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Discipline : Mathématiques
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Présentée par

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Intitulée : Stochastic integro-differential equations with nonlocal conditions and infinite delay

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This Thesis is dedicated to the memory of :,
Prof. Bénamar CHOUAF (1954 .. 2011).,

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Table des matières

Summary

Many stochastic systems arising in nature exhibit hereditary properties, that is, state depends on the past time history. The time history dependence of state renders the equation of motion of stochastic systems in the form of stochastic integro-differential equations.

The research reported in this thesis deals with the problem of stochastic integro-differential systems with delay. More precisely, existence of solution for stochastic integro-differential equations in Hilbert space with infinite delay .

We first prove the existence of mild solutions for a class of fractional neutral stochastic integro-differential equations with infinite delay. Secondly, we explore the existence results with nonlocal conditions. Our approach and technique is mainly based on fixed point theorem and C_0 semigroups theory.

Résumé

De nombreux systèmes de nature stochastique résultant présentent des propriétés héréditaires, c'est-à dire l'état dépend de l'histoire du temps passé. La dépendance de temps de l'histoire de l'état rend l'équation du mouvement des systèmes stochastiques sous la forme des équations intégro-différentielles stochastiques.

La recherche présentée dans cette thèse traite du problème des systèmes intégro-différentielles stochastiques avec retard. Plus précisément, l'existence de solution pour les équations intégro-différentielles stochastiques dans l'espace de Hilbert avec retard infini.

Nous montrons d'abord les résultats d'existence d'une solution mild pour une classe des équations stochastiques fractionnaires intégro-différentielles de type neutres avec retard infini. Puis nous explorons les résultats d'existence d'une solution mild avec des conditions non locaux. Notre approche est principalement basée sur les théorèmes du point fixe et la théorie des semigroupes C_0.

The List Of Works

1. Ait ouali, N., Kandouci, A. An existence result of mild solutions of fractional order neutral stochastic integro-differential equations with infinite delay.Journal of Numerical Mathematics and Stochastics, Vol 7 (1) : 30-47, (2015).
2. Ait ouali, N., Kandouci, A. Existence of mild solution result for fractional neutral stochastic integro-differential equations with nonlocal conditions and infinite delay. Malaya Journal of Matematik 1(1)(2015) 1-13.

Communication

An existence result of mild solutions of fractional order neutral stochastic integro-differential equations with infinite delay. IWSCA'2014 Univ.Saida, (Algeria) 28^{th}-30^{th} May 2014.

Chapitre 1

General Introduction

1.1 Fractional calculus

The traditional integral and derivative are, to say the least, a staple for the technology professional, essential as a means of understanding and working with natural and artificial systems. Fractional calculus is a field of mathematics study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is anoutgrowth of exponents with integer value.

The term fractional calculus is more than 300 years old mathematical discipline fractional calculus has its origin in the question of the extension of meaning. A well known example is the extension of meaning of real numbers to complex numbers, and another is the extension of meaning of factorials of integers to factorials of complex numbers. In generalized integration and differentiation the question of the extension of meaning is : Can the meaning of derivatives of integral order $d^n y/dx^n$ be extended to have meaning where n is any number irrational, fractional or complex ?

1.1.1 Birth of Fractional Calculus

Leibnitz invented the above notation. Perhaps, it was naive play with symbols that prompted L'Hospital to ask Leibnitz about the possibility that n be a fraction. "What if n be $\frac{1}{2}$?", asked L'Hospital. Leibnitz [36] in 1695 replied, "It will lead to a paradox." But he added prophetically, "From this apparent paradox, one day useful consequences will be drawn." In 1697, Leibnitz, referring to Wallis's infinite product for $\frac{\pi}{2}$ used the notation $d^{1/2}Y$ and stated that differential calculus might have been used to achieve the same result.

In 1819 the first mention of a derivative of arbitrary order appears in a text. The French mathematician, S. F. Lacroix [35], published a 700 page text on differential and integral calculus in which he devoted less than two pages to this topic.

Starting with

$$y = x^{n}$$

n a positive integer, he found the mth derivative to be

$$\frac{d^m y}{dx^m} = \frac{n!}{(n-m)!}x^{n-m}$$

Using Legendre's symbol Γ which denotes the generalized factorial, and by replacing m by 1/2 and n by any positive real number a,

in the manner typical of the classical formalists of this period, Lacroix obtained the formula

$$\frac{d^{\frac{1}{2}y}}{dx^{\frac{1}{2}}} = \frac{\Gamma(a+1)}{\Gamma(a+\frac{1}{2})}x^{a-\frac{1}{2}}.$$

which expresses the derivative of arbitrary order 1/2 of the function x^a. He gives the example for $y = x$ and gets

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(x) = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

because $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$ and $\Gamma(2) = 1$. This result is the same yielded by the present day Riemann-Liouville definition of a fractional derivative. It has taken 279 years since L'Hospital first raised the question for a text to appear solely devoted to this topic [62].

1.1.2 Historical foreword

1.1.3 Fractional derivative

Euler and Fourier made mention of derivatives of arbitrary order but they gave no applications or examples. So the honor of making the first application belongs to Niels Henrik Abel [35] in 1823 Abel applied the fractional calculus in the solution of an integral equation which arises in the formulation of the tautochrone problem. This problem, sometimes called the isochrone problem, is that of finding the shape of a frictionless wire lying in a vertical plane such that the time of slide of a bead placed on the wire slides to the lowest point of the wire in the same time regardless of where the beais placed. The brachistochrone problem deals with the shortest time of slide.

Abel's solution was so elegant that it is my guess it attracted the attention of Liouville [41] who made the first major attempt to give a logical definition of a fractional derivative. He published three long memoirs in 1832 and several more through 1855.

Liouville's starting point is the known result for derivatives of integral order

$$D^m e^{ax} = a^m e^{ax}$$

which he extended in a natural way to derivatives of arbitrary order

$$D^V e^{ax} = a^V e^{ax}$$

He expanded the function f(x) in the series

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x} \tag{1.1}$$

and assumed the derivative of arbitrary order f(x) to be

$$D^V f(x) = \sum_{n=0}^{\infty} c_n a_n^V e^{a_n x} \tag{1.2}$$

This formula is known as Liouville's [42] first definition and has the obvious disadvantage that V must be restricted to values such that the series converges.

Liouville's second method was applied to explicit functions of the form x^{-a}, $a > O$. He considered the integral

$$I = \int_0^\infty u^{a-1} e^{-xu} du \tag{1.3}$$

The transformation $xu = t$ gives the result

$$x^{-a} = \frac{1}{\Gamma(a)} I \tag{1.4}$$

Then, with the use of (1.1) he obtained, after operating on both sides of (1.4) with D^V, the result

$$D^V x^{-a} = \frac{(-1)^V \Gamma(a+V)}{\Gamma(a)} x^{-a-V} \tag{1.5}$$

Liouville was successful in applying these definitions to problems in potential theory. "These concepts were too narrow to last," said Emil Post [63]. The first definition is restricted to certain values of v and the second method is not suitable to a wide class of functions.

Riemann [13] in 1847 while a student wrote a paper published posthumously in which he gives a definition of a fractional operation. It is my guess that Riemann was influenced by one of Liouville's memoirs in which Liouville wrote, "The ordinary differential equation

$$\frac{d^n y}{dx^n} = 0$$

has the complementary solution

$$y_c = c_0 + c_1 x + c_2 x^2 + ... + c_{n-1} x^{n-1}$$

Thus

$$\frac{d^u}{dx^u} f(x) = 0$$

should have a corresponding complementary solution." So, I am inclined to believe Riemann saw fit to add a complementary function to his definition of a fractional integration :

$$D^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t) dt + \Psi(x)$$

Cayley [13] remarked in 1880 that Riemann's complementary function is of indeterminate nature.

The development of mathematical ideas is not without error. Peacock made several errors in the topic of fractional calculus when he misapplied the Principle of the Permanence of Equivalent Forms which is stated for algebra and which did not always apply to the theory of operators. Liouville made an error when he failed to note in his discussion of a complementary function that the specialization of one of the parameters led to an absurdity. Riemann became hopelessly entangled with an indeterminate complementary function. Two different versions of a fractional derivative yielded different results when applied to a constant. Thus, I suggest that when Oliver Heaviside published his work in the last decade of the nineteenth century, he was met with haughty silence and disdain not only because of the hilarious jibes he made at mathematicians but also because of the distrust mathematicians had in the general concept of fractional operators.

The subject of notation cannot be minimized. The succinctness of notation of fractional calculus adds to its elegance. In the papers that follow in this text, various notations are used. The notation I prefer was invented by Harold T. Davis. All the information can be conveyed by the symbols

$$ {}_cD_X^{-V} f(x), \quad v \geq 0 $$

denoting integration of arbitrary order along the x-axis. The subscripts c and x denote the limits (terminals) of integration of a definite integral which defines fractional integration.

1.1.4 Different Definitions

In this section we consider different definitions of fractional derivatives and integrals (differintegrals). For some elementary functions, explicit formula of fractional drevative and integral are presented.

L.Euler (1730)

Euler generalized the formula

$$\frac{d^n x^m}{dx^n} = m(m-1).....(m-n+1)x^{m-n}$$

by using of the following property of Gamma function,

$$\Gamma(m+1) = m(m-1)....(m-n+1)\Gamma(m-n+1)$$

to obtain

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

Gamma function is defined as follows.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dz, \quad Re(z) > 0$$

J.B.J.Fourier (1820-1822)

By means of integral representation

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} cos(px - pz)dp$$

he wrote

$$\frac{d^n f(x)}{dx^n} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} cos(px - pz + n\frac{\pi}{2})dp$$

N.H.Abel (1823-1826)

Abel considered the integral representation $\int_0^x \frac{\grave{s}(\eta)d\eta}{(x-\eta)^\alpha} = \psi(x)$ for arbitrary α and then wrote

$$s(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d^{-\alpha}\psi(x)}{dx^{-\alpha}}$$

J.Lioville (1832-1855)

I. In first definition, according to exponential representation of a function $f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}$, he generalized the formula $\frac{d^m e^{ax}}{dx^n} a^m e^{ax}$ as

$$\frac{d^v f(x)}{dx^v} = \sum_{n=0}^{\infty} c_n a_n^v e^{a_n x}.$$

II. Second type of his definition was Fractional Integral

$$\int^{\mu} \Phi(x)dx^{\mu} = \frac{1}{(-1)^{\mu}\Gamma(\mu)} \int_0^{\infty} (x+\alpha)\alpha^{\mu-1}d\alpha.$$

$$\int^{\mu} \Phi(x)dx^{\mu} = \frac{1}{\Gamma(\mu)} \int_0^{\infty} (x-\alpha)\alpha^{\mu-1}d\alpha.$$

By substituting of $\tau = x + \alpha$ and $\tau = x - \alpha$ in the above formulas respectively, he obtained

$$\int^{\mu} \Phi(x)dx^{\mu} = \frac{1}{(-1)^{\mu}\Gamma(\mu)} \int_x^{\infty} (\tau - x)^{\mu-1}\Phi(\tau)d\tau.$$

$$\int^{\mu} \Phi(x)dx^{\mu} = \frac{1}{\Gamma(\mu)} \int_x^{-\infty} (x-\tau)^{\mu-1}\Phi(\tau)d\tau.$$

III. Third definition, includes Fractional derivative,

$$\frac{d^{\mu}F(x)}{dx^{\mu}} = \frac{(-1)^{\mu}}{h^{\mu}} \left(F(x)\frac{\mu}{1}F(x+h) + \frac{\mu(\mu-1)}{1.2}F(x+2h) - \ldots.. \right).$$

$$\frac{d^{\mu}F(x)}{dx^{\mu}} = \frac{1}{h^{\mu}} \left(F(x)\frac{\mu}{1}F(x-h) + \frac{\mu(\mu-1)}{1.2}F(x-2h) - \ldots.. \right).$$

G.F.B.Riemann (1847-1876)

His definition of Fractional Integralis

$$D^{-v}f(x) = \frac{1}{\Gamma(\mu)} \int_c^x (x-t)^{v-1}f(t)dt + \Psi(t).$$

N.Ya.Sonin (1869),A.V.Letnikov (1872), H.Laurent (1884), N.Nekrasove (1888),K. Nishimoto (1987) :

They considered to the Cauchy Integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_c \frac{f(t)}{(t-z)^{n+1}}dt$$

and substituted n by ν to obtain

$$D^{\nu}f(z) = \frac{\Gamma(\nu+1)}{2\pi i} \int_c^{x^+} \frac{f(t)}{(t-z)^{\nu+1}}dt.$$

Riemann-Liouvill definition

The popular definition of fractional calculus is this which shows joining of two previous definitions.

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}} \quad (n-1 \leq \alpha < n) $$

M.Caputo(1967)

The second popular definition is

$$ D_a^C f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\alpha)^{\alpha+1-n}}, \quad (n-1 \leq \alpha < n) $$

K.S. Miller, B.Ross(1993)

They used differential operator D as

$$ D^{\bar{\alpha}} f(t) = D^{\alpha_1} D^{\alpha_2} ....D^{\alpha_n} f(t), \quad \bar{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) $$

which D^{α_i} is Riemann-Liouville or Caputo definitions.

1.2 Stochastic integro-differential equations

Stochastic differential equations are well known to model problems from many areas of science and engineering wherein, quite often the future state of such systems depends not only on the present state but also on its past history (delay) leading to stochastic functional differential equations and it has played an important role in many ways such as option pricing, forecast of the growth of population , etc., [32,48,55].

From time in memory, the theory of nonlinear functional differential or integro-differential equation is an equation which involves both integrals and derivatives of an unknown function.
The theory and application of integro-differential equation play an important role in the mathematical modeling of many fields : physical, biological phenomena and engineering sciences in which it is necessary to take into account the effect of real world problems.

The advantage of the integro-differential equations representation for a variety of problem is witnessed by its increasing frequency in the literature and in many texts on method of advanced applied mathematics.

In recent years,the theory of various integro-differential equations in Banach spaces has been studied deeply due to their important values in sciences and technologies,and many significant results have been established [see,e.g., [19, 71]] and references therein).

1.2.1 Earlier works

The study of abstract integro-differential equations has been an active topic of research in recent years because it has many applications in different areas. For instance, in the theory development in Gurtin and Pipkin [26] and Nunziato [53] for the description of heat conduction in materials with fading memory. It should be pointed out that the deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic ones. In addition, there exists an extensive literature about integrodifferential equations with nonlocal initial conditions,(cf.e.g.,[6,22,30,64,65,72]).

In addition Using the method of semigroup, existence and uniqueness of mild, strong and classical solutions of semilinear evolution equations have been discussed by Pazy [57] and the nonlocal Cauchy problem for the same equation has been studied by Byszewskii [11,12]. Balachandran and Chandrasekaran [3] studied the nonlocal Cauchy problem for semi- linear integrodifferential equation with deviating argument. Balachandran and Park [3] has been discussed about the existence of solutions and controllability of nonlinear integro-differential systems in Banach spaces. Grimmer [4] obtained the representation of solutions of integro-differential equations by using resolvent operators in a Banach space. Liu [27] discussed the Cauchy problem for integro-differential evolution equations in abstract spaces and al soin [43] he discussed nonautonomous integro-differential equations. Lin and Liu [44] studied the nonlocal Cauchy problem for semilinear integro-differential equations by using resolvent operators.

1.3 The Plan Of-The Thesis

The structure of the Thesis is as follows. Chapter one is basic introduction, dealing with development of the fractional calculus. Several definitions of fractional differintegrations and the most popular ones, are introduced here. A table in this, chapter gives the brief presentation of the thesis.

Chapter two deals with the basic concepts, notation and elementary results that are used throughout this thesis. The important functions relevant to fractional calculus basis in Section 2.1. Understanding of definitions and use of fractional calculus will be made more clear by quickly discussing some necessary but relatively simple mathematical definitions that will arise in the study of these concepts. These are The Gamma Function, The Beta Function, and the Mittag-Leffer Function and are addressed in the following four subsections (2.1.1, 2.1.2 and 2.1.3). Section 2.2 is devoted the fundamentals of the theory of a C_0 semigroups of linear operators with the goal of studying the existence of classical and mild solutions to Stochastic Differential Equations.

The chapter three is divided into two sections : In (3.1) we will give definitions and properties of Infinite-dimensional Wiener processes. We briefly recall some basic notions of the linear operators, the definition and propriety of the Gaussian random variable, the representation of the Q-Wiener process and then define cylindrical Wiener process. In (3.2) we introduce the Stochastic Differential Equations on Hilbert spaces and their solutions.

In chapter four, We study the existence of mild solutions for a class of fractional neutral stochastic integro-differential equations with infinite delay in Hilbert spaces. Our approach is based on Schaefer fixed point theorem.

$$\begin{cases} {}^cD_t^\alpha[x(t) + G(t, x_t)] = -Ax(t) + f(t, x_t) + \int_{-\infty}^t \sigma(t, s, x_s)dW(s) & t \in J = [0, b] \\ x(t) = \phi(t) & t \in (-\infty, 0] \end{cases}$$

Finally in chapter five, using a Krasnoselski-Schaefer fixed point we prove the existence and uniqueness of the mild solution for the fractional differential equation with nonlocal conditions and infinite delay of the form.

$$\begin{cases} {}^cD_t^{\alpha}[x(t) - h(t, x_t)] = A[x(t) - h(t, x_t)] + f(t, x_t) + \int_{-\infty}^{t} \sigma(t, s, x_s)dW(s) & t \in J = [0, b] \\ x(0) + \mu(x) = x_0 = \phi(t) & t \in (-\infty, 0], \end{cases}$$

Chapitre 2

The Fundamental Theory of Fractional Calculus

In this chapter, we will give the basic concepts and results concerning fractional calculus and we present a number of functions that have been found useful in the solution of the problems of fractional calculus and refer the reader to [31],[49],[58]. Also we introduce the fundamentals of semigroup theory.

