N° d'ordre :

REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBES

THÈSE DE DOCTORAT

Présentée par

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Spécialité : *Mathématiques*

Option : *Statistiques Mathématiques*

Intitulée

Une Contribution aux Equations et Inclusions Différentielles Stochastiques

Soutenue le : 31 / 01 /2016

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Je dédie ce modeste travail à
mes chers parents
ma femme et mes enfants
mes frères et sœurs
et tous mes amis .

Remerciements

Tout d'abord, mes remerciements s'adressent à mon directeur de thèse, Pr OUAHAB Abdelghani pour son aide et son encouragement prodigués tout au long de ce travail de recherche, ainsi que pour sa grande disponibilité, sa rigueur scientifique et ses précieux conseils.

Je tiens à exprimer aussi ma reconnaissance aux membres du jury : Au Pr Yousfate Abderrahmane qui me fait l'honneur de présider ce jury et aux professeurs Tomas Caraballo, Terbeche Mekki, Benchohra Mouffak et Belmekki Mohammed qui m'ont fait l'honneur d'examiner ce travail.

A tous ceux qui n'ont pas été nommément mentionnés dans cette page de remerciements mais qui ont contribué directement ou indirectement à la réalisation de cette thèse; qu'ils trouvent en cette dernière phrase l'expression de toute ma gratitude.

Contents

Introduction

Stochastic Integration and Stochastic Differential Equations have become an active area of investigation due to their applications in the fields such as mechanics, electrical engineering, medicine biology, ecology etc.

It is highly considered that the evolution of many physical systems is described by an ordinary differential equation of the form:

$$dy(t) = f(t, y(t))dt, \tag{1}$$

However, in certain circumstances, physical systems are disturbed by random noise. One way to handle these disturbances is to alter equation (1) disturbing terms of the form $gdB_Q^H(t)$, where g characterizes the noise power. This leads to an evolution equation of the form:

$$dy(t) = f(t, y(t))dt + g(t)dB_Q^H(t). \tag{2}$$

On the other hand, the Brownian motion is not differentiable, equation (2) should be discerned from the classical differential calculus. So, a meaning to $dB_Q^H(t)$ should be given in order to define $y(t)$ as the solution of the equation:

$$y(t) = y(0) + \int_0^t f(s, y(s))ds + \int_0^t g(s)dB_Q^H(s). \tag{3}$$

An equation of the form (2) is called stochastic differential equation.

In recent years, stochastic differential and partial differential equations and inclusions have been extensively studied. For instance, in [1, 2, 4, 9] it is investigated the existence of solutions of nonlinear stochastic differential inclusions by Banach fixed point theorem and semigroup approach. Jiang and Shen [50] studied the existence and uniqueness of mild solutions to neutral stochastic partial functional differential equations under some Carathéodory-type conditions on the coefficients by means of the successive approximation. The nonlinear alternative of Leray-Schauder type is used to investigate the existence of solutions for first-order semilinear stochastic functional differential equations in Hilbert spaces in [11]. Balasubramaniam [8] obtained existence of solutions of functional stochastic differential inclusions by Kakutani's fixed point theorem, Balasubramaniam *et al.* [9,10] initiated the study of existence of solutions of semilinear stochastic delay evolution inclusions in a Hilbert space by using the nonlinear alternative of Leray-Schauder type [30]. In [64] the authors study the existence results for impulsive neutral stochastic evolution inclusions in Hilbert spaces where they considered a class of first-order evolution inclusions with convex and nonconvex cases for the above problem by a fixed point theorem due to Dhage and Covitz and Nadler's theorem for contraction multivalued maps (see [27]). Very recently Boudaoui et al [19] have initialed the study the existence of mild solutions for a first-order impulsive semilinear stochastic functional differential inclusions driven by a fractional Brownian motion with infinite delay. We consider the cases in which the right hand side is convex or nonconvex-valued. The results are obtained by using two different fixed point theorems for multivalued mappings.

Meanwhile, differential equations with impulses were considered for the first time in the 1960's by Milman and Myshkis [61, 62]. After a period of active research, mostly in Eastern Europe from 1960-1970, culminating with the monograph by Halanay and Wexler [39].

Impulsive differential systems and evolution differential systems are used to describe various models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. That is why in recent years they have been the object of investigations. We refer to the monographs of Bainov and Simeonov [7], Benchohra *et al.* [12], Lakshmikantham *et al.* [56], Samoilenko and Perestyuk [76] where numerous properties of their solutions are studied, and a detailed bibliography is given. Semilinear functional differential equations and inclusions with or without impulses have been extensively studied where the operator A generates a C_0-semigroup. Existence and uniqueness, among other things, are derived; see the books of Dejabli *et al.* [29] and Graef et al [36].

In many fields of science we can describe various evolutionary process by differential equations with delay and for this reason the study of this type of equations has received great attention during the last years. The literature on differential equations with delay is very extensive, we refer to the monographs of Hale [40], Hale and Verduyn Lunel [43], Kolmanovskii and Myshkis [54], V. Lakshmikantham et al. [57] and Wu [84] and the reference therein.

There has not been very much study of stochastic functional differential equation driven by a fractional Brownian Motion. To the best of our knowledge, there exist only a few papers published in this field. Tindel et al. studied in [80] the existence, uniqueness and the spatial regularity of the solution, as well as the smoothness of the density when $H > 1/2$ under strong hypotheses. In [24] Caraballo et al. studied the existence, uniqueness and exponential asymptotic behavior of mild solutions to stochastic delay evolution equations perturbed by a fractional Brownian motion. Boufoussi and Hajji analyzed in [20] the existence and uniqueness of mild solutions for a neutral stochastic differential equation with finite delay, driven by a fractional Brownian motion in a Hilbert space, and established some sufficient conditions ensuring the exponential decay to zero in mean square for the mild solution. Boudaoui et al [17] proved the existence of mild solutions to stochastic impulsive evolution equations with time delays, driven by fractional Brownian motion with the Hurst index $H > 1/2$ is based on a fixed point theorem of Burton and Kirk [22] for the sum of a contraction map and a completely continuous one.

Recently, Boudaoui et al. [18] studied the local and global existence and attractively of mild solutions for stochastic impulsive neutral functional differential equations with infinite delay, driven by fractional Brownian motion.

In this thesis, we shall be concerned by semilinear stochastic differential equations and inclusions with impulsive and delay, some existence results, among others things, are derived, Our results are based upon very recently fixed point theorems and using semigroups theory. We have arranged this thesis as follows:
In chapter 1 we give some basic concepts about stochastic processes, martingale theory and Brownian motion, in the last section we show some recent applications of the Malliavin Calculus to develop a stochastic calculus with respect to the fractional Brownian motion

In chapter 2 we collect some preliminary materials on phase spaces used throughout this thesis, the next section is devoted to set-valued maps some notions,in section 3 we give some fixed point theorems , in the last section we present some propriety of semigroups.

In chapter 3 we give our first main existence of mild solutions to stochastic impulsive evolution equations with time delays, driven by fractional Brownian motion with the Hurst index $H > 1/2$.

$$dy(t) = [Ay(t) + f(t, y_t)]dt + g(t)dB_Q^H(t), \quad t \in J := [0, T]; \tag{4}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k)), \; k = 1, \ldots, m \tag{5}$$

$$y(t) = \phi(t), \quad \text{for a.e. } t \in [-r, 0], \tag{6}$$

in a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$, where $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$, then we assume that $f : J \times \mathcal{D}_{\mathcal{H}}^0 \to \mathcal{H}$, $g : J \to L_Q^0(\mathcal{K}, \mathcal{H})$ Here, $L_Q^0(\mathcal{K}, \mathcal{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathcal{K}$ into $\mathcal{H}$.

As for the impulse functions we will assume that $I_k \in C(\mathcal{H}, \mathcal{H})$ $(k = 1, \ldots, m)$, and $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$.

In Chapter 4, our main objective is to establish sufficient conditions for the local and global existence and attractivity of mild solutions to the following first order neutral stochastic impulsive functional equation with time delays:

$$d[y(t) - g(t, y_t)] = [Ay(t) + f(t, y_t)]dt + \sigma(t)dB_Q^H(t), \quad t \in J := [0, T], \tag{7}$$

$$\Delta y|_{t=t_k} = y(t_k^+) - y(t_k) = I_k(y(t_k)), \; k = 1, \ldots, m, \tag{8}$$

$$y(t) = \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad \text{for a.e. } t \in J_0 = (-\infty, 0], \tag{9}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$, B_Q^H is a fractional Brownian motion on a real and separable Hilbert space $\mathcal{K}$, with Hurst parameter $H \in (1/2, 1)$.

Then we assume that $g : J \times \mathcal{D}_{\mathcal{F}_0} \to \mathcal{H}$, $f : J \times \mathcal{D}_{\mathcal{F}_0} \to \mathcal{H}$ and $\sigma : J \to L_Q^0(\mathcal{K}, \mathcal{H})$.

Finally, in Chapter 5, we prove the existence of mild solutions for a first-order impulsive semilinear stochastic functional differential inclusions driven by a fractional Brownian motion with infinite delay. We consider the cases in which the right hand side is convex or nonconvex-valued

$$\begin{cases} dy(t) \in [Ay(t) + F(t, y_t)]dt + g(t)dB_Q^H(t), & t \in J = [0, T], \quad t \neq t_k, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \; k = 1, \dots, m, & \\ y(t) = \phi(t) \in \mathcal{D}, & J_0 = (-\infty, 0], \end{cases} \tag{10}$$

where $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t)$, $0 \leq t \leq T$.

Assume $F : J \times \mathcal{D} \longrightarrow \mathcal{P}(\mathcal{H})$ is a bounded, closed and convex-valued multivalued map, $g : J \to L_Q^0(\mathcal{K}, \mathcal{H})$.

Mots clé: Fractional Brownian motion, mild solutions, stochastic functional differential equation, attractivity, neutral stochastic functional differential equation, impulsive stochastic functional differential inclusion.
Classification AMS: 34A37, 34A60, 35A01, 34G20, 34G25, 34K30, 60H10, 60H15, 60H20.

Chapter 1

Some Elements of Stochastic Analysis

1.1 Notations and Definitions

Definition 1.1.1. *Let Ω be a set, a set $\mathcal{F}$ of subsets of Ω is called a σ-algebra if the following three properties are satisfied:*

(i) $\Omega \in \mathcal{F}$,

(ii) $\forall A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,

(iii) $A_n \in \mathcal{F} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

A pair $(\Omega, \mathcal{F})$ for which $\mathcal{F}$ is a σ-algebra in Ω is called a measurable space.

Definition 1.1.2. *$(E, \mathcal{O})$ is a topological space, where $\mathcal{O}$ is the set of open sets in E. then $\sigma(\mathcal{O})$ is called the Borel σ-algebra of the topological space. If $\mathcal{A} \subset \mathcal{B}$, then A is called a subalgebra of $\mathcal{B}$. A set B in $\mathcal{B}$ is also called a Borel set.*

Definition 1.1.3. *Given a measurable space $(\Omega, \mathcal{F})$. A function $P : \mathcal{F} \longrightarrow \mathbb{R}$ is called a probability measure and $(\Omega, \mathcal{F}, P)$ is called a probability space if the following three properties called Kolmogorov axioms are satisfied:*

(i) $P(A) \geq 0$ *for all* $A \in \mathcal{F}$,

(ii) $P(\Omega) = 1$,

(iii) $A_n \in \mathcal{F}$ *disjoint* $\Rightarrow P(\bigcup_n A_n) = \sum_n P(A_n)$.

The last property is called σ-additivity.

Definition 1.1.4. *A map X from a measure space $(\Omega, \mathcal{F})$ to an other measure space $(\Delta, \mathcal{B})$ is called measurable, if $X^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. The set $X^{-1}(B)$ consists of all points $x \in \Omega$ for which $X(x) \in B$. This pull back set $X^{-1}(B)$ is defined even if X is non-invertible.*

Definition 1.1.5. *A function $X : \Omega \longrightarrow \mathbb{R}$ is called a random variable, if it is a measurable map from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}$. Denote by $\mathcal{L}$ the set of all real random variables. The set $\mathcal{L}$ is an algebra under addition and multiplication: one can add and multiply random variables and gets new random variables. More generally, one can consider random variables taking values in a second measurable space $(E, \mathcal{B})$. If $E = \mathbb{R}^d$, then the random variable X is called a random vector. For a ran dom vector $X = (X_1, \cdots, X_d)$, each component X_i is a random variable.*

Example 1.1.1. *Let $\Omega = \mathbb{R}^2$ with Borel σ-algebra $\mathcal{F}$ and let*

$$P(A) = \frac{1}{2\pi} \int \int_A e^{-(x^2-y^2)/2} dxdy.$$

Any continuous function X of two variables is a random variable on Ω. For example, $X(x, y) = xy(x + y)$ is a random variable. But also $X(x, y) = 1/(x+y)$ is a random variable, even so it is not continuous. The vectorvalued function $X(x, y) = (x, y, x^3)$ is an example of a random vector.

Definition 1.1.6. *A statement S about points $\omega \in \Omega$ ft is a map from Ω to true, false. A statement is said to hold almost everywhere, if the set $P[\{\omega | S(\omega) = \text{ false } \}] = 0$. For example, the statement " let $X_n \longrightarrow X$ almost everywhere ", is a short hand notation for the statement that the set $\{x \in \Omega \mid X_n(x) \longrightarrow X(x)\}$ is measurable and has measure 1.*

Definition 1.1.7. *The algebra of all random variables is denoted by $\mathcal{L}$. It is a vector space over the field $\mathbb{R}$ of the real numbers in which one can multiply. A elementary function or step function is an element of $\mathcal{L}$ which is of the form*

$$X = \sum_{i=1}^{n} \alpha_i \cdot 1_{A_i}$$

with $\alpha_i \in \mathbb{R}$ and where $A_i \in \mathcal{F}$ are disjoint sets. Denote by $\mathcal{S}$ the algebra of step functions. For $X \in \mathcal{S}$ we can define the integral

$$E(X) = \int_\Omega XdP = \sum_{i=1}^{n} \alpha_i P(A_i).$$

Definition 1.1.8. *Define $\mathcal{L}^1 \subset \mathcal{L}$ as the set of random variables X, for which*

$$\sup_{Y \in \mathcal{S}, Y \leq X} \int YdP$$

is finite. For $X \in \mathcal{L}^1$, we can define the integral or expectation

$$E(X) = \int XdP = \sup_{Y \in \mathcal{S}, Y \leq X^+} \int YdP - \sup_{Y \in \mathcal{S}, Y \leq X^-} \int YdP,$$

where $X^+ = X \vee 0 = \max(X, 0)$ and $X^- = -X \vee 0 = \max(-X, 0)$. The vector space $\mathcal{L}^1$ is called the space of integrable random variables. Similarly, for $p \geq 1$ write $\mathcal{L}^p$ for the set of random variables X for which $E(|X|^P) < \infty$.

Definition 1.1.9. *For $X, Y \in \mathcal{L}^2$ define the covariance*

$$Cov(X, Y) := E\big[\big(X - E(X)\big)\big(Y - E(Y)\big)\big] = E(XY) - E(X)E(Y).$$

Two random variables in $\mathcal{L}^2$ are called uncorrected if $Cov(X, Y) = 0$.

Definition 1.1.10. *For $X \in \mathcal{L}^2$,we can define the variance*

$$Var(X) = Cov(x, x) = E((X - E(X))^2).$$

Definition 1.1.11. *Write $J \subset I$ if J is a finite subset of I. A family $\{\mathcal{F}_i\}_{i \in I}$ of σ-sub-algebras of $\mathcal{F}$ is called independent, if for every $J \subset I$ and every choice $A_j \in \mathcal{F}_j$ $P[\bigcap_{j \in J} A_j] = \prod_{j \in J} P(A_j)$. A family $\{X_j\}_{j \in J}$ of random variables is called independent, if $\{\sigma(X_j)\}_{j \in J}$ are independent σ-algebras. A family of sets $\{A_j\}_{j \in I}$ is called independent, if the σ-algebras $\mathcal{F}_j = \{\emptyset, A_j, A_j^c, \Omega\}$ are independent.*

Probability distribution function

The probability distribution function $F_x : \mathbb{R} \longrightarrow [0, 1]$ of a random variable X was defined as

$$F_X(x) = P(X \leq x),$$

where $P(X \leq x)$ is a short hand notation for $P(\{\omega \in \Omega \mid X(\omega) < x\})$.

1.1.1 Some inequalities

[52]
Let X and Y are random variables on $\mathcal{L}^p(\Omega, \mathcal{F}, P)$

- `Chebychev inequality:` for all $\lambda > 0$

$$P(|X| > \lambda) \leq \frac{1}{\lambda^p} E(|X|^p).$$

- `Shwartz inequality:`

$$E(XY) \leq \sqrt{E(X^2)E(Y^2)}.$$

- `Hölder inequality:`

$$\mathbb{E}(XY) \leq [E|X|^p]^{\frac{1}{p}} [\mathbb{E}|Y|^q]^{\frac{1}{q}}$$

  where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

- `Jensen inequality:` For any convex function $h : \mathbb{R} \longrightarrow \mathbb{R}$, we have

$$h(E(X)) \leq E(hE(X)).$$

- `Kolmogorov inequality:` Let $X_1, \cdots, X_n$ are independent random variables with $E(X_i) = 0$ and $var(X_i) < \infty$. If $S_n = X_1 + \cdots + X_n$ then

$$P(\max_{1 \leq k \leq n} |S_k| \geq t) \leq \frac{1}{t^2} Var S_k, \forall t > 0.$$

1.1.2 Convergence of random variables

Converges almost surely:
Sequence of random variables X_n converges almost every where or almost surely to a random variable X, if

$$P(\{\omega \in \Omega : \lim_{n \longrightarrow \infty} X_n(\omega) = X(\omega)\}) = 1.$$

Converges in probability :

Sequence of random variables X_n converges in probability to a random variable X, if

$$\lim_{n \longrightarrow \infty} P(|X_n - X| > \varepsilon) = 0$$

for all $\varepsilon > 0$.
Converges in $\mathcal{L}^p$:

sequence of $\mathcal{L}^p$ random variables X_n converges in $\mathcal{L}^p$ to a random variable X, if

$$||X_n - X|| = \mathbb{E}[|X_n - X|^p]^{\frac{1}{p}} \underset{n \longrightarrow +\infty}{\longrightarrow} 0.$$

If X_n converges in $\mathcal{L}^p \Longrightarrow X_n$ converges in probability
Converges completely :

sequence of random variables X_n converges fast in probability, or completely if

$$\sum_n P(|X_n - X| \geq \varepsilon) < \infty$$

for all $\varepsilon > 0$.
Converges in law :
A sequence of random variables X_n converges in law to a random variable X, if

$$\lim_{n \longrightarrow \infty} F_{X_n}(t) = F_X(t)$$

for all $t \in \mathbb{R}$.

1.1.3 Conditional expectation

Definition 1.1.12. *Given a random variable X with $E|X| < \infty$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and some sub-σ-field $\mathcal{G} \in \mathcal{F}$ we will define the conditional expectation as the almost surely unique random variable $E(X|\mathcal{G})$ which satisfies the following two conditions*

1. *$E(X|\mathcal{G})$ is $\mathcal{G}$-measurable*
2. *$E(XZ) = E(E(Y|\mathcal{G})Z)$ for all Z which are bounded and $\mathcal{G}$-measurable*

Remark 1.1.1. *one could replace 2. in the previous definition with:*

$$\forall G \in \mathcal{G},\ E(X 1_G) = E(E(X|\mathcal{G})1_G).$$

Definition 1.1.13. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ and X another random variable defined on $(\Omega, \mathcal{F}, P)$. Define then $E(X|Y)$ the conditional expectation of X given Y as $E(X|\sigma(Y))$.*

Properties of Conditional Expectation

1. $E(\cdot|\mathcal{G})$ is positive:
$$X \geq 0 \Longrightarrow E(X|\mathcal{G}) \geq 0)$$

2. $E(\cdot|\mathcal{G})$ is linear:
$$E(aX + bY|\mathcal{G}) = aE(X|\mathcal{G}) + bE(Y|\mathcal{G})$$

3. $E(\cdot|\mathcal{G})$ is a projection:
$$E(E(X|\mathcal{G})|\mathcal{G}) = E(X|\mathcal{G})$$

4. If $\widehat{\mathcal{G}} \subset \mathcal{G}$ then
$$E(E(X|\mathcal{G})|\widehat{\mathcal{G}}) = E(X|\widehat{\mathcal{G}}) = E(E(X|\widehat{\mathcal{G}})|\mathcal{G})$$

5. $E(\cdot|\mathcal{G})$ commutes with multiplication by $\mathcal{G}$-measurable variables:
$$E(XY|\mathcal{G}) = E(X|\mathcal{G})Y \text{ for } E|XY| < \infty \text{ and } Y\mathcal{G} - \text{measurable}$$

6. If ψ is convex and $E|\psi(X)| < \infty$ then a conditional form of Jensen's inequality holds:
$$\psi(E(X|\mathcal{G}) \leq E(\psi(X)|\mathcal{G})$$

1.1.4 Processes and filtrations

Definition 1.1.14. . *A collection of sub σ-algebras $\{\mathcal{F}_t; t \geq 0\}$ of the σ-algebra $\mathcal{F}$ is called a filtration if $s \leq t$ implies that $\mathcal{F}_s \subset \mathcal{F}_t$*

For a given stochastic process X, write F_t^X for the filtration $\sigma\{X_s; 0 \leq s \leq t\}$. Call $\{F_t^X; t \geq 0\}$ is called the natural filtration associated to the process X.

Definition 1.1.15. *A stochastic process is a parametrized collection of random variables $\{X_t\}_{t\in T}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^n$.*

The parameter space T is usually the halfline $[0, \infty)$, but it may also be an interval $[a, b]$, the non-negative integers and even subsets of $\mathbb{R}^n$ for $n \geq 1$. Note that for each $t \in T$ fixed we have a random variable

$$\omega \longrightarrow X_t(\omega); \quad \omega \in \Omega.$$

On the other hand, fixing $\omega \in \Omega$ we can consider the function

$$t \longrightarrow X_t(\omega); \quad t \in T$$

which is called a path of X_t.

Definition 1.1.16. *The stochastic process X is adapted to the filtration $\{\mathcal{F}_t\}$ if, for each $t \geq 0$, X_t is an $\mathcal{F}_t$- measurable random variable.*

Obviously, every process X is adapted to $\{\mathcal{F}_t^X\}$. Moreover, if X is adapted to $\{\mathcal{F}_t\}$ and Y is a modification of X, then Y is also adapted to $\{\mathcal{F}_t\}$ provided that $\mathcal{F}_0$ contains all the P-negligible sets in $\mathcal{F}$. Note that this requirement is not the same as saying that $\mathcal{F}_0$ is complete, since some of the P-negligible sets in $\mathcal{F}$ may not be in the completion of $\mathcal{F}_0$.

1.1.5 Stopping times

Definition 1.1.17. *A random variable $\tau : \Omega \longrightarrow [a, b]$ is called a stopping time with respect to a filtration $\{\mathcal{F}_t; a \leq t \leq b\}$ if $\{\omega; \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \in [a, b]$.*

properties

- If the filtration is right continuous then τ is a stopping time if and only if, for all t, $\{\tau < t\} \in \mathcal{F}_t$.
- Let τ_1 and τ_1 be two stopping times. Then $\tau_1 + \tau_2$, $\tau_1 \wedge \tau_2$, $\tau_1 \vee \tau_2$ are stopping times.

Definition 1.1.18. *The σ-algebra $\mathcal{F}_\tau$ for a stopping time τ is given by*

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for allt}\}.$$

The following holds:

1. τ *is measurable with respect to* $\mathcal{F}_\tau$ *.*

2. *If* $\tau(\omega) = t$ *for almost all* ω *then* $\mathcal{F}_\tau = \mathcal{F}_t$.

1.1.6 Continuous time martingales

In this section we shall consider exclusively real-valued processes $X = \{X_t; 0 \leq t < \infty\}$ on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a given filtration $\mathcal{F}_t$ and such that $E|X_t| < \infty$ holds for every $t \geq 0$.

Definition 1.1.19. *The process* $\{X_t, \mathcal{F}_t; 0 < t < \infty\}$ *is said to be a submartingale (respectively, a supermartingale) if, for every* $0 \leq s < t < \infty$, *we have, a.s.* $P : E(X_t|\mathcal{F}_s) \geq X_s$, *(respectively,* $E(X_t|\mathcal{F}_t) \leq X_s$*).*

We shall say that $\{X_t, \mathcal{F}_t, \quad 0 \leq t < \infty\}$ *is a martingale if it is both a submartingale and a supermartingale.*

Proposition 1.1.1. *[72]*

1. *A stochastic process* $X \equiv \{X_t, \mathcal{F}_t, t = 0, 1, 2, \cdots\}$ *is a submartingale if and only if* $-X$ *is a supermartingale. It is a martingale if and only if it is both a sub- and supermartingale*

2. *If* X *is a martingale,* $E(X_t\} = E(X_0\}$ *for all* t. *If* $t_1 < t_2$ *and if* X *is a submartingale, then* $E(X_{t_2}\} \leq E(X_{t_1}\}$*; if* X *is a supermartingale, then* $E(X_{t_1}\} \geq E(X_{t_2}\}$

3. *Suppose* $\{X_i, \mathcal{F}_t, t = 0, 1, 2, \cdots\}$ *is a martingale and* ϕ *is a convex function on* $\mathbb{R}$. *Then, if* $\phi(X_t)$ *is integrable for all* t,$\{\phi(X_t), \mathcal{F}_t, t = 01, 2, \cdots\}$ *is a submartingale.*

4. *Suppose that* $\{X_i, \mathcal{F}_t, t = 0, 1, 2, \cdots\}$ *is a submartingale and* ϕ *is an increasing convex function on* $\mathbb{R}$. *Then, if* $\phi(X_t)$ *is integrable for all* t, $\{\phi(X_t), \mathcal{F}_t, t = 01, 2, \cdots\}$ *is a submartingale.*

5. *Doob's maximal inequality:*

$$E\Big(\sup_{0\leq s\leq t} X_s\Big)^p \leq \Big(\frac{p}{p-1}\Big)^p E(X_t^p), \quad p > 1,$$

Definition 1.1.20. *A stochastic process* $(X_t)_{t\geq 0}$ *is called a local martingale (with respect to the filtration* $(\mathcal{F}_t)_{t\geq 0}$*) if there is a sequence of stopping times* $(\tau_n)_{n\geq 0}$ *such that:*

- *The sequence* $(\tau_n)_{n\geq 0}$ *is increasing and almost surely satisfies* $\lim\limits_{n\longrightarrow+\infty} \tau_n = +\infty$;

- *For* $n \geq 1$ *the process* $(X_{t\wedge\tau_n})_{t\geq 0}$ *is a uniformly integrable martingale with respect to the filtration* $(\mathcal{F}_t)_{t\geq 0}$

Definition 1.1.21. *Let* $(X_t)_{t\geq 0}$ *be an adapted continuous stochastic process on the filtered probability space* $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, P)$. *We say that* $(X_t)_{t\geq 0}$ *is a semimartingale with respect to the filtration* $(\mathcal{F}_t)_{t\geq 0}$ *if* $(X_t)_{t\geq 0}$ *may be written as:*
$X_t = X_0 + A_t + M_t$
where $(A_t)_{t\geq 0}$ *is a bounded variation process and* $(M_t)_{t\geq 0}$ *is a continuous local martingale such that* $M_0 = 0$.

1.1.7 Brownian motion

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A stochastic process is a measurable function $X(t,\omega)$ defined on the product space $[0,\infty) \times \Omega$. In particular,

(a) for each t, $X(t,\cdot)$ is a random variable,

(b) for each ω, $X(\cdot,\omega)$ is a measurable function (called a sample path).

Definition 1.1.22. *A stochastic process* $W(t,\omega)$ *is called a Brownian motion if it satisfies the following conditions:*

1. $W\{\omega; B(0,\omega) = 0\} = 1$.

2. *For any* $0 \leq s < t$, *the random variable* $W(t) - W(s)$ *is normally distributed with mean* 0 *and variance* $t-s$, *i.e., for any* $a < b$,
$$P\{a \leq W(t) - W(s) \leq b\} = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-x^2/2(t-s)} dx.$$

3. $W(t,\omega)$ *has independent increments, i.e., for any* $0 \leq t_1 < \cdots < t_n$, *the random variables*
$$W(t_1), W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n-1}),$$
are independent.

4. *Almost all sample paths of $W(t,\omega)$ are continuous functions, i.e.,*

$$P = \{\omega; W(\cdot,\omega) \text{ is continuous } \} = 1.$$

Suppose that $\mathcal{H}$ is a real separable Hilbert space with scalar product denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. The norm of an element $h \in \mathcal{H}$ will be denoted by $\|h\|_{\mathcal{H}}$.

Definition 1.1.23. *We say that a stochastic process $W = \{W(h), h \in \mathcal{H}\}$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$ is an is normal Gaussian process (or a Gaussian process on $\mathcal{H}$) if W is a centered Gaussian family of random variables such that $E(W(h)W(g)) = \langle h, g\rangle_{\mathcal{H}}$ for all $h, g \in \mathcal{H}$.*

1.2 Fractional Brownian Motion

The fractional Brownian motion was first introduced within a Hilbert space framework by Kolmogorov in 1940 in [53], where it was called Wiener Helix. It was further studied by Yaglom in [83]. The name fractional Brownian motion is due to Mandelbrot and Van Ness, who in 1968 provided in [59] a stochastic integral representation of this process in terms of a standard Brownian motion.
The fractional Brownian motion is a self-similar centered Gaussian process with stationary increments and variance equals t^{2H}, where H is a parameter in the interval $(0,1)$. For $H = \frac{1}{2}$ this process is a classical Brownian motion.

On the other hand, the increments of the fractional Brownian motion are not independent, except the in case of standard Brownian motion. The Hurst parameter H can be used to characterize the dependence of the increments and indicates the memory of the process. The increments over two disjoint time intervals are positively correlated when $H > \frac{1}{2}$ and are negatively correlated for $H < \frac{1}{2}$. In the first case, the dependence between two increments decay slowly so that it does sum to infinity as the time intervals grow apart, and exhibits long range dependence or the long memory property. For the latter case, the dependence is fast and is refered to as short rage dependence or short memory. Obviously, for $H = \frac{1}{2}$, the increments are independent.

The self-similarity and long range dependence properties allow us to use fractional Brownian motion as a model in different areas of applications e.g. hydrology, climatology, signal processing, network traffic analysis and mathematical finance. Besides such applications, it turns out that fractional Brownian motion is not a semimartingale nor a Markov process, except in the case when $H = \frac{1}{2}$. Hence, the classical stochastic integration theory for semimartingales is not at hand and so makes fractional Brownian motion more interesting from a purely mathematical point of view.

The financial pricing models with continuous trading, based on geometric fractional Brownian motion sometimes allow for existence of arbitrage. The existence of arbitrage essentially depends to the kind of stochastic integral in the definition of the wealth process. It can be shown that with Skorohod integration theory arbitrages disappear, but difficult to give economic interpretation, see Björk and Hult [16] and Sottinen and Valkeila [68]. On the other hand, Riemann-Stieltjes integrals seem more natural and sound better for economical interpretations. Using the Riemann-Stieltjes integration theory with adding proportional transaction costs lets us to construct a framework which acknowledges the pricing models with geometric fractional Brownian motion. First, Guasoni [37] showed that we have the absence of arbitrage in pricing model with proportional transaction costs based on geometric fractional Brownian motion with continuous trading. Moreover, in this setup, Guasoni, Rasonyi and Schachermayer [38] proved a fundamental theorem of asset pricing type result. The results by Guasoni, Rasonyi and Schachermayer open a new window to the pricing models based on fractional type processes such geometric fractional Brownian motion. In this section we will present the application of the Malliavin Calculus to develop a stochastic calculus with respect to the fractional Brownian motion.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space.

Definition 1.2.1. *The (two-sided, normalized) fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a Gaussian process $B^H = \{B^H_t, t \in \mathbb{R}\}$ on $(\Omega, \mathcal{F}, P)$, having the properties:*

1. $B^H_0 = 0$,
2. $E(B^H_t) = 0$, $t \in \mathbb{R}$,
3. $$R_H(t,s) = E(B^H_t B^H_s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), s,t \in \mathbb{R}. \quad (1.1)$$

Fractional Brownian motion has the following self-similar property: For any constant $a > 0$, the processes $\{a^{-H} B^H_{at}, t \geq 0\}$ and $\{B^H_t, t \geq 0\}$ have the same distribution. This property is an immediate consequence of the fact that the covariance function (4.11) is homogeneous of order $2H$.

From (5.1) we can deduce the following expression for the variance of the increment of the process in an interval $[s,t]$:

$$E(|B_t^H - B_s^H|^2) = |t-s|^{2H}. \tag{1.2}$$

This implies that fBm has stationary increments.

By Kolmogorov's continuity criterion and (1.2) we deduce that fBm has a version with continuous trajectories. Moreover, by Garsia-Rodemich- Rumsey Lemma (see Lemma A.3.1), we can deduce the following modulus of continuity for the trajectories of fBm: For all $\varepsilon > 0$ and $T > 0$, there exists a nonnegative random variable $G_{\varepsilon,T}$ such that $E(|G_{\varepsilon,T}|^p) < \infty$ for all $p \geq 1$, and

$$|B_t^H - B_s^H| \leq G_{\varepsilon,T}|t-s|^{H-\varepsilon},$$

for all $s,t \in [0,T]$. In other words, the parameter H controls the regularity of the trajectories, which are Hölder continuous of order $H-\varepsilon$, for any $\varepsilon > 0$.

