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Abstract

This thesis is a contribution to the study of various classes of functional and neutral functional differential equations and inclusions of fractional order with state dependent delay.
To get the existence of the mild solutions, sufficient conditions are considered in the study of different classes.
Uniqueness results are also given for some classes of these problems.
The method used is to reduce the existence of these mild solutions to the search for the existence of fixed points of appropriate operators by applying different nonlinear alternatives in Fréchet spaces to entire the existence of fixed points of the above operators which are mild solutions of our problems. This method is based on fixed point theorems and is combined with the α-resolvent families theory.

Key words and phrases:
Functional differential equations and inclusions, fractional order, mild solution, fixed point theory, α-resolvent families, Fréchet spaces, state dependent delay, Riemann-Liouville's integral and derivative, Caputo's integral and derivative.

AMS Subject Classification: 34G20, 34G25, 34K40, 47G20.

Résumé

Cette thèse est une contribution à l'étude d'une variété de classes d'équations et d'inclusions différentielles d'ordre fractionnaire ainsi que celles de type neutre avec retard dépendant de l'état.
Dans l'étude des différentes classes, des conditions suffisantes d'existence de solutions faibles sont considérées.
Pour certaines classes, on a aussi présenté des résultats d'unicité.
La méthode utilisée consiste à réduire l'éxistence des solutions à l'éxistence de points fixes pour des opérateurs appropriés en appliquant différentes altérnatives non linéaires dans des éspaces de Fréchet, de tels points fixes sont aussi solutions des problèmes posés.
Cette méthode est basée sur des théorèmes de points fixes et est combinée avec la théorie des familles α-résolvantes.

Mots et phrases cléfs:

Equations et Inclusions Différentielles Fonctionnelles, order fractionnaire, solution faible, théorie du point fixe, familles α-résolvantes, espace de Fréchet, retard dépendant de l'état, intégrale et dérivée au sens de Riemann-Liouville, intégrale et dérivée au sens de Caputo .

Classification AMS: 34G20, 34G25, 34K40, 47G20.

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Publications

1. M. Benchohra, O. Bennihi and K. Ezzinbi, Semilinear Functional Fractional Differential Equations with State Dependent Delay, *Commentationes Mathematicae*, Vol.53, No.1 (2013), 47-59.

2. M. Benchohra, O. Bennihi and K. Ezzinbi, Existence Results for Some Neutral Functional Fractional Differential Equations with State-Dependent Delay, *CUBO, A Mathematical Journal*, Vol.16, No.03, (2014), 37-53.

3. M. Benchohra, O. Bennihi and K. Ezzinbi, Fractional Differential Inclusions with State-Dependent Delay, **(submitted)**.

4. M. Benchohra, O. Bennihi and K. Ezzinbi, Fractional Neutral Differential Inclusions with State-Dependent Delay, **(submitted)**.

Introduction

Functional differential equations and inclusions arise in a variety of areas of biological, physical, and engineering applications. During the last decades, existence and uniqueness of mild, strong and classical solutions of semi linear functional differential equations (respectively inclusions) has been studied extensively by many authors using the α-resolvent families theory and fixed point arguments. We mention, for instance, the books by Abbes *et al* [5], Oldham and Spanier [81], Kolmanovskii and Myshkis [65], and the references therein.
One can see also the papers [2], [6], [9] and [26]. Nonlinear evolution equations arise not only from many fields of mathematics, but also from other branches of science such as physics, mechanics and material sciences.
Complicated situations in which the delay depends on the unknown functions have been proposed in modeling in recent years
These equations and inclusions are frequently called equations and inclusions with state-dependent delay.
Existence results and among other things were derived recently from functional differential equations and inclusions when the solution is depending on the delay. We refer the reader to the papers by Ahmed [11, 12], Adimy and Ezzinbi [8], Agarwal *et al* [10], Ait Dads and Ezzinbi [13], and Hernandez *et al* [57].
Over the past several years it has become apparent that equations and inclusions with state-dependent delay arise also in several areas such as in classical electrodynamics [44], in models of commodity price fluctuations [25], in models of blood cell productions [75] and in self similar protein dynamics [51].
Recently Li and Peng [73] studied a class of abstract homogeneous fractional evolution equations.

Baghli *et al* [17], have proved global existence and uniqueness results for an initial value problem for functional differential equations of first order with state-dependent delay.
Functional differential equations involving the Riemann-Liouville fractional derivative were considered by Benchohra *et al* [33], N'Guérékata and Mophou [80] studied semi-linear neutral fractional functional evolution equations with infinite delay using the notion of α-resolvent family.
For example, Navier-Stokes and Euler equations from fluid mechanics, nonlinear reaction-diffusion equations from heat transfers and biological sciences, nonlinear Klein-Gorden equations and nonlinear Schrödinger equations from quantum mechanics and Cahn-Hilliard equations from material science are some special examples of nonlinear evolution equations.
Complexity of nonlinear evolution equations and challenges in their theoretical study have attracted a lot of interest from many mathematicians and scientists in nonlinear sciences.
Neutral differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years.
A good guide to the literature for neutral functional differential equations is the books by Hale [53], Hale and Verduyn Lunel [55], Kolmanovskii and Myshkis [65] and the references therein.
When the delay is infinite, the notion of the phase space plays an important role in the study of both quantitative and qualitative theory.
A usual choice is a semi-norm space satisfying suitable axioms, which was introduced by Hale and Kato in [54], see also Corduneanu and Lakshmikantham [39], Kappel and Schappacher [64] and Schumacher [90, 91].
For detailed discussion and applications on this topic, we refer the reader to the books by Hino *et al* [61] and Wu [94].
Ezzinbi in [46] studied the existence of mild solutions for partial functional differential equations with infinite delay, Henriquez in [59] and Hernandez *et al* in [56] studied the existence and regularity of solutions to functional and neutral functional differential equations with unbounded delay, Balachandran and Dauer [24] have considered various classes of first and second order semi-linear ordinary functional and neutral functional differential equations on Banach spaces.
By means of fixed point arguments, Benchohra *et al* [33] have studied many classes of functional differential equations and inclusions and proposed some controllability results in [14, 29, 30, 31, 32, 37]. See also the works by Gastori [50] and Li *et al* [72].

Let us now briefly describe the organization of this thesis.
In chapter1 we collect some preliminaries, notations, definitions, theorems and other auxiliary results which will be needed in this thesis, in the first section we give some generalities, in section 2 we present some properties of phase spaces, in the third section we give some properties of fractional calculus, in section 4 we give some properties of set-valued maps and in the last section we cite some fixed point theorems.
In chapter 2 we give some results of existence and uniqueness of mild solutions for semi linear fractional functional differential equations with state dependent delay in a Fréchet space. In particular in Section 2 we studied the following problem

$$D^{\alpha}x(t) = Ax(t) + f(t, x_{\rho(t,x_t)}), t \in [0, +\infty),\ 0 < \alpha < 1, \tag{0.0.1}$$

$$x(t) = \varphi(t),\ t \in (-\infty, 0], \tag{0.0.2}$$

The Chapter 3 is devoted to the existence and uniqueness of mild solutions for a class of neutral fractional functional differential equations with state dependent delay. In Section 2 we studied the existence and uniqueness of mild solutions for the following problem

$$D^{\alpha}[x(t) - g(t, x_{\rho(t,x_t)})] = Ax(t) + f(t, x_{\rho(t,x_t)}), a.e.\ t \in [0, +\infty), 0 < \alpha < 1. \tag{0.0.3}$$

$$x_0 = \varphi, \varphi \in \mathcal{B}. \tag{0.0.4}$$

$$\tag{0.0.5}$$

Chapter 4 concerns the existence of mild solutions for a class of fractional functional differential inclusions with state-dependent delay.
In Section 2 we studied the existence and uniqueness of mild solutions for the following problem

$$D^{\alpha}x(t) \in Ax(t) + F(t, x_{\rho(t,x_t)}), t \in J := [0, +\infty), \tag{0.0.6}$$

$$x(t) = \varphi(t), t \in (-\infty, 0]. \tag{0.0.7}$$

$$\tag{0.0.8}$$

Chapter 5 is devoted to the existence of mild solutions for a class of neutral fractional functional differential inclusions with state-dependent delay.

