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Une contribution à l'étude de quelques classes de problèmes d'évolution

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Abstract

This thesis is devoted to the existence of random mild solutions for various classes of first and second order functional differential evolutions equations with random effects, finite and infinite delay in Banach space. Sufficient conditions are considered to get the existence of mild random solutions by reducing this research to the search for the existence of random fixed point of a continuous random operator with stochastic domain under rather general conditions provided that the corresponding deterministic fixed point problem is solvable.

Key words and phrases :

Random fixed point, fixed point, Functional differential equation of second order, random mild solution, finite delay, state-dependent delay, semigroup theory, cosine function, measure of noncompactness .

AMS Subject Classification : 45D05, 45E10, 45G05, 26A33, 45G05, 34A08, 34K37.

Résumé

Cette thèse présente quelques résultats d'éxistence de la solution faible aléatoire pour quelques classes d'équations d'évolution avec un effet aléatoire et avec retard infini et dépendant de l'état dans un espace de Banach. Sous des conditions convenables, nous avons prouvé l'éxistence des solutions faibles aléatoires pour des différentes classes de problèmes d'évolution. Ainsi, nos résultats sont basées sur des récents théorèmes du point fixe aléatoire et la mesure de non compacité.

Mots et Phrases Clefs :

Equations différentielles Fonctionelles-Equations différentielles Fonctionelles de second ordre - solution faible aléatoire -Fonction measurable-la mesure de non compacité - résultats d'existence - théorie du point fixed - point fixe aléatoire- espace de Banach - retard dépendant de l'état - retard infini.

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Introduction

Probabilistic operator theory is that branch of probabilistic (or stochastic) analysis which is concerned with the study of operator-valued random variables (or, simply, random operators) and their properties. The development of a theory of random operators is of interest in its own right as a probabilistic generalization of (deterministic) operator theory; and just as operator theory is of fundamental importance in the study of operator equations, the development of probabilistic operator theory is required for the study of various classes of random equations. Although several concrete examples of random operators and random operator equations have been around for a long time, the systematic study of probabilistic operator theory and its applications was initiated by the Prague school of probabilists under the direction of the late Antonin Spacek in the 1950's. They recognized that in using operator equations to model various systems (which is the heart of applied mathematics) it is usually not sufficient to consider only random initial data, it is also necessary to take into consideration the fact that the operators used to describe the behavior of systems may not be known exactly. For example, in the case of difference and differential operators the coefficients (constants or functions) might not be known exactly. One knows only their approximate values together with some measure of the possible error. In the case of integral operators, the kernel might not be known exactly; this being the case when either the integral equation is the primary model of a system, or when it is the equivalent formulation of a differential boundary value problem used as a model. In many studies workers use what might be termed mean coefficients or kernels, there by casting their problems in the framework of deterministic operator equations. The main disadvantage of this approach is that, in general, a considerable amount of 'information' is lost concerning the behavior of the system. In the theory of random operator equations the coefficients or kernels are assumed from the outset to be random variables or random functions; and the solutions obtained (if they exist) are random functions whose dynamical and statistical properties can be studied.

It is of interest to remark that the distinction between a deterministic and probabilistic approach to the formulation of operator equations lies mainly in the nature of the questions they try to answer, and in the interpretation of the results. The advantages of a probabilistic approach are that (1) it permits from the initial formulation a greater generality (and hence flexibility) than that offered by a deterministic approach, and (2) it permits the inclusion of probabilistic features in the equations, which may play an essential role in making the connection between operator equations and the real phenomena they purport to describe.

Research in probabilistic operator theory generally falls into one or more of the following areas: (1) operator-valued random variables and their properties, (2) operator-valued random functions (including semigroups of random operators) and their properties, (3) random equations whose solutions are operator-valued, (4) spectral theory of random operators, (5) measure-theoretic problems, (6) fixed point theorems, and (7) limit theorems. In this thesis I have elected to restrict my attention to just one of the areas-namely, fixed point theorems for random operators.

The importance of random fixed point theory lies in its vast applicability in probabilistic functional analysis and various probabilistic models. The introduction of randomness however leads to several new questions of measurability of solutions, probabilistic and statistical aspects of random solutions. It is well known that random fixed point theorems are stochastic generalization of classical fixed point theorems what we call as determinstic results. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by $\tilde{S}$pa$\tilde{c}$ek [81] and Han$\tilde{s}$ (see [52, 53]). The survey article by Bharucha-Reid [20] in 1976 attracted the attention of several mathematicians and gave wings to this theory. Itoh [40] extended $\tilde{S}$pa$\tilde{c}$ekŠs and Han$\tilde{s}$Šs theorems to multivalued contraction mappings. Random fixed point theorems with an application to Random differential equations in Banach spaces are obtained by Itoh [40]. Sehgal and Waters [77] had obtained several random fixed point theorems including random analogue of the classical results due to Rothe [73]. In recent past, several fixed point theorems including Kannan type [46] Chatterjeea [24] and Zamfirescu type [91] have been generalized in stochastic version (see for detail in Joshi and Bose [42], Saha et al. ([74, 75]).

On the other hand, the stochastic differential equation with delay is a special type of stochastic functional differential equations. Delay differential equations arise in many biological and physical applications, and it often forces us to consider variable or state-dependent delays. The stochastic functional differential equations with state-dependent delay have many important applications in mathematical models of real phenomena, and the study of this type of equations has received much attention in recent years. Guendouzi and Benzatout [38] studied the existence of mild solutions for a class of impulsive stochastic differential inclusions with state-dependent delay. Sakthivel and Ren [76] studied the approximate controllability of fractional differential equations with state-dependent delay.

This thesis is devoted to the existence of mild random solution for various classes of first and second order functional differential evolutions equations with random effect, finite and infinite delay in separable Banach space $(E, |.|)$. Sufficient conditions are considered to get the existence of mild random solutions by reducing this research to the search for the existence of random fixed point of a continuous random operator with stochastic domain under rather general conditions provided that the corresponding deterministic fixed point problem is solvable. We have arranged this thesis as follows:

In **Chapter 1**, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

In **Chapter 2**, In section one of chapter 2, we prove the existence of random mild solutions of the following functional differential equation with constante delay and random effects (random parameters) of the form:

$$y'(t,w) = Ay(t,w) + f(t, y_t(.,w), w), \quad \text{a.e. } t \in J := [0,T] \tag{1}$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \tag{2}$$

(with some notations to be given later), where (Ω, F, P) is a complete probability space, $w \in \Omega$ where $f : J \times \mathcal{B} \times \Omega \to E$, $\phi \in \mathcal{B} \times \Omega$ are given random functions which represent random nonlinear of the system. This problem has been considered in the paper [15].

In Section 2.6 of chapter 2, we prove the existence of mild solutions of the following functional differential equation with infinite delay and random effects (random parameters) of the form:

$$y'(t,w) = A(t)y(t,w) + f(t, y_t(.,w), w), \quad \text{a.e. } t \in J := [0,\infty) \tag{3}$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \tag{4}$$

(with some notations to be given later), where (Ω, F, P) is a complete probability space, $w \in \Omega$ where $f : J \times \mathcal{B} \times \Omega \to E$, $\phi \in \mathcal{B} \times \Omega$ are given random functions which represent random nonlinear of the system. This problem has been considered in the paper [5].

In **Chapter 3**, we prove the existence of random mild solutions of the following functional differential equation with delay and random effects (random parameters) of the form:

$$y'(t,w) = Ay(t,w) + f(t, y_{\rho(t,y_t)}(\cdot,w), w), \quad \text{a.e. } t \in J := [0,T] \tag{5}$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \tag{6}$$

where (Ω, F, P) is a complete probability space, $w \in \Omega$, $f : J \times \mathcal{B} \times \Omega \to E$, $\phi \in \mathcal{B} \times \Omega$ are given random functions which represent random nonlinear of the system. This problem has been considered in the paper [16].

In **Chapter 4**, we prove the existence of mild random solutions of the following functional evolution differential equation with delay and random effects (random paramaters) of the form:

$$y'(t,w) = A(t)y(t,w) + f(t, y_{\rho(t,y_t)}(.,w), w), \quad \text{a.e. } t \in J := [0,\infty) \tag{7}$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \tag{8}$$

(with some notations to be given later), where (Ω, F, P) is a complete probability space, $w \in \Omega$ where $f : J \times \mathcal{B} \times \Omega \to E$, $\phi \in \mathcal{B} \times \Omega$ are given random functions which represent random nonlinear of the system. This problem has been considered in the paper [18].

In **Chapter 5**, we prove the existence of random mild solutions of the following functional differential equation with delay and random effects (random paramaters) of the form:

$$y''(t,w) = Ay(t,w) + f(t, y_t(\cdot,w), w), \quad \text{a.e. } t \in J := [0,T] \tag{9}$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \quad y'(0,w) = \varphi(w) \in E, \tag{10}$$

(with some notations to be given later), where (Ω, F, P) is a complete probability space, $w \in \Omega$ where $f : J \times \mathcal{B} \times \Omega \to E$, $\phi \in \mathcal{B} \times \Omega$ are given random functions which represent random nonlinear of the system. Later, we consider the following problem

$$y''(t,w) = Ay(t,w) + f(t, y_{\rho(t,y_t)}(\cdot,w), w), \quad \text{a.e. } t \in J := [0,T] \tag{11}$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \quad y'(0,w) = \varphi(w) \in E, \tag{12}$$

(with some notations to be given later), where (Ω, F, P) is a complete probability space, $w \in \Omega$ where $f : J \times \mathcal{B} \times \Omega \to E$, $\phi \in \mathcal{B} \times \Omega$ are given random functions which represent random nonlinear of the system, $A : D(A) \subset E \to E$ as in problem (9)-(10). This problem has been considered in the paper [17].

Chapter 1

Preliminaries

In this chapter, we present some notations, definitions and auxiliary results which are used throughout this thesis.

1.1 Notations and definitions

Let $J = [0, T]$ be a real interval , $C(J, E)$ be the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty = \sup \{ \ |y(t)| \ : \ t \in J \ \}.$$

Let $B(E)$ denote the Banach space of bounded linear operators from E into E.

A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the Bochner integral properties, see the classical monograph of Yosida [90]).

Let $L^1(J, E)$ be the Banach space of measurable functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^T |y(t)| \ dt.$$

Definition 1.1 *A map $f : J \times \mathcal{B} \times \Omega \rightarrow E$ is said to be Random Carathéodory if*

(i) $t \rightarrow f(t, y, w)$ is measurable for all $y \in \mathcal{B}$,and for all $w \in \Omega$.

(ii) $y \rightarrow f(t, y, w)$ is continuous for almost each $t \in J$, and for all $w \in \Omega$.

(iii) $w \rightarrow f(t, y, w)$ is measurable for all $y \in \mathcal{B}$, and almost each $t \in J$.

For a given set V of functions $v : (-\infty, T] \longrightarrow E$, let us denote by

$$V(t) = \{v(t) : v \in V\}, \ t \in (-\infty, T]$$

and

$$V(J) = \{v(t) : v \in V, \ t \in (-\infty, T]\}.$$

1.2 Some Examples of Phase Spaces

For any continuous function y and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t+\theta) \qquad \text{for } \theta \in (-\infty, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time t. We assume that the histories y_t belongs to some abstract *phase space* $\mathcal{B}$, to be specified later.

Consider the following space

$$B_{+\infty} = \{y : (-\infty, +\infty) \to E : y|_J \in \mathcal{C}(J; E), \quad y_0 \in \mathcal{B}\},$$

where $y|_J$ is the restriction of y to J.

In this work, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [50] and follow the terminology used in [59]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(A_1) If $y : (-\infty, T) \longrightarrow E, T > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold :
(i) $y_t \in \mathcal{B}$;
(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
(iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{\ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Denote $K_T = \sup\{K(t) : t \in J\}$ and $M_T = \sup\{M(t) : t \in J\}$.

Remark 1.2 1. *(ii) is equivalent to* $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ *for every* $\phi \in \mathcal{B}$.

2. *Since* $\|\cdot\|_{\mathcal{B}}$ *is a seminorm, two elements* $\phi, \psi \in \mathcal{B}$ *can verify* $\|\phi - \psi\|_{\mathcal{B}} = 0$ *without necessarily* $\phi(\theta) = \psi(\theta)$ *for all* $\theta \leq 0$.

3. *From the equivalence of in the first remark, we can see that for all* $\phi, \psi \in \mathcal{B}$ *such that* $\|\phi - \psi\|_{\mathcal{B}} = 0$ *: We necessarily have that* $\phi(0) = \psi(0)$.

We now indicate some examples of phase spaces. For other details we refer, for instance to the book by Hino *et al* [59].

Example 1.3 *Let:*

BC *the space of bounded continuous functions defined from* $(-\infty, 0]$ *to* E;

BUC the space of bounded uniformly continuous functions defined from $(-\infty,0]$ *to* E;

$$C^{\infty} := \left\{\phi \in BC : \lim_{\theta \longrightarrow -\infty} \phi(\theta) \text{ exist in } E\right\};$$

$$C^{0} := \left\{\phi \in BC : \lim_{\theta \longrightarrow -\infty} \phi(\theta) = 0\right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\{|\phi(\theta)| : \ \theta \leq 0\}.$$

We have that the spaces BCU, C^{∞} *and* C^{0} *satisfy conditions* $(A_1)-(A_3)$. *However,* BC *satisfies* $(A_1),(A_3)$ *but* (A_2) *is not satisfied.*

Example 1.4 *The spaces* C_g, UC_g, C_g^{∞} *and* C_g^{0}.
Let g *be a positive continuous function on* $(-\infty,0]$. *We define:*

$$C_g := \left\{\phi \in C((-\infty,0],E) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty,0]\right\};$$

$$C_g^{0} := \left\{\phi \in C_g : \lim_{\theta \longrightarrow -\infty} \frac{\phi(\theta)}{g(\theta)} = 0\right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)} : \ \theta \leq 0\right\}.$$

Then we have that the spaces C_g *and* C_g^{0} *satisfy conditions* (A_3). *We consider the following condition on the function* g.

(g_1) *For all* $a>0$, $\displaystyle\sup_{0\leq t\leq a} \sup\left\{\frac{g(t+\theta)}{g(\theta)} : -\infty < \theta \leq -t\right\} < \infty.$

They satisfy conditions (A_1) *and* (A_2) *if* (g_1) *holds.*

Example 1.5 *The space* C_{γ}.
For any real positive constant γ, *we define the functional space* C_{γ} *by*

$$C_{\gamma} := \left\{\phi \in C((-\infty,0],E) : \lim_{\theta \longrightarrow -\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E\right\}$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \ \theta \leq 0\}.$$

Then in the space C_{γ} *the axioms* $(A_1)-(A_3)$ *are satisfied.*

In what follows, we assume that $\{A(t),\ t\geq 0\}$ is a family of closed densely defined linear unbounded operators on the Banach space E and with domain $D(A(t))$ independent of t.