For $H = \frac{1}{2}$, the covariance can be written as $R_{\frac{1}{2}}(t,s) = t \wedge s$, and the process B^H is a standard Brownian motion. Hence, in this case the increments of the process in disjoint intervals are independent. However, for $H \neq \frac{1}{2}$, the increments are not independent.

Set $X_n = B_n^H - B_{n-1}^H$, $n \geq 1$. Then $\{X_n, n \geq 1\}$ is a Gaussian stationary sequence with covariance function

$$\rho_H(n) = \frac{1}{2}((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}).$$

This implies that two increments of the form $B_k^H - B_{k-1}^H$ and $B_{k+n}^H - B_{k+n-1}^H$ are positively correlated (i.e. $\rho_H(n) > 0$) if $H > \frac{1}{2}$ and they are negatively correlated (i.e. $\rho_H(n) < 0$) if $H < \frac{1}{2}$. In the first case the process presents an aggregation behaviour and this property can be used to describe cluster phenomena. In the second case it can be used to model sequences with intermittency.

In the case $H > \frac{1}{2}$ the stationary sequence X_n exhibits long range dependence, that is,

$$\lim_{n\longrightarrow\infty} \frac{\rho_H(n)}{H(2H-1)n^{2H-2}} = 1$$

and, as a consequence, $\displaystyle\sum_{n=1}^{\infty} \rho_H(n) = \infty$.

In the case $H < \frac{1}{2}$ we have

$$\sum_{n=1}^{\infty} |\rho_H(n)| < \infty.$$

1.2.1 Semimartingale property

We have seen that for $H \neq \frac{1}{2}$ fBm does not have independent increments. The following proposition asserts that it is not a semimartingale.

Proposition 1.2.1. *The fBm is not a semimartingale for* $H = \frac{1}{2}$.

Proof. For $p > 0$ set

$$Y_{n,p} = n^{pH-1} \sum_{j=1}^{n} |B^H_{j/n} - B^H_{(j-1)/n}|^p.$$

By the self-similar property of fBm, the sequence $\{Y_{n,p}, n \geq 1\}$ has the same distribution as $\{\tilde{Y}_{n,p}, n \geq 1\}$, where

$$\tilde{Y}_{n,p} = n^{-1} \sum_{j=1}^{n} |B^H_j - B^H_{j-1}|^p.$$

The stationary sequence $\{B_j - B_{j-1}, j \geq 1\}$ is mixing. Hence, by the Ergodic Theorem $\tilde{Y}_{n,p}$ converges almost surely and in $\mathcal{L}^1(\Omega)$ to $E(|B^H_1|^p)$ as n tends to infinity. As a consequence, $Y_{n,p}$ converges in probability as n tends to infinity to $E(|B^H_1|^p)$. Therefore

$$V_{n,p} = \sum_{j=1}^{n} |B^H_{j/n} - B^H_{(j-1)/n}|^p$$

converges in probability to zero as n tends to infinity if $pH > 1$, and to infinity if $pH < 1$. Consider the following two cases:

i) If $H < \frac{1}{2}$, we can choose $p > 2$ such that $pH < 1$, and we obtain that the p-variation of fBm (defined as the limit in probability $\lim_{n \longrightarrow \infty} V_{n,p}$) is infinite. Hence, the quadratic variation ($p = 2$) is also infinite.

ii) If $H > \frac{1}{2}$, we can choose p such that $\frac{1}{H} < p < 2$. Then the p-variation is zero, and, as a consequence, the quadratic variation is also zero. On the other hand, if we choose p such that $1 < p < \frac{1}{H}$ we deduce that the total variation is infinite.

Therefore, we have proved that for $H \neq \frac{1}{2}$ the fractional Brownian motion cannot be a semimartingale.

$\square$

In [25] Cheridito has introduced the notion of weak semimartingale as a stochastic process $\{X_t, t \geq 0\}$ such that for each $T > 0$, the set of random variables

$$\left\{ \sum_{j=1}^{n} f_j(X_{tj} - X_{t_{j-1}}), n \geq 1, 0 \leq t_0 < \cdots < t_n \leq T, \right.$$
$$\left. |f_j| \leq 1, f_j \text{ is } \mathcal{F}^X_{t_{j-1}} - \text{measurable} \right\}$$

is bounded in $\mathcal{L}^0(\Omega)$, where for each $t \geq 0$, $\mathcal{F}^X_t$ is the σ-field generated by the random variables $\{X_s, 0 \leq s \leq t\}$. It is important to remark that this σ-field is not completed with the null sets. Then, in [25] it is proved that fBm is not a weak semimartingale if $H \neq \frac{1}{2}$.

Let us mention the following surprising result also proved in [25] . Suppose that $\{B^H_t, t \geq 0\}$ is a fBm with Hurst parameter $H \in (0, 1)$, and $\{W_t, t \geq 0\}$ is an ordinary Brownian motion. Assume they are independent. Set

$$M_t = B^H_t + W_t.$$

Then $\{M_t, t \geq 0\}$ is not a weak semimartingale if $H \in (0, \frac{1}{2}) \bigcup (\frac{1}{2}, \frac{3}{4}]$, and it is a semimartingale, equivalent in law to Brownian motion on any finite time interval $[0, T]$, if $H \in (\frac{3}{4}, 1)$.

1.2.2 Fractional integrals and derivatives

We recall the basic definitions and properties of the fractional calculus. For a detailed presentation of these notions we refer to [74].

Let $a, b \in \mathbb{R}$, $a < b$. Let $f \in L^1(a, b)$ and $\alpha > 0$. The left and right-sided fractional integrals of f of order α are defined for almost all $x \in (a, b)$ by

$$I^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha - 1} f(y) dy \tag{1.3}$$

and

$$I^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)dy \tag{1.4}$$

respectively. Let $I^{\alpha}_{a+}(L^p)$ (resp. $I^{\alpha}_{b-}(L^p)$) the image of $L^p(a,b)$ by the operator I^{α}_{a+} (resp. I^{α}_{b-}).

If $f \in I^{\alpha}_{a+}(L^p)$ (resp. $f \in I^{\alpha}_{b-}(L^p)$) and $0 < \alpha < 1$ then the left and right-sided fractional derivatives are defined by

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right), \tag{1.5}$$

and

$$D^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}} dy \right), \tag{1.6}$$

for almost all $x \in (a,b)$ (the convergence of the integrals at the singularity $y = x$ holds point-wise for almost all $x \in (a,b)$ if $p = 1$ and moreover in L^p-sense if $1 < p < \infty$).

Recall the following properties of these operators:

- If $\alpha < \frac{1}{p}$ and $q = \frac{p}{1-\alpha p}$ then

$$I^{\alpha}_{a+}(L^p) = I^{\alpha}_{b-}(L^p) \subset L^q(a,b)$$

- If $\alpha > \frac{1}{p}$ then

$$I^{\alpha}_{a+}(L^p) \cup I^{\alpha}_{b-}(L^p) \subset C^{\alpha-\frac{1}{p}}(a,b)$$

where $C^{\alpha-\frac{1}{p}}(a,b)$ denotes the space of $\left(\alpha - \frac{1}{p}\right)$-Hölder continuous functions of order $\alpha - \frac{1}{p}$ in the interval $[a,b]$.

The following inversion formulas hold:

$$I^{\alpha}_{a+}(D^{\alpha}_{a+}f) = f$$

for all $f \in I^{\alpha}_{a+}(L^p)$, and

$$D^{\alpha}_{a+}(I^{\alpha}_{a+}f) = f$$

for all $f \in L^1(a,b)$. Similar inversion formulas hold for the operators I^{α}_{b-} and D^{α}_{b-}

The following integration by parts formula holds

$$\int_a^b (D_{a^+}^\alpha f)(s) g(s) ds = \int_a^b f(s)(D_{b^-}^\alpha g)(s) ds, \tag{1.7}$$

for any $f \in I_{a^+}^\alpha(L^p)$, $g \in I_{b^-}^\alpha(L^p)$ $\frac{1}{p} + \frac{1}{q} = 1$.

1.2.3 Moving average representation

Mandelbrot and Van Ness obtained in [59] the following integral representation of fBm in terms of a Wiener process on the whole real line (see also Samorodnitsky and Taqqu [76]).

Proposition 1.2.2. *Let $\{W(A), A \in \mathcal{B}(\mathbb{R}), \mu(A) < \infty\}$ be a white noise on $\mathbb{R}$. Then*

$$B_t^H = \frac{1}{C_1(H)} \int_{\mathbb{R}} [((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}] dW_s,$$

is a fractional Brownian motion with Hurst parameter H, if

$$C_1(H) = \left(\int_0^\infty ((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}})^2 ds + \frac{1}{2H} \right)^{\frac{1}{2}}.$$

Proof. Set $f_t(s) = \big((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\big)$, $s \in \mathbb{R}$, $t \geq 0$. Notice that $\int_{\mathbb{R}} f_t^2(s) ds < \infty$. In fact, if $H \neq \frac{1}{2}$, as s tends to $-\infty$, $f_t(s)$ behaves as $(-s)^{H-\frac{3}{2}}$ which is square integrable at infinity. For $t \geq 0$ set

$$X(t) = \int_{\mathbb{R}} \big((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\big) dW(s).$$

We have

$$\begin{aligned}
E(X_t^2) &= \int_{\mathbb{R}} \big((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\big)^2 ds \\
&= t^{2H} \Big(\int_{\mathbb{R}} \big((1-u)_+^{H-1/2} - (-u)_+^{H-1/2}\big)^2 du \\
&= t^{2H} \Big(\int_{-\infty}^t \big((1-u)^{H-1/2} - (-u)^{H-1/2}\big)^2 du + \int_0^1 (1-u)^{2H-1} du
\end{aligned}$$

$$= \quad C_1^2(H)t^{2H}. \tag{1.8}$$

Similarly, for any $s < t$ we obtain

$$\begin{aligned} E(|X_t - X_s|^2) &= \int_{\mathbb{R}} \left((t-u)_+^{H-1/2} - (s-u)_+^{H-1/2}\right)^2 du \\ &= \int_{\mathbb{R}} \left((t-s-u)_+^{H-1/2} - (-u)_+^{H-1/2}\right)^2 du \\ &= C_1^2(H)|t-s|^{2H}. \end{aligned} \tag{1.9}$$

Now

$$\begin{aligned} E(X_t X_s) &= 2\{E|X_t - X_s|^2 - E(X_t^2) - E(X_s^2)\} \\ &= \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \end{aligned} \tag{1.10}$$

From (1.8), (1.9) and (1.10) we deduce that the centered Gaussian process $\{X_t, t \geq 0\}$ has the covariance R_H of a fBm with Hurst parameter H.

Notice that the above integral representation implies that the function RH defined in (4.11) is a covariance function, that is, it is symmetric and nonnegative definite.

□

It is also possible to establish the following spectral representation of fBm (see Samorodnitsky and Taqqu [76]):

$$B^{(H)}(t) = \frac{1}{C_2(H)} \int_{\mathbb{R}} \frac{e^{its} - 1}{is} |s|^{1/2-H} d\widetilde{W}(s),$$

where $\widetilde{W} = W^1 + iW^2$ is a complex Gaussian measure on $\mathbb{R}$ such that $W^1(A) = W^1(-A)$, $W^2(A) = -W^2(A)$, and $E(W^1(A)^2) = E(W^2(A)^2) = \frac{|A|}{2}$, and

$$C_2(H) = \left(\frac{\pi}{H\Gamma(2H)\sin(H\pi)}\right)^{1/2}.$$

1.2.4 Representation of fBm on an interval

Fix a time interval $[0, T]$. Consider a fBm $\{B_t^H, t \in [0, T]\}$ with Hurst parameter $H \in (0, 1)$. We denote by E the set of step functions on $[0, T]$. Let

$\mathcal{H}$ be the Hilbert space defined as the closure of ξ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $1_{[0,t]} \longrightarrow B_t^H$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $\mathcal{H}_1$ associated with B^H. We will denote this isometry by $\varphi \longrightarrow B^H(\varphi)$. Then $\{B^H(\varphi), \varphi \in \mathcal{H}\}$ is an isonormal Gaussian process associated with the Hilbert space $\mathcal{H}$ in the sense of Definition 1.1.23.

In this subsection we will establish the representation of fBm as a Volterra process using some computations inspired in the works [5] (case $H > \frac{1}{2}$).

- If $H > \frac{1}{2}$

  It is easy to see that the covariance of fBm can be written as

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |r-u|^{2H-2} du dr, \tag{1.11}$$

  where $\alpha_H = H(2H-1)$. Formula (1.11) implies that

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u du dr, \tag{1.12}$$

  for any pair of step functions φ and ψ in $\mathcal{E}$.

  We can write

$$\begin{aligned} |r-u|^{2H-2} &= \frac{(ru)^{H-\frac{1}{2}}}{\beta(2-2H, H-\frac{1}{2})} \\ &\quad \times \int_0^{r \wedge u} v^{1-2H} (r-v)^{H-\frac{3}{2}} (u-v)^{H-\frac{3}{2}} dv, \end{aligned} \tag{1.13}$$

  where β denotes the Beta function. Let us show Equation (1.13). Suppose $r > u$. By means of the change of variables $z = \frac{r-v}{u-v}$ and $x = \frac{r}{uz}$, we obtain

$$\int_0^u v^{1-2H} (r-v)^{H-\frac{3}{2}} (u-v)^{H-\frac{3}{2}} dv = (r-u)^{2H-2} \int_{\frac{r}{u}}^{\infty} (zu-r)^{1-2H} z^{H-\frac{3}{2}} dz$$

$$
\begin{aligned}
&= (ru)^{\frac{1}{2}-H}(r-u)^{2H-2}\int_0^1 (1-x)^{1-2H}x^{H-\frac{3}{2}}dx \\
&= \beta(2-2H, H-\frac{1}{2})(ru)^{\frac{1}{2}-H}(r-u)^{2H-2}.
\end{aligned}
$$

Consider the square integrable kernel

$$
K_H(t,s) = c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du, \tag{1.14}
$$

where $c_H = \left[\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}\right]^{1/2}$ and $t > s$.

Taking into account formulas (1.11) and (1.13) we deduce that this kernel verifies

$$
\begin{aligned}
\int_0^{t\wedge s} K_H(t,u)K_H(s,u)du &= c_H^2 \int_0^{t\wedge s}\Big(\int_u^t (y-u)^{H-\frac{3}{2}}y^{H-\frac{1}{2}}dy\Big) \\
&\quad \times\Big(\int_u^s (z-u)^{H-\frac{3}{2}}z^{H-\frac{1}{2}}dz\Big)u^{1-2H}du \\
&= c_H^2\beta(2-2H, H-\frac{1}{2})\int_0^t\int_0^s |y-z|^{2H-2}dzdy \\
&= R_H(t,s). \qquad (1.15)
\end{aligned}
$$

Formula (1.15) implies that the kernel R_H is nonnegative definite and provides an explicit representation for its square root as an operator.

From (1.14) we get

$$
\frac{\partial K_H}{\partial t}(t,s) = c_H\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}. \tag{1.16}
$$

Consider the linear operator K_H^* from $\mathcal{E}$ to $\mathcal{L}^2([0,T])$ defined by

$$
(K_H^*\varphi)(s) = \int_s^T \varphi(t)\frac{\partial K_H}{\partial t}(t,s)dt. \tag{1.17}
$$

Notice that

$$
(K_H^* 1_{[0,t]})(s) = K_H(t,s)1_{[0,t]}(s). \tag{1.18}
$$

The operator K_H^* is an isometry between $\mathcal{E}$ and $\mathcal{L}^2([0,T])$ that can be extended to the Hilbert space $\mathcal{H}$. In fact, for any $s,t \in [0,T]$ we have using (1.15) and (1.18)

$$\begin{aligned}\langle K_H^* 1_{[0,t]}, K_H^* 1_{[0,s]}\rangle_{\mathcal{L}^2([0,T])} &= \langle K_H(t,\cdot)1_{[0,t]}, K_H(s,\cdot)1_{[0,s]}\rangle_{\mathcal{L}^2([0,T])} \\ &= \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du \\ &= R_H(t,s) = \langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}}.\end{aligned}$$

The operator K_H^* can be expressed in terms of fractional integrals:

$$(K_H^*\varphi)(s) = c_H\Gamma(H-\frac{1}{2})s^{\frac{1}{2}-H}(I_{T_-}^{H-\frac{1}{2}}u^{H-\frac{1}{2}}\varphi(u))(s). \tag{1.19}$$

This is an immediate consequence of formulas (1.16), (1.18) and (1.4).

For any $a \in [0,T]$, the indicator function $1_[0,a]$ belongs to the image of K_H^* and applying the rules of the fractional calculus yields

$$(K_H^*)^{-1}(1_{[0,a]}) = \frac{1}{c_H\Gamma(H-\frac{1}{2})}s^{\frac{1}{2}-H}(D_{a_-}^{H-\frac{1}{2}}u^{H-\frac{1}{2}})(s)1_{[0,a]}(s). \tag{1.20}$$

Consider the process $W = \{W_t, t \in [0,T]\}$ defined by

$$W_t = B^H((K_H^*)^{-1}(1_{[0,a]})). \tag{1.21}$$

Then W is a Wiener process, and the process B^H has the integral representation

$$B_t^H = \int_0^t K_H(t,s)dW_s.$$

Indeed, for any $s,t \in [0,T]$ we have

$$E(W_tW_s) \;=\; E\Big(B^H((K_H^*)^{-1}(1_{[0,t]}))B^H((K_H^*)^{-1}(1_{[0,s]}))\Big)$$

$$
\begin{aligned}
&= \langle (K_H^*)^{-1}(1_{[0,t]}), (K_H^*)^{-1}(1_{[0,s]})\rangle_{\mathcal{H}} \\
&= \langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{L}^2([0,T])} = t \wedge s.
\end{aligned}
$$

Moreover, for any $\varphi \in \mathcal{H}$ we have

$$B^H(\varphi) = \int_0^T (K_H^*\varphi)(t)dW_t. \tag{1.22}$$

Notice that from (1.20), the Wiener process W is adapted to the filtration generated by the fBm B^H and (1.21) and (1.22) imply that both processes generate the same filtration. Furthermore, the Wiener process W that provides the integral representation (1.22) is unique. Indeed, this follows from the fact that the image of the operator K_H^* is $\mathcal{L}^2([0,T])$, because this image contains the indicator functions.

The elements of the Hilbert space $\mathcal{H}$ may not be functions but distributions of negative order (see Pipiras and Taqqu [70], [71]). In fact, from (1.20) it follows that $\mathcal{H}$ coincides with the space of distributions f such that $s^{\frac{1}{2}-H}I_{0_+}^{H-\frac{1}{2}}(f(u)u^{H-\frac{1}{2}})(s)$ is a square integrable function.

$$\|\varphi\|^2_{|\mathcal{H}|} = \alpha_H \int_0^T\int_0^T |\varphi_r||\varphi_u||r-u|^{2H-2}drdu < \infty. \tag{1.23}$$

It is not difficult to show that $|\mathcal{H}|$ is a Banach space with the norm $\|\cdot\|_{|\mathcal{H}|}$ and $\mathcal{E}$ is dense in $|\mathcal{H}|$. On the other hand, it has been shown in [71] that the space $|\mathcal{H}|$ equipped with the inner product $\langle\varphi,\psi\rangle_{\mathcal{H}}$ is not complete and it is isometric to a subspace of $\mathcal{H}$. The following estimate has been proved in [60].

Also denoting

$$L^2_{\mathcal{H}}([0,T]) = \{\varphi \in \Lambda, K_H^*\varphi \in L^2([0,T])\},$$

since $H > 1/2$, we have

$$L^{1/H}([0,T]) \subset L^2_{\mathcal{H}}([0,T]), \tag{1.24}$$

see [63]. Moreover, the following useful result holds:

Lemma 1.2.1. *[67] . For $\varphi \in L^{1/H}([0,T])$,*

$$H(2H-1)\int_0^T\int_0^T |\varphi_r||\varphi_u||r-u|^{2H-2}drdu \leq c_H\|\Phi\|^2_{L^{1/H}([0,T])}.$$

Next, we consider a fractional Brownian motion with values in a Hilbert space and give the definition of the corresponding stochastic integral.

Definition 1.2.2. *A bounded operator ξ is called a Hilbert-Schmidt operator if there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty} \in \mathcal{H}$ such that*

$$\sum_{n=1}^{\infty} \|\xi e_n\|^2 < \infty.$$

Definition 1.2.3. *Let ξ be an operator in $\mathcal{H}$.*
The adjoint operator of ξ is the operator $\xi^* : \mathcal{H} \to \mathcal{H}$ such that

$$\langle \xi f, g\rangle = \langle f, \xi^* g\rangle, \quad \forall f, g \in \mathcal{H}$$

Definition 1.2.4. – *ξ is self-adjoint when $\xi^* = \xi$, i.e.,*

$$\langle \xi f, g\rangle = \langle f, \xi g\rangle, \quad \forall f, g \in \mathcal{H}$$

– *ξ is called symmetric if $\xi \subset \xi^*$.*
– *ξ is non-negative when*

$$\langle g, \xi g\rangle \geq 0 \ \ \forall\ g \in \mathcal{H}.$$

Proposition 1.2.3. – *Every Hilbert-Schmidt operator $\xi : \mathcal{H} \to \mathcal{H}$ is compact.*
– *Any positive operator is self-adjoint also.*

Let $Q \in L(\mathcal{K}, \mathcal{H})$ be a non-negative self-adjoint operator. Denote by $L_Q^0(\mathcal{K}, \mathcal{H})$ the space of all $\xi \in L(\mathcal{K}, \mathcal{H})$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by

$$|\xi|^2_{L_Q^0(\mathcal{K},\mathcal{H})} = tr(\xi Q \xi^*).$$

Then ξ is called a Q-Hilbert-Schmidt operator from $\mathcal{K}$ to $\mathcal{H}$.

Let $\{B_n^H(t)\}_{n\in N}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions that are mutually independent on $(\Omega, \mathcal{F}, P)$. The series

$$\sum_{n=1}^{\infty} B_n^H(t)e_n, \quad t \geq 0,$$

where $\{e_n\}_{n\in N}$ is a complete orthonormal basis in $\mathcal{K}$, does not necessarily converge in the space $\mathcal{K}$. Thus, we consider a $\mathcal{K}-$valued stochastic process $B_Q^H(t)$ given formally by the following series:

$$B_Q^H(t) = \sum_{n=1}^{\infty} B_n^H(t)Q^{\frac{1}{2}}e_n, \quad t \geq 0,$$

which is well-defined as a $\mathcal{K}$-valued Q-cylindrical fractional Brownian motion.

Let $\varphi : [0, T] \mapsto L_0^Q(\mathcal{K}, \mathcal{H})$ such that

$$\sum_{n=1}^{\infty} \|K_H^*(\varphi Q^{\frac{1}{2}}e_n)\|_{L^{1/H}([0,T];\mathcal{H})} < \infty. \tag{1.25}$$

Definition 1.2.5. *Let $\varphi : [0, T] \longrightarrow L_Q^0(\mathcal{K}, \mathcal{H})$ satisfy (1.25). Then, its stochastic integral with respect to the fractional Brownian motion B_Q^H is defined, for $t \geq 0$, as*

$$\int_0^t \varphi(s)dB_Q^H(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s)Q^{1/2}e_n dB_n^H(s) = \sum_{n=1}^{\infty} \int_0^t (K_H^*(\varphi Q^{1/2}e_n))(s)dW(s).$$

Notice that if

$$\sum_{n=1}^{\infty} \|\varphi Q^{1/2}e_n\|_{L^{1/H}([0,T];\mathcal{H})} < \infty, \tag{1.26}$$

then in particular (1.25) holds, which follows immediately from (1.24).

Lemma 1.2.2. *if $\varphi : [0, T] \longrightarrow L_Q^0(\mathcal{K}, \mathcal{H})$ satisfies*

$$\int_0^T \|\varphi(s)\|^2_{L_Q^0(\mathcal{K},\mathcal{H})} ds < \infty,$$

and for any $\alpha, \beta \in [0, T]$ with $\alpha > \beta$, then the above sum in (1.26) is well defined as a $\mathcal{H}$-valued random variable and we have

$$E\Big|\int_\beta^\alpha \varphi(s)dB_Q^H(s)\Big|^2 \le 2H(\alpha-\beta)^{2H-1}\int_\beta^\alpha \|\varphi(s)\|^2_{L_Q^0(\mathcal{K},\mathcal{H})}ds.$$

Proof. Let $\{e_n\}n \in \mathbb{N}$ be the complete orthonormal basis of $\mathcal{K}$ introduced above. Applying Lemma 1.2.1 we obtain

$$\begin{aligned}
E\Big|\int_\beta^\alpha \varphi(s)dB_Q^H(s)\Big|^2 &= E\Big|\sum_{n=1}^\infty \int_\beta^\alpha \varphi(s)Q^{1/2}e_n dB_n^H(s)\Big|^2 \\
&= \sum_{n=1}^\infty E\Big|\int_\beta^\alpha \varphi(s)Q^{1/2}e_n dB_n^H(s)\Big|^2 \\
&= \sum_{n=1}^\infty H(2H-1)\int_\beta^\alpha\int_\beta^\alpha \Big|\varphi(t)Q^{1/2}e_n\Big|^2\Big|\varphi(s)Q^{1/2}e_n\Big|^2 \\
&\quad \times|t-s|^{2H-2}dtds \\
&= 2H\sum_{n=1}^\infty\left(\int_\beta^\alpha \Big|\varphi(s)Q^{1/2}e_n\Big|^{\frac{1}{H}}ds\right)^{2H} \\
&= 2H(\alpha-\beta)^{2H-1}\sum_{n=1}^\infty\int_\beta^\alpha \Big|\varphi(s)Q^{1/2}e_n\Big|^2 ds
\end{aligned}$$

□

- If $H < \frac{1}{2}$

  To find a square integrable kernel that satisfies (1.15) is more difficult than in the case $H > \frac{1}{2}$. The following proposition provides the answer to this problem.

Proposition 1.2.4. *Let $H < \frac{1}{2}$. The kernel*

$$\begin{aligned}
K_H(t,s) &= c_H\Bigg[\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} \\
&\quad -(H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\Bigg],
\end{aligned}$$

where $c_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H,H+\frac{1}{2})}}$ *satisfies*

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du. \tag{1.27}$$

In the references [26] and [70] Eq. (4.8) is proved using the analyticity of both members as functions of the parameter H. We will give here a direct proof using the ideas of [65]. Notice first that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H(H-\frac{1}{2})\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}. \tag{1.28}$$

Proof. Consider first the diagonal case $s=t$. Set $\phi(s) = \int_0^s K_H(s,u)^2 du$. We have

$$\begin{aligned}
\phi(s) &= c_H\Bigg[\int_0^s \left(\frac{s}{u}\right)^{2H-1}(s-u)^{2H-1} \\
&\quad -(2H-1)\int_0^s s^{H-\frac{1}{2}}u^{1-2H}(s-u)^{H-\frac{1}{2}} \\
&\quad \times\Big(\int_u^s v^{H-\frac{3}{2}}(v-u)^{H-\frac{1}{2}}dv\Big)du \\
&\quad +(H-\frac{1}{2})^2\int_0^s u^{1-2H}\Big(\int_u^s v^{H-\frac{3}{2}}(v-u)^{H-\frac{1}{2}}dv\Big)^2 du\Bigg].
\end{aligned}$$

□

Making the change of variables $u = sx$ in the first integral and using Fubini's theorem yields

$$\begin{aligned}
\phi(s) &= c_H^2\Bigg[s^{2H}\beta(2-2H,2H) - (2H-1)s^{H-\frac{1}{2}}\int_0^s v^{H-\frac{3}{2}} \\
&\quad \times\Big(\int_0^v u^{1-2H}(s-u)^{H-\frac{1}{2}}(v-u)^{H-\frac{1}{2}}du\Big)dv
\end{aligned}$$

$$+2(H-\frac{1}{2})^2\int_0^s\int_0^v\int_0^w u^{1-2H}(v-u)^{H-\frac{1}{2}}(w-u)^{H-\frac{1}{2}}w^{H-\frac{3}{2}}v^{H-\frac{3}{2}}dudwdv\Bigg].$$

Now we make the change of variable $u = vx$, $v = sy$ for the second term and $u = wx$, $w = vy$ for the third term and we obtain

$$\begin{aligned}\phi(s) &= c_H^2 s^{2H}\Big[\beta(2-2H,2H)-(2H-1)\Big(\frac{1}{4H}+\frac{1}{2}\Big)\\ &\quad\times\int_0^1\int_0^1 x^{1-2H}(1-xy)^{H-\frac{1}{2}}(1-x)^{H-\frac{1}{2}}dxdy\Big]\\ &= s^{2H}.\end{aligned}$$

Suppose now that $s < t$. Differentiating Equation (1.27) with respect to t, we are aimed to show that

$$H(t^{2H-1}-(t-s)^{2H-1}) = \int_0^s \frac{\partial K_H}{\partial t}(t,u)K_H(s,u)du. \tag{1.29}$$

Set $\phi(t,s) = \displaystyle\int_0^s \frac{\partial K_H}{\partial t}(t,u)K_H(s,u)du$. Using (1.28) yields

$$\begin{aligned}\phi(s) &= c_H^2(H-\frac{1}{2})\int_0^s\left(\frac{t}{u}\right)^{H-\frac{1}{2}}(t-u)^{H-\frac{3}{2}}\left(\frac{s}{u}\right)^{H-\frac{1}{2}}(s-u)^{H-\frac{1}{2}}du\\ &\quad -c_H^2(H-\frac{1}{2})^2\int_0^s\left(\frac{t}{u}\right)^{H-\frac{1}{2}}(t-u)^{H-\frac{3}{2}}u^{\frac{1}{2}-H}\\ &\quad\times\Big(\int_u^s v^{H-\frac{3}{2}}(v-u)^{H-\frac{1}{2}}dv\Big)du.\end{aligned}$$

Making the change of variables $u = sx$ in the first integral and $u = vx$ in the second one we obtain

$$\begin{aligned}\phi(s) &= c_H^2(H-\frac{1}{2})(ts)^{H-\frac{1}{2}}\gamma(\frac{t}{s}) \\ &\quad -c_H^2(H-\frac{1}{2})^2 t^{H-\frac{1}{2}}\int_0^s v^{H-\frac{3}{2}}\gamma(\frac{t}{v})dv,\end{aligned}$$

where $\gamma(y) = \int_0^1 x^{1-2H}(yx)^{H-\frac{3}{2}}(1x)^{H-\frac{1}{2}}dx$ for $y > 1$. Then, (1.29) is equivalent to

$$\begin{aligned}& c_H^2\left[(H-\frac{1}{2})s^{H-\frac{1}{2}}\gamma(\frac{t}{s}) - (H-\frac{1}{2})^2\int_0^s v^{H-\frac{3}{2}}\gamma(\frac{t}{v})dv\right] \\ =\;& H(t^{H-\frac{1}{2}} - t^{\frac{1}{2}-H}(t-s)^{2H-1}). \qquad (1.30)\end{aligned}$$

Differentiating the left-hand side of equation (1.30) with respect to t yields

$$\begin{aligned}& c_H^2(H-\frac{3}{2})\left[(H-\frac{1}{2})s^{H-\frac{3}{2}}\delta(\frac{t}{s}) - (H-\frac{1}{2})^2\int_0^s v^{H-\frac{5}{2}}\delta(\frac{t}{v})dv\right] \\ =\;& \mu(t,s), \qquad (1.31)\end{aligned}$$

where, for $y > 1$,

$$\delta(y) = \int_0^1 x^{1-2H}(y-x)^{H-\frac{5}{2}}(1-x)^{H-\frac{1}{2}}dx.$$

By means of the change of variables $z = \frac{y(1-x)}{y-x}$ we obtain

$$\delta(y) = \beta(2-2H, H+\frac{1}{2})y^{-H-\frac{1}{2}}(y-1)^{2H-2}. \qquad (1.32)$$

Finally, substituting (1.32) into (1.31) yields

$$\begin{aligned}\mu(t,s) &= c_H^2\beta(2-2H, H+\frac{1}{2})(H-\frac{3}{2})(H-\frac{1}{2}) \\ &\quad \times t^{-H-\frac{1}{2}}s(t-s)^{2H-2} + \frac{1}{2}t^{-H-\frac{1}{2}}((t-s)^{2H-1} - t^{2H-1})\end{aligned}$$

$$= H(1-2H)$$
$$\times t^{-H-\frac{1}{2}} s(t-s)^{2H-2} + \frac{1}{2}(t-s)^{2H-1} t^{-H-\frac{1}{2}} - \frac{1}{2} t^{H-\frac{3}{2}}.$$

This last expression coincides with the derivative with respect to t of the right-hand side of (1.30). This completes the proof of the equality (1.27).