Section 2 concerns the following problem

$$ {}^{c}D_{0}^{\alpha}[x(t) - g(t, x_{\rho(t,x_t)})] \in A[x(t) - g(t, x_{\rho(t,x_t)})] + F(t, x_{\rho(t,x_t)}), t \in [0, +\infty), \tag{0.0.9} $$

$$ x(t) = \varphi(t), t \in (-\infty, 0]. \tag{0.0.10} $$

(0.0.11)

Each chapter is ended by an example to illustrate our main results.

Chapter 1

Preliminaries

This chapter concerns some preliminaries, notations, definitions, theorems and other auxiliary results which will be needed in the sequel.

1.1 Generalities

By $C([0,b];E)$, $b>0$ we denote the Banach space of continuous functions from $[0,b]$ into E, with the norm

$$\|x\|_\infty = \sup_{t\in[0,b]} \|x(t)\|_E.$$

$B(E)$ is the space of bounded linear operators from E into E, with the usual supreme norm

$$\|N\|_{B(E)} = \sup\{\|N(x)\|_E : \|x\|_E = 1\}.$$

Definition 1.1.1 *An operator $T : E \to E$ is compact if the image of each bounded set $B \subset E$ is relatively compact i.e ($\overline{T(B)}$ is compact).*
T is completely continuous operator if it is continuous and compact.

Let $L^\infty(J)$ be the Banach space of measurable functions $x : J \to E$ which are essentially bounded, equipped with the norm

$$\|x\|_{L^\infty} = inf\{c > 0 : \|x\|_E \leq c,\ a.e.\ t \in J\}.$$

A measurable function $x : J \to E$ is Bochner integrable if and only if $\|x\|_E$ is Lebesgue integrable.

Let $L^1([0,b],E)$ denote the Banach space of measurable functions $x:[0,b]\to E$ which are Bochner integrable with the norm

$$\|x\|_{L^1}=\int_0^b \|x(t)\|_E dt.$$

Let (E,d) be a metric space. For any function x defined on $(-\infty,b]$ and any $t\in J$, we denote by x_t the element of $\mathcal{B}$ defined by

$$x_t(\theta)=x(t+\theta),\ \theta\in(-\infty,0].$$

The function x_t represents the history of the state from $-\infty$ up to t.

Definition 1.1.2 *The Laplace transform of a function $f\in L^1_{loc}(\mathbb{R}^+,E)$ is defined by*

$$\widehat{f}(\lambda):=\int_0^\infty e^{-\lambda t}f(t)dt, \Re(\lambda)>\omega,$$

if the integral is absolutely convergent for $Re(\lambda)>\omega$.

1.2 Some properties of phase spaces

We define the phase space $\mathcal{B}$ axiomatically, using ideas and notations developed by Hale and Kato [54]. More precisely, $\mathcal{B}$ denote the vector space of functions defined from $(-\infty,0]$ into E endowed with a norm denoted $\|.\|_{\mathcal{B}}$, such that the following axioms hold.

(A_1) If $x:(-\infty,b)\to E$, is continuous on $[0,b]$ and $x_0\in\mathcal{B}$, then for $t\in[0,b)$ the following conditions hold

- (i) $x_t\in\mathcal{B}$
- (ii) $\|x_t\|_{\mathcal{B}}\leq K(t)\sup\{|x(s)|:0\leq s\leq t\}+M(t)\|x_0\|_{\mathcal{B}}$,
- (iii) $|x(t)|\leq H\|x_t\|_{\mathcal{B}}$
  where $H\geq 0$ is a constant, $K:[0,b)\to[0,+\infty)$, $M:[0,+\infty)\to[0,+\infty)$ with K continuous and M locally bounded and H, K and M are independent of x.

(A_2) For the function x in (A_1), the function $t\to x_t$ is a $\mathcal{B}$-valued continuous function on $[0,b]$.

(A_3) The space $\mathcal{B}$ is complete.

Denote $K_b = \sup\{K(t) : t \in [0, b]\}$ and $M_b = \sup\{M(t) : t \in [0, b]\}$.

Remark 1.2.1 *(iii) is equivalent to* $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$

We provide some examples of the phase spaces. For more details we refer to the book by Hino *et al.* [61].

Example 1.2.2 *Let BC be the space of bounded continuous functions defined from $(-\infty, 0]$ to E.*
BUC the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E,

$$C^{\infty} := \left\{ \phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } E \right\}.$$

$$C^{0} = \left\{ \phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0 \right\},$$

endowed with the uniform norm

$$\|\phi\| = \sup\{|\phi(\theta)| : \theta \leq 0\}.$$

Then the spaces BUC, C^{∞} and C^{0} satisfy conditions $(A_1) - (A_3)$. However, BC satisfies $(A_1)$and(A_3) but (A_2) is not satisfied.

Example 1.2.3 *The spaces C_g, UC_g, C_g^{∞} and C_g^{0}.*
Let g be a positive continuous function on $(-\infty, 0]$. We define:

$$C_g := \left\{ \phi \in C((-\infty, 0], E) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty, 0] \right\}.$$

$$C_g^{0} := \left\{ \phi \in C_g : \lim_{\theta \to -\infty} \frac{\phi(\theta)}{g(\theta)} = 0 \right\},$$

endowed with the following norm

$$\|\phi\| = \sup \left\{ \frac{|\phi(\theta)|}{g(\theta)} : \ \theta \leq 0 \right\}.$$

Then we have that the spaces C_g and C_g^{0} satisfy conditions $(A_1) - (A_3)$. We consider the following condition on the function g.

(g_1) *For all* $a > 0$, $\sup_{0\le t\le a} \sup\left\{\frac{g(t+\theta)}{g(\theta)} : -\infty < \theta \le -t\right\} < \infty.$

The above spaces satisfy conditions (A_1) *and* (A_2) *if* (g_1) *holds.*

Example 1.2.4 *The space* C_γ*. For any real constant* $\gamma > 0$*, we define the functional space* C_γ *by*

$$C_\gamma := \left\{\phi \in C((-\infty,0],E) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \textit{exists inE}\right\}$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \ \theta \le 0\}.$$

Then in the space C_γ *the axioms* $(A_1) - (A_3)$ *are satisfied.*

Let $E = (E, \|.\|_n)$ be a Fréchet space with a family of semi-norms $\{\|.\|_n\}_{n\in\mathbb{N}}$, we say that X is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|x\|_n \le \overline{M}_n \quad \textit{for all } x \in X.$$

To E we associate a sequence of Banach spaces $\{(E^n, \|\cdot\|_n)\}$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by: $x \sim_n y$ if and only if $\|x - y\|_n = 0$ for $x, y \in E$. We denote $E^n = (E|_{\sim_n}, \|\cdot\|_n)$ the quotient space, and we set $(E^n, \|\cdot\|_n)$ the completion of E^n with respect to $\|\cdot\|_n$. To every $X \subset E$, we associate a sequence $\{X^n\}$ of subsets $X^n \subset E^n$ as follows: For every $x \in E$, we denote $[x]_n$ the equivalence class of x in E^n and we define $X^n = \{[x]_n : x \in X\}$. We denote $\overline{X^n}$, $int_n(X^n)$ and $\partial_n X^n$, respectively, the closure, the interior and the boundary of X^n with respect to $\|\cdot\|_n$ in E^n.
We assume that the family of semi-norms $\{\|.\|_n\}_{n\in\mathbb{N}}$ verifies:

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \dots \ \textit{for every } x \in X.$$

