^1(\widetilde{\widetilde{\varphi}}_x+\widetilde{\widetilde{\psi}}+l\widetilde{\widetilde{\omega}})_x)\widetilde{\widetilde{\varphi}}'\,dx$$
$$+lEh\textstyle\int_0^1(\widetilde{\widetilde{\omega}}_x-l\widetilde{\widetilde{\varphi}}')\widetilde{\widetilde{\varphi}}'\,dx+\mu_1\left(g_1(\widetilde{\varphi}'(x,t))-g_1(\varphi'(x,t)),\widetilde{\widetilde{\varphi}}'\right) \tag{3.86}$$
$$+\mu_2\left(g_2(\widetilde{z}_1(x,1,t)-g_2(z_1(x,1,t)),\widetilde{\widetilde{\varphi}}'\right)=0.$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\widetilde{\widetilde{\psi}}_t\|_2^2+EI\|\widetilde{\widetilde{\psi}}_x\|_2^2\right)+Gh\int_0^1(\widetilde{\widetilde{\varphi}}_x+\widetilde{\widetilde{\psi}}+l\widetilde{\widetilde{\omega}})\widetilde{\widetilde{\psi}}'\,dx$$
$$+\widetilde{\mu_1}\left(\widetilde{g}_1(\widetilde{\psi}'(x,t))-\widetilde{g}_1(\psi'(x,t)),\widetilde{\widetilde{\psi}}'\right) \tag{3.87}$$
$$+\widetilde{\mu_2}\left(\widetilde{g}_2(\widetilde{z}_2(x,1,t)-\widetilde{g}_2(z_2(x,1,t)),\widetilde{\widetilde{\psi}}'\right)=0.$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\widetilde{\widetilde{\omega}}_t\|_2^2\right)+Eh\int_0^1(\widetilde{\widetilde{\omega}}_x-l\widetilde{\widetilde{\varphi}})_x)\widetilde{\widetilde{\omega}}'\,dx$$
$$+lGh\textstyle\int_0^1(\widetilde{\widetilde{\varphi}}_x+\widetilde{\widetilde{\psi}}+l\widetilde{\widetilde{\omega}})\widetilde{\widetilde{\omega}}'\,dx+\widetilde{\widetilde{\mu_1}}\left(\widetilde{\widetilde{g}}_1(\widetilde{\omega}'(x,t))-\widetilde{\widetilde{g}}_1(\omega'(x,t)),\widetilde{\widetilde{\omega}}'\right)$$
$$+\widetilde{\widetilde{\mu_2}}\left(\widetilde{\widetilde{g}}_2(\widetilde{z}_3(x,1,t))-\widetilde{\widetilde{g}}_2(z_3(x,1,t)),\widetilde{\widetilde{\omega}}'\right)=0. \tag{3.88}$$

Multiplying the second equation in (3.85) by $\widetilde{\widetilde{z_1}}$ the fourth equation by $\widetilde{\widetilde{z_2}}$ and the sixth equation by $\widetilde{\widetilde{z_3}}$, integrating over $(0,1)\times(0,1)$, we get

$$\tau_1\frac{1}{2}\frac{d}{dt}\int_0^1\|\widetilde{\widetilde{z_1}}'\|_2^2\,d\rho+\frac{1}{2}(\|\widetilde{\widetilde{z}}_1(x,1,t)\|_2^2-\|\widetilde{\widetilde{\varphi}}'\|_2^2)=0. \tag{3.89}$$

$$\tau_2\frac{1}{2}\frac{d}{dt}\int_0^1\|\widetilde{\widetilde{z_2}}'\|_2^2\,d\rho+\frac{1}{2}(\|\widetilde{\widetilde{z}}_2(x,1,t)\|_2^2-\|\widetilde{\widetilde{\psi}}'\|_2^2)=0. \tag{3.90}$$

$$\tau_3\frac{1}{2}\frac{d}{dt}\int_0^1\|\widetilde{\widetilde{z_3}}'\|_2^2\,d\rho+\frac{1}{2}(\|\widetilde{\widetilde{z}}_3(x,1,t)\|_2^2-\|\widetilde{\widetilde{\omega}}'\|_2^2)=0. \tag{3.91}$$

