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Publications

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Contents

Introduction

Several definitions of fractional derivatives and integrals have been defined in the literature, including those of Riemann–Liouville, Grünwald–Letnikov, Hadamard, Riesz, Weyl and Caputo [47, 67, 70]. In 1996, Kolwankar and Gangal proposed a local fractional derivative operator that applies to highly irregular and nowhere differentiable Weierstrass functions [13, 49]. In our work we introduce the notion of fractional derivative on an arbitrary time scale $\mathbb{T}$. In the particular case $\mathbb{T} = \mathbb{R}$, one gets the local Kolwankar–Gangal fractional derivative $\lim_{h\to 0} \frac{f(t+h)-f(t)}{h^\alpha}$, which has been considered in [49, 50] as the point of departure for fractional calculus. One of the motivations to consider such local fractional derivatives is the possibility to deal with irregular signals, so common in applications of signal processing [50].

A time scale is a model of time. The calculus on time scales was initiated by Aulbach and Hilger in 1988 [12], in order to unify and generalize continuous and discrete analysis [42, 43]. It has a tremendous potential for applications and has recently received much attention [4, 25, 26, 35, 38]. The idea to join the two subjects — the fractional calculus and the calculus on time scales — and to develop a Fractional Calculus on Time Scales, was born with the PhD thesis of Bastos [17]. See also [8, 11, 18, 19, 20, 48, 69, 76] and references therein. In this PhD thesis we introduce a general fractional calculus on time scales and develop some of its basic properties.

Fractional calculus is of increasing importance in signal processing [66]. This can be explained by several factors, such as the presence of internal noises in the structural definition of the signals. Our fractional derivative depends on the graininess function of the time scale. We trust that this possibility can be very useful in applications of signal processing, providing a concept of coarse-graining in time that can be used to model white noise that occurs in signal processing or to obtain generalized entropies and new practical meanings in signal processing. Indeed, let $\mathbb{T}$ be a time scale (continuous time $\mathbb{T} = \mathbb{R}$, discrete time $\mathbb{T} = h\mathbb{Z}$, $h > 0$, or, more generally, any closed subset of the real numbers, like the Cantor set). Our results provide a mathematical framework to deal with functions (signals) f in signal processing that are not differentiable in the time scale, that is, signals f for which the equality $\Delta f(t) = f^{\Delta}(t)\Delta t$ does not hold. More precisely, we are able to model signal processes for which $\Delta f(t) = f^{(\alpha)}(t)(\Delta t)^{\alpha}$, $0 < \alpha \leq 1$.

The fractional calculus is now subject of strong current research: see, e.g., [21, 39, 40, 46, 60, 61], that refer to nonsymmetric fractional calculi. In our work we present a general symmetric fractional calculus on time scales. For the importance to study such a symmetric calculus we refer the reader to [29, 30, 31].

Mathematical models of some natural phenomena and physical problems have appeared as initial and boundary value problems including fractional order of ordinary and partial differential equations. See Lokshin and Suvorova in 1982 on modeling of irrevocability of metals [52] and Nakhashev in 1985 on modeling of liquids moving in underground layers encountered with fractional order differential equations [64]. Later, this kind of differential equations were used in electrochemistry, control, and electromagnetic field theories [37, 41]. These important applications caused that this kind of differential equations were studied by many mathematicians in recent years [34, 64, 65].

We have organized this thesis as follows:

In Chapter 1, we present some definitions and theorems which are used throughout this thesis.

In Chapter 2, we begin by recalling the main concepts and tools necessary in the sequel. Our results are then given in Section 2.2, where the notion of fractional derivative for functions defined on arbitrary time scales is introduced and the respective fractional differential calculus developed. The notion of fractional integral on time scales, and some of its basic properties, is investigated in Section 2.3.

In Chapter 3, we begin by presenting some basic notions and necessary results. Then, in Section 3.2, we define and develop the nonsymmetric fractional calculus. In order to do that, we define the nabla fractional derivative and the nabla fractional integral of order $\alpha \in]0, 1]$. In Section 3.3, we introduce and develop the symmetric fractional calculus.

In Chapter 4, we shall be concerned with the existence and uniqueness of solution to the following initial value problem:

$$ {}_{t_0}^{\mathbb{T}}D_t^{\alpha} y(t) = f(t, y(t)), \quad t \in [t_0, t_0 + a] = \mathcal{J} \subseteq \mathbb{T}, \quad 0 < \alpha < 1, $$

$$ {}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} y(t_0) = 0, $$

where ${}_{t_0}^{\mathbb{T}}D_t^{\alpha}$ is the (left) Riemann–Liouville fractional derivative operator or order α defined on $\mathbb{T}$, ${}_{t_0}^{\mathbb{T}}I_t^{1-\alpha}$ the (left) Riemann–Liouville fractional integral operator or order $1-\alpha$ defined on $\mathbb{T}$, and function $f : \mathcal{J} \times \mathbb{T} \to \mathbb{R}$ is a right-dense continuous function. Our results are based on the Banach fixed point theorem for uniqueness of solution and Schauder's fixed point theorem for existence of solution [36].

Chapter 1

Preliminaries

In this chapter, we introduce the calculus on time scales and we also present the main results on the differentiability and integration on time scales. The reader interested on the subject is referred to the books [25, 26]. For a good survey see [4].

1.1 The Time Scale Calculus

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Thus $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$, i.e., the real numbers, the integers, the natural numbers, and the nonnegative integers are examples of time scales, as are $[0,1]\bigcup[2,3]$, $[0,1]\bigcup\mathbb{N}$, and the Cantor set.

Any time scale $\mathbb{T}$ is a complete metric space with the metric (distance) $d(t;s) = |t-s|$ for $t, s \in \mathbb{T}$. Consequently, according to the well-known theory of general metric spaces, we have for $\mathbb{T}$ the fundamental concepts such as open balls (intervals), neighborhoods of points, open sets, closed sets, compact sets, and so on. In particular, for a given number $N > 0$, the N-neighborhood $\mathrm{U}_\delta(t)$ of a given point $t \in \mathbb{T}$ is the set of all points $s \in \mathbb{T}$ such that $d(t,s) < N$. By a neighborhood of a point $t \in \mathbb{T}$ it is meant an arbitrary set in $\mathbb{T}$ containing a N-neighborhood of the point t. Also we have for functions $f : \mathbb{T} \to \mathbb{R}$ the concepts of limit, continuity, and the properties of continuous functions on general complete metric spaces (note that, in particular, any function $f : \mathbb{Z} \to \mathbb{R}$ is continuous at each point of $\mathbb{Z}$). The main task is to introduce and investigate the concept of derivative for functions $f : \mathbb{Z} \to \mathbb{R}$. This proves to be possible due to the special structure of the metric space $\mathbb{T}$. In the definition of the derivative an important role is played by the so-called forward and backward jump operators [26].

Definition 1.1. *[25]. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

Remark 1.2. *In Definition 1.1, we put:*
$\inf \emptyset = \sup \mathbb{T}$ *(i.e., $\sigma(t) = t$) if $\mathbb{T}$ has a maximum t,*
$\sup \emptyset = \inf \mathbb{T}$ *(i.e., $\rho(t) = t$) if $\mathbb{T}$ has a minimum t, where $\emptyset$ denotes the empty set.*

Definition 1.3. *[25]. If $\sigma(t) > t$, then we say that t is right-scattered; if $\rho(t) < t$, then t is said to be left-scattered. Points that are simultaneously right-scattered and left-scattered are called isolated. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense; if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense.*

Definition 1.4. *[25]. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by*

$$\mu(t) := \sigma(t) - t.$$

Definition 1.5. *[25]. The backward graininess function $\nu : \mathbb{T} \to [0, \infty)$ is defined by*

$$\nu(t) := t - \rho(t).$$

Definition 1.6. *[25]. Let $\mathbb{T}$ be a time scale.*

(i) If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

(ii) If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$.

Definition 1.7. *[25]. Let $f : \mathbb{T} \to \mathbb{R}$. We define $f^\sigma : \mathbb{T} \to \mathbb{R}$ and $f^\rho : \mathbb{T} \to \mathbb{R}$ respectively by*

$$f^\sigma(t) := (f \circ \sigma)(t) = f(\sigma(t)), \qquad \text{for all } t \in \mathbb{T}$$

and

$$f^\rho(t) := (f \circ \rho)(t) = f(\rho(t)), \qquad \text{for all } t \in \mathbb{T}.$$

1.2 Differentiation on Time Scales

Several differentiation notions are possible.

1.2.1 Delta Differentiation

Definition 1.8. *[3]. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^{\kappa}$ if there exists a number $f^{\Delta}(t)$ such that, for all $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$\left|f^{\sigma}(t) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\right| \leq \varepsilon\left|\sigma(t) - s\right|$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta derivative of f at t and we say that f is delta differentiable if f is delta differentiable for all $t \in \mathbb{T}^{\kappa}$.

Theorem 1.9. *[3]. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{k}$.*

(i) If f is Δ-differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is Δ-differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite. In this case,

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is Δ-differentiable at t, then

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t).$$

Example 1.10. *. Again we consider the two cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$*

(i) If $\mathbb{T} = \mathbb{R}$, then Theorem 1.9 (iii) yields that $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if

$$f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \quad \textit{exists}$$

i.e., if and only if f is differentiable (in the ordinary sense) at t. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t)$$

by Theorem 1.9 (iii).

(ii) *If* $\mathbb{T} = \mathbb{Z}$, *then Theorem 1.9 (ii) yields that* $f : \mathbb{Z} \to \mathbb{R}$ *is delta differentiable at* $t \in \mathbb{Z}$ *if and only if*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+1) - f(t)}{1} = \Delta f(t),$$

where Δ *is the usual forward difference operator defined by the last equality above.*

(iii) *If* $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, *and* $h > 0$, *then Theorem 1.9 (ii) yields that* $f : h\mathbb{Z} \to \mathbb{R}$ *is delta differentiable at* $t \in h\mathbb{Z}$ *if and only if*

$$f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h}.$$

(iv) *If* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *with* $\overline{q^{\mathbb{Z}}} := q^{\mathbb{Z}} \bigcup \{0\}$ *and* $q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$, *then* 0 *is a right-dense minimum and every other point in* $\mathbb{T}$ *is isolated. For a function* $f : \overline{q^{\mathbb{Z}}} \to \mathbb{R}$, *we have*

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t} \quad \textit{for all} \quad t \in \mathbb{T} \backslash \{0\}$$

and

$$f^{\Delta}(0) = \lim_{s \to 0} \frac{f(0) - f(s)}{0 - s} = \lim_{s \to 0} \frac{f(s) - f(0)}{s},$$

provided the limits exist.

Theorem 1.11. *[3]. Assume* $f, g : \mathbb{T} \to \mathbb{R}$ *are* Δ*-differentiable at* $t \in \mathbb{T}^{\kappa}$. *Then,*

(i) *The sum* $f + g$ *is* Δ*-differentiable at* t *with*

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) *For any constant* $\alpha \in \mathbb{R}$, αf *is* Δ*-differentiable at* t *with*

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(iii) *The product* fg *is* Δ*-differentiable at* t *with*

$$\begin{aligned} (fg)^{\Delta}(t) &= f^{\Delta}(t) g(t) + f^{\sigma}(t) g^{\Delta}(t) \\ &= f(t) g^{\Delta}(t) + f^{\Delta}(t) g^{\sigma}(t). \end{aligned}$$

(iv) *If* $f(t) f^{\sigma}(t) \neq 0$, *then* $\frac{1}{f}$ *is* Δ*-differentiable at* t *with*

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f^{\sigma}(t) f(t)}. \tag{1.1}$$

(v) If $g(t)g^{\sigma}(t) \neq 0$, then $\frac{f}{g}$ is Δ-differentiable at t with

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g^{\sigma}(t)g(t)}. \tag{1.2}$$

Remark 1.12. *Delta derivatives of higher-order are defined in the usual way. Let $r \in \mathbb{N}$, $\mathbb{T}^{\kappa^0} := \mathbb{T}$, and $\mathbb{T}^{\kappa^i} := \left(\mathbb{T}^{\kappa^{i-1}}\right)^{\kappa}$, $i = 1, \ldots, r$. For convenience we also put $f^{\Delta^0} = f$ and $f^{\Delta^1} = f^{\Delta}$. The rth-delta derivative f^{Δ^r} is given by $f^{\Delta^r} = \left(f^{\Delta^{r-1}}\right)^{\Delta} : \mathbb{T}^{\kappa^r} \to \mathbb{R}$ provided $f^{\Delta^{r-1}}$ is delta differentiable.*

Theorem 1.13 (Chain Rule [3])**.** *Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^{\kappa}$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ with*

$$(f \circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t).$$

1.2.2 Nabla Differentiation

Definition 1.14. *[3]. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t \in \mathbb{T}_{\kappa}$ if there exists a real number $f^{\nabla}(t)$ such that, for all $\varepsilon > 0$, there exists a neighborhood V of t such that*

$$\left|f^{\rho}(t) - f(s) - f^{\nabla}(t)(\rho(t) - s)\right| \leq \varepsilon \left|\rho(t) - s\right|$$

for all $s \in V$. We call $f^{\nabla}(t)$ the nabla derivative of f at t and we say that f is nabla differentiable if f is nabla differentiable for all $t \in \mathbb{T}_{\kappa}$.

Theorem 1.15. *[3]. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_{\kappa}$. Then we have:*

(i) If f is nabla differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is left-scattered, then f is nabla differentiable at t with

$$f^{\nabla}(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$

(iii) If t is left-dense, then f is nabla differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is nabla differentiable at t, then

$$f^\rho(t) = f(t) - \nu(t)(t) f^\nabla(t).$$

Example 1.16. *If $\mathbb{T} = \mathbb{R}$, then*

$$f^\nabla(t) = f'(t).$$

If $\mathbb{T} = \mathbb{Z}$, then

$$f^\nabla(t) = f(t) - f(t-1) =: \nabla f(t).$$

Theorem 1.17. *[3]. Assume $f, g : \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}_\kappa$. Then,*

(i) The sum $f + g$ is nabla differentiable at t with

$$(f+g)^\nabla(t) = f^\nabla(t) + g^\nabla(t).$$

(ii) For any constant $\alpha \in \mathbb{R}$, αf is nabla differentiable at t with

$$(\alpha f)^\nabla(t) = \alpha f^\nabla(t).$$

(iii) The product fg is nabla differentiable at t with

$$\begin{aligned}(fg)^\nabla(t) &= f^\nabla(t)g(t) + f^\rho(t)g^\nabla(t) \\ &= f(t)g^\nabla(t) + f^\nabla(t)g^\rho(t).\end{aligned}$$

(iv) If $f(t)f^\rho(t) \neq 0$, then $\frac{1}{f}$ is nabla differentiable at t with

$$\left(\frac{1}{f}\right)^\nabla(t) = -\frac{f^\nabla(t)}{f^\rho}(t)f(t). \tag{1.3}$$

(v) If $g(t)g^\rho(t) \neq 0$, then $\frac{f}{g}$ is nabla differentiable at t with

$$\left(\frac{f}{g}\right)^\nabla(t) = \frac{f^\nabla(t)g(t) - f(t)g^\nabla(t)}{g^\rho(t)g(t)}.$$

1.2.3 Symmetric Differentiation

A third derivative, the symmetric derivative on time scales, can be seen, under certain assumptions, as a generalization of both the nabla and delta derivatives. Symmetric properties of functions are very useful in a large number of problems. Particularly in the theory of trigonometric series, applications of such properties are well known [9]. Differentiability is one of the most important properties in the theory of functions of real variables. However, even simple functions such as

$$f(t) = |t|, \quad g(t) = \begin{cases} t \sin \frac{1}{t} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases} \quad h(t) = \frac{1}{t^2}, \quad t \neq 0, \tag{1.4}$$

do not have (classical) derivative at $t = 0$. Authors like Riemann, Schwarz, Peano, Dini, and de la Vallée-Poussin, extended the classical derivative in different ways, depending on the purpose [9]. One of those notions is the symmetric derivative:

$$f^s(t) = \lim_{h \to 0} \frac{f(t+h) - f(t-h)}{2h}. \tag{1.5}$$

While the functions in (1.4) do not have ordinary derivatives at $t = 0$, they have symmetric derivatives: $f^s(0) = g^s(0) = h^s(0) = 0$. For a deeper understanding of the symmetric derivative and its properties, we refer the reader to the specialized monograph [74] and [27, 28, 32]. Here we note that the symmetric quotient $\frac{f(t+h)-f(t-h)}{2h}$ has, in general, better convergence properties than the ordinary difference quotient [45], leading naturally to the so-called h-symmetric quantum calculus [45]. In quantum calculus, the h-symmetric difference and the q-symmetric difference, $h > 0$ and $0 < q < 1$, are defined by

$$\widetilde{\mathcal{D}_h} = \frac{f(t+h) - f(t-h)}{2h} \tag{1.6}$$

and

$$\widetilde{\mathcal{D}_q} = \frac{f(qt) - f(q^{-1}t}{(q - q^{-1})t}, \quad t \neq 0, \tag{1.7}$$

respectively [45].

Definition 1.18 (See [28]). *We say that a function $f : \mathbb{T} \to \mathbb{R}$ is symmetric continuous at $t \in \mathbb{T}$ if, for any $\varepsilon > 0$, there exists a neighborhood $U_t \subset \mathbb{T}$ of t such that, for all $s \in U_t$ for which $2t - s \in U_t$, one has $|f(s) - f(2t - s)| \leq \varepsilon$.*

Note that continuity implies symmetric continuity but the reciprocal is not true [28].

Definition 1.19 (See [28]). *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}_{\kappa}$. The symmetric derivative of f at t, denoted by $f^{\diamondsuit}(t)$, is the real number, provided it exists, with the property that, for any $\varepsilon > 0$, there exists a neighborhood $U \subset \mathbb{T}$ of t such that*

$$\begin{aligned} &\left|\left[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)\right] - f^{\diamondsuit}(t)\left[\sigma(t) + 2t - 2s - \rho(t)\right]\right| \\ &\qquad\qquad\qquad\qquad \leq \varepsilon \left|\sigma(t) + 2t - 2s - \rho(t)\right| \end{aligned}$$

for all $s \in U$ for which $2t - s \in U$. A function f is said to be symmetric differentiable provided $f^{\diamondsuit}(t)$ exists for all $t \in \mathbb{T}^{\kappa}_{\kappa}$.

Some useful properties of the symmetric derivative are given in Theorem 1.20

Theorem 1.20. *[28]. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}_{\kappa}$. The following holds:*

(i) Function f has at most one symmetric derivative at t.

(ii) If f is symmetric differentiable at t, then f is symmetric continuous at t.

(iii) If f is continuous at t and t is not dense, then f is symmetric differentiable at t with

$$f^{\diamondsuit}(t) = \frac{f(\sigma(t)) - f(\rho(t))}{\sigma(t) - \rho(t)}.$$

(iv) If t is dense, then f is symmetric differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(2t-s) - f(s)}{2t - 2s}$$

exists (finite). In this case,

$$\begin{aligned} f^{\diamondsuit}(t) &= \lim_{s \to t} \frac{f(2t-s) - f(s)}{2t - 2s} \\ &= \lim_{h \to 0} \frac{f(t+h) - f(t-h)}{2h}. \end{aligned}$$

(v) If f is symmetric differentiable and continuous at t, then

$$f^{\sigma}(t) = f^{\rho}(t) + f^{\diamondsuit}(t)[\sigma(t) - \rho(t)].$$

Example 1.21. *If $\mathbb{T} = \mathbb{R}$, then the symmetric derivative coincides with the classic symmetric derivative (1.5): $f^{\diamondsuit} = f^{s}$. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then the symmetric derivative is the symmetric difference operator (1.6): $f^{\diamondsuit} = \widetilde{\mathcal{D}_h}$. If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $0 < q < 1$, then the symmetric derivative coincides with the q-symmetric difference operator (1.6): $f^{\diamondsuit} = \widetilde{\mathcal{D}_q}$.*

Remark 1.22. *Independently of the time scale* $\mathbb{T}$*, the symmetric derivative of a constant is zero and the symmetric derivative of the identity function is one.*

Remark 1.23. *An alternative way to define the symmetric derivative of* f *at* $t \in \mathbb{T}^{\kappa}_{\kappa}$ *consists in saying that the limit*

$$f^{\diamondsuit}(t) = \lim_{s\to t}\frac{f(\sigma(t) - f(s) + f(2t-s) - f^{\rho}(t)}{\sigma(t) + 2t - 2s - \rho(t)} = \lim_{h\to 0}\frac{f^{\sigma}(t) - f(t+h) + f(t-h) - f^{\rho}(t)}{\sigma(t) - 2h - \rho(t)}$$

exists.