2.1 Fractional derivation

2.1.1 Gamma function

Definition 2.1.1. *For any complex number z such as $R(z) > 0$, we define the Gamma function*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{2.1}$$

this integral converges absolutely on half complex plane or the real part is strictly positive.

The gamma function satisfies the identity

$$\Gamma(z+1) = z\Gamma(z)$$

is demonstrated by integrating by parts

$$\Gamma(z+1) = \int_0^{+\infty} e^{-t} t^{z-1} dt = -e^{-t} t^{-z}|_0^\infty + z \int_0^\infty e^{-t} t^{z-1} dt$$

if n is an integer, we get closer and closer as

$$\Gamma(z+n) = z(z+1)...(z+n-1)\Gamma(z)$$

as $\Gamma(1) = 1$, this proves that $\Gamma(n+1) = n!$

2.1.2 Beta function

Definition 2.1.2. *The Beta function is generally defined by*

$$\beta(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, \qquad (Re(z) > 0, Re(w) > 0) \tag{2.2}$$

Relation between the gamma function and the beta function
The gamma function and Beta function are linked by the following relation (see cite [31])

$$\beta(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

2.1.3 Mittag-Leffler function

The exponential function, e^z, plays an important rôle in the theory of integer order differential equations. Its one parameter generalization, the function defined by

$$E_\alpha(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(n\alpha+1)}$$

was introduced by Mittag-Leffer (1903, 1904, 1905) and was investigated by several authors among whom Wiman (1905), Pollard (1948), Humbert (1953). For $\alpha > 0$, $E_\alpha(z)$ is the simplest entire function of order $1/\alpha$ Phragmen (1004).

Definition 2.1.3. *we Call Mittag-Leffler function defined function by*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(n\alpha+\beta)} \qquad (z,\beta \in \mathbb{C}, Re(\alpha) > 0) \tag{2.3}$$

that plays an important role in the fractional calculus, was in fact introduced by Agarwal (1953). It was studied by Humbert and Agarwal (1953), but they *used the same notation and name as for the one-parameter Mittag-Leffer function. It is noted that*

$$E_{\alpha,1}(z) = E_\alpha(z) \qquad (z, \in \mathbb{C}, Re(\alpha) > 0)$$

In particular, when $\alpha = 1$ *et* $\alpha = 2$, *We have*

$$E_1(z) = e^z \qquad et \qquad E_2(z) = \cosh(\sqrt{z})$$

Definition 2.1.4. *let* $f : [a; b) \longrightarrow \mathbb{R}$ *continuous function. we call integral Riemann-Liouville of* f *the following integral :*

$$(I_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

Proposition 2.1.1. *let* $f \in C_0([a, b])$. *for* α, β *complexes such as* $Re(\alpha) > 0$ *et* $Re(\beta) > 0$ *We have*

$$I_a^\alpha (I_a^\beta f) = I_a^{\alpha+\beta} f$$

and for $Re(\alpha) > 0$ *We have*

$$\frac{d}{dx} I_a^\alpha f = I_a^\alpha f$$

Definition 2.1.5. *we call derivative of order* α *in the sense of Riemann-Liouville function defined by*

$$\begin{aligned} (D_a^\alpha f)(x) &= (\frac{d}{dx})^n [I_a^{n-\alpha} f(x)] \\ &= \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dx})^n \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt \qquad (n = [Re(\alpha)] + 1, x > a) \end{aligned}$$

Proposition 2.1.2. *The derivation operator of Riemann-Liouville* D_a^α *has the following properties :*

(1) D_a^α *is a linear operator*

(2) in general $D_a^\alpha o D_a^\beta \neq D_a^\beta o D_a^\alpha$ *and also* $\neq D_a^{\alpha+\beta}$

(3) $D_a^\alpha o L_a^\alpha = id$

Proposition 2.1.3. *Let* $Re(\alpha) > 0$ *et* $n = [Re(\alpha)] + 1$ *then*

$$(D_a^\alpha)(x) = 0 \Leftrightarrow y(x) = \sum_{j=1}^{n} c_j (x-a)^{\alpha-j}$$

where $c_j \in \mathbb{R}$, $(j = 1, ..., n)$ *are arbitrary constants.*

In particular, if $0 < Re(\alpha) \leq 1$ *we have*

$$(D_a^\alpha y)(x) = 0 \Leftrightarrow y(x) = c(x-a)^{\alpha-1} \qquad c \in \mathbb{R}$$

2.2 Elements of Semigroup Theory

Let $(X, \|.\|_X)$ and $(X, \|.\|_Y)$ be Banach spaces. Denote by $\mathcal{L}(X,Y)$ the family of bounded linear operators from X to Y. $\mathcal{L}(X,Y)$ becomes a Banach space when equipped with the norm

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{x\in X, \|x\|_X=1} \|Tx\|_Y\,, \quad T \in \mathcal{L}(X,Y)$$

$\mathcal{L}(X)$ denote the Banach space of bounded linear operators on X.
The identity operator on X is denoted by I.
Let H be a real Hilbert space. A linear operator $T \in \mathcal{L}(H)$ is called symmetric if for all $h, g \in H$

$$\langle Th, g\rangle_H = \langle h, Tg\rangle_H$$

Definition 2.2.1. *A family $S(t) \in \mathcal{L}((X), t \geq 0$ of bounded linear operators on a Banach space X is called a strongly continuous semigroup (or C_0-semigroup) if*

(i) $S(0) = I$,

(ii) (Semigroup property) $S(t+s) = S(t)S(s)$, for all s,t ≥ 0.

(iii) (Strong continuity property) $\lim_{t\to o^+} S(t)x = x$, for all x in X.

Let $S(t)$ be C_0-semigroup on a Banach space X. Then, there exist constants $\alpha \geq 0$ and $M \geq 1$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\alpha t}, \quad t \geq 0 \tag{2.4}$$

- If $M = 1$, then $S(t)$ is called a pseudo-contraction semigroup.
- If $\alpha = 0$, then $S(t)$ is called uniformly bounded, and if $\alpha = 0$ and $M = 1$ (i.e,$\|S(t)\|_{\mathcal{L}(X)} \leq 1$), then $S(t)$ is called a semigroup of contractions.
- If for every $x \in X$, the mapping $t \to S(t)x$ is differentiable for $t > 0$, then $S(t)$ is called a differentiable semigroup.
- A semigroup of linear operators $\{S(t), t \geq 0\}$ is called compact if the operators $S(t), t > 0$, are compact

Definition 2.2.2. *Let $S(t)$ be a C_0-semigroup on a Banach space X. The linear operator A with domain*

$$\mathcal{D}(A) = \left\{x \in X, \lim_{t\to 0^+} \frac{S(t)x - x}{t} \text{ exists}\right\} \tag{2.5}$$

defined by

$$Ax = \lim_{t \to 0^+} \frac{S(t)x - x}{t} \tag{2.6}$$

is called the infinitesimal generator of the semigroup $S(t)$.

A semigroup $S(t)$ *is called uniformly continuous if*

$$\lim_{t \to 0^+} \|S(t) - I\|_{\mathcal{L}(X)} = 0$$

Proposition 2.2.1. *Let* A *be an infinitesimal generator of* C_0*-semigroup* $S(t)$ *on a Banach space* X*.Then*

1. *For every* $x \in X$ *and* $t \geq 0$*, one has*

$$\lim_{t \to o} \frac{1}{h} \int_t^{t+h} S(t)xds = S(t)x$$

2. *If* $x \in \mathcal{D}(A)$, $S(t)x \in \mathcal{D}(A)$*, and*

$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax$$

3. *For* $x \in X$, $\int_0^t S(s)xds \in \mathcal{D}(A)$ *one has*

$$A\left(\int_0^t S(s)xds\right) = S(t)x - x$$

4. *If* $S(t)$ *is differentiable then for* $n = 1, 2, ..., S(t) : X \to \mathcal{D}(A^n)$ *and*

$$S^{(n)}(t) = A^n S(t) \in \mathcal{L}(X)$$

5. *If* $S(t)$ *is compact then* $S(t)$ *is continuous in the operator topology for* $t > 0$*,*
   i.e.,

$$\lim_{s \to t, t > 0} \|S(s) - S(t)\|_{\mathcal{L}(H)} = 0$$

6. *For* $x \in \mathcal{D}(A)$

$$S(t)x - S(s)x = \int_s^t S(u)Axdu = \int_s^t AS(u)xdu$$

7. $\mathcal{D}(A)$ *is dense in* X*, and* A *is a closed linear operator.*
8. *The intersection* $\cap_{n=1}^{\infty} \mathcal{D}(A^n)$ *is dense in* X.

Theorem 2.1. *Let* $f : [0, T] \to \mathcal{D}(A)$ *be measurable, and let* $\int_0^t \|f(s)\|_{\mathcal{D}(A)} < \infty$.
Then

$$\int_0^t f(s)ds \in \mathcal{D}(A) \text{ and } \int_0^t Af(s)ds = A\int_0^t f(s)ds$$

Definition 2.2.3. *The resolvent set* $\rho(A)$ *of a closed linear operator* A *on a Banach space* X *is the set of all complex numbers* λ *for which* $\lambda I - A$ *has a bounded inverse, i.e., the operator* $(\lambda I - A)^{-1} \in \mathcal{L}(X)$*. The family of bounded linear operators*

$$R(\lambda, A) = (\lambda I - A)^{-1}, \quad \lambda \in \rho(A)$$

is called the resolvent of A.

We note that $R(\lambda, A)$ *is a one-to-one transformation of* X *onto* $\mathcal{D}(A)$*, i.e.,*

$$(\lambda I - A)R(\lambda, A)x = x, \quad x \in X$$

$$R(\lambda, A)(\lambda I - A)x = x, \quad x \in \mathcal{D}(A)$$

In particular,

$$AR(\lambda, A)x = R(\lambda, A)Ax \quad x \in \mathcal{D}(A)$$

In addition, we have the following commutativity property :

$$R(\lambda_1, A)R(\lambda_2, A) = R(\lambda_2, A)R(\lambda_1, A) \quad \lambda_1, \lambda_2 \in \rho(A)$$

Proposition 2.2.2. *Let* $S(t)$ *be a* C_0*-semigroup with infinitesimal generator* A *on a Banach space* X*. If* $\alpha_0 = \lim_{t\to\infty} t^{-1} \ln \|S(t)\|_{\mathcal{L}(X)}$*, then any real number* $\lambda > \alpha_0$ *belongs to the resolvent set* $\rho(A)$ *and*

$$R(\lambda, A)x = \int_0^t e^{-\lambda t} S(t)xdt, \quad x \in X$$

Furthermore, for each $x \in X$*,*

$$\lim_{\lambda\to\infty} \|\lambda R(\lambda, A)x - x\|_X = 0$$

Theorem 2.2. *(Hille-Yosida). Let* A $\mathcal{D}(A) \subset X \to X$ *be a linear operator on a Banach pace* X*. Necessary and suffcient conditions for* A *to generate a* C_0 *semigroup* $S(t)$ *are* *(i) A is a closed operator and $\overline{D(A)} = X$;*

(ii) there exist real numbers M and α such that for every $\lambda > \alpha$, $\lambda \in \rho(A)$ (the resolvent set) and

$$\|(R(\lambda, A))^r\|_{\mathcal{L}(X)} \leq M(\lambda - \alpha)^{-r}, \quad r = 1, 2, ...$$

In this case $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\alpha t}, t \geq 0$

Chapitre 3

Stochastic Differential Equations in Infinite Dimensions

In this chapter, we introduce the notion of the standard Wiener process in infinite dimensions. we present the asymptotic behaviors of solutions to infinite dimensional stochastic differential equations.

3.1 Infinite-dimensional Wiener processes

We fix two separable Hilbert spaces $(U, \langle \ . \ \rangle_U)$ and $(H, \langle \ . \ \rangle_H)$.

3.1.1 Linear Operators

Let $(U, \| \ \|)$ be a Banach space, $\mathcal{B}(U)$ the Borel σ-field of U and $(\Omega, \mathcal{F}, \mu)$ a measure space with finite measure μ.

Proposition 3.1.1. *Let $f \in L_1(\Omega, \mathcal{F}, \mu; U)$. Then*

$$\int L \circ f d\mu = L(\int f d\mu)$$

holds for all $L \in L(U, H)$, where Y is another Banach space.

Proof. see [9], Proposition E.11, p. 356.

Proposition 3.1.2. *Let $(\Omega, \mathcal{F})$ be a measurable space and let U be a Banach space. Then :*

1. *The set of $\mathcal{F}/\mathcal{B}(U)$-measurable functions from Ω to U is closed under the formation of pointwise limits, and*
2. *The set of strongly measurable functions from Ω to U is closed under the formation of pointwise limits.*

Proof. see [9], Proposition E.1, p. 350.

Proposition 3.1.3. *Let E be a metric space with metric d and let $f : \Omega \longrightarrow E$ be strongly measurable. Then there exists a sequence $f_n, n \in \mathbb{N}$, of simple Evalued functions (i.e. f_n is $\mathcal{F}/\mathcal{B}(U)$-measurable and takes only a finite number of values) such that for arbitrary $\omega \in \Omega$ the sequence $d(f_n(\omega), f(\omega)), n \in \mathbb{N}$, is monotonely decreasing to zero.*

Proof. see [17], Lemma 1.1, p. 16.

3.1.2 Gaussian random variable

Definition 3.1.1. *A probability measure μ on $(U, \mathcal{B}(U))$ is called Gaussian if for all bounded linear mapping*

$$\begin{aligned} \upsilon' \; : \; & U \longrightarrow \mathbb{R} \\ & \mu \longmapsto \langle u, \upsilon \rangle_U, \quad u \in U \end{aligned}$$

have a Gaussian law, i.e. for all $\upsilon \in U$ there exist $m = m(\upsilon) \in \mathbb{R}$ and $\sigma = \sigma(\upsilon) > 0$ such that

$$(\mu \circ (\upsilon')^{-1})(A) = \mu \circ (\upsilon' \in A) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_A e^{-\frac{(x-m)^2}{2\sigma^2}} \underset{\cdot}{x} \text{ for all } A \in \mathcal{B}(\mathbb{R}),$$

or

$$\mu = \delta_u \text{ for one } u \in U \text{ where } \delta_u \text{ is the Dirac measur in } u.$$

Theorem 3.1. *A measure μ on $(U, \mathcal{B}(U))$ is Gaussian if and only if*

$$\hat{\mu}(u) := \int_U e^{i\langle u, \upsilon \rangle_U} \mu(d\upsilon) = e^{i\langle m, u \rangle_U - \frac{1}{2}\langle Qu, u \rangle_U}, \; u \in U.$$

where $m \in U$ and $Q \in L(U)$ is nonnegative, symmetric, with finite trace. In this case μ will be denoted by $\mathcal{N}(m, Q)$ where m is called mean and Q is called covariance (operator). The measure μ is uniquely determined by m and Q.

Proof.[17] The following result is then obvious.

Proposition 3.1.4. *Let X be a U-valued Gaussian random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. there exist $m \in U$ and $Q \in L(U)$ nonnegative, symmetric, with finite trace such that $\mathbb{P} \circ X^{-1} = \mathcal{N}(m, Q)$.*

Then $\langle X, u\rangle_U$ is normally distributed for all $u \in U$ and the following statements hold :

— $\mathbb{E}(\langle X, u\rangle_U) = \langle m, u\rangle_U$ *for all* $u \in U$,

— $\mathbb{E}(\langle X - m, u\rangle_U . \langle X - m, v\rangle_U) = \langle Qu, v\rangle_U$ *for all* $v, u \in U$,

— $\mathbb{E}(\|X - m\|_U^2) = trQ$.

The following proposition will lead to a representation of a U-valued Gaussian random variable in terms of real-valued Gaussian random variables.

Proposition 3.1.5. *If $Q \in L(U)$ is nonnegative, symmetric, with finite trace then there exists an orthonormal basis $e_k, k \in \mathbb{N}$, of U such that*

$$Q_{e_k} = \lambda_k e_k, \quad \lambda_k \geq 0, \; k \in \mathbb{N}.$$

and 0 is the only accumulation point of the sequence $(\lambda_k)_{k \in \mathbb{N}}$.

Proof. See [59], Theorem VI.21 ; Theorem VI.16 (Hilbert-Schmidt theorem). ■

Proposition 3.1.6. *[33]* ***(Representation of a Gaussian random variable)*** *Let $m \in U$ and $Q \in L(U)$ be nonnegative, symmetric, with $trQ < \infty$. In addition, we assume that $e_k, k \in \mathbb{N}$, is an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues $\lambda_k, k \in \mathbb{N}$, as in Proposition 3.1.5, numbered in decreasing order.*
Then a U-valued random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is Gaussian with $\mathbb{P} \circ X^{-1} = \mathcal{N}(m, Q)$ if and only if

$$X = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k e_k + m \quad (\textit{as objects in } L^2(\Omega, \mathcal{F}, \mathbb{P}; U)).$$

where $\beta_k, k \in \mathbb{N}$, are independent real-valued random variables $\mathbb{P} \circ \beta_k^{-1} = \mathcal{N}(0, 1)$ for all $k \in \mathbb{N}$ with $\lambda_k > 0$. The series converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$.