1.3 Fractional Order Calculus

1.3.1 The History of Fractional Order Calculus

The concept of fractional differential calculus has a long history. One may wonder what meaning may be ascribed to the derivative of a fractional order, that is $\frac{d^n y}{dx^n}$, where n is a fraction. In fact L'Hopital himself considered

this possibility in a correspondence with Leibniz. In 1695 L'Hopital wrote to Leibniz to ask, "What if n be 1/2?" From this question, the study of fractional calculus was born. Leibniz responded to the question, "$d^{\frac{1}{2}}x$ will be equal to $\sqrt{d}x$. This is an apparent paradox from which, one day, useful consequences will be drawn."
Many known mathematicians contributed to this theory over the years. Thus, 30 September 1695 is the exact date of birth of the "fractional calculus"! Therefore, the fractional calculus has its origin in the works by Leibnitz, L'Hopital (1695), Bernoulli (1697), Euler (1730), and Lagrange (1772). Some years later, Laplace (1812), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), Grünwald (1867), Letnikov (1868), Nekrasov (1888), Hadamard (1892), Heaviside (1892), Hardy (1915), Weyl (1917), Riesz (1922), P. Levy(1923), Davis (1924), Kober (1940), Zygmund (1945), Kuttner (1953), J. L. Lions (1959), and Liverman (1964)... have developed the basic concept of fractional calculus. In June 1974, Ross has organized the "First Conference on Fractional Calculus and its Applications" at the University of New Haven, and edited its proceedings [86]; Thereafter, Oldham and Spanier [81] published the first monograph devoted to "Fractional Calculus" in 1974. The integrals and derivatives of non-integer order, and the fractional integro-differential equations have found many applications in recent studies in theoretical physics, mechanics and applied mathematics.
There exists the remarkably comprehensive encyclopedic-type monograph by Samko, Kilbas and Marichev [87] which was published in Russian in 1987 and in English in 1993. (For more details see [74] and the book of Ortigueira [82]). In recent years, the theory on existence and uniqueness of solutions of linear and nonlinear fractional functional differential equations and inclusions has attracted the attention of many authors (see for example [6, 9, 26, 34, 35, 36] and the references therein), and there has been a significant development in the theory of such equations and inclusions. It is caused by its applications in the modeling of many phenomena in various fields of science and engineering such as acoustic, control theory, chaos and fractals, signal processing, porous media, electrochemistry, visco-elasticity, rheology, polymer, physics, optics, economics, astrophysics, chaotic dynamics, statistical physics, thermodynamics, proteins, biosciences, bioengineering... etc. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. See for example [18, 19, 60, 62, 88] and the references therein.

1.3.2 Some Properties of Fractional Order Calculus

Definition 1.3.1 *[63] Let $\alpha > 0$ for $h \in L^1([0,b])$, $b > 0$ the expression*

$$(I_0^\alpha h)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds, \tag{1.3.1}$$

is called the left-side mixed Riemann-Liouville integral of order α where $\Gamma(.)$ is the Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}\exp(-t)dt,\ \xi > 0$. provided the right hand-side exists on $\mathbb{R}^+$.

In particular

$$(I_0^0 h)(t) := h(t),\ (I_0^1 h)(t) := \int_0^t h(s)ds.$$

Note that, ($I_0^\alpha h$ exists for all $\alpha > 0$ when $h \in L^1([0,b])$.
Also, when $h \in \mathcal{C}([0,b],E)$ then $(I_0^\alpha h) \in \mathcal{C}([0,b],E)$.

Example 1.3.2 *Let $\beta \in (0,\infty)$. Then*

$$I_0^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta+\alpha}t^{\beta+\alpha},\ \textit{for almost all}\ \ t \in [0,b].$$

Definition 1.3.3 *[1] The Riemann-Liouville fractional derivative of order $\alpha \in (0,1]$ of a function $h \in L^1([0,b])$ is defined by*

$$\begin{aligned} D_0^\alpha h(t) &= \frac{d}{dt} I_0^{1-\alpha} h(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s)ds, \quad \textit{for almost}\ \ t \in [0,b]. \end{aligned}$$

Example 1.3.4 *Let $\lambda \in (0,\infty)$ and $\alpha \in (0,1]$, then*

$$D_0^\alpha t^\lambda = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-\alpha)} t^{\lambda-\alpha}, \quad \textit{for almost all}\ \ t \in [0,b].$$

1.4 Some Properties of Set-Valued Maps

We use the following notations:Let (E, d) be a metric space where E is separable and X be a subset of E. We denote:

$$P(E) = \{X \subset E : X \neq \emptyset\}$$

and

$$P_b(E) = \{X \subset E : X \text{ bounded }\}, \qquad P_{cl}(E) = \{X \subset E : X \text{ closed }\}.$$

$$P_{cp}(E) = \{X \subset E : X \text{ compact }\}, \qquad P_{cv}(E) = \{X \subset E : X \text{ convexe }\}.$$

$$P_{cv,cp}(E) = P_{cv}(E) \cap P_{cp}(E).$$

Let $A, B \in P(E)$. Consider $H_d : P(E) \times P(E) \to \mathbb{R}_+ \cup \{\infty\}$ the Hausdorff distance between A and B defined by:

$$H_d(A, B) = \max\left\{\sup_{a \in A} d(a, B)\ ,\ \sup_{b \in B} d(A, b)\right\},$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$ and $d(A, b) = \inf\{d(a, b) : a \in A\}$.
As usual, $d(x, \emptyset) = +\infty$.
Then $(P_{b,cl}(E), H_d)$ is a metric space, $(P_{cl}(E), H_d)$ is a generalized (complete) metric space.

Definition 1.4.1 *A multi-valued map $F : J \to P_{cl}(E)$ is said to be measurable if, for each $x \in E$, the function $g : J \to E$ defined by*

$$g(t) = d(x, F(t)) = \inf\{d(x, z) : z \in F(t)\},$$

is measurable.

Definition 1.4.2 *Let X and Y be metric spaces. A set-valued map F from X to Y is characterized by its graph $Gr(F)$, the subset of the product space $X \times Y$ defined by*

$$Gr(F) := \{(x, y) \in X \times Y : y \in F(x)\}$$

Definition 1.4.3 *1. A measurable multi-valued function $F : J \to P_{b,cl}(E)$ is said to be integrable bounded if there exists a function $g \in L^1(\mathbb{R}_+)$ such that $|f| \leq g(t)$ for almost $t \in J$ for all $f \in F(t)$.*

2. *F is bounded on bounded sets if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in E for all $B \in P_b(E)$, i.e. $\sup_{x\in B}\{\sup\{|y| : y \in F(E)\}\} < \infty$.*

3. *A set-valued map F is called upper semi-continuous (u.s.c. for short) on E if for each $x_0 \in E$ the set $F(x_0)$ is a nonempty, closed subset of E and for each open set U of E containing $F(x_0)$, there exists an open neighborhood V of x_0 such that $F(V) \subset U$. A set-valued map F is said to be upper semi-continuous if it is so at every point $x_0 \in E$.*

4. *A set-valued map F is called lower semi-continuous (l.s.c) at $x_0 \in E$ if for any $y_0 \in F(x_0)$ and any neighborhood V of y_0 there exists a neighborhood U of x_0 such that $F(x_0) \cap V \neq \emptyset$ for all $x_0 \in U$.*

5. *A set-valued map F is said to be lower semi-continuous if it is so at every point $x_0 \in E$.*

6. *F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_b(E)$. If the multi-valued map f is completely continuous with nonempty compact values, then f is upper semi-continuous if and only if f has closed graph.*

Proposition 1.4.4 *Let $F : E \to G$ be an u.s.c map with closed values. Then $Gr(F)$ is closed.*

Definition 1.4.5 *A multi-valued map $G : E \to P(E)$ has* convex (closed) values *if $G(x)$ is convex (closed) for all $x \in E$. We say that G is* bounded *on bounded sets if $G(B)$ is bounded in E for each bounded set B of E, i.e.,*

$$\sup_{x\in B}\{\sup\{\|x\|_E : x \in G(x)\}\} < \infty.$$

Finally, we say that G has a fixed point *if there exists $x \in E$ such that $x \in G(x)$.*

For each $x : (-\infty, +\infty) \to E$ let the set $S_{F,x}$ known as *the set of selectors* from F defined by

$$S_{F,x} = \{v \in L^1(J, E) : v(t) \in F(t, x_t)\ ,\ a.e.\ t \in J\}.$$

For more details on multi-valued maps we refer to the books of Deimling [41] and Górniewicz [52] and the papers of Agarwal*et al.* [10, 89].

The following definition is the appropriate concept of *admissible contraction map* in E.