Definition 1.6 *A family of bounded linear operators*

$$\{U(t,s)\}_{(t,s)\in\Delta} : U(t,s) : E \longrightarrow E \ (t,s) \in \Delta := \{(t,s) \in J \times J : 0 \leq s \leq t < +\infty\}$$

is called en evolution system if the following properties are satisfied:

1. $U(t,t) = I$ *where* I *is the identity operator in* E,
2. $U(t,s)\ U(s,\tau) = U(t,\tau)$ *for* $0 \leq \tau \leq s \leq t < +\infty$,
3. $U(t,s) \in B(E)$ *the space of bounded linear operators on* E, *where for every* $(s,t) \in \Delta$ *and for each* $y \in E$, *the mapping* $(t,s) \longrightarrow U(t,s)\ y$ *is continuous.*

More details on evolution systems and their properties could be found on the books of Ahmed [2], Engel and Nagel [34] and Pazy [71].

Lemma 1.7 (Corduneanu) *[26]*
Let $C \subset BC(J,E)$ *be a set satisfying the following conditions:*

(i) C *is bounded in* $BC(J,E)$;

(ii) *the functions belonging to* C *are equicontinuous on any compact interval of* J;

(iii) *the set* $C(t) := \{y(t) : y \in C\}$ *is relatively compact on any compact interval of* J;

(iv) *the functions from* C *are equiconvergent, i.e., given* $\varepsilon > 0$, *there corresponds* $T(\varepsilon) > 0$ *such that* $|y(t) - y(+\infty)| < \varepsilon$ *for any* $t \geq T(\varepsilon)$ *and* $y \in C$.

Then C *is relatively compact in* $BC(J,E)$.

1.3 Some fixed point theorems

Our results will be based on the following well known and some recent nonlinear alternatives of fixed point argument theory.

Theorem 1.8 ***(Schauder fixed point**)[37]*
Let B *be a closed, convex and nonempty subset of a Banach space* E. *Let* $N : B \to B$ *be a continuous mapping such that* $N(B)$ *is a relatively compact subset of* E. *Then* N *has at least one fixed point in* B.

Theorem 1.9 *(Mönch)[3, 66] Let* D *be a bounded, closed and convex subset of a Banach space such that* $0 \in D$, *and let* N *be a continuous mapping of* D *into itself. If the implication*

$$V = \overline{conv} N(V) \quad or \quad V = N(V) \cup 0 \Longrightarrow \alpha(V) = 0$$

holds for every subset V *of* D, *then* N *has a fixed point.*

Lemma 1.10 *[39] Let* D *be a bounded, closed and convex subset of Banach space* X. *If the operator* $N : D \longrightarrow D$ *is a strict set contraction, i.e there is a constant* $0 \leq \lambda < 1$ *such that* $\alpha(N(S)) \leq \lambda\alpha(S)$ *for any bounded set* $S \subset D$ *then* N *has a fixed point in* D.

1.4 Random operators

Let Y be a separable Banach space with the Borel σ-algebra B_Y, (Ω, F, P) be a complete probability space. A mapping $y : \Omega \longrightarrow Y$ is said to be a random variable with values in Y if for each $B \in B_Y, y^{-1}(B) \in F$. A mapping $T : \Omega \times Y \longrightarrow Y$ is called a random operator if $T(., y)$ is measurable for each $y \in Y$ and is generally experessed as $T(w, y) = T(w)y$; we will use these two expressions alternatively.

Lemma 1.11 *Let* $f : [0,1] \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ *be such that for all* $\omega \in \Omega$ *and* $x \in C_n([0,1]), f(., x(.), \omega)$ *is Riemann-integrable and for all* $(t, x) \in [0,1] \times \mathbb{R}^n, f(t, x, .)$ *is measurable. Let* $T : \Omega \times C_n([0,1]) \to C_n([0,1])$

$$(\omega, x) \longrightarrow \left(t \longrightarrow \int_0^t f(s, x(s), w)ds\right)$$

Then T *is random operator.*

proof *: Let* $x \in C_n([0,1])$ *and* $t \in [0,1]$ *arbitrary, but fixed. Then* $\omega \longrightarrow \int_0^t f(s, x(s), w)ds$ *is measurable as the limit of a sequence of finite sums of measurable functions. Therefore* $T(., x)(t)$ *is measurable. Then* $T(., x)$ *is measurable which means that* T *is a random operator.*

Lemma 1.12 *Let* $K : [0,1]^2 \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ *be such that* $K(., ., ., \omega)$ *is continuous in* $[0,1]^2 \times \mathbb{R}^n$ *for all* $\omega \in \Omega$ *and* $K(t, s, x, .)$ *is measurable for all* $(t, s, x) \in [0,1]^2 \times \mathbb{R}^n$*. Let* $T : \Omega \times C_n([0,1]) \to C_n([0,1])$

$$(\omega, x) \longrightarrow \left(t \longrightarrow \int_0^1 k(t, s, x(s), w)ds\right)$$

Then T *is random operator.*

Proof *: Analogous to the proof of last example.*

Next, we will give a very useful random fixed point theorem with stochastic domain.

Definition 1.13 *[33] Let* C *be a mapping from* Ω *into* 2^Y*. A mapping* $T : \{(w, y) : w \in \Omega \wedge y \in C(w)\} \longrightarrow Y$ *is called 'random operator with stochastic domain* C*' iff* C *is measurable (i.e., for all closed* $A \subseteq Y, \{w \in \Omega : C(w) \cap A \neq \emptyset\} \in F$*) and for all open* $D \subseteq Y$ *and all* $y \in Y, \{w \in \Omega : y \in C(w) \wedge T(w, y) \in D\} \in F$*.* T *we be called 'continuous' if every* $T(w)$ *is continuous. For a random operator* T*, a mapping* $y : \Omega \longrightarrow Y$ *is called 'random (stochastic) fixed point of* T*' iff for p-almost all* $w \in \Omega, y(w) \in C(w)$ *and* $T(w)y(w) = y(w)$ *and for all open* $D \subseteq Y, \{w \in \Omega : y(w) \in D\} \in F$*('y is measurable').*

Remark 1.14 *If* $C(w) \equiv Y$*, then the definition of random operator with stochastic domain coincides with the definition of random operator.*

Lemma 1.15 *[33] Let $C : \Omega \longrightarrow 2^Y$ be measurable with $C(w)$ closed, covex and solid (i.e., int $C(w) \neq \emptyset$) for all $w \in \Omega$. We assume that there exists measurable $y_0 : \Omega \longrightarrow Y$ with $y_0 \in$ int $C(w)$ for all $w \in \Omega$. Let T be a continuous random operator with stochastic domain C such that for every $w \in \Omega, \{y \in C(w) : T(w)y = y\} \neq \emptyset$. Then T has a stochastic fixed point.*

Let y be a mapping of $J \times \Omega$ into X. y is said to be a stochastic process if for each $t \in J, y(t,.)$ is measurable.

1.5 Measure of noncompactness

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 1.16 *[12] Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \longrightarrow [0,\infty)$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E.$$

The Kuratowski measure of noncompactness satisfies the following properties(for more details see [12]).

(a) $\alpha(B) = 0 \Longleftrightarrow \overline{B}$ is compact (B is relatively compact).

(b) $\alpha(B) = \alpha(\overline{B})$.

(c) $A \subset B \Longrightarrow \alpha(A) \leq \alpha(B)$.

(d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

(e) $\alpha(cB) = |c|\alpha(B); c \in \mathbb{R}$

(f) $\alpha(conv\mathrm{B}) = \alpha(B)$.

Lemma 1.17 ([60, 39]) *If $H \subset C(J,E)$ is bounded and equicontinuous, then $\alpha(H(t))$ is continuous on J and*

$$\alpha\left(\left\{\int_J x(s)ds : x \in H\right\}\right) \leq \int_J \alpha(H(s))ds,$$

Where $H(s) = \{x(s) : x \in H\}, t \in J$

Chapter 2

Functional Differential Equations with Delay and Random Effects

2.1 Introduction

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. Differential delay equations, and functional differential equations, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay [50, 51, 59, 78, 86]. An extensive theory is developed for evolution equations [2, 34]. Uniqueness and existence results have been established recently for different evolution problems in the papers by Baghli and Benchohra for finite and infinite delay in [8, 9, 10].

On the other hand, the nature of a dynamic system in engineering or natural sciences depends on the accuracy of the information we have concerning the parameters that describe that system. If the knowledge about a dynamic system is precise then a deterministic dynamical system arises. Unfortunately in most cases the available data for the description and evaluation of parameters of a dynamic system are inaccurate, imprecise or confusing. In other words, evaluation of parameters of a dynamical system is not without uncertainties. When our knowledge about the parameters of a dynamic system are of statistical nature, that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations. Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians.We refer the reader to the monographs [13, 83], the papers [27, 28, 32, 79] and the references therein. We also refer the reader to recent results [47, 49, 48]. There are real world phenomena with anomalous dynamics such as signals transmissions through strong magnetic fields, atmospheric diffusion of pollution, network traffic, the effect of speculations on the profitability of stocks in financial markets and so on where the classical models are not sufficiently good to describe these features.

2.2 Functional differential equations with constant delay and random effects

2.2.1 Introduction

In this work we prove the existence of random mild solutions of the following functional differential equation with delay and random effects (random parameters) of the form:

$$y'(t,w) = Ay(t,w) + f(t, y_t(.,w), w), \quad \text{a.e. } t \in J := [0,T] \tag{2.1}$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \tag{2.2}$$

(with some notations to be given later), where (Ω, F, P) is a complete probability space, $w \in \Omega$ where $f : J \times \mathcal{B} \times \Omega \to E$, $\phi \in \mathcal{B} \times \Omega$ are given random functions which represent random nonlinear of the system, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, of bounded linear operators in a Banach space E, $\mathcal{B}$ is the phase space to be specified later, and $(E, |.|)$ is a real separable Banach space. For any function y defined on $(-\infty, T] \times \Omega$ and any $t \in J$ we denote by $y_t(.,w)$ the element of $\mathcal{B} \times \Omega$ defined by $y_t(\theta, w) = y(t+\theta, w), \theta \in (-\infty, 0]$. Here $y_t(.,w)$ represents the history of the state from time $-\infty$, up to the present time t. We assume that the histories $y_t(.,w)$ belong to some abstract phases $\mathcal{B}$, to be specified later. To our knowledge, the literature on the local existence of random evolution equations with delay is very limited, so the present work can be considered as a contribution to this question.

2.2.2 Existence of mild solutions

Now we give our main existence result for problem (2.1)-(2.2). Before starting and proving this result, we give the definition of the mild random solution.

Definition 2.1 *A stochastic process $y : J \times \Omega \to E$ is said to be a random mild solution of problem (2.1)-(2.2) if $y(t,w) = \phi(t,w),\ t \in (-\infty, 0]$ and the restriction of $y(.,w)$ to the interval $[0,T]$ is continuous and satisfies the following integral equation:*

$$y(t,w) = T(t)\phi(0,w) + \int_0^t T(t-s) f(s, y_s(.,w), w) ds, \ t \in J. \tag{2.3}$$

We will need to introduce the following hypotheses which are be assumed there after:

(H_1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$ which is compact for $t > 0$ in the Banach space E. Let $M = \sup\{\|T\|_{B(E)} : t \geq 0\}$.

(H_2) The function $f : J \times \mathcal{B} \times \Omega \to E$ is random Carathéodory.

(H_3) There exist two functions $\psi : J\times\Omega \longrightarrow \mathbb{R}^+$ and $p : J\times\Omega \longrightarrow \mathbb{R}^+$ such that for each $w\in\Omega$, $\psi(.,w)$ is a continuous nondecreasing function and $p(.,w)$ integrable with:

$$|f(t,u,w)| \leq p(t,w)\ \psi(\|u\|_{\mathcal{B}},w) \text{ for a.e. } t\in J \text{ and each } u\in\mathcal{B},$$

(H_4) There exists a random function $R : \Omega \longrightarrow \mathbb{R}^+\backslash\{0\}$ such that:

$$M\|\phi\|_{\mathcal{B}} + M\psi(D_T,w)\|p\|_{L^1} \leq R(w),$$

where

$$D_T := K_T R(w) + M_T\|\phi\|_{\mathcal{B}}.$$

(H_5) For each $w\in\Omega, \phi(.,w)$ is continuous and for each $t, \phi(t,.)$ is measurable.

Theorem 2.2 *Suppose that hypotheses $(H_1)-(H_4)$ are valid, then the problem (2.1)-(2.2) has at least one mild random solution on $(-\infty,T]$.*

Proof. Let $Y = \{u \in C(J,E) : u(0,w) = \phi(0,w) = 0\}$ endowed with the uniform convergence topology and $N : \Omega\times Y \longrightarrow Y$ be the random operator defined by

$$(N(w)y)(t) = T(t)\ \phi(0,w) + \int_0^t T(t-s)\ f(s,\overline{y}_s,w)\ ds, \quad t\in J, \tag{2.4}$$

where $\bar{y} : (-\infty,T]\times\Omega \longrightarrow E$ such that $\bar{y}_0(.,w) = \phi(.,w)$ and $\bar{y}(.,w) = y(.,w)$ on J. Let $\bar{\phi} : (-\infty,T]\times\Omega \longrightarrow E$ be the extension of ϕ to $(-\infty,T]$ such that $\bar{\phi}(\theta,w) = \phi(0,w) = 0$ on J.
Then we show that the mapping defined by (2.4) is a random operator. To do this, we need to prove that for any $y\in Y$, $N(.)(y) : \Omega \longrightarrow Y$ is a random variable. Then we prove that $N(.)(y) : \Omega \longrightarrow Y$ is measurable since the mapping $f(t,y,.),\ t\in J,\ y\in Y$ is measurable by assumption (H_2) and (H_5).
Let $D : \Omega \longrightarrow 2^Y$ be defined by:

$$D(w) = \{y\in Y : \|y\| \leq R(w)\}.$$

$D(w)$ is bounded, closed, convex and solid for all $w\in\Omega$. Then D is measurable by Lemma 17 (see [33]).
Let $w\in\Omega$ be fixed, then for any $y\in D(w)$, and by assumption (A1), we get

$$\begin{aligned}\|\overline{y}_s\|_{\mathcal{B}} &\leq L(s)|\overline{y}(s)| + M(s)\|\overline{y}_0\|_{\mathcal{B}}\\ &\leq K_T|\overline{y}(s)| + M_T\|\phi\|_{\mathcal{B}}.\end{aligned}$$

and by (H_4) and (H_3), we have:

$$\begin{aligned}|(N(w)y)(t)| &\leq \|T(t)\||\phi(0,w)| + M\int_0^t |f(s,\overline{y}_s,w)|\ ds\\ &\leq M\|\phi\|_{\mathcal{B}} + M\int_0^t p(s,w)\ \psi\left(\|\overline{y}_s\|_{\mathcal{B}},w\right)\ ds.\end{aligned}$$

Set

$$D_T := K_T R(w) + M_T \|\phi\|_{\mathcal{B}}.$$

Then, we have

$$|(N(w)y)(t)| \le M\|\phi\|_{\mathcal{B}} + M\psi(D_T, w)\int_0^T p(s,w)\,ds.$$

Thus

$$\|(N(w)y)\| \le M\|\phi\|_{\mathcal{B}} + M\psi(D_T, w)\|p\|_{L^1} \le R(w).$$

This implies that N is a random operator with stochastic domain D and $N(w) : D(w) \longrightarrow D(w)$ for each $w \in \Omega$.