From (3.86), (3.87), (3.88), (3.89), (3.90), (3.91) and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\widetilde{\widetilde{\varphi}}_t\|_2^2+\rho_2\|\widetilde{\widetilde{\psi}}_t\|_2^2+\rho_1\|\widetilde{\widetilde{\omega}}_t\|_2^2+Gh\|\widetilde{\widetilde{\varphi}}_x+\widetilde{\widetilde{\psi}}+l\widetilde{\widetilde{\omega}}\|_2^2+Eh\|\widetilde{\widetilde{\omega}}_x-l\widetilde{\widetilde{\varphi}}'\|_2^2+EI\|\widetilde{\widetilde{\psi}}_x\|_2^2\right)\\
&\quad+\frac{1}{2}\frac{d}{dt}\left(\tau_1\int_0^1\|\widetilde{\widetilde{z}}_1'\|_2^2\,d\rho+\tau_2\int_0^1\|\widetilde{\widetilde{z}}_2'\|_2^2\,d\rho+\tau_3\int_0^1\|\widetilde{\widetilde{z}}_3'\|_2^2\,d\rho\right)\\
&\quad+\frac{1}{2}\frac{d}{dt}\left(\|\widetilde{\widetilde{z}}_1(x,1,t)\|_2^2+\|\widetilde{\widetilde{z}}_2(x,1,t)\|_2^2+\|\widetilde{\widetilde{z}}_3(x,1,t)\|_2^2\right)\\
&\quad+\mu_1\left(g_1(\widetilde{\varphi}'(x,t))-g_1(\varphi'(x,t)),\widetilde{\widetilde{\varphi}}'\right)+\widetilde{\mu_1}\left(\widetilde{g}_1(\widetilde{\psi}'(x,t))-\widetilde{g}_1(\psi'(x,t)),\widetilde{\widetilde{\psi}}'\right)\\
&\quad+\widetilde{\widetilde{\mu_1}}\left(\widetilde{\widetilde{g}}_1(\widetilde{\omega}'(x,t))-\widetilde{\widetilde{g}}_1(\omega'(x,t)),\widetilde{\widetilde{\omega}}'\right)\\
&\quad=-\mu_2\left(g_2(\widetilde{z}_1(x,1,t)-g_2(z_1(x,1,t)),\widetilde{\widetilde{\varphi}}'\right)-\widetilde{\mu_2}\left(\widetilde{g}_2(\widetilde{z}_2(x,1,t)-\widetilde{g}_2(z_2(x,1,t)),\widetilde{\widetilde{\psi}}'\right)\\
&\quad-\widetilde{\widetilde{\mu_2}}\left(\widetilde{\widetilde{g}}_2(\widetilde{z}_3(x,1,t))-\widetilde{\widetilde{g}}_2(z_3(x,1,t)),\widetilde{\widetilde{\omega}}'\right)+\frac{1}{2}\frac{d}{dt}\left(\|\widetilde{\widetilde{\varphi}}'\|_2^2)+\|\widetilde{\widetilde{\psi}}'\|_2^2)+\|\widetilde{\widetilde{\omega}}'\|_2^2)\right)\\
&\quad\le\frac{1}{2}\left(\|\widetilde{\widetilde{\varphi}}'\|_2^2)+\|\widetilde{\widetilde{\psi}}'\|_2^2)+\|\widetilde{\widetilde{\omega}}'\|_2^2\right)+\|g_2(\widetilde{z}_1(x,1,t)-g_2(z_1(x,1,t))\|_2\|\widetilde{\widetilde{\varphi}}'\|_2\\
&\quad+\|\widetilde{g}_2(\widetilde{z}_2(x,1,t)-\widetilde{g}_2(z_2(x,1,t))\|_2\|\widetilde{\widetilde{\psi}}'\|_2+\|\widetilde{\widetilde{g}}_2(\widetilde{z}_3(x,1,t))-\widetilde{\widetilde{g}}_2(z_3(x,1,t))\|_2\|\widetilde{\widetilde{\omega}}'\|_2.
\end{aligned}$$