Theorem 1.24. *[28]. Let* $f, g : \mathbb{T} \to \mathbb{R}$ *be two symmetric differentiable functions at* $t \in \mathbb{T}^{\kappa}_{\kappa}$ *and* $\lambda \in \mathbb{R}$*. The following holds:*

(i) Function $f + g$ *is symmetric differentiable at* t *with*

$$(f+g)^{\diamondsuit}(t) = f^{\diamondsuit}(t) + g^{\diamondsuit}(t).$$

(ii) Function λf *is symmetric differentiable at* t *with*

$$(\lambda f)^{\diamondsuit}(t) = \lambda f^{\diamondsuit}(t).$$

(iii) If f *and* g *are continuous at* t*, then* fg *is symmetric differentiable at* t *with*

$$(fg)^{\diamondsuit}(t) = f^{\diamondsuit}(t)g^{\sigma}(t) + f^{\rho}(t)g^{\diamondsuit}(t).$$

(iv) If f *is continuous at* t *and* $f^{\sigma}(t)f^{\rho}(t) \neq 0$*, then* $\frac{1}{f}$ *is symmetric differentiable at* t *with*

$$\left(\frac{1}{f}\right)^{\diamondsuit}(t) = -\frac{f^{\diamondsuit}(t)}{f^{\sigma}(t)f^{\rho}(t)}. \tag{1.8}$$

(v) If f *and* g *are continuous at* t *and* $g^{\sigma}(t)g^{\rho}(t) \neq 0$*, then* $\frac{f}{g}$ *is symmetric differentiable at* t *with*

$$\left(\frac{f}{g}\right)^{\diamondsuit}(t) = \frac{f^{\diamondsuit}(t)g^{\rho}(t) - f^{\rho}(t)g^{\diamondsuit}(t)}{g^{\sigma}(t)g^{\rho}(t)}. \tag{1.9}$$

Proposition 1.25. *If* f *is delta and nabla differentiable, then* f *is symmetric differentiable and, for each* $t \in \mathbb{T}^{\kappa}_{\kappa}$, $f^{\diamondsuit}(t) = \gamma(t)f^{\Delta}(t) + (1-\gamma(t))f^{\nabla}(t)$*, where*

$$\gamma(t) = \lim_{s\to t}\frac{\sigma(t) - s}{\sigma(t) + 2t - 2s - \rho(t)}. \tag{1.10}$$

Remark 1.26. *If* f *is delta and nabla differentiable and if function* $\gamma(\cdot)$ *in (1.10) is a constant,* $\gamma(t) \equiv \alpha$*, then the symmetric derivative coincides with the diamond-*α *derivative:* $f^{\diamondsuit}(t) = \alpha f^{\Delta}(t) + (1-\alpha)f^{\nabla}(t)$.

1.3 Integration on Time Scales

Similarly to differentiation, it is also possible to define different notions of integration on a time scale $\mathbb{T}$.

1.3.1 Delta Integration

Definition 1.27. *[3]. A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limit exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.*

Definition 1.28. *[3]. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{C}_{rd}$.*

Definition 1.29. *[3]. A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with (region of differentiation $\mathcal{D}$), provided $\mathcal{D} \subset \mathbb{T}^\kappa$, $\mathcal{D}$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in \mathcal{D}$.*

Theorem 1.30 (Existence of Pre-Antiderivatives [3])**.** *Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^\Delta(t) = f(t) \quad holds\ for\ all \quad t \in D.$$

Definition 1.31. *[3]. Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as in Theorem 1.30 is called a pre-antiderivative of f. We define the indefinite integral of a regulated function f by*

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f. We define the Cauchy integral by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad for\ all \quad r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^\Delta(t) = f(t) \quad holds\ for\ all \quad t \in \mathbb{T}^\kappa.$$

Theorem 1.32 (Existence of Antiderivatives [3])**.** *Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau \quad for \quad t \in \mathbb{T}$$

is an antiderivative of f.

Example 1.33. *If $\mathbb{T} = \mathbb{Z}$ and $a \neq 1$ is a constant, then*

$$\int a^t \Delta t = \frac{a^t}{a-1} + C,$$

where C is an arbitrary constant. Note that

$$\left(\frac{a^t}{a-1}\right)^{\Delta} = \Delta\left(\frac{a^t}{a-1}\right) = a^t.$$

Let $a, b \in \mathbb{T}$, $a < b$. In what follows we denote $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}$.

Theorem 1.34. *[3]. Let $a, b, c \in \mathbb{T}$, $\lambda \in \mathbb{R}$, and f, g be two rd-continuous functions. Then,*

(i) $\displaystyle\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t;$

(ii) $\displaystyle\int_a^b (\lambda f)(t)\Delta t = \lambda \int_a^b f(t)\Delta t;$

(iii) $\displaystyle\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t;$

(iv) $\displaystyle\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$

(v) $\displaystyle\int_a^a f(t)\Delta t = 0;$

(vi) if there exist $g : \mathbb{T} \to \mathbb{R}$ such that $|f(t)| \leq g(t)$ for all $t \in [a,\ b]$, then

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t;$$

(vii) if $f(t) > 0$ for all $t \in [a,\ b]$, then $\displaystyle\int_a^b f(t)\Delta t \geq 0.$

Example 1.35. *Let $a, b \in \mathbb{T}$, $a < b$, and $f \in \mathcal{C}_{rd}$.*

(i) If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, where the last integral is the usual Riemman integral.

(ii) If $\mathbb{T} = h\mathbb{Z}$ for some $h > 0$, then

$$\int_a^b f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} hf(kh).$$

1.3.2 Nabla Integration

Definition 1.36. *Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to \mathbb{R}$. We say that function f is ld-continuous if it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$. The set of ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{C}_{ld}$.*

Theorem 1.37 (See [25, 26])**.** *Every ld-continuous function $f : \mathbb{T} \to \mathbb{R}$ has a nabla antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\nabla\tau \quad \text{for} \quad t \in \mathbb{T}$$

is a nabla antiderivative of f.

Theorem 1.38. *Let $a, b, c \in \mathbb{T}$, $\lambda \in \mathbb{R}$, and f, g be two ld-continuous functions. Then,*

(i) $\int_a^b [f(t) + g(t)]\nabla t = \int_a^b f(t)\nabla t + \int_a^b g(t)\nabla t$;

(ii) $\int_a^b (\lambda f)(t)\nabla t = \lambda \int_a^b f(t)\nabla t$;

(iii) $\int_a^b f(t)\nabla t = -\int_b^a f(t)\nabla t$;

(iv) $\int_a^b f(t)\nabla t = \int_a^c f(t)\nabla t + \int_c^b f(t)\nabla t$;

(v) $\int_a^a f(t)\nabla t = 0$;

(vi) if $f(t) > 0$ for all $t \in [a,\ b]$, then $\int_a^b f(t)\nabla t \geq 0$.

Example 1.39. *Let $a, b \in \mathbb{T}$, $a < b$, and $f \in \mathcal{C}_{ld}$.*

(i) If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\nabla t = \int_a^b f(t)dt$, where the last integral is the usual Riemman integral.

(ii) If $\mathbb{T} = h\mathbb{Z}$ for some $h > 0$, then

$$\int_a^b f(t)\nabla t = \sum_{k=\frac{a}{h}+1}^{\frac{b}{h}} hf(kh).$$

For more on nabla and delta integrals and their generalizations, we refer the reader to [25, 55, 62].

Chapter 2

Fractional Calculus on Arbitrary Time Scales

The original results of this chapter are published in [21].

2.1 Introduction

In this chapter, we introduce a general notion of fractional (noninteger) derivative for functions defined on arbitrary time scales. The basic tools for the time-scale fractional calculus (fractional differentiation and fractional integration) are then developed. As particular cases, one obtains the usual time-scale Hilger derivative when the order of differentiation is one, and a local approach to fractional calculus when the time scale is chosen to be the set of real numbers.

Fractional calculus refers to differentiation and integration of an arbitrary (noninteger) order. The theory goes back to mathematicians as Leibniz (1646–1716), Liouville (1809–1882), Riemann (1826–1866), Letnikov (1837–1888), and Grünwald (1838–1920) [47, 70]. During the last two decades, fractional calculus has increasingly attracted the attention of researchers of many different fields [1, 14, 15, 53, 56, 59, 66, 77].

The time-scale calculus can be used to unify discrete and continuous approaches to signal processing in one unique setting. Interesting in applications, is the possibility to deal with more complex time domains. One extreme case, covered by the theory of time scales and surprisingly relevant also for the process of signals, appears when one fix the time scale to be the Cantor set [16, 78]. The application of the local fractional derivative

in a time scale different from the classical time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$ was proposed by Kolwankar and Gangal themselves: see [50, 51] where nondifferentiable signals defined on the Cantor set are considered.
Our objective in this chapter is to develop the basic tools of any fractional calculus: fractional differentiation (Section 2.2) and fractional integration (Section 2.3).

2.2 Fractional Differentiation

In this section, we begin by introducing a new notion: the fractional derivative of order $\alpha \in]0,1]$ for functions defined on arbitrary time scales. For $\alpha = 1$ we obtain the usual delta derivative of the time-scale calculus.

Definition 2.1. *Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa$, and $\alpha \in]0,1]$. For $\alpha \in]0,1] \cap \{1/q : q$ is a odd number$\}$ (resp. $\alpha \in]0,1] \setminus \{1/q : q$ is a odd number$\}$) we define $f^{(\alpha)}(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a δ-neighborhood $\mathcal{U} \subset \mathbb{T}$ of t (resp. left δ-neighborhood $\mathcal{U}^- \subset \mathbb{T}$ of t), $\delta > 0$, such that*

$$\left|[f(\sigma(t)) - f(s)] - f^{(\alpha)}(t)\,[\sigma(t) - s]^\alpha\right| \leq \epsilon\,|\sigma(t) - s|^\alpha$$

for all $s \in \mathcal{U}$ (resp. $s \in \mathcal{U}^-$). We call $f^{(\alpha)}(t)$ the fractional derivative of f of order α at t.

In this section we develop the basic tools of fractional differentiation. Along the text α is a real number in the interval $]0,1]$. The next theorem provides some useful relationships concerning the fractional derivative on time scales introduced in Definition 2.1.

Theorem 2.2. *Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. The following properties hold:*

(i) *Let $\alpha \in]0,1] \cap \left\{\frac{1}{q} : q \text{ is a odd number}\right\}$. If t is right-dense and if f is fractional differentiable of order α at t, then f is continuous at t.*

(ii) *Let $\alpha \in]0,1] \setminus \left\{\frac{1}{q} : q \text{ is a odd number}\right\}$. If t is right-dense and if f is fractional differentiable of order α at t, then f is left-continuous at t.*

(iii) *If f is continuous at t and t is right-scattered, then f is fractional differentiable of order α at t with*

$$f^{(\alpha)}(t) = \frac{f^\sigma(t) - f(t)}{(\mu(t))^\alpha}.$$

(iv) *Let* $\alpha \in]0,1] \cap \left\{\frac{1}{q} : q \text{ is a odd number}\right\}$. *If* t *is right-dense, then* f *is fractional differentiable of order* α *at* t *if, and only if, the limit*

$$\lim_{s\to t} \frac{f(t)-f(s)}{(t-s)^{\alpha}}$$

exists as a finite number. In this case,

$$f^{(\alpha)}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{(t-s)^{\alpha}}.$$

(v) *Let* $\alpha \in]0,1] \setminus \left\{\frac{1}{q} : q \text{ is a odd number}\right\}$. *If* t *is right-dense, then* f *is fractional differentiable of order* α *at* t *if, and only if, the limit*

$$\lim_{s\to t^-} \frac{f(t)-f(s)}{(t-s)^{\alpha}}$$

exists as a finite number. In this case,

$$f^{(\alpha)}(t) = \lim_{s\to t^-} \frac{f(t)-f(s)}{(t-s)^{\alpha}}.$$

(vi) *If* f *is fractional differentiable of order* α *at* t, *then* $f(\sigma(t)) = f(t) + (\mu(t))^{\alpha} f^{(\alpha)}(t)$.

Proof. (*i*) Assume that f is fractional differentiable at t. Then, there exists a neighborhood $\mathcal{U}$ of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^{(\alpha)}(t)\,[\sigma(t)-s]^{\alpha}\right| \leq \epsilon\,|\sigma(t)-s|^{\alpha}$$

for $s \in \mathcal{U}$. Therefore, for all $s \in \mathcal{U} \cap]t-\epsilon, t+\epsilon[$,

$$\begin{aligned}|f\,(t) - f\,(s)| &\leq \left|[f^{\sigma}(t) - f(s)] - f^{(\alpha)}(t)\,[\sigma(t)-s]^{\alpha}\right| \\ &\quad + \left|[f^{\sigma}(t) - f(t)] - f^{(\alpha)}(t)\,[\sigma(t)-t]^{\alpha}\right| + \left|f^{(\alpha)}(t)\right| \left|[\sigma(t)-s]^{\alpha} - [\sigma(t)-t]^{\alpha}\right|\end{aligned}$$

and, since t is a right-dense point,

$$\begin{aligned}|f\,(t) - f\,(s)| &\leq \left|[f^{\sigma}(t) - f(s)] - f^{(\alpha)}(t)\,[\sigma(t)-s]^{\alpha}\right| + \left|f^{(\alpha)}(t)\,[t-s]^{\alpha}\right| \\ &\leq \epsilon\,|t-s|^{\alpha} + \left|f^{(\alpha)}(t)\right|\,|t-s|^{\alpha} \\ &\leq \epsilon^{\alpha}\left[\epsilon + \left|f^{(\alpha)}(t)\right|\right].\end{aligned}$$

It follows the continuity of f at t.

(ii) The proof is similar to the proof of (i), where instead of considering the neighborhood $\mathcal{U}$ of t we consider a left neighborhood $\mathcal{U}^-$ of t.

(iii) Assume that f is continuous at t and t is right-scattered. By continuity,

$$\lim_{s\to t}\frac{f^\sigma(t)-f(s)}{(\sigma(t)-s)^\alpha}=\frac{f^\sigma(t)-f(t)}{(\sigma(t)-t)^\alpha}=\frac{f^\sigma(t)-f(t)}{(\mu(t))^\alpha}.$$

Hence, given $\epsilon>0$ and $\alpha\in]0,1]\cap\{1/q : q \text{ is a odd number}\}$, there is a neighborhood $\mathcal{U}$ of t (or $\mathcal{U}^-$ if $\alpha\in]0,1]\setminus\{1/q : q \text{ is a odd number}\}$) such that

$$\left|\frac{f^\sigma(t)-f(s)}{(\sigma(t)-s)^\alpha}-\frac{f^\sigma(t)-f(t)}{(\mu(t))^\alpha}\right|\leq\epsilon$$

for all $s\in\mathcal{U}$ (resp. $\mathcal{U}^-$). It follows that

$$\left|[f^\sigma(t)-f(s)]-\frac{f^\sigma(t)-f(t)}{(\mu(t))^\alpha}(\sigma(t)-s)^\alpha\right|\leq\epsilon|\sigma(t)-s|^\alpha$$

for all $s\in\mathcal{U}$ (resp. $\mathcal{U}^-$). Hence, we get the desired result:

$$f^{(\alpha)}(t)=\frac{f^\sigma(t)-f(t)}{(\mu(t))^\alpha}.$$

(iv) Assume that f is fractional differentiable of order α at t and t is right-dense. Let $\epsilon>0$ be given. Since f is fractional differentiable of order α at t, there is a neighborhood $\mathcal{U}$ of t such that

$$\left|[f^\sigma(t)-f(s)]-f^{(\alpha)}(t)(\sigma(t)-s)^\alpha\right|\leq\epsilon|\sigma(t)-s|^\alpha$$

for all $s\in\mathcal{U}$. Since $\sigma(t)=t$,

$$\left|[f(t)-f(s)]-f^{(\alpha)}(t)(t-s)^\alpha\right|\leq\epsilon|t-s|^\alpha$$

for all $s\in\mathcal{U}$. It follows that

$$\left|\frac{f(t)-f(s)}{(t-s)^\alpha}-f^{(\alpha)}(t)\right|\leq\epsilon$$

for all $s\in\mathcal{U}$, $s\neq t$. Therefore, we get the desired result:

$$f^{(\alpha)}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{(t-s)^\alpha}.$$

Now assume that

$$\lim_{s\to t}\frac{f(t)-f(s)}{(t-s)^\alpha}$$

exists and is equal to L and t is right-dense. Then, there exists $\mathcal{U}$ such that

$$\left|\frac{f(t)-f(s)}{(t-s)^{\alpha}}-L\right|\leq\epsilon$$

for all $s\in\mathcal{U}$. Because t is right-dense,

$$\left|\frac{f^{\sigma}(t)-f(s)}{(\sigma\,(t)-s)^{\alpha}}-L\right|\leq\epsilon.$$

Therefore,

$$|[f^{\sigma}(t)-f(s)]-L\,(\sigma(t)-s)^{\alpha}|\leq\epsilon|\sigma\,(t)-s|^{\alpha},$$

which lead us to the conclusion that f is fractional differentiable of order α at t and $f^{(\alpha)}(t)=L$.

(v) The proof is similar to the proof of (iv), where instead of considering the neighborhood $\mathcal{U}$ of t we consider a left-neighborhood $\mathcal{U}^{-}$ of t.

(vi) If $\sigma(t)=t$, then $\mu(t)=0$ and

$$f^{\sigma}(t))=f(t)=f(t)+(\mu(t))^{\alpha}f^{(\alpha)}(t).$$

On the other hand, if $\sigma(t)>t$, then by (iii)

$$f^{\sigma}(t)=f(t)+(\mu(t))^{\alpha}\cdot\frac{f^{\sigma}(t)-f(t)}{(\mu(t))^{\alpha}}=f(t)+(\mu(t))^{\alpha}f^{(\alpha)}(t).$$

The proof is complete. □

Remark 2.3. *In a time scale $\mathbb{T}$, due to the inherited topology of the real numbers, a function f is always continuous at any isolated point t.*

Proposition 2.4. *If $f:\mathbb{T}\to\mathbb{R}$ is defined by $f(t)=c$ for all $t\in\mathbb{T}$, $c\in\mathbb{R}$, then $f^{(\alpha)}(t)\equiv 0$.*

Proof. If t is right-scattered, then, by Theorem 2.2 (iii), one has

$$f^{(\alpha)}(t)=\frac{f(\sigma(t))-f(t)}{(\mu(t))^{\alpha}}=\frac{c-c}{(\mu(t))^{\alpha}}=0.$$

Assume t is right-dense. Then, by Theorem 2.2 (iv) and (v), it follows that

$$f^{(\alpha)}(t)=\lim_{s\to t}\frac{c-c}{(t-s)^{\alpha}}=0.$$

This concludes the proof. □

Proposition 2.5. *If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then*

$$f^{(\alpha)}(t) = \begin{cases} (\mu(t))^{1-\alpha} & \text{if } \alpha \neq 1, \\ 1 & \text{if } \alpha = 1. \end{cases}$$

Proof. From Theorem 2.2 (vi) it follows that $\sigma(t) = t + (\mu(t))^{\alpha} f^{(\alpha)}(t)$, that is, $\mu(t) = (\mu(t))^{\alpha} f^{(\alpha)}(t)$. If $\mu(t) \neq 0$, then $f^{(\alpha)}(t) = (\mu(t))^{1-\alpha}$ and the desired relation is proved. Assume now that $\mu(t) = 0$, that is, $\sigma(t) = t$. In this case t is right-dense and by Theorem 2.2 (iv) and (v) it follows that

$$f^{(\alpha)}(t) = \lim_{s \to t} \frac{t - s}{(t - s)^{\alpha}}.$$

Therefore, if $\alpha = 1$, then $f^{(\alpha)}(t) = 1$; if $0 < \alpha < 1$, then $f^{(\alpha)}(t) = 0$. The proof is complete. □

Let us consider now the two classical cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$, $h > 0$.