Step 1: N is continuous.

Let y^n be a sequence such that $y^n \longrightarrow y$ in Y. Then

$$\begin{aligned} |(N(w)y^n)(t) - (N(w)y)(t)| &= \Big| T(t)\phi(0,w) + \int_0^t T(t-s)[f(s,\overline{y}^n_s, w) - f(s,\overline{y}_s, w)]\ ds \Big| \\ &\le M\int_0^t |f(s,\overline{y}^n_s, w) - f(s,\overline{y}_s, w)|\ ds. \end{aligned}$$

Since $f(s,.,w)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$\|f(.,y^n_., w) - f(.,y_., w)\|_{L^1} \longrightarrow 0 \ \text{ as } \ n \longrightarrow +\infty.$$

Thus N is continuous.

Step 2: We prove that for every $w \in \Omega, \{y \in D(w) : N(w)y = y\} \neq \emptyset$. To prove this, we apply Schauder's theorem.

(a) N maps bounded sets into equicontinuous sets in $D(w)$.
Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_2 > \tau_1$, $D(w)$ be a bounded set, and $y \in D(w)$. Then

$$\begin{aligned} |(N(w)y)(\tau_2) - (N(w)y)(\tau_1)| &\le |T(\tau_2) - T(\tau_1)|\|\phi\|_{\mathcal{B}} \\ &+ \Big|\int_0^{\tau_1}[T(\tau_2 - s) - T(\tau_1 - s)]f(s,\overline{y}_s, w)\,ds\Big| \\ &+ \Big|\int_{\tau_1}^{\tau_2} T(\tau_2 - s)f(s,\overline{y}_s, w)\,ds\Big| \\ &\le |T(\tau_2) - T(\tau_1)|\|\phi\|_{\mathcal{B}} \\ &+ \int_0^{\tau_1}|T(\tau_2 - s) - T(\tau_1 - s)||f(s,\overline{y}_s, w)|\,ds \\ &+ \int_{\tau_1}^{\tau_2}|T(\tau_2 - s)||f(s,\overline{y}_s, w)|\,ds \\ &\le |T(\tau_2) - T(\tau_1)|\|\phi\|_{\mathcal{B}} \\ &+ \psi(D_T, w)\int_0^{\tau_1}|T(\tau_2 - s) - T(\tau_1 - s)|p(s,w)ds \\ &+ M\psi(D_T, w)\int_{\tau_1}^{\tau_2} p(s,w)ds. \end{aligned}$$

The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$, since $T(t)$ is uniformly continuous. As N is bounded and equicontinuous together with the Arzelá-Ascoli theorem it suffices to show that the operator N maps $D(w)$ into a precompact set in E.

(b) Let $t \in [0, T]$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in D(w)$ we define

$$(N_\epsilon(w)y)(t) = T(t)\phi(0,w) + T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)f(s, y_s, w)\, ds.$$

Since $T(t)$ is a compact operator, the set $Z_\epsilon(t,w) = \{(N_\epsilon(w)y)(t) : y \in D(w)\}$ is pre-compact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\begin{aligned} |(N(w)y)(t) - (N_\epsilon(w)y)(t)| &\le \int_{t-\epsilon}^{t} \|T(t-s)\| |f(s, y_s, w)| ds \\ &\le M\psi(D_T, w)\int_{t-\epsilon}^{t} p(s,w) ds. \end{aligned}$$

Therefore the set $Z(t,w) = \{(N(w)y)(t) : y \in D(w)\}$ is precompact in E.

A consequence of Steps 1-2 and (a), (b), we can conclude that $N(w) : D(w) \to D(w)$ is continuous and compact. From Schauder's theorem, we deduce that $N(w)$ has a fixed point $y(w)$ in $D(w)$.

Since $\bigcap_{w\in\Omega} D(w) \neq \emptyset$, the hypothesis that a measurable selector of int$D$ exists holds. By Lemma 1.15, the random operator N has a stochastic fixed point $y^*(w)$, which is a random mild solution of the random problem (2.1)-(2.2).

2.2.3 An example

Consider the following functional partial differential equation:

$$\frac{\partial}{\partial t} z(t,x,w) = \frac{\partial^2}{\partial x^2} z(t,x,w) + C_0(w)K(w)e^{-t}\int_{-\infty}^{0} \frac{\exp(z(t+s,x,w))}{1+s^2} ds, \tag{2.5}$$

$$x \in [0,\pi],\ t \in [0,T]$$

$$z(t,0,w) = z(t,\pi,w) = 0,\ t \in [0,T], \tag{2.6}$$

$$z(s,x,w) = z_0(s,x,w),\ s \in (-\infty, 0],\ x \in [0,\pi], \tag{2.7}$$

where K and C_0 are a real-valued random variable.
Let $E = L^2[0,\pi]$, (Ω, F, P) be a complete probability space, and define $A : E \to E$ by $Av = v''$ with domain

$$D(A) = \{v \in E, v, v' \text{are absolutely continuous},\ v'' \in E,\ v(0) = v(\pi) = 0\}.$$

Then

$$Av = \sum_{n=1}^{\infty} n^2(v, v_n)v_n, v \in D(A)$$

where $\omega_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, n = 1, 2, \ldots$ is the orthogonal set of eigenvectors in A. It is well know (see [71]) that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in E and is given by

$$T(t)v = \sum_{n=1}^{\infty} \exp(-n^2 t)(v, v_n)v_n, v \in E.$$

Since the analytic semigroup $T(t)$ is compact, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Let $\mathcal{B} = BCU(\mathbb{R}^-; E)$ be the space of bounded uniformly continuous functions endowed with the following norm:

$$\|\phi\| = \sup_{s \leq 0} |\phi(s)| \qquad \text{for } \phi \in \mathcal{B}$$

If we put $\phi \in BCU(\mathbb{R}^-; E), x \in [0, \pi]$ and $w \in \Omega$

$$y(t, x, w) = z(t, x, w), t \in [0, T]$$
$$\phi(s, x, w) = z_0(s, x, w),\ s \in (-\infty, 0].$$

Set

$$f(t, \varphi(x), w) = \int_{-\infty}^{0} e^{-t} \varphi(s, x, w) ds,$$

with

$$\varphi(s, x, w) = \exp(z(t + s, x, w)).$$

The function $f(t, \varphi(x), w)$ is Carathéodory, and satisfies (H_2) with

$$p(t, w) = K(w)\frac{\pi}{2}e^{-t} \text{ and } \psi(x, w) = |C_0(w)|e^x.$$

Then the problem (2.1)-(2.2) in an abstract formulation of the problem (2.5)-(2.7), and conditions $(H_1) - (H_5)$ are satisfied. Theorem 2.2 implies that the random problem (2.5)-(2.7) has at least one random mild solutions.

2.3 Functional evolution equations with infinite delay and random effects

2.3.1 Introduction

Functional evolution equations have a very important role to describe meany phenomena of physics, mechanics, biology etc; For more details on this theory and on its applications we refer to the monographs of Hale and Verduyn Lunel [51], Kolmanovskii and Myshkis [43], and Wu [86] and the reference therein. Recently, many authors have study the existence of various model of semilinear evolution equations with finite and infinite delay in the Fréchet space, for instance we refer to Baghli and Benchohra [8, 9, 10]. In the other hand, different fields of engineering problems which is currently interest in unbounded domains. As a result it has received the attention of the researches see [4, 69, 70].

In this work we prove the existence of random mild solutions of the following functional differential equation with delay and random effects (random parameters) of the form:

$$y'(t,w) = A(t)y(t,w) + f(t, y_t(.,w), w), \quad \text{a.e. } t \in J := [0,\infty) \tag{2.8}$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \tag{2.9}$$

(with some notations to be given later), where (Ω, F, P) is a complete probability space, $w \in \Omega$ where $f : J \times \mathcal{B} \times \Omega \to E$, $\phi \in \mathcal{B} \times \Omega$ are given random functions which represent random nonlinear of the system, $\{A(t)\}_{0\leq t<+\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $0 \leq s \leq t < +\infty$, $\mathcal{B}$ is the phase space to be specified later, and $(E, |.|)$ is a real separable Banach space. For any function y defined on $(-\infty, T] \times \Omega$ and any $t \in J$ we denote by $y_t(.,w)$ the element of $\mathcal{B} \times \Omega$ defined by $y_t(\theta, w) = y(t+\theta, w), \theta \in (-\infty, 0]$. Here $y_t(.,w)$ represents the history of the state from time $-\infty$, up to the present time t. We assume that the histories $y_t(.,w)$ belong to some abstract phases $\mathcal{B}$, to be specified later. To our knowledge, the literature on the local existence of random evolution equations with delay is very limited, so the present work can be considered as a contribution to this question.

2.3.2 Existence of mild solutions

Now we give our main existence result for problem (2.8)-(2.9). Before starting and proving this result, we give the definition of the random mild solution.

Definition 2.3 *A stochastic process $y : J \times \Omega \to E$ is said to be a random mild solution of problem (2.8)-(2.9) if $y(t,w) = \phi(t,w),\ t \in (-\infty, 0]$ and the restriction of $y(.,w)$ to the interval $[0,\infty)$ is continuous and satisfies the following integral equation:*

$$y(t,w) = U(t,0)\phi(0,w) + \int_0^t U(t,s)f(s, y_s(.,w), w)ds,\ t \in J. \tag{2.10}$$

We will need to introduce the following hypotheses which are be assumed there after:

(H_1) There exists a constant $M \geq 1$ and $\alpha > 0$ such that

$$\|U(t,s)\|_{B(E)} \leq Me^{-\alpha(t-s)} \quad \text{for every } (s,t) \in \Delta.$$

(H_2) The function $f : J \times \mathcal{B} \times \Omega \to E$ is random Carathéodory.

(H_3) There exist two functions $\psi : J\times\Omega \longrightarrow \mathbb{R}^+$ and $p : J\times\Omega \longrightarrow \mathbb{R}^+$ such that for each $w\in\Omega$, $\psi(.,w)$ is a continuous nondecreasing function and $p(.,w)$ integrable with:

$$|f(t,u,w)| \leq p(t,w)\ \psi(\|u\|_{\mathcal{B}},w) \text{ for a.e. } t\in J \text{ and each } u\in\mathcal{B},$$

(H_4) There exists a random function $R : \Omega \longrightarrow \mathbb{R}^+\backslash\{0\}$ such that:

$$M\|\phi\|_{\mathcal{B}} + M\psi(D_T,w)\|p\|_{L^1} \leq R(w),$$

where

$$D_T := K_T R(w) + M_T\|\phi\|_{\mathcal{B}}.$$

(H_5) For each $w\in\Omega, \phi(.,w)$ is continuous and for each $t, \phi(t,.)$ is measurable.

(H_6) For each $(t,s)\in\Delta$ we have: $\displaystyle\lim_{t\longrightarrow+\infty}\int_0^t e^{-\alpha(t-s)}p(s,w)ds = 0.$

Theorem 2.4 *Suppose that hypotheses* $(H_1)-(H_4)$ *are valid, then the problem (2.1)-(2.2) has at least one mild random solution on* $(-\infty,\infty)$.

Proof. Let Y be the space defined by

$$Y = \{y : \mathbb{R} \longrightarrow E \text{ such that } y|_J \in BC(J,E) \text{ and } y_0\in\mathcal{B}\},$$

we denote by $y|_J$ the restriction of y to J, endowed with the uniform convergence topology and $N : \Omega\times Y \longrightarrow Y$ be the random operator defined by

$$(N(w)y)(t) = \begin{cases} \phi(t,w), & \text{if } t\in(-\infty,0] \\ U(t,0)\phi(0,w) + \displaystyle\int_0^t U(t,s)f(s,y_s(.,w),w)ds, & \text{if } t\in J, \end{cases} \tag{2.11}$$

Then we show that the mapping defined by (2.11) is a random operator. To do this, we need to prove that for any $y\in Y$, $N(.)(y) : \Omega \longrightarrow Y$ is a random variable. Then we prove that $N(.)(y) : \Omega \longrightarrow Y$ is measurable since the mapping $f(t,y,.),\ t\in J,\ y\in Y$ is measurable by assumption (H_2) and (H_5).

Let $D : \Omega \longrightarrow 2^Y$ be defined by:

$$D(w) = \{y\in Y : \|y\| \leq R(w)\}.$$

$D(w)$ is bounded, closed, convex and solid for all $w\in\Omega$. Then D is measurable by Lemma 17 (see [33]).
Let $w\in\Omega$ be fixed, then for any $y\in D(w)$, and by assumption (A1), we get

$$\begin{aligned} \|y_s\|_{\mathcal{B}} &\leq L(s)|y(s)| + M(s)\|y_0\|_{\mathcal{B}} \\ &\leq K_T|y(s)| + M_T\|\phi\|_{\mathcal{B}}. \end{aligned}$$

and by (H_4) and (H_3), we have:

$$\begin{aligned} |(N(w)y)(t)| &\leq \|U(t,0)\|_{B(E)}|\phi(0,w)| + \int_0^t \|U(t,s)\|_{B(E)}|f(s,y_s,w)|\, ds \\ &\leq Me^{-\alpha t}\|\phi\|_{\mathcal{B}} + M \int_0^t e^{-\alpha(t-s)}p(s,w)\ \psi\left(\|y_s\|_{\mathcal{B}},w\right)\ ds. \end{aligned}$$

Set

$$D_T := K_T R(w) + M_T\|\phi\|_{\mathcal{B}}.$$

Then, we have

$$|(N(w)y)(t)| \leq M\|\phi\|_{\mathcal{B}} + M\psi(D_T,w)\|p\|_{L^1}\, ds.$$

Thus

$$\|(N(w)y)\| \leq M\|\phi\|_{\mathcal{B}} + M\psi(D_T,w)\|p\|_{L^1} \leq R(w).$$

This implies that N is a random operator with stochastic domain D and $F(w) : D(w) \longrightarrow D(w)$ for each $w \in \Omega$.

Step 1: N is continuous.
Let y^n be a sequence such that $y^n \longrightarrow y$ in Y. Then

$$\begin{aligned} |(N(w)y^n)(t) - (N(w)y)(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)}\ |f(s,y^n_s,w) - f(s,y_s,w)|\, ds. \\ &\leq M\int_0^t e^{-\alpha(t-s)}\ |f(s,y^n_s,w) - f(s,y_s,w)|\, ds. \end{aligned}$$

Since $f(s,.,w)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$\|f(.,y^n_.,w) - f(.,y_.,w)\|_{L^1} \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

Thus N is continuous.

Step 2: We prove that for every $w \in \Omega, \{y \in D(w) : N(w)y = y\} \neq \emptyset$. To prove this, we apply Schauder's theorem.
$N(D(w))$ **is relatively compact:** To prove the compactness, we will use Corduneanu's lemma.