The kernel K_H can also be expressed in terms of fractional derivatives:

$$K_H(t,s) = c_H \Gamma(H+\frac{1}{2}) s^{\frac{1}{2}-H} D_{t_-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}})(s). \tag{1.33}$$

Consider the linear operator K_H^* from $\mathcal{E}$ to $L^2([0,T])$ defined by

$$(K_H^* \varphi)(s) = K_H(T,s)\varphi(s) + \int_s^T (\varphi(t)-\varphi(s)) \frac{\partial K_H}{\partial r}(t,s) dt. \tag{1.34}$$

Notice that

$$(K_H^* 1_{[0,t]}(s) = K_H(t,s) 1_{[0,t]}(s). \tag{1.35}$$

From (1.27) and (1.35) we deduce as in the case $H > \frac{1}{2}$ that the operator K_H^* is an isometry between $\mathcal{E}$ and $L^2([0,T])$ that can be extended to the Hilbert space $\mathcal{H}$.

The operator K_H^* can be expressed in terms of fractional derivatives:

$$(K_H^* \varphi)(s) = dH s^{\frac{1}{2}-H} (D_{T_-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} \varphi(u))(s), \tag{1.36}$$

where $dH = c_H \Gamma(H+\frac{1}{2})$. This is an immediate consequence of (1.34) and the equality

$$D_{t_-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}}(s) 1_{[0,t]}(s) = \left(D_{T_-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} 1_{[0,t]}(u) \right)(s)$$

As a consequence

$$C^\gamma([0,T]) \subset \mathcal{H} \subset L^2([0,T]),$$

if $\gamma > \frac{1}{2} - H$.

Using the alternative expression for the kernel K_H given by

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1(\frac{t}{s}), \tag{1.37}$$

where

$$F_1(z) = c_H(\frac{1}{2} - H)\int_0^{s-1} \theta^{H-\frac{3}{2}}(1-(\theta+1)^{H-\frac{1}{2}})d\theta,$$

one can show that $\mathcal{H} = I_{T_-}^{\frac{1}{2}-H}(L^2)$ (see [26]). In fact, from (1.34) and (1.37) we obtain, for any function φ in $I_{T_-}^{\frac{1}{2}-H}(L^2)$

$$\begin{aligned}
(K_H^*\varphi)(s) &= c_H^2(T-s)^{H-\frac{1}{2}}\varphi)(s) \\
&\quad + c_H(H-\frac{1}{2})\int_s^T (\varphi(r)-\varphi(s))(r-s)^{H-\frac{3}{2}}dr \\
&\quad + s^{H-\frac{3}{2}}\int_s^T \varphi(r)F_1'\frac{r}{s}dr \\
&= c_H\Gamma(\frac{1}{2}+H)D_{T_-}^{\frac{1}{2}-H}\varphi(s) + \Lambda\varphi(s)
\end{aligned}$$

where the operator

$$\Lambda\varphi(s) = c_H(\frac{1}{2}-H)\int_s^T \varphi(r)(r-s)^{H-\frac{3}{2}}\left(1-\left(\frac{r}{s}\right)^{H-\frac{1}{2}}\right)dr$$

is bounded in L^2.

On the other hand, (1.36) implies that

$$\mathcal{H} = \{f : \exists\phi \in L^2(0,T) : f(s) = d_H^{-1}s^{\frac{1}{2}-H}(I_{T_-}^{\frac{1}{2}-H}u^{\frac{1}{2}-H}\phi(u))(s)\},$$

with the inner product

$$\langle f, g\rangle_{\mathcal{H}} = \int_0^T \phi(s)\varphi(s)ds,$$

if

$$f(s) = d_H^{-1} s^{\frac{1}{2}-H} (I_{T-}^{\frac{1}{2}-H} u^{\frac{1}{2}-H} \phi(u))(s)$$

and

$$g(s) = d_H^{-1} s^{\frac{1}{2}-H} (I_{T-}^{\frac{1}{2}-H} u^{\frac{1}{2}-H} \varphi(u))(s).$$

Consider process $W = \{W_t, t \in [0,T]\}$ defined by

$$W_t = B((K_H^*)^{-1}(1_{[0,t]})).$$

As in the case $H > \frac{1}{2}$, we can show that W is a Wiener process, and the process B has the integral representation

$$B_t = \int_0^t K_H(t,s) dW_s.$$

Chapter 2

Some Elements of Functional Analysis

In this chapter, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis

Let $J := [a, b]$ be an interval of $\mathbb{R}$. Let $(E, |\cdot|)$ be a real Banach space. $C(J, E)$ is the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup_{t\in J} |y(t)|$$

$L^1([a, b], E)$ denotes the Banach space of measurable functions $y : [a, b] \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see for instance, Yosida [31]).

$L^1(J, E)$ denotes the Banach space of functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_a^b |y(t)|dt$$

We need the following definitions in the sequel.

Definition 2.0.6. *A map $f : J \times E \to E$ is said to be L^p-Carathéodory ($p \geq 1$)if*

(i) *$t \mapsto f(t, v)$ is measurable for each $v \in E$;*

(ii) *$v \mapsto F(t, v)$ is continuous for almost all $t \in J$;*

(iii) *for each $q > 0$, there exists $\alpha_q \in L^p(J, \mathbb{R}^+)$ such that*

$$\|f(t,v)\|^p \leq \alpha_q(t), \text{ for all } \|v\|_E^p \leq q \text{ and for a.e. } t \in J.$$

Definition 2.0.7. *A map f is said compact if the image is relatively compact. f is said completely continuous if is continuous and the image of every bounded is relatively compact.*

2.1 Some Properties of Phase Spaces

In this thesis, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale & Kato [41] and follow the terminology used in [48]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on $[0, b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0, b)$ the following conditions hold :

- (i) $y_t \in \mathcal{B}$;
- (ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
- (iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{\, |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

2.1.1 Examples of phase spaces

In this subsection, we present some examples of phase spaces

Example 2.1.1. *The spaces BC, BUC, C^∞ and C^0.*

Let: BC denote the space of bounded continuous functions defined from $(-\infty, 0]$ to $\mathbb{R}$;
BUC denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to $\mathbb{R}$;

$C^\infty := \{\phi \in BC \ : \ \lim\limits_{\theta\to-\infty}\phi(\theta)$ *exist in* $\mathbb{R}\}$; $C^0 := \{\phi \in BC \ : \ \lim\limits_{\theta\to-\infty}\phi(\theta) = 0\}$; , *endowed with the uniform norm*

$$\|\phi\|_\infty = \sup\{|\phi(\theta)| : \theta \in (-\infty, 0]\}.$$

Then, the spaces BUC, C^∞ *and* C^0 *satisfy conditions* (A_1)–(A_3), *whereas,* BC *satisfies* (A_2), (A_3), *but not* (A_1).

Example 2.1.2. *The spaces* C_g, UC_g, C_g^∞ *and* C_g^0.

Let g be a positive continuous function on $(-\infty, 0]$. *We define:*

$$C_g := \left\{\phi \in C((-\infty,0],\mathbb{R}) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty,0]\right\};$$

$C_g^0 := \left\{\phi \in C_g : \lim\limits_{\theta\to-\infty}\dfrac{\phi(\theta)}{g(\theta)} = 0\right\}$;, *endowed with the uniform norm*

$$\|\phi\|_\infty = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)} : \theta \in (-\infty, 0]\right\}.$$

Then, the spaces C_g *and* C_g^0 *satisfy condition* (A_3).

If we impose the following condition on the function g:

(g_1) *For all* $a > 0$ $\sup\limits_{0\le t\le a}\sup\left\{\dfrac{g(t+\theta)}{g(\theta)} : -\infty < \theta \le -t\right\} < \infty.$

then, the spaces C_g *and* C_g^0 *satisfy conditions* (A_1) *and* (A_3).

Example 2.1.3. *The space* C_γ.

For any real positive constant γ, *we define the functional space* C_γ *by*

$$C_\gamma := \{\phi \in C((-\infty,0],\mathbb{R}) \ : \ \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exist in } \mathbb{R}\};$$

endowed with the following norm

$$\|\phi\|_\infty = \sup\{e^{\gamma\theta}|\phi(\theta)| : \theta \le 0\}.$$

Then in the space C_γ *the axioms* (A_1)–(A_3) *are satisfied.*

We may consider the following examples of phase spaces satisfying all above properties.

Example 2.1.4. *For $r > 0$, let*

$$\mathcal{D} = \{\phi\colon\ [-r, 0] \to E,\ \phi \text{ is continuous everywhere except for a finite number of points } t \text{ at which } \phi(t^-) \text{ and } \phi(t^+) \text{ exist and satisfy } \phi(t^-) = \phi(t)\},$$

where E is a Banach space. Considering $\mathcal{D}$ as the subspace of the space of measurable functions, we may treat it as the normed space with the norm,

$$\|\phi\| = \int_{-r}^{0} \|\phi(\theta)\| d\theta.$$

1. *For $\nu > 0$ let $\mathcal{B} = PC_\nu = \{\phi : (-\infty, 0] \to E$ such that $\phi \in \mathcal{D}([-r, 0], E)$ for each $r > 0$ and $\int_{-\infty}^{0} e^{\nu\theta}\|\phi(\theta)\|d\theta < \infty$. Then we set*

$$\|\phi\|_{\mathcal{B}} = \int_{-\infty}^{0} e^{\nu\theta}\|\phi(\theta)\|d\theta$$

2. *(Spaces of "fading memory") Let*

$$\mathcal{B} = PC_\nu = \{\phi\colon\ (-\infty, 0] \to E,\ \text{ such that } \phi \in \mathcal{D}([-r, 0], E) \text{ for some } r > 0 \text{ and } \phi \text{ is Lebesgue measurable on } (-\infty, -r) \text{ and there exists a positive Lebesgue integrable function } \rho : (-\infty, -r) \to \mathbb{R}_+ \text{ such that } \rho\phi \in L^1((-\infty, -r), E)\},$$

   and moreover, there exists a locally bounded function $P : (-\infty, 0] \to \mathbb{R}_+$ such that, for all $\zeta \leq 0$ $\rho(\zeta + \theta) < P(\zeta)\rho(\theta)$ a.e. $\theta \in (-\infty, -r)$. Then

$$\|\phi\|_{\mathcal{B}} = \int_{-\infty}^{-r} \rho(\theta)\|\phi(\theta)\|d\theta + \int_{-r}^{0} \|\phi(\theta)\|d\theta.$$

   A simple example of such a space is defined for $\rho(\theta) = e^{\mu\theta}$, $\mu \in \mathbb{R}$.

2.2 Some Properties of Set-Valued Maps

Let (X, d) be a metric space and Y be a subset of X. We denote:

- $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$ and
- $\mathcal{P}_p(X) = \{Y \in P(X) : Y$ has the property "p"$\}$, where p could be: *cl*=closed, *b*=bounded, *cp*=compact, *cv*=convex, etc.

Thus

- $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
- $\mathcal{P}_{b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
- $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$,
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$,
- $\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$, etc.

A multivalued map (multimap) F of a set X into a set Y is a correspondence which associates to very $x \in X$ a non-empty subset $F(x) \subset Y$, called the value of x. We will write this correspondence as

$$F : X \to \mathcal{P}(Y).$$

Definition 2.2.1. *A multimap $F : X \to \mathcal{P}(Y)$ is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$.*
F is bounded on bounded sets if $F(B) = \cup_{x \in B} F(x)$ is bounded in Y for all $B \in \mathcal{P}_b(X)$

$$(\text{i.e. } \sup_{x \in B}\{\sup\{|y| : y \in F(x)\}\} < \infty).$$

The set $\Gamma_F \subset X \times Y$, defined by

$$\Gamma_F = \{(x, y) : x \in X, y \in F(x)\}$$

is said to be graph of F.
F is called closed graph if Γ_F is closed in $X \times Y$.

Definition 2.2.2. *Let X, Y be Hausdorff topological spaces and $F : X \to \mathcal{P}(Y)$ is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of X and if for each open set N of X containing $F(x_0)$, there exists an open neighbourhood N_0 of x_0 such that $F(N_0) \subseteq N$.*

G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in Y

Lemma 2.2.1. *[49] If $F : X \to \mathcal{P}(Y)$ is closed graph and locally compact (i.e., for every $x \in X$, there exists a $U \in \mathcal{N}(x)$ such that $\overline{F(U)} \in \mathcal{P}_{cp}(Y)$, then $F(.)$ is upper semicontinuous.*

F is said to be completely continuous if $F(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in \mathcal{P}_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [a,b] \to \mathcal{P}_{cl}(\mathbb{R}^n)$ is said to be measurable if for every $y \in \mathbb{R}^n$, the function $t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable.

Definition 2.2.3. *A multivalued map $F : [a,b] \times E \to \mathcal{P}(E)$ is said to be L^p-Carathéodory if*

(i) $t \longmapsto F(t,y)$ *is measurable for all* $y \in E$,

(ii) $y \longmapsto F(t,y)$ *is upper semicontinuous for almost each* $t \in [a,b]$,

(iii) for each $q > 0$, *there exists* $\varphi_q \in L^p([a,b], \mathbb{R}_+)$ *such that*

$$\|F(t,y)\|^p = \sup\{|v|^p : v \in F(t,y)\} \leq \varphi_q(t) \ \text{ for all } |y|^p \leq q \text{ and a.e. } t \in [a,b].$$

Definition 2.2.4. *A multivalued map $F : [a,b] \times E \to \mathcal{P}(E)$ is said to be Carathéodory if (i) and (ii) hold.*

For each $y \in C([a,b], E)$, define the set of selections of F by

$$S_{F,y} = \{v \in L^p([a,b], E) : v(t) \in F(t, y(t)) \ \text{a.e. } t \in [a,b]\}.$$

Definition 2.2.5. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([a,b], E))$ be a multivalued operator. We say N has property (BC) if*

1) N is lower semi-continuous (l.s.c.),

2) N has nonempty closed and decomposable values.

Let $F : [a,b] \times E \to \mathcal{P}(E)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C([a,b], E) \to \mathcal{P}(L^p([a,b], E))$$

by letting

$$\mathcal{F}(y) = \{w \in L^p([a,b], E) : w(t) \in F(t, y(t)) \ \text{ a.e. } t \in [a,b]\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 2.2.6. *Let $F : [a,b] \times E \to \mathcal{P}(E)$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

We need The following lemma in the sequel:

Lemma 2.2.2. *[32]. Let $F : J \times \mathcal{D} \to \mathcal{P}(E)$ be a multivalued map with nonempty, compact values. Assume that*

(2.2.2.1) $F : J \times E \longrightarrow \mathcal{P}(E)$ *is a nonempty compact valued multivalued map such that*
a) $(t, y) \mapsto F(t, y)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable;*
b) $y \mapsto F(t, u)$ *is lower semi-continuous for a.e.* $t \in J$;

(2.2.2.2) for each $r > 0$, *there exists a function* $h_r \in L^p(J, \mathbb{R}^+)$ *such that*

$$\begin{aligned}\|F(t,u)\| : \;=\; & \sup\{|v| : v \in F(t,u)\} \le h_r(t) \text{ for a.e. } t \in J \text{ and for} \\ & u \in \mathcal{D} \text{ with } \|u\|_{\mathcal{D}} \le r.\end{aligned}$$

Then F is of l.s.c. type.

Next we state a selection theorem due to Bressan and Colombo.

Theorem 2.2.1. *[21] Let Y be separable metric space and let $N : Y \to \mathcal{P}(L^p(J, E))$ be a multi-valued operator which has property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $g : Y \to L^p(J, E)$ such that $g(u) \in N(u)$ for every $u \in Y$.*

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}^n_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space see [55].

Definition 2.2.7. *A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x, y), \quad \text{for each } x,\ y \in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

The following lemmas will be used in the sequel:

Lemma 2.2.3. *[58] Let I be a compact interval and E be a Hilbert space. Let F be an L^p-Carathéodory multi-valued map with $N_{F,y} \neq \emptyset$. and let Γ be a linear continuous mapping from $L^p(I, E)$ to $C(I, E)$. Then, the operator*

$$\Gamma \circ N_F : C(I, E) \longrightarrow \mathcal{P}_{cp,c}(E), \quad y \longmapsto (\Gamma \circ N_F)(y) = \Gamma(N_F, y),$$

is a closed graph operator in $C(I, E) \times C(I, E)$, where $N_{F,y}$ is known as the selectors set from F and given by
$f \in N_{F,y} = \{f \in L^p([0, T], \mathcal{H}) : f(t) \in F(t, y) \quad for\ a.e.t \in [0, T]\}$.

For more details on multivalued maps and the proof of the known results cited in this section we refer interested reader to the books of Deimling [27], Gorniewicz [35], Hu and Papageorgiou [49], Smirnov [77] and Tolstonogov [81].

2.3 Fixed Point Theorems

Fixed point theory plays an important role in our existence results, therefore we state the following fixed point theorems.

Theorem 2.3.1 (Banach's fixed point theorem, [30])**.** *Let C be a nonempty closed subset of a Banach space E and let $G : C \to C$ be a contraction, then G has unique fixed point.*

Theorem 2.3.2 (Burton-Kirk's fixed point theorem [6])**.** *Let E be a Banach space, and $G_1, G_2 : E \to E$ be two operators satisfying:*

1. *G_1 is a contraction, and*
2. *G_2 is completely continuous*

Then, either the operator equation $y = G_1(y) + G_2(y)$ possesses a solution, or the set $\Xi = \left\{ y \in E : \lambda G_1(\frac{y}{\lambda}) + \lambda G_2(y) = y,\ for\ some\ \lambda \in (0, 1) \right\}$ is unbounded.

Next we state a well known result, the Nonlinear Alternative. By $\overline{U}$ and ∂U we denote the closure of U and the boundary of U respectively

Lemma 2.3.1. *(Nonlinear Alternative [30]). Let E be a Banach space with $C \subset E$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $G : \overline{U} \to C$ is a compact map. Then either,*

(i) there is a point $y \in \partial U$ and $\lambda \in (0, 1)$ with $y = \lambda G(y)$, or

(ii) G has a fixed point in $\overline{U}$.

The multivalued version of Nonlinear Alternative

Lemma 2.3.2. *[30]*

Let E be a Banach space with $C \subset E$ convex. Assume U is a relatively open subset of C, with $0 \in U$, and let $G : E \longrightarrow \mathcal{P}_{cp,c}(E)$ be an upper semicontinuous and compact map. Then either,

(a) *G has a fixed point in $\overline{U}$, or*

(b) *there is a point $y \in \partial U$ and $\lambda \in (0, 1)$, with $y \in \lambda G(y)$.*

2.4 Semigroups

2.4.1 C_0-Semigroups

Let E be a Banach space and $B(E)$ be the Banach space of linear bounded operators.

Definition 2.4.1. *Asemigroup of classe C_0 is a one parameter family $\{S(t) \mid t \geq 0\} \subset B(E)$ satisfying the conditions:*

(i) $S(t) \circ S(s) = S(t + s)$, for $t, s \geq 0$,

(ii) $S(0) = I$,

(iii) the map $t \to S(t)(x)$ is strongly continuous, for each $x \in E$, i.e;

$$\lim_{t\to 0} S(t)x = x, \ \forall x \in E.$$

A semigroup of bounded linear operators $S(t)$, is uniformly continuous if

$$\lim_{t\to 0} \|S(t) - I\| = 0.$$

Here I denotes the identity operator in E.

We note that if a semigroup $S(t)$ is class (C_0) then satisfies the growth condition

> $\|S(t)\|_{B(E)} \leq M \cdot \exp(\beta t)$, for $0 \leq t < \infty$, with some constants $M > 0$ and β.

If, in particular $M = 1$ and $\beta = 0$, i.e; $\|S(t)\|_{B(E)} \leq 1$, for $t \geq 0$, then the semigroup $S(t)$ is called a *contraction semigroup* (C_0).

Definition 2.4.2. *Let $S(t)$ be a semigroup of class (C_0) defined on E. The infinitesimal generator A of $S(t)$ is the linear operator defined by*

$$A(x) = \lim_{h \to 0} \frac{S((h)(x) - x)}{h}, \quad \textit{for } x \in D(A),$$

where $D(A) = \{x \in E \mid \lim_{h \to 0} \frac{S(h)(x) - x}{h}$ exists in $E\}$.

Let us recall the following property:

Proposition 2.4.1. *The infinitesimal generator A is closed linear and densely defined operator in E. If $x \in D(A)$, then $S(t)(x)$ is a C^1-map and*

$$\frac{d}{dt} S(t)(x) = A(S(t)(x)) = S(t)(A(x)) \quad \textit{on } [0, \infty).$$

Theorem 2.4.1. (Hille and Yosida) *[69]. Let A be a densely defined linear operator with domain and range in a Banach space E. Then A is the infinitesimal generator of uniquely determined semigroup $S(t)$ of class (C_0) satisfying*

$$\|S(t)\|_{B(E)} \leq M \exp(\omega t), \quad t \geq 0,$$

where $M > 0$ and $\omega \in \mathbb{R}$ if and only if $(\lambda I - A)^{-1} \in B(E)$ and $\|(\lambda I - A)^{-n}\| \leq M/(\lambda - \omega)^n$, $n = 1, 2, \ldots$, for all $\lambda \in \mathbb{R}$.

For more details on strongly operators, we refer the reader to the books of Goldstein [34], Hekkila and Lakshmikantham [44] and to the papers of Travis and Webb [78, 79], and for properties on semigroup theory we refer the interested reader to the books of Goldstein [34] and Pazy [69].

2.4.2 Analytic semigroups

Definition 2.4.3. *Let $\Delta = \{z : \varphi_1 < argz << \varphi_2, \varphi_1 < 0 << \varphi_2\}$ and for $z \in \Delta$ let $S(z)$ be a bounded linear operator. The family $S(z)$, $z \in \Delta$ is an analytic semigroup in Δ if*

(i) *$z \to S(z)$ is analytic in Δ.*

(ii) *$S(0) = I$ and $\lim_{z \longrightarrow 0} S(z)x = x$ for every $x \in E$.*

(iii) *$S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \Delta$.*

A semigroup $S(t)$ will be called analytic if it is analytic in some sector Δ containing the nonnegative real axis.

Clearly, the restriction of an analytic semigroup to the real axis is a C_0 semigroup. We will be interested below in the possibility of extending a given C_0 semigroup to an analytic semigroup in some sector Δ around the nonnegative real axis.

Theorem 2.4.2. *[69] Let $S(t)$ be a uniformly bounded C_0 semigroup. Let A be the infinitesimal generator of $S(t)$ and assume $0 \in \rho(A)$. The following statements are equivalent:*

(a) *$S(t)$ can be extended to an analytic semigroup in a sector $\Delta_\delta = \{z : |argz| < \delta\}$ and $\|S(z)\|$ is uniformly bounded in every closed subsector $\Delta_{\delta'}$, $\delta' < \delta$, of Δ_δ.*

(b) *There exists a constant C such that for every $\sigma > 0$, $\tau \neq 0$*

$$\|R(\sigma + it : A)\| \leq \frac{C}{\tau}.$$

(c) *There exist $0 < \delta < \pi/2$ and $M > 0$ such that*

$$\rho(A) \supset \Sigma = \{\lambda : |arg\lambda| < \frac{\pi}{2} + \delta\} \cup \{0\}$$

and

$$\|R(\lambda : A)\| \leq \frac{M}{|\lambda|} \text{ for } \lambda \in \Sigma, \lambda \neq 0.$$

(d) *$S(t)$ is differentiable for $t > 0$ and there is a constant C such that*

$$\|AS(t)\| \leq \frac{C}{t}, \quad t > 0.$$

2.4.3 Fractional Powers of Closed Operators

For our definition we will make the following assumption.

Assumption 2.4.1. *Let A be a densely defined closed linear operator for which*

$$\rho(A) \supset \Sigma^{+} = \{\lambda : 0 < \omega < |arg\lambda| \leq \pi\} \cup V$$

where V is a neighborhood of zero, and

$$\|R(\lambda : A)\| \leq \frac{M}{1+|\lambda|} \text{ for } \lambda \in \Sigma^{+}.$$

If $M = 1$ and $w = \pi/2$ then ^{-}A is the infinitesimal generator of a C_0 semigroup. If $w < \pi/2$ then, by Theorem 2.4.2, $-A$ is the infinitesimal generator of an analytic semigroup. The assumption that $0 \in \rho(A)$ and therefore a whole neighborhood V of zero is in $\rho(A)$ was made mainly for convenience. Most of the results on fractional powers that we will obtain in this section remain true even if $0 \in \rho(A)$.

Definition 2.4.4. *Let A satisfy Assumption 2.4.1 with $w < \pi/2$. For every $\alpha > 0$ we define*

$$A^{\alpha} = (A^{-\alpha})^{-1}.$$

For $\alpha = 0$, $A^{\alpha} = I$.

Theorem 2.4.3. *[69] Let A^{α} be defined by Definition 2.4.4 then,*

(a) *A^{α} is a dosed operator with domain $D(A^{\alpha}) = R(A^{-\alpha})$ = the range of $A^{-\alpha}$.*

(b) *$\alpha \geq \beta > 0$ implies $D(A^{\alpha}) \subset D(A^{\beta})$.*

(c) *$\overline{D(A^{\alpha})} = E$ for every $\alpha \geq 0$.*

(d) *If α, β are real then*

$$A^{\alpha+\beta}x = A^{\alpha} \cdot A^{\beta}x$$

for every $x \in D(A^{\gamma})$ where $\gamma = \max(\alpha, \beta, \alpha+\beta)$.

Theorem 2.4.4. *[69] Let $-A$ be the infinitesimal generator of an analytic semigroup $S(t)$. if $0 \in \rho(A)$ then,*

(a) *$S(t) : E \to D(A^{\alpha})$ for every $t > 0$ and $\alpha \geq 0$.*

(b) *For every $x \in D(A^\alpha)$ we have $S(t)A^\alpha x = A^\alpha S(t)x$.*

(c) *For every $t > 0$ the operator $A^\alpha S(t)$ is bounded and*

$$\|A^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\delta t}.$$

(d) *Let $0 < \alpha \leq 1$ and $x \in D(A^\alpha)$ then*

$$\|S(t)x - x\| \leq C_\alpha t^\alpha \|A^\alpha x\|.$$

Chapter 3

Stochastic Delay Evolution Equations with Impulses

In this chapter, our main objective is to establish sufficient conditions for the existence of mild solutions of the following first order stochastic impulsive functional equation with time delays, driven by fractional Brownian motion with the Hurst index $H > 1/2$:

$$dy(t) = [Ay(t) + f(t, y_t)]dt + g(t)dB_Q^H(t), \quad t \in J := [0, T]; \tag{3.1}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k)), \;\; k = 1, \ldots, m \tag{3.2}$$

$$y(t) = \phi(t), \quad \text{for a.e. } t \in [-r, 0], \tag{3.3}$$

in a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$, where $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$, is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t)$, $0 \leq t \leq T$. B_Q^H is a fractional Brownian motion on a real and separable Hilbert space $\mathcal{K}$, with Hurst parameter $H \in (1/2, 1)$, and with respect to a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ furnished with a family of right continuous and increasing σ-algebras $\{\mathcal{F}_t, t \in J\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. Also $r > 0$ is the maximum delay, and the impulse times t_k satisfy $0 = t_0 < t_1 < t_2 < \ldots, t_m < T$. As for y_t we mean the segment solution which is define in the usual way, that is, if $y(\cdot, \cdot) : [-r, T] \times \Omega \to \mathcal{H}$, then for any $t \geq 0$, $y_t(\cdot, \cdot) : [-r, 0] \times \Omega \to \mathcal{H}$ is given by

$$y_t(\theta, \omega) = y(t + \theta, \omega), \text{ for } \theta \in [-r, 0], \; \omega \in \Omega.$$

Before describe the properties fulfilled by the operators f, g and I_k, we need to introduce some notation and describe some spaces.

Let $\mathcal{D}_{\mathcal{H}}$ the following Banach space defined by

$$\mathcal{D}_{\mathcal{H}} = \{\phi\colon [-r,0] \to \mathcal{H},\ \phi \text{ is continuous everywhere except for a finite number of points } t \text{ at which } \phi(t^-) \text{ and } \phi(t^+) \text{ exist and satisfy } \phi(t^-) = \phi(t)\},$$

endowed with the L^2-norm:

$$\|\phi\|^2 = \int_{-r}^{0} \|\phi(t)\|^2 dt.$$

Also we define $\mathcal{D}^0_{\mathcal{H}}$ as the space of all piecewise continuous processes $\phi : [-r,0] \times \Omega \to \mathcal{H}$ such that $\phi(\theta,\cdot)$ is $\mathcal{F}_0$–measurable for each $\theta \in [-r,0]$ and $\int_{-r}^{0} E\|\phi(t)\|^2 dt < \infty$. In the space $\mathcal{D}^0_{\mathcal{H}}$, we consider the norm:

$$\|\phi\|^2_{\mathcal{D}^0_{\mathcal{H}}} = \int_{-r}^{0} E\|\phi(t)\|^2 dt.$$

Now, for a given $T > 0$, we define

$$\begin{aligned}\mathcal{D}^T_{\mathcal{H}} = \big\{y\colon\ & [-r,T] \times \Omega \to \mathcal{H},\ y_k \in C(J_k,\mathcal{H}) \text{ for } k = 1,\dots m,\ y_0 \in \mathcal{D}^0_{\mathcal{H}},\\ & \text{and there exist } y(t_k^-) \text{ and } y(t_k^+) \text{ with } y(t_k) = y(t_k^-),\ \ k = 1,\cdots,m,\\ & \sup_{t\in[0,T]} E(|y(t)|^2) < \infty \text{ and } \int_{-r}^{0} E\|\phi(t)\|^2 dt < \infty\big\},\end{aligned}$$

endowed with the norm

$$\|y\|_{\mathcal{D}^T_{\mathcal{H}}} = \sup_{t\in[0,T]} (E(|y(t)|^2))^{\frac{1}{2}} + \|y\|_{\mathcal{D}^0_{\mathcal{H}}},$$

where y_k denotes the restriction of y to $J_k = (t_{k-1}, t_k]$, $k = 1, 2, \cdots, m$, and $J_0 = [-r, 0]$.

Then we will consider our initial data $\phi \in \mathcal{D}^0_{\mathcal{H}}$.

Let $\mathcal{K}$ be another real separable Hilbert and suppose that B^H_Q is a $\mathcal{K}$-valued fractional Brownian motion with increment covariance given by a non-negative trace class operator Q (see next section for more details), and let us denote by $L(\mathcal{K},\mathcal{H})$ the space of all bounded, continuous and linear operators from $\mathcal{K}$ into $\mathcal{H}$.

Then we assume that $f : J\times\mathcal{D}^0_{\mathcal{H}} \to \mathcal{H}$, $g : J \to L^0_Q(\mathcal{K},\mathcal{H})$ Here, $L^0_Q(\mathcal{K},\mathcal{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathcal{K}$ into $\mathcal{H}$.

As for the impulse functions we will assume that $I_k \in C(\mathcal{H},\mathcal{H})$ $(k = 1,\dots,m)$, and $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k+h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k-h)$.

3.1 Existence result

New, we are able to state and prove our main theorem for the initial value problem (4.1)-(4.3). Before starting and proving this one, we give the definition of mild solution to our problem.

Definition 3.1.1. *Given $\phi \in \mathcal{D}^0_{\mathcal{H}}$, an $\mathcal{H}-$valued stochastic process $\{y(t), t \in [-r,T]\}$ is called a mild solution of the problem (4.1)-(4.3) if $y(t)$ is measurable and $\mathcal{F}_t$-adapted, for each $t > 0$, $y(t) = \phi(t)$ on $[-r,0]$ and y satisfies the integral equation*

$$\begin{aligned} y(t) &= S(t)\phi(0) + \int_0^t S(t-s)f(s,y_s)ds + \int_0^t S(t-s)g(s)dB^H_Q(s) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(y(t_k)),\ t\in[0,T]. \end{aligned} \tag{3.4}$$

Notice that this concept of solution can be considered as more general than then classical concept of solution to equation (4.1)–(4.3). A continuous solution of (5.9) is called a mild solution of (4.1)-(4.3).

Our main result in this section is based upon the fixed point theorem 2.3.2 due to Burton and Kirk [22].

We are now in a position to state and prove our existence result for the problem (4.1)-(4.3). First we will list the following hypotheses which will be imposed in our main theorem.

- (H1) operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t)\}$, $t \in J$ which is compact for $t > 0$ in $\mathcal{H}$, and there exists a constant M such that $\|S(t)\|^2 \le M$ for all $t \in [0,T]$.

- (H2) There exist constants $c_k > 0$, $k = 1,\dots,m$ with $8Mm\sum_{k=1}^{m} c_k < 1$

such that

$$|I_k(y) - I_k(x)|^2 \leq c_k |y - x|^2, \text{ for all } y, x \in \mathcal{H}.$$

- (H3) For each $t \in J$ the functions $f(t, \cdot) : \mathcal{D}^0_{\mathcal{H}} \longrightarrow \mathcal{H}$ is continuous, and for each $y \in \mathcal{H}$ the function $f(\cdot, y) : J \to \mathcal{H}$ is strongly $\mathcal{F}_t$-measurable.
- (H4)The function $g : J \longrightarrow L_Q(\mathcal{K}, \mathcal{H})$ satisfies

$$\int_0^T \|g(s)\|^2_{L^0_Q} ds < \infty.$$

- (H5) For the initial value $\phi \in \mathcal{D}^0_{\mathcal{H}}$, there exists a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that $E|f(t, y)|^2 \leq p(t)\psi(\|y\|^2_{\mathcal{D}^0_{\mathcal{H}}})$, for a.e. $t \in J$ and $y \in \mathcal{D}^0_{\mathcal{H}}$

  with
$$\int_{TC_0}^{\infty} \frac{du}{\psi(u)} > TC_1 \int_0^T p(t)dt,$$
  where
$$C_0 = \frac{4M\left(E|\phi(0)|^2 + 2m\sum_{k=1}^{m} E|I_k(0)|^2 + 2HT^{2H-1}\int_0^T \|g(s)\|^2_{L^0_Q} ds\right)}{\left(1 - 8Mm\sum_{k=1}^{m} c_k\right)},$$

$$C_1 = \frac{4MT}{\left(1 - 8Mm\sum_{k=1}^{m} c_k\right)}.$$

Theorem 3.1.1. *Assume that hypotheses (H1), (H2), (H3), (H4) and (H5) hold. Then, problem (4.1)-(4.3) possesses at least one mild solution on* $[-r, T]$.