Corollary 2.6. *Function $f : \mathbb{R} \to \mathbb{R}$ is fractional differentiable of order α at point $t \in \mathbb{R}$ if, and only if, the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{(t - s)^{\alpha}}$$

exists as a finite number. In this case,

$$f^{(\alpha)}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{(t - s)^{\alpha}}. \tag{2.1}$$

Proof. Here $\mathbb{T} = \mathbb{R}$ and all points are right-dense. The result follows from Theorem 2.2 (iv) and (v). Note that if $\alpha \in]0, 1] \setminus \left\{ \frac{1}{q} : q \text{ is a odd number} \right\}$, then the limit only makes sense as a left-side limit. □

Remark 2.7. *Definition 2.1 corresponds to the well-known Kolwankar–Gangal approach to fractional calculus [49, 75].*

Corollary 2.8. *Let $h > 0$. If $f : h\mathbb{Z} \to \mathbb{R}$, then f is fractional differentiable of order α at $t \in h\mathbb{Z}$ with*

$$f^{(\alpha)}(t) = \frac{f(t + h) - f(t)}{h^{\alpha}}.$$

Proof. Here $\mathbb{T} = h\mathbb{Z}$ and all points are right-scattered. The result follows from Theorem 2.2 (iii). □

We now give an example using a more sophisticated time scale: the Cantor set.

Example 2.9. *Let $\mathbb{T}$ be the Cantor set. It is known (see Example 1.47 of [25]) that $\mathbb{T}$ does not contain any isolated point, and that*

$$\sigma(t) = \begin{cases} t + \frac{1}{3^{m+1}} & \text{if } t \in L, \\ t & \text{if } t \in \mathbb{T} \setminus L, \end{cases}$$

where

$$L = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} : m \in \mathbb{N} \text{ and } a_k \in \{0,2\} \text{ for all } 1 \leq k \leq m \right\}.$$

Thus,

$$\mu(t) = \begin{cases} \frac{1}{3^{m+1}} & \text{if } t \in L, \\ 0 & \text{if } t \in \mathbb{T} \setminus L. \end{cases}$$

Let $f : \mathbb{T} \to \mathbb{R}$ be continuous and $\alpha \in]0,1]$. It follows from Theorem 2.2 that the fractional derivative of order α of a function f defined on the Cantor set is given by

$$f^{(\alpha)}(t) = \begin{cases} \left[f\left(t + \frac{1}{3^{m+1}}\right) - f(t)\right] 3^{(m+1)\alpha} & \text{if } t \in L, \\ \lim\limits_{s \rightsquigarrow t} \dfrac{f(t) - f(s)}{(t-s)^{\alpha}} & \text{if } t \in \mathbb{T} \setminus L, \end{cases}$$

where $\lim_{s \rightsquigarrow t} = \lim_{s \to t}$ if $\alpha = \frac{1}{q}$ with q an odd number, and $\lim_{s \rightsquigarrow t} = \lim_{s \to t^-}$ otherwise.

For the fractional derivative on time scales to be useful, we would like to know formulas for the derivatives of sums, products and quotients of fractional differentiable functions. This is done according to the following theorem.

Theorem 2.10. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are fractional differentiable of order α at $t \in \mathbb{T}^{\kappa}$. Then,*

(i) *the sum $f + g : \mathbb{T} \to \mathbb{R}$ is fractional differentiable at t with $(f+g)^{(\alpha)}(t) = f^{(\alpha)}(t) + g^{(\alpha)}(t)$;*

(ii) *for any constant λ, $\lambda f : \mathbb{T} \to \mathbb{R}$ is fractional differentiable at t with $(\lambda f)^{(\alpha)}(t) = \lambda f^{(\alpha)}(t)$;*

(iii) *if f and g are continuous, then the product $fg : \mathbb{T} \to \mathbb{R}$ is fractional differentiable at t with*

$$\begin{aligned} (fg)^{(\alpha)}(t) &= f^{(\alpha)}(t)g(t) + f(\sigma(t))g^{(\alpha)}(t) \\ &= f^{(\alpha)}(t)g(\sigma(t)) + f(t)g^{(\alpha)}(t); \end{aligned}$$

(iv) *if f is continuous and $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is fractional differentiable at t with*

$$\left(\frac{1}{f}\right)^{(\alpha)}(t) = -\frac{f^{(\alpha)}(t)}{f(t)f(\sigma(t))};$$

(v) *if f and g are continuous and $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is fractional differentiable at t with*

$$\left(\frac{f}{g}\right)^{(\alpha)}(t) = \frac{f^{(\alpha)}(t)g(t) - f(t)g^{(\alpha)}(t)}{g(t)g(\sigma(t))}.$$

Proof. Let us consider that $\alpha \in]0,1] \cap \left\{\frac{1}{q} : q \text{ is a odd number}\right\}$. The proofs for the case $\alpha \in]0,1] \setminus \left\{\frac{1}{q} : q \text{ is a odd number}\right\}$ are similar: one just needs to choose the proper left-sided neighborhoods. Assume that f and g are fractional differentiable at $t \in \mathbb{T}^{\kappa}$. (i) Let $\epsilon > 0$. Then there exist neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of t for which

$$\left|f(\sigma(t)) - f(s) - f^{(\alpha)}(t)[\sigma(t) - s]^{\alpha}\right| \leq \frac{\epsilon}{2}|\sigma(t) - s|^{\alpha} \quad \textit{for all} \quad s \in \mathcal{U}_1$$

and

$$\left|g(\sigma(t)) - g(s) - g^{(\alpha)}(t)[\sigma(t) - s]^{\alpha}\right| \leq \frac{\epsilon}{2}|\sigma(t) - s|^{\alpha} \quad \textit{for all} \quad s \in \mathcal{U}_2.$$

Let $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$. Then

$$\begin{aligned}
&\left|(f+g)(\sigma(t)) - (f+g)(s) - \left[f^{(\alpha)}(t) + g^{(\alpha)}(t)\right](\sigma(t) - s)^{\alpha}\right| \\
&\quad = \left|f(\sigma(t)) - f(s) - f^{(\alpha)}(t)[\sigma(t) - s]^{\alpha} + g(\sigma(t)) - g(s) - g^{(\alpha)}(t)[\sigma(t) - s]^{\alpha}\right| \\
&\quad \leq \left|f(\sigma(t)) - f(s) - f^{(\alpha)}(t)[\sigma(t) - s]^{\alpha}\right| + \left|g(\sigma(t)) - g(s) - g^{(\alpha)}(t)[\sigma(t) - s]^{\alpha}\right| \\
&\quad \leq \frac{\epsilon}{2}|\sigma(t) - s|^{\alpha} + \frac{\epsilon}{2}|\sigma(t) - s|^{\alpha} = \epsilon|\sigma(t) - s|^{\alpha}
\end{aligned}$$

for all $s \in \mathcal{U}$. Therefore, $f + g$ is fractional differentiable at t and

$$(f+g)^{(\alpha)}(t) = f^{\alpha}(t) + g^{(\alpha)}(t).$$

(ii) Let $\epsilon > 0$. Then there exists a neighborhood $\mathcal{U}$ of t with

$$\left|f(\sigma(t)) - f(s) - f^{(\alpha)}(t)[\sigma(t) - s]^{\alpha}\right| \leq \epsilon|\sigma(t) - s|^{\alpha} \text{ for all } s \in \mathcal{U}.$$

It follows that

$$\left|(\lambda f)(\sigma(t)) - (\lambda f)(s) - \lambda f^{(\alpha)}(t)[\sigma(t) - s]^{\alpha}\right| \leq \epsilon|\lambda|\,|\sigma(t) - s|^{\alpha} \text{ for all } s \in \mathcal{U}.$$

Therefore, λf is fractional differentiable at t and $(\lambda f)^{\alpha} = \lambda f^{(\alpha)}$ holds at t.
(iii) If t is right-dense, then

$$\begin{aligned}(fg)^{(\alpha)}(t) &= \lim_{s\to t} \frac{(fg)(t) - (fg)(s)}{(t-s)^{\alpha}} \\ &= \lim_{s\to t} \frac{f(t)-f(s)}{(t-s)^{\alpha}} g(t) + \lim_{s\to t} \frac{g(t)-g(s)}{(t-s)^{\alpha}} f(s) \\ &= f^{(\alpha)}(t)g(t) + g^{(\alpha)}(t)f(t) \\ &= f^{(\alpha)}(t)g(t) + f(\sigma(t))g^{(\alpha)}(t).\end{aligned}$$

If t is right-scattered, then

$$\begin{aligned}(fg)^{(\alpha)}(t) &= \frac{(fg)^{\sigma}(t) - (fg)(t)}{(\mu(t))^{\alpha}} \\ &= \frac{f^{\sigma}(t) - f(t)}{(\mu(t))^{\alpha}} g(t) + \frac{g^{\sigma}(t)-g(t)}{(\mu(t))^{\alpha}} f^{\sigma}(t) \\ &= f^{(\alpha)}(t)g(t) + f(\sigma(t))g^{(\alpha)}(t).\end{aligned}$$

The other product rule formula follows by interchanging in $(fg)^{(\alpha)}(t) = f^{(\alpha)}(t)g(t) + f(\sigma(t))g^{(\alpha)}(t)$ the functions f and g. *(iv)* We use the fractional derivative of a constant (Proposition 2.4) and Theorem 2.10 *(iii)* just proved: from Proposition 2.4 we know that

$$\left(f \cdot \frac{1}{f}\right)^{(\alpha)}(t) = (1)^{(\alpha)}(t) = 0$$

and, therefore, by (iii)

$$\left(\frac{1}{f}\right)^{(\alpha)}(t) f(\sigma(t)) + f^{(\alpha)}(t)\frac{1}{f(t)} = 0.$$

Since we are assuming $f(\sigma(t)) \neq 0$,

$$\left(\frac{1}{f}\right)^{(\alpha)}(t) = -\frac{f^{(\alpha)}(t)}{f(t)f(\sigma(t))}.$$

For the quotient formula *(v)*, we use *(ii)* and *(iv)* to calculate

$$\begin{aligned}\left(\frac{f}{g}\right)^{(\alpha)}(t) &= \left(f \cdot \frac{1}{g}\right)^{(\alpha)}(t) \\ &= f(t)\left(\frac{1}{g}\right)^{(\alpha)}(t) + f^{(\alpha)}(t)\frac{1}{g(\sigma(t))} \\ &= -f(t)\frac{g^{(\alpha)}(t)}{g(t)g(\sigma(t))} + f^{(\alpha)}(t)\frac{1}{g(\sigma(t))} \\ &= \frac{f^{(\alpha)}(t)g(t) - f(t)g^{(\alpha)}(t)}{g(t)g(\sigma(t))}.\end{aligned}$$

This conclude the proof. □

The following theorem is proved in [25] for $\alpha = 1$. Here we show its validity for $\alpha \in]0,1[$.

Theorem 2.11. *Let c be a constant, $m \in \mathbb{N}$, and $\alpha \in]0,1[$.*

(i) *If $f(t) = (t-c)^m$, then*

$$f^{(\alpha)}(t) = (\mu(t))^{1-\alpha} \sum_{\nu=0}^{m-1} (\sigma(t)-c)^{\nu} (t-c)^{m-1-\nu}.$$

(ii) *If $g(t) = \frac{1}{(t-c)^m}$, then*

$$g^{(\alpha)}(t) = -(\mu(t))^{1-\alpha} \sum_{\nu=0}^{m-1} \frac{1}{(\sigma(t)-c)^{m-\nu}(t-c)^{\nu+1}},$$

provided $(t-c)(\sigma(t)-c) \neq 0$.

Proof. We prove the first formula by induction. If $m = 1$, then $f(t) = t - c$ and

$$f^{(\alpha)}(t) = (\mu(t))^{1-\alpha}$$

holds from Propositions 2.4 and 2.5 and Theorem 2.10 (i). Now assume that

$$f^{(\alpha)}(t) = (\mu(t))^{1-\alpha} \sum_{\nu=0}^{m-1} (\sigma(t)-c)^{\nu}(t-c)^{m-1-\nu}$$

holds for $f(t) = (t-c)^m$ and let $F(t) = (t-c)^{m+1} = (t-c)f(t)$. We use the product rule (Theorem 2.10 (iii)) to obtain

$$\begin{aligned}
F^{(\alpha)}(t) &= (t-c)^{(\alpha)} f(\sigma(t)) + f^{(\alpha)}(t)(t-c) = (\mu(t))^{1-\alpha} f(\sigma(t)) + f^{(\alpha)}(t)(t-c) \\
&= (\mu(t))^{1-\alpha}(\sigma(t)-c)^m + (\mu(t))^{1-\alpha}(t)(t-c) \sum_{\nu=0}^{m-1} (\sigma(t)-c)^{\nu}(t-c)^{m-1-\nu} \\
&= (\mu(t))^{1-\alpha} \left[(\sigma(t)-c)^m + \sum_{\nu=0}^{m-1} (\sigma(t)-c)^{\nu}(t-c)^{m-\nu} \right] \\
&= (\mu(t))^{1-\alpha} \sum_{\nu=0}^{m} (\sigma(t)-c)^{\nu}(t-c)^{m-\nu}.
\end{aligned}$$

Hence, by mathematical induction, part (i) holds. For $g(t) = \frac{1}{(t-c)^m} = \frac{1}{f(t)}$, we apply Theorem 2.10 (iv) to obtain

$$\begin{aligned} g^{(\alpha)}(t) &= -\frac{f^{(\alpha)}(t)}{f(t)f(\sigma(t))} = -(\mu(t))^{1-\alpha}\frac{\sum_{\nu=0}^{m-1}(\sigma(t)-c)^{\nu}(t-c)^{m-1-\nu}}{(t-c)^m(\sigma(t)-c)^m} \\ &= -(\mu(t))^{1-\alpha}\sum_{\nu=0}^{m-1}\frac{1}{(t-c)^{\nu+1}(\sigma(t)-c)^{m-\nu}}, \end{aligned}$$

provided $(t-c)\,(\sigma(t)-c) \neq 0$. □

Let us illustrate Theorem 2.11 in special cases.

Example 2.12. *Let* $\alpha \in]0,1[$.

(i) *If* $f(t) = t^2$, *then* $f^{(\alpha)}(t) = (\mu(t))^{1-\alpha}[\sigma(t)+t]$.

(ii) *If* $f(t) = t^3$, *then* $f^{(\alpha)}(t) = (\mu(t))^{1-\alpha}[t^2 + t\sigma(t) + (\sigma(t))^2]$.

(iii) *If* $f(t) = \frac{1}{t}$, *then* $f^{(\alpha)}(t) = -\frac{(\mu(t))^{1-\alpha}}{t\sigma(t)}$.

From the results already obtained, it is not difficult to see that the fractional derivative does not satisfy a chain rule like $(f \circ g)^{(\alpha)}(t) = f^{(\alpha)}(g(t))g^{(\alpha)}(t)$:

Example 2.13. *Let* $\alpha \in]0,1[$. *Consider* $f(t) = t^2$ *and* $g(t) = 2t$. *Then,*

$$(f \circ g)^{(\alpha)}(t) = \left(4t^2\right)^{(\alpha)} = 4(\mu(t))^{1-\alpha}\,(\sigma(t)+t) \tag{2.2}$$

while

$$f^{(\alpha)}(g(t))g^{(\alpha)}(t) = (\mu(2t))^{1-\alpha}\,(\sigma(2t)+2t)\,2(\mu(t))^{1-\alpha} \tag{2.3}$$

and, for example for $\mathbb{T} = \mathbb{Z}$, *it is easy to see that* $(f \circ g)^{(\alpha)}(t) \neq f^{(\alpha)}(g(t))g^{(\alpha)}(t)$.

Note that when $\alpha = 1$ and $\mathbb{T} = \mathbb{R}$ our derivative $f^{(\alpha)}$ reduces to the standard derivative f' and, in this case, both expressions (2.2) and (2.3) give $8t$, as expected. In the fractional case $\alpha \in]0,1[$ we are able to prove the following result, valid for an arbitrary time scale $\mathbb{T}$.

Theorem 2.14 (Chain rule)**.** *Let* $\alpha \in]0,1[$. *Assume* $g : \mathbb{R} \to \mathbb{R}$ *is continuous,* $g : \mathbb{T} \to \mathbb{R}$ *is fractional differentiable of order* α *at* $t \in \mathbb{T}^\kappa$, *and* $f : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable. Then there exists* c *in the real interval* $[t, \sigma(t)]$ *with*

$$(f \circ g)^{(\alpha)}(t) = f'(g(c))g^{(\alpha)}(t). \tag{2.4}$$

Proof. Let $t \in \mathbb{T}^\kappa$. First we consider t to be right-scattered. In this case

$$(f \circ g)^{(\alpha)}(t) = \frac{f(g(\sigma(t))) - f(g(t))}{(\mu(t))^{(\alpha)}}.$$

If $g(\sigma(t)) = g(t)$, then we get $(f \circ g)^{(\alpha)}(t) = 0$ and $g^{(\alpha)}(t) = 0$. Therefore, (2.4) holds for any c in the real interval $[t, \sigma(t)]$ and we can assume $g(\sigma(t)) \neq g(t)$. By the mean value theorem,

$$\begin{aligned}(f \circ g)^{(\alpha)}(t) &= \frac{f(g(\sigma(t))) - f(g(t))}{g(\sigma(t)) - g(t)} \cdot \frac{g(\sigma(t)) - g(t)}{(\mu(t))^{(\alpha)}} \\ &= f'(\xi) g^{(\alpha)}(t),\end{aligned}$$

where ξ is between $g(t)$ and $g(\sigma(t))$. Since $g : \mathbb{R} \to \mathbb{R}$ is continuous, there is a $c \in [t, \sigma(t)]$ such that $g(c) = \xi$, which gives us the desired result. Now consider the case when t is right-dense. In this case

$$\begin{aligned}(f \circ g)^{(\alpha)}(t) &= \lim_{s \to t} \frac{f(g(t)) - f(g(s))}{g(t) - g(s)} \cdot \frac{g(t) - g(s)}{(t - s)^{(\alpha)}} \\ &= \lim_{s \to t} \left\{ f'(\xi_s) . \frac{g(t) - g(s)}{(t - s)^{(\alpha)}} \right\}\end{aligned}$$

by the mean value theorem, where ξ_s is between $g(s)$ and $g(t)$. By the continuity of g we get that $\lim_{s \to t} \xi_s = g(t)$, which gives us the desired result. □

Example 2.15. *Let $\mathbb{T} = \mathbb{Z}$, for which $\sigma(t) = t + 1$ and $\mu(t) \equiv 1$, and consider the same functions of Example 2.13: $f(t) = t^2$ and $g(t) = 2t$. We can find directly the value c, guaranteed by Theorem 2.14 in the interval $[4, \sigma(4)] = [4, 5]$, so that*

$$(f \circ g)^{(\alpha)}(4) = f'(g(c)) g^{(\alpha)}(4). \tag{2.5}$$

From (2.2) it follows that $(f \circ g)^{(\alpha)}(4) = 36$. Because $g^{(\alpha)}(4) = 2$ and $f'(g(c)) = 4c$, equality (2.5) simplifies to $36 = 8c$, and so $c = \frac{9}{2}$.

We end Section 2.2 explaining how to compute fractional derivatives of higher-order. As usual, we define the derivative of order zero as the identity operator: $f^{(0)} = f$.

Definition 2.16. *Let β be a nonnegative real number. We define the fractional derivative of f of order β by*

$$f^{(\beta)} := \left(f^{\Delta^N}\right)^{(\alpha)},$$

where $N := \lfloor \beta \rfloor$ (that is, N is the integer part of β) and $\alpha := \beta - N$.

Note that the α of Definition 2.16 is in the interval $[0, 1]$. We illustrate Definition 2.16 with some examples.

Example 2.17. *If $f(t) = c$ for all $t \in \mathbb{T}$, c a constant, then $f^{(\beta)} \equiv 0$ for any $\beta \in \mathbb{R}_0^+$.*

Example 2.18. *Let $f(t) = t^2$, $\mathbb{T} = h\mathbb{Z}$, $h > 0$, and $\beta = 1.3$. Then, by Definition 2.16, we have $f^{(1.3)} = \left(f^{\Delta}\right)^{(0.3)}$. It follows from $\sigma(t) = t + h$ that $f^{(1.3)}(t) = (2t + h)^{(0.3)}$. Proposition 2.4 and Theorem 2.10 (i) and (ii) allow us to write that $f^{(1.3)}(t) = 2(t)^{(0.3)}$. We conclude from Proposition 2.5 with $\mu(t) \equiv h$ that $f^{(1.3)}(t) = 2h^{0.7}$.*

2.3 Fractional Integration

The two major ingredients of any calculus are differentiation and integration. Now we introduce the fractional integral on time scales.