(a) Firstly, it is clear that the assumption (i) is holds. Then we will demonstrate that $N(D(w))$ is equicontinuous set for each closed bounded interval $[0,T]$ in J. Let

$\tau_1, \tau_2 \in [0, T]$ with $\tau_2 > \tau_1$, $D(w)$ be a bounded set, and $y \in D(w)$. Then

$$\begin{aligned}
|(N(w)y)(\tau_2) - (N(w)y)(\tau_1)| &\leq \|U(\tau_2, 0) - U(\tau_1, 0)\|_{B(E)} \|\phi\|_{\mathcal{B}} \\
&+ \Big| \int_0^{\tau_1} [U(\tau_2, s) - U(\tau_1, s)] f(s, y_s, w)\, ds \Big| \\
&+ \Big| \int_{\tau_1}^{\tau_2} U(\tau_2, s) f(s, y_s, w)\, ds \Big| \\
&\leq \|U(\tau_2, 0) - U(\tau_1, 0)\|_{B(E)} \|\phi\|_{\mathcal{B}} \\
&+ \int_0^{\tau_1} |U(\tau_2, s) - U(\tau_1, s)| |f(s, y_s, w)|\, ds \\
&+ \int_{\tau_1}^{\tau_2} |U(\tau_2, s)| |f(s, y_s, w)|\, ds \\
&\leq \|U(\tau_2, 0) - U(\tau_1, 0)\|_{B(E)} \|\phi\|_{\mathcal{B}} \\
&+ \psi(D_T, w) \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} p(s, w) ds \\
&+ M\psi(D_T, w) e^{-\alpha(\tau_2 - s)} \int_{\tau_1}^{\tau_2} p(s, w) ds.
\end{aligned}$$

The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$. As N is bounded and equicontinuous.

(b) Now we will prove that $Z(t, w) = \{(N(w)y)(t) : y \in D(w)\}$ is precompact in E. Let $t \in [0, T]$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in D(w)$ we define

$$(N_\epsilon(w)y)(t) = U(t, 0)\phi(0, w) + U(t, t - \epsilon) \int_0^{t-\epsilon} U(t - \epsilon, s) f(s, y_s, w)\, ds.$$

Since $U(t, s)$ is a compact operator and the set $Z_\epsilon(t, w) = \{(N_\epsilon(w)y)(t) : y \in D(w)\}$ is the imageof bounded set of E then $Z_\epsilon(t, w)$ is pre-compact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\begin{aligned}
|(N(w)y)(t) - (N_\epsilon(w)y)(t)| &\leq \int_{t-\epsilon}^{t} \|U(t, s)\|_{B(E)} |f(s, y_s, w)| ds \\
&\leq M\psi(D_T, w) e^{-\alpha(t-s)} \int_{t-\epsilon}^{t} p(s, w) ds.
\end{aligned}$$

Therefore the set $Z(t, w) = \{(N(w)y)(t) : y \in D(w)\}$ is precompact in E.

(c) Finally, it remains to show that N is equiconvergent.

Let $y \in D(w)$, then from (H_1), (H_3) we have

$$|(N(w)y)(t)| \leq Me^{-\alpha t}\|\phi\|_{\mathcal{B}} + M\int_0^t e^{-\alpha(t-s)}p(s,w)\ \psi\left(D_T,w\right)\ ds.$$

It follows immediately by (H_6) that $|(N(w)y)(t)| \longrightarrow 0$ as $t \longrightarrow +\infty$.
Then

$$\lim_{t\longrightarrow+\infty}|(N(w)y)(t) - (N(w)y)(+\infty)| = 0,$$

which implies that N is equiconvergent.

A consequence of Steps 1-2 and (a), (b), (c), we can conclude that $N(w) : D(w) \to D(w)$ is continuous and compact. From Schauder's theorem, we deduce that $N(w)$ has a fixed point $y(w)$ in $D(w)$.

Since $\bigcap_{w\in\Omega} D(w) \neq \emptyset$, the hypothesis that a measurable selector of int$D$ exists holds. By Lemma 1.15, the random operator N has a stochastic fixed point $y^*(w)$, which is a random mild solution of the random problem (2.8)-(2.9).

2.3.3 An example

Consider the following functional partial differential equation:

$$\frac{\partial}{\partial t}z(t,x,w) = a(t,x)\frac{\partial^2}{\partial x^2}z(t,x,w) + C_0(w)K(w)e^{-t}\int_{-\infty}^0 \frac{\exp(z(t+s,x,w))}{1+s^2}ds, \quad (2.12)$$

$$x \in [0,\pi],\ t \in [0,+\infty)$$

$$z(t,0,w) = z(t,\pi,w) = 0,\ t \in [0,+\infty), \quad (2.13)$$

$$z(s,x,w) = z_0(s,x,w),\ s \in (-\infty,0],\ x \in [0,\pi], \quad (2.14)$$

where $a(t,\xi)$ is a continuous function which is uniformly Hölder continuous in t, K and C_0 are a real-valued random variable.
Let $E = L^2[0,\pi]$, (Ω, F, P) be a complete probability space, and define $A(t)$ by

$$A(t)v = a(t,\xi)v''$$

with domain

$$D(A) = \{v \in E, v, v'\text{are absolutely continuous},\ v'' \in E,\ v(0) = v(\pi) = 0\}.$$

Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumption (H_1) (see [36, 45]).
Let $\mathcal{B} = BCU(\mathbb{R}^-; E)$ be the space of bounded uniformly continuous functions endowed with the following norm:

$$\|\phi\| = \sup_{s\leq 0}|\phi(s)| \qquad \text{for } \phi \in \mathcal{B}$$

If we put $\phi \in BCU(\mathbb{R}^-; E), x \in [0, \pi]$ and $w \in \Omega$

$$y(t, x, w) = z(t, x, w), t \in [0, T]$$
$$\phi(s, x, w) = z_0(s, x, w), \ s \in (-\infty, 0].$$

Set

$$f(t, \varphi(x), w) = \int_{-\infty}^{0} e^{-t} \varphi(s, x, w) ds,$$

with

$$\varphi(s, x, w) = \exp(z(t + s, x, w)).$$

The function $f(t, \varphi(x), w)$ is Carathéodory, and satisfies (H_2) with

$$p(t, w) = K(w)\frac{\pi}{2}e^{-t} \text{ and } \psi(x, w) = |C_0(w)|e^x.$$

Then the problem (2.8)-(2.9) in an abstract formulation of the problem (2.12)-(2.14), and conditions $(H_1) - (H_5)$ are satisfied. Theorem 2.4 implies that the random problem (2.12)-(2.14) has at least one random mild solutions.

Chapter 3

Functional Differential Equations with State-Dependent Delay and Random Effects

3.1 Introduction

Functional evolution equations with state-dependent delay appear frequently in mathematical modeling of several real world problems and for this reason the study of this type of equations has received great attention in the last few years, see for instance [30, 55, 56, 1]. An extensive theory is developed for evolution equations [2, 34]. Uniqueness and existence results have been established recently for different evolution problems in the papers by Baghli and Benchohra for finite and infinite delay in [8, 9, 10].

On the other hand, the nature of a dynamic system in engineering or natural sciences depends on the accuracy of the information we have concerning the parameters that describe that system. If the knowledge about a dynamic system is precise then a deterministic dynamical system arises. Unfortunately in most cases the available data for the description and evaluation of parameters of a dynamic system are inaccurate, imprecise or confusing. In other words, evaluation of parameters of a dynamical system is not without uncertainties. When our knowledge about the parameters of a dynamic system are of statistical nature, that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations. Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians; see [61, 62, 63, 82, 89] and references therein. Between them differential equations with random coefficients (see, [82, 25]) offer a natural and rational approach (see [79], Chapter 1), since sometimes we can get the random distributions of some main disturbances by historical experiences and data rather than take all random disturbances into account and assume the noise to be white noises.

3.2 Functional differential equations with state-dependent delay and random effects

3.2.1 Introduction

In this work we prove the existence of random mild solutions of the following functional differential equation with delay and random effects (random parameters) of the form:

$$y'(t,w) = Ay(t,w) + f(t, y_{\rho(t,y_t)}(\cdot,w), w), \quad \text{a.e. } t \in J := [0,T] \tag{3.1}$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \tag{3.2}$$

where (Ω, F, P) is a complete probability space, $w \in \Omega$, $f : J \times \mathcal{B} \times \Omega \to E$, $\phi \in \mathcal{B} \times \Omega$ are given random functions which represent random nonlinear of the system, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t),\ t \in J$, of bounded linear operators in a Banach space E, $\mathcal{B}$ is the phase space to be specified later, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$, and $(E, |.|)$ is a real separable Banach space. For any function y defined on $(-\infty, T] \times \Omega$ and any $t \in J$ we denote by $y_t(\cdot, w)$ the element of $\mathcal{B} \times \Omega$ defined by $y_t(\theta, w) = y(t+\theta, w), \theta \in (-\infty, 0]$. Here $y_t(\cdot, w)$ represents the history of the state from time $-\infty$, up to the present time t. We assume that the histories $y_t(\cdot, w)$ to some abstract phases $\mathcal{B}$, to be specified later. To our knowledge, the literature on the local existence of random evolution equations with delay is very limited, so the present work can be considered as a contribution to this question.

3.2.2 Existence of mild solutions

Now we give our main existence result for problem (3.1)-(3.2). Before starting and proving this result, we give the definition of the random mild solution.

Definition 3.1 A stochastic process $y : J \times \Omega \to E$ is said to be random mild solution of problem (3.1)-(3.2) if $y(t,w) = \phi(t),\ t \in (-\infty, 0]$ and the restriction of $y(., w)$ to the interval $[0,T]$ is continuous and satisfies the following integral equation:

$$y(t,w) = T(t)\phi(0,w) + \int_0^t T(t-s) f(s, y_{\rho(s,y_s)}(\cdot, w), w) ds,\ t \in J. \tag{3.3}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \le 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \longrightarrow (-\infty, T]$ is continuous. Additionally, we introduce following hypothesis:

(H_ϕ) The function $t \longrightarrow \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \longrightarrow (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \le L^\phi(t) \|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.2 The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see for instance [59].

Lemma 3.3 *([54], Lemma 2.4) If $y : (-\infty, T] \longrightarrow E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_T + L^\phi)\|\phi\|_{\mathcal{B}} + K_T \sup\{|y(\theta)|; \theta \in [0, max\{0, s\}]\}, \ s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup_{t \in \mathcal{R}(\rho^-)} L^\phi(t)$.

We will need to introduce the following hypotheses which are assumed there after:

(H_1) The operator solution $T(t)_{t\in J}$ is uniformly continuous for $t > 0$. Let $M = \sup\{\|T\|_{B(E)} : t \geq 0\}$.

(H_2) The function $f : J \times \mathcal{B} \times \Omega \to E$ is random Carathéodory.

(H_3) There exists two functions $\psi : J \times \Omega \longrightarrow \mathbb{R}^+$ and $p : J \times \Omega \longrightarrow \mathbb{R}^+$ such that for each $w \in \Omega$, $\psi(., w)$ is a continuous nondecreasing function and $p(., w)$ integrable with:

$$|f(t, u, w)| \leq p(t, w) \ \psi(\|u\|_{\mathcal{B}}, w) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B},$$

(H_4) There exists a functions $L : J \times \Omega \longrightarrow \mathbb{R}^+$ with $L(., w) \in L^1(J, \mathbb{R}^+)$ for each $w \in \Omega$ such that for any bounded $B \subseteq E$.

$$\alpha(f(t, B, w)) \leq l(t, w)\alpha(B),$$

(H_5) There exist a random function $R : \Omega \longrightarrow \mathbb{R}^+\backslash\{0\}$ such that:

$$M\|\phi\|_{\mathcal{B}} + M \ \psi\left((M_T + L^\phi)\|\phi\|_{\mathcal{B}} + K_T R(w), w\right) \int_0^T p(s, w)ds \leq R(w),$$

(H_6) For each $w \in \Omega, \phi(., w)$ is continuous and for each $t, \phi(t, .)$ is measurable.

Theorem 3.4 *Suppose that hypotheses (H_ϕ) and $(H_1) - (H_6)$ are valid, then the random of delay problem (3.1)-(3.2) has at least one mild random solution on $(-\infty, T]$.*

Proof 3.5 *Let $Y = \{u \in C(J, E) : u(0, w) = \phi(0, w) = 0\}$ endowed with the uniform convergence topology and $N : \Omega \times Y \longrightarrow Y$ be the random operator defined by*

$$(N(w)y)(t) = T(t) \ \phi(0, w) + \int_0^t T(t - s) \ f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w) \ ds, \quad t \in J, \tag{3.4}$$

where $\bar{y} : (-\infty, T] \times \Omega \longrightarrow E$ is such that $\bar{y}_0(\cdot, w) = \phi(\cdot, w)$ and $\bar{y}(\cdot, w) = y(\cdot, w)$ on J. Let $\bar{\phi} : (-\infty, T] \times \Omega \longrightarrow E$ be the extension of ϕ to $(-\infty, T]$ such that $\bar{\phi}(\theta, w) = \phi(0, w) = 0$ on J.

Then we show that the mapping defined by (3.4) is a random operator. To do this, we need to prove that for any $y \in Y$, $N(.)(y) : \Omega \longrightarrow Y$ is a random variable. Then we prove that $N(.)(y) : \Omega \longrightarrow Y$ is measurable. as a mapping $f(t, y, \cdot)$, $t \in J$, $y \in Y$ is measurable by assumptions (H_2) and (H_6).

Let $D : \Omega \longrightarrow 2^Y$ be defined by:

$$D(w) = \{y \in Y : \|y\| \leq R(w)\}.$$

The set $D(w)$ bounded, closed, convex and solid for all $w \in \Omega$. Then D is measurable by Lemma 17 in [33].
Let $w \in \Omega$ be fixed. If $y \in D(w)$, from Lemma 3.3 it follows that

$$\|\bar{y}_{\rho(t,\bar{y}_t)}\|_{\mathcal{B}} \leq (M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_T R(w)$$

and for each $y \in D(w)$, by (H_3) and (H_5), we have for each $t \in J$

$$\begin{aligned}
|(N(w)y)(t)| &\leq M\|\phi\|_{\mathcal{B}} + M\int_0^t |f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)|ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M\int_0^t p(s, w)\ \psi\left(\|\overline{y}_{\rho(s,\overline{y}_s)}\|_{\mathcal{B}}, w\right) ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M\int_0^t p(s, w)\ \psi\left((M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_T R(w), w\right) ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M\ \psi\left((M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_T R(w), w\right) \int_0^T p(s, w)ds \\
&\leq R(w).
\end{aligned}$$

This implies that N is a random operator with stochastic domain D and $N(w) : D(w) \to D(w)$ for each $w \in \Omega$.