Proof. Transform the problem (4.1)-(4.3) into a fixed point problem. Consider the two operators:

$$\Phi_1, \Phi_2 : \mathcal{D}^T_{\mathcal{H}} \to \mathcal{D}^T_{\mathcal{H}}$$

defined by:

$$\Phi_1(y)(t) = \begin{cases} \phi(t), & \text{if } \ t \in [-r, 0]; \\ S(t)\phi(0) + \displaystyle\int_0^t S(t-s)g(s)dB_Q^H(s) + \sum_{0<t_k<t} S(t-t_k)I_k(y(t_k)), & \text{if } \ t \in [0, T], \end{cases}$$

and

$$\Phi_2(y)(t) = \begin{cases} \phi(0), & \text{if } \ t \in [-r, 0]; \\ S(t)\phi(0) + \displaystyle\int_0^t S(t-s)f(s, y_s)ds, & \text{if } \ t \in [0, T]. \end{cases}$$

Then the problem of finding the solution of problem (4.1)-(4.3) is reduced to finding the solution of the operator equation $\Phi_1(y)(t) + \Phi_2(y)(t) = y(t)$, $t \in [-r, T]$. We shall show that the operators Φ_1 and Φ_2 satisfy all the conditions of Theorem 2.3.2. The proof will be given in several steps.

Step 1 : Φ_1 is a contraction.

Let $x, y \in \mathcal{D}_{\mathcal{H}}^T$. Then for $t \in J$

$$\begin{aligned} E|\Phi_1(y)(t) - \Phi_1(x)(t)|^2 &\leq E\left|\sum_{0\leq t_k\leq t} S(t-t_k)(I_k(y(t_k^-)) - I_k(x(t_k^-))\right|^2 \\ &\leq Mm \sum_{0\leq t_k\leq t} E\left|(I_k(y(t_k^-)) - I_k(x(t_k^-))\right|^2 \\ &\leq Mm \sum_{0\leq t_k\leq t} c_k E\left|(y(t_k^-) - x(t_k^-))\right|^2 \\ &\leq Mm \sum_{k=0}^m c_k \|y - x\|^2_{\mathcal{D}_{\mathcal{H}}^T}, \\ &\leq 8Mm \sum_{k=0}^m c_k \|y - x\|^2_{\mathcal{D}_{\mathcal{H}}^T}, \end{aligned}$$

which is a contraction.

Step 2 : Φ_2 is continuous.

Let y_n be a sequence such that $y_n \to y$ in $\mathcal{D}^T_{\mathcal{H}}$. Then, for $t \in J$, and thanks to (H1) and (H3),

$$\begin{aligned}
&E|\Phi_2(y_n)(t) - \Phi_2(y)(t)|^2 \\
&\qquad \leq E\Big|\int_0^t S(t-s)(f(s,(y_n)_s) - f(s,y_s))ds\Big|^2 \\
&\qquad \leq E\Big(\int_0^t |S(t-s)||f(s,(y_n)_s) - f(s,y_s)|ds\Big)^2 \\
&\qquad \leq E\Big[\int_0^t |S(t-s)|^2 ds\Big(\int_0^t |f(s,(y_n)_s) - f(s,y_s)|^2 ds\Big)\Big].
\end{aligned}$$

Hence

$$\sup_{t\in[0,T]} E|\Phi_2(y_n)(t) - \Phi_2(y)(t)|^2 \leq MT\int_0^T E|f(s,(y_n)_s) - f(s,y_s)|^2 ds \to 0 \text{ as } n \to +\infty.$$

Thus Φ_2 is continuous.

Step3. Φ_2 maps bounded sets into bounded sets in $\mathcal{D}^T_{\mathcal{H}}$.

Indeed, it is enough to show that for any $q > 0$, there exists a positive constant l such that for each $y \in \mathcal{B}_q = \{y \in \mathcal{D}^T_{\mathcal{H}} : \|y\|^2_{\mathcal{D}^T_{\mathcal{H}}} \leq q\}$, one has $||\Phi_2(y)||^2_{\mathcal{D}^T_{\mathcal{H}}} \leq l$.
Let $y \in \mathcal{B}_q$, then for each $t \in [0,T]$, we have

$$\begin{aligned}
|\Phi_2 y(t)|^2 &\leq \Big|S(t)\phi(0) + \int_0^t S(t-s)f(s,y_s)ds\Big|^2 \\
&\leq 2M|\phi(0)|^2 + 2M\Big|\int_0^t f(s,y_s)ds\Big|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
E|\Phi_2 y(t)|^2 &\leq 2ME|\phi(0)|^2 + 2TM\int_0^t p(s)\psi(\|y_s\|^2_{\mathcal{D}^0_{\mathcal{H}}})ds \\
&\leq 2ME|\phi(0)|^2 + 2MT\psi(q)\int_0^T p(s)ds.
\end{aligned}$$

Then we have

$$E|\Phi_2 y(t)|^2 \leq 2ME|\phi(0)|^2 + 2MT\psi(q)\|p\|_{L^1} = l.$$

Step 4 : Φ_2 maps bounded sets into equicontinuous sets of $\mathcal{D}^T_{\mathcal{H}}$.

Let $0 < \epsilon \leq \tau_1 < \tau_2 \in J$, $\tau_1, \tau_2 \neq t_i$, $i = 1, \cdots, m$, and $\mathcal{B}_q$ be a bounded set of $\mathcal{D}^T_{\mathcal{H}}$ as in Step 2. Let $y \in \mathcal{B}_q$ then for each $t \in J$ we have

$$\begin{aligned} E|(\Phi_2 y)(\tau_2) - (\Phi_2 y)(\tau_1)|^2 &\leq 2(|[S(\tau_2) - S(\tau_1)]|E|\phi(0)|)^2 \\ &+6T\psi(q)\int_0^{\tau_1-\epsilon} |S(\tau_2 - s) - S(\tau_1 - s)|^2 p(s)ds \\ &+6T\psi(q)\int_{\tau_1-\epsilon}^{\tau_1} |S(\tau_2 - s) - S(\tau_1 - s)|^2 p(s)ds \\ &+6T\psi(q)\int_{\tau_1}^{\tau_2} |S(\tau_2 - s)|^2 p(s)ds. \end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and ϵ sufficiently small, since $S(t)$ is strongly continuous operator and the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology [69]. This proves the equicontinuity for the case where $t \neq t_i$ $i = 1, \cdots, m$. It remains to check the equicontinuity at $t = t_i$.

First we prove equicontinuity at $t = t_i^-$, for a fixed $i \in \{1, \cdots, m\}$. Fix $\delta_1 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$. For $0 < h < \delta_1$ we have

$$\begin{aligned} E|(\Phi_2 y)(t_i - h) - (\Phi_2 y)(t_i)|^2 &\leq 2(|[S(t_i) - S(t_i - h)]|E|\phi(0)|)^2 \\ &+4TE\int_0^{t_i-h} |(S(t_i - h - s) - S(t_i - s))f(s, y_s)|^2 ds \\ &+4T\psi(q)\int_{t_i-h}^{t_i} Mp(s)ds, \end{aligned}$$

and the right-hand side obviously tends to zero as $h \to 0$.

Next we prove equicontinuity at $t = t_i^+$. Fix $\delta_2 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$. For $0 < h < \delta_1$ we have that

$$\begin{aligned} E|(\Phi_2 y)(t_i) - (\Phi_2 y)(t_i - h)|^2 &\leq 2|[S(t_i + h) - S(t_i)]E(\phi(0))|^2 \\ &+4TE\int_0^{t_i} |(S(t_i + h - s) - S(t_i - s))f(s, y_s)|^2 ds \\ &+4T\psi(q)\int_{t_i}^{t_i+h} Mp(s)ds, \end{aligned}$$

and once more, the right-hand side tends to zero as $h \to 0$. The cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ follows from the uniform continuity of ϕ on the interval $[-r, 0]$.

As a consequence of Steps 2 to 3, together with the Arzelá-Ascoli theorem, it suffices to show that Φ_2 maps $\mathcal{B}_q$ into a precompact set in $\mathcal{H}$.

Let $0 < t < T$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in \mathcal{B}_q$ we define

$$(\Phi_{2\epsilon} y)(t) = S(t)\phi(0) + S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)f(s, y_s)ds$$

Since $S(t)$ is a compact operator, the set

$$Y_\epsilon(t) = \{\Phi_{2\epsilon}(y)(t) : \quad y \in \mathcal{B}_q\}$$

is precompact in H for every ϵ, $0 < \epsilon < t$. Moreover, for every $y \in \mathcal{B}_q$ we have

$$\begin{aligned} E|(\Phi_2 y)(t) - (\Phi_{2\epsilon}(y)(t)|^2 &\leq T\int_{t-\epsilon}^t |S(t,s)|^2 p(s)\psi(q)ds \\ &\leq TM\int_{t-\epsilon}^t p(s)\psi(q)ds. \end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $Y_\epsilon(t) = \{\Phi_{2\epsilon}(y)(t) : \quad y \in \mathcal{B}_q\}$. Hence the set $Y(t) = \{\Phi_2(y)(t) : \quad y \in \mathcal{B}_q\}$ is precompact in $\mathcal{H}$, and therefore, the operator $\Phi_2 : \mathcal{D}_{\mathcal{H}}^T \to \mathcal{D}_{\mathcal{H}}^T$ is completely continuous.

Step 5 : A priori bounds.

Now it remains to show that the set

$$\Xi = \{y \in \mathcal{D}_{\mathcal{H}}^T : y = \lambda\Phi_2(y) + \lambda\Phi_1\left(\frac{y}{\lambda}\right), \text{for some } \ 0 < \lambda < 1\}$$

is bounded.

Let $y \in \Xi$. Then $y = \lambda\Phi_2(y) + \lambda\Phi_1(\frac{y}{\lambda})$ for some $0 < \lambda < 1$. Thus for each $t \in J$

$$\begin{aligned} y(t) &= \lambda\Big[S(t)\phi(0) + \textstyle\int_0^t S(t-s)f(s,y_s)ds + \int_0^t S(t-s)g(s)dB_Q^H(s) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k\left(\frac{y}{\lambda}(t_k)\right)\Big], \end{aligned}$$

and

$$y(t) = \phi(t), \text{ for each } t \in [-r, 0].$$

This implies, for each $t \in J$,

$$\begin{aligned}
E|y(t)|^2 &\leq 4\lambda M E|\phi(0)|^2 + 4\lambda MT \textstyle\int_0^t p(s)\psi(\|y_s\|^2_{\mathcal{D}^0_{\mathcal{H}}})ds + 8\lambda M H t^{2H-1} \int_0^t \|g(s)\|^2_{L^0_Q} ds \\
&\quad + 4\lambda M m \sum_{k=1}^m E\left(I_k(\frac{y}{\lambda}(t_k))\right)^2 \\
&\leq 4\lambda M E|\phi(0)|^2 + 4\lambda MT \int_0^t p(s)\psi(\|y_s\|^2_{\mathcal{D}_{\mathcal{H}}})ds + 8\lambda M H t^{2H-1} \int_0^t \|g(s)\|^2_{L^0_Q} ds \\
&\quad + 8\lambda M m \sum_{k=1}^m E(I_k(\frac{y}{\lambda}(t_k)) - I_k(0))^2 + 8\lambda M m \sum_{0=1}^m E(I_k(0))^2 \\
&\leq 4M\left(E|\phi(0)|^2 + 2m\sum_{0=1}^m E|I_k(0)|^2 + 2HT^{2H-1}\int_0^T \|g(s)\|^2_{L^0_Q} ds\right) \\
&\quad + 4MT \textstyle\int_0^t p(s)\psi(\|y_s\|^2_{\mathcal{D}^0_{\mathcal{H}}})ds + 8Mm \sum_{k=1}^m c_k E(|y(t_k)|)^2.
\end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{E|y(t)|^2 : 0 \leq s \leq t\}, \quad t \in J.$$

Since $\|y_s\|^2_{\mathcal{D}^0_{\mathcal{H}}} = \int_{-r}^0 E|y(s+\theta)|^2 d\theta = \int_{-r+s}^s E|y(\theta)|^2 d\theta$ then

$$\|y_s\|^2_{\mathcal{D}^0_{\mathcal{H}}} \leq \int_{-r}^0 E|\phi(\theta)|^2 d\theta + \int_0^s E|\phi(\theta)|^2 d\theta \leq \int_{-r}^0 E|\phi(\theta)|^2 d\theta + T\mu(t).$$

Hence

$$\|y_s\|^2_{\mathcal{D}^0_{\mathcal{H}}} \leq \|\phi\|^2_{\mathcal{D}^0_{\mathcal{H}}} + T\mu(t)$$

Then, for $t \in J$ we have

$$\begin{aligned}
\mu(t) \leq\ & 4M\left(E|\phi(0)|^2 + 2m\sum_{k=1}^m E|I_k(0)|^2 + 2HT^{2H-1}\int_0^T \|g(s)\|^2_{L^0_Q} ds)\right) \\
& + 4MT\int_0^t p(s)\psi(T\mu(s) + \|\phi\|^2_{\mathcal{D}^0_{\mathcal{H}}})ds + 8Mm\sum_{k=1}^m c_k \mu(t).
\end{aligned} \tag{3.5}$$

Then

$$\begin{aligned}
\left(1 - 8Mm\sum_{k=1}^m c_k\right)\mu(t) \leq\ & 4M\left(E|\phi(0)|^2 + 2m\sum_{k=1}^m E|I_k(0)|^2 + 2HT^{2H-1}\int_0^T \|g(s)\|^2_{L^2_Q} ds\right) \\
& + 4MT\int_0^t p(s)\psi(T\mu(s) + \|\phi\|^2_{\mathcal{D}^0_{\mathcal{H}}})ds.
\end{aligned} \tag{3.6}$$

Thus we have

$$T\mu(t) + \|\phi\|^2_{\mathcal{D}^0_{\mathcal{H}}} \le \|\phi\|^2_{\mathcal{D}^0_{\mathcal{H}}} + TC_0 + TC_1 \int_0^t p(s)\psi(T\mu(s) + \|\phi\|^2_{\mathcal{D}^0_{\mathcal{H}}})ds. \quad (3.7)$$

Let us denote the right-hand side of the inequality (4.8) by $v(t)$. Then we have

$$v(0) = TC_0 + \|\phi\|^2_{\mathcal{D}^0_{\mathcal{H}}},\ T\mu(t) + \|\phi\|^2_{\mathcal{D}^0_{\mathcal{H}}} \le v(t), \quad t \in J,$$

and

$$v'(t) = TC_1 p(t)\psi(T\mu(t) + \|\phi\|^2_{\mathcal{D}^0_{\mathcal{H}}}), \quad t \in J.$$

Using the increasing character of ψ we obtain

$$v'(t) \le TC_1 p(t)\psi(v(t)), \quad \text{for } \ a.e.\ \ t \in J$$

This implies for each $t \in J$ we have

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le TC_1 \int_0^T p(s)ds < \int_{TC_0}^{\infty} \frac{du}{\psi(u)}.$$

Consequently, there exists a constant K such that

$$T\mu(t) + \|\phi\|^2_{\mathcal{D}^0_{\mathcal{H}}} \le v(t) \le K,\ t \in J.$$

Now from the definition of μ it follows that

$$E|y|^2 \le \mu(T) \le \frac{K}{T}, \quad \text{for all} \ \ y \in \Xi.$$

This shows that the set Ξ is bounded. And, as a consequence of Theorem (2.3.2) we deduce that $\Phi_1 + \Phi_2$ has a fixed point which is a mild solution of problem (4.1)-(4.3).

□

3.2 Application

Consider the following stochastic partial differential equation with delays and impulsive effects

$$\begin{cases} \frac{\partial}{\partial t}u(t,\xi) &= \frac{\partial^2}{\partial \xi^2}u(t,\xi) + F(t, u(t-r,\xi)) \\ & +\sigma(t)\frac{dB_Q^H}{dt}, \quad t \geq 0, \quad t \neq t_k, \quad 0 \leq \xi \leq \pi, \\ u(t_k^+,\xi) &- \ u(t_k^-,\xi) = \alpha_k u(t_k^-,\xi), \quad k = 1, \cdots, m, \\ u(t,0) &= u(t,\pi) = 0, t \geq 0, \\ u(t,\xi) &= \phi(t,\xi), -r \leq t \leq 0, 0 \leq \xi \leq \pi, \end{cases} \tag{3.8}$$

where $r > 0$, $\alpha_k > 0$, B_Q^H denotes a fractional Brownian motion, $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let

$$y(t)(\xi) = u(t,\xi) \quad t \in J, \quad \xi \in [0,\pi],$$

$$I_k(y(t_k)) = \alpha_k u(t_k^-,\xi), \quad \xi \in [0,\pi], \quad k = 1, \cdots, m,$$

$$f(t,\phi)(\xi) = F(t, \phi(-r,\xi)), \quad \theta \in [-r,0], \quad \xi \in [0,\pi],$$

$$\phi(\theta)(\xi) = \phi(\theta,\xi), \quad \theta \in [-r,0], \quad \xi \in [0,\pi],$$

Take $\mathcal{K} = \mathcal{H} = L^2([0,\pi])$. We define the operator A by $Au = \frac{\partial^2}{\partial \xi^2}u.$ with domain $D(A) = \{u \in \mathcal{H}, \frac{\partial u}{\partial \xi}, \frac{\partial^2 u}{\partial \xi^2} \in \mathcal{H} \quad \text{and} \quad u(0) = u(\pi) = 0\}$.

Then, it is well known that

$$Az = -\sum_{n=1}^{\infty} n^2 \langle z, e_n \rangle e_n, \quad z \in \mathcal{H},$$

and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t)\}_{t\geq 0}$ on $\mathcal{H}$, which is given by

$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}\langle u, e_n\rangle e_n$, $u \in \mathcal{H}$, and $e_n(u) = (2/\pi)^{1/2}\sin(nu), n = 1, 2, \cdots$, is the orthogonal set of eigenvectors of A. It is well known that $\{S(t)\}$, $t \in J$, is compact, and there exists a constant $M \geq 1$ such that $\|S(t)\|^2 \leq M$.

In order to define the operator $Q : \mathcal{K} \longrightarrow \mathcal{K}$, we choose a sequence $\{\sigma_n\}_{n\geq 1} \subset \mathbb{R}^+$, set $Qe_n = \sigma_n e_n$, and assume that

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty.$$

Define the process $B_Q^H(s)$ by

$$B_Q^H = \sum_{n=1}^{\infty} \sqrt{\sigma_n}\gamma_n^H(t)e_n,$$

where $H \in (1/2, 1)$, and $\{\gamma_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional fractional Brownian motions mutually independent.

Assume now that

(i) There exist some positive number $c_k, \quad k \in \{1, \cdots, m\}$ such that

$$|I_k(\xi_1) - I_k(\xi_2)| \le c_k|\xi_1 - \xi_2|,$$

for any $\xi_1, \xi_2 \in \mathbb{R}$

(ii) Assume that there exists an integrable function $\eta : [0, T] \longrightarrow \mathbb{R}^+$ such that

$$E|F(t, u(\omega)|^2 \le \eta(t)\psi(E|u(\omega)|^2)$$

for any $t \in [0, T]$ and any random variable $u(\cdot) \in L^2(\Omega)$, where $\psi : [0, \infty) \longrightarrow (0, \infty)$ is continuous and nondecreasing with

$$\int_1^{\infty} \frac{ds}{\psi(s)} = +\infty.$$

(iii) The function $g : [0, T] \longrightarrow L_Q^2(\mathcal{K}, \mathcal{H})$ is bounded, that is, there exists a positive constant L such that

$$\int_0^T \|g(s)\|_{L_Q^2}^2 < L, \quad \forall T > 0.$$

Thus the problem (4.19) can be written in the abstract form

$$\begin{cases} dy(t) = [Ay(t) + f(t, y_t)]dt + g(t)dB_Q^H(t), \quad t \in J := [0, T] \quad ; \\ y(t_k^+) - y(t_k) = I_k(y(t_k)), \ k = 1, \ldots, m; \\ y(t) = \phi(t), \quad \text{for} \quad \text{a.e. } t \in [-r, 0]. \end{cases}$$

Thanks to these assumptions, it is straightforward to check that (H1)-(H4) hold provided $8Mm\sum_{k=1}^{m} c_k < 1$. If we impose suitable conditions on the initial value ϕ ensuring that (H5) also holds, then assumptions in Theorem (5.1) are fulfilled, and we can conclude that the system (4.19) possesses a unique mild solution on $[-r, T]$.

Chapter 4

Impulsive Neutral Functional Differential Equations with Unbounded Delay

The existence of neutral stochastic functional differential equation driven by a fractional Brownian motion have attracted great interest of researchers. For example, Boufoussi and Hajji [20] analyzed the existence and uniqueness of mild solutions for a neutral stochastic differential equation with finite delay,driven by a fractional Brownian motion in a Hilbert space, and established some sufficient conditions ensuring the exponential decay to zero in mean square for the mild solution. In [23] Caraballo and Diop, studied the existence and uniqueness of mild solutions to neutral stochastic delay functional integro-differential equations perturbed by a fractional Brownian motion. The existence and stability of second order stochastic differential equations driven by a fractional Brownian motion has been examined by Revathi *et al.* [73].

In this chapter, we establish sufficient conditions for the local and global existence and attractivity of mild solutions to the following first order neutral stochastic impulsive functional equation with time delays:

$$d[y(t) - g(t, y_t)] = [Ay(t) + f(t, y_t)]dt + \sigma(t)dB_Q^H(t), \quad t \in J := [0, T], \tag{4.1}$$

$$\Delta y|_{t=t_k} = y(t_k^+) - y(t_k) = I_k(y(t_k)), \;\; k = 1, \dots, m, \tag{4.2}$$

$$y(t) = \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, \quad \text{for a.e. } t \in J_0 = (-\infty, 0], \tag{4.3}$$

in a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$, B_Q^H is a fractional Brownian motion on a real and separable Hilbert space $\mathcal{K}$, with Hurst parameter $H \in (1/2, 1)$, and with respect to a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ furnished with a family of right continuous and increasing σ-algebras $\{\mathcal{F}_t, t \in J\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. The impulse times t_k satisfy $0 = t_0 < t_1 < t_2 < \ldots, t_m < T$ (if $T = \infty$, t_k satisfies $0 = t_0 < t_1 < t_2 < \ldots, t_m < \cdots$). As for y_t, we mean the segment solution which is defined in the usual way, that is, if $y(\cdot,\cdot) : (-\infty, T] \times \Omega \to \mathcal{H}$, then for any $t \geq 0$, $y_t(\cdot,\cdot) : (-\infty, 0] \times \Omega \to \mathcal{H}$ is given by:

$$y_t(\theta, \omega) = y(t + \theta, \omega), \text{ for } \theta \in (-\infty, 0], \ \omega \in \Omega.$$

Before describing the properties fulfilled by the operators f, g, σ and I_k, we need to introduce some notation and describe some spaces.

In this work, we will employ an axiomatic definition of the phase space $\mathcal{D}_{\mathcal{F}_0}$ introduced by Hale and Kato [41].

Definition 4.0.1. *$\mathcal{D}_{\mathcal{F}_0}$ is a linear space of a family of $\mathcal{F}_0$-measurable functions from $(-\infty, 0]$ into $\mathcal{H}$ endowed with a norm $\|\cdot\|_{\mathcal{D}_{\mathcal{F}_0}}$, which satisfies the following axioms.*

(A-1) *If $y : (-\infty, T] \longrightarrow \mathcal{H}$, $T > 0$, is such that $y_0 \in \mathcal{D}_{\mathcal{F}_0}$, then for every $t \in [0, T)$ the following conditions hold:*

- **(i)** $y_t \in \mathcal{D}_{\mathcal{F}_0}$,
- **(ii)** $\|y(t)\| \leq \mathcal{L}\|y_t\|_{\mathcal{D}_{\mathcal{F}_0}}$,
- **(iii)** $\|y_t\|_{\mathcal{D}} \leq K(t) \sup\{\|y(s)\| : 0 \leq s \leq t\} + N(t)\|y_0\|_{\mathcal{D}_{\mathcal{F}_0}}$,
  where $\mathcal{L} > 0$ is a constant; $K, N : [0, \infty) \longrightarrow [0, \infty)$, K is continuous, N is locally bounded and K, N are independent of $y(\cdot)$.

(A-2) *For the function $y(\cdot)$ in $(A-1)$, y_t is a $\mathcal{D}_{\mathcal{F}_0}$-valued function on $[0, T)$.*

(A-3) *The space $\mathcal{D}_{\mathcal{F}_0}$ is complete.*

Denote

$$\widetilde{K} = \sup\{K(t) : t \in J\} \text{ and } \widetilde{N} = \sup\{N(t) : t \in J\}.$$

Now, for a given $T > 0$, we define

$$\mathcal{D}_{\mathcal{F}_T} = \big\{ y\colon (-\infty, T] \times \Omega \to \mathcal{H},\ y_k \in C(J_k, \mathcal{H}) \text{ for } k = 1, \dots m,\ y_0 \in \mathcal{D}_{\mathcal{F}_0},\ \text{and there exist } y(t_k^-) \text{ and } y(t_k^+) \text{ with } y(t_k) = y(t_k^-),\ k = 1, \cdots, m, \text{ and } \sup_{t \in [0,T]} E(|y(t)|^2) < \infty \big\},$$

endowed with the norm

$$\|y\|_{\mathcal{D}_{\mathcal{F}_T}} = \|\phi\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{0 \le s \le T} (E\|y(s)\|^2)^{\frac{1}{2}},$$

where y_k denotes the restriction of y to $J_k = (t_{k-1}, t_k]$, $k = 1, 2, \cdots, m$, and $J_0 = (-\infty, 0]$.

Then we will consider our initial data $\phi \in \mathcal{D}_{\mathcal{F}_0}$.

Let $\mathcal{K}$ be another real separable Hilbert and suppose that B_Q^H is a $\mathcal{K}$-valued fractional Brownian motion with increment covariance given by a non-negative trace class operator Q (see next section for more details), and let us denote by $L(\mathcal{K}, \mathcal{H})$ the space of all bounded, continuous linear operators from $\mathcal{K}$ into $\mathcal{H}$.

Then we assume that $g : J \times \mathcal{D}_{\mathcal{F}_0} \to \mathcal{H}$, $f : J \times \mathcal{D}_{\mathcal{F}_0} \to \mathcal{H}$ and $\sigma : J \to L_Q^0(\mathcal{K}, \mathcal{H})$. Here, $L_Q^0(\mathcal{K}, \mathcal{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathcal{K}$ into $\mathcal{H}$.

As for the impulse functions, we will assume that $I_k \in C(\mathcal{H}, \mathcal{H})$ ($k = 1, \dots, m$), and $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$.

Assume that $S(t)$ is an analytic semigroup with infinitesimal generator A such that $0 \in \rho(A)$ (the resolvent set of A). Then, it is possible to define the fractional power $(-A)^\alpha$, $0 < \alpha \le 1$ as a closed linear invertible operator with its domain $D((-A)^\alpha)$ being dense in $\mathcal{H}$. We denote by H_α the Banach space $D((-A)^\alpha)$ endowed with the norm $\|y\|_\alpha = \|(-A)^\alpha y\|$, which is equivalent to the graph norm of $(-A)^\alpha$. In the sequel, $\mathcal{H}_\alpha$ represents the space $D((-A)^\alpha)$ with the norm $\|\cdot\|_\alpha$. Then, we have the following well-known properties that appear in ([69]).

Lemma 4.0.1. (i) *If $0 < \beta < \alpha \le 1$, then $\mathcal{H}_\alpha \subset \mathcal{H}_\beta$ and the embedding is compact whenever the resolvent operator of A is compact.*

(ii) *For each $0 < \alpha \le 1$, there exists a positive constant C_α such that*

$$\|(-A)^\alpha S(t)\| \le \tfrac{C_\alpha}{t^\alpha} e^{-\lambda t}, \quad t > 0, \quad \lambda > 0.$$

Lemma 4.0.2. *([45]) Let $v(\cdot)$,$w(\cdot) : [0,T] \longrightarrow [0,\infty)$ be continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $\theta > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} ds, \qquad t \in J,$$

then

$$v(t) \leq e^{\theta^n \Gamma(\alpha)^n t^{n\alpha}/\Gamma(n\alpha)} \sum_{j=0}^{n-1} \left(\frac{\theta T^\alpha}{\alpha} \right)^j w(t),$$

for every $t \in [0,T]$ and every $n \in \mathbb{N}$ such that $n\alpha > 1$, where $\Gamma(\cdot)$ is the Gamma function.

4.1 Existence result

Now we first define the concept of mild solution to our problem.

Definition 4.1.1. *Given $\phi \in \mathcal{D}_{\mathcal{F}_0}$, a $\mathcal{H}-$valued stochastic process $\{y(t), t \in (-\infty, T]\}$ is called a mild solution of the problem (4.1)-(4.3) if $y(t)$ is measurable and $\mathcal{F}_t$-adapted, for each $t > 0$, $y(t) = \phi(t)$ on $(-\infty, 0]$, $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, 2, \cdots, m$, the restriction of $y(\cdot, \cdot)$ to $[0,T) - \{t_1, t_2, \cdots, t_m\}$ is continuous, and for each $s \in [0,t)$, the function $AS(t-s)g(s, x_s)$ is integrable, and y satisfies the integral equation*

$$\begin{aligned} y(t) &= S(t)[\phi(0) - g(s,\phi)] + g(t, y_t) + \int_0^t AS(t-s)g(s,y_s)ds \\ &\quad + \int_0^t S(t-s)f(s,y_s)ds + \int_0^t S(t-s)\sigma(s)dB_Q^H(s) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-)), \ t \in J. \end{aligned} \tag{4.4}$$

Notice that this concept of solution can be considered as more general than the classical concept of solution to equation (4.1)-(4.3). A continuous solution of (5.9) is called a mild solution of (4.1)-(4.3).

Our main result in this section is based upon the fixed point theorem 2.3.2 due to Burton and Kirk [22].

We are now in a position to state and prove our local existence result for the problem (4.1)-(4.3). First we will list the following hypotheses which will be imposed in our main theorem.

- (H1) A is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t)$, $t \geq 0$ and there exists a constant M such that $\{\|S(t)\|^2 \leq M\}$ for all $t \geq 0$ and $\|(-A)^{1-\beta}S(t)\| \leq \frac{M_{1-\beta}}{t^{1-\beta}}$, for all $t > 0$.

- (H2) There exist constants $0 < \beta < 1$, $L_g \geq 0$ and a bounded continuous function $\zeta : J \longrightarrow \mathbb{R}^+$ such that g is $\mathcal{H}_\beta$-valued, $(-A)^\beta g$ is continuous, and

  (i) $E|(-A)^\beta g(t,y)|^2 \leq \zeta(t)\|y\|^2_{\mathcal{D}_{\mathcal{F}_0}}, \quad t \in J, \quad y \in \mathcal{D}_{\mathcal{F}_0}$

  (ii) $E|(-A)^\beta g(t,y_1) - (-A)^\beta g(t,y_2)|^2 \leq L_g\|y_1 - y_2\|^2_{\mathcal{D}_{\mathcal{F}_0}}, \quad t \in J,$

  $$y_1 \text{ and } y_2 \in \mathcal{D}_{\mathcal{F}_0} \text{ with } L_0 = 4\widetilde{K}^2 L_g\left(\|(-A)^{-\beta}\|^2 + \frac{(C_{1-\beta}T^\beta)^2}{2\beta-1}\right) < 1.$$

- (H3) For all $y \in \mathcal{H}$, there exist constants $d_k > 0$, $k = 1, \dots, m, \dots$ for each $|I_k(y)| \leq d_k$, and $\sum_{k=0}^{\infty} d_k < \infty$.

- (H4) f is a L^2-Caratheodory map and for every $t \in [0,T]$ the function $t \to f(t,y_t)$, $y_t \in \mathcal{D}_{\mathcal{F}_0}$, is mesurable.