Definition 2.19. *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. We define the indefinite fractional integral of f of order β, $0 \leq \beta \leq 1$, by*

$$\int f(t)\Delta^{\beta}t := \left(\int f(t)\Delta t\right)^{(1-\beta)},$$

where $\int f(t)\Delta t$ is the usual indefinite integral of time scales [25].

Remark 2.20. *It follows from Definition 2.19 that $\int f(t)\Delta^1 t = \int f(t)\Delta t$ and $\int f(t)\Delta^0 t = f(t)$.*

Definition 2.21. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Let*

$$F^{\beta}(t) = \int f(t)\Delta^{\beta}t$$

denote the indefinite fractional integral of f of order β with $0 \leq \beta \leq 1$. We define the Cauchy fractional integral by

$$\int_a^b f(t)\Delta^{\beta}t := F^{\beta}(t)\big|_a^b = F^{\beta}(b) - F^{\beta}(a), \quad a, b \in \mathbb{T}.$$

The next theorem gives some properties of the fractional integral of order β.

Theorem 2.22. *If $a, b, c \in \mathbb{T}$, $\xi \in \mathbb{R}$, and $f, g \in \mathcal{C}_{rd}$ with $0 \leq \beta \leq 1$, then*

(i) $\int_a^b [f(t) + g(t)]\Delta^{\beta}t = \int_a^b f(t)\Delta^{\beta}t + \int_a^b g(t)\Delta^{\beta}t$;

(ii) $\int_a^b (\xi f)(t)\Delta^\beta t = \xi \int_a^b f(t)\Delta^\beta t$;

(iii) $\int_a^b f(t)\Delta^\beta t = -\int_b^a f(t)\Delta^\beta t$;

(iv) $\int_a^b f(t)\Delta^\beta t = \int_a^c f(t)\Delta^\beta t + \int_c^b f(t)\Delta^\beta t$;

(v) $\int_a^a f(t)\Delta^\beta t = 0$.

Proof. The equalities follow from Definition 2.19 and Definition 2.21, analogous properties of the delta integral of time scales, and the properties of Section 2.2 for the fractional derivative on time scales. (i) From Definition 2.21

$$\int_a^b (f+g)(t)\Delta^\beta t = \int (f(t)+g(t))\,\Delta^\beta t\Big|_a^b$$

and, from Definition 2.19,

$$\int_a^b (f+g)(t)\Delta^\beta t = \left(\int (f(t)+g(t))\,\Delta t\right)^{(1-\beta)}\Bigg|_a^b .$$

It follows from the properties of the delta integral and Theorem 2.10 (i) that

$$\int_a^b (f+g)(t)\Delta^\beta t = \left(\int f(t)\Delta t\right)^{(1-\beta)} + \left(\int g(t)\Delta t\right)^{(1-\beta)}\Bigg|_a^b .$$

Using again Definition 2.19 and Definition 2.21, we arrive to the intended relation:

$$\begin{aligned}\int_a^b (f+g)(t)\Delta^\beta t &= \int f(t)\Delta^\beta t + \int g(t)\Delta^\beta t\Big|_a^b\\ &= F^\beta(t) + G^\beta(t)\big|_a^b = F^\beta(b) + G^\beta(b) - F^\beta(a) - G^\beta(a)\\ &= \int_a^b f(t)\Delta^\beta t + \int_a^b g(t)\Delta^\beta t.\end{aligned}$$

(ii) From Definition 2.21 and Definition 2.19 one has

$$\int_a^b (\xi f)(t)\Delta^\beta t = \int (\xi f)(t)\Delta^\beta t\Big|_a^b = \left(\int (\xi f)(t)\Delta t\right)^{(1-\beta)}\Bigg|_a^b .$$

It follows from the properties of the delta integral and Theorem 2.10 (ii) that

$$\int_a^b (\xi f)(t)\Delta^\beta t = \xi\left(\int f(t)\Delta t\right)^{(1-\beta)}\Bigg|_a^b .$$

We conclude the proof of (ii) by using again Definition 2.19 and Definition 2.21:

$$\begin{aligned}\int_a^b (\xi f)(t)\Delta^\beta t &= \xi \int f(t)\Delta^\beta t\Big|_a^b = \xi F^\beta(t)\big|_a^b = \xi\left(F^\beta(b) - F^\beta(a)\right)\\ &= \xi \int_a^b f(t)\Delta^\beta t.\end{aligned}$$

The last three properties are direct consequences of Definition 2.21:

(iii)

$$\begin{aligned}\int_a^b f(t)\Delta^\beta t &= F^\beta(b) - F^\beta(a) = -\left(F^\beta(a) - F^\beta(b)\right)\\ &= -\int_b^a f(t)\Delta^\beta t.\end{aligned}$$

(iv)

$$\begin{aligned}\int_a^b f(t)\Delta^\beta t &= F^\beta(b) - F^\beta(a) = F^\beta(c) - F^\beta(a) + F^\beta(b) - F^\beta(c)\\ &= \int_a^c f(t)\Delta^\beta t + \int_c^b f(t)\Delta^\beta t.\end{aligned}$$

(v)

$$\int_a^a f(t)\Delta^\beta t = F^\beta(a) - F^\beta(a) = 0.$$

The proof is complete. □

We end with a simple example of a discrete fractional integral.

Example 2.23. *Let $\mathbb{T} = \mathbb{Z}$, $0 \le \beta \le 1$, and $f(t) = t$. Using the fact that in this case*

$$\int t\Delta t = \frac{t^2}{2} + C$$

with C a constant, we have

$$\int_1^{10} t\,\Delta^\beta t = \int t\,\Delta^\beta t\Big|_1^{10} = \left(\int t\,\Delta t\right)^{(1-\beta)}\Bigg|_1^{10} = \left(\frac{t^2}{2} + C\right)^{(1-\beta)}\Bigg|_1^{10}.$$

It follows from Example 2.12 (i) with $\mu(t) \equiv 1$, Theorem 2.10 (i) and (ii) and Proposition 2.4 that

$$\int_1^{10} t\,\Delta^\beta t = \frac{1}{2}\,(2t+1)\Big|_1^{10} = \frac{21}{2} - \frac{3}{2} = 9.$$

Chapter 3

Nonsymmetric and Symmetric Fractional Calculi on Time Scales

The results of this chapter are original and are published in [22].

3.1 Introduction

In this chapter, we introduce a nabla, a delta, and a symmetric fractional calculus on arbitrary nonempty closed subsets of the real numbers. These fractional calculi provide a study of differentiation and integration of noninteger order on discrete, continuous, and hybrid settings. Main properties of the new fractional operators are investigated, and some fundamental results presented, illustrating the interplay between discrete and continuous behaviors.

The notion of *derivative* is at the core of any calculus. One can interpret the derivative in a geometrical way, as the slope of a curve, or, physically, as a rate of change. But what if we generalize the notion of derivative and we study the limit

$$\lim_{s\to t}\frac{f(s)-f(t)}{(s-t)^{\alpha}}$$

for $\alpha \in]0,1]$ (derivative of order α)? In this chapter we discuss this question in the general framework of the calculus on time scales, which might best be understood as the continuum bridge between discrete time and continuous time theories, offering a rich formalism for studying hybrid discrete-continuous dynamical systems [25, 26, 58].

A time scale is a model of time, where the continuous and the discrete are considered and merged into a single theory. Time scales were first introduced by Aulbach and

Hilger in 1988 [12]. They have found applications in many different fields that require simultaneous modeling of discrete and continuous data [35, 38, 73].

In order to define an inverse operator of our new derivative, the antiderivative, we apply some ideas from fractional calculus, which is a branch of mathematical analysis that studies the possibility of taking real number powers of the differentiation operator [1, 47, 70]. The fractional calculus goes back to Leibniz (1646–1716) himself. However, it was only in the last 20 years that fractional calculus has gained an increasingly attention of researchers. In October 2009, Science Watch of Thomson Reuters identified it as an *Emerging Research Front* and gave an award to Metzler and Klafter for their paper [59]. Here we consider fractional calculus in the more general setting of time scales.
We develop two types of fractional calculi on arbitrary time scales: nonsymmetric (Section 3.2) and symmetric (Section 3.3). The new calculi provide, as particular cases, discrete, quantum, continuous and hybrid fractional derivatives and integrals.

3.2 Nonsymmetric Fractional Calculus

In this section, we begin by introducing a new notion: the nabla fractional derivative of order $\alpha \in]0,1]$ for functions defined on arbitrary time scales. For $\alpha = 1$ we obtain the usual nabla derivative of the time-scale calculus. Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$, and $\delta > 0$. We define the right δ-neighborhood of t as $\mathcal{U}^+ := [t, t+\delta[\cap \mathbb{T}$ and the left δ-neighborhood of t as $\mathcal{U}^- :=]t-\delta, t] \cap \mathbb{T}$.

Definition 3.1 (The nabla fractional derivative). *Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}_k$. For $\alpha \in]0,1] \cap \{1/q : q$ is a odd number$\}$ (resp. $\alpha \in]0,1] \setminus \{1/q : q$ is a odd number$\}$) we define $f^{\nabla^\alpha}(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a δ-neighborhood $\mathcal{U} \subset \mathbb{T}$ of t (resp. right δ-neighborhood $\mathcal{U}^+ \subset \mathbb{T}$ of t), $\delta > 0$, such that*

$$\left|[f(s) - f^\rho(t))] - f^{\nabla^\alpha}(t)\,[s - \rho(t)]^\alpha\right| \leq \varepsilon\,|s - \rho(t)|^\alpha$$

for all $s \in \mathcal{U}$ (resp. $s \in \mathcal{U}^+$). We call $f^{\nabla^\alpha}(t)$ the nabla fractional derivative of f of order α at t.

Recall the notion of delta fractional derivative considered in Chapter 2.

Definition 3.2 (The delta fractional derivative). *Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa$. For $\alpha \in]0,1] \cap \{1/q : q$ is a odd number$\}$ (resp. $\alpha \in]0,1] \setminus \{1/q : q$ is a odd number$\}$) we define*

$f^{\Delta^\alpha}(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a δ-neighborhood $\mathcal{U} \subset \mathbb{T}$ of t (resp. left δ-neighborhood $\mathcal{U}^- \subset \mathbb{T}$ of t), $\delta > 0$, such that

$$\left|[f^\sigma(t) - f(s)] - f^{\Delta^\alpha}(t)\,[\sigma(t) - s]^\alpha\right| \leq \varepsilon\,|\sigma(t) - s|^\alpha$$

for all $s \in \mathcal{U}$ (resp. $s \in \mathcal{U}^-$). We call $f^{\Delta^\alpha}(t)$ the delta fractional derivative of f of order α at t.

Throughout this section we only consider the nonsymmetric fractional calculus as the calculus derived from the nabla fractional derivative. The delta fractional calculus associated with the delta fractional derivative was considered in Chapter 2.

Along the text α is a real number in the interval $]0,1]$. The next theorem provides some useful properties of the nabla fractional derivative on time scales.

Theorem 3.3. *Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}_k$. The following properties hold:*

(i) Let $\alpha \in]0,1] \cap \left\{\frac{1}{q} : q \text{ is a odd number}\right\}$. If t is left-dense and if f is nabla fractional differentiable of order α at t, then f is continuous at t.

(ii) Let $\alpha \in]0,1] \setminus \left\{\frac{1}{q} : q \text{ is a odd number}\right\}$. If t is left-dense and if f is nabla fractional differentiable of order α at t, then f is right-continuous at t.

(iii) If f is continuous at t and t is left-scattered, then f is nabla fractional differentiable of order α at t with

$$f^{\nabla^\alpha}(t) = \frac{f(t) - f^\rho(t)}{[t - \rho(t)]^\alpha}.$$

(iv) Let $\alpha \in]0,1] \cap \left\{\frac{1}{q} : q \text{ is a odd number}\right\}$. If t is left-dense, then f is nabla fractional differentiable of order α at t if, and only if, the limit

$$\lim_{s \to t} \frac{f(s) - f(t)}{(s-t)^\alpha}$$

exists as a finite number. In this case,

$$f^{\nabla^\alpha}(t) = \lim_{s \to t} \frac{f(s) - f(t)}{(s-t)^\alpha}.$$

(v) Let $\alpha \in]0,1] \setminus \left\{\frac{1}{q} : q \text{ is a odd number}\right\}$. If t is left-dense, then f is nabla fractional differentiable of order α at t if, and only if, the limit

$$\lim_{s \to t^+} \frac{f(s) - f(t)}{(s-t)^\alpha}$$

exists as a finite number. In this case,

$$f^{\nabla^\alpha}(t) = \lim_{s \to t^+} \frac{f(s) - f(t)}{(s-t)^\alpha}.$$

(vi) If f is nabla fractional differentiable of order α at t, then

$$f(t) = f^\rho(t) + [t - \rho(t)]^\alpha f^{\nabla^\alpha}(t).$$

Proof. (i) Assume that f is fractional differentiable at t. Then, there exists a neighborhood $\mathcal{U}$ of t such that

$$\left|[f(s) - f^\rho(t)] - f^{\nabla^\alpha}(t)\,[s - \rho(t)]^\alpha\right| \leq \varepsilon\,|s - \rho(t)|^\alpha$$

for $s \in \mathcal{U}$. Therefore, for all $s \in \mathcal{U} \cap\,]t - \varepsilon, t + \varepsilon[$,

$$\begin{aligned} |f\,(t) - f\,(s)| &\leq \left|[f(s) - f^\rho(t)] - f^{\nabla^\alpha}(t)\,[s - \rho(t)]^\alpha\right| \\ &\quad + \left|[f(t) - f^\rho(t)] - f^{\nabla^\alpha}(t)\,[t - \rho(t)]^\alpha\right| \\ &\quad + \left|f^{\nabla^\alpha}(t)\right| \left|[s - \rho(t)]^\alpha - [t - \rho(t)]^\alpha\right| \end{aligned}$$

and, since t is a left-dense point,

$$\begin{aligned} |f\,(t) - f\,(s)| &\leq \left|[f(s) - f^\rho(t)] - f^{\nabla^\alpha}(t)\,[s - \rho(t)]^\alpha\right| + \left|f^{\nabla^\alpha}(t)\,[s - t]^\alpha\right| \\ &\leq \varepsilon\,|s - t|^\alpha + \left|f^{\nabla^\alpha}(t)\,[s - t]^\alpha\right| \\ &\leq \varepsilon^\alpha \left[\varepsilon + \left|f^{\nabla^\alpha}(t)\right|\right]. \end{aligned}$$

We conclude that f is continuous at t. (ii) The proof is similar to the proof of (i), where instead of considering the neighborhood $\mathcal{U}$ of t we consider a right neighborhood $\mathcal{U}^+$ of t. (iii) Assume that f is continuous at t and t is left-scattered. By continuity,

$$\lim_{s \to t} \frac{f(s) - f^\rho(t)}{[s - \rho(t)]^\alpha} = \frac{f(t) - f^\rho(t)}{[t - \rho(t)]^\alpha}.$$

Hence, given $\varepsilon > 0$ and $\alpha \in]0, 1] \cap \{1/q : q \text{ is a odd number}\}$, there is a neighborhood $\mathcal{U}$ of t (or $\mathcal{U}^+$ if $\alpha \in]0, 1] \setminus \{1/q : q \text{ is a odd number}\}$) such that

$$\left|\frac{f(s) - f^\rho(t)}{[s - \rho(t)]^\alpha} - \frac{f(t) - f^\rho(t)}{[t - \rho(t)]^\alpha}\right| \leq \varepsilon$$

for all $s \in \mathcal{U}$ (resp. $\mathcal{U}^+$). It follows that

$$\left|[f(s) - f^\rho(t)] - \frac{f(t) - f^\rho(t)}{[t - \rho(t)]^\alpha}\,[s - \rho(t)]^\alpha\right| \leq \varepsilon\,|s - \rho(t)|^\alpha$$

for all $s \in \mathcal{U}$ (resp. $\mathcal{U}^+$). Hence, we get the desired result:

$$f^{\nabla^\alpha}(t) = \frac{f(t) - f^\rho(t)}{[t - \rho(t)]^\alpha}.$$

(iv) Assume that f is nabla fractional differentiable of order α at t and t is left-dense. Let $\varepsilon > 0$ be given. Since f is nabla fractional differentiable of order α at t, there is a neighborhood $\mathcal{U}$ of t such that

$$\left|[f(s) - f^\rho(t)] - f^{\nabla^\alpha}(t)\,[s - \rho(t)]^\alpha\right| \leq \varepsilon\,|s - \rho(t)|^\alpha$$

for all $s \in \mathcal{U}$. Since $\rho(t) = t$,

$$\left|[f(s) - f(t)] - f^{\nabla^\alpha}(t)\,[s - t]^\alpha\right| \leq \varepsilon|s - t|^\alpha$$

for all $s \in \mathcal{U}$. It follows that

$$\left|\frac{f(s) - f(t)}{[s - t]^\alpha} - f^{\nabla^\alpha}(t)\right| \leq \varepsilon$$

for all $s \in \mathcal{U}$, $s \neq t$. Therefore, we get the desired result:

$$f^{\nabla^\alpha}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{(t - s)^\alpha}.$$

Now assume that

$$\lim_{s \to t} \frac{f(s) - f(t)}{(s - t)^\alpha}$$

exists and is equal to L and t is left-dense. Then, there exists a neighborhood $\mathcal{U}$ of t such that

$$\left|\frac{f(s) - f(t)}{(s - t)^\alpha} - L\right| \leq \varepsilon$$

for all $s \in \mathcal{U} \backslash \{t\}$. Because t is left-dense,

$$\left|\frac{f(s) - f^\rho(t)}{[s - \rho(t)]^\alpha} - L\right| \leq \varepsilon.$$

Therefore,

$$|[f(s) - f^\rho(t)] - L\,[s - \rho(t)]^\alpha| \leq \varepsilon|s - \rho(t)|^\alpha$$

for all $s \in U$ (note that the inequality is trivially verified for $s = t$). Hence, f is nabla fractional differentiable of order α at t and

$$f^{\nabla^\alpha}(t) = \lim_{s \to t} \frac{f(s) - f(t)}{(s - t)^\alpha}.$$

(v) The proof is similar to the proof of (iv), where instead of considering the neighborhood $\mathcal{U}$ of t we consider a right-neighborhood $\mathcal{U}^+$ of t. (vi) If $\rho(t) = t$, then

$$f^\rho(t) = f(t) = f(t) + [t - \rho(t)]^\alpha f^{\nabla^\alpha}(t) .$$

On the other hand, if $t > \rho(t)$, then, by (iii),

$$f(t) = f^\rho(t) + [t - \rho(t)]^\alpha \frac{f(t) - f^\rho(t)}{[t - \rho(t)]^\alpha} = f^\rho(t) + [t - \rho(t)]^\alpha f^{\nabla^\alpha}(t).$$

The proof is complete. □

Next result relates different orders of the nabla fractional derivative of a function.

Theorem 3.4. *Let $\alpha, \beta \in]0, 1]$ with $\beta \geq \alpha$ and let $f : \mathbb{T} \to \mathbb{R}$ be a continuous function. If f is nabla fractional differentiable of order β at $t \in \mathbb{T}$, then f is nabla fractional differentiable of order α at t.*

Proof. If t is left-scattered, then, by Theorem 3.3 (iii), f is nabla fractional differentiable of any order $\alpha \in]0, 1]$. If t is left-dense, then, by Theorem 3.3 (iv, v),

$$f^{\nabla^\beta}(t) = \lim_{s \to t} \frac{f(s) - f(t)}{(s - t)^\beta}.$$

Since

$$f^{\nabla^\beta}(t) = \lim_{s \to t} \frac{\frac{f(s)-f(t)}{(s-t)^\alpha}}{(s - t)^{\beta-\alpha}},$$

we have

$$f^{\nabla^\alpha}(t) = \lim_{s \to t} (s - t)^{\beta-\alpha} f^{\nabla^\beta}(t),$$

which proves existence of the nabla fractional derivative of f of order α at $t \in \mathbb{T}$. □

Proposition 3.5. *If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = c$ for all $t \in \mathbb{T}$, $c \in \mathbb{R}$, then $f^{\nabla^\alpha} \equiv 0$.*

Proof. If t is left-scattered, then, by Theorem 3.3 (iii), one has

$$f^{\nabla^\alpha}(t) = \frac{f(t) - f^\rho(t)}{[t - \rho(t)]^\alpha} = \frac{c - c}{[t - \rho(t)]^\alpha} = 0.$$

Assume t is left-dense. Then, by Theorem 3.3 (iv) and (v), it follows that

$$f^{\nabla^\alpha}(t) = \lim_{s \to t} \frac{c - c}{[t - \rho(t)]^\alpha} = 0.$$

This concludes the proof. □

Proposition 3.6. *If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then*

$$f^{\nabla^\alpha}(t) = \begin{cases} [t-\rho(t)]^{1-\alpha} & \text{if } \alpha \neq 1, \\ 1 & \text{if } \alpha = 1. \end{cases}$$

Proof. Clearly, the function is nabla differentiable, which is the same as saying that function f is nabla fractional differentiable of order 1. Then, by Theorem 3.4, the function is nabla fractional differentiable of order α, with $\alpha \in]0,1]$. From Theorem 3.3 (vi) it follows that

$$t - \rho(t) = [t-\rho(t)]^\alpha f^{\nabla^\alpha}(t).$$

If $t - \rho(t) \neq 0$, then

$$f^{\nabla^\alpha}(t) = [t-\rho(t)]^{1-\alpha}$$

and the desired relation is proved. Assume now that $t - \rho(t) = 0$, that is, $\rho(t) = t$. In this case t is left-dense and by Theorem 3.3 (iv) and (v) it follows that

$$f^{\nabla^\alpha}(t) = \lim_{s \to t} \frac{s-t}{(s-t)^\alpha}.$$

Therefore, if $\alpha = 1$, then $f^{\nabla^\alpha}(t) = 1$; if $0 < \alpha < 1$, then $f^{\nabla^\alpha}(t) = 0$. □

Let us now consider the particular case $\mathbb{T} = \mathbb{R}$.