Step 1: *N is continuous.*
Let y^n be a sequence such that $y^n \longrightarrow y$ in Y. Then

$$\begin{aligned}
|(N(w)y^n)(t) - (N(w)y)(t)| &= \Big|T(t)\phi(0, w) \\
&+ \int_0^t T(t-s)\ \left[f(s, \overline{y^n}_{\rho(s,\overline{y^n}_s)}, w) - f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)\right] ds\Big| \\
&\leq M\int_0^t |f(s, \overline{y^n}_{\rho(s,\overline{y^n}_s)}, w) - f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)|ds.
\end{aligned}$$

Since $f(s, \cdot, w)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$|(N(w)y^n)(t) - (N(w)y)(t)| \longrightarrow 0 \ \ as\ \ n \longrightarrow +\infty.$$

Thus N is continuous.

Step 2: *We prove that for every $w \in \Omega, \{y \in D(w) : N(w)y = y\} \neq \emptyset$. For this we apply the Mönch fixed point theorem.*

(a) N maps bounded sets into equicontinuous sets in $D(w)$.
Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_2 > \tau_1$, $D(w)$ be a bounded set, and $y \in D(w)$. Then

$$\begin{aligned}
|(N(w)y)(\tau_2) - (N(w)y)(\tau_1)| &\leq |T(\tau_2) - T(\tau_1)| \|\phi\|_{\mathcal{B}} \\
&+ \Big| \int_0^{\tau_1} [T(\tau_2 - s) - T(\tau_1 - s)] f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w) ds \Big| \\
&+ \Big| \int_{\tau_1}^{\tau_2} T(\tau_2 - s) f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w) \Big| ds \\
&\leq |T(\tau_2) - T(\tau_1)| \|\phi\|_{\mathcal{B}} \\
&+ \int_0^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)| |f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)| \, ds \\
&+ \int_{\tau_1}^{\tau_2} |T(\tau_2 - s) f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)| \, ds \\
&\leq |T(\tau_2) - T(\tau_1)| \|\phi\|_{\mathcal{B}} + \psi\left((M_T + L^\phi)\|\phi\|_{\mathcal{B}} + K_T R(w), w\right) \\
&\qquad \int_0^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)| p(s, w) ds \\
&+ M\psi\left((M_T + L^\phi)\|\phi\|_{\mathcal{B}} + K_T R(w), w\right) \int_{\tau_1}^{\tau_2} p(s, w) ds.
\end{aligned}$$

The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$, since $T(t)$ is uniformly continuous.
Next, let $w \in \Omega$ be fixed (therefore we do not write 'w' in the sequel) but arbitrary.

(b) Now let V be a subset of $D(w)$ such that $V \subset \overline{conv}\,(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $v \longrightarrow v(t) = \alpha(V(t))$ is continuous on $(-\infty, T]$. By (H_4), Lemma 1.17 and the properties of the measure α we have for

each $t \in (-\infty, T]$

$$\begin{aligned}
v(t) &\leq \alpha\left(N(V)\right)(t) \cup \{0\}) \\
&\leq \alpha\left(N(V(t))\right. \\
&\leq \alpha\Big(T(t)\ \phi(0) + \int_0^t T(t-s)\ f(s, \overline{y}_{\rho(s,\overline{y}_s)})\ ds\Big) \\
&\leq \alpha\Big(T(t)\ \phi(0)\Big) + \alpha\Big(\int_0^t T(t-s)\ f(s, \overline{y}_{\rho(s,\overline{y}_s)})\ ds\Big) \\
&\leq M\int_0^t l(s)\alpha(\big\{\overline{y}_{\rho(s,y_s)} : \overline{y} \in V\big\})ds \\
&\leq M\int_0^t l(s)K(s) \sup_{0\leq\tau\leq s} \alpha(V(\tau))ds \\
&\leq \int_0^t l(s)K(s)\alpha(V(s))ds \\
&\leq M\int_0^t v(s)\ l(s)K(s)ds \\
&= M\int_0^t l(s)K(s)v(s)ds.
\end{aligned}$$

Gronwall's lemma implies that $v(t) = 0$ *for each* $t \in J$*, and then* $V(t)$ *is relatively compact in* E*. In view of the Ascoli-Arzelà theorem,* V *is relatively compact in* $D(w)$*. Applying now Theorem 1.9 we conclude that* N *has a fixed point* $y(w) \in D(w)$*.*

Since $\bigcap_{w\in\Omega} D(w) \neq \emptyset$*, then the hypothesis that a measurable selector of* $intD$ *exists holds. By Lemma 1.15, the random operator* N *has a stochastic fixed point* $y^*(w)$*, which is a mild solution of the random problem (3.1)-(3.2).*

Proposition 3.6 *Assume that* $(H_\phi), (H_1), (H_2), (H_5), (H_6)$ *are satisfied, then a slight modification of the proof (i.e. use the Darbo's fixed point theorem) guarantees that* (H_4) *could be replaced by*

$(H_4)^*$ *There exists a nonnegative function* $l(., w) \in L^1(J, \mathbb{R}^+)$ *for each* $w \in \Omega$*, such that*

$$\alpha(f(t, B, w)) \leq l(t, w)\alpha(B), \quad t \in J.$$

Proof 3.7 *Consider the Kuratowski measure of noncompactness* α_C *defined on the family of bounded subsets of the space* $C(J, E)$ *by*

$$\alpha_C(H) = \sup_{t\in J} e^{-\tau L(t)}\alpha(H(t)),$$

where $L(t) = \int_0^t \widetilde{l}(s)ds,\ \ \widetilde{l}(t) = Ml(t)K(t),\ \tau > 1.$

We show that the operator $N : D(w) \longrightarrow D(w)$ *is a strict set contraction for each* $w \in \Omega$. *We know that* $N : D(w) \longrightarrow D(w)$ *is bounded and continuous, we need to prove that there exists a constant* $0 \leq \lambda < 1$ *such that* $\alpha_C(NH) \leq \lambda \alpha_C(H)$ *for* $H \subset D(w)$. *For each* $t \in J$ *we have*

$$\alpha((NH)(t)) \leq M \int_0^t \alpha(f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)) : \overline{y} \in H)ds.$$

This implies by $(H_4)^*$ *and Theorem* 2.1 *in [41]*

$$\begin{aligned}
\alpha((NH)(t)) &\leq \int_0^t M\ l(s)\alpha(\overline{y}_{\rho(s,\overline{y}_s)} : \overline{y} \in H)ds \\
&\leq \int_0^t Ml(s)K(s) \sup_{0\leq\tau\leq s} \alpha(H(\tau))ds \\
&\leq \int_0^t Ml(s)K(s)\alpha(H(s))ds \\
&= \int_0^t \widetilde{l}(s)\alpha(H(s))ds \\
&= \int_0^t e^{\tau L(s)}e^{-\tau L(s)}\widetilde{l}(s)\alpha(H(s))ds \\
&\leq \int_0^t \widetilde{l}(s)\ e^{\tau L(s)} \sup_{s\in[0,t]} e^{-\tau L(s)}\alpha(H(s))ds \\
&\leq \sup_{t\in[0,T]} e^{-\tau L(t)}\alpha(H(t)) \int_0^t \widetilde{l}(s)e^{\tau L(s)}ds \\
&= \alpha_C(H) \int_0^t \left(\frac{e^{\tau L(s)}}{\tau}\right)' ds \\
&\leq \alpha_C(H)\frac{1}{\tau}e^{\tau L(t)}.
\end{aligned}$$

Therefore,

$$\alpha_C(NH) \leq \frac{1}{\tau}\alpha_C(H).$$

So, the operator N *is a set contraction. As a consequence of Theorem 1.10, we deduce that* N *has a fixed point* $y(w) \in D(w)$. *Since* $\bigcap_{w\in\Omega} D(w) \neq \emptyset$, *the hypothesis that a measurable selector of intD exists holds. By Lemma 1.15, the random operator* N *has a stochastic fixed point* $y^*(w)$, *which is a mild solution of the random problem (3.1)-(3.2).*

3.2.3 An example

Consider the following functional partial differential equation:

$$\frac{\partial}{\partial t}z(t,x,w) = \frac{\partial^2}{\partial x^2}z(t,x,w) + C_0(w)b(t)\int_{-\infty}^t F(z(t+\sigma(t,z(t+s,x,w)),x,w))ds, \quad (3.5)$$

$$x \in [0,\pi],\ t \in [0,T],\ w \in \Omega$$

$$z(t,0,w) = z(t,\pi,w) = 0,\ t \in [0,T], w \in \Omega \tag{3.6}$$

$$z(s,x,w) = z_0(s,x,w),\ s \in (-\infty,0],\ x \in [0,\pi],\ w \in \Omega, \tag{3.7}$$

where C_0 are a real-valued random variable, $b \in L^1(J;\mathbb{R}_+)$, $F : \mathbb{R} \to \mathbb{R}$ is continuous, $z_0 :) - \infty, 0] \times [0,\pi] \times \Omega \to \mathbb{R}$ and $\sigma : J \times \mathbb{R} \to \mathbb{R}$ are given functions.

Suppose that $E = L^2[0,\pi]$, (Ω, F, P) is a complete probability space. Let A be $A : E \to E$ by $Av = v''$ with domain

$$D(A) = \{v \in E, v, v'\text{are absolutely continuous},\ v'' \in E,\ v(0) = v(\pi) = 0\}.$$

Then

$$Av = \sum_{n=1}^{\infty} n^2 (v, v_n) v_n, v \in D(A)$$

where $\omega_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, n = 1, 2, \ldots$ is the orthogonal set of eigenvectors in A. It is well know (see [71]) that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in E and is given by

$$T(t)v = \sum_{n=1}^{\infty} \exp(-n^2 t)(v, v_n) v_n, v \in E.$$

Since the analytic semigroup $T(t)$ is compact, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Let $\mathcal{B} = BCU(\mathrm{I\!R}^-; E)$ be the space of uniformly bounded continuous functions endowed with the following norm:

$$\|\phi\| = \sup_{s \leq 0} |\phi(s)|, \qquad \text{for } \phi \in \mathcal{B}.$$

If we put $\phi \in BUC(\mathrm{I\!R}^-; E), x \in [0,\pi]$ and $w \in \Omega$

$$y(t,x,w) = z(t,x,w),\ t \in [0,T]$$
$$\phi(s,x,w) = z_0(s,x,w),\ s \in (-\infty,0].$$

Set

$$f(t,\phi(x),w) = C_0(w) b(t) \int_{-\infty}^{t} F(z(t + \sigma(t, z(t+s,x,w)), x, w)) ds,$$

and

$$\rho(t,\phi)(x) = \sigma(t, z(t,x,w)).$$

Let $\phi \in \mathcal{B}$ be such that (H_ϕ) holds, and let $t \to \phi_t$ be continuous on $\mathcal{R}(\rho^-)$, and let f satisfies the conditions $(H_3), (H_4), (H_5)$

Then the problem (3.1)-(3.2) is as in an abstract formulation of the problem (3.5)-(3.7), and conditions $(H_1) - (H_6)$ are satisfied. Theorem 3.4 implies that the random problem (3.5)-(3.7) has at least one random mild solution.

Chapter 4

Functional Evolution Equations with infinite State-Dependent Delay and Random Effect

4.1 Introduction

In this work we prove the existence of random mild solutions of the following functional evolution differential equation with delay and random effects (random paramaters) of the form:

$$y'(t,w) = Ay(t,w) + f(t, y_{\rho(t,y_t)}(.,w), w), \quad \text{a.e. } t \in J := [0,\infty) \tag{4.1}$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \tag{4.2}$$

(with some notations to be given later), where (Ω, F, P) is a complete probability space, $w \in \Omega$ where $f : J \times \mathcal{B} \times \Omega \to E$, $\phi \in \mathcal{B} \times \Omega$ are given random functions which represent random nonlinear of the system, $\{A(t)\}_{0 \leq t < +\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t,s)\}_{(t,s) \in J \times J}$ for $0 \leq s \leq t < +\infty$, $\mathcal{B}$ is the phase space to be specified later, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$, and $(E, |.|)$ is a real separable Banach space. For any function y defined on $(-\infty, T] \times \Omega$ and any $t \in J$ we denote by $y_t(.,w)$ the element of $\mathcal{B} \times \Omega$ defined by $y_t(\theta, w) = y(t+\theta, w), \theta \in (-\infty, 0]$. Here $y_t(.,w)$ represents the history of the state from time $-\infty$, up to the present time t. We assume that the histories $y_t(.,w)$ to some abstract phases $\mathcal{B}$, to be specified later. To our knowledge, the literature on the local existence of random evolution equations with delay is very limited, so the present section can be considered as a contribution to this question.

4.2 Functional evolution equations with infinite state-dependent delay and random effect

4.2.1 Existence of mild solutions

Now we give our main existence result for problem (4.1)-(4.2). Before starting and proving this result, we give the definition of the random mild solution.

Definition 4.1 *A stochastic process $y : J \times \Omega \to E$ is said to be random mild solution of problem (4.1)-(4.2) if $y(t, w) = \phi(t, w),\ t \in (-\infty, 0]$ and the restriction of $y(., w)$ to the interval $[0, T]$ is continuous and satisfies the following integral equation:*

$$y(t, w) = U(t, 0)\phi(0, w) + \int_0^t U(t, s)f(s, y_{\rho(s,y_s)}(., w), w)ds,\ t \in J. \tag{4.3}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s, \varphi) : (s, \varphi) \in J \times \mathcal{B}, \rho(s, \varphi) \le 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \longrightarrow (-\infty, \infty)$ is continuous. Additionally, we introduce following hypothesis:

(H_ϕ) The function $t \longrightarrow \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \longrightarrow (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \le L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 4.2 *The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see for instance [59].*

Lemma 4.3 *([54], Lemma 2.4) If $y : (-\infty, T] \longrightarrow E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \le (M_T + L^\phi)\|\phi\|_{\mathcal{B}} + K_T \sup\{|y(\theta)|; \theta \in [0, max\{0, s\}]\},\ s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup_{t \in \mathcal{R}(\rho^-)} L^\phi(t)$.

We will need to introduce the following hypothesis which are be assumed there after:

(H_1) There exists a constant $M \ge 1$ and $\alpha > 0$ such that

$$\|U(t, s)\|_{B(E)} \le Me^{-\alpha(t-s)} \quad \text{for every } (s, t) \in \Delta.$$

(H_2) The function $f : J \times \mathcal{B} \times \Omega \to E$ is Carathéodory.

(H_3) There exists a functions $\psi : J \times \Omega \longrightarrow R^+$ and $p : J \times \Omega \longrightarrow R^+$ such that for each $w \in \Omega$, $\psi(., w)$ is a continuous nondecreasing function and $p(., w)$ integrable with:

$$|f(t, u, w)| \leq p(t, w)\ \psi(\|u\|_{\mathcal{B}}, w) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B},$$

(H_4) For each $(t, s) \in \Delta$ we have: $\lim\limits_{t \longrightarrow +\infty} \int_0^t e^{-\alpha(t-s)} p(s, w) ds = 0.$

(H_5) There existe a random function $R : \Omega \longrightarrow \mathbb{R}^+/\{0\}$ such that:

$$M\|\phi\|_{\mathcal{B}} + M\ \psi\left((M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_T R(w), w\right) \|p\|_{L^1} \leq R(w),$$

(H_6) For each $w \in \Omega, \phi(., w)$ is continuous and for each $t, \phi(t, .)$ is measurable.