Definition 1.4.6 *[47] A multi-valued map $F : E \to \mathcal{P}(\mathcal{E})$ is called an* admissible contraction *if for each $n \in \mathbb{N}$ there exists a constant $k_n \in (0,1)$ such that*

i) $H_d(F(x), F(y)) \leq k_n \|x - y\|_n$ *for all* $x, y \in E$,

ii) for every $x \in E$ and every $\epsilon \in (0,\infty)^n$, there exists $y \in F(x)$ such that $\|x - y\|_n \leq \|x - F(x)\|_n + \epsilon_n$ *for every* $n \in \mathbb{N}$.

1.5 Some properties of α-resolvent families

In order to define the mild solutions of the considered problems, we recall the following definitions and theorems

Definition 1.5.1 *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space E. We call A the generator of an α-resolvent family or solution operator if there exists $\omega > 0$ and a strongly continuous function $T_\alpha : \mathbb{R}^+ \to L(E)$ such that*

$$\{\lambda : Re(\lambda) > \omega\} \subset \rho(A),$$

where $\rho(A)$, is the resolvant set of A, and

$$(\lambda^\alpha - A)^{-1}x = \int_0^\infty \exp^{-\lambda t} T_\alpha(t)x dt, \quad Re(\lambda) > \omega, \ x \in E.$$

In this case, $T_\alpha(t)$ is called the solution operator generated by A.
The following result is a direct consequence of (Proposition 3.1 and Lemma 2.2) in [71].

Proposition 1.5.2 *Let $T_\alpha(t) \in L(E)$ be the solution operator with generator A. Then the following conditions are satisfied:*

1. *$T_\alpha(t)$ is strongly continuous for $t \geq 0$ and $T_\alpha(0) = I$.*

2. $T_\alpha(t)D(A) \subset D(A)$ *and* $AT_\alpha(t)x = T_\alpha(t)Ax$ *for all* $x \in D(A)$, $t \geq 0$.
3. *For every* $x \in D(A)$ *and* $t \geq 0$,
$$T_\alpha(t)x = x + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AT_\alpha(s)xds.$$
4. *Let* $x \in D(A)$. *Then*
$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} T_\alpha(s)xds \in D(A).$$
and
$$T_\alpha(t)x = x + A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} T_\alpha(s)xds$$

Remark 1.5.3 *The concept of the solution operator, as defined above, is closely related to the concept of a resolvent family. (see Prüss [85].) Because of the uniqueness of the Laplace transform, in the border case* $\alpha = 1$, *the family* $T_\alpha(t)$ *corresponds to the* C_0*- semigroup (see [45]), where as in the case* $\alpha = 2$ *a solution operator corresponds to the concept of cosine family (see Arendt* et al. *[15]).*

For more details on the α-resolvent families, we refer to [80] and the references therein.

1.6 Some Fixed Point Theorems

In the beginning, let us give the definition of a contraction on a space E.

Definition 1.6.1 *[49] A function* $f : E \to E$ *is said to be a contraction if for every* $n \in \mathbb{N}$ *there exists* $k_n \in [0, 1)$ *such that:*
$$\|f(x) - f(y)\|_n \leq k_n\|x - y\|_n \text{ for all } x, y \in E.$$

Hereafter the fixed point theorems used in this thesis:

Theorem 1.6.2 *[49]. Let* E *be a Fréchet space and* X *a closed subset of* E *such that* $0 \in X$ *and let* $N : X \to E$ *be a contraction map such that* $N(X)$ *is bounded. Then one of the following statements holds:*

- *N has a unique fixed point in E.*
- *There exists $0 \leq \lambda < 1$, $n \in \mathbb{N}$ and $x \in \partial_n X^n$: $\|x - \lambda N(x)\|_n = 0$.*

For multi-valued maps, our results are based on the following nonlinear alternative due to Frigon [48] for admissible contractive multi-valued maps in Fréchet spaces.

Theorem 1.6.3 *[48] Let E be a* Fréchet space and U an open neighborhood of the origin in E and let $N : \overline{U} \to \mathcal{P}(E)$ be *an admissible multi-valued contraction.* Assume that N is bounded, then one of the following statements holds:

- $(S1)$ N has a fixed point,
- $(S2)$ There exists $\lambda \in [0, 1)$ and $x \in \partial U$ such that $x \in \lambda N(x)$.

Chapter 2

Fractional Functional Differential Equations with State Dependent Delay

2.1 Introduction

In this chapter, we establish the existence and uniqueness of the mild solution defined on the semi-infinite positive real interval $[0, +\infty)$ for a class of semi-linear fractional functional differential equations with state dependent delay. This problem was studied by Darwish and N'touyas in [40].
Consider the following problem

$$D^{\alpha}x(t) = Ax(t) + f(t, x_{\rho(t,x_t)}), t \in [0, +\infty), 0 < \alpha < 1, \tag{2.1.1}$$

$$x(t) = \varphi(t), t \in (-\infty, 0], \tag{2.1.2}$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of an α-resolvent family $(T_{\alpha}(t))_{t\geq 0}$ defined on a real Banach space E, D^{α} is understood here in the Riemann-Liouville sense, $f : J \times \mathcal{B} \to E$, $\rho : J \times \mathcal{B} \to \mathbb{R}$ are appropriate given functions. φ belongs to the abstract*phase space* $\mathcal{B}$ with $\varphi(0) = 0$.

2.2 Existence Results

Let $f : J \times \mathcal{B} \to E$ be a continuous function and $\varphi(0) = 0$.

Definition 2.2.1 *[1] A function x is said to be a mild solution of* (2.1.1)-(2.1.2) *if x satisfies*

$$x(t) = \begin{cases} \varphi(t), t \in (-\infty, 0], \\ \int_0^t T_\alpha(t-s) f(s, x_{\rho(s,x_s)})\, ds, t \in J. \end{cases} \tag{2.2.3}$$

Set $\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \leq 0\}$. We assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Moreover we assume the following assumption and hypothesis:

- (H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\varphi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that
$$\|\varphi_t\|_\mathcal{B} \leq L^\varphi(t)\|\varphi\|_\mathcal{B} \text{ for every } t \in \mathcal{R}(\rho^-)$$
- (H_1) There exists a constant $M > 0$ such that
$$\|T_\alpha(t)\|_{B(E)} \leq \widehat{M}, t \in J.$$
- (H_2) There exists a function $p \in L^1_{loc}(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : [0, +\infty) \to (0, \infty)$ such that:
$$|f(t,u)| \leq p(t)\psi(\|u\|_\mathcal{B}) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$
- (H_3) For all $n > 0$, there exists $l_n \in L^1_{loc}(J, \mathbb{R}^+)$ such that:
$$|f(t,u) - f(t,v)| \leq l_R(t)\|u - v\|_\mathcal{B} \text{ for all } t \in [0,n] \textit{ and } u, v \ \in \mathcal{B}.$$

Remark 2.2.2 *The assumption (H_φ) is frequently verified by continuous and bounded functions. for more details, see Hino et al [61].*

Define the following space

$$B_{+\infty} = \{x : \mathbb{R} \to E : x|_{[0,b]} \text{ continuous for } b > 0 \text{ and } x_0 \in \mathcal{B}\},$$

where $x|_{[0,b]}$ is the restriction of x to the real compact interval $[0,b]$.
Let us fix $r > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms by:

$$\|x\|_n := \sup\{e^{-rL^*_n(t)}|x(t)| : t \in [0,n]\}$$

where $L^*_n(t) = \int_0^t \bar{l}_n(s)ds$, $\bar{l}_n(t) = K_n\widehat{M}l_n(t)$ and l_n is the function from (H_3).
Then $B_{+\infty}$ is a Fréchet space with these semi-norms family $\|.\|_n$.

Lemma 2.2.3 *[61], (Lemma 2.4) Let $x : (-\infty, b] \to E$ is a function such that $x_0 = \varphi$, then*

$$\|x_s\|_{\mathcal{B}} \leq (M_b + L^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \sup\{|x(\theta)|,\ \theta \in [0, max(0,s)]\},\ s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^{\varphi} = \sup_{t\in\mathcal{R}(\rho^-)} L^{\varphi}(t)$.