Corollary 3.7. *Function $f : \mathbb{R} \to \mathbb{R}$ is nabla fractional differentiable of order α at point $t \in \mathbb{R}$ if, and only if, the limit*

$$\lim_{s \to t} \frac{f(s) - f(t)}{(s-t)^\alpha}$$

exists as a finite number. In this case,

$$f^{\nabla^\alpha}(t) = \lim_{s \to t} \frac{f(s) - f(t)}{(s-t)^\alpha}.$$

Proof. Here $\mathbb{T} = \mathbb{R}$ and all points are left-dense. The result follows from Theorem 3.3 (iv) and (v). Note that if $\alpha \in]0,1] \setminus \left\{\frac{1}{q} : q \text{ is a odd number}\right\}$, then the limit only makes sense as a right-side limit. □

The next result shows that there are functions which are nabla fractional differentiable but are not nabla differentiable.

Proposition 3.8. *If $f : \mathbb{R}_0^+ \to \mathbb{R}$ is defined by $f(t) = \sqrt{t}$ for all $t \in \mathbb{R}_0^+$, then*

$$f^{\nabla^{1/2}}(t) = \begin{cases} 0 & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}$$

Proof. Here $\mathbb{T} = \mathbb{R}_0^+$. In this time scale every point t is left-dense and by Theorem 3.3 (v) with $\alpha = 1/2$ it follows that

$$f^{\nabla^{1/2}}(t) = \lim_{s \to t^+} \frac{\sqrt{s} - \sqrt{t}}{\sqrt{s-t}} = \lim_{s \to t^+} \frac{\sqrt{s-t}}{\sqrt{s} + \sqrt{t}} = 0$$

for $t \neq 0$. If $t = 0$, then

$$f^{\nabla^{1/2}}(t) = \lim_{s \to 0^+} \frac{\sqrt{s}}{\sqrt{s}} = 1.$$

This concludes the proof. □

For the fractional derivative on time scales to be useful, we would like to know formulas for the derivatives of sums, products and quotients of fractional differentiable functions. This is done according to the following theorem.

Theorem 3.9. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are nabla fractional differentiable of order α at $t \in \mathbb{T}_k$. Then,*

(i) the sum $f + g : \mathbb{T} \to \mathbb{R}$ is nabla fractional differentiable at t with

$$(f+g)^{\nabla^\alpha}(t) = f^{\nabla^\alpha}(t) + g^{\nabla^\alpha}(t);$$

(ii) for any constant $\lambda \in \mathbb{R}$, $\lambda f : \mathbb{T} \to \mathbb{R}$ is nabla fractional differentiable at t with

$$(\lambda f)^{\nabla^\alpha}(t) = \lambda f^{\nabla^\alpha}(t);$$

(iii) if f and g are continuous, then the product $fg : \mathbb{T} \to \mathbb{R}$ is nabla fractional differentiable at t with

$$\begin{aligned}(fg)^{\nabla^\alpha}(t) &= f^{\nabla^\alpha}(t)\, g(t) + f^\rho(t)\, g^{\nabla^\alpha}(t) \\ &= f^{\nabla^\alpha}(t) g^\rho(t) + f(t) g^{\nabla^\alpha}(t);\end{aligned}$$

(iv) if f is continuous and $f^\rho(t) f(t) \neq 0$, then $\frac{1}{f}$ is nabla fractional differentiable at t with

$$\left(\frac{1}{f}\right)^{\nabla^\alpha}(t) = -\frac{f^{\nabla^\alpha}(t)}{f^\rho(t) f(t)};$$

(v) if f and g are continuous and $g^\rho(t)g(t) \neq 0$, then $\frac{f}{g}$ is fractional differentiable at t with

$$\left(\frac{f}{g}\right)^{\nabla^\alpha}(t) = \frac{f^{\nabla^\alpha}(t)g(t) - f(t)g^{\nabla^\alpha}(t)}{g^\rho(t)g(t)}.$$

Proof. Let us consider that $\alpha \in]0,1] \cap \left\{\frac{1}{q} : q \text{ is a odd number}\right\}$. The proofs for the case $\alpha \in]0,1] \setminus \left\{\frac{1}{q} : q \text{ is a odd number}\right\}$ are similar: one just needs to choose the proper right-sided neighborhoods. Assume that f and g are nabla fractional differentiable of order α at $t \in \mathbb{T}_k$. (i) Let $\varepsilon > 0$. Then there exist neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of t for which

$$\left|[f(s) - f^\rho(t)] - f^{\nabla^\alpha}(t)\,[s-\rho(t)]^\alpha\right| \leq \frac{\varepsilon}{2}\,|s-\rho(t)|^\alpha \quad for\ all \quad s \in \mathcal{U}_1$$

and

$$\left|[g(s) - g^\rho(t)] - g^{\nabla^\alpha}(t)\,[s-\rho(t)]^\alpha\right| \leq \frac{\varepsilon}{2}\,|s-\rho(t)|^\alpha \quad for\ all \quad s \in \mathcal{U}_2.$$

Let $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$. Then

$$\begin{aligned}
&\left|(f+g)(s) - (f+g)^\rho(t) - \left[f^{\nabla^\alpha}(t) + g^{\nabla^\alpha}(t)\right][s-\rho(t)]^\alpha\right| \\
&= \left|f(s) - f^\rho(t) - f^{\nabla^\alpha}(t)\,[s-\rho(t)]^\alpha + g(s) - g^\rho(t) - g^{\nabla^\alpha}(t)[s-\rho(t)]^\alpha\right| \\
&\leq \left|[f(s) - f^\rho(t)] - f^{\nabla^\alpha}(t)\,[s-\rho(t)]^\alpha\right| + \left|[g(s) - g^\rho(t)] - g^{\nabla^\alpha}(t)\,[s-\rho(t)]^\alpha\right| \\
&\leq \varepsilon|s-\rho(t)|^\alpha
\end{aligned}$$

for all $s \in \mathcal{U}$. Therefore, $f+g$ is fractional differentiable of order α at t and

$$(f+g)^{\nabla^\alpha}(t) = f^{\nabla^\alpha}(t) + g^{\nabla^\alpha}(t).$$

(ii) Let $\varepsilon > 0$. Then there exists a neighborhood $\mathcal{U}$ of t with

$$\left|[f(s) - f^\rho(t)] - f^{\nabla^\alpha}(t)\,[s-\rho(t)]^\alpha\right| \leq \varepsilon\,|s-\rho(t)|^\alpha \text{ for all } s \in \mathcal{U}.$$

It follows that

$$\left|[(\lambda f)(s) - (\lambda f)^\rho(t)] - \lambda f^{\nabla^\alpha}(t)\,[s-\rho(t)]^\alpha\right| \leq \varepsilon|\lambda|\ |s-\rho(t)|\,|^\alpha \text{ for all } s \in \mathcal{U}.$$

Therefore, λf is fractional differentiable of order α at t and $(\lambda f)^{\nabla^\alpha}(t) = \lambda f^{\nabla^\alpha}(t)$ holds at t. (iii) If t is left-dense, then

$$\begin{aligned}
(fg)^{\nabla^\alpha}(t) &= \lim_{s\to t}\frac{(fg)(s) - (fg)(t)}{(s-t)^\alpha} \\
&= \lim_{s\to t}\frac{f(s)-f(t)}{(s-t)^\alpha}g(s) + \lim_{s\to t}\frac{g(s)-g(t)}{(s-t)^\alpha}f(t) \\
&= f^{\nabla^\alpha}(t)g(t) + g^{\nabla^\alpha}(t)f(t) \\
&= f^{\nabla^\alpha(t)}g(t) + f^\rho(t)\,g^{\nabla^\alpha}(t).
\end{aligned}$$

If t is left-scattered, then

$$\begin{aligned}(fg)^{\nabla^\alpha}(t) &= \frac{(fg)(t)-(fg)^\rho(t)}{[t-\rho(t)]^\alpha}\\ &= \frac{f(t)-f^\rho(t)}{[t-\rho(t)]^\alpha}g(t)+\frac{g(t)-g^\rho(t)}{[t-\rho(t)]^\alpha}f^\rho(t)\\ &= f^{\nabla^\alpha}(t)g(t)+f^\rho(t)g^{\nabla^\alpha}(t).\end{aligned}$$

The other product rule formula follows by interchanging the role of functions f and g. (iv) Using the fractional derivative of a constant (Proposition 3.5) and Theorem 3.9 (iii), we know that

$$\left(f\cdot\frac{1}{f}\right)^{\nabla^\alpha}(t)=(1)^{(\alpha)}(t)=0.$$

Therefore,

$$\left(\frac{1}{f}\right)^{\nabla^\alpha}(t)f^\rho(t)+f^{\nabla^\alpha}(t)\frac{1}{f(t)}=0.$$

Since we are assuming $f^\rho(t)\neq 0$,

$$\left(\frac{1}{f}\right)^{\nabla^\alpha}(t)=-\frac{f^{\nabla^\alpha}(t)}{f^\rho(t)f(t)},$$

as intended. (v) Follows trivially from the previous properties:

$$\begin{aligned}\left(\frac{f}{g}\right)^{\nabla^\alpha}(t) &= \left(f\cdot\frac{1}{g}\right)^{\nabla^\alpha}(t)\\ &= f(t)\left(\frac{1}{g}\right)^{\nabla^\alpha}(t)+f^{\nabla^\alpha}(t)\frac{1}{g^\rho(t)}\\ &= -f(t)\frac{g^{\nabla^\alpha}(t)}{g^\rho(t)g(t)}+f^{\nabla^\alpha}(t)\frac{1}{g^\rho(t)}\\ &= \frac{f^{\nabla^\alpha}(t)g(t)-f(t)g^{\nabla^\alpha}(t)}{g(t)g(\sigma(t))}.\end{aligned}$$

The proof is complete. □

The next result provides examples of how to use the algebraic properties of the nabla fractional derivatives of order α.

Theorem 3.10. *Let c be a constant, $m\in\mathbb{N}$, and $\alpha\in\,]0,1[$.*

(i) If $f(t) = (t-c)^m$, *then*

$$f^{\nabla^\alpha}(t) = \begin{cases} [t-\rho(t)]^{1-\alpha} \sum_{\nu=0}^{m-1} [\rho(t)-c]^\nu (t-c)^{m-1-\nu} & \text{if } \alpha \neq 1, \\ \sum_{\nu=0}^{m-1} [\rho(t)-c]^\nu (t-c)^{m-1-\nu} & \text{if } \alpha = 1. \end{cases}$$

(ii) If $g(t) = \dfrac{1}{(t-c)^m}$, *then*

$$g^{\nabla^\alpha}(t) = \begin{cases} -[t-\rho(t)]^{1-\alpha} \sum_{\nu=0}^{m-1} \dfrac{1}{[\rho(t)-c]^{m-\nu}(t-c)^{\nu+1}} & \text{if } \alpha \neq 1, \\ -\sum_{\nu=0}^{m-1} \dfrac{1}{[\rho(t)-c]^{m-\nu}(t-c)^{\nu+1}} & \text{if } \alpha = 1, \end{cases}$$

provided $[\rho(t)-c](t-c) \neq 0$.

Proof. We use mathematical induction. First, let us consider the case $\alpha \neq 0$. If $m = 1$, then $f(t) = t - c$ and $f^{\nabla^\alpha}(t) = [t-\rho(t)]^{1-\alpha}$ holds from Propositions 3.5 and 3.6 and Theorem 3.9 (*i*). Now, assume that

$$f^{\nabla^\alpha}(t) = [t-\rho(t)]^{1-\alpha} \sum_{\nu=0}^{m-1} [\rho(t)-c]^\nu (t-c)^{m-1-\nu}$$

holds for $f(t) = (t-c)^m$ and let $F(t) = (t-c)^{m+1} = (t-c)f(t)$. By Theorem 3.9 (*iii*), we have

$$\begin{aligned} F^{\nabla^\alpha}(t) &= (t-c)^{\nabla^\alpha} f^\rho(t) + f^{\nabla^\alpha}(t)(t-c) \\ &= [t-\rho(t)]^{1-\alpha} f^\rho(t) + f^{\nabla^\alpha}(t)(t-c) \\ &= [t-\rho(t)]^{1-\alpha} \left[[\rho(t)-c]^m + \sum_{\nu=0}^{m-1} [\rho(t)-c]^\nu (t-c)^{m-\nu} \right] \\ &= [t-\rho(t)]^{1-\alpha} \sum_{\nu=0}^{m} [\rho(t)-c]^\nu (t-c)^{m-\nu}. \end{aligned}$$

Hence, by mathematical induction, part (*i*) holds. For $g(t) = \dfrac{1}{(t-c)^m} = \dfrac{1}{f(t)}$, we apply Theorem 3.9 (*iv*) to obtain

$$\begin{aligned} g^{\nabla^\alpha}(t) &= -\frac{f^{\nabla^\alpha}(t)}{f^\rho(t)f(t)} = -[t-\rho(t)]^{1-\alpha} \frac{\sum_{\nu=0}^{m-1} [\rho(t)-c]^\nu (t-c)^{m-1-\nu}}{[\rho(t)-c]^m (t-c)^m} \\ &= -[t-\rho(t)]^{1-\alpha} \sum_{\nu=0}^{m-1} \frac{1}{[\rho(t)-c]^{m-\nu}(t-c)^{\nu+1}}, \end{aligned}$$

provided $[\rho(t) - c]\,(t - c) \neq 0$. For $\alpha = 1$ the proofs are similar bearing in mind that $(t)^{\nabla^1} = (t)^{\nabla} = 1$. □

Now we introduce the nabla fractional integral on time scales.

Definition 3.11 (The indefinite nabla fractional integral)**.** *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. We define the indefinite nabla fractional integral of f of order β, $0 \leq \beta \leq 1$, by*

$$\int f(t)\nabla^{\beta} t := \left(\int f(t)\nabla t\right)^{\nabla^{(1-\beta)}}$$

with $\int f(t)\nabla t$ the usual indefinite nabla integral of time scales [25].

Remark 3.12. *It follows from Definition 3.11 that*

$$\int f(t)\nabla^{1} t = \int f(t)\nabla t, \quad \int f(t)\nabla^{0} t = f(t).$$

Definition 3.13 (The definite nabla fractional integral)**.** *Assume $f : \mathbb{T} \to \mathbb{R}$ is a ld-continuous function. Let*

$$F^{\nabla^{\beta}}(t) = \int f(t)\nabla^{\beta} t$$

denote the indefinite nabla fractional integral of f of order β with $0 \leq \beta \leq 1$, and let $a, b \in \mathbb{T}$. We define the Cauchy nabla fractional integral from a to b by

$$\int_a^b f(t)\nabla^{\beta} t := F^{\nabla^{\beta}}(t)\Big|_a^b = F^{\nabla^{\beta}}(b) - F^{\nabla^{\beta}}(a).$$

The next theorem gives some algebraic properties of the nabla fractional integral.

Theorem 3.14. *If $a, b, c \in \mathbb{T}$, $\lambda \in \mathbb{R}$, and $f, g \in \mathcal{C}_{ld}$ with $0 \leq \beta \leq 1$, then*

(i) $\displaystyle\int_a^b [f(t) + g(t)]\nabla^{\beta} t = \int_a^b f(t)\nabla^{\beta} t + \int_a^b g(t)\nabla^{\beta} t;$

(ii) $\displaystyle\int_a^b (\lambda f)(t)\nabla^{\beta} t = \lambda \int_a^b f(t)\nabla^{\beta} t;$

(iii) $\displaystyle\int_a^b f(t)\nabla^{\beta} t = -\int_b^a f(t)\nabla^{\beta} t;$

(iv) $\displaystyle\int_a^b f(t)\nabla^{\beta} t = \int_a^c f(t)\nabla^{\beta} t + \int_c^b f(t)\nabla^{\beta} t;$

(v) $\displaystyle\int_a^a f(t)\nabla^\beta t = 0.$

Proof. The equalities follow from Definitions 3.11 and 3.13, analogous properties of the nabla integral on time scales, and the properties of Section 3.2 for the fractional nabla derivative on time scales. (i) From Definition 3.13, we have

$$\begin{aligned}
\int_a^b (f+g)(t)\nabla^\beta t &= \int [f(t)+g(t)]\,\nabla^\beta t\Big|_a^b \\
&= \left(\int [f(t)+g(t)]\,\nabla t\right)^{\nabla^{(1-\beta)}}\Bigg|_a^b \\
&= \left[\left(\int f(t)\Delta t\right)^{\nabla^{(1-\beta)}} + \left(\int g(t)\Delta t\right)^{\nabla^{(1-\beta)}}\right]\Bigg|_a^b \\
&= \int_a^b f(t)\nabla^\beta t + \int_a^b g(t)\nabla^\beta t.
\end{aligned}$$

(ii) From Definitions 3.13 and 3.11, one has

$$\begin{aligned}
\int_a^b (\lambda f)(t)\nabla^\beta t &= \int (\lambda f)(t)\nabla^\beta t\Big|_a^b = \left(\int (\lambda f)(t)\nabla t\right)^{\nabla^{(1-\beta)}}\Bigg|_a^b \\
&= \lambda\left(\int f(t)\nabla t\right)^{\nabla^{(1-\beta)}}\Bigg|_a^b = \lambda\int_a^b f(t)\nabla^\beta t.
\end{aligned}$$

The last properties (iii), (iv) and (v) are direct consequences of Definition 3.13: (iii)

$$\int_a^b f(t)\nabla^\beta t = F^\beta(b) - F^\beta(a) = -\left(F^\beta(a) - F^\beta(b)\right) = -\int_b^a f(t)\nabla^\beta t;$$

(iv)

$$\begin{aligned}
\int_a^b f(t)\nabla^\beta t &= F^{\nabla^\beta}(b) - F^{\nabla^\beta}(a) = F^{\nabla^\beta}(c) - F^{\nabla^\beta}(a) + F^{\nabla^\beta}(b) - F^{\nabla^\beta}(c) \\
&= \int_a^c f(t)\nabla^\beta t + \int_c^b f(t)\nabla^\beta t;
\end{aligned}$$

(v)

$$\int_a^a f(t)\nabla^\beta t = F^{\nabla^\beta}(a) - F^{\nabla^\beta}(a) = 0.$$

This concludes the proof. □

We end this section with a simple example of a discrete fractional integral of order α.

Example 3.15. *Let $\mathbb{T} = \mathbb{Z}$, $0 \leq \beta < 1$, and $f(t) = t$. Using the fact that*

$$\int t\nabla t = \frac{t^2}{2} + C,$$

where C is a constant, we have

$$\int_1^{10} t\,\nabla^\beta t = \int t\,\nabla^\beta t\Big|_1^{10} = \left(\int t\,\nabla t\right)^{\nabla(1-\beta)}\Bigg|_1^{10} = \left(\frac{t^2}{2} + C\right)^{(1-\beta)}\Bigg|_1^{10}.$$

It follows that

$$\int_1^{10} t\,\nabla^\beta t = [t - \rho(t)]^{(1-\alpha)}[\rho(t) + t]\big|_1^{10} = \frac{1}{2}\,(2t-1)\Big|_1^{10} = \frac{19}{2} - \frac{1}{2} = 9.$$

The fundamental concepts of the nabla fractional calculus of order α, which are the differentiation and integration of noninteger order using the nabla operator, were presented in this section. The properties of the delta fractional calculus of order α are similar to the properties of the nabla case and were discussed in Chapter 2. In the next section we will use both nabla and delta approaches, the delta and nabla fractional calculi of order α, to obtain useful results of the symmetric fractional calculus of order α and extend the results of [28].