- (H5) The function $\sigma : J \longrightarrow L_Q(\mathcal{K},\mathcal{H})$ satisfies

  $$\int_0^T \|\sigma(s)\|^2_{L_Q} ds < \infty.$$

- (H6) For the initial value $\phi \in \mathcal{D}_{\mathcal{F}_0}$, there exists a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$, $\psi(0) = 0$ and $p \in L^1(J,\mathbb{R}_+)$ such that $E|f(t,y)|^2 \leq p(t)\psi(\|y\|^2_{\mathcal{D}_{\mathcal{F}_0}})$, for a.e. $t \in J$ and $y \in \mathcal{D}_{\mathcal{F}_0}$ with

  $$\int_{\eta K_0}^{\infty} \frac{du}{\psi(u)} > \eta K_2 \int_0^T p(s)ds,$$

  where $K_0 = \frac{4\widetilde{K}^2 ME|\widehat{\phi}(0)|^2 + 4\widetilde{N}^2\|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 4\widetilde{K}^2 F}{1-24\widetilde{K}^2(-A)^{-\beta}\zeta^*}$, $K_2 = \frac{24T\widetilde{K}^2 M}{1-24\widetilde{K}^2(-A)^{-\beta}\zeta^*}$, and

  $$\eta = e^{K_1^n(\Gamma(2\beta-1))^n T^{n(2\beta-1)}/\Gamma(n(2\beta-1))} \sum_{j=0}^{n-1}\left(\frac{K_1 T^{2\beta-1}}{2\beta-1}\right).$$

Theorem 4.1.1. *Assume that hypotheses (H1)-(H6) hold. If* $12\widetilde{K}^2(-A)^{-\beta}\theta_1 < 1$, *then, problem (4.1)-(4.3) possesses at least one mild solution on* $(-\infty, T]$.

Proof. Transform the problem (4.1)-(4.3) into a fixed point problem. Consider the operator $\Phi : \mathcal{D}_{\mathcal{F}_T} \to \mathcal{D}_{\mathcal{F}_T}$ defined by

$$\Phi(y)(t) = \begin{cases} \phi(t), \text{ if } t \in (-\infty, 0], \\ S(t)[\phi(0) - g(0,\phi)] + g(t, y_t) + \displaystyle\int_0^t AS(t-s)g(s, y_s)ds \\ + \displaystyle\int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dB_Q^H(s) \\ + \displaystyle\sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-)), \quad \text{if } t \in J. \end{cases}$$

For $\phi \in \mathcal{D}_{\mathcal{F}_0}$, we define $\widehat{\phi}$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ S(t)\phi(0), & t \in J; \end{cases}$$

then $\widehat{\phi} \in \mathcal{D}_{\mathcal{F}_T}$.
Let $y(t) = z(t) + \widehat{\phi}(t), -\infty < t \leq T$. It is evident that z satisfies $z_0 = 0, t \in (-\infty, 0]$ and

$$\begin{aligned} z(t) = \ & -S(t)g(0,\phi) + g(t, z_t + \widehat{\phi}_t) + \int_0^t AS(t-s)g(s, z_s + \widehat{\phi}_s) \\ & + \int_0^t S(t-s)f(s, z_s + \widehat{\phi}_s) + \int_0^t S(t-s)g(s)dB_Q^H(s) \\ & + \sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-) + \widehat{\phi}(t_k^-)), t \in J. \end{aligned}$$

Set $\mathcal{D}'_{\mathcal{F}_T} = \{y \in \mathcal{D}_{\mathcal{F}_T}, \quad \text{such that} \quad y_0 = 0 \in \mathcal{D}_{\mathcal{F}_0}\}$; for any $y \in \mathcal{D}_{\mathcal{F}_0}$ we have

$$\|y\|_{\mathcal{F}_T} = \|y_0\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{t\in J}(E\|y(t)\|^2)^{\frac{1}{2}} = \sup_{t\in J}(E\|y(t)\|^2)^{\frac{1}{2}},$$

Then $(\mathcal{D}'_{\mathcal{F}_T}, \|\cdot\|_{\mathcal{F}_T})$ is a Banach space.

Let $\mathcal{B}_q = \left\{ y \in \mathcal{D}'_{\mathcal{F}_T}, \quad \|y\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq q, \ q \geq 0 \right\}$. Clear that the set $\mathcal{B}_q$ is a bounded closed convex set in $\mathcal{D}'_{\mathcal{F}_T}$ for each $q \geq 0$ and for each $y \in \mathcal{B}_q$. we have

$$\begin{aligned}\|z_t+\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq \ & 2(\|z_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}})\\ & \leq 4(\widetilde{N}^2\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\widetilde{K}^2(q+ME|\phi(0)|^2))\\ & = q'.\end{aligned}$$

Let the operator $\widehat{\Phi}:\mathcal{D}'_{\mathcal{F}_T}\to\mathcal{D}'_{\mathcal{F}_T}$ be defined by

$$\widehat{\Phi}(z)=\begin{cases}0, \text{ if } t\in(-\infty,0],\\ -S(t)g(0,\phi)+g(t,z_t+\widehat{\phi}_t)+\displaystyle\int_0^t AS(t-s)g(s,z_s+\widehat{\phi}_s)\\ +\displaystyle\int_0^t S(t-s)f(s,z_s+\widehat{\phi}_s)+\int_0^t S(t-s)g(s)dB_Q^H(s)\\ +\displaystyle\sum_{0<t_k<t}S(t-t_k)I_k(z(t_k^-)+\widehat{\phi}(t_k^-)), & t\in J.\end{cases}$$

Now, consider the two operators $\widehat{\Phi}_1,\widehat{\Phi}_2$ defined by:

$$\widehat{\Phi}_1(z)(t)=\begin{cases}0, & \text{if } \ t\in(-\infty,0],\\ -S(t)g(0,\phi)+g(t,z_t+\widehat{\phi}_t)+\displaystyle\int_0^t AS(t-s)g(s,z_s+\widehat{\phi}_s)\\ +\displaystyle\int_0^t S(t-s)g(s)dB_Q^H(s), & \text{if } \ t\in[0,T],\end{cases}$$

and

$$\widehat{\Phi}_2(z)(t)=\begin{cases}0, & \text{if } \ t\in(-\infty,0],\\ \displaystyle\int_0^t S(t-s)f(s,z_s)+\sum_{0<t_k<t}S(t-t_k)I_k(z(t_k^-)+\widehat{\phi}(t_k^-)), & \text{if } \ t\in[0,T].\end{cases}$$

Clear that

$$\widehat{\Phi}_1+\widehat{\Phi}_2=\widehat{\Phi}.$$

Then the problem of finding a solution of (4.1)-(4.3) is reduced to finding a solution of the operator equation $y(t)=\widehat{\Phi}_1(y)(t)+\widehat{\Phi}_2(y)(t)$, $t\in(-\infty,T]$. We shall show that the operators $\widehat{\Phi}_1$ and $\widehat{\Phi}_2$ satisfy all the conditions of Theorem 2.3.2. The proof will be given in several steps.

Step 1: $\widehat{\Phi_1}$ is a contraction.

Let $v, u \in \mathcal{D}'_{\mathcal{F}_T}$. Then, for $t \in J$,

$$\begin{aligned}
E|\widehat{\Phi}_1(v)(t) - \widehat{\Phi}_1(u)(t)|^2 &\leq 2E|g(t, v_t + \widehat{\phi}_t) - g(t, u_t + \widehat{\phi}_t)|^2 \\
&\quad +2E\Big|\int_0^t AS(t-s)(g(s, v_s + \widehat{\phi}_s) - g(s, u_s + \widehat{\phi}_s))\Big|^2 \\
&\leq 2L_g\|(-A)^{-\beta}\|^2\|v(t) - u(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\
&\quad +2T\int_0^t \frac{C^2_{1-\beta}}{t - s^{2(1-\beta)}} L_g\|v(s) - u(s)\|^2_{\mathcal{D}_{\mathcal{F}_0}} ds, \\
&\leq 2L_g\bigg(\|(-A)^{-\beta}\|^2 + \tfrac{(C_{1-\beta}T^\beta)^2}{2\beta-1}\bigg) \times \Big[2\widetilde{K}^2 \sup_{0\leq s\leq T} E|v(s) - u(s)|^2 \\
&\quad +2\widetilde{N}\|v_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 2\widetilde{N}\|u_0\|^2_{\mathcal{D}_{\mathcal{F}_0}}\Big] \\
&\leq L_0 \sup_{0\leq s\leq T} E|v(s) - u(s)|^2.
\end{aligned}$$

Since $\|u_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} = 0$, $\|v_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} = 0$. Taking the supremum over t, we obtain

$$\|\widehat{\Phi}_1(v) - \widehat{\Phi}_1(u)\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq L_0\|v - u\|^2_{\mathcal{D}'_{\mathcal{F}_T}},$$

where $L_0 = 4\widetilde{K}^2 L_g\bigg(\|(-A)^{-\beta}\|^2 + \frac{(C_{1-\beta}T^\beta)^2}{2\beta-1}\bigg) < 1$. Thus $\widehat{\Phi}_1$ is a contraction.

Next, we prove that the operator $\widehat{\Phi}_2$ is completely continuous.
Step 2: $\widehat{\Phi}_2$ is continuous.

Let z^n be a sequence such that $z^n \to z$ in $\mathcal{D}'_{\mathcal{F}_T}$. Then, for $t \in J$, and thanks to $(H1), (H3)$ and $(H4)$, I_k , $k = 1, 2, \cdots, m$, is continuous.
By the dominated convergence theorem, we have

$$\begin{aligned}
&E|\widehat{\Phi}_2(z^n)(t) - \widehat{\Phi}_2(z)(t)|^2 \\
&\qquad \leq 2E\Big|\int_0^t S(t-s)(f(s, (z^n_s + \widehat{\phi}_s)) - f(s, z_s + \widehat{\phi}_s))ds\Big|^2 \\
&\qquad +2E\Big|\sum_{0\leq t_k\leq t} |S(t-t_k)|^2 |I_k(z^n_{t_k^-} + \widehat{\phi}(t_k^-)) - I_k(z_{t_k^-} + \widehat{\phi}(t_k^-))\Big|^2 \\
&\qquad \leq 2MT\int_0^t E\Big|(f(s, (z^n_s + \widehat{\phi}_s)) - f(s, z_s + \widehat{\phi}_s))ds\Big|^2 \\
&\qquad +2ME\Big|I_k(z^n(t_k^-) + \widehat{\phi}(t_k^-)) - I_k(z(t_k^-) + \widehat{\phi}(t_k^-))\Big|^2 \to 0 \text{ as } n \to +\infty.
\end{aligned}$$

Thus, Φ_2 is continuous.

Step 3: $\widehat{\Phi}_2$ maps bounded sets into bounded sets in $\mathcal{D}'_{\mathcal{F}_T}$.

Indeed, it is enough to show that for any $q > 0$, there exists a positive constant l such that for each $z \in \mathcal{B}_q = \{z \in \mathcal{D}'_{\mathcal{F}_T} : \|z\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq q\}$, one has $\|\widehat{\Phi}_2(z)\|^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq l$.
Let $z \in \mathcal{B}_q$; then for each $t \in [0,T]$, we have

$$\begin{aligned} |\widehat{\Phi}_2 z(t)|^2 &\leq \left| \int_0^t S(t-s) f(s, z_s + \widehat{\phi}_s) ds + \sum_{0 \leq t_k \leq t} S(t-t_k) I_k(z^n_{t_k^-} + \widehat{\phi}(t_k^-)) \right|^2 \\ &\leq 2M \Big| \int_0^t f(s, z_s + \widehat{\phi}_s) ds \Big|^2 + 2M \sum_{0 \leq t_k \leq t} \Big| I_k(z^n(t_k^-) + \widehat{\phi}(t_k^-)) \Big|^2. \end{aligned}$$

Thus,

$$\begin{aligned} E|\widehat{\Phi}_2 z(t)|^2 &\leq 2TM \int_0^t p(s) \psi(\|z_s + \widehat{\phi}_s\|^2_{\mathcal{D}'_{\mathcal{F}_T}}) ds \\ &\quad + 2M \sum_{0 \leq t_k \leq t} E|I_k(z(t_k^-) + \widehat{\phi}(t_k^-))|^2 \\ &\leq 2MT\psi(q') \int_0^T p(s) ds + 2M \Big(\sum_{k=1}^m d_k \Big)^2. \end{aligned}$$

Then we have

$$E|\widehat{\Phi}_2 z(t)|^2 \leq 2MT\psi(q') \|p\|_{L^1} + 2M \Big(\sum_{k=1}^m d_k \Big)^2 = l.$$

Step 4: $\widehat{\Phi}_2$ maps bounded sets into equicontinuous sets in $\mathcal{D}'_{\mathcal{F}_T}$.
Let $0 < \epsilon \leq \tau_1 < \tau_2 \in J$, $\tau_1, \tau_2 \neq t_i$, $i = 1, \cdots, m$, and $\mathcal{B}_q$ be a bounded

set of $\mathcal{D}'_{\mathcal{F}_T}$ as in Step 3. Let $z \in \mathcal{B}_q$; then we have

$$\begin{aligned}
E|(\widehat{\Phi}_2 y)(\tau_2) - (\widehat{\Phi}_2 y)(\tau_1)|^2 &\leq 6T\int_0^{\tau_1-\epsilon} |S(\tau_2-s) - S(\tau_1-s)|^2 E|f(s, z_s + \widehat{\phi}_s)|^2 ds \\
&+6T\int_{\tau_1-\epsilon}^{\tau_1} |S(\tau_2-s) - S(\tau_1-s)|^2 E|f(s, z_s + \widehat{\phi}_s)|^2 ds \\
&+6T\int_{\tau_1}^{\tau_2} |S(\tau_2-s)|^2 E|f(s, z_s + \widehat{\phi}_s)|^2 ds \\
&+4\sum_{0<t_k<\tau_1} |S(\tau_2-t_k) - S(\tau_1-t_k)|^2 E|I_k(z^n(t_k^-) + \widehat{\phi}(t_k^-))|^2 \\
&+4\sum_{\tau_1\leq t_k<\tau_2} |S(\tau_2-t_k)|^2 |I_k(z^n(t_k^-) + \widehat{\phi}(t_k^-))|^2 \\
&\leq 6T\int_0^{\tau_1-\epsilon} |S(\tau_2-s) - S(\tau_1-s)|^2 \alpha_{q'}(s) ds \\
&+6T\int_{\tau_1-\epsilon}^{\tau_1} |S(\tau_2-s) - S(\tau_1-s)|^2 \alpha_{q'}(s) ds \\
&+6TM\int_{\tau_1}^{\tau_2} \alpha_{q'}(s) ds \\
&+4\sum_{0<t_k<\tau_1} |S(\tau_2-t_k) - S(\tau_1-t_k) d_k|^2 \\
&+4M\Big(\sum_{\tau_1\leq t_k<\tau_2} d_k\Big)^2.
\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and ϵ sufficiently small, since the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology [69]. This proves the equicontinuity. Here, we consider the case $0 < \tau_1 < \tau_2 \leq T$, since the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2 \leq T$ are easier to handle.

Step 5: $(\widehat{\Phi}_2 \mathcal{B}_q)(t)$ is precompact in $\mathcal{H}$

As a consequence of Steps 2 to 4, together with the Arzelá-Ascoli theorem, it suffices to show that $\widehat{\Phi}_2$ maps $\mathcal{B}_q$ into a precompact set in $\mathcal{H}$.

Let $0 < t < T$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in \mathcal{B}_q$, we define

$$(\widehat{\Phi}_{2\epsilon} z)(t) = S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon) f(s, z_s + \widehat{\phi}_s) ds + S(\epsilon)\sum_{0<t_k<t-\epsilon} S(t-t_k-\epsilon) I_k(z(t_k^-) + \widehat{\phi}(t_k^-)).$$

Since $S(t)$ is a compact operator, the set

$$Y_\epsilon(t) = \{\widehat{\Phi}_{2\epsilon}(z)(t) : \quad z \in \mathcal{B}_q\}$$

is precompact in $\mathcal{H}$ for every ϵ and $0<\epsilon<t$. Moreover, for every $z\in\mathcal{B}_q$ we have

$$\begin{aligned} E|(\widehat{\Phi}_2 y)(t)-(\widehat{\Phi}_{2\epsilon}(y)(t)|^2 &\leq 4T\int_{t-\epsilon}^{t}|S(t,s)|^2E|f(s,z_s+\widehat{\phi}_s)|^2ds \\ &\quad +4\sum_{t-\epsilon<t_k<t}|S(t-t_k)|^2E|I_k(z(t_k^-)+\widehat{\phi}(t_k^-))|^2 \\ &\leq 4TM\int_{t-\epsilon}^{t}\alpha_{q'}(s)ds+4M\Big(\sum_{0<t_k<t-\epsilon}d_k\Big)^2. \end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $Y_\epsilon(t)=\{\widehat{\Phi}_{2\epsilon}(z)(t):\ \ z\in\mathcal{B}_q\}$. Hence the set $Y(t)=\{\widehat{\Phi}_2(z)(t):\ \ y\in\mathcal{B}_q\}$ is precompact in $\mathcal{H}$, and therefore, the operator $\widehat{\Phi}_2:\mathcal{D}'_{\mathcal{F}_T}\to\mathcal{D}'_{\mathcal{F}_T}$ is completely continuous.

Step 5 : A priori bounds.

Now it remains to show that the set

$$\Xi=\{z\in\mathcal{D}'_{\mathcal{F}_T}:z=\lambda\widehat{\Phi}_2(z)+\lambda\widehat{\Phi}_1\left(\frac{z}{\lambda}\right),\text{for some }\ 0<\lambda<1\}$$

is bounded. For each $t\in J$

$$\begin{aligned} z(t) &= -S(t)g(0,\phi)+g(t,z_t+\widehat{\phi}_t)+\int_0^t AS(t-s)g(s,z_s+\widehat{\phi}_s)ds \\ &\quad +\textstyle\int_0^t S(t-s)f(s,z_s+\widehat{\phi}_s)ds+\int_0^t S(t-s)\sigma(s)dB_Q^H(s) \\ &\quad +\sum_{0<t_k<t}S(t-t_k)I_k(z(t_k)+\widehat{\phi}(t_k)). \end{aligned}$$

This implies, for each $t\in J$,

$$\begin{aligned} E|z(t)|^2 \leq& 6(-A)^{-\beta}M\zeta(t)\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+6(-A)^{-\beta}\zeta(t)\|z_t+\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ &+6T\int_0^t\frac{C^2_{1-\beta}}{(t-s)^{2(1-\beta)}}\zeta(s)\|z_s+\widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}}ds+6MT\int_0^t p(s)\psi(\|z_s+\widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}})ds \\ &+12MHt^{2H-1}\int_0^t\|\sigma(s)\|^2_{L_Q^0}ds+6M\Big(\sum_{k=1}^m d_k\Big)^2 \\ \leq& F+6(-A)^{-\beta}\zeta(t)\|z_t+\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ &+6T\int_0^t\frac{C^2_{1-\beta}}{(t-s)^{2(1-\beta)}}\zeta(s)\|z_s+\widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}}ds \end{aligned}$$

$$+ 6MT \int_0^t p(s)\psi(\|z_s + \widehat{\phi}_s\|^2_{\mathcal{D}_{\mathcal{F}_0}})ds, \tag{4.5}$$

where

$$F = 6(-A)^{-\beta} M\zeta^* \|\phi\|_{\mathcal{D}_{\mathcal{F}_0}} + 12MHT^{2H-1} \int_0^T \|\sigma(s)\|^2_{L_Q^0} ds + 6M\Big(\sum_{k=1}^m d_k\Big)^2,$$

and

$$\zeta^* = \sup_{t \in J} |\zeta(t)|.$$

But

$$\begin{aligned} \|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} &\leq 2(\|z_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\ &\leq 4\widetilde{K}^2 \sup_{s\in[0,T]} E|z(s)|^2 + 4\widetilde{K}^2 ME|\widehat{\phi}(0)|^2 + 4\widetilde{N}^2 \|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ &\leq 4\widetilde{K}^2 \sup_{s\in[0,T]} E|z(s)|^2 + 4\alpha^2(M+1)\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}, \end{aligned}$$

where

$$\alpha^2 = \max\{\widetilde{K}^2, \widetilde{N}^2\}.$$

If we set $w(t)$ the right hand side of the above inequality we have that

$$\|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq w(t),$$

and therefore (4.5) becomes

$$\begin{aligned} E|z(t)|^2 \leq& F + 6(-A)^{-\beta}\zeta(t)w(t) + 6T \int_0^t \frac{C^2_{1-\beta}}{(t-s)^{2(1-\beta)}} \zeta(s)w(s)ds \\ &+ 6MT \int_0^t p(s)\psi(w(s))ds. \end{aligned} \tag{4.6}$$

Using (4.6) in the definition of w, we have that

$$\begin{aligned} w(t) \leq\ & 4\widetilde{K}^2 \Bigg(F + 6(-A)^{-\beta}\zeta^* w(t) + 6T \int_0^t \frac{C^2_{1-\beta}}{(t-s)^{2(1-\beta)}} \zeta^* w(s)ds \\ & +6MT \int_0^t p(s)\psi(w(s))ds \Bigg) + 4\widetilde{K}^2 ME|\widehat{\phi}(0)|^2 + 4\widetilde{N}^2 \|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}. \end{aligned} \tag{4.7}$$

Thus, we obtain

$$w(t) \leq K_0 + K_1 \int_0^t \frac{w(s)}{(t-s)^{2(1-\beta)}} ds + K_2 \int_0^t p(s)\psi(w(s))ds. \tag{4.8}$$

where

$$K_0 = \frac{4\widetilde{K}^2 ME|\widehat{\phi}(0)|^2 + 4\widetilde{N}^2 \|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 4\widetilde{K}^2 F}{1 - 24\widetilde{K}^2(-A)^{-\beta}\zeta^*},$$

$$K_1 = \frac{24T\widetilde{K}^2 C^2_{1-\beta}\zeta^*}{1 - 24\widetilde{K}^2(-A)^{-\beta}\zeta^*},$$

and

$$K_2 = \frac{24T\widetilde{K}^2 M}{1 - 24\widetilde{K}^2(-A)^{-\beta}\zeta^*}.$$

By Lemma 4.0.2, we get

$$w(t) \leq \eta \left(K_0 + K_2 \int_0^t p(s)\psi(w(s))ds \right), \tag{4.9}$$

where

$$\eta = e^{C_1^n (\Gamma(2\beta-1))^n T^{n(2\beta-1)}/\Gamma(n(2\beta-1))} \sum_{j=0}^{n-1} \left(\frac{K_2 T^{2\beta-1}}{2\beta - 1} \right).$$

Let us denote the right-hand side of the inequality (4.9) by $v(t)$. Then we have

$$v(0) = \eta K_0, \ w(t) \leq v(t), \quad t \in J,$$

and

$$v'(t) = \eta K_2 p(t)\psi(w(t)), \quad t \in J.$$

Using the increasing character of ψ, we obtain

$$v'(t) \leq \eta K_2 p(t)\psi(v(t)), \quad \text{for } \ a.e. \ \ t \in J.$$

This implies, for each $t \in J$, we have

$$\int_{v(0)}^{v(t)} \frac{ds}{\psi(s)} \leq \eta K_2 \int_0^T p(s)ds \Rightarrow \Gamma(v(t)) \leq \eta K_2 \|p\|_{L^1},$$

where Γ is nondegreasing function defined by

$$\Gamma(z) = \int_{\eta K_0}^{z} \frac{du}{\psi(u)}.$$

Hence

$$v(t) \leq \Gamma^{-1}\left(\eta K_2 \|p\|_{L^1}\right) := K \Rightarrow w(t) \leq v(t) \leq K,\ t \in J.$$

From equation (4.5), we obtain that

$$\begin{aligned} E|z(t)|^2 \leq & F + 6(-A)^{-\beta}\zeta^* K + 6T\int_0^t \frac{C_{1-\beta}^2}{(t-s)^{2(1-\beta)}}\zeta^* K \\ & + 6MT\int_0^T p(s)\psi(K)ds = L, \end{aligned} \tag{4.10}$$

Thus

$$\|z\|^2_{\mathcal{D}_{\mathcal{F}'_T}} \leq L.$$

As a consequence of Theorem 2.3.2, we deduce that $\widehat{\Phi}$ has a fixed point, since $y(t) = z(t) + \widehat{\phi}(t)$, $t \in (-\infty, T]$. Then y is a fixed point of the operator Φ which is a mild solution of the problem (4.1)-(4.3).

□

4.2 Global existence and uniqueness result

In this section we establish a result for the global existence of mild solutions ($T = \infty$) for our semilinear equations with infinite delay. In order to achieve this end, we need to impose some stronger assumptions, but will obtain the mild solutions defined in $\mathbb{R}$, something that will be necessary for the study of attractivity of solutions in the next section.

We will need to introduce the following hypotheses which are assumed thereafter:

- $(H1')$The semigroup $S(t)$ satisfies the additional condition:

$$\exists\lambda > 0 \text{ and } \exists M > 0 \text{ such that } \|S(t)\| \leq Me^{-\lambda t}.$$

- $(H2^{'})$ There exist constants, L_f and a function $p \in L^1(\mathbb{R}^+, \mathbb{R}^+)$ such that:

  (i) $E|f(t,y) - f(t,x)|^2 \leq L_f \|y - x\|^2_{\mathcal{D}_{\mathcal{F}_0}}, \quad t \in J = [0, \infty);$

  (ii) $E|f(t,y)|^2 \leq p(t)(\|y\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 1)$ for a.e. $t \in [0, \infty)$ and each $y \in \mathcal{D}_{\mathcal{F}_0}$;

  (iii) For every $t \in [0, \infty)$ the function $t \to f(t, y_t)$, $y_t \in \mathcal{D}_{\mathcal{F}_0}$ is mesurable.

- $(H3^{'})$ The function $\sigma : J = [0, \infty) \longrightarrow L^0_Q(\mathcal{K}, \mathcal{H})$ satisfies

$$\int_0^\infty e^{2\gamma s} \|\sigma(s)\|^2_{L^0_Q} ds < \infty, \qquad \gamma > 0.$$

Theorem 4.2.1. *Assume that* $f(t,0) = g(t,0) = 0$, $\forall t \geq 0$. *Assume that hypotheses* $(H2)(ii)$, $(H3)$ *and* $(H1^{'})$*-*$(H3^{'})$ *hold. If*

$$L_1 = 6\widetilde{K}^2 \left(L_g |(-A)^{-\beta}|^2 + M^2_{1-\beta} L_g \frac{\Gamma^2(\beta)}{\lambda^{2\beta}} + M^2 L_f \lambda^{-2} \right) < 1,$$

then, there exists unique mild solution to (4.1)-(4.3) defined on $(-\infty, \infty)$.

Proof. We shall consider the space

$$\begin{aligned} \mathcal{D}_{\mathcal{F}_\infty} = \big\{ & y\colon (-\infty, \infty) \times \Omega \to \mathcal{H},\ y_k \in C(J_k, \mathcal{H}) \text{ for } k = 1, \ldots,\ y_0 \in \mathcal{D}_{\mathcal{F}_0}, \\ & \text{and there exist } y(t_k^-) \text{ and } y(t_k^+) \text{ with } y(t_k) = y(t_k^-),\ k = 1, 2, \cdots, \\ & \text{and } \sup_{t \in [0,\infty)} E(|y(t)|^2) < \infty \big\}, \end{aligned}$$

where y_k denotes the restriction of y to $J_k = (t_{k-1}, t_k]$, $k = 1, 2, \cdots$; $\lim_{k \to \infty} t_k = \infty$ and $J_0 = (-\infty, 0]$. Then we will consider our initial data $\phi \in \mathcal{D}_{\mathcal{F}_0}$.

Consider the set $Z^0_{\mathcal{F}_\infty} = \{ y \in \mathcal{D}_{\mathcal{F}_\infty} : \sup_{t \in J} E\|y\|^2 < \infty \}$ endowed with the norm

$$\|y\|_{Z^0_{\mathcal{F}_\infty}} = \|\phi\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{0 \leq s \leq \infty} (E\|y(s)\|^2)^{\frac{1}{2}}.$$

We transform the problem (4.1)-(4.3) into a fixed point problem. Consider the operator $\Psi : Z^0_{\mathcal{F}_\infty} \to Z^0_{\mathcal{F}_\infty}$ defined by

$$\Psi(y)(t) = \begin{cases} \phi(t), \text{ if } t \in (-\infty, 0], \\ S(t)[\phi(0) - g(0,\phi)] + g(t, y_t) + \displaystyle\int_0^t AS(t-s)g(s, y_s)ds \\ + \displaystyle\int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dB_Q^H(s) \\ + \displaystyle\sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-)), \text{ if } t \in [0,\infty). \end{cases}$$

For $\phi \in \mathcal{D}_{\mathcal{F}_0}$, we define $\widehat{\phi}$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ S(t)\phi(0), & t \in [0, \infty). \end{cases}$$

Then $\widehat{\phi} \in \mathcal{D}_{\mathcal{F}_\infty}$. Let $y(t) = z(t) + \widehat{\phi}(t), -\infty < t < \infty$. It is evident that z satisfies $z_0 = 0, t \in (-\infty, 0]$ and

$$\begin{aligned} z(t) = \;& -S(t)g(0,\phi) + g(t, z_t + \widehat{\phi}_t) + \int_0^t AS(t-s)g(s, z_s + \widehat{\phi}_s) \\ & + \int_0^t S(t-s)f(s, z_s + \widehat{\phi}_s) + \int_0^t S(t-s)g(s)dB_Q^H(s) \\ & + \sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-) + \widehat{\phi}(t_k^-)), t \in J. \end{aligned}$$

Set

$$\begin{aligned} Z^1_{\mathcal{F}_\infty} = \big\{ & z\colon (-\infty,\infty)\times\Omega \to \mathcal{H},\ z_k \in C(J_k, \mathcal{H}) \text{ for } k = 1, \dots m,\ z \in Z^0_{\mathcal{F}_\infty} \\ & \text{and } z_0 = 0, \text{ and there exist } z(t_k^-) \text{ and } z(t_k^+) \text{ with } z(t_k) = z(t_k^-),\ \ k \geq 1, \\ & \text{and } \sup_{t\in[0,\infty)} E(|z(t)|^2) < \infty \big\}. \end{aligned}$$

For any $z \in Z^1_{\mathcal{F}_\infty}$, we have

$$\|z\|_{Z^1_{\mathcal{F}_\infty}} = \|z_0\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{t\in J}(E|z(t)|^2)^{\frac{1}{2}} = \sup_{t\in J}(E|z(t)|^2)^{\frac{1}{2}}.$$

Thus, $(Z^1_{\mathcal{F}_\infty}, \|.\|_{Z^1_{\mathcal{F}_\infty}})$ is a Banach space.

Let the operator $\widehat{\Psi}: Z^1_{\mathcal{F}_\infty} \to Z^1_{\mathcal{F}_\infty}$ be defined by

$$\widehat{\Psi}(z) = \begin{cases} 0, \text{ if } t \in (-\infty, 0], \\ -S(t)g(0,\phi) + g(t, z_t + \widehat{\phi}_t) + \int_0^t AS(t-s)g(s, z_s + \widehat{\phi}_s) \\ + \int_0^t S(t-s)f(s, z_s + \widehat{\phi}_s) + \int_0^t S(t-s)g(s)dB^H_Q(s) \\ + \sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-) + \widehat{\phi}(t_k^-)), t \in [0, +\infty). \end{cases}$$

The problem of finding a solution of problem (4.1)-(4.3) is reduced to finding a solution of the operator equation $\widehat{\Psi}(z)(t) = z(t)$, $t \in [0, \infty)$.

Consider the set

$$\mathcal{B} = \{z \in Z^1_{\mathcal{F}_\infty} : \exists a > 0,\ M^* = M^*(\phi, a) \text{ such that } E|z(t)|^2 \leq M^* e^{-at}, \quad t \geq 0\};$$

then, $\mathcal{B} \subset Z^1_{\mathcal{F}_\infty}$ is closed.

Now, we will show that by using the Banach fixed point theorem, the operator $\widehat{\Psi}$ has a fixed point.

Step 1: We first verify $\widehat{\Psi}(\mathcal{B}) \subset \mathcal{B}$. We denote by $M_i^*, i = 1, 2, \cdots$ finite positive constants depending on ϕ and a.

For any $z \in \mathcal{B}$, we have

$$\begin{aligned} \widehat{\Psi}(z)(t) =& - S(t)g(0,\phi(0)) + g(t, z_t + \widehat{\phi}_t) \\ &+ \int_0^t AS(t-s)g(s, z_t + \widehat{\phi}_t)ds + \int_0^t S(t-s)f(s, z_t + \widehat{\phi}_t)ds \\ &+ \int_0^t S(t-s)\sigma(s)dB^H(s) + \sum_{0<t_k<t} S(t-t_k)I_k(z(t_k) + \widehat{\phi}(t_k)) \\ =:& \sum_{1\leq i\leq 6} \eta_i(t). \end{aligned}$$

By assumption $(H1^{'})$, we have

$$E|\eta_1(t)|^2 \leq M^2 E|g(0,\phi(0))|^2 e^{-\lambda t} \leq M_1^* e^{-\lambda t}. \tag{4.11}$$

To estimate $\eta_i(t)$, $i = 2, \cdots, 5$, we observe that for $z \in \mathcal{D}'_{\mathcal{F}_\infty}$, the following useful estimate holds

$$E\|z(t)\|^2 \leq 2\widetilde{K}^2 M^* e^{-at} + 2\widetilde{N}\|z_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} e^{-at}$$

$$\leq 2\widetilde{K}^2 M^* e^{-at},$$

where $\|z_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} = 0$.