3.1.3 The definition of the standard Q-Wiener process

After these preparations we will give the definition of the standard Q-Wiener process. To this end we fix an element $Q \in L(U)$, nonnegative, symmetric and with finite trace and a positive real number T.

Definition 3.1.2. *A U-valued stochastic process $W(t), t \in [0,T]$, on a probability space*
$(\Omega, \mathcal{F}, \mathrm{IP})$ *is called normal (standard) Q-Wiener process if :*

— $W(0) = 0$
— *W has IP-a.s. continuous trajectories,*
— *the increments of W are independent, i.e. the random variables*

$$W(t_1), W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n-1})$$

are independent for all $0 \le t_1 < \ldots < t_n \le T, n \in \mathrm{IN}$,
— *the increments have the following Gaussian laws :*

$$\mathrm{IP} \circ (W(t) - W(s))^{-1} = \mathcal{N}(0, (t-s)Q) \text{ for all } 0 \le s \le t \le T.$$

Proposition 3.1.7. *For arbitrary trace class symmetric nonnegative operator Q on a separable Hilbert space U there exists a Q-Wiener process $W(t), t \ge 0$.*

Proof. See [17], Proposition 4.2, p. 88. ■

Proposition 3.1.8. *[33] Let $T > 0$ and $W(t), t \in [0,T]$, be a U-valued Q-Wiener process with respect to a normal filtration $\mathcal{F}_t, t \in [0,T]$, on a probability space $(\Omega, \mathcal{F}, \mathrm{IP})$ Then $W(t)$,*
$t \in [0,T]$, is a continuous square integrable $\mathcal{F}_t$-martingale, i.e. $W \in \mathcal{M}_T^2(U)$.

3.1.4 Representation of the Q-Wiener process

Proposition 3.1.9. *[33] Let e_k, $k \in \mathrm{IN}$, be an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues $\lambda_k, k \in \mathrm{IN}$. Then a U-valued stochastic process $W(t)$, $t \in [0,T]$, is a Q-Wiener process if and only if*

$$W(t) = \sum_{k \in \mathrm{IN}} \sqrt{\lambda_k} \beta_k(t) e_k, \quad t \in [0,T], \tag{3.1}$$

where $\beta_k, k \in \{n \in \mathbb{N} \mid \lambda_n > 0\}$, *are independent real-valued Brownian motions on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *The series even converges in* $L^2(\Omega, \mathcal{F}, \mathbb{P}; C([0,T], U))$, *and thus always has a* $\mathbb{P}$*-a.s. continuous modification. (Here the space* $C([0,T], U)$ *is equipped with the sup norm).*

Definition 3.1.3. ***(Normal filtration).*** *A filtration* $\mathcal{F}_t, t \in [0,T]$, *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is called normal if :*

— $\mathcal{F}_0$ *contains all elements* $A \in \mathcal{F}$ *with* $\mathbb{P}(A) = 0$ *and*

— $\mathcal{F}_t = \mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s$ *for all* $t \in [0,T]$.

Definition 3.1.4. *(**Q-Wiener process with respect to a filtration**).* *A* Q*-Wiener process* $W(t), t \in [0,T]$, *is called a* Q*-Wiener process with respect to a filtration* $\mathcal{F}_t, t \in [0,T]$, *if :*

— $W(t), t \in [0,T]$, *is adapted to* $\mathcal{F}_t, t \in [0,T]$ *and*

— $W(t) - W(s)$ *is independent of* $\mathcal{F}_s$ *for all* $0 \leq s \leq t \leq T$.

In fact it is possible to show that any U-valued Q-Wiener process $W(t), t \in [0,T]$, is a Q-Wiener process with respect to a normal filtration.

3.1.5 Cylindrical Wiener Processes

Definition 3.1.5. *We call a family* $\{\tilde{W}_t\}_{t\geq 0}$ *defined on a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *a cylindrical Wiener process in a Hilbert space* K *if :*

1. *For an arbitrary* $t \geq 0$, *the mapping* $\tilde{W}_t : K \to L^2(\Omega, \mathcal{F}, \mathbb{P})$*is linear.*
2. *For an arbitrary* $k \in K$, $\tilde{W}_t(k)$ *is an* $\mathcal{F}_t$*-Brownian motion.*
3. *For an arbitrary* $k, \grave{k} \in K$ *and* $t \geq 0$, $E(\tilde{W}_t(k)\tilde{W}_t(\grave{k})) = t\langle\, k, \grave{k}\, \rangle_K$

For every $t > 0$, $\tilde{W}_t/\sqrt{t}$ *is a standard cylindrical Gaussian random variable.*

3.2 Hilbert-Schmidt operator

Definition 3.2.1. *(**Hilbert-Schmidt operator**). Let* $e_k, k \in \mathbb{N}$, *be an orthonormal basis of* U. *An operator* $A \in L(U,H)$ *is called Hilbert Schmidt if*

$$\sum_{k\in\mathbb{N}} \langle A_{e_k}, A_{e_k} \rangle < \infty.$$

(i) The definition of Hilbert-Schmidt operator and the number

$$\|A\|_{L_2(U,H)} := \Big(\sum_{k\in\mathbb{N}} \|A_{e_k}\|^2\Big)^{\frac{1}{2}}$$

does not depend on the choice of the orthonormal basis $e_k, k \in \mathbb{N}$, and we have that $\|A\|_{L_2(U,H)} = \|A^*\|_{L_2(H,U)}$. For simplicity we also write $\|A\|_{L_2(U,H)}$ instead of $\|A\|_{L_2}$.

(ii) $\|A\|_{L(U,H)} \le \|A\|_{L_2(U,H)}$.

(iii) Let G be another Hilbert space and $B_1 \in L(H,G), B_2 \in L(G,U), A \in L_2(U,H)$. Then $B_1A \in L_2(U,G)$ and $AB_2 \in L_2(G,H)$ and

$$\|B_1A\|_{L(U,G)} \le \|B_1\|_{L(H,G)}\|A\|_{L_2(U,H)},$$

$$\|AB_2\|_{L(G,H)} \le \|A\|_{L_2(U,H)}\|B_2\|_{L(G,U)},$$

Proposition 3.2.1. *[33] Let $B, A \in L_2(U,H)$ and let $e_k, k \in \mathbb{N}$, be an orthonormal basis of U. If we define*

$$\langle A, B\rangle_{L_2} := \sum_{k\in\mathbb{N}} \langle Ae_k, Be_k\rangle$$

we obtain that $(L_2(U,H), \langle\ ,\ \rangle_{L_2})$ is a separable Hilbert space.

If $f_k, k \in \mathbb{N}$, is an orthonormal basis of H we get that $f_j \otimes e_k := f_j\langle e_k, .\rangle_U, j,k \in \mathbb{N}$, is an orthonormal basis of $L_2(U,H)$.

Besides we recall the following fact.

Proposition 3.2.2. *If $Q \in L(U)$ is nonnegative and symmetric then there exists exactly one element $Q^{\frac{1}{2}} \in L(U)$ nonnegative and symmetric such that $Q^{\frac{1}{2}} \circ Q^{\frac{1}{2}} = Q$.*

If, in addition, $trQ < \infty$ we have that $Q^{\frac{1}{2}} \in L_2(U)$ where $\|Q^{\frac{1}{2}}\|^2_{L_2} = trQ$ and of course $L \circ Q^{\frac{1}{2}} \in L_2(U,H)$ for all $L \in L(U,H)$.

Proof. [59], Theorem VI.9, p. 196. ∎

Proposition 3.2.3. *[33] Let $T \in L(U)$ and T^{-1} the pseudo inverse of T.*

1. *If we define an inner product on $T(U)$ by*

$$\langle x, y\rangle_{T(U)} := \langle T^{-1}x, T^{-1}y\rangle_U \text{ for all } x, y \in T(U),$$

then $(T(U), \langle\ ,\ \rangle_{T(U)})$ is a Hilbert space.

2. *Let $e_k, k \in \mathbb{N}$, be an orthonormal basis of $(KerT)^{\perp}$. Then $Te_k, k \in \mathbb{N}$, is an orthonormal basis of $(T(U), \langle\ ,\ \rangle_{T(U)})$.*

3.3 Stochastic Differential Equations and their Solutions

Let K and H be real separable Hilbert spaces, W_t be a K-valued Q-Wiener process on a complete filtred probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, \mathbb{P})$ with the filtration $\mathcal{F}_t$ satisfying the usual conditions.
We consider semilinear SDEs (SSDEs for short) on $[0, T]$ in H. The general form of such SSDE is

$$\begin{cases} dX_t &= [AX(t) + F(t, X)]dt + B(t, X)dW_t \in H \\ X_0 &= \alpha_0 \end{cases} \tag{3.2}$$

Here, $A : \mathcal{D}(A) \subset H \to H$ is the generator of a C_0-semigroup of operators $\{S(t), t \geq 0\}$ on H. The coefficients F and B are, in general, nonlinear mappings,

$$F : \Omega \times [0, T] \times C([0, T], H) \to H$$

$$B : \Omega \times [0, T] \times C([0, T], H) \to \mathcal{L}_2(K_Q, H).$$

The initial condition α_0 is an $\mathcal{F}_0$ mesurable H-valued random variable.

We will study the existence and uniqueness problem under various regularity assumptions on the coefficients of 3.2 that include :

(A1) F and B are jointly measurable, and for every $0 \leq t \leq T$, they are measurable with respect to the product σ-field $\mathcal{F}_t \otimes \varphi_t$, where φ_t is a σ-field generated by cylinders with bases over $[0, t]$.

(A2) F and B are jointly continuous.

(A3) There exists a constant l such that for all $x \in C([0, T], H)$

$$\|F(w, t, x)\|_H + \|B(w, t, x)\|_{\mathcal{L}_2(K_Q,H)} \leq l(1 + \sup_{0\leq s\leq T} \|x(s)\|_H)$$

for $w \in \Omega$ and $0 \leq t \leq T$.

For every $t \in [0, T]$, we define the following operator θ_t on $C([0, T], H)$

$$\theta_t x(s) = \begin{cases} x(s) & 0 \le s \le t \\ \\ x(t) & t < s \le T \end{cases}$$

Assumption (A1) implies that

$$F(w, t, x) = F(w, t, x_1) \text{ and } B(w, t, x) = B(w, t, x_1)$$

if $x = x_1$ on $[0, T]$. Because $\theta_t x$ is a Borel function of t with values in $C([0, T], H)$, $F(w, t, \theta_t x)$ and $B(w, t, \theta_t x)$ also are Borel functions in t. With this notation, 3.2 can be rewritten as

$$\begin{cases} dX_t &= [AX_t + F(t, \theta_t X)]dt + B(t, \theta_t X)dW_t \in H \\ X_0 &= \alpha_0 \end{cases} \tag{3.3}$$

We will say that F and B satisfy the Lipschitz condition if

(A.4) For all $x, y \in C([0, T], H), w \in \Omega, 0 \le t \le T$, there exists $\mathcal{K} > 0$ such that

$$\|F(w, t, x) - F(w, t, y)\|_H + \|B(w, t, x) - B(w, t, y)\|_{\mathcal{L}_2(K_Q, H)} \le \mathcal{K} \sup_{0 \le s \le T} \|x(s) - y(s)\|_H$$

There exist different notions of a solution to the semilinear SDE (3.2), and we define strong, weak, mild solutions

3.3.1 Strong solution

Definition 3.3.1. *A stochastic process $X(t)$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \le T}, \mathbb{P})$ and adapted to the filtration $\{\mathcal{F}_t\}_{t \le T}$ is a strong solution of (3.2) if*

1. *$X(.) \in C([0, T], H)$.*
2. *$X(t, w) \in \mathcal{D}(A) dt \otimes d\mathbb{P}$-almost everywhere.*
3. *the following conditions hold :*

$$\mathbb{P}\left(\int_0^T \|AX(t)\|_H \, dt < +\infty\right) = 1$$

$$\mathbb{P}\left(\int_0^T \|F(t, X)\|_H + \|B(t, X)\|^2_{\mathcal{L}_2(K_Q, H)} \, dt < \infty\right) = 1$$

4. *for every* $t \leq T$, $\mathbb{P}$*-a.s,*

$$X(t) = \alpha_0 + \int_0^t (AX(s) + F(s,X))ds + \int_0^t B(s,X)dW_s \tag{3.4}$$

3.3.2 Weak solution

Definition 3.3.2. *A stochastic process* $X(t)$ *defined on a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, \mathbb{P})$ *and adapted to the filtration* $\{\mathcal{F}_t\}_{t\leq T}$ *is a weak solution of (3.2) if*

1. *the following conditions hold :*

$$\mathbb{P}\left(\int_0^T \|X(t)\|_H \, dt < +\infty\right) = 1 \tag{3.5}$$

$$\mathbb{P}\left(\int_0^T \|F(t,X)\|_H + \|B(t,X)\|^2_{\mathcal{L}_2(K_Q,H)} \, dt < \infty\right) = 1 \tag{3.6}$$

2. *for every* $h \in \mathcal{D}(A^*)$ *and* $t \leq T$, $\mathbb{P}$*-a.s,*

$$\langle X(t), h\rangle_H = \langle \alpha_0, h\rangle_H + \int_0^t (\langle X(s), A^* h\rangle_h \tag{3.7}$$

$$+ \langle F(s,X), h\rangle_H)ds + \int_0^t \langle h, B(s,X)dW_s\rangle_H \,. \tag{3.8}$$

3.3.3 Mild solution

Definition 3.3.3. *A stochastic process* $X(t)$ *defined on a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, \mathbb{P})$ *and adapted to the filtration* $\{\mathcal{F}_t\}_{t\leq T}$ *is a mild solution of (3.2) if*

1. *conditions (3.5) and (3.6) hold;*
2. *for all* $t \leq T$, $\mathbb{P}$*-a.s.,*

$$X(t) = S(t)\alpha_0 + \int_0^t S(t-s)F(s,X)ds + \int_0^t S(t-s)B(s,X)dW_s. \tag{3.9}$$

3.3.4 Martingale solution

Definition 3.3.4. *We say that a process* X *is martingale solution of the equation*

$$\begin{cases} dX_t &= [AX_t + F(t,X)]dt + B(t,X)dW_t \\ X_0 &= x \in H \text{ deterministic} \end{cases} \tag{3.10}$$

if there exists a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, \mathbb{P})$ *and, on this probability space, a Q-Wiener process* W_t *relative to the filtration* $\{\mathcal{F}_t\}_{t\leq T}$*, such that* X_t *is a mild solution of 3.10.*

Unlike the strong solution, where the filtred probability space and the wiener process are given, a martingale solution is a system $((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, \mathbb{P}), W, X)$ *where the filtered probability space and the Wiener process are part of the solution.*

If $A = 0, S(t) = I_H$*, we obtain the SDE*

$$\begin{cases} dX_t &= F(t,X)dt + B(t,X)dW_t \\ X_0 &= x \in H \ \textit{deterministic} \end{cases} \tag{3.11}$$

and a martingale solution of (3.11) is called a weak solution (in the stochastic sense see [73])

Chapitre 4

An existence result of mild solutions of fractional order neutral stochastic integro-differential equations with infinite delay

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An existence result of mild solutions of fractional order neutral stochastic integro-differential equations with infinite delay

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Abstract

The main purpose of this paper to study the existence of mild solutions for a class of fractional neutral stochastic integro-differential equations with infinite delay in Hilbert spaces. Using fractional calculations, Schaefer fixed point theorem, stochastic analysis techniques. Under non-Lipschitz conditions, we obtain a sufficient condition for the existence results. An example is provided to illustrate the application of the obtained results..

Keywords : Infinite delay, Stochastic fractional differential equations, mild solution, fixed point method.

4.1 Introduction

It is well known that the fractional calculus is a classical mathematical notion, and is a generalization of ordinary differentiation and integration to arbitrary (non- integer) order. Nowadays, studying fractional-order calculus has become an active research field ([7], [25], [[31]], [67], [68], [75]). Much effort has been devoted to apply the fractional calculus to networks control E.g., Chen et al [16], Delshad et al [18], Wang and Zhang [66] and studied the synchronization for fractional-order complex dy- namical networks ; Zhang et al [74] investigated a fractional order three-dimensional Hop field neural network and pointed out that chaotic behaviors can emerge in a fractional network.

In fact, the fractional differential equations are valuable tools in the modeling of many phenomena in various fields of science and engineering; so, they attracted many researchers (cf., e.g., [2]-[50] and references therein). On the other hand, the integro-differential equations arise in various applications such as viscoelasticity, heat equations, and many other physical phenomena (cf., e.g., [38]-[70] and references therein).

One of the emerging branches of this study is the theory of fractional evolution equations, say, evolution equations where the integer derivative with respect to time is replaced by a derivative of fractional order. The increasing interest in this class of equations is motivated both by their application to problems from fluid dynamic traffic model, viscoelasticity, heat conduction in materials with memory, electrodynamics with memory, and also because they can be employed to approach nonlinear conservation laws (see [61] and references therein). In addition, neutral stochastic differential equations with infinite delay have become important in recent years as mathematical models of phenomena in both science and engineering. For instance, in the theory development in Gurtin and Pipkin [26] and Nunziato [54] for the description of heat conduction in materials with fading memory. It should be pointed out that the deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic ones. We mention here the recent papers [20],[21] concerning the existence of mild solutions of fractional stochastic systems.