Using condition (3.4) and Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\widetilde{\widetilde{\varphi}}_t\|_2^2+\rho_2\|\widetilde{\widetilde{\psi}}_t\|_2^2+\rho_1\|\widetilde{\widetilde{\omega}}_t\|_2^2+Gh\|\widetilde{\widetilde{\varphi}}_x+\widetilde{\widetilde{\psi}}+l\widetilde{\widetilde{\omega}}\|_2^2+Eh\|\widetilde{\widetilde{\omega}}_x-l\widetilde{\widetilde{\varphi}}'\|_2^2+EI\|\widetilde{\widetilde{\psi}}_x\|_2^2\right)\\
&\quad+\frac{1}{2}\frac{d}{dt}\left(\tau_1\int_0^1\|\widetilde{\widetilde{z}}_1'\|_2^2\,d\rho+\tau_2\int_0^1\|\widetilde{\widetilde{z}}_2'\|_2^2\,d\rho+\tau_3\int_0^1\|\widetilde{\widetilde{z}}_3'\|_2^2\,d\rho\right)\\
&\quad\le c\left(\|\widetilde{\widetilde{\varphi}}'\|_2^2)+\|\widetilde{\widetilde{\psi}}'\|_2^2)+\|\widetilde{\widetilde{\omega}}'\|_2^2\right).
\end{aligned}$$

where c is a positive constant. Then integrating over $(0,t)$, using Gronwall's lemma, we conclude that

$$\begin{aligned}
&\rho_1\|\widetilde{\widetilde{\varphi}}_t\|_2^2+\rho_2\|\widetilde{\widetilde{\psi}}_t\|_2^2+\rho_1\|\widetilde{\widetilde{\omega}}_t\|_2^2+Gh\|\widetilde{\widetilde{\varphi}}_x+\widetilde{\widetilde{\psi}}+l\widetilde{\widetilde{\omega}}\|_2^2+Eh\|\widetilde{\widetilde{\omega}}_x-l\widetilde{\widetilde{\varphi}}'\|_2^2+EI\|\widetilde{\widetilde{\psi}}_x\|_2^2\\
&\quad+\tau_1\int_0^1\|\widetilde{\widetilde{z}}_1'\|_2^2\,d\rho+\tau_2\int_0^1\|\widetilde{\widetilde{z}}_2'\|_2^2\,d\rho+\tau_3\int_0^1\|\widetilde{\widetilde{z}}_3'\|_2^2\,d\rho=0
\end{aligned}$$

3.4 Asymptotic Behavior

First we state and prove a lemma that will be needed to establish the asymptotic behavior.

Lemma 3.5 *There exists a positive constant C such that the following inequality holds for every $(\varphi,\psi,\omega)\in(H_0^1(0,L))^3$*

$$\int_0^L(|\varphi_x|^2+|\psi_x|^2+|\omega_x|^2)\,dx\le C\int_0^L(EI|\psi_x|^2+Gh|\varphi_x+\psi+l\omega|^2+Eh|\omega_x-l\varphi|^2)\,dx. \quad (3.92)$$