Theorem 4.4 *Suppose that hypotheses (H_ϕ) and $(H_1) - (H_6)$ are valid, then the random of delay problem (4.1)-(4.2) has at least one mild random solution on $) - \infty, \infty)$.*

Proof. Let Y is the space defined by

$$Y = \{y : \mathbb{R} \longrightarrow E \text{ such that } y|_J \in BC(J, E) \text{ and } y_0 \in \mathcal{B}\},$$

we denote by $y|_J$ the restriction of y to J, endowed with the uniform convergence topology and $N : \Omega \times Y \longrightarrow Y$ be the random operator defined by:

$$(N(w)y)(t) = \begin{cases} \phi(t, w), & \text{if } t \in (-\infty, 0] \\ U(t, 0)\phi(0, w) + \displaystyle\int_0^t U(t, s) & \\ f(s, y_{\rho(s, y_s)}(., w), w) ds, & \text{if } t \in J, \end{cases} \tag{4.4}$$

Then we show that the mapping defined by (4.4) is a random operator. To do this, we need to prove that for any $y \in Y$, $N(.)(y) : \Omega \longrightarrow Y$ is a random variable. Then we prove that $N(.)(y) : \Omega \longrightarrow Y$ is measurable. as a mapping $f(t, y, .), t \in J, y \in Y$ is measurable by assumption (H_2) and (H_6).

Let $D : \Omega \longrightarrow 2^Y$ be defined by:

$$D(w) = \{y \in Y : \|y\| \leq R(w)\}.$$

With $D(w)$ bounded, closed, convex and solid for all $w \in \Omega$. Then D is measurable by lemma 17 in [33].

Let $w \in \Omega$ be fixed, If $y \in D(w)$, from Lemma 4.3 follows that

$$\|y_{\rho(t, y_t)}\|_{\mathcal{B}} \leq (M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_T R(w)$$

and For each $y \in D(w)$, by (H3) and (H5), we have for each $t \in J$

$$\begin{aligned}
|(N(w)y)(t)| &\leq M\|U(t,0)\|_{B(E)}\|\phi\|_{\mathcal{B}} + M\int_0^t \|U(t,s)\|_{B(E)}|f(s, y_{\rho(s,y_s)}, w)|\ ds \\
&\leq Me^{-\alpha t}\|\phi\|_{\mathcal{B}} + M\int_0^t e^{-\alpha(t-s)}p(s,w)\ \psi\left(\|y_{\rho(s,y_s)}\|_{\mathcal{B}}, w\right)\ ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M\int_0^t p(s,w)\ \psi\left((M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_T R(w), w\right)\ ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M\ \psi\left((M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_T R(w), w\right)\|p\|_{L^1} \\
&\leq R(w).
\end{aligned}$$

This implies that N is a random operator with stochastic domain D and $N(w) : D(w) \longrightarrow D(w)$ for each $w \in \Omega$.

Step 1: F is continuous.
Let y^n be a sequence such that $y^n \longrightarrow y$ in Y. Then

$$\begin{aligned}
|(N(w)y^n)(t) - (N(w)y)(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)}\left|f(s, y^n_{\rho(s,y^n_s)}, w) - f(s, y_{\rho(s,y_s)}, w)\right|\ ds \\
&\leq M\int_0^t e^{-\alpha(t-s)}\left|f(s, y^n_{\rho(s,y^n_s)}, w) - f(s, y_{\rho(s,y_s)}, w)\right|\ ds.
\end{aligned}$$

Since $f(s,.,w)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$\||(N(w)y^n)(t) - (N(w)y)(t)|\| \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

Thus N is continuous.

Step 2:we prove that for every $w \in \Omega, \{y \in D(w) : N(w)y = y\} \neq \emptyset$. For prove this we apply Schauder's theorem.
$N(D(w))$ **is relatively compact:** To prove the compactness, we will use Corduneanu's lemma.

(a) Firstly, it is clear that the assumption (i) is holds. Then we will demonstrate that $N(D(w))$ is equicontinuous set for each closed bounded interval $[0,T]$ in J. Let

$\tau_1, \tau_2 \in [0,T]$ with $\tau_2 > \tau_1$, $D(w)$ be a bounded set, and $y \in D(w)$. Then

$$\begin{aligned}
|(N(w)y)(\tau_2) - (N(w)y)(\tau_1)| &\leq \|U(\tau_2,0) - U(\tau_1,0)\|_{B(E)}\|\phi\|_{\mathcal{B}} \\
&+ \Big|\int_0^{\tau_1}[U(\tau_2,s) - U(\tau_1,s)]f(s, y_{\rho(s,y_s)}, w)ds\Big| \\
&+ \Big|\int_{\tau_1}^{\tau_2} U(\tau_2,s)f(s, y_{\rho(s,y_s)}, w)\Big|ds \\
&\leq \|U(\tau_2,0) - U(\tau_1,0)\|_{B(E)}\|\phi\|_{\mathcal{B}} \\
&+ \int_0^{\tau_1} |U(\tau_2,s) - U(\tau_1,s)||f(s, y_{\rho(s,y_s)}, w)|\ ds \\
&+ \int_{\tau_1}^{\tau_2} |U(\tau_2,s)f(s, y_{\rho(s,y_s)}, w)|\,ds \\
&\leq |U(\tau_2,0) - U(\tau_1,0)|\|\phi\|_{\mathcal{B}} + \psi\left((M_b + L^{\phi})\|\phi\|_{\mathcal{B}} + K_bR(w), w\right) \\
&\quad \int_0^{\tau_1} |U(\tau_2,s) - U(\tau_1,s)|p(s,w)ds \\
&+ M\psi\left((M_b + L^{\phi})\|\phi\|_{\mathcal{B}} + K_bR(w), w\right)\int_{\tau_1}^{\tau_2} p(s,w)ds.
\end{aligned}$$

The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$, As N is bounded and equicontinuous.

Next, let $w \in \Omega$ be fixed (therefore we do not write 'w' in the sequel) but arbitrary.

(b) Now we will prove that $Z(t,w) = \{(N(w)y)(t) : y \in D(w)\}$ is precompact in E. Let $t \in [0,T]$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in D(w)$ we define

$$(N_\epsilon(w)y)(t) = U(t,0)\phi(0,w) + U(t,t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon,s)f(s, y_{\rho(s,y_s)}, w)\,ds.$$

Since $U(t,s)$ is a compact operator and the set $Z_\epsilon(t,w) = \{(N_\epsilon(w)y)(t) : y \in D(w)\}$ is the imageof bounded set of E then $Z_\epsilon(t,w)$ is pre-compact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\begin{aligned}
|(N(w)y)(t) - (N_\epsilon(w)y)(t)| &\leq \int_{t-\epsilon}^{t} \|U(t,s)\|_{B(E)}|f(s, y_{\rho(s,y_s)}, w)|ds \\
&\leq M\psi\left((M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_TR(w), w\right)e^{-\alpha(t-s)}\int_{t-\epsilon}^{t} p(s,w)ds.
\end{aligned}$$

The right-hand side tends to zero as $\epsilon \longrightarrow 0$, then $N(w)y$ converge uniformly to $N_\epsilon(w)y$ which implies that $Z(t,w) = \{(N(w)y)(t) : y \in D(w)\}$ is precompact in E.

(c) Finally, it remains to show that N is equiconvergent.

Let $y \in D(w)$, then from (H_1), (H_3) we have

$$|(N(w)y)(t)| \leq Me^{-\alpha t}\|\phi\|_{\mathcal{B}}+M\,\psi\left((M_T+L^{\phi})\|\phi\|_{\mathcal{B}}+K_T R(w),w\right)\int_0^t e^{-\alpha(t-s)}p(s,w)\,ds,$$

it follows immediately by (H_4) that $|(N(w)y)(t)| \longrightarrow 0$ as $t \longrightarrow +\infty$.

Then

$$\lim_{t\longrightarrow+\infty}|(N(w)y)(t)-(N(w)y)(+\infty)|=0,$$

which implies that N is equiconvergent.

A consequence of Steps 1-2 and (a), (b), (c), we can conclude that $N(w) : D(w) \rightarrow D(w)$ is continuous and compact. From Schauder's theorem, we deduce that $N(w)$ has a fixed point $y(w)$ in $D(w)$.

Since $\bigcap_{w\in\Omega} D(w) \neq \emptyset$, the hypothesis that a measurable selector of int$D$ exists holds. By lemma 1.15, the random operator N has a stochastic fixed point $y^*(w)$, which is a mild solution of the random problem (4.1)-(4.2).

4.2.2 An example

Consider the following functional partial differential equation:

$$\frac{\partial}{\partial t}z(t,x,w)=a(t,x)\frac{\partial^2}{\partial x^2}z(t,x,w)+C_0(w)b(t)F(z(t+\sigma(t,z(t+s,x,w)),x,w)), \quad (4.5)$$

$$x\in[0,\pi],\ t\geq 0,\ w\in\Omega$$

$$z(t,0,w)=z(t,\pi,w)=0,\ t\geq 0, w\in\Omega \quad (4.6)$$

$$z(s,x,w)=z_0(s,x,w),\ s\in(-\infty,0],\ x\in[0,\pi], w\in\Omega \quad (4.7)$$

where $a(t,\xi)$ is a continuous function which is uniformly Hölder continuous in t, Where C_0 are a real-valued random variable, $b \in L^1(J;\mathbb{R}_+)$, $F : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $z_0 :) - \infty, 0] \times [0,\pi] \times \Omega \rightarrow \mathbb{R}$ and $\sigma : J \times \mathbb{R} \rightarrow \mathbb{R}$ are given functions.

Suppose that $E = L^2[0,\pi]$, (Ω, F, P) is a complete probability space. Take and define $A : E \rightarrow E$ by $Av = v''$ with domain:

$$D(A)=\{v\in E, v, v'\text{are absolutely continuous},\ v''\in E,\ v(0)=v(\pi)=0\}.$$

Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumption (H_1) (see [36, 45]).

Let $\mathcal{B} = BCU(\mathbb{R}^-; E)$: the space of uniformly bounded continuous functions endowed with the following norm:

$$\|\phi\| = \sup_{s \leq 0} |\phi(s)|, \qquad \text{for } \phi \in \mathcal{B}$$

If we put $\phi \in BCU(\mathbb{R}^-; E), x \in [0, \pi]$ and $w \in \Omega$

$$y(t, x, w) = z(t, x, w), t \geq 0$$

$$\phi(s, x, w) = z_0(s, x, w), \ s \in (-\infty, 0],$$

Set

$$f(t, \phi(x), w) = C_0(w)b(t)F(z(t + \sigma(t, z(t + s, x, w)), x, w)),$$

and

$$\rho(t, \phi)(x) = \sigma(t, z(t, x, w)).$$

Let $\phi \in \mathcal{B}$ be such that (H_ϕ) holds, and let $t \to \phi_t$ be continuous on $\mathcal{R}(\rho^-)$.

Then the problem (4.1)-(4.2) in an abstract formulation of the problem (4.5)-(4.7), and conditions $(H_1) - (H_6)$ are satisfied. Theorem 4.4 implies that the random problem (4.5)-(4.7) has at least one random mild solutions.

Chapter 5

Second Order Functional Differential Equations with Delay and Random Effect

5.1 Functional differential equations with constant delay and random effect

5.1.1 Introduction

The importance of random fixed point theory lies in its vast applicability in probabilistic functional analysis and various probabilistic models. The introduction of randomness however leads to several new questions of measurability of solutions, probabilistic and statistical aspects of random solutions. It is well known that random fixed point theorems are stochastic generalization of classical fixed point theorems what we call as determinstic results. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by $\tilde{S}$pa$\tilde{c}$ek [81] and Han$\tilde{s}$ (see [52, 53]). The survey article by Bharucha-Reid [20] in 1976 attracted the attention of several mathematicians and gave wings to this theory. Itoh [40] extended $\tilde{S}$pa$\tilde{c}$ekŠs and Han$\tilde{s}$Šs theorems to multivalued contraction mappings. Random fixed point theorems with an application to Random differential equations in Banach spaces are obtained by Itoh [40]. Sehgal and Waters [77] had obtained several random fixed point theorems including random analogue of the classical results due to Rothe [73]. In recent past, several fixed point theorems including Kannan type [46] Chatterjeea [24] and Zamfirescu type [91] have been generalized in stochastic version (see for detail in Joshi and Bose [42], Saha et al. ([74, 75]).

In this work we prove the existence of random mild solutions of the following functional differential equation with delay and random effects (random paramaters) of the form:

$$y''(t,w) = Ay(t,w) + f(t, y_t(\cdot,w), w), \quad \text{a.e. } t \in J := [0,T] \tag{5.1}$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \quad y'(0,w) = \varphi(w) \in E, \tag{5.2}$$

(with some notations to be given later), where (Ω, F, P) is a complete probability space, $w \in \Omega$ where $f : J \times \mathcal{B} \times \Omega \to E$, $\phi \in \mathcal{B} \times \Omega$ are given random functions which represent random nonlinear of the system, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on E, we denote by $(S(t))_{t\in\mathbb{R}}$ the sine function associated with $(C(t))_{t\in\mathbb{R}}$, which is defined by $S(t)x = \int_0^t C(s)xds$ for $x \in E$ and $t \in \mathbb{R}$, $\mathcal{B}$ is the phase space to be specified later, and $(E, |.|)$ is a real separable Banach space. For any function y defined on $(-\infty, T] \times \Omega$ and any $t \in J$ we denote by $y_t(., w)$ the element of $\mathcal{B} \times \Omega$ defined by $y_t(\theta, w) = y(t + \theta, w), \theta \in (-\infty, 0]$. Here $y_t(., w)$ represents the history of the state from time $-\infty$, up to the present time t. We assume that the histories $y_t(., w)$ to some abstract phases $\mathcal{B}$, to be specified later. Later, we consider the following problem

$$y''(t, w) = Ay(t, w) + f(t, y_{\rho(t,y_t)}(\cdot, w), w), \quad \text{a.e. } t \in J := [0, T] \tag{5.3}$$

$$y(t, w) = \phi(t, w), \quad t \in (-\infty, 0], \quad y'(0, w) = \varphi(w) \in E, \tag{5.4}$$

(with some notations to be given later), where (Ω, F, P) is a complete probability space, $w \in \Omega$ where $f : J \times \mathcal{B} \times \Omega \to E$, $\phi \in \mathcal{B} \times \Omega$ are given random functions which represent random nonlinear of the system, $A : D(A) \subset E \to E$ as in problem (5.1)-(5.2), $\mathcal{B}$ is the phase space to be specified later, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$, and $(E, |.|)$ is a real separable Banach space. For any function y defined on $(-\infty, T] \times \Omega$ and any $t \in J$ we denote by $y_t(., w)$ the element of $\mathcal{B} \times \Omega$ defined by $y_t(\theta, w) = y(t+\theta, w), \theta \in (-\infty, 0]$. Here $y_t(., w)$ represents the history of the state from time $-\infty$, up to the present time t. We assume that the histories $y_t(., w)$ to some abstract phases $\mathcal{B}$, to be specified later. The main results are based upon Schauder's fixed theorem and random fixed point theorem combined with the family of cosine operators.