The aim of this paper is to establish the existence of mild solutions of fractional order neutral stochastic integro-differential equations with infinite delay of the form

$$\begin{cases} {}^cD_t^\alpha[x(t)+G(t,x_t)] = -Ax(t)+f(t,x_t)+\int_{-\infty}^t \sigma(t,s,x_s)dW(s) & t\in J=[0,b] \\ x(t)=\phi(t) & t\in(-\infty,0] \end{cases} \tag{4.1}$$

Where $0<\alpha<1$, ${}^cD^\alpha$ denotes the Caputo fractional derivative operator of order α. Here, $x(.)$ takes value in a real separable Hilbert space $\mathbb{H}$ with inner product $(\cdot,\cdot)_{\mathbb{H}}$ and the norm $\|\cdot\|_{\mathbb{H}}$. The operator $-A : \mathcal{D}(-A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup of a bounded linear operator $S(t), t \geq 0$, on $\mathbb{H}$. The history $x_t : (-\infty, 0] \to \mathcal{C}_h$, $x_t = \{x(t+\theta), \theta \in (-\infty, 0]\}$ belong to the phase space $\mathcal{C}_h$ which will be described axiomatically in Section 2. Let $\mathbb{K}$ be another separable Hilbert space with inner product $(.,.)_{\mathbb{K}}$ and the norm $\|\cdot\|_{\mathbb{K}}$. Suppose $\{W(t), t \geq 0\}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$ which is generated by the Wiener process W. We are also employing the same notation $\|\cdot\|$ for the norm of $\mathcal{L}(\mathbb{K}, \mathbb{H})$, where $\mathcal{L}(\mathbb{K}, \mathbb{H})$ denotes the space of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$ .The initial data $\phi = \{\phi(t), t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$- measurable, $\mathcal{C}_h$-valued random variable independent of W with finite second moments, and $G : J \times \mathcal{C}_h \to \mathbb{H}$, $f : J \times \mathbb{H} \to \mathbb{H}$, $\sigma : J \times J \times \mathbb{H} \to \mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$ are appropriate functions where $\mathcal{L}_2^0(\mathbb{K}, \mathbb{H}))$ denotes the space of all Q-Hilbert Schmidt operators from $\mathbb{K}$ into $\mathbb{H}$.

The article is organized as follows. In section 2, for convenience of readers, we briefly present some basic notations and preliminaries. The existence of a mild solution to (4.1) by Schaefer fixed point theorem is proved in Section 3. In the last section, An example is given to illustrate the result obtained.

4.2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual conditions (i.e right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). An $\mathbb{H}$ valued random variable is an $\mathcal{F}$ mesurable function $x(t) : \Omega \to \mathbb{H}$ and a collection of random variables $V = \{x(t, w) : \Omega \to \mathbb{H}, t \in J\}$ is called a stochastic process. Generally we just write $x(t)$ instead of $x(t, w)$ and $x(t) : J \to \mathbb{H}$ in the space of V. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of $\mathbb{K}$. Suppose that $\{W(t), t \geq 0\}$ is a cylindrical $\mathbb{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $Tr(Q) = \sum_{i=1}^{\infty} \lambda_i < \infty$, which satisfies that $Qe_i = \lambda_i e_i$ $i = 1, 2, ...,$ and a sequence of independent

Brownian motions $\{\beta_i\}_{i\geq 1}$ such that

$$(W(t),e)_{\mathbb{K}} = \sum_{i=1}^{\infty} \sqrt{\lambda_i}(e_i,e)_{\mathbb{K}}\beta_i(t) \quad e\in\mathbb{K} \quad t\geq 0$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K},\mathbb{H})$ be the space of all Hilbert Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ to $\mathbb{H}$ with the inner product $\langle\varphi,\phi\rangle_{\mathcal{L}_2^0} = tr[\varphi Q\phi^*]$.

Let $-A$ be the infinitesimal generator of ananalytic semigroup $\{S(t)\}_{t\geq 0}$ of uniformly bounded linear operators on $\mathbb{H}$. For the semigroup $S(t)$, there is an $M\geq 1$ such that $\|S(t)\|\leq M$. We suppose that $0\in\rho(-A)$, the resolvent set of $-A$. Then, for $\alpha\in(0,1]$, it is possible to define the fractional power operator A^α as a closed linear operator on its domain $\mathcal{D}(A^\alpha)$. Furthermore, the subspace $\mathcal{D}(A^\alpha)$ is dense in $\mathbb{H}$ and the expression

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x\in\mathcal{D}(A^\alpha)$$

defines a norm on $\mathbb{H}_\alpha = \mathcal{D}(A^\alpha)$. The following properties are well known.

Lemma 4.2.0.1. [57] *Suppose that the preceding conditions are satisfied.*

i *If $0<\beta<\alpha\leq 1$, then $\mathbb{H}_\alpha\subset\mathbb{H}_\beta$ and the embedding is compact whenever the resolvent operator of A is compact.*

ii *For every $\alpha\in(0,1]$, there exists a positive constant C_α such that*

$$\|A^\alpha S(t)\|\leq\frac{C_\alpha}{t^\alpha}, \quad t>0$$

Now, we present the abstract space phase $\mathcal{C}_h$. Assume that $h:(-\infty,0]\to(0,+\infty)$ with $l=\int_{-\infty}^0 h(t)dt<+\infty$ a continuous function.
Recall that the abstract phase space $\mathcal{C}_h$ is defined by

$$\mathcal{C}_h = \{\varphi:(-\infty,0]\to\mathbb{H},\ \text{for any } a>0, (\mathbb{E}\,|\varphi(\theta)|^2)^{1/2} \text{ is bounded and measurable}$$
$$\text{function on } [-a,0] \text{ and } \int_{-\infty}^0 h(s)\sup_{s\leq\theta\leq 0}(\mathbb{E}\,|\varphi(\theta)|^2)^{1/2}ds<+\infty\}.$$

If $\mathcal{C}_h$ is endowed with the norm

$$\|\varphi\|_{\mathcal{C}_h} = \int_{-\infty}^0 h(s)\sup_{s\leq\theta\leq 0}(\mathbb{E}\,|\varphi(\theta)|^2)^{\frac{1}{2}}ds, \quad \varphi\in\mathcal{C}_h$$

then $(\mathcal{C}_h,\|.\|_{\mathcal{C}_h})$ is a Banach space (see [37]).

Now, we consider the space,

$$\mathcal{C}_h' = \{x : (-\infty, b] \to \mathbb{H}, x_0 = \phi \in \mathcal{C}_h\}$$

Set $\|.\|_b$ be a seminorm defined by

$$\|x\|_b = \|x_0\|_{\mathcal{C}_h} + \sup_{s\in[0,b]} (E\,|x(s)|^2)^{\frac{1}{2}},\ \ x \in \mathcal{C}_h'$$

We have the following useful lemma appeared in [37].

Lemma 4.2.0.2. [14] *Assume that* $x \in \mathcal{C}_h'$*, then for all* $t \in J$*,* $x_t \in \mathcal{C}_h$*, Moreover,*

$$l(E\,|x(t)|^2)^{\frac{1}{2}} \le \|x_t\|_{\mathcal{C}_h} \le l \sup_{s\in[0,t]} (E\,|x(s)|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{C}_h}$$

where $l = \int_{-\infty}^{0} h(s)ds < \infty$

Let us now recall some basic definitions and results of fractional calculus.

Definition 4.2.1. *The fractional integral of order* α *with the lower limit 0 for a function* f *is defined as*

$$I^\alpha f = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds \quad t > 0 \quad \alpha > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$*, where* $\Gamma(.)$ *is the gamma function.*

Definition 4.2.2. *The Caputo derivative of order* α *with the lower limit 0 for a function* f *can be written as*

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t),\ t > 0,\ 0 \le n-1 < \alpha < n$$

If f is an abstract function with values in $\mathbb{H}$, then the integrals appearing in the above definitions are taken in Bochner's sense (see [47]).
At the end of this section, we recall the fixed point theorem of Schaefer which is used to establish the existence of the mild solution to the system (4.1).

Lemma 4.2.0.3. [29] *Let* $v(.), w(.) : [0, b] \to [0, \infty)$ *be continuous function. If* $w(.)$ *is nondecreasing and there exist two constants* $\theta \geq 0$ *and* $0 < \alpha < 1$ *such that*

$$v(t) \leq w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} ds, \ \ t \in J$$

then

$$v(t) \leq e^{\theta^n (\Gamma(\alpha))^n t^{n\alpha} / \Gamma(n\alpha)} \sum_{j=0}^{n-1} \left(\frac{\theta b^\alpha}{\alpha} \right)^j w(t),$$

for every $t \in [0, b]$ *and every* $n \in N$ *such that* $n\alpha > 1$ *and* $\Gamma(.)$ *is the Gamma function.*

Lemma 4.2.0.4. *Let* X *be a Banach space and* $\Phi : X \to X$ *be a completely continuous map. If the set*

$$U = \{x \in X : \lambda x = \Phi x \text{ for some } \lambda > 1\}$$

is bounded, then Φ *has a fixed point.*

4.3 Global Existence of a Mild Solution

Motivated by [21, 51], we give the following definition of mild solution of the system 4.1.

Definition 4.3.1. *An* $\mathbb{H}$*-valued stochastic process* $\{x(t), t \in (-\infty, b]\}$ *is said to be a mild solution of the system* 4.1 *if*

- *x(t) is* $\mathcal{F}_t$*-adapted and measurable,* $t \geq 0$.
- *x(t) is continuous on* $[0, b]$ *almost surely and for each* $s \in [0, t)$*, the function* $(t-s)^{\alpha-1} A T_\alpha(t-s) G(s, x_s) ds$ *is integrable such that the following stochastic integral equation is verified :*

$$\begin{aligned} x(t) = S_\alpha(t)[\phi(0) + G(0, \phi)] - G(t, x_t) - \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) G(s, x_s) ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s, \tau, x_\tau) dW(\tau) \right] ds \end{aligned}$$

— $x(t) = \phi(t)$ *on* $(-\infty, 0]$ *satisfying* $\|\phi\|^2_{\mathcal{C}_h} < \infty$.

where

$$S_\alpha(t)x = \int_0^\infty \zeta_\alpha(\theta)S(t^\alpha\theta)xd\theta, \quad T_\alpha(t)x = \alpha\int_0^\infty \theta\zeta_\alpha(\theta)S(t^\alpha\theta)xd\theta$$

and ζ_α *a probability density function defined on* $(0, \infty)$

The following properties of $S_\alpha(t)$ and $T_\alpha(t)$ appeared in [75] are useful.

Lemma 4.3.0.5. *The operators* $S_\alpha(t)$ *and* $T_\alpha(t)$ *have the following properties*

i) *For any fixed* $t \geq 0$, $S_\alpha(t)$ *and* $T_\alpha(t)$ *are linear and bounded operators such that for any* $x \in \mathbb{H}$

$$\|S_\alpha(t)x\|_{\mathbb{H}} \leq M\,\|x\|_{\mathbb{H}} \text{ and } \|T_\alpha(t)x\|_{\mathbb{H}} \leq \frac{M_\alpha}{\Gamma(1+\alpha)}\,\|x\|_{\mathbb{H}}$$

ii) $S_\alpha(t)$ *and* $T_\alpha(t)$ *are strongly continuous and compact.*

iii) *For any* $x \in \mathbb{H}$, $\beta \in (0,1)$ *and* $\eta \in (0,1]$ *we have*

$$AT_\alpha(t)x = A^{1-\beta}T_\alpha(t)A^\beta x \text{ and } \|A^\eta T_\alpha(t)\| \leq \frac{\alpha C_\eta \Gamma(2-\eta)}{t^{\alpha\eta}\Gamma(1+\alpha(1-\eta))}, \quad t \in [0, b]$$

In order to obtain our existence results,we need the following assumptions.

(H_0) : -A is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t)$ in $\mathbb{H}$, $0 \in \rho(-A)$, S(t) is compact for $t > 0$. and there exists a positive constant M such that $\|S(t)\| \leq M$.

(H_1) : The function $G : J \times \mathcal{C}_h \to \mathbb{H}$ is continuous and there exist some constants $L_G > 0$, $\beta \in (0,1)$, such that G is $\mathbb{H}_\beta$-valued and

$$E\left\|A^\beta G(t,x) - A^\beta G(t,y)\right\|^2_{\mathbb{H}} \leq L_G\,\|x-y\|^2_{\mathcal{C}_h}, x, y \in \mathcal{C}_h, \quad t \in J$$

$$E\left\|A^\beta G(t,x)\right\|^2_{\mathbb{H}} \leq L_G(1 + \|x\|^2_{\mathcal{C}_h})$$

(H_2) : For each $\varphi \in \mathcal{C}_h$,

$$K(t) = \lim_{a\to\infty}\int_{-a}^0 \sigma(t,s,\varphi)dW(s)$$

exists and is continuous. Further, there exists a positive constant M_k such that

$$E\,\|K(t)\|^2_{\mathbb{H}} \leq M_k$$

(H_3) $f : J \times \mathcal{C}_h \to \mathbb{H}$ satisfies the following :

i) $f(t,.) : \mathcal{C}_h \to \mathbb{H}$ is continuous for each $t \in J$ and for each $x \in \mathcal{C}_h$, $f(.,x) : J \to \mathbb{H}$ is strongly measurable ;

ii) there is a positive integrable function $P_f \in L^1([0,b])$ and a continuous nondecreasing function $\Omega_1 : [0,\infty) \to (0,\infty)$ such that for every $(t,x) \in J \times \mathcal{C}_h$, we have

$$E\,\|f(t,x)\|_{\mathbb{H}}^2 \leq P_f(t)\Omega_1(\|x\|_{\mathcal{C}_h}^2), \quad \lim_{r\to\infty}\inf \frac{\Omega_1(r)}{r}ds = \Lambda < \infty$$

(H_4) $\sigma : J \times J \times \mathcal{C}_h \to L(\mathbb{K},\mathbb{H})$ satisfies the following :

i) for each $(t,s) \in D = J \times J\times, \sigma(t,s,.) : \mathcal{C}_h \to L(\mathbb{K},\mathbb{H})$ is continuous and for each $x \in \mathcal{C}_h$, $\sigma(.,.,x) : D \to L(\mathbb{K},\mathbb{H})$ is strongly measurable ;

ii) there is a positive integrable function $P_\sigma \in L^1([0,b])$ and a continuous nondecreasing function $\Omega_2 : [0,\infty) \to (0,\infty)$ such that for every $(t,s,x) \in J \times J \times \mathcal{C}_h$, we have

$$\int_0^t E\,\|\sigma(t,s,x)\|_{\mathcal{L}_2^0}^2\,ds \leq P_\sigma(t)\Omega_2(\|x\|_{\mathcal{C}_h}^2), \quad \lim_{r\to\infty}\inf \frac{\Omega_2(r)}{r}ds = \vartheta < \infty$$

(H_5) :

$$Q_0 = 2l^2\{5\left\|A^{-B}\right\|^2 L_g\} \tag{4.2}$$

$$Q_1 = 2\,\|\phi\|_{\mathcal{C}_h}^2 + 2l^2\bar{H} \tag{4.3}$$

$$Q_2 = 10l^2\frac{\alpha^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}\frac{L_G b^{\alpha\beta}}{\alpha\beta} \tag{4.4}$$

$$Q_3 = 10bl^2\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^{2\alpha}}{\alpha^2} \tag{4.5}$$

$$Q_4 = 20bl^2\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^{2\alpha}}{\alpha^2}Tr(Q) \tag{4.6}$$

$$N_1 = \frac{Q_1}{1-Q_0}, \quad N_2 = \frac{Q_2}{1-Q_0}, \quad N_3 = \frac{Q_3}{1-Q_0}, \quad N_4 = \frac{Q_4}{1-Q_0} \tag{4.7}$$

$$\bar{H} = 10M^2(C_1+C_2) + 5\left\|A^{-B}\right\|^2 L_g \quad +5\frac{\alpha^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}\frac{L_G b^{2\alpha\beta}}{\alpha\beta^2} + 10b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^{2\alpha}}{\alpha^2}M_k \tag{4.8}$$

(H_6) :

$$\int_0^b \pi(s)ds \leq \int_{C_0N_1}^{\infty} \frac{ds}{\Omega_1(s)+\Omega_2(s)}$$

where

$$\pi(t) = \max\left\{C_0N_2t^{\alpha-1}P_f(t), C_0N_3t^{\alpha-1}P_\sigma(t)\right\}$$

The main object of this paper is to explain and prove the following theorem.