3.3 Symmetric Fractional Calculus

In this section, we introduce the notion of symmetric fractional derivative of order $\alpha \in]0,1]$ on time scales.

Definition 3.16 (The symmetric fractional derivative)**.** *Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa_\kappa$, and $\alpha \in\ ]0,1]$. The symmetric fractional derivative of f at t, denoted by $f^{\Diamond^\alpha}(t)$, is the real number (provided it exists) with the property that, for any $\varepsilon > 0$, there exists a neighborhood $U \subset \mathbb{T}$ of t such that*

$$\left|[f^\sigma(t) - f(s) + f(2t-s) - f^\rho(t)] - f^{\Diamond^\alpha}(t)\,[\sigma(t) + 2t - 2s - \rho(t)]^\alpha\right| \leq \varepsilon\,|\sigma(t) + 2t - 2s - \rho(t)|^\alpha$$

for all $s \in U$ for which $2t - s \in U$. A function f is said to be symmetric fractional differentiable of order α provided $f^{\Diamond^\alpha}(t)$ exists for all $t \in \mathbb{T}^\kappa_\kappa$.

Remark 3.17. *If $\alpha = 1$, then the symmetric fractional derivative is the symmetric derivative on time scales [28].*

Some useful properties of the symmetric derivative are given in Theorem 3.18.

Theorem 3.18. *Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}_\kappa^\kappa$ and $\alpha \in]0, 1]$. The following holds:*

(i) Function f has at most one symmetric fractional derivative of order α.

(ii) If f is symmetric fractional differentiable of order α at t and t is dense or isolated, then f is symmetric continuous at t (Definition 1.18).

(iii) If f is continuous at t and t is not dense, then f is symmetric differentiable of order α at t with

$$f^{\diamondsuit^\alpha}(t) = \frac{f^\sigma(t) - f^\rho(t)}{[\sigma(t) - \rho(t)]^\alpha}.$$

(iv) If t is dense, then f is symmetric fractional differentiable of order α at t if and only if the limit

$$\lim_{s \to t} \frac{f(2t - s) - f(s)}{2^\alpha (t - s)^\alpha}$$

exists as a finite number. In this case,

$$f^{\diamondsuit^\alpha}(t) = \lim_{s \to t} \frac{f(2t - s) - f(s)}{2^\alpha (t - s)^\alpha} = \lim_{h \to 0} \frac{f(t + h) - f(t - h)}{2^\alpha h^\alpha}.$$

(v) If f is symmetric differentiable of order α and continuous at t, then

$$f^\sigma(t) = f^\rho(t) + f^{\diamondsuit^\alpha}(t)\,[\sigma(t) - \rho(t)]^\alpha.$$

Proof. (i) Suppose that f has two symmetric derivatives of order α at t, $f_1^{\diamondsuit^\alpha}(t)$ and $f_2^{\diamondsuit^\alpha}(t)$. Then, there exists a neighborhood U_1 of t such that

$$\left|[f^\sigma(t) - f(s) + f(2t - s) - f^\rho(t)] - f_1^{\diamondsuit^\alpha}(t)\,[\sigma(t) + 2t - 2s - \rho(t)]^\alpha\right| \\ \leq \frac{\varepsilon}{2}\,|\sigma(t) + 2t - 2s - \rho(t)|^\alpha$$

for all $s \in U_1$ for which $2t - s \in U_1$, and a neighborhood U_2 of t such that

$$\left|[f^\sigma(t) - f(s) + f(2t - s) - f^\rho(t)] - f_2^{\diamondsuit^\alpha}(t)\,[\sigma(t) + 2t - 2s - \rho(t)]^\alpha\right| \\ \leq \frac{\varepsilon}{2}\,|\sigma(t) + 2t - 2s - \rho(t)|^\alpha$$

for all $s \in U_2$ for which $2t-s \in U_2$. Therefore, for all $s \in U_1 \cap U_2$ for which $2t-s \in U_1 \cap U_2$,

$$\begin{aligned}
&\left| f_1^{\Diamond^\alpha}(t) - f_2^{\Diamond^\alpha}(t) \right| = \left| \left[f_1^{\Diamond^\alpha}(t) - f_2^{\Diamond^\alpha}(t) \right] \frac{[\sigma(t)+2t-2s-\rho(t)]^\alpha}{[\sigma(t)+2t-2s-\rho(t)]^\alpha} \right| \\
&\leq \frac{\left| [f^\sigma(t) - f(s) + f(2t-s) - f^\rho(t)] - f_2^{\Diamond^\alpha}(t)[\sigma(t)+2t-2s-\rho(t)]^\alpha \right|}{|\sigma(t)+2t-2s-\rho(t)|^\alpha} \\
&\quad + \frac{\left| [f^\sigma(t) - f(s) + f(2t-s) - f^\rho(t)] - f_1^{\Diamond^\alpha}(t)[\sigma(t)+2t-2s-\rho(t)]^\alpha \right|}{|\sigma(t)+2t-2s-\rho(t)|^\alpha} \\
&\leq \varepsilon.
\end{aligned}$$

(ii) From hypothesis, for any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\begin{aligned}
&\left| [f^\sigma(t) - f(s) + f(2t-s) - f^\rho(t)] - f^{\Diamond^\alpha}(t)[\sigma(t)+2t-2s-\rho(t)]^\alpha \right| \\
&\qquad\qquad \leq \varepsilon |\sigma(t)+2t-2s-\rho(t)|^\alpha
\end{aligned}$$

for all $s \in U$ for which $2t-s \in U$. Note that

$$\begin{aligned}
&|f(s) - f(2t-s)| \\
&\leq \left| [f^\sigma(t) - f(s) + f(2t-s) - f^\rho(t)] - f^{\Diamond^\alpha}(t)[\sigma(t)+2t-2s-\rho(t)]^\alpha \right| \\
&\quad + \left| [f^\sigma(t) - f^\rho(t)] - f^{\Diamond^\alpha}(t)[\sigma(t)+2t-2s-\rho(t)]^\alpha \right| \\
&\leq \left| [f^\sigma(t) - f(s) + f(2t-s) - f^\rho(t)] - f^{\Diamond^\alpha}(t)[\sigma(t)+2t-2s-\rho(t)]^\alpha \right| \\
&\quad + \left| [f^\sigma(t) - f(t) + f(t) - f^\rho(t)] - f^{\Diamond^\alpha}(t)[\sigma(t)+2t-2t-\rho(t)]^\alpha \right| \\
&\quad + \left| f^{\Diamond^\alpha}(t)[\sigma(t)+2t-2s-\rho(t)]^\alpha - f^{\Diamond^\alpha}(t)[\sigma(t)-\rho(t)]^\alpha \right| \\
&\leq \varepsilon |\sigma(t)+2t-2s-\rho(t)|^\alpha + \varepsilon |\sigma(t)+2t-2t-\rho(t)|^\alpha \\
&\quad + \left| f^{\Diamond^\alpha}(t)[\sigma(t)+2t-2s-\rho(t)]^\alpha - f^{\Diamond^\alpha}(t)[\sigma(t)-\rho(t)]^\alpha \right|.
\end{aligned}$$

If t is dense, then

$$\begin{aligned}
|f(s) - f(2t-s)| &\leq \varepsilon 2^\alpha |t-s|^\alpha + \left| f^{\Diamond^\alpha}(t) \right| 2^\alpha |t-s|^\alpha \\
&\leq \varepsilon^\alpha 2^\alpha \left(\varepsilon + \left| f^{\Diamond^\alpha}(t) \right| \right)
\end{aligned}$$

for all $s \in U \cap \,]t-\varepsilon, t+\varepsilon[$, which proves the result for a point t which is dense. If t is isolated, then the function is continuous at t (because of the inherited topology) and therefore the function is symmetric continuous at t. (iii) Suppose that $t \in \mathbb{T}_\kappa^\kappa$ is not dense and f is continuous at t. Then,

$$\lim_{s \to t} \frac{f^\sigma(t) - f(s) + f(2t-s) - f^\rho(t)}{[\sigma(t)+2t-2s-\rho(t)]^\alpha} = \frac{f^\sigma(t) - f^\rho(t)}{[\sigma(t)-\rho(t)]^\alpha}.$$

Hence, for any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left|\frac{f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)}{[\sigma(t) + 2t - 2s - \rho(t)]^{\alpha}} - \frac{f^{\sigma}(t) - f^{\rho}(t)}{[\sigma(t) - \rho(t)]^{\alpha}}\right| \leq \varepsilon$$

for all $s \in U$ for which $2t - s \in U$. It follows that

$$\left|[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)] - \frac{f^{\sigma}(t) - f^{\rho}(t)}{[\sigma(t) - \rho(t)]^{\alpha}}[\sigma(t) + 2t - 2s - \rho(t)]^{\alpha}\right| \leq \varepsilon\left|\sigma(t) + 2t - 2s - \rho(t)\right|^{\alpha},$$

which proves that

$$f^{\diamondsuit^{\alpha}}(t) = \frac{f^{\sigma}(t) - f^{\rho}(t)}{[\sigma(t) - \rho(t)]^{\alpha}}.$$

(iv) Assume that f is symmetric fractional differentiable of order α at t and t is dense. Let $\varepsilon > 0$ be given. Then, there exists a neighborhood U of t such that

$$\left|[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)] - f^{\diamondsuit^{\alpha}}(t)[\sigma(t) + 2t - 2s - \rho(t)]^{\alpha}\right| \leq \varepsilon\left|\sigma(t) + 2t - 2s - \rho(t)\right|^{\alpha}$$

for all $s \in U$ for which $2t - s \in U$. Since t is dense,

$$\left|[-f(s) + f(2t-s)] - f^{\diamondsuit^{\alpha}}(t)[2t - 2s]^{\alpha}\right| \leq \varepsilon\left|2t - 2s\right|^{\alpha}$$

for all $s \in U$ for which $2t - s \in U$. It follows that

$$\left|\frac{f(2t-s) - f(s)}{(2t-2s)^{\alpha}} - f^{\diamondsuit^{\alpha}}(t)\right| \leq \varepsilon$$

for all $s \in U$ with $s \neq t$. Therefore, we get the desired result:

$$f^{\diamondsuit^{\alpha}}(t) = \lim_{s \to t} \frac{f(2t-s) - f(s)}{2^{\alpha}(t-s)^{\alpha}}.$$

Conversely, let us suppose that t is dense and the limit

$$\lim_{s \to t} \frac{f(2t-s) - f(s)}{2^{\alpha}(t-s)^{\alpha}} =: L$$

exists. Then, there exists a neighborhood U of t such that $\left|\frac{f(2t-s)-f(s)}{2^{\alpha}(t-s)^{\alpha}} - L\right| \leq \varepsilon$ for all $s \in U$ for which $2t - s \in U$. Because t is dense, we have

$$\left|\frac{f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)}{[\sigma(t) + 2t - 2s - \rho(t)]^{\alpha}} - L\right| \leq \varepsilon.$$

Therefore,

$$\left|\left[f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)\right]-L\left[\sigma(t)+2t-2s-\rho(t)\right]^{\alpha}\right| \\ \leq \varepsilon\left|\sigma(t)+2t-2s-\rho(t)\right|^{\alpha},$$

which leads us to the conclusion that f is symmetric differentiable of order α and $f^{\diamondsuit^{\alpha}}(t) = L$. Note that if we use the substitution $s = t + h$, then

$$f^{\diamondsuit^{\alpha}}(t)=\lim_{h\to 0}\frac{f(t+h)-f(t-h)}{2^{\alpha}h^{\alpha}}.$$

(v) If t is a dense point, then $\sigma(t)=\rho(t)$ and

$$f^{\sigma}(t)=f^{\rho}(t)+f^{\diamondsuit^{\alpha}}(t)\left[\sigma(t)-\rho(t)\right]^{\alpha}.$$

If t is not dense, and since f is continuous, then

$$f^{\diamondsuit^{\alpha}}(t)=\frac{f^{\sigma}(t)-f^{\rho}(t)}{\left[\sigma(t)-\rho(t)\right]^{\alpha}} \Leftrightarrow f^{\sigma}(t)=f^{\rho}(t)+f^{\diamondsuit}(t)\left[\sigma(t)-\rho(t)\right]^{\alpha}.$$

This concludes the proof. □

Remark 3.19. *An alternative way to define the symmetric fractional derivative of f of order $\alpha \in]0,1]$ at $t \in \mathbb{T}_{\kappa}^{\kappa}$ consists in saying that the limit*

$$\begin{aligned} f^{\diamondsuit^{\alpha}}(t) &= \lim_{s\to t}\frac{f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)}{\left[\sigma(t)+2t-2s-\rho(t)\right]^{\alpha}} \\ &= \lim_{h\to 0}\frac{f^{\sigma}(t)-f(t+h)+f(t-h)-f^{\rho}(t)}{\left[\sigma(t)-2h-\rho(t)\right]^{\alpha}} \end{aligned}$$

exists. Similarly, we can say that the nabla fractional derivative of f of order α is defined by

$$f^{\nabla^{\alpha}}(t)=\lim_{s\to t}\frac{f(s)-f^{\rho}(t)}{\left[s-\rho(t)\right]^{\alpha}}$$

and the delta fractional derivative of f of order α is defined by

$$f^{\Delta^{\alpha}}(t)=\lim_{s\to t}\frac{f^{\sigma}(t)-f(s)}{\left[\sigma(t)-s\right]^{\alpha}}.$$

Remark 3.20. *A function $f:\mathbb{R}\to\mathbb{R}$ is symmetric fractional differentiable of order α at point $t\in\mathbb{R}$ if, and only if, the limit*

$$\lim_{h\to 0}\frac{f(t+h)-f(t-h)}{2^{\alpha}h^{\alpha}}$$

exists as finite number. In this case,

$$f^{\diamondsuit^\alpha}(t) = \lim_{h\to 0} \frac{f(t+h) - f(t-h)}{2^\alpha h^\alpha}.$$

If $\alpha = 1$, then we obtain the classical symmetric derivative in the real numbers [74], defined by

$$f^{\diamondsuit}(t) = \lim_{h\to 0} \frac{f(t+h) - f(t-h)}{h}.$$

Remark 3.21. *Let $h > 0$. If a function $f : h\mathbb{Z} \to \mathbb{R}$ is symmetric differentiable of order α for $t \in h\mathbb{Z}$, then*

$$f^{\diamondsuit^\alpha}(t) = \frac{f(t+h) - f(t-h)}{2^\alpha h^\alpha}.$$

If $\alpha = 1$, then we obtain the symmetric h-derivative in the quantum set $h\mathbb{Z}$ [45], defined by

$$f^{\diamondsuit}(t) = \frac{f(t+h) - f(t-h)}{h}.$$

Let us now see some examples.

Proposition 3.22. *If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = c$ for all $t \in \mathbb{T}$, $c \in \mathbb{R}$, then $f^{\diamondsuit^\alpha}(t) = 0$ for any $t \in \mathbb{T}^\kappa_\kappa$.*

Proof. Trivially, we have

$$f^{\diamondsuit^\alpha}(t) = \lim_{s\to t} \frac{f^\sigma(t) - f(s) + f(2t-s) - f^\rho(t)}{[\sigma(t) + 2t - 2s - \rho(t)]^\alpha} = 0.$$

The proof is complete. □

Proposition 3.23. *If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then*

$$f^{\diamondsuit^\alpha}(t) = \begin{cases} [\sigma(t) - \rho(t)]^{1-\alpha} & \text{if } \alpha \neq 1 \\ 1 & \text{if } \alpha = 1 \end{cases}$$

for all $t \in \mathbb{T}^\kappa_\kappa$.

Proof. If t is not dense, then by Theorem 3.18 (iii)

$$f^{\diamondsuit^\alpha}(t) = \frac{f^\sigma(t) - f^\rho(t)}{[\sigma(t) - \rho(t)]^\alpha} = [\sigma(t) - \rho(t)]^{1-\alpha}.$$

If t is dense, then by Theorem 3.18 (iv)

$$f^{\diamondsuit^\alpha}(t) = \lim_{s\to t} \frac{f(2t-s) - f(s)}{2^\alpha (t-s)^\alpha} = \lim_{s\to t} \frac{2(t-s)}{2^\alpha (t-s)^\alpha}.$$

Thus, if $\alpha = 1$, then $f^{\diamondsuit^\alpha}(t) = 1$; if $0 < \alpha < 1$, then $f^{\diamondsuit^\alpha}(t) = 0$. □

Next proposition shows a function that is not differentiable at point $t = 0$ in the sense of classical (integer-order) calculus and standard (nonsymmetric) calculus on time scales [25, 26], but is symmetric fractional differentiable of order $\alpha \in]0, 1]$.

Proposition 3.24. *Let $\mathbb{T}$ be a time scale with $0 \in \mathbb{T}^{\kappa}_{\kappa}$ and $f : \mathbb{T} \to \mathbb{R}$ be defined by $f(t) = |t|$. Then,*

$$f^{\diamondsuit^{\alpha}}(0) = \begin{cases} 0 & \text{if } 0 \text{ is dense} \\ \dfrac{\sigma(0) + \rho(0)}{[\sigma(0) - \rho(0)]^{\alpha}} & \text{otherwise} \end{cases}$$

for any $\alpha \in]0, 1]$.

Proof. We know (see Remark 3.19) that

$$f^{\diamondsuit}(0) = \lim_{h \to 0} \frac{f^{\sigma}(0) - f(0 + h) + f(0 - h) - f^{\rho}(0)}{[\sigma(0) - 2h - \rho(0)]^{\alpha}} = \lim_{h \to 0} \frac{\sigma(0) + \rho(0)}{[\sigma(0) - 2h - \rho(0)]^{\alpha}}.$$

The result follows immediately from this equality. □

We now give some algebric properties of the symmetric fractional derivative.