By assumption $(H2)(ii)$ we have

$$\begin{aligned}\mathbb{E}|\eta_2(h)|^2 \leq & |(-A)^{-\beta}|^2 E|(-A)^{\beta} g(t, z_t + \widehat{\phi}_t) - (-A)^{\beta} g(t,0)|^2 \\ & \leq |(-A)^{-\beta}|^2 L_g E|z(t)|^2 \\ & \leq 2\widetilde{K}^2 M^* \|(-A)^{-\beta}\|^2 e^{-at}.\end{aligned}$$

Then we have

$$E|\eta_2(t)|^2 \leq M_2^* e^{-at}. \tag{4.12}$$

Using Lemma 4.0.1, Hölder's inequality and assumption (H3) we obtain that

$$\begin{aligned}E|\eta_3(t)|^2 \leq & E\Big|\int_0^t AS(t-s) g(t, z_s + \widehat{\phi}_s)\Big|^2 \\ & \leq \int_0^t \Big|(-A)^{1-\beta} S(t-s)\Big| ds \int_0^t \Big|(-A)^{1-\beta} S(t-s)\Big| E|(-A)^{\beta} g(s, z_s + \widehat{\phi}_s)|^2 ds \\ & \leq M_{1-\beta}^2 L_g \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} ds \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} E|z(s)|^2 ds. \\ & \leq 2M_{1-\beta}^2 L_g \frac{\Gamma(\beta)}{\lambda^{\beta}} \widetilde{K}^2 M^* e^{-at} \int_0^t (t-s)^{\beta-1} e^{(a-\lambda)(t-s)} ds \\ & \leq 2M_{1-\beta}^2 L_g \frac{\Gamma^2(\beta)}{\lambda^{\beta}(\lambda-a)^{\beta}} \widetilde{K}^2 M^* e^{-at}.\end{aligned}$$

We therefore have

$$E|\eta_3(t)|^2 \leq M_3^* e^{-at}. \tag{4.13}$$

Similarly, by assumption $(H2')$,

$$\begin{aligned}\mathbb{E}|\eta_4(t)|^2 \leq & E\Big|\int_0^t S(t-s) f(t, z_s + \widehat{\phi}_s) ds\Big|^2 \\ & \leq M^2 L_f \int_0^t e^{-\lambda(t-s)} ds \int_0^t e^{-\lambda(t-s)} E|z(s)|^2 ds \\ & \leq 2M^2 L_f \lambda^{-1} \widetilde{K}^2 M^* \int_0^t e^{-\lambda(t-s)} e^{-as} ds\end{aligned}$$

$$\leq 2M^2 L_f \lambda^{-1}(\lambda - a)^{-1}\widetilde{K}^2 M^* e^{-at}$$
$$\leq M_4^* e^{-at}.$$

Therefore,

$$E\|\eta_4(t)\|^2 \leq M_4^* e^{-at}. \tag{4.14}$$

Now, for the term $\eta_5(t)$, we have

$$\eta_5(t) \leq 2M^2 H t^{2H-1} \int_0^t e^{-2\lambda(t-s)} \|\sigma(s)\|_{L_Q}^2 ds. \tag{4.15}$$

From this inequality we can ensure that

$$E|\eta_5(t)|^2 \leq 2M^2 H t^{2H-1} e^{-2\lambda_1 t} \int_0^t e^{2\lambda_2 s} \|\sigma(s)\|_{L_Q}^2 ds, \tag{4.16}$$

where $\lambda_1 = \lambda \wedge \lambda_2$. Indeed, if $\lambda < \lambda_2$, then $\lambda_1 = \lambda$ and we have

$$\begin{aligned} E|\eta_5(t)|^2 \leq & 2M^2 H t^{2H-1} e^{-2\lambda t} \int_0^t e^{2\lambda s} \|\sigma(s)\|_{L_Q}^2 ds \\ \leq & 2M^2 H t^{2H-1} e^{-2\lambda_1 t} \int_0^t e^{2\lambda_2 s} \|\sigma(s)\|_{L_Q}^2 ds. \end{aligned}$$

If $\lambda_2 < \lambda$, then $\lambda_1 = \lambda_2$ and we have

$$\begin{aligned} E|\eta_5(t)|^2 \leq & 2M^2 H t^{2H-1} e^{-2\lambda_2 t} \int_0^t e^{-2(\lambda-\lambda_2)(t-s)} e^{2\lambda_2 S} \|\sigma(s)\|_{L_Q}^2 ds \\ \leq & 2M^2 H t^{2H-1} e^{-2\lambda_1 t} \int_0^\infty e^{2\lambda_2 s} \|\sigma(s)\|_{L_Q}^2 ds. \end{aligned}$$

Since $\sup\limits_{t\geq 0}(t^{2H-1}e^{-\lambda_1 t}) < \infty$, this, together with (4.16), gives us

$$E|\eta_5(t)|^2 \leq M_5^* e^{-\lambda_2 t}.$$

From $(H3)$ and Hölder's inequality, we obtain the following estimate for $\eta_6(t)$

$$\begin{aligned} E|\eta_6(t)|^2 \;\leq\; & E\Big|\sum_{0<t_k<t} S(t-t_k) I_k(z(t_k) + \widehat{\phi}(t_k))\Big|^2 \\ & + E\Big(\Big|\sum_{0<t_k<t} S(t-t_k)\Big|\Big|I_k(z(t_k) + \widehat{\phi}(t_k))\Big|\Big)^2 \\ \leq\; & M^2 e^{-at} \sum_{k=1}^\infty d_k \sum_{k=1}^\infty d_k e^{-2\lambda t_k} \leq M_6^* e^{-at}. \end{aligned}$$

We therefore have

$$E\|\eta_4(t)\|^2 \leq M_6^* e^{-at}. \tag{4.17}$$

Combining (4.11)-(4.17) we see that there exist $\overline{M}^* > 0$ and $\overline{a} > 0$ such that

$$E\|\widehat{\Psi}(t)\|^2 \leq \overline{M}^* e^{-\overline{a}t}, \qquad t \geq 0$$

Hence, we can conclude that $\widehat{\Psi}(\mathcal{B}) \subset \mathcal{B}$.

Step 2: Now, we prove that $\widehat{\Psi}$ is a contracting mapping in $\mathcal{B}$.

For every $z_1, z_2 \in \mathcal{B}$ and $t \in [0, \infty)$, we obtain

$$\begin{aligned}
&E|\widehat{\Psi}(z_1)(t) - \widehat{\Psi}(z_2)(t)|^2 \\
&\quad \leq 3E|g(t, z_{1t} + \widehat{\phi}_t) - g(t, z_{2t} + \widehat{\phi}_t)|^2 \\
&\qquad + 3E\bigg|\int_0^t AS(t-s)(g(s, z_{1s} + \widehat{\phi}_s) - g(s, z_{2s} + \widehat{\phi}_s))\bigg|^2 ds \\
&\qquad + 3E\bigg|\int_0^t S(t-s)(f(s, z_{1s} + \widehat{\phi}_s) - f(s, z_{2s} + \widehat{\phi}_s))\bigg|^2 ds \\
&\quad \leq 3L_g\|(-A)^{-\beta}\|^2 E\|z_1(t) - z_2(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\
&\qquad + 3M_{1-\beta}^2 L_g \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} ds \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} E\|z_1(t) - z_2(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}} ds, \\
&\qquad + 3M^2 L_f \int_0^t e^{-\lambda(t-s)} ds \int_0^t e^{-\lambda(t-s)} E\|z_1(t) - z_2(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}} ds \\
&\quad \leq 3L_g\|(-A)^{-\beta}\|^2 E\|z_1(t) - z_2(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\
&\qquad + 3M_{1-\beta}^2 L_g \frac{\Gamma^2(\beta)}{\lambda^{2\beta}} E\|z_1(t) - z_2(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\
&\qquad + 3M^2 L_f \lambda^{-2} E\|z_1(t) - z_2(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\
&\quad \leq \left(3L_g\|(-A)^{-\beta}\|^2 + 3M_{1-\beta}^2 L_g \frac{\Gamma^2(\beta)}{\lambda^{2\beta}} + 3M^2 L_f \lambda^{-2}\right) \\
&\qquad \times \left[2\widetilde{K}^2 \sup_{t\geq 0} E|z_1(t) - z_2(t)|^2 + 2\widetilde{N}^2\|z_{10}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 2\widetilde{N}^2\|z_{20}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\right] \\
&\quad \leq L_1 \sup_{t\geq 0} E|z_1(t) - z_2(t)|^2.
\end{aligned}$$

Hence, $\widehat{\Psi}$ is a contraction mapping on $\mathcal{B}$ and therefore $\widehat{\Psi}$ has a unique fixed point, since $y(t) = z(t) + \widehat{\phi}(t)$, $t \in (-\infty, \infty)$. Then y is a fixed point of the operator Ψ which is a mild solution of the problem (4.1)-(4.3). This completes the proof.

□

4.3 Attractivity of solutions

In this section we study the local attractivity of solutions of the problem (4.1)-(4.3)

Definition 4.3.1. *([28]) We say that solutions of (4.1)-(4.3) are locally attractive if there exists a closed ball $\bar{\mathcal{B}}(z^*, \rho)$ in the space $Z^1_{\mathcal{F}_\infty}$ for some $z^* \in Z^0_{\mathcal{F}_\infty}$ such that for arbitrary solutions z and $\tilde{z}$ of (4.1)-(4.3) belonging to $\bar{\mathcal{B}}(z^*, \rho)$ we have*

$$\lim_{t \longrightarrow +\infty} E|z(t) - \tilde{z}(t)|^2 = 0.$$

Under the assumptions of Section 3 and 4 , let z^* be a solution of (4.1)-(4.3) and $\bar{B}(z^*, \rho)$ the closed ball in $\mathcal{D}_{\mathcal{F}'_T}$ with ρ satisfying

$$\rho \geq \frac{6M^2\lambda^{-1}\|p\|_{L^1}}{1 - 12L_g\left(\|(-A)^{-\beta}\|^2 + \frac{C^2_{1-\beta}\Gamma^2(\beta)}{\lambda^{2\beta}}\right) \times \widetilde{K}^2 - 24M^2\lambda^{-1}\widetilde{K}^2\|p\|_{L^1}}.$$

Moreover, we assume that

$$\lim_{t \longrightarrow \infty} \zeta(t) = 0, \quad \lim_{t \longrightarrow \infty} \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{1-\beta}}\zeta(s) = 0 \text{ and } \lim_{t \longrightarrow \infty} \int_0^t e^{-\lambda(t-s)}p(s)d(s) = 0. \tag{4.18}$$

Then, for $z \in \bar{B}(z^*, \rho)$ by $(H1^{'})$, $(H2^{'})$ and $(H2)$ we have

$$\begin{aligned} E|z(t) - z^*(t)|^2 &= E|\widehat{\Psi}_1(z)(t) - \widehat{\Psi}_1(z^*)(t)|^2 \\ &\leq 3E|g(t, z_t + \widehat{\phi}_t) - g(t, z^*_t + \widehat{\phi}^*_t)|^2 \\ &\quad +3\left|\int_0^t AS(t-s)(g(s, z_s + \widehat{\phi}_s) - g(s, z^*_s + \widehat{\phi}^*_s))\right|^2 \\ &\quad +3E\left|\int_0^t S(t-s)f(s, z_s + \widehat{\phi}_s) - f(s, z^*_s + \widehat{\phi}^*_s)\right|^2 ds \end{aligned}$$

$$
\begin{aligned}
&\leq 3L_g\|(-A)^{-\beta}\|^2\|(z(t)+\widehat{\phi}(t))-(z^*(t)+\widehat{\phi}^*(t))\|^2_{\mathcal{D}_{\mathcal{F}_0}}\\
&+3\frac{\Gamma(\beta)}{\lambda^\beta}\int_0^t\frac{C^2_{1-\beta}}{(t-s)^{\beta-1}}e^{-2\lambda(t-s)}L_g\|(z(t)+\widehat{\phi}(t))-(z^*(t)+\widehat{\phi}^*(t))\|^2_{\mathcal{D}_{\mathcal{F}_0}}ds,\\
&+3\lambda^{-1}M^2\int_0^t e^{-\lambda(t-s)}p(s)\Big[\|z+\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|z^*+\widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}+2\Big]\\
&\leq 12L_g\left(\|(-A)^{-\beta}\|^2+\frac{C^2_{1-\beta}\Gamma^2(\beta)}{\lambda^{2\beta}}\right)\times\widetilde{K}^2\rho\\
&+6M^2\lambda^{-1}(4\widetilde{K}^2\rho+1)\|p\|_{L^1}\leq\rho,
\end{aligned}
$$

Therefore, we have $\Phi(\bar{B}(z^*,\rho))\subset\bar{B}(z^*,\rho)$.

So, for each solution of problem (4.1)-(4.3) $z\in\bar{B}(z^*,\rho)$ and $t\in J=[0,\infty)$, we have

$$
\begin{aligned}
E|z(t)-z^*(t)|^2 &= E|\widehat{\Phi}(z)(t)-\widehat{\Phi}(z^*)(t)|^2\\
&\leq 3E|g(t,z_t+\widehat{\phi}_t)-g(t,z^*_t+\widehat{\phi}^*_t)|^2\\
&+3E\left|\int_0^t AS(t-s)(g(s,z_s+\widehat{\phi}_s)-g(s,z^*_s+\widehat{\phi}^*_s))ds\right|^2\\
&+3E\left|\int_0^t S(t-s)f(s,z_s+\widehat{\phi}^*_s)-f(s,z^*_s+\widehat{\phi}^*_s)ds\right|^2\\
&\leq 6\|(-A)^{-\beta}\|^2\zeta(t)(\|z(t)+\widehat{\phi}(t))\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|z^*(t)+\widehat{\phi}^*(t)\|^2_{\mathcal{D}_{\mathcal{F}_0}})\\
&+6C^2_{1-\beta}\tfrac{\Gamma(\beta)}{\lambda^\beta}\int_0^t\frac{e^{-\lambda(t-s)}}{(t-s)^{1-\beta}}\zeta(s)(\|z(s)+\widehat{\phi}(s)\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|z^*(s))+\widehat{\phi}^*(s)\|^2_{\mathcal{D}_{\mathcal{F}_0}}ds,\\
&+6\lambda^{-1}M^2\int_0^t e^{-\lambda(t-s)}p(s)\Big[\|z+\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|z^*+\widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}+2\Big]\\
&\leq 12\|(-A)^{-\beta}\|^2(4(\alpha^2(M+1)\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\widetilde{K}^2\rho)\zeta(t)\\
&+12C^2_{1-\beta}\tfrac{\Gamma(\beta)}{\lambda^\beta}(4(\alpha^2(M+1)\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\widetilde{K}^2\rho))\int_0^t\frac{e^{-\lambda(t-s)}}{(t-s)^{1-\beta}}\zeta(s)ds\\
&+12\lambda^{-1}M^2(4(\alpha^2(M+1)\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\widetilde{K}^2\rho+2)\int_0^t e^{-\lambda(t-s)}p(s)ds,
\end{aligned}
$$

where

$$\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}=\max\{\|\widehat{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}},\|\widehat{\phi}^*\|^2_{\mathcal{D}_{\mathcal{F}_0}}\}.$$

Hence, from (4.18), we conclude that

$$\lim_{t\longrightarrow\infty}E|z(t)-\widetilde{z}(t)|^2=0.$$

Consequently, the solutions of problem (4.1)-(4.3) are locally attractive.

4.4 An example

Consider the following stochastic partial differential equation with delays and impulsive effects

$$\begin{cases} d[u(t,\xi)+G(t,u(t-h,\xi))] = \dfrac{\partial^2}{\partial \xi^2}u(t,\xi)+F(t,u(t-h,\xi)) \\ \qquad\qquad + \sigma(t)\dfrac{dB_Q^H}{dt}, \quad t \geq 0, \quad t \neq t_k, \quad 0 \leq \xi \leq \pi, \\ u(t_k^+,\xi)-u(t_k^-,\xi) = \alpha_k u(t_k^-,\xi), \qquad k=1,\cdots,m, \\ u(t,0)=u(t,\pi)=0, \qquad t \geq 0, \\ u(t,\xi)=\phi(t,\xi), \qquad -\infty \leq t \leq 0, 0 \leq \xi \leq \pi, \end{cases} \tag{4.19}$$

where $\alpha_k > 0$, B_Q^H denotes a fractional Brownian motion and $G, F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions.
Let

$$\begin{aligned} y(t)(\xi) &= u(t,\xi) \;\; t \in J = [0,T], \;\; \xi \in [0,\pi], \\ I_k(y(t_k)) &= \alpha_k u(t_k^-,\xi), \;\; \xi \in [0,\pi], \;\; k=1,\cdots,m, \\ g(t,\phi)(\xi) &= G(t,\phi(-h,\xi)), \;\; \theta \in (-\infty,0], \;\; \xi \in [0,\pi], \\ f(t,\phi)(\xi) &= F(t,\phi(-h,\xi)), \;\; \theta \in (-\infty,0], \;\; \xi \in [0,\pi], \\ \phi(\theta)(\xi) &= \phi(\theta,\xi), \;\; \theta \in (-\infty,0], \;\; \xi \in [0,\pi]. \end{aligned}$$

Take $\mathcal{K} = \mathcal{H} = L^2([0,\pi])$. We define the operator A by $Au = u''$, with domain

$$D(A) = \{u \in \mathcal{H}, u'' \in \mathcal{H} \quad \text{and} \quad u(0)=u(\pi)=0\}.$$

Then, it is well known that

$$Az = -\sum_{n=1}^{\infty} e^{-n^2t}\langle z, e_n\rangle e_n, \quad z \in \mathcal{H},$$

and A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{H}$, which is given by $S(t)u = \sum_{n=1}^{\infty} e^{-n^2t}\langle u, e_n\rangle e_n$, $u \in \mathcal{H}$, where $e_n(u) = (2/\pi)^{1/2}\sin(nu), n = 1,2,\cdots$, is the orthogonal set of eigenvectors of A. Due to the fact that the semigroup $\{S(t)\}$ is analytic and compact, there exists a constant $M \geq 1$ such that $\|S(t)\|^2 \leq M$ for all $t \in J$.

In order to define the operator $Q : \mathcal{K} \longrightarrow \mathcal{K}$, we choose a sequence $\{\sigma_n\}_{n\geq 1} \subset \mathbb{R}^+$, set $Qe_n = \sigma_n e_n$, and assume that

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty.$$

Define the process $B_Q^H(s)$ by

$$B_Q^H = \sum_{n=1}^{\infty} \sqrt{\sigma_n} \gamma_n^H(t) e_n,$$

where $H \in (1/2, 1)$, and $\{\gamma_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional fractional Brownian motions that are mutually independent. Assume now that

(i) There exist positive number, $d_k, \quad k = 1, \cdots, m$ such that

$$|I_k(\xi)| \leq d_k,$$

for any $\xi \in \mathbb{R}$

(ii) The function $g : [0, T] \times \mathcal{H} \longrightarrow \mathcal{H}$ defined by $g(t, u)(.) = G(t, u(.))$ is continuous and we impose suitable conditions on G to satisfy assumption $(H2)$.

(iii) Assume that there exists an integrable function $\eta : [0, T] \longrightarrow \mathbb{R}^+$ such that

$$E|F(t, u(\omega)|^2 \leq \eta(t)\psi(E|u(\omega)|^2)$$

for any $t \in [0, T]$ and any random variable $u(\cdot) \in L^2(\Omega)$, where $\psi : [0, \infty) \longrightarrow (0, \infty)$ is continuous and nondecreasing with

$$\int_1^{\infty} \frac{ds}{\psi(s)} = +\infty.$$

(iv) The function $g : [0, T] \longrightarrow L_Q^2(\mathcal{K}, \mathcal{H})$ is bounded, that is, there exists a positive constant L such that

$$\int_0^T \|\sigma(s)\|_{L_Q^2}^2 < L.$$

The problem (4.19) can be written in the abstract form

$$\begin{cases} d[y(t)+g(t,y_t)] = [Ay(t)+f(t,y_t)]dt+\sigma(t)dB_Q^H(t), \quad t\in J:=[0,T], \\ y(t_k^+)-y(t_k) = I_k(y(t_k)),\ k=1,\dots,m, \\ y(t)=\phi(t), \quad \text{for} \ \text{ a.e. } t\in(-\infty,0]. \end{cases}$$

Thanks to those assumptions, it is straightforward to check that all the conditions of Theorem 5.1 hold. Hence, we can conclude that the problem (4.19) has at least one mild solution on $(-\infty,T]$.

In the case of $t\in J=[0,+\infty)$ we observe that

(i') $\|S(t)\|\le e^{-\pi^2 t}$, and $\|(-A)^{\frac{3}{4}}\|=1$;

(ii') The function $f:[0,\infty)\times\mathcal{H}\longrightarrow\mathcal{H}$ defined by $f(t,u)(.)=F(t,u(.))$ is continuous and it is easy to impose suitable conditions on F to make assumption $(H2')$ hold;

(iii') The function $g:[0,T]\longrightarrow L_Q^2(\mathcal{K},\mathcal{H})$ is bounded, that is, there exists a positive constant L such that

$$\int_0^\infty e^{\gamma s}\|\sigma(s)\|^2_{L_Q^0}<L;$$

(iv') There exist positive number, $d_k,\quad k=1,\cdots,m,\cdots$ such that

$$|I_k(\xi)|\le d_k \text{ and } \sum_{k=1}^{\infty} d_k<\infty$$

for any $\xi\in\mathbb{R}$.

Thus, the problem (4.19) can be written in the abstract form

$$\begin{cases} d[y(t)+g(t,y_t)] = [Ay(t)+f(t,y_t)]dt+\sigma(t)dB_Q^H(t), \quad t\in J:=[0,\infty); \\ y(t_k^+)-y(t_k) = I_k(y(t_k)),\ k=1,\dots,m,\dots, \\ y(t)=\phi(t), \quad \text{for} \ \text{ a.e. } t\in(-\infty,0]. \end{cases}$$

Thanks to these assumptions, it is straightforward to check that $(H1')$-$(H3')$, (H2) and (H3) hold. The assumptions in Theorem 5.3 are fulfilled, and conclude that system (4.19) has a unique mild solution on $(-\infty,\infty)$, which implies that the mild solution of (4.19) is locally attractive.

Chapter 5

Impulsive stochastic functional differential inclusions with delay

In this chapter we prove the existence of mild solutions for a first-order impulsive semilinear stochastic functional differential inclusions driven by a fractional Brownian motion with infinite delay. We consider the cases in which the right hand side is convex or nonconvex-valued. The results are obtained by using two different fixed point theorems for multivalued mappings.

We are interested in the existence problem of the following stochastic differential inclusions:

$$\begin{cases} dy(t) \in [Ay(t) + F(t, y_t)]dt + g(t)dB_Q^H(t), & t \in J = [0, T], \quad t \neq t_k, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \ k = 1, \ldots, m, & \\ y(t) = \phi(t) \in \mathcal{D}, & J_0 = (-\infty, 0], \end{cases} \tag{5.1}$$

in a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, where $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t)$, $0 \leq t \leq T$.

Assume $F : J \times \mathcal{D} \longrightarrow \mathcal{P}(\mathcal{H})$ is a bounded, closed and convex-valued multivalued map, $g : J \to L_Q^0(\mathcal{K}, \mathcal{H})$. Here, $L_Q^0(\mathcal{K}, \mathcal{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathcal{K}$ into $\mathcal{H}$.

As for the impulse functions we will assume that $I_k \in C(\mathcal{H}, \mathcal{H})$ $(k = 1, 2, \ldots, m)$ are bounded. Moreover, the fixed times t_k satisfy $0 < t_1 < t_2 < \ldots < t_m < T$, $y(t_k^-)$ and $y(t_k^+)$ denotes the left and right limits of $y(t)$ at $t = t_k$. As for y_t we mean the segment solution which is define in the usual way, that is, if $y(\cdot,\cdot) : (-\infty, T] \times \Omega \to \mathcal{H}$, then for any $t \geq 0$, $y_t(\cdot,\cdot) : (-\infty, 0] \times \Omega \to \mathcal{H}$ is given by

$$y_t(\theta, \omega) = y(t+\theta, \omega), \text{ for } \theta \in (-\infty, 0], \ \omega \in \Omega,$$

belong to some abstract phase space $\mathcal{D}$, that is a phase space defined axiomatically. Thus $\mathcal{D}$ is a linear space of functions mapping $[0,T] \times \Omega$ into $\mathcal{H}$ endowed with a seminorm $\|\cdot\|_{\mathcal{D}_T}$. Consider the following space

$$\mathcal{D}_T := \Big\{ y : [0,T] \to \mathcal{H}, \quad \text{such that} \quad y|_{J_k} \in C(J_k, \mathcal{H}) \quad \text{and there exist} \quad y(t_k^+), \\ \text{and} \quad y(t_k^-) \text{ with} \quad y(t_k) = y(t_k^-), \quad k = 1, \cdots, m, \sup_{t \in [0,T]} E(|y(t)|^2) < \infty \Big\}$$

where $y|_{J_k}$ is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \cdots, m$. We endow $\mathcal{D}_T$ with a norm $\|.\|_{\mathcal{D}_T}$ on $\mathcal{D}_T$ defined by

$$\|y\|_{\mathcal{D}_T} = \sup_{0 \leq s \leq T} (E\|y(s)\|^2)^{\frac{1}{2}}$$

We will assume that $\mathcal{D}$ satisfies the following axioms suggested by Hale and Kato [41] and Hino *et al.* [48]. The axioms of the space $\mathcal{D}$ are established for $\mathcal{F}_0$-measurable functions from J_0 into $\mathcal{H}$, endowed with the seminorm $\|\cdot\|_{\mathcal{D}}$. We will assume that $\mathcal{D}$ satisfies the following axioms:

(A-1) If $y : (-\infty, T] \longrightarrow \mathcal{H}$, $T > 0$ is such that $y|_{[0,T]} \in \mathcal{D}$ and $y_0 \in \mathcal{D}$, then for every $t \in [0,T)$ the following conditions hold

- **(i)** y_t is in $\mathcal{D}$
- **(ii)** $\|y(t)\| \leq L\|y_t\|_{\mathcal{D}}$
- **(iii)** $\|y_t\|_{\mathcal{D}} \leq K(t) \sup \|y(s)\| : 0 \leq s \leq t + N(t)\|y_0\|_{\mathcal{D}}$, where $L > 0$ is a constant; $K, N : [0,\infty) \longrightarrow [0,\infty)$, K is continuous, N is locally bounded and L, K, N are independent of $y(\cdot)$.

(A-2) For the function $y(\cdot)$ in (A-1), y_t is a $\mathcal{D}$-valued function $[0,T)$.

(A-3) The space $\mathcal{D}$ is complete.

Set
$$\mathcal{D}_T^* = \{y : (-\infty, T] \longrightarrow \mathcal{H}, y \in \mathcal{D}_T \cap \mathcal{D}\}$$
$\|.\|_{\mathcal{D}_T^*}$ on $\mathcal{D}_T^*$, it is defined by

$$\|y\|_{\mathcal{D}_T^*} = \sup_{0\leq s\leq T}(E\|y(s)\|^2)^{\frac{1}{2}} = \|\phi\|_{\mathcal{D}} + \sup_{0\leq s\leq T}(E\|y(s)\|^2)^{\frac{1}{2}}, y \in \mathcal{D}_T^*.$$

Denote

$$\widetilde{K} = \sup\{K(t) : t \in J\} \text{ and } \widetilde{M} = \sup\{M(t) : t \in J\}.$$

Differential inclusions are widely studied by many authors by their numerous applications in various fields of science. For this reason, in the literature there are many papers study the existence, uniqueness, regularity, controllability and behavior of the solution for various classes of semilinear differential inclusion with finite an infinite delay, for instance we refer to [1,2,3,12,13,33].

5.1 Existence results for the convex case

In this section, we will show same results concerning the existence results of mild solutions for convex case of system (5.1)in the convex case.

First, we define what we mean by a mild solution.

Definition 5.1.1. *A stochastic process $y : (-\infty, T] \times \Omega \longrightarrow \mathcal{H}$ is called a mild solution of the system (5.1) if*

- *$y(t)$ is measurable and $\mathcal{F}_t$-adapted, for each $t \geq 0$;*
- *$y(t) \in \mathcal{H}$ has càdlàg paths on $t \in [0, T]$ a.s., for every $0 \leq s < t \leq T$, there exist $f \in N_{F,y}$ such that the following integral equation holds*

$$\begin{aligned} y(t) = \;& S(t)\phi(0) + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dB_Q^H(s) \\ &+ \sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-)), \quad t \in J. \end{aligned} \tag{5.2}$$

- $y_0(\cdot) = \phi \in \mathcal{D}$ *on* J_0 *satisfies* $\|\phi\|_{\mathcal{D}} < \infty$.

We are now in a position to state and prove our existence result for the problem (5.1). First we will list the following hypotheses which will be imposed in our main theorem.

(H1) Operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t)\}$, $t \in J$ which is compact for $t > 0$ in $\mathcal{H}$ such that $\|S(t)\|^2 \leq M$ for some $M \geq 0$ and for each $t \in [0, T]$.

(H2) The function $g : J \longrightarrow L^0_Q(\mathcal{K}, \mathcal{H})$ satisfies

$$\int_0^T \|g(s)\|^2_{L^0_Q} ds = \Lambda < \infty, \quad t \in J$$

(H3) There exist constants $c_k > 0$, $k = 1, \ldots, m$ for which $|I_k(y)|^2 \leq c_k$, for all $y \in \mathcal{H}$.

(H4) $F : [0, T] \times \mathcal{D} \longrightarrow \mathcal{P}_{cp,c}(\mathcal{H})$ is an L^2-Carathédory function and for every $t \in [0, T]$ the multifunction $t \to F(t, y_t)$, $y_t \in \mathcal{D}$ is measurable.

(H5) There exist $\eta \in L^2(J, \mathbb{R}^+)$ and $p : \mathbb{R}^+ \longrightarrow (0, \infty)$ is continuous and increasing such that

$$E|F(t, \Theta)|^2 = \{\sup E(|f|)^2 : f \in F(t, \Theta)\}\} \leq \eta(t) p(\|\Theta\|^2_{\mathcal{D}}), \quad t \in J, \quad \Theta \in \mathcal{D},$$

where

$$8\widetilde{K}^2 M(t_k - t_{k-1}) \int_{t_{k-1}}^{t_k} \eta(s) ds < \int_{N_{k-1}}^{\infty} \frac{du}{p(u)}, \quad k = 1, \cdots, m+1,$$

and

$$N_0 = v_0(0) = 16\widetilde{K}^2 M H t_1^{2H-1} \int_0^{t_1} \|g(s)\|^2_{L^0_Q} ds + C,$$

$$C = 4\widetilde{K}^2 M E|\widehat{\phi}(0)|^2 + 4\widetilde{M}^2 \|\widehat{\phi}\|^2_{\mathcal{D}}$$

and for $k = 2, \cdots, m+2$,

$$\begin{aligned} N_{k-1} = \ & 4\widetilde{K}^2 \Big[4M \sup_{y \in (-\infty, t_{k-1}]} E|I_{k-1}(y_{t_{2-1}}(t_{k-1}))|^2 \\ & + 8\widetilde{K}^2 M (t_{k-1} - t_{k-2})^{2H-1} \int_{t_{k-2}}^{t_{k-1}} \|g(s)\|^2_{L^0_Q} ds \Big] + C, \end{aligned}$$

$$\widehat{M}_{k-2} = \Gamma_{k-1}^{-1}\Big(8\widetilde{K}^2 M(t_{k-1}-t_{k-2})\int_{t_{k-2}}^{t_{k-1}}\eta(s)ds\Big)$$

with

$$\Gamma_l(x) = \int_{N_{l-1}}^{x}\frac{du}{p(u)} \quad x \geq N_{l-1}, \quad l \in \{1,\cdots,m+1\}.$$

Theorem 5.1.1. *Assume that hypotheses $(H1)-(H5)$ hold. Then the problem (5.1) has at least one integral solution on $(-\infty, T]$.*

Proof. The proof will be given in several steps.

Step 1. Consider the problem (5.1) on $(-\infty, t_1]$

$$\begin{array}{ll} dy(t) \in [Ay(t) + F(t,y_t)]dt + g(t)dB_Q^H(t), & \text{if } t \in [0,t_1], \\ y(t) = \phi(t), & \text{if } t \in (-\infty, 0]. \end{array} \tag{5.3}$$

Let

$$\mathcal{D}_{t_0} = \{y \in C([0,t_1],\mathcal{H}) : \sup_{t\in[0,t_1]} E(|y(t)|^2) < \infty\}.$$

Set

$$\mathcal{D}_{t_0}^* = \mathcal{D} \cap \mathcal{D}_{t_0}.$$

We transform the problem (5.3) into a fixed point problem. Consider the multivalued operator $\Phi_0 : \mathcal{D}_{t_0}^* \to \mathcal{P}(\mathcal{D}_{t_0}^*)$ defined by

$$\Phi_0(y) = \left\{ \rho \in \mathcal{D}_{t_0}^* : \ \rho(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if} t \in (-\infty, 0], \\ S(t)\phi(0) + \displaystyle\int_0^t S(t-s)f(s)ds & \\ + \displaystyle\int_0^t S(t-s)g(s)dB^H(s), & \text{if } t \in [0,t_1] \end{array} \right. \right\},$$

where $f \in N_{F,y} = \{f \in L^2([0,t_1],\mathcal{H}) : f(t) \in F(t,y_t) \ \text{ for } \ \text{a.e.} t \in [0,t_1]\}$.

We will prove that Φ has a fixed point.