Theorem 4.1. *Assume that assumptions $(H_0)-(H_6)$ hold. Then there exists a mild solution to the system 4.1.*

We transform the problem 4.1 into a fixed point problem.
Consider the map $\mathcal{D} : \mathcal{C}_h' \to \mathcal{C}_h'$ defined by

$$(\mathcal{D}x)(t) = \begin{cases} \phi(t) & t \in (-\infty, 0] \\ S_\alpha(t)[\phi(0)+G(0,\phi)] - G(t,x_t) - \int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)ds \\ +\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x_s)ds \\ +\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,x_\tau)dW(\tau)\right]ds & t \in J \end{cases} \tag{4.9}$$

By virtue of lemma 4.3.0.5, it follows that

$$\begin{aligned} &E\left\|\int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)ds\right\|_{\mathbb{H}}^2 \\ &\leq E\left[\int_0^t \left\|(t-s)^{\alpha-1}A^{1-\beta}T_\alpha(t-s)A^\beta G(s,x_s)\right\|_{\mathbb{H}} ds\right]^2 \\ &\leq \frac{\alpha^2 C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} E\left[\int_0^t \left\|(t-s)^{\alpha\beta-1}A^\beta G(s,x_s)\right\|_{\mathbb{H}} ds\right]^2 \end{aligned}$$

applying the Hölder inequality and assumption (H_1), we further derive that

$$
\begin{aligned}
&E\left\|\int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)ds\right\|_{\mathbb{H}}^2\\
&\leq \frac{\alpha^2 C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)})\int_0^t (t-s)^{\alpha\beta-1}ds\int_0^t (t-s)^{\alpha\beta-1}E\left\|A^\beta G(s,x_s)\right\|_{\mathbb{H}}^2 ds\\
&\leq \frac{\alpha^2 C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}\frac{b^{\alpha\beta}}{\alpha\beta}\int_0^t (t-s)^{\alpha\beta-1}E\left\|A^\beta G(s,x_s)\right\|_{\mathbb{H}}^2 ds\\
&\leq \frac{\alpha^2 C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}\frac{L_G b^{\alpha\beta}}{\alpha\beta}\int_0^t (t-s)^{\alpha\beta-1}(1+\|x_s\|_{\mathcal{C}_h}^2)ds
\end{aligned}
$$

which deduces that $(t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)$ is integrable on J by Bochner's theorem and (see [47] and lemma 4.2.0.2).

We shall show that $\mathcal{D}$ has a fixed point, which is then a mild solution for the system 4.1. For $\phi \in \mathcal{C}_h$, define

$$
\tilde{\phi}(t) = \begin{cases} \phi(t) & t \in (-\infty, 0] \\ S_\alpha(t)\phi(0) & t \in J \end{cases} \tag{4.10}
$$

Then $\tilde{\phi} \in \mathcal{C}_h'$. Let $x(t) = \tilde{\phi}(t) + z(t)$, $-\infty < t \leq b$. It is easy to see that x satisfies 4.1 if and only if z satisfies $z_0 = 0$ and

$$
\begin{aligned}
z(t) &= S_\alpha(t)G(0,\phi) - G(t,\tilde{\phi}_t + z_t) - \int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,\tilde{\phi}_s + z_s)ds\\
&+ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,\tilde{\phi}_s + z_s)ds\\
&+ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau)dW(\tau)\right]ds
\end{aligned}
$$

Let

$$
\mathcal{C}_h'' = \{z \in \mathcal{C}_h', z_0 = 0 \in \mathcal{C}_h\}
$$

For any $z \in \mathcal{C}_h''$, we have

$$
\|z\|_b = \|z_0\|_{\mathcal{C}_h} + \sup_{s\in[0,b]}(E\|z(s)\|^2)^{\frac{1}{2}} = \sup_{s\in[0,b]}(E\|z(s)\|^2)^{\frac{1}{2}}
$$

Thus $(\mathcal{C}''_h, \|.\|_b)$ is a Banach space, set

$$B_q = \{y \in \mathcal{C}''_h, \|y\|_b^2 \leq q\}, \text{ for some } q \geq 0$$

then, $B_q \subset \mathcal{C}''_h$ is uniformly bounded.
Moreover, for $z \in B_q$, from Lemma 4.3.0.5, we have

$$\begin{aligned}
\left\|z_t + \tilde{\phi}_t\right\|_{\mathcal{C}_h}^2 &\leq 2(\|z_t\|_{\mathcal{C}_h}^2 + \left\|\tilde{\phi}_t\right\|_{\mathcal{C}_h}^2) \\
&\leq 4(l^2 \sup_{s\in[0,t]} E\,\|z(s)\|^2 + \|z_0\|_{\mathcal{C}_h}^2 + l^2 \sup_{s\in[0,t]} E\left\|\tilde{\phi}(s)\right\|^2 + \left\|\tilde{\phi}_0\right\|_{\mathcal{C}_h}^2) \\
&\leq 4l^2(q + M^2 E\,\|\phi(0)\|_{\mathbb{H}}^2) + 4\,\|\phi\|_{\mathcal{C}_h}^2 \\
&= \grave{q}
\end{aligned}$$

Define the operator $\Pi : C''_h \to C''_h$ by

$$(\Pi z)(t) = \begin{cases} 0 & t \in (-\infty, 0] \\ S_\alpha(t)G(0,\phi) - G(t, \tilde{\phi}_t + z_t) - \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) G(s, \tilde{\phi}_s + z_s) ds & \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, \tilde{\phi}_s + z_s) ds & \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau) dW(\tau)\right] ds & t \in J \end{cases}$$

Observe that Π is well defined on B_q for each $q > 0$. Obviously, the operator $\mathcal{D}$ having a fixed point if and only if Π has a fixed point.

We shall prove that the operator Π is a completely continuous operator

Step 1 :

We first show that Π maps B_q into an equicontinuous family. Let $z \in B_q$ and

$t_1, t_2 \in J$ and $\epsilon > 0$. Then if $0 < \epsilon < t_1 < t_2 < b$.

$$
\begin{aligned}
&E\left\|(\Pi z)(t_1) - (\Pi z)(t_2)\right\|_{\mathbb{H}}^2 \\
&\leq 5\left\|S_\alpha(t_1) - S_\alpha(t_2)\right\|_{\mathbb{H}}^2 E\left\|G(0,\phi)\right\|_{\mathbb{H}}^2 \\
&+ 5\left\|A^{-B}\right\|^2 L_G \left\|(z_{t_1} - z_{t_2})\right\|_{\mathcal{C}_h}^2 \\
&+ 15\int_0^{t_1-\epsilon} \left\|\left[(t_1-s)^{\alpha-1}A^{1-B}T_\alpha(t_1-s) - (t_2-s)^{\alpha-1}A^{1-B}T_\alpha(t_2-s)\right]\right\|_{\mathbb{H}}^2 L_G(1+\left\|\tilde{\phi}_s + z_s\right\|_{\mathcal{C}_h}^2)ds \\
&+ 15\int_{t_1-\epsilon}^{t_1} \left\|\left[(t_1-s)^{\alpha-1}A^{1-B}T_\alpha(t_1-s) - (t_2-s)^{\alpha-1}A^{1-B}T_\alpha(t_2-s)\right]\right\|_{\mathbb{H}}^2 L_G(1+\left\|\tilde{\phi}_s + z_s\right\|_{\mathcal{C}_h}^2)ds \\
&+ 15\int_{t_1}^{t_2} \left\|\left[(t_2-s)^{\alpha-1}A^{1-B}T_\alpha(t_2-s)\right]\right\|_{\mathbb{H}}^2 L_G(1+\left\|\tilde{\phi}_s + z_s\right\|_{\mathcal{C}_h}^2)ds \\
&+ 15b\int_0^{t_1-\epsilon} \left\|\left[(t_1-s)^{\alpha-1}T_\alpha(t_1-s) - (t_2-s)^{\alpha-1}T_\alpha(t_2-s)\right]\right\|_{\mathbb{H}}^2 P_f(s)\Omega_1(\left\|\tilde{\phi}_s + z_s^q\right\|_{\mathcal{C}_h}^2)ds \\
&+ 15b\int_{t_1-\epsilon}^{t_1} \left\|\left[(t_1-s)^{\alpha-1}T_\alpha(t_1-s) - (t_2-s)^{\alpha-1}A^{1-B}T_\alpha(t_2-s)\right]\right\|_{\mathbb{H}}^2 P_f(s)\Omega_1(\left\|\tilde{\phi}_s + z_s^q\right\|_{\mathcal{C}_h}^2)ds \\
&+ 15b\int_{t_1}^{t_2} \left\|\left[(t_2-s)^{\alpha-1}T_\alpha(t_2-s)\right]\right\|_{\mathbb{H}}^2 P_f(s)\Omega_1(\left\|\tilde{\phi}_s + z_s^q\right\|_{\mathcal{C}_h}^2)ds \\
&+ 15b\int_0^{t_1-\epsilon} \left\|\left[(t_1-s)^{\alpha-1}T_\alpha(t_1-s) - (t_2-s)^{\alpha-1}T_\alpha(t_2-s)\right]\right\|_{\mathbb{H}}^2 (2M_k + 2Tr(Q)P_\sigma(s) \\
&\Omega_2(\left\|\tilde{\phi}_s + z_s^q\right\|_{\mathcal{C}_h}^2)ds \\
&+ 15b\int_{t_1-\epsilon}^{t_1} \left\|\left[(t_1-s)^{\alpha-1}AT_\alpha(t_1-s) - (t_2-s)^{\alpha-1}A^{1-B}T_\alpha(t_2-s)\right]\right\|_{\mathbb{H}}^2 \\
&(2M_k + 2Tr(Q)P_\sigma(s)\Omega_2(\left\|\tilde{\phi}_s + z_s^q\right\|_{\mathcal{C}_h}^2)ds \\
&+ 15b\int_{t_1}^{t_2} \left\|\left[(t_2-s)^{\alpha-1}AT_\alpha(t_2-s)\right]\right\|_{\mathbb{H}}^2 (2M_k + 2Tr(Q)P_\sigma(s)\Omega_2(\left\|\tilde{\phi}_s + z_s^q\right\|_{\mathcal{C}_h}^2)ds
\end{aligned}
$$

The right hand side is independent of $z \in B_q$ and tends to zero as $t_2 - t_1 \to 0$ and ϵ sufficiently small, since the compacteness of $S_\alpha(t)$ and $T_\alpha(t)$ for $t > 0$ implies the continuity in the uniform operator topology.

Thus Π maps B_q into an equicontinuous family of functions. It is easy to see that the family B_q is uniformly bounded.

Step 2 :

Next, we show that $\overline{\Pi B_q}$ is compact. Since we have shown that ΠB_q is an equicontinuous collection, it suffices by $Arzela-Ascoli$ theorem to show that Π maps B_q into a precompact set in $\mathbb{H}$.

Let $0 \leq t \leq b$ be fixed and ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_q$, we define

$$\begin{aligned}
(\Pi z_\epsilon)(t) &= S_\alpha(t)G(0,\phi) - G(t,\tilde{\phi}_t + z_t) - \int_0^{t-\epsilon} (t-s)^{\alpha-1} AT_\alpha(t-s)G(s,\tilde{\phi}_s + z_s)ds \\
&+ \int_0^{t-\epsilon} (t-s)^{\alpha-1} T_\alpha(t-s) f(s,\tilde{\phi}_s + z_s)ds \\
&+ \int_0^{t-\epsilon} (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^{s} \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau)dW(\tau)\right] ds \\
&= S_\alpha(t)G(0,\phi) - G(t,\tilde{\phi}_t + z_t) - T_\alpha(\epsilon)\int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} AT_\alpha(t-s-\epsilon)G(s,\tilde{\phi}_s + z_s)ds \\
&+ T_\alpha(\epsilon)\int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} T_\alpha(t-s-\epsilon) f(s,\tilde{\phi}_s + z_s)ds \\
&+ T_\alpha(\epsilon)\int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} T_\alpha(t-s-\epsilon) \left[\int_{-\infty}^{s} \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau)dW(\tau)\right] ds
\end{aligned}$$

Since $S_\alpha(t)$ and $T_\alpha(t)$ are compact, the set $V_\epsilon(t) = \{(\Pi_\epsilon z)(t) : z \in B_q\}$ is precompact in $\mathbb{H}$, for every $\epsilon\,;\, 0 < \epsilon < t$. Moreover, for every $z \in B_q$ we have

$$\begin{aligned}
E\left\|(\Pi z)(t) - (\Pi_\epsilon z)(t))\right\|^2 &\leq 3\frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \frac{L_G b^{\alpha\beta}}{\alpha\beta} \int_{t-\epsilon}^{t} (t-s)^{\alpha\beta-1}(1+\grave{q})ds \\
&+ 3b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha} \int_{t-\epsilon}^{t} (t-s)^{\alpha-1} P_f(s)\Omega_1(\grave{q})ds \\
&+ 3b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha} \int_{t-\epsilon}^{t} (t-s)^{\alpha-1} (2M_k + 2Tr(Q)P_\sigma(s)\Omega_2(\grave{q}))ds
\end{aligned}$$

Therefore,

$$E\left\|(\Pi z)(t) - (\Pi_\epsilon z)(t))\right\|^2 \to 0 \qquad \text{as } \epsilon \to 0$$

and there are precompact sets arbitrarily close to the set $\{(\Pi z)(t) : z \in B_q\}$ Thus, the set $\{(\Pi_\epsilon z)(t) : z \in B_q\}$ is precompact in $\mathbb{H}$.

Step 3 :

It remains to show that $\Pi : \mathcal{C}_h^{''} \to \mathcal{C}_h^{''}$ is continuous. Let $\{z^n\}_{n=0}^{\infty}$ be a sequence in $\mathcal{C}_h^{''}$ such that $z^n \to z$ in $\mathcal{C}_h^{''}$. Then, there is a number $q \geq 0$ such that $|z^{(n)}(t)| \leq q$ for all n and a.e. $t \in J$, so $z^{(n)} \in B_q$ and $z \in B_q$.

$$A^{\beta}G(t, z_t^{(n)} + \tilde{\phi}_t) \to A^{\beta}G(t, z_t + \tilde{\phi}_t)$$

$$f(t, z_t^{(n)} + \tilde{\phi}_t) \to f(t, z_t + \tilde{\phi}_t)$$

$$\sigma(s, \tau, z_\tau^{(n)} + \tilde{\phi}_\tau) \to \int_0^t \sigma(s, \tau, z_\tau + \tilde{\phi}_\tau)$$

for $t \in J$, and since

$$E\left\|[A^B G(t, z_s^{(n)}) - A^B G(t, z_t)]\right\|^2 \leq 2\alpha_{q'}(t)$$

$$E\left\|[f(t, z_s^{(n)}) - f(t, z_t)]\right\|^2 \leq 2P_f(t)\Omega_1(q^{'})$$

$$E\left\|[\sigma(s, \tau, z_\tau^{(n)}) - \sigma(s, \tau, z_\tau^{(n)})]\right\|^2 \leq 2P_\sigma(t)\Omega_2(q^{'})$$

By the dominated convergence theorem that

$$\begin{aligned}
E\left\|\Pi z_t^{(n)} - \Pi z_t\right\|^2 \leq & 4\sup_{t\in J} E\left\|[G(t, z_t) - G(t, z_t^{(n)})]\right\|^2 \\
& + 4\sup_{t\in J} E\left\|\int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s)[G(t, z_s) - G(t, z_s^{(n)})]ds\right\|^2 \\
& + 4b\sup_{t\in J} E\left\|\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[f(t, z_s^{(n)})) - f(t, z_s)]ds\right\|^2 \\
& + 4b\sup_{t\in J} E\left\|\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\left[\int_{-\infty}^s [\sigma(s, \tau, z_\tau^{(n)}) - \sigma(s, \tau, z_\tau) dw(\tau)]\right] ds\right\|^2 \\
& \leq 4\left\|A^{-B}\right\|^2 E\left\|[A^B G(t, z_s^{(n)}) - A^B G(t, z_t)]\right\|^2 \\
& + 4\frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \frac{b^{2\alpha\beta}}{(\alpha\beta)^2} \int_0^t E\left\|[A^B G(s, z_s^{(n)}) - A^B G(s, z_s)]\right\|^2 ds \\
& + 4b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2} \int_0^t E\left\|[f(t, z_s^{(n)}) - f(t, z_s)]\right\|^2 ds \\
& + 4b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2} \int_0^t E\left\|\left[\int_{-\infty}^s [\sigma(s, \tau, z_\tau^{(n)}) - \sigma(s, \tau, z_\tau)]\right] dw(\tau)\right\|^2 ds \\
& \to 0 \text{ as } n \to \infty
\end{aligned}$$

Thus Π is continuous. This completes the proof that Π is completely continuous.