Theorem 3.25. *Let $f, g : \mathbb{T} \to \mathbb{R}$ be two symmetric fractional differentiable functions of order α at $t \in \mathbb{T}^{\kappa}_{\kappa}$ and let $\lambda \in \mathbb{R}$. The following holds:*

(i) Function $f + g$ is symmetric fractional differentiable of order α at t with

$$(f + g)^{\diamondsuit^{\alpha}}(t) = f^{\diamondsuit^{\alpha}}(t) + g^{\diamondsuit^{\alpha}}(t).$$

(ii) Function λf is symmetric fractional differentiable of order α at t with

$$(\lambda f)^{\diamondsuit^{\alpha}}(t) = \lambda f^{\diamondsuit^{\alpha}}(t).$$

(iii) If f and g are continuous at t, then fg is symmetric fractional differentiable of order α at t with

$$(fg)^{\diamondsuit^{\alpha}}(t) = f^{\diamondsuit^{\alpha}}(t)\, g^{\sigma}(t) + f^{\rho}(t)\, g^{\diamondsuit^{\alpha}}(t).$$

(iv) If f is continuous at t and $f^{\sigma}(t) f^{\rho}(t) \neq 0$, then $1/f$ is symmetric fractional differentiable of order α at t with

$$\left(\frac{1}{f}\right)^{\diamondsuit^{\alpha}}(t) = -\frac{f^{\diamondsuit^{\alpha}}(t)}{f^{\sigma}(t)\, f^{\rho}(t)}.$$

(v) *If f and g are continuous at t and $g^{\sigma}(t)\, g^{\rho}(t) \neq 0$, then f/g is symmetric fractional differentiable of order α at t with*

$$\left(\frac{f}{g}\right)^{\diamondsuit^{\alpha}}(t) = \frac{f^{\diamondsuit^{\alpha}}(t)\, g^{\rho}(t) - f^{\rho}(t)\, g^{\diamondsuit^{\alpha}}(t)}{g^{\sigma}(t)\, g^{\rho}(t)}.$$

Proof. (i) For $t \in \mathbb{T}^{\kappa}_{\kappa}$ we have

$$\begin{aligned}(f+g)^{\diamondsuit^{\alpha}}(t) &= \lim_{s\to t} \frac{(f+g)^{\sigma}(t) - (f+g)(s) + (f+g)(2t-s) - (f+g)^{\rho}(t)}{[\sigma(t) + 2t - 2s - \rho(t)]^{\alpha}} \\ &= \lim_{s\to t} \frac{f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)}{[\sigma(t) + 2t - 2s - \rho(t)]^{\alpha}} \\ &\quad + \lim_{s\to t} \frac{g^{\sigma}(t) - g(s) + g(2t-s) - g^{\rho}(t)}{[\sigma(t) + 2t - 2s - \rho(t)]^{\alpha}} \\ &= f^{\diamondsuit^{\alpha}}(t) + g^{\diamondsuit^{\alpha}}(t).\end{aligned}$$

(ii) Let $t \in \mathbb{T}^{\kappa}_{\kappa}$ and $\lambda \in \mathbb{R}$. Then,

$$\begin{aligned}(\lambda f)^{\diamondsuit^{\alpha}}(t) &= \lim_{s\to t} \frac{(\lambda f)^{\sigma}(t) - (\lambda f)(s) + (\lambda f)(2t-s) - (\lambda f)^{\rho}(t)}{[\sigma(t) + 2t - 2s - \rho(t)]^{\alpha}} \\ &= \lambda \lim_{s\to t} \frac{f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)}{[\sigma(t) + 2t - 2s - \rho(t)]^{\alpha}} \\ &= \lambda f^{\diamondsuit^{\alpha}}(t).\end{aligned}$$

(iii) Let us assume that $t \in \mathbb{T}^{\kappa}_{\kappa}$ and f and g are continuous at t. If t is dense, then

$$\begin{aligned}(fg)^{\diamondsuit^{\alpha}}(t) &= \lim_{h\to 0} \frac{(fg)(t+h) - (fg)(t-h)}{2^{\alpha} h^{\alpha}} \\ &= \lim_{h\to 0} \frac{f(t+h) - f(t-h)}{2^{\alpha} h^{\alpha}} g(t+h) + \lim_{h\to 0} \frac{g(t+h) - g(t-h)}{2^{\alpha} h^{\alpha}} f(t-h) \\ &= f^{\diamondsuit^{\alpha}}(t)\, g^{\sigma}(t) + f^{\rho}(t)\, g^{\diamondsuit^{\alpha}}(t).\end{aligned}$$

If t is not dense, then

$$\begin{aligned}(fg)^{\diamondsuit^{\alpha}}(t) &= \frac{(fg)^{\sigma}(t) - (fg)^{\rho}(t)}{[\sigma(t) - \rho(t)]^{\alpha}} = \frac{f^{\sigma}(t) - f^{\rho}(t)}{[\sigma(t) - \rho(t)]^{\alpha}} g^{\sigma}(t) + \frac{g^{\sigma}(t) - g^{\rho}(t)}{[\sigma(t) - \rho(t)]^{\alpha}} f^{\rho}(t) \\ &= f^{\diamondsuit^{\alpha}}(t)\, g^{\sigma}(t) + f^{\rho}(t)\, g^{\diamondsuit^{\alpha}}(t).\end{aligned}$$

We just proved the intended equality. (iv) Using the relation $\left(\frac{1}{f} \times f\right)(t) = 1$ we can write that

$$0 = \left(\frac{1}{f} \times f\right)^{\diamondsuit^{\alpha}}(t) = f^{\diamondsuit^{\alpha}}(t) \left(\frac{1}{f}\right)^{\sigma}(t) + f^{\rho}(t) \left(\frac{1}{f}\right)^{\diamondsuit^{\alpha}}(t).$$

Therefore,

$$\left(\frac{1}{f}\right)^{\Diamond^\alpha}(t) = -\frac{f^{\Diamond^\alpha}(t)}{f^\sigma(t)\, f^\rho(t)}.$$

(v) Let $t \in \mathbb{T}_\kappa^\kappa$. Then,

$$\begin{aligned}\left(\frac{f}{g}\right)^{\Diamond^\alpha}(t) &= \left(f \times \frac{1}{g}\right)^{\Diamond^\alpha}(t) = f^{\Diamond^\alpha}(t)\left(\frac{1}{g}\right)^{\sigma}(t) + f^\rho(t)\left(\frac{1}{g}\right)^{\Diamond^\alpha}(t)\\ &= \frac{f^{\Diamond^\alpha}(t)}{g^\sigma(t)} + f^\rho(t)\left(-\frac{g^{\Diamond^\alpha}(t)}{g^\sigma(t)\, g^\rho(t)}\right) = \frac{f^{\Diamond^\alpha}(t)\, g^\rho(t) - f^\rho(t)\, g^{\Diamond^\alpha}(t)}{g^\sigma(t)\, g^\rho(t)}.\end{aligned}$$

The proof is complete. □

Example 3.26. *The symmetric fractional derivative of $f(t) = t^2$ of order α is*

$$f^{\Diamond^\alpha}(t) = \begin{cases} [\sigma(t) - \rho(t)]^{1-\alpha}[\sigma(t) + \rho(t)] & \text{if} \quad \alpha \neq 1\\ \sigma(t) + \rho(t) & \text{if} \quad \alpha = 1.\end{cases}$$

Example 3.27. *The symmetric derivative of $f(t) = 1/t$ of order α is*

$$f^{\Diamond^\alpha}(t) = \begin{cases} -\dfrac{[\sigma(t) - \rho(t)]^{1-\alpha}}{\sigma(t)\rho(t)} & \text{if} \quad \alpha \neq 1\\ -\dfrac{1}{\sigma(t)\rho(t)} & \text{if} \quad \alpha = 1.\end{cases}$$

The next result gives a relation between the nonsymmetric and symmetric fractional derivatives.

Proposition 3.28. *If f is both delta and nabla fractional differentiable of order α, then f is symmetric fractional differentiable of order α with*

$$f^{\Diamond^\alpha}(t) = \gamma_1(t)\, f^{\Delta^\alpha}(t) + \gamma_2(t)\, f^{\nabla^\alpha}(t)$$

for each $t \in \mathbb{T}_\kappa^\kappa$, where

$$\gamma_1(t) := \lim_{s\to t}\left[\frac{\sigma(t) - s}{\sigma(t) + 2t - 2s - \rho(t)}\right]^\alpha$$

and

$$\gamma_2(t) := \lim_{s\to t}\left[\frac{(2t - s) - \rho(t)}{\sigma(t) + 2t - 2s - \rho(t)}\right]^\alpha.$$

Proof. Note that

$$\begin{aligned}
f^{\diamondsuit^{\alpha}}(t) &= \lim_{s\to t}\frac{f^{\sigma}(t)-f(s)+f(2t-s)-f^{\rho}(t)}{[\sigma(t)+2t-2s-\rho(t)]^{\alpha}}\\
&= \lim_{s\to t}\left(\frac{[\sigma(t)-s]^{\alpha}}{[\sigma(t)+2t-2s-\rho(t)]^{\alpha}}\frac{f^{\sigma}(t)-f(s)}{[\sigma(t)-s]^{\alpha}}\right.\\
&\qquad\left.+\frac{[(2t-s)-\rho(t)]^{\alpha}}{[\sigma(t)+2t-2s-\rho(t)]^{\alpha}}\frac{f(2t-s)-f^{\rho}(t)}{[(2t-s)-\rho(t)]^{\alpha}}\right)\\
&= \lim_{s\to t}\left(\left[\frac{\sigma(t)-s}{\sigma(t)+2t-2s-\rho(t)}\right]^{\alpha}f^{\Delta}(t)\right.\\
&\qquad\left.+\left[\frac{(2t-s)-\rho(t)}{\sigma(t)+2t-2s-\rho(t)}\right]^{\alpha}f^{\nabla}(t)\right).
\end{aligned}$$

If $t\in\mathbb{T}$ is dense, then

$$\gamma_1(t)=\lim_{s\to t}\left[\frac{\sigma(t)-s}{\sigma(t)+2t-2s-\rho(t)}\right]^{\alpha}=\lim_{s\to t}\left[\frac{t-s}{2t-2s}\right]^{\alpha}=\frac{1}{2^{\alpha}}$$

and

$$\gamma_2(t)=\lim_{s\to t}\left[\frac{(2t-s)-\rho(t)}{\sigma(t)+2t-2s-\rho(t)}\right]^{\alpha}=\lim_{s\to t}\left[\frac{t-s}{2t-2s}\right]^{\alpha}=\frac{1}{2^{\alpha}}.$$

On the other hand, if $t\in\mathbb{T}$ is not dense, then

$$\gamma_1(t)=\lim_{s\to t}\left[\frac{\sigma(t)-s}{\sigma(t)+2t-2s-\rho(t)}\right]^{\alpha}=\left[\frac{\sigma(t)-t}{\sigma(t)-\rho(t)}\right]^{\alpha}$$

and

$$\gamma_2(t)=\lim_{s\to t}\left[\frac{(2t-s)-\rho(t)}{\sigma(t)+2t-2s-\rho(t)}\right]^{\alpha}=\lim_{s\to t}\left[\frac{t-\rho(t)}{\sigma(t)-\rho(t)}\right]^{\alpha}.$$

Hence, functions γ_1, $\gamma_2:\mathbb{T}\to\mathbb{R}$ are well defined and, if f is delta and nabla differentiable, then $f^{\diamondsuit^{\alpha}}(t)=\gamma_1(t)f^{\Delta}(t)+\gamma_2(t)f^{\nabla}(t)$. □

Remark 3.29. *Suppose that f is delta and nabla fractional differentiable of order α. If point $t\in\mathbb{T}_{\kappa}^{\kappa}$ is right-scattered and left-dense, then its fractional symmetric derivative of order α is equal to its delta fractional derivative of order α. If t is left-scattered and right-dense, then its symmetric fractional derivative of order α is equal to its nabla fractional derivative of order α.*

Due to Proposition 3.28, we can now define a symmetric integral of noninteger order.

Definition 3.30 (The symmetric fractional integral)**.** *Assume function* $f : \mathbb{T} \to \mathbb{R}$ *is simultaneously rd- and ld-continuous. Let* $a, b \in \mathbb{T}$ *and* $F^{\Delta^\beta}(t) = \int f(t)\Delta^\beta t$ *and* $F^{\nabla^\beta}(t) = \int f(t)\nabla^\beta t$ *denote the indefinite delta and nabla fractional integrals of* f *of order* β*, respectively. Then we define the Cauchy symmetric fractional integral of* f *of order* $\beta \in]0, 1]$ *by*

$$\begin{aligned}\int_a^b f(t)\diamondsuit^\beta t &= \gamma_1(t)\, F^{\Delta^\beta}(t)\Big|_a^b + \gamma_2(t)\, F^{\nabla^\beta}(t)\Big|_a^b \\ &= \gamma_1(b)\, F^{\Delta^\beta}(b) - \gamma_1(a)\, F^{\Delta^\beta}(a) + \gamma_2(b)\, F^{\nabla^\beta}(b) - \gamma_2(a)\, F^{\nabla^\beta}(a).\end{aligned}$$

Finally, we present some algebraic properties of the symmetric fractional integral.

Theorem 3.31. *Let* $a, b, c \in \mathbb{T}$ *and* $\lambda \in \mathbb{R}$*. If* $f, g \in \mathcal{C}_{ld}$ *and* $f, g \in \mathcal{C}_{rd}$ *with* $0 \leq \beta \leq 1$*, then*

(i) $\displaystyle\int_a^b [f(t) + g(t)]\diamondsuit^\beta t = \int_a^b f(t)\diamondsuit^\beta t + \int_a^b g(t)\diamondsuit^\beta t;$

(ii) $\displaystyle\int_a^b (\lambda f)(t)\diamondsuit^\beta t = \lambda \int_a^b f(t)\diamondsuit^\beta t;$

(iii) $\displaystyle\int_a^b f(t)\diamondsuit^\beta t = -\int_b^a f(t)\diamondsuit^\beta t;$

(iv) $\displaystyle\int_a^b f(t)\diamondsuit^\beta t = \int_a^c f(t)\diamondsuit^\beta t + \int_c^b f(t)\diamondsuit^\beta t;$

(v) $\displaystyle\int_a^a f(t)\diamondsuit^\beta t = 0.$

Proof. Equalities (i)–(v) follow from Definition 3.30 and analogous properties of the nabla and delta fractional integrals (cf. Theorem 3.14). □

Chapter 4

Fractional Riemann–Liouville Calculus on Time Scales

In this chapter, we introduce the concept of fractional derivative of Riemann–Liouville on a time scale $\mathbb{T}$. The notion is nonlocal, which contrasts with Chapters 2 and 3. Fundamental properties of the new operator are proved, as well as an existence and uniqueness result for a fractional initial value problem on an arbitrary time scale. The results of this chapter are original and are published in [23].

4.1 Introduction

Fractional calculus is nowadays one of the most intensively developing areas of mathematical analysis (see, e.g., [1, 6, 44, 53, 54, 55] and references therein), including several definitions of fractional operators like Riemann–Liouville, Caputo, and Grünwald–Letnikov. Operators for fractional differentiation and integration have been used in various fields, such as signal processing, hydraulics of dams, temperature field problem in oil strata, diffusion problems, and waves in liquids and gases [21, 71, 72].

We consider the following initial value problem:

$$ {}_{t_0}^{\mathbb{T}}D_t^{\alpha} y(t) = f(t, y(t)), \quad t \in [t_0, t_0 + a] = \mathcal{J} \subseteq \mathbb{T}, \quad 0 < \alpha < 1, \tag{4.1} $$

$$ {}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} y(t_0) = 0, \tag{4.2} $$

where ${}_{t_0}^{\mathbb{T}}D_t^{\alpha}$ is the (left) Riemann–Liouville fractional derivative operator or order α defined on $\mathbb{T}$, ${}_{t_0}^{\mathbb{T}}I_t^{1-\alpha}$ the (left) Riemann–Liouville fractional integral operator or order $1-\alpha$ defined on $\mathbb{T}$, and function $f : \mathcal{J} \times \mathbb{T} \to \mathbb{R}$ is a right-dense continuous function. Our main results give necessary and sufficient conditions for the existence and uniqueness of solution to problem (4.1)–(4.2). Before that, we need to fix some notations and recall some results.

We use $\mathcal{C}(\mathcal{J}, \mathbb{R})$ to denote the Banach space of continuous functions y with the norm $\|y\|_\infty = \sup\{|y(t)| : t \in \mathcal{J}\}$, where $\mathcal{J}$ is an interval.

Proposition 4.1 (See [5]). *Suppose $\mathbb{T}$ is a time scale and f is an increasing continuous function on the time-scale interval $[a, b]$. If F is the extension of f to the real interval $[a, b]$ given by*

$$F(s) := \begin{cases} f(s) & \text{if } s \in \mathbb{T}, \\ f(t) & \text{if } s \in (t, \sigma(t)) \notin \mathbb{T}, \end{cases}$$

then

$$\int_a^b f(t)\Delta t \leq \int_a^b F(t)dt.$$

We also make use of the classical gamma and beta functions.

Definition 4.2 (Gamma function). *For complex numbers with a positive real part, the gamma function $\Gamma(t)$ is defined by the following convergent improper integral:*

$$\Gamma(t) := \int_0^\infty x^{t-1} e^{-x} dx.$$

Definition 4.3 (Beta function). *The beta function, also called the Euler integral of first kind, is the special function $\mathrm{B}(x, y)$ defined by*

$$\mathrm{B}(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x > 0, \quad y > 0.$$

Remark 4.4. *The gamma function satisfies the following useful property: $\Gamma(t+1) = t\Gamma(t)$. The beta function can be expressed through the gamma function by $\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.*

The main results of this chapter are based on Banach's and Schauder's fixed point theorems.

Definition 4.5. *Let (X, d) be a metric space. Then a map $T : X \longrightarrow X$ is called a contraction mapping on X if there exists $\gamma \in [0, 1]$ such that*

$$d(T(x), T(y)) \leq \gamma d(x, y) \quad \textit{for all } x, y \textit{ in } X.$$

Theorem 4.6. *[36]. Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \longrightarrow X$. Then T admits a unique fixed point x^* in X (i.e., $T(x^*) = x^*$).*

Theorem 4.7 (Nonlinear Alternative of Leray–Schauder Type [36])**.** *Let X be a Banach space, C a closed, convex and nonempty subset of X, U an open subset of C and $0 \in X$. Suppose that $N : \overline{U} \to C$ is a continuous, compact map.*
Then either,

1. *N has a fixed point in U, or*

2. *There exists $\lambda \in (0, 1)$ and $x \in \partial U$ (the boundary of U in C) with $x = \lambda N(x)$.*

4.2 Fractional Riemann–Liouville Integral and Derivative

In this section, we introduce a new notion of fractional derivative on time scales. Before that, we define the fractional integral on a time scale $\mathbb{T}$. This is in contrast with [21, 22, 24] and in contrast with previous chapters, where first a notion of fractional differentiation on time scales is introduced and only after that, with the help of such concept, the fraction integral is defined.

Definition 4.8 (Fractional integral on time scales)**.** *Suppose $\mathbb{T}$ is a time scale, $[a, b]$ is an interval of $\mathbb{T}$, and h is an integrable function on $[a, b]$. Let $0 < \alpha < 1$. Then the (left) fractional integral of order α of h is defined by*

$$ {}_a^{\mathbb{T}}I_t^{\alpha} h(t) := \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s, $$

where Γ is the gamma function.

Definition 4.9 (Riemann–Liouville fractional derivative on time scales)**.** *Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$, $0 < \alpha < 1$, and $h : \mathbb{T} \to \mathbb{R}$. The (left) Riemann–Liouville fractional derivative of order α of h is defined by*

$$ {}_a^{\mathbb{T}}D_t^{\alpha} h(t) := \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t-s)^{-\alpha} h(s)\Delta s \right)^{\Delta}. \tag{4.3} $$

Remark 4.10. *If $\mathbb{T} = \mathbb{R}$, then Definition 4.9 gives the classical (left) Riemann–Liouville fractional derivative [68]. For different extensions of the fractional derivative to time scales, using the Caputo approach instead of the Riemann–Liouville, see [5, 20]. For local approaches to fractional calculus on time scales we refer the reader to [21, 22, 24]. Here we are only considering left operators. The corresponding right operators are easily obtained by changing the limits of integration in Definitions 4.8 and 4.9 from a to t (left of t) into t to b (right of t), as done in the classical fractional calculus [68]. Here we restrict ourselves to the delta approach to time scales. Analogous definitions are, however, trivially obtained for the nabla approach to time scales by using the duality theory of [33].*

Along the work, we consider the order α of the fractional derivatives in the real interval $(0, 1)$. We can, however, easily generalize our definition of fractional derivative to any positive real α. Indeed, let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then there exists $\beta \in (0, 1)$ such that $\alpha = \lfloor \alpha \rfloor + \beta$, where $\lfloor \alpha \rfloor$ is the integer part of α, and we can set

$${}_a^{\mathbb{T}}D_t^{\alpha} h := {}_a^{\mathbb{T}}D_t^{\beta} h^{\Delta^{\lfloor \alpha \rfloor}}.$$

Fractional operators of negative order are defined as follows.

Definition 4.11. *If $-1 < \alpha < 0$, then the (Riemann–Liouville) fractional derivative of order α is the fractional integral of order $-\alpha$, that is,*

$${}_a^{\mathbb{T}}D_t^{\alpha} := {}_a^{\mathbb{T}}I_t^{-\alpha}.$$

Definition 4.12. *If $-1 < \alpha < 0$, then the fractional integral of order α is the fractional derivative of order $-\alpha$, that is,*

$${}_a^{\mathbb{T}}I_t^{\alpha} := {}_a^{\mathbb{T}}D_t^{-\alpha}.$$

4.2.1 Properties of the Time-Scale Fractional Operators

In this subsection we prove some fundamental properties of the fractional operators on time scales.

Proposition 4.13. *Let $\mathbb{T}$ be a time scale with derivative Δ, and $0 < \alpha < 1$. Then,*

$${}_a^{\mathbb{T}}D_t^{\alpha} = \Delta \circ {}_a^{\mathbb{T}}I_t^{1-\alpha}.$$

Proof. Let $h : \mathbb{T} \to \mathbb{R}$. From 4.3 we have

$$\begin{aligned} {}_a^{\mathbb{T}}D_t^{\alpha} h(t) &= \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t-s)^{-\alpha} h(s) \Delta s \right)^{\Delta} \\ &= \left({}_a^{\mathbb{T}}I_t^{1-\alpha} h(t) \right)^{\Delta} = \left(\Delta \circ {}_a^{\mathbb{T}}I_t^{1-\alpha} \right) h(t). \end{aligned}$$

The proof is complete. □

Proposition 4.14. *For any function h integrable on $[a, b]$, the Riemann–Liouville Δ-fractional integral satisfies*

$${}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\beta} = {}_a^{\mathbb{T}}I_t^{\alpha+\beta}$$

for $\alpha > 0$ and $\beta > 0$.