Theorem 2.2.4 *Assume that (H_{φ}) and $(H_1)-(H_3)$ hold and moreover for each $n \in \mathbb{N}$*

$$\int_{c_n}^{+\infty} \frac{ds}{\psi(s)} > k_n \widehat{M} \int_0^n p(s)ds, \tag{2.2.4}$$

with $c_n = (M_n + L^{\varphi} + K_n\widehat{M}H)\|\varphi\|_{\mathcal{B}}$. Then the problem (2.1.1)-(2.1.2) has a unique mild solution on $(-\infty, +\infty)$.

Proof 2.2.5 *We transform the problem (2.1.1)-(2.1.2) into a fixed point theorem. In fact, we define the operator $N : B_{+\infty} \to B_{+\infty}$ defined by*

$$N(x)(t) = \begin{cases} \varphi(t), t \in (-\infty, 0], \\ \int\limits_0^t T_{\alpha}(t-s) f(s, x_{\rho(s,x_s)})\ ds, t \in J. \end{cases} \tag{2.2.5}$$

It is clear that the fixed points of the operator N are mild solutions of the problem (2.1.1)-(2.1.2).
For $\varphi \in \mathcal{B}$, we define the function $y : \mathbb{R} \to E$ by

$$y(t) = \begin{cases} \varphi(t), t \in (-\infty, 0], \\ 0, t \in J. \end{cases} \tag{2.2.6}$$

Then $y_0 = \varphi$.
For each function $z \in B_{+\infty}$ with $z(0) = 0$ we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, t \in (-\infty, 0], \\ z(t), t \in J. \end{cases} \tag{2.2.7}$$

If $x(t)$ satisfies (2.2.1), we can decompose it as $x(t) = y(t) + z(t)$ for $t \geq 0$, which implies that $x_t = y_t + z_t$ for every $t \geq 0$. The function z satisfies

$$z(t) = \int_0^t T_{\alpha}(t-s) f(s, y_{\rho(s,y_s+z_s)} + z_{\rho(s,y_s+z_s)})ds\ for\ t \in J.$$

Let

$$B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0 \in \mathcal{B}\}.$$

For any $z \in B^0_{+\infty}$ *we have*

$$\|z\|_\infty = \|z_0\|_{\mathcal{B}} + \sup\{|z(s)| : 0 \le s < +\infty\} = \sup\{|z(s)| : 0 \le s < +\infty\}.$$

Thus $(B^0_{+\infty}, \|.\|_{+\infty})$ *is a Banach space.*
We define the operator $G : B^0_{+\infty} \to B^0_{+\infty}$ *by:*

$$G(z)(t) = \int_0^t T_\alpha(t-s) f(s, y_{\rho(s,y_s+z_s)} + z_{\rho(s,y_s+z_s)}) ds, t \in J.$$

The operator N *has a fixed point is equivalent to say that* G *has one, so it turns to prove that* G *has a fixed point.*
Let $z \in B^0_{+\infty}$ *be such that* $z = \lambda G(z)$ *for some* $\lambda \in [0,1)$*. By hypotheses* (H_1)*,* (H_2)*,* (H_φ) *and* (2.2.3)*, we have for each* $t \in [0,n]$

$$\begin{aligned} |z(t)| &\le \int_0^t \|T_\alpha(t-s)\|_{B(E)} |f(s, y_{\rho(s,y_s+z_s)} + z_{\rho(s,y_s+z_s)})| ds \\ &\le \widehat{M} \int_0^t p(s) \psi(\|y_{\rho(s,y_s+z_s)} + z_{\rho(s,y_s+z_s)}\|_{\mathcal{B}}) ds \\ &\le \widehat{M} \int_0^t p(s) \psi(K_n u(s) + (M_n + L^\varphi + K_n \widehat{M} H)\|\varphi\|_{\mathcal{B}}) ds, \end{aligned}$$

where

$$u(s) = \sup\{|z(\theta)| : \theta \in [0,s]\}.$$

Set

$$c_n := (M_n + L^\varphi + K_n \widehat{M} H)\|\varphi\|_{\mathcal{B}}.$$

Then, for $t \in [0,n]$*, we have*

$$u(t) \le \widehat{M} \int_0^t [p(s)\psi(K_n u(s) + c_n)] ds.$$

Thus

$$K_n u(t) + c_n \le c_n + K_n \widehat{M} \int_0^t p(s) \psi(K_n u(s) + c_n) ds.$$

Now define the function μ *by*

$$\mu(t) = \sup\{K_n u(s) + c_n : 0 \le s \le t\}, \; t \in [0,n].$$

Let $t^* \in [0,t]$ be such that

$$\mu(t) = K_n u(t^*) + c_n\|\varphi\|_{\mathcal{B}}.$$

Then by the previous inequality, we have

$$\mu(t) \leq c_n + K_n\widehat{M}\int_0^t p(s)\psi(\mu(s))ds,\ t \in [0,n].$$

Set

$$v(t) = c_n + K_n\widehat{M}\int_0^t p(s)\psi(\mu(s))ds.$$

Then we have $\mu(t) \leq v(t)$ for all $t \in [0,n]$.
By the definition of v, we have

$$v(0) = c_n \text{ and } v'(t) = K_n\widehat{M}p(t)\psi(\mu(t)) \text{ a.e. } t \in [0,n].$$

Using the fact that ψ is non-decreasing, we get that

$$v'(t) \leq K_n\widehat{M}p(t)\psi(v(t)) \text{ a.e. } t \in [0,n].$$

This implies that for each $t \in [0,n]$ we have

$$\int_{c_n}^{v(t)} \frac{ds}{\psi(s)} \leq K_n\widehat{M}\int_0^t p(s)ds \leq K_n\widehat{M}\int_0^n p(s)ds < \int_{c_n}^{+\infty} \frac{ds}{\psi(s)}.$$

Thus for $t \in [0,n]$ there exists a constant $\mathbf{A_n}$ such that $v(t) \leq \mathbf{A_n}$ and hence $\mu(t) \leq \mathbf{A_n}$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq \mathbf{A_n}$.
Set

$$Z = \{z \in B^0_{+\infty} : \sup_{0\leq t\leq n} |z(t)| \leq \mathbf{A_n} + 1,\ for\ all\ n \in \mathbb{N}\}.$$

It is clear that Z is closed subset of $B^0_{+\infty}$.
We claim show that $G : Z \to B^0_{+\infty}$ is a contraction operator.
In fact, let $z, \overline{z} \in Z$, thus using (H_1) and (H_3) for each $t \in [0,n]$ and $n \in \mathbb{N}^*$

$$\begin{aligned} |G(z)(t) - G(\overline{z})(t)| &\leq \int_0^t \|T_\alpha(t-s)\|_{B(E)}|f(s, y_{\rho(s,y_s+z_s)} + z_{\rho(s,y_s+z_s)}) \\ &\quad - f(s, y_{\rho(s,y_s+z_s)} + \overline{z}_{\rho(s,z_s+y_s)})|\, ds \\ &\leq \int_0^t \widehat{M}l_n(s)\|z_{\rho(s,y_s+z_s)} - \overline{z}_{\rho(s,y_s+z_s)}\|_{\mathcal{B}}ds. \end{aligned}$$

Using (H_φ) *and Lemma* (2.2.3)*, we obtain that*

$$\begin{aligned}|G(z)(t) - G(\overline{z})(t)| &\leq \int_0^t \widehat{M} l_n(s) K_n |z(s) - \overline{z}(s)| ds \\ &\leq \int_0^t \Big[\bar{l}_n(s) e^{r l_n^*(s)}\Big] \Big[e^{-r l_n^*(s)} |z(s) - \overline{z}(s)|\Big] ds \\ &\leq \int_0^t \Big[\frac{e^{r l_n^*(s)}}{r}\Big]' ds \|z - \overline{z}\|_n \\ &\leq \frac{1}{r} e^{r l_n^*(t)} \|z - \overline{z}\|_n.\end{aligned}$$