Proof. we will argue by contradiction. Indeed, let us suppose that (3.92) is not true. So, we can find a sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in\mathbb{N}}$ in $(H_0^1(0,L))^3$ satisfying

$$\int_0^L (EI|\psi_{\nu x}|^2 + Gh|\varphi_{\nu x} + \psi + l\omega\nu|^2 + Eh|\omega_{\nu x} - l\varphi\nu|^2)\, dx \leq \frac{1}{\nu} \tag{3.93}$$

and

$$\int_0^L (|\varphi_{\nu x}|^2 + |\psi_{\nu x}|^2 + |\omega_{\nu x}|^2)\, dx = 1. \tag{3.94}$$

From (3.94), the sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in\mathbb{N}}$ is bounded in $(H_0^1(0,L))^3$, Since the embedding $H_0^1(0,L) \hookrightarrow L^2(0,L)$ is compact, then the sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in\mathbb{N}}$ converge strongly in $(L^2(0,L))^3$.
From (3.93)

$$\varphi_{\nu x} \to 0 \text{ strongly in } L^2(0,L). \tag{3.95}$$

Using Poincaré's inequality we can conclude that

$$\varphi_\nu \to 0 \text{ strongly in } L^2(0,L). \tag{3.96}$$

Now, setting $\varphi_\nu \to \varphi$ and $\omega_\nu \to \omega$ strongly in $L^2(0,L)$.
From (3.93), we have

$$\varphi_{\nu x} + \psi_\nu + l\omega_\nu \to 0 \text{ strongly in } L^2(0,L). \tag{3.97}$$

Then

$$\varphi_{\nu x} + \psi_\nu + l\omega_\nu = \varphi_{\nu x} + \psi_\nu + l(\omega_\nu - \omega) + l\omega \to 0 \text{ strongly in } L^2(0,L). \tag{3.98}$$

which implies that

$$\varphi_{\nu x} \to -l\omega \text{ strongly in } L^2(0,L). \tag{3.99}$$

Then, $\{\varphi_\nu\}_n$ is a Cauchy sequence in $H^1(0,L)$. Therefore $\{\varphi_\nu\}_n$ converge to a function φ_1 in $H^1(0,L)$. Consequently $\{\varphi_\nu\}_n$ converge to φ_1 in $L^2(0,L)$. Thus by the uniqueness of the limit $\varphi_1 = \varphi$. Moreover $\varphi \in H_0^1(0,L)$.
From (3.100) we deduce that

$$\varphi_x + l\omega = 0 \text{ a.e } x \in (0,L). \tag{3.100}$$

Similarly, we have

$$\omega_x - l\varphi = 0 \text{ a.e } x \in (0,L) \tag{3.101}$$

and $\omega \in H_0^1(0,L)$.
(3.100) and (3.101) provides us $\varphi = \omega = 0$, contradicting (3.94). □

From now on, we denote by c various positive constants which may be different at different occurrences.

Multiplying the first equation in (3.8) by $\frac{\varphi(E)}{E}\varphi$, the third equation by $\frac{\varphi(E)}{E}\psi$ and the five equation by $\frac{\varphi(E)}{E}\omega$. We obtain

$$0 = \int_S^T \frac{\varphi(E)}{E} \int_0^L \varphi\left(\rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1\varphi_t + \mu_2 z_1(x,1,t)\right) dxdt$$

$$\begin{aligned} 0 &= \left[\frac{\varphi(E)}{E}\rho_1\int_0^L \varphi\varphi_t dx\right]_S^T - \int_S^T \rho_1\left(\frac{\varphi(E)}{E}\right)'\int_0^L \varphi\varphi_t dxdt - \rho_1\int_S^T \frac{\varphi(E)}{E}\|\varphi_t\|_2^2 dt \\ &\quad -\int_S^T \frac{\varphi(E)}{E}\int_0^L \varphi_x Gh(\varphi_x+\psi+l\omega)dxdt - \int_S^T \frac{\varphi(E)}{E}\int_0^L \varphi(lEh)(\omega_x - l\varphi)dxdt \\ &\quad +\mu_1\int_S^T \frac{\varphi(E)}{E}\int_0^L \varphi_t\varphi dxdt + \mu_2\int_S^T \frac{\varphi(E)}{E}\int_0^L \varphi z_1(x,1,t)dxdt \end{aligned}$$