The cosine function theory is related to abstract linear second order differential equations in the same manner that the semigroup theory of bounded linear operators is related to first order partial differential equations and itŠs equally appealing devoted their generality and simplicity. For basic concepts and applications of this theory, we refer to the reader to Fattorini [35], Travis and Weeb [85].

Our purpose in this work is to consider a simultaneous generalization of the classical second order abstract Cauchy problem studied by Travis and Weeb in [84, 85]. Additionally, we observe that the ideas and techniques in this paper permit the reformulation of the problems studied in [11, 19, 44, 67, 68] to the context of Ş partial Ť second order differential equations, see ([84], pp. 557) and the referred papers for details.

Complicated situations in which the delay depends on the unknown functions have been studied in the recent years (see for instance [6, 72, 87, 88] and the references therein). Over the past several years it has become apparent that equations with state-dependent delay arise also in several areas such as classical electrodynamics [31], in population models [21], models of commodity price fluctuations [22, 64], and models of blood cell productions [65]. These equations are frequently called equations with state-dependent delay.

The literature devoted differential equations with state-dependent delay is concerned fundamentally with first order functional differential equations for which the state belong to some finite dimensional space, see among another works [14, 23, 29]. The problem of the existence of solutions for first and second order partial functional differential with state-dependent delay have treated recently in [55, 58, 56, 72]. The literature relative second order differential system with state-dependent delay is very restrict, and related this matter we only cite [80] for ordinary differential system and [57] for abstract partial differential systems.

To the best of our knowledge, the study of the existence of solutions for abstract second order functional differential equations with state-dependent delay on unbounded interval is an untreated topic in the literature and this fact, is the main motivation of the present work.

5.1.2 Existing result for the constante delay case

In this section we give our main existence result for problem (5.1)-(5.2). Before starting and proving this result, we give the definition of a random mild solution.

Definition 5.1 *A stochastic process $y : J\times\Omega \to E$ is said to be a random mild solution of problem (5.1)-(5.2) if $y(t,w) = \phi(t,w),\ t \in (-\infty, 0]$, $y'(0,w) = \varphi(w)$ and the restriction of $y(.,w)$ to the interval $[0,T]$ is continuous and satisfies the following integral equation:*

$$y(t,w) = C(t)\phi(0,w) + S(t)\varphi(w) + \int_0^t C(t-s)f(s,y_s,w)ds,\ t \in J.$$

Let

$$M = \sup\{\|C(t)\|_{B(E)} : t \geq 0\},\ M' = \sup\{\|S(t)\|_{B(E)} : t \geq 0\}.$$

Let us introduce the following hypotheses.

(H_1) $C(t)$ is compact for $t > 0$ in the Banach space E.

(H_2) The function $f : J \times \mathcal{B} \times \Omega \to E$ is Carathéodory.

(H_3) There exist a functions $\psi : J \times \Omega \longrightarrow \mathbb{R}^+$ and $p : J \times \Omega \longrightarrow \mathbb{R}^+$ such that for each $w \in \Omega$, $\psi(.,w)$ is a continuous nondecreasing function and $p(.,w)$ integrable with:

$$|f(t,u,w)| \leq p(t,w)\ \psi(\|u\|_{\mathcal{B}}, w) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B},$$

(H_4) There existe a random function $R : \Omega \longrightarrow \mathbb{R}^+/\{0\}$ such that:

$$M\|\phi\|_{\mathcal{B}} + M'\|\varphi\| + M\psi(D_T, w)\|p\|_{L^1} \leq R(w)$$

where

$$D_T := K_T R(w) + M_T\|\phi\|_{\mathcal{B}}.$$

(H_5) For each $w \in \Omega, \phi(., w)$ is continuous and for each $t, \phi(t, .)$ is measurable, and for each $w \in \Omega$, $\varphi(w)$ is measurable.

Theorem 5.2 *Suppose that hypotheses $(H_1)-(H_5)$ are valid, then the problem (5.1)-(5.2) has at least one mild random solution on $(-\infty, T]$.*

Proof. Consider the random operator: $N : \Omega \times BC \longrightarrow BC$ defined by:

$$(N(w)y)(t) = \begin{cases} \phi(t, w), \quad \text{if } t \in (-\infty, 0], \\ C(t)\ \phi(0, w) + S(t)\varphi(w) \\ + \displaystyle\int_0^t C(t-s)\ f(s, y_s(\cdot, w), w)\ ds, \text{if } t \in J. \end{cases} \tag{5.5}$$

Then we show that the mapping defined by (5.5) is a random operator. To do this, we need to prove that for any $y \in BC$, $N(.)(y) : \Omega \longrightarrow BC$ is a random variable. Then we prove that $N(.)(y) : \Omega \longrightarrow BC$ is measurable. as a mapping $f(t, y, .), t \in J, y \in BC$ is measurable by assumption (H_2) and (H_5).

Let $D : \Omega \longrightarrow 2^{BC}$ be defined by:

$$D(w) = \{y \in BC : \|y\|_{BC} \leq R(w)\}.$$

$D(w)$ is bounded, closed, convex and solid for all $w \in \Omega$. Then D is measurable by Lemma 17 (see [33]).
Let $w \in \Omega$ be fixed, then for any $y \in D(w)$,and by assumption (A1), we get

$$\begin{aligned} \|y_s\|_{\mathcal{B}} &\leq L(s)|y(s)| + M(s)\|y_0\|_{\mathcal{B}} \\ &\leq K_T|y(s)| + M_T\|\phi\|_{\mathcal{B}}. \end{aligned}$$

and by (H_4) and (H_3), we have:

$$\begin{aligned} |(N(w)y)(t)| &\leq M\|\phi\|_{\mathcal{B}} + M'|\varphi| + M\int_0^t |f(s, \overline{y}_s, w)|\ ds \\ &\leq M\|\phi\|_{\mathcal{B}} + M'|\varphi| + M\ \int_0^t p(s, w)\ \psi\left(\|y_s\|_{\mathcal{B}}, w\right)\ ds. \end{aligned}$$

Set

$$D_T := K_T R(w) + M_T\|\phi\|_{\mathcal{B}}.$$

Then, we have

$$|(N(w)y)(t)| \leq M\|\phi\|_{\mathcal{B}} + M'|\varphi| + M\psi(D_T, w)\int_0^T p(s, w)\, ds.$$

Thus

$$\|(N(w)y)\|_{BC} \leq M\|\phi\|_{\mathcal{B}} + M'|\varphi| + M\psi(D_T, w)\|p\|_{L^1} \leq R(w).$$

This implies that N is a random operator with stochastic domain D and $F(w) : D(w) \longrightarrow D(w)$ for each $w \in \Omega$.

Step 1: N is continuous.
Let y^n be a sequence such that $y^n \longrightarrow y$ in BC. Then

$$\begin{aligned} |(N(w)y^n)(t) - (N(w)y)(t)| &= \Big| \int_0^t C(t-s)[f(s, y_s^n, w) - f(s, y_s, w)]\, ds\Big| \\ &\leq M \int_0^t |f(s, y_s^n, w) - f(s, y_s, w)|\, ds. \end{aligned}$$

Since $f(s, ., w)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$\|f(., y_.^n, w) - f(., y_., w)\|_{L^1} \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

Thus N is continuous.

Step 2: we prove that for every $w \in \Omega, \{y \in D(w) : N(w)y = y\} \neq \emptyset$. For prove this we apply Schauder's theorem.

(a) N maps bounded sets into equicontinuous sets in $D(w)$.
Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_2 > \tau_1$, $D(w)$ be a bounded set, and $y \in D(w)$. Then

$$\begin{aligned} & |(N(w)y)(\tau_2) - (N(w)y)(\tau_1)| \\ \leq\ & \|C(\tau_2) - C(\tau_1)\|_{B(E)}\|\phi\|_{\mathcal{B}} + \|S(\tau_2) - S(\tau_1)\|_{B(E)}|\varphi| \\ +\ & \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}|f(s, y_s, w)|ds \\ +\ & \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)}|f(s, y_s, w)|ds \\ \leq\ & \|C(\tau_2) - C(\tau_1)\|_{B(E)}\|\phi\|_{\mathcal{B}} + \|S(\tau_2) - S(\tau_1)\|_{B(E)}|\varphi| \\ +\ & \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}|f(s, y_s, w)|\, ds \\ +\ & \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)}|f(s, y_s, w)|\, ds \\ \leq\ & \|C(\tau_2) - C(\tau_1)\|_{B(E)}\|\phi\|_{\mathcal{B}} + \|S(\tau_2) - S(\tau_1)\|_{B(E)}|\varphi| \\ +\ & \psi(D_T, w)\int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}p(s, w)ds \\ +\ & M\psi(D_T, w)\int_{\tau_1}^{\tau_2} p(s, w)ds. \end{aligned}$$

The right-hand of the above inequality tends to zero as $\tau_2-\tau_1 \longrightarrow 0$, the right-hand side of the above inequality tends to zero, since $C(t), S(t)$ are a strongly continuous operator and the compactness of $C(t), S(t)$ for $t>0$, implies the continuity in the uniform operator topology (see [84, 85]).

(b) Let $t \in [0,T]$ be fixed and let $y \in D(w)$: by assumption (H_3) the function $f(t, y_t, w)$ is bounded and since $C(t)$ is compact, the set

$$\left\{\int_0^t C(t-s)\ f(s, y_s, w)ds\right\}$$

is precompact in E, then the set

$$\left\{C(t)\ \phi(0,w)+S(t)\varphi(w)+\int_0^t C(t-s)\ f(s, y_s, w)ds\right\}$$

is precompact in E.

A consequence of Steps 1-2 and (a), (b), we can conclude that $N(w): D(w) \rightarrow D(w)$ is continuous and compact. From Schauder's theorem, we deduce that $N(w)$ has a fixed point $y(w)$ in $D(w)$.

Since $\bigcap_{w\in\Omega} D(w) \neq \emptyset$, the hypothesis that a measurable selector of int$D$ exists holds. By Lemma 1.15, the random operator N has a stochastic fixed point $y^*(w)$, which is a random mild solution of the random problem (5.1)-(5.2).

5.1.3 The state-dependent delay case

In this section we give our main existence result for problem (5.3)-(5.4). Before starting and proving this result, we give the definition of the random mild solution.

Definition 5.3 *A stochastic process $y: J\times\Omega \rightarrow E$ is said to be random mild solution of problem (5.3)-(5.4) if $y(t,w)=\phi(t),\ t\in(-\infty,0]$ and the restriction of $y(.,w)$ to the interval $[0,T]$ is continuous and satisfies the following integral equation:*

$$y(t,w)=C(t)\phi(0,w)+S(t)\varphi(w)+\int_0^t C(t-s)f(s, y_{\rho(s,y_s)}(s,w), w)ds,\ t\in J. \quad (5.6)$$

Set

$$\mathcal{R}(\rho^-)=\{\rho(s,\varphi):(s,\varphi)\in J\times\mathcal{B}, \rho(s,\varphi)\leq 0\}.$$

We always assume that $\rho: J\times\mathcal{B}\longrightarrow(-\infty,T]$ is continuous. Additionally, we introduce following hypothesis:

(H_ϕ) The function $t\longrightarrow\phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi:\mathcal{R}(\rho^-)\longrightarrow(0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}}\leq L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t\in\mathcal{R}(\rho^-).$$

Remark 5.4 *The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see for instance [59].*

Lemma 5.5 *([54], Lemma 2.4) If $y : (-\infty, T] \longrightarrow E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_T + L^\phi)\|\phi\|_{\mathcal{B}} + K_T \sup\{|y(\theta)|; \theta \in [0, max\{0, s\}]\}, \ s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup\limits_{t \in \mathcal{R}(\rho^-)} L^\phi(t)$.

We will need to introduce the following hypothesis which are be assumed there after:

(H_1) $C(t)$ is compact for $t > 0$ in the Banach space E.

(H_2) The function $f : J \times \mathcal{B} \times \Omega \to E$ is Carathéodory.

(H_3) There exists two functions $\psi : J \times \Omega \longrightarrow \mathbb{R}^+$ and $p : J \times \Omega \longrightarrow \mathbb{R}^+$ such that for each $w \in \Omega$, $\psi(.,w)$ is a continuous nondecreasing function and $p(.,w)$ integrable with:

$$|f(t,u,w)| \leq p(t,w)\ \psi(\|u\|_{\mathcal{B}}, w) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B},$$

(H_4) There exist function $L : J \times \Omega \longrightarrow \mathbb{R}^+$ with $L(.,w) \in L^1(J, \mathbb{R}^+)$ for each $w \in \Omega$ such that for any bounded $B \subseteq E$.

$$\alpha(f(t,B,w)) \leq l(t,w)\alpha(B),$$

(H_5) There exist a random function $R : \Omega \longrightarrow \mathbb{R}^+/\{0\}$ such that:

$$M\|\phi\|_{\mathcal{B}} + M'\|\varphi\| + M\ \psi\left((M_T + L^\phi)\|\phi\|_{\mathcal{B}} + K_T R(w), w\right) \int_0^T p(s,w)ds \leq R(w),$$

(H_6) For each $w \in \Omega, \phi(.,w)$ is continuous and for each $t, \phi(t,.)$ is measurable and for each $w \in \Omega$, $\varphi(w)$ is measurable.

Theorem 5.6 *Suppose that hypotheses (H_ϕ) and $(H_1) - (H_6)$ are valid, then the random of delay problem (5.3)-(5.4) has at least one mild random solution on $(-\infty, T]$.*

Proof. Consider the random operator: $N : \Omega \times BC \longrightarrow BC$ defined by:

$$(N(w)y)(t) = \begin{cases} \phi(t,w), & \text{if } t \in (-\infty, 0], \\ C(t)\ \phi(0,w) + S(t)\varphi(w) \\ + \displaystyle\int_0^t C(t-s) f(s, y_{\rho(s,y_s)}(\cdot, w), w)\ ds, \text{if } t \in J. \end{cases} \tag{5.7}$$

Then we show that the mapping defined by (5.7) is a random operator. To do this, we need to prove that for any $y \in BC$, $N(.)(y) : \Omega \longrightarrow BC$ is a random variable. Then we prove that $N(.)(y) : \Omega \longrightarrow BC$ is measurable. as a mapping $f(t, y, .), t \in J, y \in BC$ is measurable by assumption (H_2) and (H_6).

Let $D : \Omega \longrightarrow 2^{BC}$ be defined by:

$$D(w) = \{y \in BC : \|y\|_{BC} \leq R(w)\}.$$

With $D(w)$ bounded, closed, covex and solid for all $w \in \Omega$. Then D is measurable by lemma 17 in [33].