Let $\widehat{\phi} : (-\infty, t_1] \longrightarrow \mathcal{H}$ be the function defined by

$$\widehat{\phi}(t) = \left\{ \begin{array}{ll} \phi(t), & t \in (-\infty, 0], \\ S(t)\phi(0), & t \in [0,t_1]; \end{array} \right.$$

Then $\widehat{\phi}$ is an element of $\mathcal{D}_{t_0}^*$ and $\widehat{\phi}_0 = \phi$.

Let $y(t) = z(t) + \widehat{\phi}(t), -\infty < t \leq t_1$. Obviously, if y satisfies the integral equation

$$y(t) = S(t)\phi(0) + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dB_Q^H(s), t \in [0,t_1]. \tag{5.4}$$

then z satisfies $z_0 = 0, t \in (-\infty, 0]$ and

$$z(t) = \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dB_Q^H(s), t \in [0, t_1], \tag{5.5}$$

where $f(t) \in F(t, z_t + \widehat{\phi}_t)$ for a.e. $t \in [0, t_1]$.

Set $\mathcal{D}_{t_0}^0 = \{z \in \mathcal{D}_{t_0}^*, \quad \text{such that} \quad z_0 = 0 \in \mathcal{D}\}$ and for any $z \in \mathcal{D}_{t_0}^0$ we have

$$\|y\|_{\mathcal{D}_{t_0}^0} = \|z_0\|_{\mathcal{D}} + \sup_{t\in[0,t_1]} (E\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t\in[0,t_1]} (E\|y(t)\|^2)^{\frac{1}{2}}.$$

Thus $(\mathcal{D}_{t_0}^0, \|.\|_{\mathcal{D}_{t_0}^0})$ is a Banach space. Let the operator $\widehat{\Phi}_0 : \mathcal{D}_{t_0}^0 \to \mathcal{P}(\mathcal{D}_{t_0}^0)$ be defined by

$$\widehat{\Phi}_0(z) = \left\{ \widehat{\rho} \in \mathcal{D}_{t_0}^0 \ \widehat{\rho}(t) = \left\{ \begin{array}{ll} 0, & \text{if } t \in (-\infty, 0], \\ \displaystyle\int_0^t S(t-s)f(s)ds & \\ \displaystyle + \int_0^t S(t-s)g(s)dB^H(s), & \text{if } t \in [0, t_1]. \end{array} \right. \right\}.$$

Clearly, that the operator Φ_0 has a fixed point is equivalent to $\widehat{\Phi}_0$ having a fixed point, and so we turn our attention to proving that $\widehat{\Phi}_0$ does in fact have a fixed point. We shall show that $\widehat{\Phi}_0$ satisfies assumptions of Lemma 2.3.2.

Claim 1. $\widehat{\Phi}_0(z)$ is convex for each $z \in \mathcal{D}_{t_0}^0$.

Let $\widehat{\rho}_1$,$\widehat{\rho}_2 \in \widehat{\Phi}_0(z)$, then there exist $f_1,\ f_2 \in N_{F,z+\widehat{\phi}}$ such that, for each $t \in [0, t_1]$ we have

$$\widehat{\rho}_i(t) = \int_0^t S(t-s)f_i(s)ds + \int_0^t S(t-s)g(s)dB^H(s).$$

Let $0 \leq \delta \leq 1$. Then, for each $t \in [0, t_1]$, we have

$$(\delta\widehat{\rho}_1+(1-\delta)\widehat{\rho}_2)(t) = \int_0^t S(t-s)[\delta f_1(s)+(1-\delta)f_2(s)]ds+\int_0^t S(t-s)g(s)dB^H(s)$$

Since $N_{F,z+\widehat{\phi}}$ is convex ($F(t,z)$ has convex values), one has

$$\delta\widehat{\rho}_1 + (1-\delta)\widehat{\rho}_2 \in \widehat{\Phi}_0(z)$$

Claim 2. $\widehat{\Phi}_0$ maps bounded sets into bounded sets in $\mathcal{D}^0_{t_0}$.

Indeed, it is enough to show that there exists a positive constant ζ such that for each $\widehat{\rho} \in \widehat{\Phi}_0$, $w \in \mathcal{B}_q = \{z \in \mathcal{D}^0_{t_0}, \|z\|^2_{\mathcal{D}^0_{t_0}} \leq q\}$ one has $\|\widehat{\rho}\|^2_{\mathcal{D}^0_{t_0}} \leq \zeta$.

Let $\widehat{\rho} \in \widehat{\Phi}_0$, then there exists $f \in N_{F,z+\widehat{\phi}}$ such that for $t \in [0, t_1]$, we have

$$\widehat{\rho}(t) = \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dB^H(s).$$

From (A1) we have

$$\begin{aligned} \|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}} \leq\ & 2(\|z_t\|^2_{\mathcal{D}} + \|\widehat{\phi}_t\|^2_{\mathcal{D}}) \\ & \leq 4((k(t))^2 \sup_{s\in[0,t]} E|z(s)|^2 + (M(t))^2\|z_0\|^2_{\mathcal{D}}) \\ & +4((K(t))^2 \sup_{s\in[0,t]} E\|\widehat{\phi}(s)\|^2 + (M(t))^2\|\widehat{\phi}_0\|^2_{\mathcal{D}}) \\ & \leq 4(\widetilde{M}^2\|\phi\|^2_{\mathcal{D}} + \widetilde{K}^2(q + ME|\phi(0)|^2)) = q'. \end{aligned}$$

From $(H1) - (H4)$, we obtain for $t \in [0, t_1]$,

$$\begin{aligned} E|\widehat{\rho}(t)|^2 &= E\Big|\int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dB^H(s)\Big|^2 \\ &\leq 2E\Big|\int_0^t S(t-s)f(s)ds\Big|^2 + 2E\Big|\int_0^t S(t-s)g(s)dB^H(s)\Big|^2 \\ &\leq 2Mt_1\int_0^t E|f(s)|^2ds + 4MHt_1^{2H-1}\int_0^{t_1}\|g(s)\|^2_{L^0_Q}ds \\ &\leq 2Mt_1p(q')\int_0^t \eta(s)ds + 4MHt_1^{2H-1}\Lambda \\ &:= \zeta, \end{aligned}$$

Claim 3. $\widehat{\Phi}_0$ maps bounded sets into equicontinuous sets of $\mathcal{D}^0_{t_0}$.

Let $0 < \tau_1 < \tau_2 \in [0, t_1]$, $\mathcal{B}_q$ be a bounded set of $\mathcal{D}^0_{t_0}$ as in Claim 2. For each $z \in \mathcal{B}_q$ and $\widehat{\rho} \in \widehat{\Phi}_0$, there exists $f \in N_{F,z+\widehat{\phi}}$ such that

$$\widehat{\rho}(t) = \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dB^H(s), \quad t \in [0, t_1].$$

Then we have

$$\begin{aligned}
E|\widehat{\rho}(\tau_2)-\widehat{\rho}(\tau_1)|^2 &\leq 4t_1p(q^{'})\int_0^{\tau_2}|S(\tau_2-s)-S(\tau_1-s)|^2\eta(s)ds\\
&+4t_1p(q^{'})\int_{\tau_1}^{\tau_2}|S(\tau_1-s)|^2\eta(s)ds\\
&+4\int_0^{\tau_2}|S(\tau_2-s)-S(\tau_1-s)|^2E|g(s)dB^H(s)|^2\\
&+4\int_{\tau_1}^{\tau_2}|S(\tau_1-s)|^2E|g(s)dB^H(s)|^2
\end{aligned}$$

The right-hand side of the above inequality tends to zero as $\tau_2 \longrightarrow \tau_1$, since $S(t)$ is strongly continuous operator and the compactness of $S(t)$ for $t>0$ implies the continuity in the uniform operator topology [69]. Thus, the set $\{\widehat{\Phi}_0(z) : z \in \mathcal{B}_q\}$ is equicontinuous.

As a consequence of Claims 2 and 3, together with the Arzelá-Ascoli theorem, it suffices to show that $\widehat{\Phi}_0$ maps $\mathcal{B}_q$ into a precompact set in $\mathcal{H}$.

Let $0<t<t_1$ be fixed and let ϵ be a real number satisfying $0<\epsilon<t$. For $z \in \mathcal{B}_q$ we define

$$(\widehat{\Phi}_{0\epsilon}z)(t) = S(\epsilon)\int_0^{t-\epsilon}S(t-s-\epsilon)f(s)ds + S(\epsilon)\int_0^{t-\epsilon}S(t-s-\epsilon)g(s)dB^H(s),$$

where $f \in N_{F,z+\widehat{\phi}}$.

Since $S(t)$ is a compact operator, the set

$$V_\epsilon(t) = \{\widehat{\Phi}_{0\epsilon}(z)(t) : \quad z \in \mathcal{B}_q\}$$

is precompact in $\mathcal{H}$ for every ϵ, $0<\epsilon<t$. Moreover, for every $z \in \mathcal{B}_q$ we have

$$\begin{aligned}
E|(\widehat{\Phi}_0 z)(t)-(\widehat{\Phi}_{0\epsilon}(z)(t)|^2 &\leq 2t_1\int_{t-\epsilon}^{t}|S(t,s)|^2\eta(s)p(q)ds\\
&+2\int_{t-\epsilon}^{t}S(t-\epsilon)E|g(s)dB^H(s)|^2\\
&\leq 2t_1M\int_{t-\epsilon}^{t}\eta(s)p(q^{'})ds\\
&+4MHt_1^{2H-1}\int_0^{t_1}\|g(s)\|^2_{L_Q^0}ds.
\end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $V_\epsilon(t)$=$\{\widehat{\Phi}_{0\epsilon}(z)(t): \quad z \in \mathcal{B}_q\}$. Hence the set $V(t) = \{\widehat{\Phi}_0(z)(t): \quad z \in \mathcal{B}_q\}$ is precompact in $\mathcal{H}$, Hence, the Arzelá-Ascoli shows that $\widehat{\Phi}_0$ is a compact multi-valued map.

Claim 4. $\widehat{\Phi}_0$ has a closed graph.

Let $z_n \longrightarrow z_*$, $\widehat{\rho}_n \in \widehat{\Phi}_0(z_n)$ and $\widehat{\rho}_n \longrightarrow \widehat{\rho}_*$ as $n \longrightarrow \infty$, we shall prove that $\widehat{\rho}_* \in \widehat{\Phi}_0(z_*)$. Since $\widehat{\rho}_n \in \widehat{\Phi}_0(z_n)$, then there exists $f_n \in N_{F,z_n+\widehat{\phi}}$ such that

$$\widehat{\rho}_n(t) = \int_0^t S(t-s)f_n(s)ds + \int_0^t S(t-s)g(s)dB^H(s), \quad t \in [0,t_1].$$

We must prove that there exists $f_* \in N_{F,z_*+\widehat{\phi}}$ such that

$$\widehat{\rho}_*(t) = \int_0^t S(t-s)f_*(s)ds + \int_0^t S(t-s)g(s)dB^H(s), \quad t \in [0,t_1].$$

Consider the linear continuous operator

$$\Gamma : L^2([0,t_1],\mathcal{H}) \longrightarrow \mathcal{D}^0_{t_0}, \qquad \Gamma(f)(t) = \int_0^t S(t-s)f(s)ds.$$

From lemma (2.2.3), it follows that $\Gamma \circ N_F$ is a closed graph operator. Moreover, we have that

$$\rho_n(t) - \int_0^t S(t-s)g(s)dB^H(s) \in \Gamma(N^1_{F,z_n+\widehat{\phi}}).$$

Since $z_n \longrightarrow z_*$ and $\widehat{\rho}_n \longrightarrow \widehat{\rho}_*$, there is $f_* \in N_{F,z_*+\widehat{\phi}}$ such that

$$\widehat{\rho}_*(t) = \int_0^t S(t-s)f_*(s)ds + \int_0^t S(t-s)g(s)dB^H(s), \quad t \in [0,t_1].$$

Therefore $\widehat{\Phi}_0$ is a completely continuous

Claim 5. There exist a priori bounds on solutions.

Let z be a possible solution of the equation $z \in \lambda\widehat{\Phi}_0(z)$ and $z_0 = \phi$, for some $\lambda \in (0,1)$. Then,

$$\begin{aligned}E|z(t)|^2 =&E\Big|\int_0^t S(t-s)f(s)ds+\int_0^t S(t-s)g(s)dB^H(s)\Big|^2, \quad t\in[0,t_1].\\ \le&2Mt_1\int_0^t \eta(s)p(\|z_s+\widehat{\phi}_s\|^2_{\mathcal{D}})\\ &+4MHt_1^{2H-1}\int_0^{t_1}\|g(s)\|^2_{L_Q^0}ds,\ t\in[0,t_1]\end{aligned}$$

$$\begin{aligned}E|z(t)|^2 &\le& 2Mt_1\int_0^t \eta(s)p(\|z_s+\widehat{\phi}_s\|^2_{\mathcal{D}})\\ &&+4MHt_1^{2H-1}\int_0^{t_1}\|g(s)\|^2_{L_Q^0}ds,\ t\in[0,t_1].\end{aligned} \tag{5.6}$$

But

$$\begin{aligned}\|z_t+\widehat{\phi}_t\|^2_{\mathcal{D}} &\le& 2(\|z_t\|^2_{\mathcal{D}}+\|\widehat{\phi}_t\|^2_{\mathcal{D}}\\ &\le& 4\widetilde{K}^2\sup_{s\in[0,t_1]}E|z(s)|^2+4\widetilde{K}^2ME|\widehat{\phi}(0)|^2+4\widetilde{M}^2\|\phi\|^2_{\mathcal{D}}\end{aligned}$$

If we set $w_0(t)$ the right hand side of the above inequality we have that

$$\|z_t+\widehat{\phi}_t\|^2_{\mathcal{D}}\le w_0(t)$$

and therefore (5.6) becomes

$$E|z(t)|^2\le 2Mt_1\int_0^t \eta(s)p(w_0(s))+4MHt_1^{2H-1}\int_0^{t_1}\|g(s)\|^2_{L_Q^0}ds,\ t\in[0,t_1]. \tag{5.7}$$

Using (5.7) in the definition of w_0, we have that

$$\begin{aligned}w_0(t) &\le& 4\widetilde{K}^2\Big[2Mt_1\int_0^t \eta(s)p(w_0(s))+4MHt_1^{2H-1}\int_0^{t_1}\|g(s)\|^2_{L_Q^0}ds\Big]\\ &&+4\widetilde{K}^2ME|\widehat{\phi}(0)|^2+4\widetilde{M}^2\|\widehat{\phi}\|^2_{\mathcal{D}},\ t\in[0,t_1].\end{aligned} \tag{5.8}$$

Denoting by $v_0(t)$ the right-hand side of the above inequality we have

$$w_0(t)\le v_0(t)\quad t\in[0,t_1],$$

$$v_0(0) = 16\widetilde{K}^2 MHt_1^{2H-1}\int_0^{t_1}\|g(s)\|^2_{L_Q^0}ds] + 4\widetilde{K}^2 ME|\widehat{\phi}(0)|^2 + 4\widetilde{M}^2\|\widehat{\phi}\|^2_{\mathcal{D}}$$

and

$$v_0'(t) = 8\widetilde{K}^2 Mt_1\eta(t)p(w_0(t)) \quad t \in [0, t_1].$$

By using the nondecreasing character of p we obtain

$$v_0'(t) \le 8\widetilde{K}^2 Mt_1\eta(t)p(v_0(t)) \quad t \in [0, t_1].$$

Then for each $t \in [0, t_1]$ we have

$$\Gamma_1(v_0(t)) = \int_{v_0(0)}^{v_0(t)}\frac{du}{p(u)} \le 8\widetilde{K}^2 Mt_1\int_0^1\eta(s)ds < \int_{v_0(0)}^{\infty}\frac{du}{p(u)}.$$

In view of (H4), we deduce

$$v_0(t) \le \Gamma_1^{-1}(8\widetilde{K}^2 Mt_1\int_0^{t_1}\eta(s)ds) = M_0^2.$$

Since for every $t \in [0, t_1]$, $\|z(t)\|_{\mathcal{D}^0_{t_0}} \le M_0$

Set

$$U_0 = \{z \in \mathcal{D}^0_{t_0} : \sup_{t\in[0,t_1]}(E\|y(t)\|^2)^{\frac{1}{2}} < M_0 + 1\}.$$

From the choice of U_0, there is no $z \in \partial U_0$ such that $z \in \lambda\widehat{\Phi}_0(z)$ for some $\lambda \in (0, 1)$. As a consequence of Lemma 2.3.2, we deduce that $\widehat{\Phi}_0$ has a fixed point $z_0 \in \overline{U}_0$. Hence, Φ_0 has a fixed point y that is a solution to the problem (5.1). Denote this solution by y_0.

Step 2. Consider now the problem,

$$\begin{array}{l} dy(t) \in [Ay(t) + F(t, y_t)]dt + g(t)dB_Q^H(t), \quad \text{if } t \in (t_1, t_2], \\ y(t_1^+) - y_0(t_1^-) = I_1(y_0(t_1^-)), \quad y(t) = y_0(t) \quad \text{if } t \in (-\infty, t_1]. \end{array} \tag{5.9}$$

Let

$$\mathcal{D}_{t_1} = \{y \in C([t_1, t_2], \mathcal{H}) : y(t_1^+) \text{ exists }, \sup_{t\in[t_1,t_2]} E(|y(t)|^2) < \infty)\}.$$

Set

$$\mathcal{D}^*_{t_1} = \mathcal{D} \cap \mathcal{D}_{t_0} \cap \mathcal{D}_{t_1}.$$

Consider the operator $\Phi_1 : \mathcal{D}^*_{t_1} \longrightarrow \mathcal{P}(\mathcal{D}^*_{t_1})$ defined by,

$$\Phi_1(y) = \left\{ \rho_1 \in \mathcal{D}^*_{t_1} : \ \rho_1(t) = \left\{ \begin{array}{l} y_0(t), \qquad \text{if } \ t \in (-\infty, t_1], \\ y_0(t_1^-) + S(t-t_1)I_1(y_0(t_1^-)) + \int_{t_1}^t S(t-s)f(s)ds \\ + \int_{t_1}^t S(t-s)g(s)dB^H(s), \qquad \text{if } \ t \in (t_1, t_2] \end{array} \right. \right\}$$

where $f \in N_{F,y} = \{f \in L^2([t_1, t_2], \mathcal{H}) : f(t) \in F(t, y_t) \ \ for \ a.e.t \in [t_1, t_2]\}$.
Let $\widehat{\phi}(\cdot) : (-\infty, t_2] \longrightarrow \mathcal{H}$ be the function defined by

$$\widehat{\phi}(t) = \left\{ \begin{array}{ll} y_0(t), & \text{if;} t \in (-\infty, t_1], \\ y_0(t_1^-) + S(t-t_1)I_1(y_0(t_1^-)), & \text{if } t \in (t_1, t_2] \end{array} \right.$$

Then $\widehat{\phi}_{t_1}$ is an element of $\mathcal{D}^*_{t_1}$ and $\widehat{\phi}_{t_1} = y_0$.

Let $y(t) = z(t) + \widehat{\phi}(t), t_1 < t \leq t_2$. Obviously, if y satisfies the integral equation

$$\begin{aligned} y(t) \ &= \ y_0(t_1^-) + S(t-t_1)I_1(y_0(t_1^-)) + \int_{t_1}^t S(t-s)f(s)ds \\ &+ \int_{t_1}^t S(t-s)g(s)dB^H_Q(s), t \in [t_1, t_2], \end{aligned} \tag{5.10}$$

that z satisfies $z(t_1) = 0, t \in (-\infty, t_1]$ and

$$\begin{aligned} z(t) \ &= \ \int_{t_1}^t S(t-s)f(s)ds + \int_{t_1}^t S(t-s)g(s)dB^H_Q(s) \\ &+ S(t-t_1)I_1(z_0(t_1^-) + \widehat{\phi}(t_1^-)), \quad t \in [t_1, t_2]. \end{aligned} \tag{5.11}$$

Set $\mathcal{D}^1_{t_1} = \big\{z \in \mathcal{D}^*_{t_1}, \quad \text{such that} \quad z_{t_1} = 0\big\}$

Let the operator $\widehat{\Phi}_1 : \mathcal{D}^1_{t_1} \to \mathcal{P}(\mathcal{D}^1_{t_1})$ be defined by

$$\widehat{\Phi}_1(z) = \left\{ \widehat{\rho} \in \mathcal{D}^1_{t_1} : \ \widehat{\rho}(t) = \left\{ \begin{array}{l} 0, \qquad \text{if } t \in (-\infty, t_1], \\ \int_{t_1}^t S(t-s)f(s)ds + \int_{t_1}^t S(t-s)g(s)dB^H(s) \\ + S(t-t_1)(I_1(z(t_1^-) + \phi(t_1^-)), \qquad \text{if } \ t \in [t_1, t_2]. \end{array} \right. \right\}$$

where

$$f \in N_{F,z+\widehat{\phi}} = \{f \in L^2([t_1,t_2],\mathcal{H}) : f(t) \in F(t, z_{t_1} + \widehat{\phi}_{t_1})\text{a.e.}t \in [t_1,t_2]\}.$$

From (H4), for each $t \in [t_1, t_2]$, we have

$$\begin{aligned}
&E|z(t)|^2\\
=&E\Big|\int_{t_1}^{t} S(t-s)f(s)ds + \int_{t_1}^{t} S(t-s)g(s)dB^H(s) + S(t-t_1)I_1(z_0(t_1^-)+\widehat{\phi}(t_1^-))\Big|^2,\\
\leq&2M(t_2-t_1)\int_{t_1}^{t}\eta(s)p(\|z_s+\widehat{\phi}_s\|^2_{\mathcal{D}})\\
&+8MH(t_2-t_1)^{2H-1}\int_{t_1}^{t_2}\|g(s)\|^2_{L_Q^0}ds,\\
&+4M\sup_{t\in(-\infty,t_1]}E(I_1(z_0(t_1))+\widehat{\phi}(t_1))|^2,\ t\in[t_1,t_2]. \qquad (5.12)
\end{aligned}$$

If we set $w_1(t)$ the right hand side of the above inequality we have that

$$\|z_t + \widehat{\phi}_t\|^2_{\mathcal{D}} \leq w_1(t),$$

and therefore (5.12) becomes

$$\begin{aligned}
E|z(t)|^2 \leq&2M(t_2-t_1)\int_{t_1}^{t}\eta(s)p(w_1(s)) + 8MH(t_2-t_1)^{2H-1}\int_{t_1}^{t_2}\|g(s)\|^2_{L_Q^0}ds,\\
&+4M\sup_{t\in(-\infty,t_1]}E(I_1(z_0(t_1))+\widehat{\phi}(t_1))|^2,\ t\in[t_1,t_2]. \qquad (5.13)
\end{aligned}$$

Using (5.13) in the definition of w_1, we have that

$$\begin{aligned}
w_1(t) \leq&4\widetilde{K}^2\Big[2M(t_2-t_1)\int_{t_1}^{t}\eta(s)p(w_1(s)) + 8MH(t_2-t_1)^{2H-1}\int_{t_1}^{t_2}\|g(s)\|^2_{L_Q^0}ds,\\
&+4M\sup_{t\in(-\infty,t_1]}E(I_1(z_0(t_1))\Big] + 4M\widetilde{K}^2E|\widehat{\phi}(0)|^2 + 4\widetilde{M}^2\|\widehat{\phi}\|^2_{\mathcal{D}},\ t\in[t_1,t_2],\\
=&N_1 + 8\widetilde{K}^2M(t_2-t_1)\int_{t_1}^{t}\eta(s)p(w_1(s)). \qquad (5.14)
\end{aligned}$$

Denoting by $v_1(t)$ the right-hand side of the above inequality we have

$$w_1(t) \leq v_1(t) \quad t \in [t_1,t_2],$$

$$v_1(t_1) = N_1$$

and

$$v_1'(t) = 8\widetilde{K}^2 M(t_2 - t_1)\eta(t)p(w_1(t)) \quad t \in [t_1, t_2].$$

By using the nondecreasing character of p we deduce

$$v_1'(t) \leq 8\widetilde{K}^2 M(t_2 - t_1)\eta(t)p(v_1(t)) \quad t \in [t_1, t_2].$$

Then, for each $t \in [0, t_1]$, we have

$$\int_{N_1}^{v_1(t)} \frac{du}{p(u)} \leq 8\widetilde{K}^2 M(t_2 - t_1) \int_{t_1}^{t_2} \eta(s)ds.$$

In view of (H4), we obtain

$$v_1(t) \leq \Gamma_2^{-1}(8\widetilde{K}^2 M(t_2 - t_1) \int_{t_1}^{t_2} \eta(s)ds) = M_1^2.$$

Since for every $t \in [t_1, t_2]$, $\|z(t)\|_{\mathcal{D}_{t_1}^2} \leq M_1$, set

$$U_1 = \{z \in \mathcal{D}_{t_1}^1 : \sup_{t \in [t_1, t_2]} (E\|z(t)\|^2)^{\frac{1}{2}} < M_1 + 1\}.$$

As in Step 1 we can show that $\widehat{\Phi} : \overline{U}_1 \longrightarrow \mathcal{P}_{cv}(\mathcal{D}_{t_1}^1)$ is a compact multivalued map and u.s.c.

From the choice of U_1, there is no $z \in \partial U_1$ such that $z \in \lambda\widehat{\Phi}_1(z)$ for some $\lambda \in (0, 1)$. As a consequence of Lemma 2.3.2, we deduce that $\widehat{\Phi}_1$ has a fixed point $z_1 \in \overline{U}_1$. Hence, Φ_1 has a fixed point y that is a solution to the problem (5.1). Denote this solution by y_1.

Step3. We continue this process and take into account that $z_m := z|_{[t_m,T]}$ is a fixed point of the operator $\widehat{\Phi}_m : \mathcal{D}_{t_m}^m \to \mathcal{P}(\mathcal{D}_{t_m}^m)$ defined by

$$\widehat{\Phi}_m(z) = \left\{ \widehat{\rho} \in \mathcal{D}_{t_m}^m \ \widehat{\rho}(t) = \left\{ \begin{array}{l} 0, \qquad \text{if } \ t \in (-\infty, t_m], \\ \displaystyle\int_{t_m}^t S(t-s)f(s)ds + \int_{t_m}^t S(t-s)g(s)dB^H(s) \\ +S(t-t_m)(I_m(z_{m-1}(t_m^-) + \phi(t_m^-)), \quad \text{if } \ t \in [t_m, T], \end{array} \right. \right\}$$

where

$$f \in N_{z+\widehat{\phi}} = \{f \in L^2([t_m, T], \mathcal{H}) : f(t) \in F(t, z_{t_t} + \widehat{\phi}_t)\text{a.e.}t \in [t_m, T]\}.$$

Let

$$\mathcal{D}_{t_m} = \{z \in ([t_m, T], \mathcal{H}), z(t_m^+) \text{ exists }, \sup_{t\in[t_m,T]} E(|y(t)|^2) < \infty)\}.$$

Set

$$\mathcal{D}_{t_m}^* = \mathcal{D} \cap \mathcal{D}_{m-1} \cap \mathcal{D}_m,$$

and $\mathcal{D}_{t_m}^m = \left\{z \in \mathcal{D}_{t_m}^*, \quad \text{such that} \quad z_{t_m} = 0\right\}$. Then, there exists a fixed point z_m of $\widehat{\Phi}_m$, The fixed point z of the operator $\widehat{\Phi}$ is then defined by

$$z(t) = \begin{cases} z_0(t), & \text{if } t \in (-\infty, t_1], \\ z_1(t), & \text{if } t \in (t_1, t_2], \\ . \\ . \\ . \\ z_m(t), & \text{if } t \in (t_m, t_{m+1}], \end{cases}$$

Hence, Φ has a fixed point y that is a solution to the problem (5.1). This completes the proof.

□

5.2 Existence results for the nonconvex case

In this section we present a result for the problem (5.1) in the spirit of the nonlinear alternative of Leray-Schauder type (Lemma 2.3.2), for single-valued maps, combined with a selection theorem due to Bressan and Colombo [21] for lower semicontinuous multivalued maps with decomposable values.

Let $\mathcal{A}$ be a subset of $J \times \mathcal{D}$. $\mathcal{A}$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if $\mathcal{A}$ belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$ where $\mathcal{J}$ is Lebesgue measurable in J and $\mathcal{D}$ is Borel measurable in $\mathcal{D}$. A subset $\mathcal{A}$ of $L^2(J, \mathcal{H})$ is decomposable if for all w, $v \in \mathcal{A}$ and $\mathcal{J} \in J$ measurable, $w\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in A$, where χ stands for the characteristic function.

Let $G : \mathcal{H} \longrightarrow \mathcal{P}(\mathcal{H})$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in \mathcal{H} : G(y) \cap B \neq \emptyset\}$ is open for any open set B in $\mathcal{H}$.

Definition 5.2.1. *Let Y be a separable metric space and let $N : Y \longrightarrow \mathcal{P}(L^2(J, \mathcal{H})$ be a multivalued operator. We say that N has property (BC) if*

1) *N is lower semi-continuous (l.s.c.);*

2) *N has nonempty closed and decomposable values.*

Let $F : J \times \mathcal{D} \longrightarrow \mathcal{P}(\mathcal{H}))$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\Psi : D_T^* \longrightarrow \mathcal{P}(L^2(J, \mathcal{H})),$$

by letting

$$\Psi(y) = \{f \in L^2(J, \mathcal{H}) : f(t) \in F(t, y_t) \text{ for a.e. } t \in J\}.$$

The operator Ψ is called the Niemytzki operator associated to F.

Definition 5.2.2. *Let $F : J \times \mathcal{H} \longrightarrow \mathcal{P}(\mathcal{H})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator Ψ is lower semi-continuous and has nonempty closed and decomposable values.*

Next we state a selection theorem due to Bressan and Colombo [21].

Theorem 5.2.1. *Let Y be a separable metric space and let $N : Y \longrightarrow \mathcal{P}(L^2(J, \mathcal{H}))$ be a multivalued operator which has property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $\widetilde{g} : Y \longrightarrow L^2(J, \mathcal{H})$ such that $\widetilde{g}(y) \in N(y)$ for every $y \in Y$.*

Let us introduce the following hypotheses which are assumed hereafter:

(H6) $F : J \times \mathcal{D} \longrightarrow \mathcal{P}(\mathcal{H})$ is a nonempty compact valued multivalued map such that:

- **a)** $(t, y) \longmapsto F(t, y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable and for every $t \in J$, the multifunction $t \to F(t, y_t)$ is measurable.
- **b)** $(t, y) \longmapsto F(t, y)$ is lower semi-continuous for a.e. $t \in J$.

(H7) $E|F(t, \Theta)|^2 = \sup\{E(|f|)^2 : f \in F(t, \Theta)\}\} \leq \eta(t)p(\|\Theta\|_{\mathcal{D}}^2), \quad t \in J, \quad \Theta \in \mathcal{D}$,
where $\eta \in L^2(J, \mathbb{R}^+)$ and $p : \mathbb{R}^+ \longrightarrow (0, \infty)$ is continuous and increasing with

$$8\widetilde{K}^2 MT \int_{t_0}^{T} \eta(s)ds < \int_{N}^{\infty} \frac{du}{p(u)},$$

$$N = 4\widetilde{K}^2\Big[4Mm\sum_{k=1}^{m} c_k + 8\widetilde{K}^2 MT^{2H-1}\int_0^T \|g(s)\|^2_{L_Q^0} ds\Big] + C.$$

The following lemma is crucial in the proof of our main theorem.

Lemma 5.2.1. *[32] Let $F : J \times \mathcal{D} \longrightarrow \mathcal{P}_{cp}(\mathcal{H})$ be a multivalued map. Assume that $(H6)$ and $(H7)$ hold. Then F is of l.s.c. type.*

Theorem 5.2.2. *Assume that hypotheses $(H1)-(H3)$ and $(H6)-(H7)$ hold. Then the impulsive initial value problem (5.1) has at least one solution.*

Proof. Consider the operator $G : \mathcal{D}_T^* \to \mathcal{P}(\mathcal{D}_T^*)$ defined by

$$G(y) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ S(t)\phi(0) + \displaystyle\int_0^t S(t-s)f(s)ds & \\ + \displaystyle\int_0^t S(t-s)g(s)dB^H(s) + \sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-)), & \text{if } t \in [0,T] \end{cases}$$

Let $\widehat{\phi} : (-\infty, T] \longrightarrow \mathcal{H}$ be the function defined by

$$\widehat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ S(t)\phi(0), & t \in [0,T]; \end{cases}$$

Then $\widehat{\phi}$ is an element of $\mathcal{D}_T^*$ and $\widehat{\phi}_0 = \phi$. It is evident that z satisfies $z_0 = 0, t \in (-\infty, 0]$ and

$$z(t) = \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dB_Q^H(s)$$
$$+ \sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-) + \widehat{\phi}(t_k^-)), t \in J.$$

where $f(t) \in F(t, z_t + \widehat{\phi}_t)$ for a.e. $t \in [0,T]$.
Set $\mathcal{D}_T^{**} = \{z \in \mathcal{D}_T^*, \quad \text{such that} \quad z_0 = 0 \in \mathcal{D}\}$ and for any $z \in \mathcal{D}_T^{**}$ we have

$$\|z\|_{\mathcal{D}_T^{**}} = \|z_0\|_{\mathcal{D}} + \sup_{t\in[0,T]} (E\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t\in[0,T]} (E\|z(t)\|^2)^{\frac{1}{2}}$$

thus $(\mathcal{D}_T^{**}, \|.\|_{\mathcal{D}_T^{**}})$ is a Banach space.