$$0 = \int_S^T \frac{\varphi(E)}{E}\int_0^L \psi\left(\rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x+\psi+l\omega) + \widetilde{\mu_1}\psi_t + \widetilde{\mu_2}z_2(x,1,t)\right)dxdt$$

$$\begin{aligned} 0 &= \left[\frac{\varphi(E)}{E}\rho_2\int_0^L \psi\psi_t dx\right]_S^T - \int_S^T \rho_2\left(\frac{\varphi(E)}{E}\right)'\int_0^L \psi\psi_t dxdt - \rho_2\int_S^T \frac{\varphi(E)}{E}\|\psi_t\|_2^2 dt \\ &\quad +\int_S^T \frac{\varphi(E)}{E}EI\|\psi_x\|_2^2 dt + \int_S^T \frac{\varphi(E)}{E}\int_0^L \psi\psi Gh(\varphi_x+\psi+l\omega))dxdt \\ &\quad +\widetilde{\mu_1}\int_S^T \frac{\varphi(E)}{E}\int_0^L \psi\psi_t dxdt + \widetilde{\mu_2}\int_S^T \frac{\varphi(E)}{E}\int_0^L \psi z_2(x,1,t)dxdt \end{aligned}$$

$$0 = \int_S^T \frac{\varphi(E)}{E}\int_0^L \omega\Big(\rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x+\psi+l\omega) + \widetilde{\widetilde{\mu_1}}\omega_t + \widetilde{\widetilde{\mu_2}}z_3(x,1,t)\Big)dxdt$$

$$\begin{aligned} 0 &= \left[\frac{\varphi(E)}{E}\rho_1\int_0^L \omega\omega_t dx\right]_S^T - \int_S^T \rho_1\left(\frac{\varphi(E)}{E}\right)'\int_0^L \omega\omega_t dxdt \\ &\quad -\rho_1\int_S^T \frac{\varphi(E)}{E}\|\omega_t\|_2^2 dt + \int_S^T \frac{\varphi(E)}{E}\int_0^L Eh\omega_x(\omega_x - l\varphi)dxdt \\ &\quad +\int_S^T \frac{\varphi(E)}{E}\int_0^L \omega(lGh)(\varphi_x+\psi+l\omega)dxdt \\ &\quad +\widetilde{\widetilde{\mu_1}}\int_S^T \frac{\varphi(E)}{E}\int_0^L \omega_t\omega dxdt + \widetilde{\widetilde{\mu_2}}\int_S^T \frac{\varphi(E)}{E}\int_0^L \omega z_3(x,1,t)dxdt \end{aligned}$$