Proof. By definition,

$$\begin{aligned} &\left({}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\beta} \right)(h(t)) = {}_a^{\mathbb{T}}I_t^{\alpha} \left({}_a^{\mathbb{T}}I_t^{\beta}(h(t)) \right) \\ &\quad = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left({}_a^{\mathbb{T}}I_t^{\beta}(h(s)) \right) \Delta s \\ &\quad = \frac{1}{\Gamma(\alpha)} \int_a^t \left((t-s)^{\alpha-1} \frac{1}{\Gamma(\beta)} \int_a^s (s-u)^{\beta-1} h(u) \Delta u \right) \Delta s \\ &\quad = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_a^s (t-s)^{\alpha-1} (s-u)^{\beta-1} h(u) \Delta u \Delta s \\ &\quad = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left[\int_a^s (t-s)^{\alpha-1} (s-u)^{\beta-1} h(u) \Delta u + \int_s^t (t-s)^{\alpha-1} (s-u)^{\beta-1} h(u) \Delta u \right] \Delta s \\ &\quad = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left[\int_a^t (t-s)^{\alpha-1} (s-u)^{\beta-1} h(u) \Delta u \right] \Delta s. \end{aligned}$$

From Fubini's theorem, we interchange the order of integration to obtain

$$\begin{aligned} \left({}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\beta} \right)(h(t)) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left[\int_a^t (t-s)^{\alpha-1} (s-u)^{\beta-1} h(u) \Delta s \right] \Delta u \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left[\int_a^t (t-s)^{\alpha-1} (s-u)^{\beta-1} \Delta s \right] h(u) \Delta u \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left[\int_u^t (t-s)^{\alpha-1} (s-u)^{\beta-1} \Delta s \right] h(u) \Delta u. \end{aligned}$$

By setting $s = u + r(t-u)$, $r \in \mathbb{R}$, we obtain that

$$\begin{aligned}
&\left({}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\beta}\right)(h(t)) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left[\int_0^1 (1-r)^{\alpha-1}(t-u)^{\alpha-1} r^{\beta-1}(t-u)^{\beta-1}(t-u)dr\right] h(u)\Delta u \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-r)^{\alpha-1} r^{\beta-1} dr \int_a^t (t-u)^{\alpha+\beta-1} h(u)\Delta u \\
&= \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-u)^{\alpha+\beta-1} h(u)\Delta u = \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-u)^{\alpha+\beta-1} h(u)\Delta u \\
&= {}_a^{\mathbb{T}} I_t^{\alpha+\beta} h(t).
\end{aligned}$$

The proof is complete. □

Proposition 4.15. *For any function h integrable on $[a,b]$ one has ${}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} h = h$.*

Proof. By Propositions 4.13 and 4.14, we have

$${}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} h(t) = \left[{}_a^{\mathbb{T}}I_t^{1-\alpha}\left({}_a^{\mathbb{T}}I_t^{\alpha}(h(t))\right)\right]^{\Delta} = \left[{}_a^{\mathbb{T}}I_t h(t)\right]^{\Delta} = h(t).$$

The proof is complete. □

Corollary 4.16. *For $0 < \alpha < 1$, we have ${}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}D_t^{-\alpha} = Id$ and ${}_a^{\mathbb{T}}I_t^{-\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} = Id$, where Id denotes the identity operator.*

Proof. From Definition 4.12 and Proposition 4.15, we have that

$${}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}D_t^{-\alpha} = {}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} = Id;$$

from Definition 4.11 and Proposition 4.15, we have that ${}_a^{\mathbb{T}}I_t^{-\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} = {}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} = Id$. □

Definition 4.17. *For $\alpha > 0$, let ${}_a^{\mathbb{T}}I_t^{\alpha}([a,b])$ denote the space of functions that can be represented by the Riemann–Liouville Δ integral of order α of some $\mathcal{C}([a,b])$-function.*

Theorem 4.18. *Let $f \in \mathcal{C}([a,b])$ and $\alpha > 0$. In order that $f \in {}_a^{\mathbb{T}}I_t^{\alpha}([a,b])$, it is necessary and sufficient that*

$${}_a^{\mathbb{T}}I_t^{1-\alpha} f \in C^1([a,b]) \tag{4.4}$$

and

$$\left({}_a^{\mathbb{T}}I_t^{1-\alpha} f(t)\right)\Big|_{t=a} = 0. \tag{4.5}$$

Proof. Assume $f \in {}_a^{\mathbb{T}}I_t^\alpha([a,b])$, $f(t) = {}_a^{\mathbb{T}}I_t^\alpha g(t)$ for some $g \in \mathcal{C}([a,b])$, and

$$ {}_a^{\mathbb{T}}I_t^{1-\alpha}(f(t)) = {}_a^{\mathbb{T}}I_t^{1-\alpha}\left({}_a^{\mathbb{T}}I_t^\alpha g(t)\right). $$

From Proposition 4.14, we have

$$ {}_a^{\mathbb{T}}I_t^{1-\alpha}(f(t)) = {}_a^{\mathbb{T}}I_t g(t) = \int_a^t g(s)\Delta s. $$

Therefore, ${}_a^{\mathbb{T}}I_t^{1-\alpha} f \in \mathcal{C}^1([a,b])$ and

$$ \left({}_a^{\mathbb{T}}I_t^{1-\alpha} f(t)\right)\Big|_{t=a} = \int_a^a g(s)\Delta s = 0. $$

Conversely, assume that $f \in \mathcal{C}([a,b])$ satisfies (4.4) and (4.5). Then, by Taylor's formula applied to function ${}_a^{\mathbb{T}}I_t^{1-\alpha} f$, one has

$$ {}_a^{\mathbb{T}}I_t^{1-\alpha} f(t) = \int_a^t \frac{\Delta}{\Delta s}{}_a^{\mathbb{T}}I_t^{1-\alpha} f(s)\Delta s, \quad \forall t \in [a,b]. $$

Let $\varphi(t) := \frac{\Delta}{\Delta s}{}_a^{\mathbb{T}}I_t^{1-\alpha} f(t)$. Note that $\varphi \in \mathcal{C}([a,b])$ by (4.4). Now, by Proposition 4.14, we have

$$ {}_a^{\mathbb{T}}I_t^{1-\alpha}(f(t)) = {}_a^{\mathbb{T}}I_t^1 \varphi(t) = {}_a^{\mathbb{T}}I_t^{1-\alpha}\left[{}_a^{\mathbb{T}}I_t^\alpha(\varphi(t))\right] $$

and thus

$$ {}_a^{\mathbb{T}}I_t^{1-\alpha}(f(t)) - {}_a^{\mathbb{T}}I_t^{1-\alpha}\left[{}_a^{\mathbb{T}}I_t^\alpha(\varphi(t))\right] \equiv 0. $$

Then,

$$ {}_a^{\mathbb{T}}I_t^{1-\alpha}\left[f - {}_a^{\mathbb{T}}I_t^\alpha(\varphi(t))\right] \equiv 0. $$

From the uniqueness of solution to Abel's integral equation [44], this implies that

$$ f - {}_a^{\mathbb{T}}I_t^\alpha \varphi \equiv 0. $$

Thus, $f = {}_a^{\mathbb{T}}I_t^\alpha \varphi$ and $f \in {}_a^{\mathbb{T}}I_t^\alpha[a,b]$. □

Theorem 4.19. *Let $\alpha > 0$ and $f \in \mathcal{C}([a,b])$ satisfy the condition in Theorem 4.18. Then,*

$$ \left({}_a^{\mathbb{T}}I_t^\alpha \circ {}_a^{\mathbb{T}}D_t^\alpha\right)(f) = f. $$

Proof. By Theorem 4.18 and Proposition 4.14, we have:

$$ {}_a^{\mathbb{T}}I_t^\alpha \circ {}_a^{\mathbb{T}}D_t^\alpha f(t) = {}_a^{\mathbb{T}}I_t^\alpha \circ {}_a^{\mathbb{T}}D_t^\alpha\left({}_a^{\mathbb{T}}I_t^\alpha \varphi(t)\right) = {}_a^{\mathbb{T}}I_t^\alpha \varphi(t) = f(t). $$

The proof is complete. □

4.2.2 Application to a Fractional Riemann–Liouville IVP on Time Scales

In this subsection we prove existence of solution to the fractional order initial value problem (4.1)–(4.2) defined on a time scale. For this, let $\mathbb{T}$ be a time scale and $\mathcal{J} = [t_0, t_0+a] \subset \mathbb{T}$. Then the function $y \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ is a solution of problem (4.1)–(4.2) if

$$ {}_{t_0}^{\mathbb{T}}D_t^{\alpha} y(t) = f(t,y) \text{ on } \mathcal{J}, $$

$$ {}_{t_0}^{\mathbb{T}}I_t^{\alpha} y(t_0) = 0. $$

To establish this solution, we need to prove the following lemma and theorem.

Lemma 4.20. *Let $0 < \alpha < 1$, $\mathcal{J} \subseteq \mathbb{T}$, and $f : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$. Function y is a solution of problem (4.1)–(4.2) if and only if this function is a solution of the following integral equation:*

$$ y(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, y(s)) \Delta s. $$

Proof. By Theorem 4.19, ${}_{t_0}^{\mathbb{T}}I_t^{\alpha} \circ \left({}_{t_0}^{\mathbb{T}}D_t^{\alpha}(y(t))\right) = y(t)$. From 4.3 we have

$$ y(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, y(s)) \Delta s. $$

The proof is complete. □

Our first result is based on the Banach fixed point theorem [36].

Theorem 4.21. *Assume $\mathcal{J} = [t_0, t_0+a] \subseteq \mathbb{T}$. The initial value problem (4.1)–(4.2) has a unique solution on $\mathcal{J}$ if the function $f(t,y)$ is a right-dense continuous bounded function such that there exists $M > 0$ for which $|f(t,y(t))| < M$ on $\mathcal{J}$ and the Lipshitz condition*

$$ \exists\, L > 0 : \forall\, t \in \mathcal{J} \text{ and } x, y \in \mathbb{R}, \quad \|f(t,x) - f(t,y)\| \le L\|x-y\| $$

holds.

Proof. Let $\mathcal{S}$ be the set of rd-continuous functions on $\mathcal{J} \subseteq \mathbb{T}$. For $y \in \mathcal{S}$, define

$$ \|y\| = \sup_{t \in \mathcal{J}} \|y(t)\|. $$

It is easy to see that $\mathcal{S}$ is a Banach space with this norm. The subset of $\mathcal{S}(\rho)$ and the operator T are defined by

$$ \mathcal{S}(\rho) = \{X \in \mathcal{S} : \|X_s\| \le \rho\} $$

and

$$\mathrm{T}(y) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, y(s)) \Delta s.$$

Then,

$$|\mathrm{T}(y(t))| \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} M \Delta s \leq \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Delta s.$$

Since $(t-s)^{\alpha-1}$ is an increasing monotone function, by using Proposition 4.1 we can write that

$$\int_{t_0}^{t} (t-s)^{\alpha-1} \Delta s \leq \int_{t_0}^{t} (t-s)^{\alpha-1} ds.$$

Consequently,

$$|\mathrm{T}(y(t))| \leq \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} ds \leq \frac{M}{\Gamma(\alpha)} \frac{a^\alpha}{\alpha} = \rho.$$

By considering $\rho = \frac{Ma^\alpha}{\Gamma(\alpha+1)}$, we conclude that T is an operator from $\mathcal{S}(\rho)$ to $\mathcal{S}(\rho)$. Moreover,

$$\begin{aligned}
\|\mathrm{T}(x) - \mathrm{T}(y)\| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} |f(s, x(s)) - f(s, y(s))| \Delta s \\
&\leq \frac{L\|\|x-y\|_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Delta s \\
&\leq \frac{L\|\|x-y\|_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} ds \\
&\leq \frac{L\|x-y\|_\infty}{\Gamma(\alpha)} \frac{a^\alpha}{\alpha} = \frac{La^\alpha}{\Gamma(\alpha+1)} \|x-y\|_\infty
\end{aligned}$$

for $x, y \in \mathcal{S}(\rho)$. If $\frac{La^\alpha}{\Gamma(\alpha+1)} \leq 1$, then it is a contraction map. This implies the existence and uniqueness of solution to problem (4.1)–(4.2). □

Theorem 4.22. *Suppose $f : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ is a rd-continuous bounded function such that there exists $M > 0$ with $|f(t, y)| \leq M$ for all $t \in \mathcal{J}$, $y \in \mathbb{R}$. Then problem* (4.1)–(4.2) *has a solution on $\mathcal{J}$.*

Proof. We use Schauder's fixed point theorem [36] to prove that T has a fixed point. The proof is given in several steps.

Step 1: T is continuous. Let y_n be a sequence such that $y_n \to y$ in $\mathcal{C}(\mathcal{J}, \mathbb{R})$. Then, for each $t \in \mathcal{J}$,

$$\begin{aligned}
&|T(y_n)(t) - T(y)(t)| \\
&\quad\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \left| f(s, y_n(s)) - f(s, y(s)) \right| \Delta s \\
&\quad\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \sup_{s\in\mathcal{J}} \left| f(s, y_n(s)) - f(s, y(s)) \right| \Delta s \\
&\quad\le \frac{\|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Delta s \\
&\quad\le \frac{\|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} ds \\
&\quad\le \frac{\|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_\infty}{\Gamma(\alpha)} \frac{a^\alpha}{\alpha} \\
&\quad\le \frac{a^\alpha \left\|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\right\|_\infty}{\Gamma(\alpha+1)}.
\end{aligned}$$

Since f is a continuous function, we have

$$|T(y_n)(t) - T(y)(t)|_\infty \le \frac{a^\alpha}{\Gamma(\alpha+1)} \left\|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\right\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2: the map T sends bounded sets into bounded set in $\mathcal{C}(\mathcal{J}, \mathbb{R})$. Indeed, it is enough to show that for any ρ there exists a positive constant l such that, for each

$$y \in B_\rho = \{y \in \mathcal{C}(\mathcal{J}, \mathbb{R}) : \|y\|_\infty \le \rho\},$$

we have $\|\mathrm{T}(y)\|_\infty \le l$. By hypothesis, for each $t \in \mathcal{J}$ we have

$$\begin{aligned}
|\mathrm{T}(y)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} |f(s, y(s))| \Delta s \\
&\le \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Delta s \\
&\le \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} ds \\
&\le \frac{M a^\alpha}{\alpha \Gamma(\alpha)} \\
&= \frac{M a^\alpha}{\Gamma(\alpha+1)} = l.
\end{aligned}$$

Step 3: the map T sends bounded sets into equicontinuous sets of $\mathcal{C}(\mathcal{J}, \mathbb{R})$. Let $t_1, t_2 \in$

$\mathcal{J}, t_1 < t_2$, B_ρ be a bounded set of $\mathcal{C}(\mathcal{J},\mathbb{R})$ as in Step 2, and $y \in B_\rho$. Then,

$$\begin{aligned}
&|T(y)(t_2) - T(y)(t_1)| \\
&\quad \le \frac{1}{\Gamma(\alpha)}\left|\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}f(s,y(s))\Delta s - \int_{t_0}^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))\Delta s\right| \\
&\quad \le \frac{1}{\Gamma(\alpha)}\Big|\int_{t_0}^{t_1}((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}+(t_2-s)^{\alpha-1})f(s,y(s))\Delta s \\
&\qquad\qquad - \int_{t_0}^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))\Delta s\Big| \\
&\quad \le \frac{M}{\Gamma(\alpha)}\left|\int_{t_0}^{t_1}((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1})\Delta s + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\Delta s\right| \\
&\quad \le \frac{M}{\Gamma(\alpha)}\left|\int_{t_0}^{t_1}((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1})ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right| \\
&\quad \le \frac{M}{\Gamma(\alpha+1)}[(t_2-t_1)^\alpha + (t_1-t_0)^\alpha - (t_2-t_0)^\alpha] + \frac{M}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha \\
&\quad = \frac{2M}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha + \frac{M}{\Gamma(\alpha+1)}[(t_1-t_0)^\alpha - (t_2-t_0)^\alpha].
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzela–Ascoli theorem, we conclude that $T : \mathcal{C}(\mathcal{J},\mathbb{R}) \to \mathcal{C}(\mathcal{J},\mathbb{R})$ is completely continuous.

Step 4: a priori bounds. Now it remains to show that the set

$$\Omega = \{y \in \mathcal{C}(\mathcal{J},\mathbb{R}) : y = \lambda \mathrm{T}(y), 0 < \lambda < 1\}$$

is bounded. Let $y \in \Omega$. Then $y = \lambda\mathrm{T}(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in \mathcal{J}$, we have

$$y(t) = \lambda\left[\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,y(s))\Delta s\right].$$

We complete this step by considering the estimation in Step 2. As a consequence of Schauder's fixed point theorem, we conclude that T has a fixed point, which is solution of problem (4.1)–(4.2). □

Conclusion and Further Work

In this PhD thesis the study of differentiation and integration of non-integer order is extended, via the recent and powerful calculus on time scales, to include, in a single theory, the discrete fractional difference calculus and the local continuous fractional differential calculus. We introduced some fundamental concepts and proved some basic properties, and much remains to be done in order to develop the theory here initiated: to prove concatenation properties of derivatives and integrals, to consider partial fractional operators on time scales, to introduce a suitable fractional exponential on time scales, to study boundary value problems for fractional differential equations on time scales, to investigate the usefulness of the new fractional calculus in applications to real world problems where the time scale is partially continuous and partially discrete with a time-varying graininess function, etc. Both non-symmetric and symmetric fractional derivatives and integrals on an arbitrary nonempty closed subset of the real numbers are introduced and their fundamental properties derived. It is shown that a function may be fractional differentiable but not differentiable; and that a function may be symmetric fractional differentiable but not fractional differentiable. A relation between the non-symmetric and symmetric fractional derivatives is also derived.

For further work, let us note that much remains to be carried out in order to develop the theory here initiated. In particular, it would be interesting to investigate the usefulness of the new fractional calculi in applications to real world problems, where the time scale is partially continuous and partially discrete with a time-varying graininess function. This and other questions will be subject to future research. There are several possibilities, since it is possible to develop fractional calculi on time scales in different directions, e.g., instead of following the more common delta approach to time scales, one can develop a nabla [7, 57], a diamond [55, 63], or a symmetric [31, 28] time scale fractional calculus. The richness of time scales together with the richness of fractional calculus will continue to motivate further research [24].

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Abstract

In this PhD thesis, we introduce new general notions of fractional derivative and integration for functions defined on arbitrary time scales, like nonsymetric and symmetric fractional calculus on arbitrary time scales. Main properties of the new fractional operators are investigated and some fundamental results presented, illustrating the interplay between discrete and continuous behaviors. Finally, we introduce the concept of fractional derivative of Riemann–Liouville on time scales. Fundamental properties of the new operator are proved, as well as an existence and uniqueness result for a fractional initial value problem on an arbitrary time scale.

Keywords and phrases:

Factional differentiation, fractional integration, calculus on time scales, discrete and continuous fractional calculi, nonsymmetric and symmetric fractional calculi, fractional order derivatives, dynamic equations, initial value problems, time scales.

AMS (MOS) Subject Classifications: 26A33, 26E70, 34N05.

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Résumé

Dans cette thèse, nous introduisons des nouvelles notions générales de la dérivée et intégration fractionnaire des fonctions définies sur des échelles de temps arbitraires, ainsi que les calculs fractionnaires symétriques et non-symétriques sur des échelles de temps arbitraires. Plusieurs propriétés de nouveaux opérateurs fractionnaires sont établies, et quelques résultats fondamentaux sont présentés, illustrant la relation entre le comportement continu et discret. Et enfin, nous introduisons le concept de dérivée fractionnaire de Riemann–Liouville sur des échelles de temps. Nous prouvons les propriétés fondamentales du nouvel opérateur, ainsi que l'existence et l'unicité du problème à valeur initial fractionnaire sur une échelle de temps arbitraire.

Mots et Phrases Clefs:

Différentiation fractionnaire, intégration fractionnaire, calculs sur les échelles de temps, calcul fractionnaire discret et continue, le calcul fractionnaire symétrique et non-symétrique, dérivées d'ordre fractionnaire, équations dynamiques, problèmes à valeurs initiales, échelles de temps.

Classification AMS: 26A33, 26E70, 34N05.