Let $w \in \Omega$ be fixed, If $y \in D(w)$, from Lemma 5.5 follows that

$$\|y_{\rho(t,y_t)}\|_{\mathcal{B}} \leq (M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_T R(w)$$

and For each $y \in D(w)$, by (H_3) and (H_5), we have for each $t \in J$

$$\begin{aligned}
|(N(w)y)(t)| &\leq M\|\phi\|_{\mathcal{B}} + M'|\varphi| + M\int_0^t |f(s, y_{\rho(s,y_s)}, w)|\, ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M'|\varphi| + M\int_0^t p(s,w)\ \psi\left(\|y_{\rho(s,y_s)}\|_{\mathcal{B}}, w\right)\ ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M'|\varphi| + M\int_0^t p(s,w)\ \psi\left((M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_T R(w), w\right)\ ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M'|\varphi| + M\ \psi\left((M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_T R(w), w\right)\int_0^T p(s,w)ds \\
&\leq R(w).
\end{aligned}$$

This implies that N is a random operator with stochastic domain D and $N(w) : D(w) \to D(w)$ for each $w \in \Omega$.

Step 1: N is continuous.
Let y^n be a sequence such that $y^n \longrightarrow y$ in BC. Then

$$\begin{aligned}
|(N(w)y^n)(t) - (N(w)y)(t)| &= \left|\int_0^t C(t-s)\ \left[f(s, y^n_{\rho(s,y^n_s)}, w) - f(s, y_{\rho(s,y_s)}, w)\right]\ ds\right| \\
&\leq M\int_0^t |f(s, y^n_{\rho(s,y^n_s)}, w) - f(s, y_{\rho(s,y_s)}, w)|\, ds.
\end{aligned}$$

Since $f(s, ., w)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$\|(N(w)y^n)(t) - (N(w)y)(t)\|_{BC} \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

Thus N is continuous.

Step 2: We prove that for every $w \in \Omega, \{y \in D(w) : N(w)y = y\} \neq \emptyset$. To prove this, we apply the Mönch fixed point theorem.

(a) N maps bounded sets into equicontinuous sets in $D(w)$.
Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_2 > \tau_1$, $D(w)$ be a bounded set, and $y \in D(w)$. Then

$$\begin{aligned}
|(N(w)y)(\tau_2) - (N(w)y)(\tau_1)| &\leq \|C(\tau_2) - C(\tau_1)\|_{B(E)}\|\phi\|_{\mathcal{B}} + \|S(\tau_2) - S(\tau_1)\|_{B(E)}|\varphi| \\
&+ \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)}|f(s, y_{\rho(s,y_s)}, w)|ds \\
&+ \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)}|f(s, y_{\rho(s,y_s)}, w)|ds \\
&\leq \|C(\tau_2) - C(\tau_1)\|_{B(E)}\|\phi\|_{\mathcal{B}} + \|S(\tau_2) - S(\tau_1)\|_{B(E)}|\varphi| \\
&+ \psi\left((M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_T R(w)\right) \\
&\quad \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} p(s,w)ds \\
&+ M\psi\left((M_T + L^{\phi})\|\phi\|_{\mathcal{B}} + K_T R(w), w\right) \int_{\tau_1}^{\tau_2} p(s,w)ds.
\end{aligned}$$

The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$, since $C(t), S(t)$ are a strongly continuous operator and the compactness of $C(t), S(t)$ for $t > 0$, implies the continuity in the uniform operator topology (see [84, 85]).

Next, let $w \in \Omega$ be fixed (therefore we do not write 'w' in the sequel) but arbitrary.

(b) Now let V be a subset of $D(w)$ such that $V \subset \overline{conv}\,(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $v \longrightarrow v(t) = \alpha(V(t))$ is continuous on $(-\infty, T]$. By (H_4), Lemma 1.17 and the properties of the measure α we have for each $t \in (-\infty, T]$

$$
\begin{aligned}
v(t) &\leq \alpha\left(N(V)\right)(t) \cup \{0\}) \\
&\leq \alpha\left(N(V(t))\right) \\
&\leq \alpha\Big(C(t)\ \phi(0) + S(t)\varphi(w) + \int_0^t C(t-s)\ f(s, y_{\rho(s,y_s)})\ ds\Big) \\
&\leq \alpha\Big(C(t)\ \phi(0)\Big) + \alpha\Big(S(t)\varphi(w)\Big) + \alpha\Big(\int_0^t C(t-s)\ f(s, y_{\rho(s,y_s)})\ ds\Big) \\
&\leq M\int_0^t l(s)\alpha(\big\{y_{\rho(s,y_s)} : y \in V\big\})ds \\
&\leq M\int_0^t l(s)K(s)\sup_{0\leq\tau\leq s}\alpha(V(\tau))ds \\
&\leq \int_0^t l(s)K(s)\alpha(V(s))ds \\
&\leq M\int_0^t v(s)\ l(s)K(s)ds \\
&= M\int_0^t l(s)K(s)v(s)ds.
\end{aligned}
$$

Gronwall's Lemma implies that $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in $D(w)$. Applying now Theorem 1.9 we conclude that N has a fixed point $y(w) \in D(w)$.

Since $\bigcap_{w\in\Omega} D(w) \neq \emptyset$, the hypothesis that a measurable selector of int$D$ exists holds. By lemma 1.15, the random operator N has a stochastic fixed point $y^*(w)$, which is a mild solution of the random problem (5.3)-(5.4).

Proposition 5.7 Assume that $(H_\phi), (H_1), (H_2), (H_5), (H_6)$ are satisfied, then a slight modification of the proof (i.e. use the Darbo's fixed point theorem) guarantees that (H_4) could be replaced by

$(H_4)^*$ There exists a nonnegative function $l(.,w) \in L^1(J, \mathbb{R}^+)$ for each $w \in \Omega$, such that

$$\alpha(f(t, B, w)) \leq l(t, w)\alpha(B), \quad t \in J,$$

and consider the Kuratowski measure of noncompactness α_C defined on the family of bounded subsets of the space $C(J, E)$ by

$$\alpha_C(H) = \sup_{t\in J} e^{-\tau L(t)}\alpha(H(t)),$$

where $L(t) = \displaystyle\int_0^t \widetilde{l}(s)ds, \quad \widetilde{l}(t) = Ml(t)K(t), \ \tau > 1.$

Proof. We show that the operator $N : D(w) \longrightarrow D(w)$ is a strict set contraction for each $w \in \Omega$. We know that $N : D(w) \longrightarrow D(w)$ is bounded and continuous, we need to prove that there exists a constant $0 \leq \lambda < 1$ such that $\alpha_C(NH) \leq \lambda \alpha_C(H)$ for $H \subset D(w)$. For each $t \in J$ we have

$$\alpha((NH)(t)) \leq M \int_0^t \alpha(f(s, y_{\rho(s,y_s)}, w) : y \in H) ds.$$

This implies by $(H_4)^*$ and Theorem 2.1 in [41]

$$\begin{aligned}
\alpha((NH)(t)) &\leq \int_0^t M\ l(s)\alpha\left(\left\{y_{\rho(s,y_s)} : y \in H\right\}\right) ds \\
&\leq \int_0^t Ml(s)K(s) \sup_{0\leq\tau\leq s} \alpha(H(\tau))ds \\
&\leq \int_0^t Ml(s)K(s)\alpha(H(s))ds \\
&= \int_0^t \widetilde{l}(s)\alpha(H(s))ds \\
&= \int_0^t e^{\tau L(s)}e^{-\tau L(s)}\widetilde{l}(s)\alpha(H(s))ds \\
&\leq \int_0^t \widetilde{l}(s)\ e^{\tau L(s)} \sup_{s\in[0,t]} e^{-\tau L(s)}\alpha(H(s))ds \\
&\leq \sup_{t\in[0,T]} e^{-\tau L(t)}\alpha(H(t)) \int_0^t \widetilde{l}(s)e^{\tau L(s)}ds \\
&= \alpha_C(H) \int_0^t \left(\frac{e^{\tau L(s)}}{\tau}\right)' ds \\
&\leq \alpha_C(H)\frac{1}{\tau}e^{\tau L(t)}.
\end{aligned}$$

Therefore,

$$\alpha_C(NH) \leq \frac{1}{\tau}\alpha_C(H).$$

So, the operator N is a set contraction. As a consequence of Theorem 1.10, we deduce that N has a fixed point $y(w) \in D(w)$.

Since $\bigcap_{w\in\Omega} D(w) \neq \emptyset$, the hypothesis that a measurable selector of int$D$ exists holds. By lemma 1.15, the random operator N has a stochastic fixed point $y^*(w)$, which is a mild solution of the random problem (5.3)-(5.4).

5.1.4 Examples

Example 1. Consider the functional partial differential equation of second order

$$\frac{\partial^2}{\partial t^2}z(t,x,w) = \frac{\partial^2}{\partial x^2}z(t,x,w) + f(t, z(t,x,w), w),\ x \in [0,\pi],\ t \in J, w \in \Omega, \qquad (5.8)$$

$$z(t,0,w) = z(t,\pi,w) = 0,\ t \in [0,+\infty), w \in \Omega, \tag{5.9}$$

$$z(t,x,w) = \phi(t,w), \quad \frac{\partial z(0,x,w)}{\partial t} = v(x,w),\ t \in (-\infty,0],\ x \in [0,\pi], w \in \Omega, \tag{5.10}$$

where $J := [0,+\infty)$ and $f : J \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ is a given map. Take $E = L^2[0,\pi]$, (Ω, F, P) is a complete probability space and define $A : E \to E$ by $Av = v''$ with domain

$$D(A) = \{v \in E; v, v'\text{are absolutely continuous},\ v'' \in E,\ v(0) = v(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$on E, respectively. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$ with corresponding normalized eigenvectors $z_n(\tau) := (\frac{2}{\pi})^{\frac{1}{2}} \sin n\tau$, and the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E.

(b) If $y \in E$, then $Ay = -\sum_{n=1}^{\infty} n^2 < y, z_n > z_n$.

(c) For $y \in E, C(t)y = \sum_{n=1}^{\infty} \cos(nt) < y, z_n > z_n$, and the associated sine family is

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < y, z_n > z_n$$

which implies that the operator $S(t)$ is compact for all $t > 0$ and that

$$\|C(t)\| = \|S(t)\| \leq 1,\ \text{for all } t \geq 0.$$

(d) If Φ denotes the group of translations on E defined by

$$\Phi(t)y(\xi,w) = \tilde{y}(\xi + t, w),$$

where $\tilde{y}$ is the extension of y with period 2π, then

$$C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t)); A = B^2,$$

where B is the infinitesimal generator of the group Φ on

$$X = \{y(.,w) \in H^1(0,\pi) : y(0,w) = y(\pi,w) = 0\}.$$

For more details, see [35].

Then the problem (5.1)-(5.2) in an abstract formulation of the problem (5.8)-(5.10). If conditions $(H_1)-(H_5)$ are satisfied, theorem 5.2 implies that the problem (5.8)-(5.10) has at least one random mild solutions in BC.

Example 2. Take $E = L^2[0, \pi]; \mathcal{B} = C_0 \times L^2(g, E)$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E; \omega, \omega' \text{are absolutely continuous}, \ \omega'' \in E, \ \omega(0) = \omega(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t \in \mathbb{R}}$ on E, respectively. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$ with corresponding normalized eigenvectors

$$z_n(\tau) := (\frac{2}{\pi})^{\frac{1}{2}} \sin n\tau,$$

and the following properties hold.

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E.

(b) If $y \in E$, then $Ay = -\sum_{n=1}^{\infty} n^2 < y, z_n > z_n$.

(c) For $y \in E, C(t)y = \sum_{n=1}^{\infty} \cos(nt) < y, z_n > z_n$, and the associated sine family is

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < y, z_n > z_n$$

which implies that the operator $S(t)$ is compact, for all $t \in J$ and that

$$\|C(t)\| = \|S(t)\| \leq 1, \text{ for all } t \in \mathbb{R}.$$

(d) If Φ denotes the group of translations on E defined by

$$\Phi(t)y(\xi, w) = \tilde{y}(\xi + t, w),$$

where $\tilde{y}$ is the extension of y with period 2π. Then

$$C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t)); A = B^2,$$

where B is the infinitesimal generator of the group Φ on

$$X = \{y(., w) \in H^1(0, \pi) : y(0, w) = x(\pi, w) = 0\}.$$

For more details, see [35].

Consider the functional partial differential equation of second order

$$\frac{\partial^2}{\partial t^2} z(t, x, w) = \frac{\partial^2}{\partial x^2} z(t, x, w) + C_0(w) \int_{-\infty}^{0} a(s-t) z(s - \rho_1(t)\rho_2(|z(t)|), x, w) ds, x \in [0, \pi], \ t \in J, w \in \Omega, \tag{5.11}$$

$$z(t, 0, w) = z(t, \pi, w) = 0, \ t \in J, w \in \Omega \tag{5.12}$$

$$z(t,x,w) = \phi(t,w), \quad \frac{\partial z(0,x,w)}{\partial t} = v(x,w),\ t \in (-\infty, 0],\ x \in [0,\pi], w \in \Omega, \tag{5.13}$$

where C_0 are a real-valued random variable, $J := [0, +\infty)$, $\rho_i : [0,\infty) \to [0,\infty)$, $a : \mathbb{R} \to \mathbb{R}$ be continuous, and

$$L_f = \left(\int_{-\infty}^{0} \frac{a^2(s)}{g(s)} ds \right)^{\frac{1}{2}} < \infty.$$

We define the functions $f : J \times \mathcal{B} \times \Omega \to E, \rho : J \times \mathcal{B} \to \mathbb{R}$ by

$$f(t, \psi(x), w) = C_0(w) \int_{-\infty}^{0} a(s)\psi(s,x)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2(\|\psi(0)\|).$$

We have $\|f(t,.,.)\|_{\mathcal{B}} \leq L_f$.

Let $\phi \in \mathcal{B}$ be sush that (H_ϕ) holds, and let $t \to \phi_t$ be continuous on $\mathcal{R}(\rho^-)$.

Then the problem (5.3)-(5.4) in an abstract formulation of the problem (5.11)-(5.13). If conditions $(H_1) - (H_6)$ are satisfied, theorem 5.6 implies that the problem (5.11)-(5.13) has at least one random mild solution in BC.

Conclusion and Perspective

In this thesis, we have presented some results to the theory of existence of random mild solutions of some classes of semilinear functional differential equations on finite and infinite intervals with random effect and infinite delay in a Banach space. The results are based on the semigroup theory, measure of noncompactness, the random fixed point and deterministic fixed point theorems; in particular we have used Schauder's theorem, Mönch theorem, Darbo theorem.

It would be interesting, for a future research, to consider the existence of random mild solution for the neutral functional differential equations:

$$\begin{cases} \dfrac{d}{dt}[y(t,w) - g(t, y_t, w)] - Ay(t,w) = f(t, y_t, w), & t \in J := [0,T], w \in \Omega \\ y(t,w) = \phi(t,w) & t \in (-\infty, 0], \end{cases} \tag{5.14}$$

where g, f are given functions from $J \times \mathcal{B} \times \Omega$ into E, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, of bounded linear operators in a Banach space E.

And we plan to consider the problems considered in this thesis in the case when the operator A is not densly defined and generates an integrated semigroup, in this case we look for the existence of random integral solution.

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