Taking their sum, we obtain

$$\begin{aligned} 0 &= \left[\frac{\varphi(E)}{E}\rho_1\int_0^L \varphi\varphi_t dx\right]_S^T + \left[\frac{\varphi(E)}{E}\rho_2\int_0^L \psi\psi_t dx\right]_S^T + \left[\frac{\varphi(E)}{E}\rho_1\int_0^L \omega\omega_t dx\right] \\ &\quad -\int_S^T \rho_1\left(\frac{\varphi(E)}{E}\right)'\int_0^L (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t + \rho_1\omega\omega_t)dxdt \\ &\quad -2\rho_1\int_S^T \frac{\varphi(E)}{E}\|\varphi_t\|_2^2 dt - 2\rho_2\int_S^T \frac{\varphi(E)}{E}\|\psi_t\|_2^2 dt - 2\rho_1\int_S^T \frac{\varphi(E)}{E}\|\omega_t\|_2^2 dt \\ &\quad +\int_S^T \frac{\varphi(E)}{E}\left(\rho_1\|\varphi_t\|_2^2 + \rho_2\|\psi_t\|_2^2 + \rho_1\|\omega_t\|_2^2 + Gh\|\varphi_x+\psi+l\omega\|_2^2 + EI\|\psi_t\|_2^2 + Eh\|\omega_x - l\psi\|_2^2\right) \\ &\quad +\mu_1\int_S^T \frac{\varphi(E)}{E}\int_0^L \varphi_t\varphi dxdt + \mu_2\int_S^T \frac{\varphi(E)}{E}\int_0^L \varphi z_1(x,1,t)dxdt \\ &\quad +\widetilde{\mu_1}\int_S^T \frac{\varphi(E)}{E}\int_0^L \psi\psi_t dxdt + \widetilde{\mu_2}\int_S^T \frac{\varphi(E)}{E}\int_0^L \psi z_2(x,1,t)dxdt \\ &\quad +\widetilde{\widetilde{\mu_1}}\int_S^T \frac{\varphi(E)}{E}\int_0^L \omega_t\omega dxdt + \widetilde{\widetilde{\mu_2}}\int_S^T \frac{\varphi(E)}{E}\int_0^L \omega z_3(x,1,t)dxdt \end{aligned} \tag{3.102}$$

Similarly, we multiply the equation of (3.7) by $\frac{\varphi(E)}{E}\xi_i e^{-2\tau_i\rho}z_i(x,\rho,t)$ and get

$$\begin{aligned}
0 &= \int_S^T \frac{\varphi(E)}{E}\int_0^L\int_0^1 e^{-2\tau_i\rho}\xi_i z_i(\tau_i z_{it}+z_{i\rho})\,d\rho dx dt\\
&= \left[\frac{1}{2}\xi_i\tau_i\frac{\varphi(E)}{E}\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx\right]_S^T-\frac{\tau_i\xi_i}{2}\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx dt\\
&\quad+\int_S^T\frac{\varphi(E)}{E}\xi_i\int_0^L\int_0^1\frac{e^{-2\tau_i\rho}}{2}\frac{d}{d\rho}(z_i^2)\,d\rho dx dt
\end{aligned}$$

$$\begin{aligned}
0 &= \left[\frac{1}{2}\xi_i\tau_i\frac{\varphi(E)}{E}\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx\right]_S^T-\frac{\tau_i\xi_i}{2}\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx dt\\
&\quad+\frac{\xi_i}{2}\int_S^T\frac{\varphi(E)}{E}\int_0^L\int_0^1\left[\frac{d}{d\rho}\left(e^{-2\tau_i\rho}z_i^2\right)+2\tau_i e^{-2\tau_i\rho}z_i^2\right]d\rho dx dt
\end{aligned}$$

$$\begin{aligned}
0 = &\left[\tfrac{1}{2}\xi_i\tau_i\frac{\varphi(E)}{E}\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx\right]_S^T-\tfrac{\tau_i\xi_i}{2}\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx dt\\
&+\tfrac{\xi_i}{2}\int_S^T\frac{\varphi(E)}{E}\int_0^L\left[e^{-2\tau_i}z_i^2(x,1,t)-z_i^2(x,0,t)\right]dx dt\\
&+\xi_i\tau_i\int_S^T\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx dt
\end{aligned} \tag{3.103}$$