Now, we shall prove that the set

$$U = \{x \in \mathcal{C}_h' : \lambda x = \Pi x \text{ for some } \lambda > 1\}$$

is bounded.
Let $x \in U$. Then $\lambda x = \Pi x$ for some $\lambda > 1$. Then

$$\begin{aligned} x(t) &= \lambda^{-1} \left(S_\alpha(t)[\phi(0) + G(0,\phi)]\right) - \lambda^{-1} G(t, x_t) \\ &- \lambda^{-1} \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) G(s, x_s) ds \\ &+ \lambda^{-1} \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds \\ &+ \lambda^{-1} \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s, \tau, x_\tau) dW(\tau)\right] ds \end{aligned}$$

$$\begin{aligned} E\|x(t)\|^2 &\le 5E\|S_\alpha(t)(\phi(0) + G(0,\phi))\|_{\mathbb{H}}^2 + 5\|G(t,x_t)\|_{\mathbb{H}}^2 \\ &+ 5E\left\|\int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) G(s,x_s) ds\right\|_{\mathbb{H}}^2 \\ &+ 5E\left\|\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s,x_s) ds\right\|_{\mathbb{H}}^2 \\ &+ 5E\left\|\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s,\tau,x_\tau) dW(\tau)\right] ds\right\|_{\mathbb{H}}^2 \\ &\le 10M^2(C_1 + C_2) + 5\left\|A^{-B}\right\|^2 L_g(\|x\|_{\mathcal{C}_h}^2 + 1) \\ &+ 5\frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \frac{L_G b^{\alpha\beta}}{\alpha\beta} \int_0^t (t-s)^{\alpha\beta-1}(1 + \|x_s\|_{\mathcal{C}_h}^2) ds \\ &+ 5b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha} \int_0^t (t-s)^{\alpha-1} P_f(s) \Omega_1(\|x_s\|_{\mathcal{C}_h}^2) ds \\ &+ 5b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha} \int_0^t (t-s)^{\alpha-1}(2M_k + 2Tr(Q) P_\sigma(s) \Omega_2(\|x_s\|_{\mathcal{C}_h}^2) ds \end{aligned}$$

Now, we consider the function μ defined by

$$\mu(t) = \sup\{E\|x(s)\|^2, 0 \le s \le t\}, 0 \le t \le b$$

From, Lemma [4.2.0.2] and the above inequality, we have

$$E\|x(t)\|^2 = 2\|\phi\|^2_{\mathcal{C}_h} + 2l^2 \sup_{0\le s\le t}(E\|x(s)\|^2)$$

Therefore, we get

$$\mu(t) \le 2\|\phi\|^2_{\mathcal{C}_h} + 2l^2\{\bar{H} + 5\left\|A^{-B}\right\|^2 L_g\mu(t) + 5\frac{\alpha^2 C^2_{1-\beta}\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}\frac{L_G b^{\alpha\beta}}{\alpha\beta}\int_0^t (t-s)^{\alpha\beta-1}\mu(s)ds$$
$$+ 5b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}P_f(s)\Omega_1(\mu(s))ds$$
$$+ 10b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}Tr(Q)P_\sigma(s)\Omega_2(\mu(s))ds\}$$

where $\bar{H}$ is given in (4.8). Thus, we have

$$\mu(t) \le N_1 + N_2\int_0^t \frac{\mu(s)}{t-s)^{1-\alpha\beta}}ds + N_3\int_0^t P_f(s)\Omega_1(\mu(s))ds$$
$$+N_4\int_0^t P_\sigma(s)\Omega_2(\mu(s))ds$$

where N_1, N_2, N_3, N_4 are given in (4.7). By Lemma [4.2.0.3], we have

$$\mu(t) \le C_0(N_1 + N_3\int_0^t P_f(s)\Omega_1(\mu(s))ds + N_4\int_0^t P_\sigma(s)\Omega_2(\mu(s))ds)$$

Where

$$C_0 = e^{N_2^n(\Gamma(\alpha\beta))^n b^{n\alpha\beta}/\Gamma(n\alpha\beta)}\sum_{j=0}^{n-1}\left(\frac{N_2 b^{\alpha\beta}}{\alpha\beta}\right)^j$$

Denoting by $\nu(t)$ the right hand side of the last inequality, we have $\nu(0) = C_0N_1$

$$\grave{\nu}(t) \le C_0\left(N_3P_f(t)\Omega_1(\mu(t)) + N_4P_\sigma(t)\Omega_2(\mu(t))\right)$$

$$\grave{\nu}(t) \le C_0\left(N_3P_f(t)\Omega_1(\nu(t)) + N_4P_\sigma(t)\Omega_2(\nu(t))\right)$$

Or equivalently by (H_6), we have

$$\int_{\nu(0)}^{\nu(t)}\frac{ds}{\Omega_1(s)+\Omega_2(s)} \le \int_0^b \pi(s)ds < \int_{C_0N_1}^{\infty}\frac{ds}{\Omega_1(s)+\Omega_2(s)}$$

This inequality implies that there is a constant K such that $\nu(t) \leq K$, $t \in J$ and hence $\mu(t) \leq K$, $t \in J$. Furthermore, we get $\|x_t\|^2_{\mathcal{C}_h} \leq \mu(t) \leq \nu(t) \leq K$, $t \in J$.

As a consequence of lemma [4.2.0.4] we deduce that Π has a fixed point, which is a mild solution of (4.1)

4.4 Example

Consider the following fractional neutral stochastic partial differential equation with infinite delays of the form :

$$\begin{cases} {}^cD_t^\alpha[u(t,x) - G(t,u(t-h,x))] = \frac{\partial^2}{\partial x^2}u(t,x) + f(t,u(t-h,x)) & \\ + \int_{-\infty}^t \sigma(s,u(s-h,x))dW(s) & 0 \leq x \leq \pi, h > 0, t \in J = [0,b] \\ u(t,0) = u(t,\pi) = 0 & t \in [0,b] \\ u(t,x) = \phi(t,x) & t \in (-\infty,0], \end{cases} \tag{4.11}$$

Where $\alpha \in (0,1)$, and $W(t)$ is a standard cylindrical Wiener process defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

To rewrite this system into the abstract form (4.1), let $\mathbb{H} = L^2([0,\pi])$ with the norm $\|.\|$. Define $A : \mathbb{H} \to \mathbb{H}$ by $A(t)Z = z''$ with the domain

$$\mathcal{D}(A) = \Big\{x(.) \in \mathbb{H} : x, x' \text{ are absolutely continuous }, x'' \in \mathbb{H}, x(0) = x(\pi) = 0\Big\}$$

then A generates a symmetric C_0-semigroup e^{-tA} in $\mathbb{H}$ and there exists a complete orthonormal
set $\{z_n, n = 1, 2, ...\}$ of eigenvectors of A with

$$z_n(s) = \sqrt{\frac{2}{\pi}} sin(ns), \quad n = 1, 2, ...$$

Then the operator $A^{-\frac{1}{2}}$ is given by

$$A^{-\frac{1}{2}}\zeta = \sum_{n=1}^{\infty} n \langle \zeta, z_n \rangle z_n$$

on the space $\mathcal{D}(A^{-\frac{1}{2}}) = \{\zeta(.) \in \mathbb{H} : \sum_{n=1}^{\infty} n \langle \zeta, z_n \rangle z_n \in \mathbb{H}\}$.

Now, we give a special $\mathcal{C}_h$ space. Let $h(s) = e^{2s}, s < 0$, then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2}$.

Let

$$\|\varphi\|_{\mathcal{C}_h} = \int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} E\left(\|\varphi(\theta)\|^2\right)^{\frac{1}{2}} ds$$

Then $(\mathcal{C}_h, \|.\|_{\mathcal{C}_h})$ is a Banach space.

Hence, for $(t, \varphi) \in [0, b] \times \mathcal{C}_h$, where $\varphi(\theta)(\zeta) = \phi(\theta, \zeta), (\theta, \zeta) \in (-\infty, 0] \times [0, \pi]$.

Set $u(t)(\zeta) = u(t, \zeta)$, and define the functions $G, f : J \times \mathcal{C}_h \to \mathbb{H}, \sigma : J \times \mathcal{C}_h \to \mathcal{L}_2^0(\mathbb{H}, \mathbb{H})$ for the infinite delay as follows :

$$(-A)^{\frac{1}{2}} G(t, \varphi)(x) = \int_{-\infty}^{0} \mu_1(\theta)\varphi(\theta)(x)d\theta$$

$$f(t, \varphi)(x) = \int_{-\infty}^{0} \mu_2(t, x, \theta)G_1(\varphi(\theta)(x))d\theta$$

$$\sigma(t, \varphi)(x) = \int_{-\infty}^{0} \mu_3(t, x, \theta)G_2(\varphi(\theta)(x))d\theta$$

hence, we can impose some hypotheses on $\mu_i, i = 1, 2, 3$, and G_k, k = 1, 2 (see [5]), to satisfy the assumptions stated in theorem [4.1] ; we omit it here. Thus, there exists a mild solution for the system (4.11).

Chapitre 5

Existence of mild solution result for fractional neutral stochastic integro-differential equations with nonlocal conditions and infinite delay

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Existence of mild solution result for fractional neutral stochastic integro-differential equations with nonlocal conditions and infinite delay

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Abstract

We investigate in this paper the existence of mild solutions for the fractional differential equations of neutral type with nonlocal conditions and infinite delay in Hilbert spaces by employing fractional calculus and Krasnoselski-Schaefer fixed point theorem. Finally an example is provided to illustrate the application of the obtained results.

Keywords : Infinite delay, Stochastic fractional differential equations, mild solution, fixed point theorem.

5.1 Introduction

The main purpose of this paper is to prove the Existence of the mild solution for fractional differential equations of neutral type with infinite delay in Hilbert spaces of the form.

$$\begin{cases} {}^cD_t^\alpha[x(t)-h(t,x_t)] = A[x(t)-h(t,x_t)] + f(t,x_t) + \int_{-\infty}^t \sigma(t,s,x_s)dW(s) & t \in J=[0,b] \\ x(0)+\mu(x) = x_0 = \phi(t) & t \in (-\infty,0], \end{cases} \tag{5.1}$$

Here, $x(.)$ takes value in a real separable Hilbert space $\mathbb{H}$ with inner product $(\cdot,\cdot)_{\mathbb{H}}$ and the norm $\|\cdot\|_{\mathbb{H}}$. The fractional derivative ${}^cD^{\alpha}, \alpha \in (0,1)$, is understood in the Caputo sense. The operator A generates a strongly continuous semigroup of bounded linear operators $S(t), t \geq 0$, on $\mathbb{H}$. Let $\mathbb{K}$ be another separable Hilbert space with inner product $(.,.)_{\mathbb{K}}$ and the norm $\|\cdot\|_{\mathbb{K}}$. W is a given $\mathbb{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The histories $x_t : \Omega \to \mathcal{C}_v$ defined by $x_t = \{x(t+\theta), \theta \in (-\infty, 0]\}$ belong to the phase space $\mathcal{C}_v$, which will be defined in section 2. The initial data $\phi = \{\phi(t), t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$- measurable, $\mathcal{C}_v$-valued random variable independent of W with finite second moments, and $h : J \times \mathcal{C}_{\vartheta} \to \mathbb{H}$, $h : J \times \mathcal{C}_v \to \mathbb{H}$, $\sigma : J \times J_1 \times \mathcal{C}_v \to \mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$ are appropriate functions, where $J_1 = (-\infty, b]$ and $\mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$ denotes the space of all Q-Hilbert Schmidt operators from $\mathbb{K}$. into $\mathbb{H}$. $\mu : C(J, \mathbb{H}) \to \mathbb{H}$ is bounded and the initial data x_0 is an $\mathcal{F}$ adapted $\mathbb{H}$-valued random variable independent of Wiener process W.

The fractional differential equations arise in many engineering and scientific disciplines as the mathematica modeling of systems and processes in the fields of physics,
chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology,etc.,
involves derivatives of fractional order. It is worthwhile mentioning that several important problems of the theory of ordinary and delay differential equations lead to investigations of functional differential equations of various types (see the books by Hale and Verduyn Lunel [28] , Wu [69] , Liang et al [39], Liang and Xiao [40], and the references therein).

In particular the nonlocal condition problems for some fractional differential equations have been attractive to many researchers Mophou et al [50] studied existence of mild solution for some fractional differential equations with nonlocal condition. Chang et al [15] investigate the fractional order integro-differential equations with nonlocal conditions in the Riemann-Liouville fractional derivative sense.

In this paper, we prove the existence theorem of mild solution for neutral differential equation with nonlocal conditions and infinite delay by using the Krasnoselski-Schaefer fixed point theorem. An example is provided to illustrate the application of the obtained results.

5.2 Preliminaries

Next we mention a few results and notations needed to establish our results. Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}})$ be two real separable Hilbert spaces. We denote by $\mathcal{L}(\mathbb{K}, \mathbb{H})$ the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|.\|$. In this article, we use the symbol $\|.\|$ to denote norms of operators regardless of the spaces involved when no confusion possibly arises.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. $W = (W_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with the covariance operator Q such that $trQ < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k, k = 1, 2, ...,$ and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$(W(t), e)_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}(e_k, e)_{\mathbb{K}}\beta_k(t) \quad e \in \mathbb{K} \quad t \geq 0$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K}, \mathbb{H})$ be the space of all Hilbert Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ to $\mathbb{H}$ with the inner product $\langle \varphi, \phi \rangle_{\mathcal{L}_2^0} = tr[\varphi Q \phi^*]$.

The semigroup $S(\cdot)$ is uniformly bounded. That is to say, $\|S(t)\| \leq M$ for some constant $M \geq 1$ and every $t \geq 0$.

Assume that $\upsilon : (-\infty, 0] \to (0, +\infty)$ with $l = \int_{-\infty}^{0} \upsilon(t)dt < +\infty$ a continuous function.

Recall that the abstract phase space $\mathcal{C}_\upsilon$ is defined by

$$\mathcal{C}_\upsilon = \{\varphi : (-\infty, 0] \to \mathbb{H}, \text{ for any } a > 0, (\mathbb{E}\,|\varphi(\theta)|^2)^{1/2} \text{ is bounded and measurable}$$
$$\text{function on } [-a, 0] \text{ and } \int_{-\infty}^{0} \upsilon(s) \sup_{s\leq\theta\leq 0} (\mathbb{E}\,|\varphi(\theta)|^2)^{1/2} ds < +\infty\}.$$

If $\mathcal{C}_v$ is endowed with the norm

$$\|\varphi\|_{\mathcal{C}_v} = \int_{-\infty}^{0} v(s) \sup_{s\le\theta\le 0} (\mathbb{E}\,|\varphi(\theta)|^2)^{\frac{1}{2}} ds, \qquad \varphi \in \mathcal{C}_v$$

then $(\mathcal{C}_v, \|.\|_{\mathcal{C}_v})$ is a Banach space (see [37]).

Let us now recall some basic definitions and results of fractional calculus.

Definition 5.2.1. [47] *The fractional integral of order α with the lower limit 0 for a function f is defined as*

$$I^\alpha f = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds \quad t > 0 \quad \alpha > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(.)$ is the gamma function.

Definition 5.2.2. *The Caputo derivative of order α with the lower limit 0 for a function f can be written as*

$$^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t),\ t > 0,\ 0 \le n-1 < \alpha < n$$

The Caputo derivative of a constant equal to zero. If f is an abstract function with values in $\mathbb{H}$, then the integrals appearing in the above definitions are taken in Bochner's sense (see [47]).

Lemma 5.2.0.6. [10] *Let H be a Hilbert space and Φ_1, Φ_2 two operators on H such that*

- **i)** *Φ_1 is a contraction and*
- **ii)** *Φ_2 is completely continuous.*

Then either

- **a)** *the operator equation $\Phi_1 x + \Phi_2 x = x$ has a solution or*
- **b)** *$G = \{x \in \mathbb{H} : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x\}$ is unbounded for $\lambda \in (0,1)$.*

Lemma 5.2.0.7. [29] *Let $v(.), w(.) : [0,b] \to [0,\infty)$ be continuous function. If $w(.)$ is nondecreasing and there exist two constants $\theta \ge 0$ and $0 < \alpha < 1$ such that*

$$v(t) \le w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} ds, \ \ t \in J$$

then

$$v(t) \leq e^{\theta^n (\Gamma(\alpha))^n t^{n\alpha}/\Gamma(n\alpha)} \sum_{j=0}^{n-1} \left(\frac{\theta b^\alpha}{\alpha}\right)^j w(t),$$

for every $t \in [0, b]$ *and every* $n \in N$ *such that* $n\alpha > 1$.

5.3 Existence results

Definition 5.3.1. *An* $\mathbb{H}$*- valued stochastic process* $\{x(t), t \in (-\infty, b]\}$ *is a mild solution of the system 5.1 if* $x(0) + \mu(x) = x_0 = \phi(t)$ *on* $(-\infty, 0]$ *satisfying* $\|\phi\|^2_{\mathcal{C}_v} < +\infty$*, the process* x *satisfies the following integral equation*

$$x(t) = S_\alpha(t)[\phi(0) - \mu(x) - h(0, \phi)] + h(t, x_t) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds$$
$$+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s, \tau, x_\tau) dW(\tau)\right] ds$$

where

$$S_\alpha(t)x = \int_0^\infty \zeta_\alpha(\theta) S(t^\alpha \theta) x d\theta, \quad T_\alpha(t)x = \alpha \int_0^\infty \theta \zeta_\alpha(\theta) S(t^\alpha \theta) x d\theta$$

and ζ_α *is a probability density function defined on* $(0, \infty)$

The following properties of $S_\alpha(t)$ and $T_\alpha(t)$ appeared in [75] are useful.

Lemma 5.3.0.8. *The operators* $S_\alpha(t)$ *and* $T_\alpha(t)$ *have the following properties*

i) *For any fixed* $t \geq 0$*,* $S_\alpha(t)$ *and* $T_\alpha(t)$ *are linear and bounded operators such that for any* $x \in \mathbb{H}$

$$\|S_\alpha(t)x\|_{\mathbb{H}} \leq M \|x\|_{\mathbb{H}} \text{ and } \|T_\alpha(t)x\|_{\mathbb{H}} \leq \frac{M_\alpha}{\Gamma(1+\alpha)} \|x\|_{\mathbb{H}}$$

ii) $S_\alpha(t)$ *and* $T_\alpha(t)$ *are strongly continuous and compact.*

To study existence of mild solutions of 5.1, we introduce the following hypotheses.