Therefore,

$$\|G(z) - G(\overline{z})\|_n \leq \frac{1}{r}\|z - \overline{z}\|_n.$$

Then the operator G *is a contraction for all* $n \in \mathbb{N}$*. By the choice of* Z *there is no* $z \in \partial Z$ *such that* $z = \lambda G(z), \lambda \in (0, 1)$*. Then the second statement in theorem* (1.6.2) *dose not hold. The nonlinear alternative of Frigon-Granas shows that the first statement holds. Thus, we deduce that the operator* G *has a unique fixed-point* z^**. Then* $x^* = y^* + z^*, t \in (-\infty, +\infty)$ *is a fixed point of the operator* N*, which is the unique mild solution of the problem* (2.1.1)-(2.1.2)*.*

2.3 Example

To illustrate our results,we propose the following system

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha}(u, \xi) = \frac{\partial^2}{\partial \xi^2} u(t, \xi) \\ + \int_{-\infty}^0 a_1(s - t) u\Big[s - \rho_1(t)\rho_2\Big(\int_0^\pi a_2(\theta)|u(t, \theta)|^2 d\theta\Big), \xi\Big] ds, t \geq 0, \xi \in [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, t \geq 0, \\ u(\theta, \xi) = u_0(\theta, \xi), -\infty < \theta \leq 0, \xi \in [0, \pi]. \end{cases} \tag{2.3.8}$$

Where $a_1 : [0, +\infty) \to \mathbb{R}$, $\rho_1 : [0, +\infty) \to \mathbb{R}$ and $\rho_2 : [0, +\infty) \to \mathbb{R}$ are integrable functions, a_2 is a real function defined on $(-\infty, 0]$ and $u_0 : (-\infty, 0] \times [0, \pi] \to \mathbb{R}$.

To represent this system in the abstract form (2.1.1)-(2.1.2), we choose the space

$E = L^2([0,\pi],\mathbb{R})$ and the operator $A : D(A) \subset E \to E$ given by $A\omega = \omega''$ with domain

$$D(A) := H^2(0,t) \cap H_0^1(0,t).$$

It is well known that A is an infinitesimal generator of an α-resolvent family $(T_\alpha(t))_{t\geq 0}$ on E. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, with $n \in \mathbb{N}^*$ and corresponding normalized eigenfunctions given by

$$z_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi).$$

In addition, $\{z_n : n \in \mathbb{N}*\}$ is an ortho-normal basis of E. and

$$T_\alpha(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}(x, z_n)z_n,$$

for $x \in E$ and $t \geq 0$. It follows from this representation that $(T_\alpha(t))_{t\geq 0}$ is compact for every $t > 0$ and that

$$\|T_\alpha(t)\| \leq e^{-t} \text{ for every } t \geq 0.$$

Theorem 2.3.1 *Let $\mathcal{B} = BUC(\mathbb{R}^-; E)$ and $\phi \in \mathcal{B}$. assume that condition (H_ϕ) holds, $\rho_i : [0,+\infty) \to [0,\infty), i = 1,2$ are continuous and the functions $a_i : \mathbb{R} \to \mathbb{R}$ are continuous, for $i = 1,2$. Then there exists a unique mild solution of* (2.3.8).

Proof 2.3.2 *From the above assumptions, we have that*

$$f(t,\psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s,\xi)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\Big(\int_0^{\pi} a_2(\theta)|\psi(0,\xi)|^2 d\theta\Big),$$

are well defined functions, which permit to transform system (2.3.8) *into the abstract system* (2.1.1)-(2.1.2). *Moreover, the function f is linear and bounded. Now, the existence of a mild solution can be deduced by a direct application of Theorem* (2.2.4). *By Remark* (2.2.2), *we obtain the following result: There exists a unique mild solution of* (2.3.8) *on* $(-\infty,+\infty)$.

Chapter 3

Fractional Neutral Functional Differential Equations with State Dependent Delay

3.1 Introduction

In this chapter, we discuss the existence of the unique mild solution defined on the semi-infinite positive real interval $[0, +\infty)$ for a class of neutral fractional functional differential equations with state dependent delay . Baghli *et al* [17] studied the existence and uniqueness of mild solutions for neutral partial functional equations of entire order with state-dependent delay in a real Banach space $(E, |.|)$ when the delay is infinite. Our contribution is to introduce a new approach based on the notion of semi norms in Fréchet spaces. In particular, we consider the following initial value problem

$$D^{\alpha}[x(t)-g(t, x_{\rho(t,x_t)})] = Ax(t)+f(t, x_{\rho(t,x_t)}), t \in [0, +\infty), 0 < \alpha < 1, \quad (3.1.1)$$

$$x_0 = \varphi, \varphi \in \mathcal{B}, \quad (3.1.2)$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of an α-resolvent family $(T_{\alpha}(t))_{t\geq 0}$ defined on a real Banach space E, D^{α} is understood here in the Riemann-Liouville sense, $f : J \times \mathcal{B} \to E$, $\rho : J \times \mathcal{B} \to \mathbb{R}$ and $g : J \times \mathcal{B} \to E$ are appropriate given functions and satisfy some conditions that will be specified later, φ belongs to an abstract space denoted $\mathcal{B}$ and called *phase space* with $\varphi(0) - g(0, \varphi) = 0$. This chapter is arranged as follows: In Section

2, existence results are presented , and in Section 3, an example is given to illustrate the abstract theory.

3.2 Existence Results

Before starting and proving the existence results, let us give the definition of mild solution to the neutral partial evolution problem (3.1.1)-(3.1.2). Throughout this work, the function $f : J \times \mathcal{B} \to E$ will be continuous.

Definition 3.2.1 *A function x is said to be a mild solution of* (3.1.1)-(3.1.2) *if x satisfies*

$$x(t) = \begin{cases} \varphi(t), \ t \in (-\infty, 0], \\ g(t, x_{\rho(t,x_t)}) + \int\limits_0^t T_\alpha(t-s)A(s)g(s, x_{\rho(s,x_s)})ds \\ + \int\limits_0^t T_\alpha(t-s)f(s, x_{\rho(s,x_s)})ds, \ t \in J. \end{cases} \tag{3.2.3}$$

Set $\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \ \rho(s,\varphi) \leq 0\}$. We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous.
Let $\widehat{M}$ be such that $\widehat{M} = \sup_{t\in J} |T_\alpha(t)|$ then

$$\|T_\alpha(t)\|_{B(E)} \leq \widehat{M}, \quad t \in J.$$

Additionally, we introduce the following assumption and hypothesis:

- (H_φ)The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\varphi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

  $$\|\varphi_t\|_{\mathcal{B}} \leq L^\varphi(t)\|\varphi\|_{\mathcal{B}} \text{ for every } t \in \mathcal{R}(\rho^-)$$

  Remark 3.2.2 *The condition (H_φ), is frequently verified by continuous and bounded functions. For more details, see Hino et all [**?**], Proposition 7.1.1).*

- (H_1) There exist a function $p \in L^1_{loc}(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : [0, +\infty) \to (0, \infty)$ such that

  $$|f(t,u)| \leq p(t)\psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$

- (H_2) For all $n > 0$, there exists $l_n \in L^1_{loc}(J, \mathbb{R}^+)$ such that:

$$|f(t,u) - f(t,v)| \leq l_n(t)\|u - v\|_{\mathcal{B}} \text{ for all } t \in [0,n] \text{ and } u, v \in \mathcal{B}.$$

- (H_3) There exists a constant $\bar{M}_0 > 0$ such that

$$\|A^{-1}\|_{B(E)} \leq \bar{M}_0 \quad \text{for all } t \in J.$$

- (H_4) There exists a constant $L_* > 0$ such that

$$|Ag(s,\varphi) - Ag(\bar{s},\bar{\varphi})| \leq L_* \left(|s - \bar{s}| + \|\varphi - \bar{\varphi}\|_{\mathcal{B}}\right)$$

  for all $s, \bar{s} \in J$ and $\varphi, \bar{\varphi} \in \mathcal{B}$.