From (3.102) and (3.103) we get

$$\begin{aligned}
A\int_S^T\varphi(E)dt \;\le\; &-\left[\rho_1\frac{\varphi(E)}{E}\int_0^L\varphi\varphi_t\,dx\right]_S^T-\left[\rho_2\frac{\varphi(E)}{E}\int_0^L\psi\psi_t dx\right]_S^T-\left[\rho_1\frac{\varphi(E)}{E}\int_0^L\omega\omega_t\,dx\right]_S^T\\
&+\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_0^L(\rho_1\varphi\varphi_t+\rho_2\psi\psi_t+\rho_1\omega\omega_t)\,dx dt\\
&+2\int_S^T\frac{\varphi(E)}{E}\left(\rho_1\|\varphi_t\|_2^2+\rho_2\|\psi_t\|_2^2+\rho_1\|\omega_t\|_2^2\right)dt\\
&-\mu_1\int_S^T\frac{\varphi(E)}{E}\int_0^L\varphi_t\varphi dx dt-\mu_2\int_S^T\frac{\varphi(E)}{E}\int_0^L\varphi z_1(x,1,t)\,dx dt\\
&-\widetilde{\mu_1}\int_S^T\frac{\varphi(E)}{E}\int_0^L\psi\psi_t dx dt-\widetilde{\mu_2}\int_S^T\frac{\varphi(E)}{E}\int_0^L\psi z_2(x,1,t)\,dx dt\\
&-\widetilde{\widetilde{\mu_1}}\int_S^T\frac{\varphi(E)}{E}\int_0^L\omega_t\omega dx dt-\widetilde{\widetilde{\mu_2}}\int_S^T\frac{\varphi(E)}{E}\int_0^L\omega z_3(x,1,t)\,dx dt\\
&-\sum_{i=1}^3\left[\frac{1}{2}\xi_i\tau_i\frac{\varphi(E)}{E}\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx\right]_S^T+\sum_{i=1}^3\frac{\tau_i\xi_i}{2}\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx dt\\
&-\sum_{i=1}^3\frac{\xi_i}{2}\int_S^T\frac{\varphi(E)}{E}e^{-2\tau_i}\int_0^L z_i^2(x,1,t)\,dx dt\\
&+\sum_{i=1}^3\frac{\xi_i}{2}\int_S^T\frac{\varphi(E)}{E}\|z_i(x,0,t)\|_2^2\,dt
\end{aligned} \tag{3.104}$$

where $A = 2\min\{1, 2\tau_1 e^{-2\tau_1}, 2\tau_2 e^{-2\tau_2}, 2\tau_3 e^{-2\tau_3}\}$. Since E is non-increasing, we find that

$$-\left[\frac{\varphi(E)}{E}\int_0^L \varphi\varphi_t\,dx\right]_S^T = \frac{\varphi(E(S))}{E(S)}\int_0^L \varphi(S)\varphi'(S)\,dx - \frac{\varphi(E(T))}{E(T)}\int_0^L \varphi(T)\varphi'(T)\,dx$$
$$\leq C\varphi(E(S))$$

$$\left|\int_S^T \left(\frac{\varphi(E)}{E}\right)'\int_0^L (\rho_1\varphi\varphi' + \rho_2\psi\psi' + \rho_1\omega\omega')\,dxdt\right| \leq c\int_S^T (-E')\frac{\varphi(E)}{E}\,dt$$
$$\leq c\varphi(E(S))$$

$$\left|\frac{1}{2}\xi_i\tau_i\frac{\varphi(E)}{E}\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,dxd\rho\right| \leq c\varphi(E(S)) \quad \forall t \geq S$$

$$\int_S^T \frac{\varphi(E)}{E}\int_0^L u'^2\,dxdt \leq c\int_S^T \frac{\varphi(E)}{E}(-E')\,dt$$
$$\leq c\varphi(E(S))$$

$$\int_S^T \frac{\varphi(E)}{E}\xi_i\int_0^L e^{-2\tau_i}z_i^2(x,1,t)\,dxdt \leq c\int_S^T \frac{\varphi(E)}{E}(-E')\,dt$$
$$\leq c\varphi(E(S))$$

$$\frac{1}{2}\int_S^T \frac{\varphi(E)}{E}\xi_i\int_0^L z_i^2(x,0,t))\,dxdt = \frac{1}{2}\int_S^T \frac{\varphi(E)}{E}\xi_i\int_0^L \varphi'^2\,dxdt$$
$$\leq c\varphi(E(S)),$$