(H_1) : The function $h, f : J \times \mathcal{C}_v \to \mathbb{H}$ are continuous and there exist some constants M_h, M_f, such that

$$E \|h(t, x) - h(t, y)\|^2_{\mathbb{H}} \leq M_h \|x - y\|^2_{\mathcal{C}_v}, x, y \in \mathcal{C}_v, \ t \in J$$

$$E\left\|h(t,x)\right\|_{\mathbb{H}}^2 \leq M_h(1+\|x\|_{\mathcal{C}_v}^2)$$

$$E\left\|f(t,x)-f(t,y)\right\|_{\mathbb{H}}^2 \leq M_f\left\|x-y\right\|_{\mathcal{C}_v}^2, x,y\in\mathcal{C}_v,\ \ t\in J$$

$$E\left\|f(t,x)\right\|_{\mathbb{H}}^2 \leq M_f(1+\|x\|_{\mathcal{C}_v}^2)$$

(H_2) : μ is continuous and there exists some positive constants M_μ such that

$$E\left\|\mu(x)-\mu(y)\right\|_{\mathbb{H}}^2 \leq M_\mu\left\|x-y\right\|_{\mathcal{C}_v}^2, x,y\in\mathcal{C}_v,\ \ t\in J$$

$$E\left\|\mu(x)\right\|_{\mathbb{H}}^2 \leq M_\mu(1+\|x\|_{\mathcal{C}_v}^2)$$

(H_3) : For each $\varphi\in\mathcal{C}_v$,

$$k(t)=\lim_{a\to\infty}\int_{-a}^{0}\sigma(t,s,\varphi)dW(s)$$

exists and is continuous. Further, there exists a positive constant M_k such that

$$E\left\|k(t)\right\|_{\mathbb{H}}^2 \leq M_k$$

(H_4) The function $\sigma : J\times J_1\times\mathcal{C}_v\to L(\mathbb{K},\mathbb{H})$ satisfies the following :

i) for each $(t,s)\in J\times J\times, \sigma(t,s,.) : \mathcal{C}_v\to L(\mathbb{K},\mathbb{H})$ is continuous and for each $x\in\mathcal{C}_v$, $\sigma(.,.,x) : J\times J\to L(\mathbb{K},\mathbb{H})$ is strongly measurable ;

ii) there is a positive integrable function $m\in L^1([0,b])$ and a continuous nondecreasing function M_σ : $[0,\infty)\to(0,\infty)$ such that for every $(t,s,x)\in J\times J\times\mathcal{C}_v$, we have

$$\int_0^t E\left\|\sigma(t,s,x)\right\|_{\mathcal{L}_2^0}^2 ds \leq m(t)M_\sigma(\|x\|_{\mathcal{C}_v}^2),\quad \lim_{r\to\infty}\inf\frac{M_\sigma(r)}{r}ds=\Delta<\infty$$

iii) For any $x,y\in\mathcal{C}_v, t\geq 0$, there exists a positive constant L_σ such that

$$\int_0^t E\left\|\sigma(t,s,x)-\sigma(t,s,y)\right\|_{\mathcal{L}_2^0}^2 ds \leq L_\sigma\left\|x-y\right\|_{\mathcal{C}_v}^2$$

(H_5) :

$$N_0 = 2l^2\{12M^2M_\mu+4M_h\} \tag{5.2}$$

$$N_1 = 2\left\|\phi\right\|_{\mathcal{C}_v}^2+2l^2\bar{F} \tag{5.3}$$

$$N_2 = 8l^2\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^\alpha}{\alpha}M_f \tag{5.4}$$

$$N_3 = 16bl^2 \left(\frac{M_\alpha}{\Gamma(1+\alpha)} \right)^2 \frac{b^\alpha}{\alpha} Tr(Q) \tag{5.5}$$

$$K_1 = \frac{N_1}{1-N_0}, \quad K_2 = \frac{N_2}{1-N_0}, \quad K_3 = \frac{N_3}{1-N_0} \tag{5.6}$$

$$\bar{F} = 12M^2(C_1+C_2) + 12M^2 M_\mu + 4M_h + 4\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2} M_f + 8b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2} M_k \tag{5.7}$$

Now, we consider the space,

$$\mathcal{C}_v^{'} = \{x : (-\infty, b] \to \mathbb{H}, x_0 = \phi \in \mathcal{C}_v\}$$

Set $\|.\|_b$ be a seminorm defined by

$$\|x\|_b = \|x_0\|_{\mathcal{C}_v} + \sup_{s\in[0,b]} (E\,|x(s)|^2)^{\frac{1}{2}}, \;\; x \in \mathcal{C}_v^{'}$$

We have the following useful lemma appeared in [37].

Lemma 5.3.0.9. [14] *Assume that* $x \in \mathcal{C}_v^{'}$*, then for all* $t \in J$*,* $x_t \in \mathcal{C}_v$*, Moreover,*

$$l(E\,|x(t)|^2)^{\frac{1}{2}} \le \|x_t\|_{\mathcal{C}_v} \le l \sup_{s\in[0,t]} (E\,|x(s)|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{C}_v}$$

where $l = \int_{-\infty}^{0} v(s)ds < \infty$

The main object of this paper is to explain and prove the following theorem.

Theorem 5.1. *Assume that assumptions* $(H_0)-(H_5)$ *hold. Then there exists a mild solution*

Proof Consider the map $\Pi : \mathcal{C}_v^{'} \to \mathcal{C}_v^{'}$ defined by

$$(\Pi x)(t) = \begin{cases} \phi(t) & t \in (-\infty, 0] \\ S_\alpha(t)[\phi(0) - \mu(x) - h(0,\phi)] + h(t,x_t) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s,x_s)ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^{s} \sigma(s,\tau,x_\tau)dW(\tau)\right] ds \;\; t \in J \end{cases} \tag{5.8}$$

In what follows, we shall show that the operator Π has a fixed point, which is then a mild solution for system 5.1.
For $\phi \in \mathcal{C}_v$, define

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & t \in (-\infty, 0] \\ S_\alpha(t)\phi(0) & t \in J \end{cases} \tag{5.9}$$

Then $\tilde{\phi} \in \mathcal{C}_v^{'}$. Let $x(t) = \tilde{\phi}(t) + z(t)$, $-\infty < t \leq b$. It is easy to see that x satisfies 5.1 if and only if z satisfies $z_0 = 0$ and

$$z(t) = S_\alpha(t)\left[-\mu(\tilde{\phi} + z) - h(0,\phi)\right] + h(t, \tilde{\phi}_t + z_t) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, \tilde{\phi}_s + z_s) ds$$
$$+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s, \tau, \tilde{\phi}_\tau + z_\tau) dW(\tau)\right] ds$$

Let

$$\mathcal{C}_v^{''} = \{z \in \mathcal{C}_v^{'}, z_0 = 0 \in \mathcal{C}_v\}$$

For any $z \in \mathcal{C}_v^{''}$, we have

$$\|z\|_b = \|z_0\|_{\mathcal{C}_v} + \sup_{s\in[0,b]} (E\|z(s)\|^2)^{\frac{1}{2}} = \sup_{s\in[0,b]} (E\|z(s)\|^2)^{\frac{1}{2}}$$

Thus $(\mathcal{C}_v^{''}, \|.\|_b)$ is a Banach space, set

$$B_q = \{z \in \mathcal{C}_v^{''}, \|z\|_b^2 \leq q\}, \text{ for some } q \geq 0$$

then, $B_q \subset \mathcal{C}_v^{''}$ is uniformly bounded.

then, for each q, B_q is clearly a bounded closed convex set in $\mathcal{C}^{''}$. For $z \in B_q$, from Lemma 5.3.0.8, we have

$$\begin{aligned} \left\|z_t + \tilde{\phi}_t\right\|_{\mathcal{C}_v}^2 &\leq 2(\|z_t\|_{\mathcal{C}_v}^2 + \left\|\tilde{\phi}_t\right\|_{\mathcal{C}_v}^2) \\ &\leq 4(l^2 \sup_{s\in[0,t]} E\|z(s)\|^2 + \|z_0\|_{\mathcal{C}_v}^2 + l^2 \sup_{s\in[0,t]} E\left\|\tilde{\phi}(s)\right\|^2 + \left\|\tilde{\phi}_0\right\|_{\mathcal{C}_v}^2) \\ &\leq 4l^2(q + M^2 E\|\phi(0)\|_{\mathbb{H}}^2) + 4\|\phi\|_{\mathcal{C}_v}^2 \\ &= \grave{q} \end{aligned}$$

Define the operator $\Phi : C_v^{''} \to C_v^{''}$ by

$$(\Phi z)(t) = \begin{cases} 0 \quad t \in (-\infty, 0] \\ S_\alpha(t)[-\mu(\tilde{\phi} + z) - h(0, \phi)] + h(t, \tilde{\phi}_t + z_t) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, \tilde{\phi}_s + z_s) ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s, \tau, \tilde{\phi}_\tau + z_\tau) dW(\tau) \right] ds \;\; t \in J \end{cases}$$

Observe that Φ is well defined on B_q for each $q > 0$.

Now we will show that the operator Φ has a fixed point on B_q, which implies that E.q 5.1 has a mild solution. To this end, we decompose Φ as $\Phi = \Phi_1 + \Phi_2$, where the operators Φ_1 and Φ_2 are defined on B_q, respectively, by

$$(\Phi_1 z)(t) = S_\alpha(t)[-\mu(\tilde{\phi} + z) - h(0, \phi)] + h(t, \tilde{\phi}_t + z_t)$$

$$\begin{aligned}(\Phi_2 z)(t) = &\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, \tilde{\phi}_s + z_s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s, \tau, \tilde{\phi}_\tau + z_\tau) dW(\tau) \right] ds\end{aligned}$$

Thus, the theorem follows from the next theorem

Theorem 5.2. *If assumption $(H_1) - (H_5)$ hold, then Φ_1 is a contraction and Φ_2 is completely continuous.*

Proof To prove that Φ_1 is a contraction on $\mathcal{C}''_v$, we take $u, v \in \mathcal{C}''_v$. Then for each $t \in J$ we have

$$\begin{aligned} E\left\|\Phi_1 u(t) - \Phi_1 v(t)\right\|^2_{\mathbb{H}} &\leq 2E\left\|S_\alpha(t)(\mu(\tilde{\phi} + u) - \mu(\tilde{\phi} + v))\right\|^2_{\mathbb{H}} \\ &+ 2E\left\|h(t, \tilde{\phi}_t + u_t) - h(t, \tilde{\phi}_t + v_t)\right\|^2_{\mathbb{H}} \\ &\leq 2M^2 M_\mu \left\|u - v\right\|^2_{\mathcal{C}_v} + 2M_h \left\|u_t - v_t\right\|^2_{\mathcal{C}_v} \\ &\leq 2(M^2 M_\mu + M_h) \left\|u_t - v_t\right\|^2_{\mathcal{C}_v} \\ &\leq 2(M^2 M_\mu + M_h) \\ &[2l^2 \sup_{s \in [0,t]} E\left\|u(s) - v(s)\right\|^2 + 2\left\|u_0\right\|^2_{\mathcal{C}_v} + 2\left\|v_0\right\|^2_{\mathcal{C}_v}] \\ &\leq 4l^2 (M^2 M_\mu + M_h) E\left\|u(s) - v(s)\right\|^2 \\ &\leq \sup_{s \in [0,b]} L_0 E\left\|u(s) - v(s)\right\|^2 \end{aligned}$$

where we have used the fact that $\|u_0\|^2_{\mathcal{C}_v} = 0$, $\|v_0\|^2_{\mathcal{C}_v} = 0$.
Thus,

$$\|\Phi_1 u - \Phi_1 v\| \leq L_0 \|u - v\|$$

and by assumption $0 \leq L_0 \leq 1$ it is clear that Φ_1 is contraction.

Now, we show that the operator Φ_2 is completely continuous, firstly we prove that $\Phi_2 : \mathcal{C}''_h \to \mathcal{C}''_h$ is continuous.

Let $\{z^n(t)\}_{n=0}^{\infty}$,with $z^n \to z$ in $\mathcal{C}''_h$. Then, there is a number $q \geq 0$ such that $|z^n(t)| \leq q$, for all n and a.e. $t \in J$. So $z^{(n)} \in B_q$ and $z \in B_q$.

$$f(t, z_t^{(n)} + \tilde{\phi}_t) \to f(t, z_t + \tilde{\phi}_t)$$
$$\sigma(s, \tau, z_\tau^{(n)} + \tilde{\phi}_\tau) \to \sigma(s, \tau, z_\tau + \tilde{\phi}_\tau)$$

for $t \in J$, and since

$$E \left\| [f(t, z_t^{(n)} + \tilde{\phi}_t) - f(t, z_t + \tilde{\phi}_t)] \right\|^2 \leq 2M_{q'}(t)$$

$$E \left\| [\sigma(s, \tau, z_\tau^{(n)} + \tilde{\phi}_\tau) - \sigma(s, \tau, z_\tau^{(n)} + \tilde{\phi}_\tau)] \right\|^2 \leq 2m(t)M_\sigma(q')$$

By the dominated convergence theorem we obtain continuity of Φ_2

$$\begin{aligned} E \left\| \Phi z_t^{(n)} - \Phi_t z \right\|^2 \leq & 2 \sup_{t \in J} E \left\| \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[f(t, z_s^{(n)}) - f(t, z_s)] ds \right\|^2 \\ & + 2b \sup_{t \in J} E \left\| \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s [\sigma(s, \tau, z_\tau^{(n)}) - \sigma(s, \tau, z_\tau)] dw(\tau) \right] ds \right\|^2 \\ & \leq 2 \left(\frac{M_\alpha}{\Gamma(1+\alpha)} \right)^2 \frac{b^{2\alpha}}{\alpha^2} \int_0^t E \left\| [f(t, z_s^{(n)}) - f(t, z_s)] \right\|^2 ds \\ & + 2b \left(\frac{M_\alpha}{\Gamma(1+\alpha)} \right)^2 \frac{b^{2\alpha}}{\alpha^2} \int_0^t E \left\| \left[\int_{-\infty}^s [\sigma(s, \tau, z_\tau^{(n)}) - \sigma(s, \tau, z_\tau)] dw(\tau) \right] ds \right\|^2 \\ & \to 0 \text{ as } n \to \infty \end{aligned}$$

Next, we prove that Φ_2 maps bounded sets into bounded sets in $\mathcal{C}''_v$.
For each $z \in B_q$ from [5.3.0.9], we have

$$\left\| z_t + \tilde{\phi}_t \right\|^2_{\mathcal{C}_v} \leq 4l^2(q + M^2 E \|\phi(0)\|^2_{\mathbb{H}}) + 4 \|\phi\|^2_{\mathcal{C}_v} = q'$$

$$
\begin{aligned}
E\|\Phi_2 z(t)\|_{\mathbb{H}}^2 &\leq 2E\left\|(t-s)^{\alpha-1}T_\alpha(t-s)f(s,\tilde{\phi}_s+z_s)\right\|_{\mathbb{H}}^2 \\
&+ 2E\left\|(t-s)^{\alpha-1}T_\alpha(t-s)[\int_{-\infty}^{s}\sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)]ds\right\|_{\mathbb{H}}^2 \\
&\leq 2\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2\frac{b^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}M_f(1+\left\|\tilde{\phi}_s+z_s\right\|_{\mathcal{C}_v}^2)ds \\
&+ \left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2\frac{b^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}(2M_k+2Tr(Q)m(s)M_\sigma(\left\|\tilde{\phi}_s+z_s\right\|_{\mathcal{C}_v}^2)ds. \\
&\leq 2\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2\frac{b^{2\alpha}}{\alpha^2}M_f(1+q^{'}) \\
&+ 2\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2\frac{b^{2\alpha}}{\alpha^2}(M_k+Tr(Q)M_\sigma(q^{'})\sup_{t\in J}m(s)) \\
&\leq r
\end{aligned}
$$

Which implies that for each $z \in B_q$, $|\Phi_2 z\|_b^2 \leq r$.