Define the following space

$$B_{+\infty} = \{x : \mathbb{R} \to E : x|_{[0,b]} \text{ continuous for } b > 0 \text{ and } x_0 \in \mathcal{B}\},$$

where $x|_{[0,b]}$ is the restriction of x to the real compact interval $[0, b]$.
Let us fix $r > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi norms by:

$$\|x\|_n := \sup\{e^{-rL_n^*(t)}|x(t)| : t \in [0,n]\}$$

where $L_n^*(t) = \int_0^t \bar{l}_n(s)ds$, $\bar{l}_n(t) = K_n\widehat{M}l_n(t)$ and l_n is the function given in (H_2).
Then $B_{+\infty}$ is a Fréchet space with those semi norms family $\|.\|_n$.

Lemma 3.2.3 *[56], (Lemma 2.4) If $x : (-\infty, b] \to E$ is a function such that $x_0 = \varphi$, then*

$$\|x_s\|_{\mathcal{B}} \leq (M_b + L^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \sup\{|x(\theta)|,\ \theta \in [0, max(0,s)]\},\ s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^{\varphi} = \sup_{t\in\mathcal{R}(\rho^-)} L^{\varphi}(t)$.

Theorem 3.2.4 *Suppose the hypothesis (H_{φ}) and $(H_1) - (H_4)$ are satisfied and moreover for each $n \in \mathbb{N}$*

$$\int_{\delta_n}^{+\infty} \frac{ds}{s + \psi(s)} > \frac{\widehat{M}K_n}{1 - \bar{M}_0 L K_n} \int_0^n \max(L, p(s))ds \qquad (3.2.4)$$

with

$$\delta_n = c_n + K_n L \frac{\bar{M}_0(1+\widehat{M}) + \widehat{M}n + \bar{M}_0[c_n + \widehat{M}\|\varphi\|_{\mathcal{B}}]}{1 - \bar{M}_0 L K_n},$$

and $c_n = (M_n + L^{\varphi} + K_n \widehat{M} H)\|\varphi\|_{\mathcal{B}}$, *then the problem* (3.1.1)-(3.1.2) *has a unique mild solution.*

Proof 3.2.5 *Define the operator* $N : B_{+\infty} \to B_{+\infty}$ *by:*

$$N(x)(t) = \begin{cases} \varphi(t), t \leq 0 \\ g(t, x_{\rho(t,x_t)}) \\ + \int_0^t T_\alpha(t-s)A(s)g(s, x_{\rho(s,x_s)})ds \\ + \int_0^t T_\alpha(t-s)f(s, x_{\rho(s,x_s)})ds, \ t \in J. \end{cases} \tag{3.2.5}$$

Then, fixed points of the operator N *are mild solutions of the problem* (3.1.1)-(3.1.2).

For $\varphi \in \mathcal{B}$, *we consider the function* $x : \mathbb{R} \to E$ *defined as follows by*

$$y(t) = \begin{cases} \varphi(t), t \leq 0, \\ 0, t \in J. \end{cases}$$

Then $y_0 = \varphi$. *For each function* $z \in B_{+\infty}$ *with* $z(0) = 0$, *we consider the function* $\bar{z}$ *by*

$$\bar{z}(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ z(t), & \text{if } t \in J. \end{cases}$$

If $x(\cdot)$ *satisfies* (3.2.1), *we decompose it as* $x(t) = z(t) + y(t)$, $t \geq 0$, *which implies* $x_t = z_t + y_t$, *for every* $t \in J$ *and the function* $z(\cdot)$ *satisfies* $z_0 = 0$ *and for* $t \in J$, *we get*

$$\begin{aligned} z(t) &= g(t, z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}) \\ &+ \int_0^t T_\alpha(t-s)A(s)g(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)})ds \\ &+ \int_0^t T_\alpha(t-s)f(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)})ds. \end{aligned}$$

Define the operator $F : B^0_{+\infty} \to B^0_{+\infty}$ *by :*

$$\begin{array}{rl} F(z)(t) & = g(t, z_{\rho(t,z_s+y_s)} + y_{\rho(t,z_s+y_s)}) \\ & + \int_0^t T_\alpha(t-s)A(s)g(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)})ds \\ & + \int_0^t T_\alpha(t-s)f(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)})ds. \end{array} \tag{3.2.6}$$

Obviously the operator N *has a fixed point is equivalent to* F *has one, so it turns to prove that* F *has a fixed point. Let* $z \in B^0_{+\infty}$ *be such that* $z = \lambda F(z)$ *for some* $\lambda \in [0,1)$*. Then, using* $(H1)-(H4)$*, we have for each* $t \in [0,n]$

$$\begin{array}{rcl} |z(t)| & \leq & |g(t, z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)})| \\ & + & |\displaystyle\int_0^t T_\alpha(t-s)Ag(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)})ds| \\ & + & |\displaystyle\int_0^t T_\alpha(t-s)f(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)})ds| \\ & \leq & \|A^{-1}\|_{B(E)}\|Ag(t, z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)})\| \\ & + & \displaystyle\int_0^t \|T_\alpha(t-s)\|_{B(E)}\|Ag(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)})\|ds \\ & + & \displaystyle\int_0^t \|T_\alpha(t-s)\|_{B(E)}|f(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)})|ds \\ & \leq & \bar{M}_0L(\|z_{\rho(t,z_s+y_s)} + y_{\rho(t,z_s+y_s)}\|_{\mathcal{B}} + 1) + \widehat{M}\bar{M}_0L(\|\varphi\|_{\mathcal{B}} + 1) \\ & + & \widehat{M}\displaystyle\int_0^t L(\|z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)}\|_{\mathcal{B}} + 1)ds \\ & + & \widehat{M}\displaystyle\int_0^t p(s)\psi(\|z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)}\|_{\mathcal{B}})ds \\ & \leq & \bar{M}_0L\|z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}\|_{\mathcal{B}} + \bar{M}_0L(1+\widehat{M}) + \widehat{M}Ln + \widehat{M}\bar{M}_0L\|\varphi\|_{\mathcal{B}} \\ & + & \widehat{M}L\displaystyle\int_0^t \|z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)}\|_{\mathcal{B}}ds \\ & + & \widehat{M}\displaystyle\int_0^t p(s)\psi(\|z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)}\|_{\mathcal{B}})ds. \end{array}$$

Using the assumption (A_1), we get

$$\begin{aligned}\|z_{\rho(t,z_s+y_s)}+y_{\rho(t,z_s+y_s)}\|_{\mathcal{B}} &\leq \|z_{\rho(t,z_s+y_s)}\|_{\mathcal{B}}+\|y_{\rho(t,z_s+y_s)}\|_{\mathcal{B}}\\ &\leq K(s)|z(s)|+M(s)\|Z_0\|_{\mathcal{B}}+K(s)|y(s)|+M(s)\|y_0\|_{\mathcal{B}}\\ &\leq K_n|z(s)|+M_nM|\varphi(0)|+M_n\|\varphi\|_{\mathcal{B}}\\ &\leq K_n|z(s)|+M_nMH\|\varphi\|_{\mathcal{B}}+M_n\|\varphi\|_{\mathcal{B}}\\ &\leq K_n|z(s)|+(K_nMH+M_n)\|\varphi\|_{\mathcal{B}}.\end{aligned}$$

Set $c_n=(K_nMH+M_n)\|\varphi\|_{\mathcal{B}}$ *we obtain*

$$\begin{aligned}|z(t)| &\leq \bar{M}_0L(K_n|z(t)|+c_n)+\bar{M}_0L(1+\widehat{M})+\widehat{M}Ln+\widehat{M}\bar{M}_0L\|\varphi\|_{\mathcal{B}}\\ &+ \widehat{M}L\int_0^t(K_n|z(s)|+c_n)ds+\widehat{M}\int_0^t p(s)\psi(K_n|z(s)|+c_n)ds\\ &\leq \bar{M}_0LK_n|z(t)|+\bar{M}_0L(1+\widehat{M})+\widehat{M}Ln+\bar{M}_0Lc_n+\widehat{M}\bar{M}_0L\|\varphi\|_{\mathcal{B}}\\ &+ \widehat{M}L\int_0^t(K_n|z(s)|+c_n)ds+\widehat{M}\int_0^t p(s)\psi(K_n|z(s)|+c_n)ds.\end{aligned}$$