We define the Niemytzki operator associated to F,

$$\Psi : \mathcal{D}_T^{**} \longrightarrow \mathcal{P}(L^2(J, \mathcal{H})),$$

by letting

$$\Psi(z) = \{f \in L^2(J, \mathcal{H}) : f(t) \in F(t, z_t + \widehat{\phi}_t) \text{ for a.e. } t \in J\}$$

be a selection set of Ψ. From $(H6)$ and $(H7)$, and thanks to Lemma 5.2.1, we deduce that F is of lower semi-continuous type. Then, from Theorem 5.2.1 there exists a continuous function $f : \mathcal{D}_T^{**} \longrightarrow L^2(J, \mathcal{H})$ such that $f(z) \in \Psi(z + \widehat{\phi})$ for all $z \in \mathcal{D}_T^{**}$.

Consider the problem,

$$\begin{cases} dz(t) \in [Az(t) + f(z_t + \widehat{\phi}_t)]dt + g(t)dB_Q^H(t), & t \in J = [0, T], \quad t \neq t_k, \\ z(t_k^+) - z(t_k^-) = I_k(z(t_k^-) + \widehat{\phi}(t_k^-)), \ k = 1, \dots, m, & \\ z(t) = 0, & J_0 = (-\infty, 0], \end{cases} \tag{5.15}$$

and consider the operator $\widehat{G}_1 : \mathcal{D}_T^{**} \to \mathcal{D}_T^{**}$ defined by

$$\widehat{G}_1(z) = \left\{ \begin{array}{l} 0, \qquad \text{if } t \in (-\infty, 0], \\ \displaystyle\int_0^t S(t-s) f(z_s + \widehat{\phi}_s) ds + \int_0^t S(t-s) g(s) dB^H(s) \\ \displaystyle + \sum_{0 < t_k < t} S(t - t_k) I_k(z(t_k^-) + \widehat{\phi}(t_k^-)), \quad \text{if } \ t \in [0, T], \end{array} \right\},$$

where $f(z) \in \Psi(z + \widehat{\phi})$.

Clearly, if $z \in \mathcal{D}_T^{**}$ is fixed point of $\widehat{G}_1$, then z is fixed fixed point of $\widehat{G}$. Thus, there exists $y \in \mathcal{D}_T^{*}$ such that y is a fixed point of the operator G,

It is also straightforward that $\widehat{G}_1$ is completely continuous and there exists $M_T > 0$ such that for every solution of $z = \lambda \widehat{G}_1(z)$, for some $\lambda \in (0, 1)$, we have $\|z\|_{\mathcal{D}_T^{**}} \leq M_T$. Set

$$U = \{z \in \mathcal{D}_T^{**} : \|z\|_{\mathcal{D}_T^{**}} < M_T + 1\}.$$

We see that U is an open set in $\mathcal{D}_T^{**}$. From the choice of U there is no $z \in \partial U$ such that $z = \lambda \widehat{G}_1(z)$ for some $\lambda \in (0, 1)$. As a consequence the lemma 2.3.2, we deduce that $\widehat{G}_1(z)$ has a fixed point z in $\overline{U}$. Hence, G has a fixed point y that is a solution to the problem (5.1)

□

Now we present a second result for the problem (5.1) with a nonconvex valued right-hand side. Our considerations are based on a fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 (see also Deimling [27], Theorem 11.1).

Lemma 5.2.2. *[27] Let (X, d) be a complete metric space. If the multivalued operator $G : X \longrightarrow \mathcal{P}_{cl}(X)$ is a contraction , then G has at least one fixed point.*

Let us introduce the following hypotheses:

(H8) $F : J \times \mathcal{D} \longrightarrow \mathcal{P}_{cp}(\mathcal{H}); (t, y) \longrightarrow F(t, y)$ is measurable for each $y \in \mathcal{D}$.

(H9) There exists a function $l_f \in L^2(J, \mathbb{R}^+)$ (denote $l^* = \int_0^T l^2(t)dt$) ,such that

$$EH_d^2(F(t, x), F(t, y)) \leq l_f(t)\|x - y\|_{\mathcal{D}}^2 \text{ for each } x, y \in \mathcal{D}, t \in J.$$

Theorem 5.2.3. *Assume that hypotheses $(H1)$, $(H2)$ and $(H8)$ hold. Then the problem (5.1) has at least one mild solution.*

Although the proof follows the same steps than the one of theorem 5.2.2, the technical details necessary for the proof are different.

Proof. The proof will be given in several steps.

Consider the problem (5.1) on $(-\infty, t_1]$:

$$\begin{array}{ll} dy(t) \in [Ay(t) + F(t, y_t)]dt + g(t)dB_Q^H(t), & \text{if } t \in [0, t_1], \\ y(t) = \phi(t), & \text{if } t \in (-\infty, 0]. \end{array} \tag{5.16}$$

Let

$$\mathcal{D}_{t_0} = \{y \in C([0, t_1], \mathcal{H}) : \sup_{t \in [0, t_1]} E(|y(t)|^2) < \infty\},$$

and set

$$\mathcal{D}_{t_0}^* = \mathcal{D} \cap \mathcal{D}_{t_0}.$$

We transform problem (5.3) into a fixed point one. Consider the multivalued operator $\Phi_0 : \mathcal{D}_{t_0}^* \to \mathcal{P}(\mathcal{D}_{t_0}^*)$ defined by

$$\Phi_0(y) = \left\{ \rho \in \mathcal{D}_{t_0}^* : \quad \rho(t) = \left\{ \begin{array}{l} \phi(t), \qquad \text{if } t \in (-\infty, 0], \\ S(t)\phi(0) + \displaystyle\int_0^t S(t - s)f(s)ds \\ + \displaystyle\int_0^t S(t - s)g(s)dB^H(s), \quad \text{if } t \in [0, t_1], \end{array} \right. \right\}$$

where $f \in N_{F,y} = \{f \in L^2([0,t_1],\mathcal{H}) : f(t) \in F(t,y_t) \quad \text{for a.e. } t \in [0,t_1]\}$.

We will now prove that Φ has a fixed point. Let $\widehat{\phi} : (-\infty, t_1] \longrightarrow \mathcal{H}$ be the function defined by

$$\widehat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ S(t)\phi(0), & t \in [0,t_1]; \end{cases}$$

Then $\widehat{\phi}$ is an element of $\mathcal{D}^*_{t_0}$ and $\widehat{\phi}_0 = \phi$. Let $y(t) = z(t) + \widehat{\phi}(t), -\infty < t \le t_1$.

Obviously, if y satisfies the integral equation

$$y(t) = S(t)\phi(0) + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dB^H_Q(s), t \in [0,t_1], \tag{5.17}$$

then z satisfies $z_0 = 0, t \in (-\infty, 0]$ and

$$z(t) = \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dB^H_Q(s), t \in [0,t_1], \tag{5.18}$$

where $f(t) \in F(t, z_t + \widehat{\phi}_t)$ for a.e. $t \in [0,t_1]$.
Set $\mathcal{D}^0_{t_0} = \left\{z \in \mathcal{D}^*_{t_0}, \text{ such that } z_0 = 0 \in \mathcal{D}\right\}$ and for any $z \in \mathcal{D}^0_{t_0}$ we have

$$\|y\|_{\mathcal{D}^0_{t_0}} = \|z_0\|_{\mathcal{D}} + \sup_{t\in[0,t_1]} (E\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t\in[0,t_1]} (E\|y(t)\|^2)^{\frac{1}{2}}$$

thus $(\mathcal{D}^0_{t_0}, \|.\|_{\mathcal{D}^0_{t_0}})$ is a Banach space. Let the operator $\widehat{\Phi}_0 : \mathcal{D}^0_{t_0} \to \mathcal{P}(\mathcal{D}^0_{t_0})$ defined by

$$\widehat{\Phi}_0(z) = \left\{ \widehat{\rho} \in \mathcal{D}^0_{t_0} : \ \widehat{\rho}(t) = \left\{ \begin{array}{ll} 0, & \text{if;} t \in (-\infty, 0], \\ \displaystyle\int_0^t S(t-s)f(s)ds & \\ \displaystyle+ \int_0^t S(t-s)g(s)dB^H(s), & \text{if } t \in [0,t_1]. \end{array} \right. \right\}$$

We shall show that $\widehat{\Phi}_0$ satisfies the assumptions of Lemma 5.2.2. The proof will be given in two claims.

Claim 1. $\widehat{\Phi}_0(z) \in \mathcal{P}_{cl}(\mathcal{D}^0_{t_0})$ for each $z \in \mathcal{D}^0_{t_0}$.

Let $z_n \in \widehat{\Phi}_0(z)$ and $\|z_n - z\|^2_{\mathcal{D}^0_{t_0}} \longrightarrow 0$. $z \in \mathcal{D}^0_{t_0}$ and there exist $f \in N_{F,z+\widehat{\phi}}$, such that

$$z_n(t) = \int_0^t S(t-s)f_n(s)ds + \int_0^t S(t-s)g(s)dB^H_Q(s).$$

Since $F(t, z(t) + \widehat{\phi}(t))$ is compact values and from (H8), we may pass to a subsequence if necessary to get that f_n converges to f in $L^2(J, \mathcal{H})$. Then, for each $t \in [0, t_1]$,

$$E\Big|z_n(t) \quad - \int_0^t S(t-s)f(s)ds - \int_0^t S(t-s)g(s)dB^H_Q(s)\Big|^2 \\ \longrightarrow 0, \text{ as } n \longrightarrow 0.$$

So we have that there exists a $f(\cdot) \in N_{F,z+\widehat{\phi}}$ such that

$$z(t) = \int_0^t S(t-s)f(s)ds + S(t-s)g(s)dB^H_Q(s)$$

which implies $z \in \widehat{G}(z)$.

Claim 2. There exists $\gamma < 1$, such that

$$EH^2_d(\widehat{\Phi}_0(z_1), \widehat{\Phi}_0(z_2)) \leq \gamma\|z_1 - z_2\|_{\mathcal{D}^0_{t_0}}$$

for any $z_1, z_2 \in \mathcal{D}^0_{t_0}$. For all $h_1 \in \widehat{\Phi}_0(z_1)$, there exists $f_1(\cdot) \in N_{F,z_1+\widehat{\phi}}$, such that

$$h_1(t) = \int_0^t S(t-s)f_1(s)ds + \int_0^t S(t-s)g(s)dB^H_Q(s)$$

For $H_d(F(t, z_1(t)+\widehat{\phi}(t)), F(t, z_2(t)+\widehat{\phi})) \leq l(t)|z_1(t)-z_2(t)|$, there exists $f_2(t) \in N_{F,z_2}$, such that

$$E|f_1(t) - f_2(t)|^2 \leq l_f(t)\|z_1(t) - z_2(t)\|, \text{ a.e. } t \in [0, t_1].$$

Define

$$h_2(t) = \int_0^t S(t-s)f_2(s)ds + \int_0^t S(t-s)g(s)dB^H_Q(s)$$

and we have

$$\begin{aligned} E|h_1(t)-h_2(t)|^2 &\leq E\Big|\int_0^t S(t-s)(f_1(s)-f_2(s))ds\Big|^2 \\ &\leq E\Big|\int_0^t S(t-s)(f_1(s)-f_2(s))ds\Big|^2 \\ &\leq MT\int_0^t l^2(s)E|z_1-z_2|^2ds \\ &\leq \int_0^t \bar{l}(s)E|z_1-z_2|^2ds \\ &\leq \tfrac{1}{\tau}e^{\tau L(t)}\|z_1-z_2\|_*^2. \end{aligned}$$

Thus,

$$e^{-\tau L(t)}E|h_1(t)-h_2(t)|^2 \leq \frac{1}{\tau}\|z_1-z_2\|_*^2$$

Therefore,

$$\|h_1(t)-h_2(t)\|_*^2 \leq \frac{1}{\tau}\|z_1-z_2\|_*^2,$$

where $L(t) = \displaystyle\int_0^t L(s)ds$, $\bar{l}(t) = MTl^2(t)$ and $\|\cdot\|_*$ denote the Bielecki-type [15] norm on $C([0,t_1],\mathcal{H})$ defined by

$$\|y\|_*^2 = \sup_{t\in[0,t_1]} E|y(t)|^2e^{-\tau L(t)}, \qquad \tau > 1$$

By an analogous relation, obtained by interchanging the roles of z_1 and z_2, it follows that

$$EH_d^2(\widehat{\Phi}_0(z_1)-\widehat{\Phi}_0(z_2)) \leq \frac{1}{\tau}\|z_1-z_2\|_*^2$$

So, $\widehat{\Phi}_0$ is a contraction, and thus, by Lemma 5.2.2, $\widehat{\Phi}_0$ has a fixed point z, so the problem (5.16) has at least one solution. Denote this solution by y_0.

Step 2. Consider now the problem,

$$\begin{array}{ll} dy(t) \in [Ay(t)+F(t,y_t)]dt + g(t)dB_Q^H(t), & \text{if } t\in(t_1,t_2], \\ y(t_1^+)-y_0(t_1^-) = I_1(y_0(t_1^-)), \quad y(t) = y_0(t) & \text{if;} t\in(-\infty,t_1]. \end{array} \tag{5.19}$$

Let

$$\mathcal{D}_{t_1} = \{y \in C([t_1,t_2],\mathcal{H}) : y(t_1^+) \text{ exists }, \sup_{t\in[t_1,t_2]} E(|y(t)|^2) < \infty)\},$$

and set

$$\mathcal{D}_{t_1}^* = \mathcal{D} \cap \mathcal{D}_{t_0} \cap \mathcal{D}_{t_1}.$$

Consider the operator $\Phi_1 : \mathcal{D}_{t_1}^* \longrightarrow \mathcal{P}(\mathcal{D}_{t_1}^*)$ defined by,

$$\Phi_1(y) = \left\{ \rho_1 \in \mathcal{D}_{t_1}^* : \ \rho_1(t) = \left\{ \begin{array}{ll} y_0(t), & \text{if } t \in (-\infty, t_1], \\ y_0(t_1^-) + S(t-t_1)I_1(y_0(t_1^-)) & \\ + \displaystyle\int_{t_1}^t S(t-s)f(s)ds & \\ + \displaystyle\int_{t_1}^t S(t-s)g(s)dB^H(s), & \text{if } t \in (t_1,t_2], \end{array} \right. \right\},$$

where $f \in N_{F,y} = \{f \in L^2([t_1,t_2],\mathcal{H}) : f(t) \in F(t,y_t) \ \text{ for } \ \text{a.e.} t \in [t_1,t_2]\}$.

Let $\widehat{\phi}(\cdot) : (-\infty, t_2] \longrightarrow \mathcal{H}$ be the function defined by

$$\widehat{\phi}(t) = \left\{ \begin{array}{ll} y_0(t), & \text{if;} t \in (-\infty, t_1], \\ y_0(t_1^-) + S(t-t_1)I_1(y_0(t_1^-)), & \text{if } t \in (t_1,t_2]. \end{array} \right.$$

Then $\widehat{\phi}_{t_1}$ is an element of $\mathcal{D}_{t_1}^*$ and $\widehat{\phi}_{t_1} = y_0$.

Let $y(t) = z(t) + \widehat{\phi}(t), t_1 < t \leq t_2$

Obviously, if y satisfies the integral equation

$$\begin{aligned} y(t) \ &= \ y_0(t_1^-) + S(t-t_1)I_1(y_0(t_1^-)) + \int_{t_1}^t S(t-s)f(s)ds \\ &\quad + \int_{t_1}^t S(t-s)g(s)dB_Q^H(s), t \in [t_1,t_2], \end{aligned} \tag{5.20}$$

then z satisfies $z(t_1) = 0, t \in (-\infty, t_1]$ and

$$\begin{aligned} z(t) \ &= \ \int_{t_1}^t S(t-s)f(s)ds + \int_{t_1}^t S(t-s)g(s)dB_Q^H(s) \\ &\quad + S(t-t_1)I_1(z_0(t_1^-) + \widehat{\phi}(t_1^-)), \quad t \in [t_1,t_2]. \end{aligned} \tag{5.21}$$

Set $\mathcal{D}^1_{t_1} = \{z \in \mathcal{D}^*_{t_1}, \quad \text{such that} \quad z_{t_1} = 0\}$, and let the operator $\widehat{\Phi}_1 : \mathcal{D}^1_{t_1} \to \mathcal{P}(\mathcal{D}^1_{t_1})$ be defined by

$$\widehat{\Phi}_1(z) = \left\{ \widehat{\rho} \in \mathcal{D}^1_{t_1} : \ \widehat{\rho}(t) = \left\{ \begin{array}{l} 0, \qquad \text{if } t \in (-\infty, t_1], \\ \displaystyle\int_{t_1}^{t} S(t-s)f(s)ds + \int_{t_1}^{t} S(t-s)g(s)dB^H(s) \\ +S(t-t_1)(I_1(z(t_1^-)) + \phi(t_1^-)), \quad \text{if } t \in [t_1, t_2]. \end{array} \right. \right\},$$

where

$$f \in N_{F,z+\widehat{\phi}} = \{f \in L^2([t_1,t_2],\mathcal{H}) : f(t) \in F(t, z_{t_1} + \widehat{\phi}_{t_1}) \text{ a.e.} t \in [t_1,t_2]\}.$$

Similar to Step 1, we can prove that the problem (5.19) has at least one solution, which we denote by y_1.

Step 3. We continue this process and take into account that $z_m := z|_{[t_m,T]}$ is a fixed point of the operator $\widehat{\Phi}_m : \mathcal{D}^m_{t_m} \to \mathcal{P}(\mathcal{D}^m_{t_m})$ defined by

$$\widehat{\Phi}_m(z) = \left\{ \widehat{\rho} \in \mathcal{D}^m_{t_m} \ \widehat{\rho}(t) = \left\{ \begin{array}{l} 0, \qquad \text{if } t \in (-\infty, t_m], \\ \displaystyle\int_{t_m}^{t} S(t-s)f(s)ds + \int_{t_m}^{t} S(t-s)g(s)dB^H(s) \\ +S(t-t_m)(I_m(z_{m-1}(t_m^-)) + \phi(t_m^-)), \quad \text{if } t \in [t_m, T]. \end{array} \right. \right\}$$

where

$$f \in N_{z+\widehat{\phi}} = \{f \in L^2([t_m,T],\mathcal{H}) : f(t) \in F(t, z_{t_t} + \widehat{\phi}_t) \text{ a.e.} t \in [t_m,T]\}.$$

Let

$$\mathcal{D}_{t_m} = \{z \in ([t_m,T],\mathcal{H}), z(t_m^+) \text{ existe }, \sup_{t\in[t_m,T]} E(|y(t)|^2) < \infty)\},$$

set

$$\mathcal{D}^*_{t_m} = \mathcal{D} \cap \mathcal{D}_{m-1} \cap \mathcal{D}_m$$

and $\mathcal{D}^m_{t_m} = \{z \in \mathcal{D}^*_{t_m}, \quad \text{such that} \quad z_{t_m} = 0\}$.

Then, there exists a fixed point z_m of $\widehat{\Phi}_m$ and the fixed point z of the operator $\widehat{\Phi}$ is then defined by

$$z(t) = \left\{ \begin{array}{ll} z_0(t), & \text{if } t \in (-\infty, t_1], \\ z_1(t), & \text{if } t \in (t_1, t_2], \\ . & \\ . & \\ . & \\ z_m(t), & \text{if } t \in (t_m, t_{m+1}], \end{array} \right.$$

has at least one solution, which we denote by

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in (-\infty, t_1], \\ y_1(t), & \text{if } t \in (t_1, t_2], \\ . \\ . \\ . \\ y_m(t), & \text{if } t \in (t_m, t_{m+1}], \end{cases}$$

□

Conclusion and Perspective

In this thesis, we have presented some results to the theory of existence and attractivity of mild solutions of some classes of stochastic semilinear functional and neutral functional differential equations and inclusions driven by fractional Brownian motion with the Hurst index $H > \frac{1}{2}$ with finite and infinite delay .

We plan to extend the results presented in this thesis by considering stochastic semilinear functional differential equations and inclusions and functional fractional differential equations and inclusions with not instantaneous impulses. This study has been started recently by Hernández and O'Regan [46] and Yan and Lu [82].

Bibliography

[1] N. U. Ahmed, Nonlinear stochastic differential inclusions on Bananch space, *Stochastic Anal. Appl.* **12** (1994), 1-10.

[2] N. U. Ahmed, Existence of solutions of nonlinear stochastic differential inclusions on Banach space , in: V. Lakshmikantham (Ed.), Proceedings of the First World Congress of Nonlinear Analysis, Tampa, FL, August (1992) 19-26, de Gruyter, Berlin, (1996), pp. 1699-1712.

[3] A. Arara, M. Benchohra, L. Górniewicz and A. Ouahab, Controllability results for semilinear functional differential inclusions with unbounded delay, *Math. Bulletin*, **3** (2006), 157–183.

[4] E. Alòs, O. Mazet and D. Nualart, *Stochastic calculus with respect to Gaussian processes. Ann Probab*, **29** (1999), 766-801.

[5] E. Alòs and D. Nualart, An extension of Itô's formula for anticipating processes. *J. Theoret. Probab.* 11 (1998) 493-514.

[6] C. Avramescu, Some remarks on a fixed point theorem of Krasnoselskii, *Electron. J. Qual. Theory Differ. Equ.* **5** (2003), 1-15.

[7] D.D. Bainov and P.S. Simeonov, *Systems with Impulsive Effect*, Horwood, Chichester, 1989.

[8] P. Balasubramaniam, Existence of solutions of functional stochastic differential inclusions, *Tamkang J. Math.* **33** (2002), 35-43.

[9] P. Balasubramaniam, S. K. Ntouyas and D. Vinayagam, Existence of solutions of semilinear stochastic delay evolution inclusions in a Hilbert space. *J. Math. Anal. Appl.* **305** (2005), 438-451.

[10] P. Balasubramaniam and D. Vinayagam, Existence of solutions of nonlinear neutral stochastic differential inclusions in a Hilbert Space. *Stochastic Analysis and Applications*, **23** (2005), 137-151.

[11] Benchohra M, Ntouyas S K, Ouahab A. On nondensely defined semilinear stochastic functional differential equations with nonlocal conditions. *J Appl Math Stoch Anal*, 2006, Art ID 69584

[12] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive differential equations and inclusions*, Vol. 2, Hindawi Publishing Corporation, New York, 2006.

[13] M. Benchohra and S. K. Ntouyas, Existence of mild solutions for certain delay semilinear evolution inclusion with nonlocal conditions, *Dynam. systems Appl.* **9** (3) (2000), 405–412.

[14] K. Bichteler, *Stochastic Integration with jumps*, Cambridge University Press, 2002.

[15] A. Bielecki, Une remarque sur la métode de Banach-Cacciopoli-Tikhonov dans la théorie des équations différentielles ordinaires, *Bull. Acad. Polon. Sci.* **4** (1956), 261-264.

[16] T. Björk and H. Hult, A note on Wick products and the fractional Black-Scholes model. *Finance and Stochastics*, **9** (2005) 197-209.

[17] A. Boudaoui, T. Caraballo and A. Ouahab, Existence of mild solutions to stochastic delay evolution equations with a fractional Brownian motion and impulses, *Stoch. Anal. Appl.* **33** (2015), 244-258.

[18] A. Boudaoui, T. Caraballo and A. Ouahab, Impulsive neutral functional differential equations driven by a fractional Brownian motion with unbounded delay. *Applicable Analysis* (2015), http://www.tandfonline.com/doi/full/10.1080/00036811.2015.1086756.

.

[19] A. Boudaoui, T. Caraballo and A. Ouahab, Impulsive stochastic functional differential inclusions driven by a fractional Brownian motion with infinite delay. *Mathematical methods in the Applied Sciences* (2015), http://onlinelibrary.wiley.com/doi/10.1002/mma.3580/abstract.

[20] B. Boufoussi and S. Hajji, Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space. *Statist Probab Lett*, **82** (2012), 1549-1558.

[21] A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.* **90** (1988), 69-86.

[22] T. A. Burton and C. Kirk, A fixed point theorem of Krasnoselskii type, *Math. Nachrichten*, **189** (1998), 23–31.

[23] T. Caraballo and M. Abdoul Diop, Neutral stochastic delay partial functional integro-differential equations driven by a fractional Brownian motion *Front. Math. China* **8** (2013), 745-760.

[24] T. Caraballo, M. J. Garrido-Atienza and T. Taniguchi, The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion. *Nonlinear Anal.*, **74** (2011), 3671-3684.

[25] P.Cheridito, Mixed fractional Brownian motion. *Bernoulli* 7, 913-934, 2001.

[26] L. Decreusefond and A. S. Üstünel, Stochastic analysis of the fractional Brownian motion. *Potential Analysis* **10** (1998) 177–214.

[27] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York, 1992.

[28] B.C. Dhage and V. Lakshmikantham, On global existence and attractivity results for nonlinear functional integral equations, *Nonlinear Anal.*, **72** (2010) 2219–2227.

[29] S. Djebali, L. Gorniewicz and A. Ouahab, *Solutions Sets for Differential Equations and Inclusions*, De Gruyter Series in Nonlinear Analysis and Applications **18**. Berlin: de Gruyter, 2013.

[30] J. Dugundji and A. Granas, *Fixed point theory*, Springer-Verlag, New York, 2003.

[31] L. Erbe, H. I. Freedman, X. Z. Liu and J. H. Wu, Comparison principales for impulsive parabolic equations with applications to models of singles species growth, *J. Austral. Math. Soc.* Ser. B **32** (1991), 382-400.

[32] M. Frigon and A. Granas, Théorèmes d'existence pour des inclusions différentielles sans convexit, *C. R. Acad. Sci. Paris, Ser. 1* **310** (1990), 819-822.

[33] R. K. George, D. N. Chalishajar and A. K. Nandakumaran, Controllability of second order semi-linear neutral functional differential inclusions in Banach spaces, *Mediterr. J. Math.*, **1** (2004), 463-–477.

[34] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Univ. Press, New York, 1985.

[35] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Mathematics and its Applications, 495, Kluwer Academic Publishers, Dordrecht, 1999.

[36] J. R. Graef, J. Henderson and A. Ouahab, Impulsive differential inclusions. A fixed point approach. De Gruyter Series in Nonlinear Analysis and Applications 20. *Berlin*: de Gruyter, 2013.

[37] P. Guasoni, No arbitrage under transaction costs, with fractional Brownian motion and beyond. *Math. Finance* **16** (2006) no. 3, 569-582.

[38] P. Guasoni, M. Rasonyi, W. Schachermayer, The fundamental theorem of asset pricing for continuous processes under small transaction costs. *Ann. Finance.* **6** (2010), no. 2, 157-191.

[39] A. Halanay and D. Wexler, *Teoria calitativa a systeme cu impulduri*, Editura Republicii Socialiste Romania, Bucharest, 1968.

[40] J. K. Hale, *Theory of Functional Differential Equations,* Springer-Verlag, New York, 1977.

[41] J.K. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.*, **21** (1978), 11-41.

[42] J. K. Hale and K. R. Meyer, A class of functional equations of neutral type, *Mem. Amer. Math. Soc.* **76** (1967), 1-65.

[43] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equation,* Applied Mathematical Sciences 99, Springer-Verlag, New York, 1993.

[44] S. Heikkila, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker, Inc., New York, 1994.

[45] E. Hernández, Existence results for partial neutral functional integrodifferential equations with unbounded delay, *J. Math. Anal. Appl.* **292** (2004), 194-210.

[46] E. Hernández, D. O'Regan, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.* S 0002–9939 (2012) 11613-11612.

[47] E. Hernández, A. Prokopczyk and L. Ladeira, A note on partial functional differential equations with state-dependent delay, *Nonlin. Anal.* **7** (2006), 510–519.

[48] Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Infinite Delay*, in: Lecture Notes in Mathematics, vol. 1473, Springer-Verlag, Berlin, 1991.

[49] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis*, Volume I: Theory, Kluwer Academic Publishers, Dordrecht, 1997.

[50] F. Jiang and Y. Shen, A note on the existence and uniqueness of mild solutions to neutral stochastic partial functional differential equations with non-Lipschitz coefficients. *Comput Math Appl*, **61** (2011), 1590-1594.

[51] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht 1991.

[52] O. Knill, *Probability and Stochastic Processes with Applications*, Overseas Press, 2009.

[53] A.N. Kolmogorov, Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *C.R.(Doklady) Acad. URSS* **26**, 115-118, 1940.

[54] V. Kolmanovskii, and A. Myshkis, *Introduction to the Theory and Application of Functional-Differential Equations,* Kluwer Academic Publishers, Dordrecht, 1999.

[55] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, The Netherlands, 1991.

[56] V.Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations. World Scientific*, Singapore 1989.

[57] V. Lakshmikantham, L. Wen and B. Zhang, *Theory of Differential Equations with Unbounded Delay*, Kluwer Acad. Publ., Dordrecht, 1994.

[58] A. Lasota and Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Pol. Sci. Ser. Sci. Math.Astronom. Phys.* **13** (1965), 781-786.

[59] B. B. Mandelbrot and J.W. Van Ness, Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **10**, 422-437, 1968

[60] J. Memin, Y. Mishura, and E. Valkeila, Inequalities for the moments of Wiener integrals with respecto to fractional Brownian motions. *Statist. Prob. Letters* **55** (2001) 421-430.

[61] V.D. Milman and A.A. Myshkis, On the stability of motion in the presence of impulses, *Sib. Math. J.* (in Russian) **1** (1960) 233-237.

[62] V. D. Milman and A. A. Myshkis, Randorn impulses in linear dynamical systems, in "Approximante Methods for Solving Differential Equations," *Publishing house of the Academy of Sciences of Ukainian SSR, Kiev*, in Russian, (1963) 64-81.

[63] Y. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Topics.* Lecture Notes in Mathematics, Vol 1929. Berlin: Springer-Verlag, 2008.

[64] H. W. Ning and B. Liu, Existence results for impulsive neutral stochastic evolution inclusions in Hilbert Space, *Acta Mathematica Sinica,* **27** (2011), 1405-1418.

[65] I. Norros, E. Valkeila, and J. Virtamo, An elementary approach to a Girsanov formula and other analytical resuls on fractional Brownian motion. *Bernoulli* **5** (1999) 571–587.

[66] D. Nualart, Stochastic integration with respect to fractional Brownian motion and applications. Stochastic Models (Mexico City, 2002), Contemp. Math. 336 *Amer. Math. Soc. Providence, RI*, 3-39, 2003.

[67] D. Nualart, *The Malliavin Calculus and Related Topics*. 2nd ed. Berlin: Springer- Verlag, 2006 .

[68] T. Sottinen, E. Valkeila, On arbitrage and replication in the fractional Black-Scholes pricing model. *Statistics & Decisions*, **21** (2003), 93-107.

[69] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Springer-Verlag, New York, 1983.

[70] V. Pipiras and M. S. Taqqu, Integration questions related to fractional Brownian motion. *Probab. Theory Rel. Fields* 118 (2000) 121-291.

[71] V. Pipiras and M. S. Taqqu, Are classes of deterministic integrands for fractional Brownian motion on a interval complete? *Bernoulli* 7 (2001) 873–897

[72] P. Protter, *Stochastic Integration and Differential Equations*, Springer Verlag, 1990.

[73] P. Revathi, R. Sakthivel, D. Y. Song, Y. Ren and P. Zhang, Existence and stability results for second-order stochastic equations driven by fractional Brownian motion, *Transport Theory Statist. Phys.* **42** (2013), 299–317.

[74] S. G. Samko, A. A. Kilbas O. L. and Marichev: *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, 1993.

[75] G. Samorodnitsky, and M. S. Taqqu, *Stable non-Gaussian random processes. Stochastic models with infinite variance*. Stochastic Modeling. Chapman and Hall, New York, 1994.

[76] A.M. Samoilenko and N.A. Perestyuk, *Impulsive Differential Equations.* World Scientific, Singapore 1995.

[77] G. V. Smirnov, *Introduction to the theory of differential inclusions*, Graduate studies in mathematics, vol. 41, American Mathematical Society, Providence, 2002.

[78] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, *Trans. Amer. Math. Soc.*, **200** (1974), 395–418.

[79] C. C. Travis and G.F. Webb, Second order differential equations in Banach spaces, in: Nonlinear Equations in Abstract Spaces, *Proc. Internat. Sympos. (Univ. Texas, Arlington, TX, 1977), Academic Press, New-York*, 1978, 331–361.

[80] S. Tindel, C. Tudor and F. Viens, Stochastic evolution equations with fractional Brownian motion. *Probab Theory Related Fields*,**127** (2003), 186-204.

[81] A. Tolstonogov, *Differential Inclusions in a Banach Space*, Kluwer Academic Publishers, Dordrecht, (2000).

[82] Z. Yan and F. Lu, Existence results for a new class of fractional impulsive partial interro-differential equation with infinity delay. *J. Applied Analysis and Computation.* **5**, 3, (2015), 329-346.

[83] A.M. Yaglom, Correlation theory of processes with random stationary nth increments. *AMS Transl.* **2**, 8, (1958), 87-141.

[84] J. Wu, *Theory and Application of Partial Functional Differential Equations,* Springer-Verlag, New York, 1996.