Next, we establish the compactness of Φ_2. We employ the Arzela-Ascoli theorem to show the set $V(t) = \{(\Phi_2 z)(t), z \in B_q\}$ is relatively compact in $\mathbb{H}$. Le $0 < t \leq b$ be fixed and ϵ be a real number satisfying $0 < \epsilon \leq t$. For $\delta > 0$, for $z \in B_q$, We define

$$
\begin{aligned}
(\Phi_2^{\epsilon,\delta}z)(t) &= \alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta))S((t-s)^\alpha\theta)f(s,\tilde{\phi}_s+z_s)ds \\
&+ \alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta))S((t-s)^\alpha\theta)[\int_{-\infty}^{s}\sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)]ds \\
&= S(\epsilon^\alpha\delta)\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta))S((t-s)^\theta-\epsilon^\alpha\delta)f(s,\tilde{\phi}_s+z_s)ds \\
&+ S(\epsilon^\alpha\delta)\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta))S((t-s)^\theta-\epsilon^\alpha\delta)[\int_{-\infty}^{s}\sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dW(\tau)]ds
\end{aligned}
$$

Since $S(t), t > 0$, is a compact operator, the set $V_{\epsilon,\delta} = \{\Phi_2^{\epsilon,\delta}(t), z \in B_q\}$ is relatively compact in $\mathbb{H}$ for every $\epsilon \in (0,t)$, $\delta > 0$. Moreover, for each $z \in B_q$, we have

$$
\begin{aligned}
&E\left\|(\Phi_2 z)(t)-(\Phi_2^{\epsilon,\delta}z)(t)\right\|_{\mathbb{H}}^2\\
&\leq 4\alpha^2 E\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)f(s,z_s+\tilde{\phi}_s)dt\theta ds\right\|_{\mathbb{H}}^2\\
&+4\alpha^2 E\left\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)f(s,z_s+\tilde{\phi}_s)d\theta ds\right\|_{\mathbb{H}}^2\\
&+4\alpha^2 E\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)\left[\int_{-\infty}^s \sigma(s,\tau,z_\tau+\tilde{\phi}_\tau)dW(\tau)\right]d\theta ds\right\|_{\mathbb{H}}^2\\
&+4\alpha^2 E\left\|\int_{t-\epsilon}^t\int_\infty^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S((t-s)^\alpha\theta)\left[\int_{-\infty}^s \sigma(s,\tau,z_\tau+\tilde{\phi}_\tau)dW(\tau)\right]d\theta ds\right\|_{\mathbb{H}}^2\\
&\leq 4M^2b^{2\alpha}M_f(1+q^{'})\left(\int_0^\delta \theta\eta_\alpha(\theta)d\theta\right)^2+\frac{4M^2\epsilon^{2\alpha}M_f(1+q^{'})}{\Gamma^2(1+\alpha)}\\
&+4\alpha M^2b^\alpha\int_0^t(t-s)^{\alpha-1}(2M_k+2Tr(Q)M_\sigma(q^{'})m(s))ds\left(\int_0^\delta \theta\eta_\alpha(\theta)d\theta\right)^2\\
&+\frac{4\alpha M^2\epsilon^\alpha}{\Gamma^2(1+\alpha)}\int_{t-\epsilon}^t(t-s)^{\alpha-1}(2M_k+2Tr(Q)M_\sigma(q^{'})m(s))ds
\end{aligned}
$$

where we have used the equality (see [46, 67])

$$\int_0^\infty \theta^\varsigma\eta_\alpha(\theta)=\frac{\Gamma(1+\varsigma)}{\Gamma(1+\alpha\varsigma)}$$

We see that for each $z\in B_q$

$$E\left\|(\Phi_2 z)(t)-(\Phi_2^{\epsilon,\delta})\right\|_{\mathbb{H}}^2\to 0 \text{ as } \epsilon^+\to 0,\delta\longrightarrow 0.$$

Since the right-hand side of the above inequality can be made arbitrarily small, there is relatively compact $V_{\epsilon,\delta}$ arbitrarily close to the set $V(t)$. Hence, the set $V(t)$ is relatively compact in B_q. It remains to showt hat Φ_2 maps is bounded set into equicontinuous sets of $\mathcal{C}_v^{''}$.
Let $0<\epsilon<t<b$ and $\delta>0$ such that $\|T_\alpha(s_1)-T_\alpha(s_2)\|\leq\epsilon$, for every $s_1,s_2\in J$.

With $|s_1 - s_2| < \delta$. For $z \in B_q$, we have

$$
\begin{aligned}
& E\left\|\Phi_2 z(t+h) - \Phi_2 z(t)\right\|_{\mathbb{H}}^2 \\
& \leq 6E\left\|\int_0^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}]T_\alpha(t+h-s)f(s,\tilde{\phi}_s + z_s)ds\right\|_{\mathbb{H}}^2 \\
& + 6E\left\|\int_t^{t+h} (t+h-s)^{\alpha-1}T_\alpha(t+h-s)f(s,\tilde{\phi}_s + z_s)ds\right\|_{\mathbb{H}}^2 \\
& + 6E\left\|\int_0^t (t-s)^{\alpha-1}[T_\alpha(t+h-s) - T_\alpha(t-s)]f(s,\tilde{\phi}_s + z_s)ds\right\|_{\mathbb{H}}^2 \\
& + 6E\left\|\int_0^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}]T_\alpha(t+h-s)[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau)dW(\tau)ds]\right\|_{\mathbb{H}}^2 \\
& + 6E\left\|\int_t^{t+h} (t+h-s)^{\alpha-1}T_\alpha(t+h-s)[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau)dW(\tau)ds]\right\|_{\mathbb{H}}^2. \\
& + 6E\left\|\int_0^t (t-s)^{\alpha-1}[T_\alpha(t+h-s) - T_\alpha(t-s)][\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau)dW(\tau)ds]\right\|_{\mathbb{H}}^2 \\
& \leq 6\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \int_0^t \left|(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right|^2 M_f(1+q')ds \\
& + 6\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \int_t^{t+h} \left|(t+h-s)^{\alpha-1}\right|^2 M_f(1+q')ds \\
& + 6\epsilon^2 \int_0^t \left|(t-s)^{\alpha-1}\right|^2 M_f(1+q')ds + 6\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \int_0^t \left|(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right|^2 \\
& \times (2M_k + 2Tr(Q)m(s)M_\sigma(q'))ds \\
& + 6\left\{\frac{M_\alpha}{\Gamma(1+\alpha)}\right\}^2 \int_0^t \left|(t+h-s)^{\alpha-1}\right|^2 (2M_k + 2Tr(Q)m(s)M_\sigma(q'))ds \\
& + 6\epsilon^2 \int_0^t \left|(t-s)^{\alpha-1}\right| (2M_k + 2Tr(Q)m(s)M_\sigma(q'))ds
\end{aligned}
$$

It is known that the compactness of $T_\alpha(t), t > 0$ implies the continuity in the uniform operator topology. Therefore, for ϵ sufficiently small, the right-hand side of the above inequality tends to zero as $h \to 0$. Thus, the set $\{\Phi_2 z, z \in B_q\}$ is equicontinuous.
This completes the proof that Φ_2 is completely continuous.

To apply the Krasnoselski-Schaefer theorem, it remains to show that the set

$$G = \{x \in \mathbb{H} : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x\} \text{ is bounded for } \lambda \in (0,1)$$

.

We consider the following nonlinear operator equation,

$$\begin{aligned} x(t) &= \lambda\left(S_\alpha(t)[\phi(0) - \mu(x) - h(0,\phi)]\right) + \lambda h(t, x_t) \\ &+ \lambda \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds \\ &+ \lambda \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s,\tau,x_\tau) dW(\tau)\right] ds \end{aligned}$$

Theorem 5.3. *If hypothesis $(H_1)-(H_5)$ are satisfied, then there exist an a priori bound $\rho \geq 0$ such that $\|x_t\|^2_{\mathcal{C}_v} \leq \rho, t \in J$, where ρ depends only on b and on the function $\pi(s)$ and $M_\sigma(s)$.*

Proof

$$\begin{aligned} E\|x(t)\|^2 &\leq 4E\|S_\alpha(t)(\phi(0) - \mu(x) - h(0,\phi))\|^2_{\mathbb{H}} + 4E\|h(t,x_t)\|^2_{\mathbb{H}} \\ &+ 4E\left\|\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s,x_s) ds\right\|^2_{\mathbb{H}} \\ &+ 4E\left\|\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,x_\tau) dW(\tau)\right] ds\right\|^2_{\mathbb{H}} \\ &\leq 12M^2(C_1 + C_2 + M_\mu) + 12M^2(1 + \|x\|^2_{\mathcal{C}_v}) \\ &+ 4\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2} M_f(1 + \|x\|^2_{\mathcal{C}_v}) \\ &+ 4b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha} \int_0^t (t-s)^{\alpha-1}(2M_k + 2Tr(Q)m(s)M_\sigma(\|x_s\|^2_{\mathcal{C}_v}) ds \end{aligned}$$

Now, we consider the function ν defined by

$$\vartheta(t) = \sup\{E\|x(s)\|^2, 0 \leq s \leq t\}, 0 \leq t \leq b$$

From lemma [5.3.0.9] and the above inequality, we have

$$E\|x(t)\|^2 = 2\|\phi\|^2_{\mathcal{C}_v} + 2l^2 \sup_{0\leq s\leq t}(E\|x(s)\|^2)$$

Therefore, we get

$$\vartheta(t) \leq 2\left\|\phi\right\|^2_{\mathcal{C}_v} + 2l^2\{\bar{F} + 12M^2\vartheta(t) + 4\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2} M_f \vartheta(t)$$
$$+ 8b\left(\frac{M_\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1} Tr(Q) m(s) M_\sigma(\vartheta(s))ds\}$$

where $\bar{F}$ is given in (5.7). Thus, we have

$$\vartheta(t) \leq K_1 + K_2\int_0^t \frac{\vartheta(s)}{(t-s)^{1-\alpha}}ds + K_3\int_0^t m(s)M_\sigma(\vartheta(s))ds$$

where K_1, K_2, K_3 are given in (5.6). By Lemma [5.2.0.7], we have

$$\vartheta \leq B_0(K_1 + K_3\int_0^t m(s)M_\sigma(\vartheta(s))ds)$$

Where

$$B_0 = e^{K_2^n(\Gamma(\alpha))^n b^{n\alpha}/\Gamma(n\alpha)} \sum_{j=0}^{n-1}\left(\frac{K_2 b^\alpha}{\alpha}\right)^j$$

Denoting by $\nu(t)$ the right hand side of the last inequality, we have $\nu(0) = B_0K_1$

$$\grave{\nu}(t) \leq B_0K_3 m(t)M_\sigma\vartheta(t)$$
$$\grave{\nu}(t) \leq B_0K_3 m(t)M_\sigma(\vartheta(t))$$

This implies

$$\int_{\nu(0)}^{\nu(t)} \frac{ds}{M_\sigma(s)} \leq \int_0^b \pi(s)ds < \int_{B_0K_1}^{\infty} \frac{ds}{M_\sigma(s)}$$

This inequality implies that there is a constant ρ such that $\nu(t) \leq \rho$, $t \in J$ and hence $\vartheta(t) \leq \rho$, $t \in J$. Furthermore, we get $\|x_t\|^2_{\mathcal{C}_v} \leq \vartheta(t) \leq \nu(t) \leq \rho$, $t \in J$, where ρ depends only on b and on the function $\pi(s)$ and $M_\sigma(s)$.

Theorem 5.4. *Assume that the hypotheses $(H_1) - (H_5)$ hold. Then problem has at least one mild solution on J.*

Proof

Let us take the set

$$D(\Phi) = \{z \in \mathcal{C}_v'' : z = \lambda\Phi_1(\frac{z}{x}) + \lambda\Phi_2 z \text{ for some } \lambda \in [0,1]\} \tag{5.10}$$

Then, for any $z \in D(\Phi)$, we have by theorem [5.3] that $\|x\|^2_{\mathcal{C}_v} \leq \rho, t \in J$, and hence

$$\begin{aligned}
\|z\|_b^2 &= \|z_0\|^2_{\mathcal{C}_v} + \sup\{E\|z(t)\|^2 ; 0 \leq t \leq b\} \\
&= \sup\{E\|z(t)\|^2 : 0 \leq t \leq b\} \\
&\leq \sup\{E\|x(t)\|^2 : 0 \leq t \leq b\} + \sup\{E\left\|\tilde{\phi}(t)\right\|^2 : 0 \leq t \leq b\} \\
&\leq \sup\{l^- \|x(t)\|^2_{\mathcal{C}_v} : 0 \leq t \leq b\} + \sup\{\|s_\alpha(t)\phi(0)\| : 0 \leq t \leq b\} \\
&\leq l^- \rho + M_1 \|\phi(0)\|^2
\end{aligned}$$

This implies that D is bounded on J. Consequently by Lemma 5.2.0.6, the operator Φ has a fixed point $z \in \mathcal{C}''_h$. So Eq.(5.1) has a mild solution.Theorem is proved.

5.4 Example

As an application of the above result,consider the following fractional order neutral stochastic partial differential system with non local conditions and infinite delay in Hilbert space.

$$\begin{cases}
{}^cD_t^\alpha[z(t,x) - \int_{-\infty}^t e^{4(s-t)}z(s,x)ds] = \frac{\partial^2}{\partial x^2}[z(t,x) - \int_{-\infty}^t e^{4(s-t)}z(s,x)ds] + \eta(t,x) \\
+ \int_{-\infty}^0 \hat{a}(s)\sin z(t+s,x)ds + \int_{-\infty}^t \int_{-\infty}^t \sigma(t,x,s-t)dsd\beta(s,x) \quad t \in J = [0,b] \\
z(t,0) = z(t,\pi) = 0 \quad t \in J \\
z(0,x) + \int_0^\pi k_1(x,y)z(t,y)dy = x_0 = \varphi(t,x) \quad t \in (-\infty, 0],
\end{cases} \tag{5.11}$$

Where ${}^cD^\alpha$ is a Caputo fractional partial derivative of order $\alpha \in (0,1)$, and $K_1(x,y) \in \mathbb{H} = L^2([0,\pi] \times [0,\pi])$ and $\int_{-\infty}^0 |\hat{a}(s)|\, ds < +\infty$. $\beta(t)$ is a one-dimensional standard Wiener process on filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.
To rewrite this system into the abstract form (5.1), let $\mathbb{H} = L^2([0,\pi])$ with

the norm $\|.\|$. Define $A : \mathbb{H} \to \mathbb{H}$ by $A(t)z = z''$ with the domain

$$\mathcal{D}(A) = \left\{ z \in \mathbb{H} : z, z' \text{ are absolutely continuous }, z'' \in \mathbb{H}, z(0) = z(\pi) = 0 \right\}$$

It is well known that A generates a strongly continuous semigroup T(.),which is compact,analytic and self adjoint.

Then

$$Az = \sum_{n=1}^{\infty} n^2 \langle z, z_n \rangle z_n, \quad z \in D(A)$$

where $z_n(s) = \sqrt{\frac{2}{\pi}} \sin(ns), n = 1, 2, ...$ is the orthonormal set of eigenvector of A. It is well known that A is thein infinitesimal generator of ananalytic semigroup T(t)in H and is given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} \langle z, z_n \rangle z_n$$

Then the operator $A^{-\frac{1}{2}}$ is given by

$$A^{-\frac{1}{2}} z = \sum_{n=1}^{\infty} n \langle z, z_n \rangle z_n$$

on the space $\mathcal{D}(A^{-\frac{1}{2}}) = \{z(.) \in \mathbb{H} : \sum_{n=1}^{\infty} n \langle \zeta, z_n \rangle z_n \in \mathbb{H}\}$.

Now, we present a special $\mathcal{C}_v$ space. Let $\vartheta(s) = e^{2s}, s < 0$, then $l = \int_{-\infty}^{0} \vartheta(s) ds = \frac{1}{2}$.

Let

$$\|\varphi\|_{\mathcal{C}_v} = \int_{-\infty}^{0} h(s) \sup_{s \le \theta \le 0} E \left(\|\varphi(\theta)\|^2 \right)^{\frac{1}{2}} ds$$

Then $(\mathcal{C}_v, \|.\|_{\mathcal{C}_v})$ is a Banach space.

For $(t, \varphi) \in J \times \mathcal{C}_v$ where $\varphi(\theta)(x) = \varphi(\theta, x), (\theta, x) \in (-\infty, 0] \times [0, \pi]$, and define the Lipschitz continuous functions $h, f : J \times \mathcal{C}_v \to H$, $\sigma : J \times \mathcal{C}_v \to L_Q(\mathbb{H})$, for the infinite delay as follows

$$h(t, \varphi)(x) = \int_{-\infty}^{0} e^{-4\theta} \varphi(\theta)(x) d\theta$$

$$f(t, \varphi)(x) = \int_{-\infty}^{0} \hat{a}(\theta) \sin(\varphi(\theta)(x)) d\theta$$

$$\sigma(t, \varphi)(x) = \int_{-\infty}^{0} \varsigma(t, x, \theta) \sigma(\varphi(\theta)(x)) d\theta$$

Then, the equation (5.11) can be rewritten as the abstract form as the system 5.1. Thus, under the appropriate condition so the functions h,f, and σ are satisfies the hypotheses $(H_1) - (H_5)$. All conditions of the Theorem 5.2 are satisfied, therefore the system (5.11) has a mild solution.

Chapitre 6

Conclusion and Perspectives

The theme treated in this thesis is of great importance, particularly the existence of mild solutions for stochastic integro-differential equations with nonlocal conditions and infinite delay in Hilbert spaces. Our techniques rely on the fractional calculus, methods and results for infinite dimensional SDEs, properties of the semigroup methods , and fixed point theorem . Our methods not only present a new way to study such problems under non-Lipschitz conditions , but also provide new theory results appeared in thesis previously are generalized to the fractional stochastic systems settings and the case of nonlocal conditions and infinite delay.

On the other hand, the advantage of the integro-differential equations representation for a variety of problem is witnessed by its increasing frequency in the literature and in many texts on method of advanced applied mathematics.

In this perspective , we address the problem still posed :

1. **Concept of controllability (K.Balachandran and E.R.Anandhi) :** In particular, the concept of controllability plays an important role in both the deterministic and the stochastic control theory. In recent years, controllability problems for various types of non linear dynamical systems in infinite dimensional spaces by using different kinds of approaches have been considered in many publications.
2. **The theory of impulsive integro-differential equations (S.K.Ntouyas and X.Xiang) :** The theory of impulsive integro-differential equa-

tions has become an active are a of investigation due to its applications in fields such as mechanics, electrical engineering , medicine, biology,and ecology.

3. The technique of resolvent operators for integro-differential equations (E.Hernández,P.C.J.Dos Santos and J.H.Liu).

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