$$\left|\frac{\tau_i\xi_i}{2}\int_S^T \left(\frac{\varphi(E)}{E}\right)'\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2 dxd\rho\,dt\right| \leq c\int_S^T (-E')\frac{\varphi(E)}{E}\,dt$$
$$\leq c\varphi(E(S))$$

$$\begin{aligned}\left|\int_S^T \frac{\varphi(E)}{E}\int_0^L \varphi\varphi'\,dxdt\right| &\leq \varepsilon \int_S^T \frac{\varphi(E)}{E}\int_0^L \varphi^2\,dxdt + c(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_0^L \varphi'^2\,dxdt\\ &\leq \varepsilon c\int_0^L \varphi(E)\,dt + c(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_0^L \varphi'^2\,dxdt\\ &\leq \varepsilon c\int_0^L \varphi(E)\,dt + c(\varepsilon)\int_S^T \frac{\varphi(E)}{E}(-E')\,dt\\ &\leq \varepsilon c\int_0^L \varphi(E)\,dt + c(\varepsilon)E(S)^{q+1}\end{aligned} \tag{3.105}$$

and

$$\begin{aligned}\left|\int_S^T \frac{\varphi(E)}{E}\int_0^L \varphi z_1(x,1,t)\,dxdt\right| &\leq \varepsilon_1 \int_S^T \frac{\varphi(E)}{E}\int_0^L \varphi^2\,dxdt + c(\varepsilon_1)\int_S^T \frac{\varphi(E)}{E}\int_0^L z_1(x,1,t)^2\,dxdt\\ &\leq \varepsilon_1 c\int_{\mathbb{R}^n} \varphi(E)\,dt + c(\varepsilon_1)\int_S^T \frac{\varphi(E)}{E}\int_0^L z_1(x,1,t)^2\,dxdt\\ &\leq \varepsilon_1 c\int_0^L \varphi(E)\,dt + c(\varepsilon_1)\int_S^T \frac{\varphi(E)}{E}(-E')\,dt\\ &\leq \varepsilon_1 c\int_0^L \varphi(E)\,dt + c(\varepsilon_1)\varphi(E(S)).\end{aligned} \tag{3.106}$$

$$\left|\int_S^T \frac{\varphi(E)}{E}\int_0^L \psi\psi'\,dxdt\right| \leq \varepsilon' c\int_0^L \varphi(E)\,dt + c(\varepsilon')\varphi(E(S)) \tag{3.107}$$

$$\left|\int_S^T \frac{\varphi(E)}{E}\int_0^L \psi z_2(x,1,t)\,dxdt\right| \leq \varepsilon_1' c\int_0^L \varphi(E)\,dt + c(\varepsilon_1')\varphi(E(S)) \tag{3.108}$$

$$\left|\int_S^T \frac{\varphi(E)}{E}\int_0^L \omega\omega'\,dxdt\right| \leq \varepsilon'' c\int_0^L \varphi(E)\,dt + c(\varepsilon'')\varphi(E(S)) \tag{3.109}$$

$$\left|\int_S^T \frac{\varphi(E)}{E}\int_0^L \omega z_3(x,1,t)\,dxdt\right| \leq \varepsilon_1'' c\int_0^L E^{q+1}\,dt + c(\varepsilon_1'')\varphi(E(S)) \tag{3.110}$$

Choosing $\varepsilon, \varepsilon_1, \varepsilon', \varepsilon_1', \varepsilon''$ and ε_1'' small enough, we deduce from (1.104), (3.105), (3.106), (3.107), (3.108), (3.109) and (3.110) that

$$\int_S^T \varphi(E)\,dt \leq c\varphi(E(S)),$$

where c is a positive constant independent of $E(0)$. Hence, we deduce from Lemma 3.5 that

$$E(t) \leq cE(0)e^{-\omega t}, \qquad t \geq 0.$$

This ends the proof of Theorem 3.1. □

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