Then

$$\begin{aligned}(1-\bar{M}_0LKn)|z(t)| &\leq L(\bar{M}_0(1+\widehat{M})+\widehat{M}n+\bar{M}_0c_n+\widehat{M}\bar{M}_0\|\varphi\|_{\mathcal{B}})\\ &+ \widehat{M}L\int_0^t(K_n|z(s)|+c_n)ds+\widehat{M}\int_0^t p(s)\psi(K_n|z(s)|+c_n)ds.\end{aligned}$$

Set

$$\delta_n:=c_n+\frac{LK_n}{1-\bar{M}_0LK_n}[\bar{M}_0(1+\widehat{M})+\widehat{M}n+\bar{M}_0c_n+\widehat{M}\bar{M}_0\|\varphi\|_{\mathcal{B}}].$$

Thus

$$\begin{aligned}K_n|z(t)|+c_n &\leq \delta_n+\frac{\widehat{M}LK_n}{1-\bar{M}_0LK_n}\int_0^t(K_n|z(s)|+c_n)ds\\ &+ \frac{\widehat{M}K_n}{1-\bar{M}_0LK_n}\int_0^t p(s)\psi(K_n|z(s)|+c_n)ds.\end{aligned}$$

We consider the function μ defined by

$$\mu(t):=\sup\{\ K_n|z(s)|+c_n:0\leq s\leq t\ \},0\leq t<+\infty.$$

Let $t^\star \in [0,t]$ be such that $\mu(t^\star) = K_n|z(t^\star)| + c_n$. By the previous inequality, we have

$$\mu(t) \leq \delta_n + \frac{\widehat{M}K_n}{1-\bar{M}_0 L K_n}\left[\int_0^t L\mu(s)ds + \int_0^t p(s)\psi(\mu(s))ds\right] \text{ for } t \in [0,n].$$

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have

$$\mu(t) \leq v(t) \text{ for all } t \in [0,n].$$

By the definition of v, we have $v(0) = \delta_n$ and

$$v'(t) = \frac{\widehat{M}K_n}{1-\bar{M}_0 L K_n}[L\mu(t) + p(t)\psi(\mu(t))] \text{ a.e. } t \in [0,n].$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq \frac{\widehat{M}K_n}{1-\bar{M}_0 L K_n}[Lv(t) + p(t)\psi(v(t)] \text{ a.e. } t \in [0,n].$$

Using the condition (2.2.4), this implies that for each $t \in [0,n]$, we have

$$\begin{aligned}\int_{\delta_n}^{v(t)} \frac{ds}{s+\psi(s)} &\leq \frac{\widehat{M}K_n}{1-\bar{M}_0 L K_n}\int_0^t \max(L, p(s))ds \\ &\leq \frac{\widehat{M}K_n}{1-\bar{M}_0 L K_n}\int_0^n \max(L, p(s))ds \\ &< \int_{\delta_n}^{+\infty} \frac{ds}{s+\psi(s)}.\end{aligned}$$

Thus, for every $t \in [0,n]$, there exists a constant Λ_n such that $v(t) \leq \Lambda_n$ and hence $\mu(t) \leq \Lambda_n$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq \Lambda_n$.

Now, we show that $F : Z \to B^0_{+\infty}$ is a contraction operator.

Set $z, \bar{z} \in Z$, thus for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$\begin{aligned}
& |F(z)(t) - F(\bar{z})(t)| \leq |g(t, z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}) - g(t, \bar{z}_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)})| \\
+ & \int_0^t \|T_\alpha(t-s)\|_{B(E)} |A(s)[g(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)}) - g(s, \bar{z}_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)})]| ds \\
+ & \int_0^t \|T_\alpha(t-s)\|_{B(E)} |f(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)}) - f(s, \bar{z}_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)})| ds \\
\leq & \|A^{-1}\|_{B(E)} \ |Ag(t, z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}) - Ag(t, \bar{z}_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)})| \\
+ & \int_0^t \widehat{M} \ |Ag(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)}) - Ag(s, \bar{z}_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)})| ds \\
+ & \int_0^t \widehat{M} \ |f(s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)}) - f(s, \bar{z}_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)})| ds \\
\leq & \bar{M}_0 L_\star \|z_{\rho(t,z_t+y_t)} - \bar{z}_{\rho(t,z_t+y_t)}\|_{\mathcal{B}} + \int_0^t \widehat{M} L_\star \|z_{\rho(s,z_s+y_s)} - \bar{z}_{\rho(s,z_s+y_s)}\|_{\mathcal{B}} ds \\
+ & \int_0^t \widehat{M} l_n(s) \|z_{\rho(s,z_s+y_s)} - \bar{z}_{\rho(s,z_s+y_s)}\|_{\mathcal{B}} ds \\
\leq & \bar{M}_0 L_\star \|z_{\rho(t,z_t+y_t)} - \bar{z}_{\rho(t,z_t+y_t)}\|_{\mathcal{B}} + \int_0^t \widehat{M} [L_\star + l_n(s)] \|z_{\rho(s,z_s+y_s)} - \bar{z}_{\rho(s,z_s+y_s)}\|_{\mathcal{B}} ds.
\end{aligned}$$

Since $\|z_{\rho(t,z_t+y_t)}\|_{\mathcal{B}} \leq K_n |z(t)| + c_n$ we obtain

$$|F(z)(t) - F(\bar{z})(t)| \leq \bar{M}_0 L_* K_n |z(t) - \bar{z}(t)| + \int_0^t \widehat{M} [L_* + l_n(s)] K_n |z(s) - \bar{z}(s)| ds.$$

Let us take here $\bar{l}_n(t) = \widehat{M} K_n [L_* + l_n(t)]$ for the family semi norms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$, then

$$\begin{aligned}
|F(z)(t) - F(\bar{z})(t)| & \leq \bar{M}_0 L_* K_n |z(t) - \bar{z}(t)| + \int_0^t \bar{l}_n(s) \ |z(s) - \bar{z}(s)| ds \\
& \leq [\bar{M}_0 L_* K_n e^{\tau L_n^*(t)}][e^{-\tau L_n^*(t)} |z(t) - \bar{z}(t)|] \\
& + \int_0^t [\bar{l}_n(s) e^{\tau L_n^*(s)}][e^{-\tau L_n^*(s)} |z(s) - \bar{z}(s)|] ds \\
& \leq \bar{M}_0 L_* K_n e^{\tau \ L_n^*(t)} \|z - \bar{z}\|_n + \int_0^t [\frac{e^{\tau L_n^*(s)}}{\tau}]' \ ds \|z - \bar{z}\|_n \\
& \leq [\bar{M}_0 L_* K_n + \frac{1}{\tau}] e^{\tau L_n^*(t)} \|z - \bar{z}\|_n.
\end{aligned}$$

Therefore,

$$\|F(z) - F(\bar{z})\|_n \le [\bar{M}_0 L_* K_n + \frac{1}{\tau}]\|z - \bar{z}\|_n.$$

So, for an appropriate choice of L_* and τ such that

$$\bar{M}_0 L_* K_n + \frac{1}{\tau} < 1,$$

the operator F is a contraction for all $n \in \mathbb{N}$. By the choice of Z there is no $z \in \partial Z^n$ such that $z = \lambda\ F(z)$ for some $\lambda \in (0,1)$. Then the statement $S2$ in Theorem 1.6.2 does not hold. A consequence of the nonlinear alternative of Frigon and Granas shows that the statement $S1$ holds. We deduce that the operator F has a unique fixed-point $z^\star$. Then $x^\star(t) = z^\star(t) + y^\star(t)$, $t \in (-\infty, +\infty)$ is a fixed point of the operator N, which is the unique mild solution of the problem (3.1.1)-(3.1.2).

3.3 An Example

To illustrate our results, we give an example

Example 3.3.1 *Consider the neutral evolution equation*

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha}[u(t,\xi) - \int_{-\infty}^{0} a_3(s-t)u(s-\rho_1(t)\rho_2(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta),\xi)ds] \\ = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \int_{-\infty}^{0} a_1(s-t)u(s-\rho_1(t)\rho_2(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta),\xi)ds, \quad t \ge 0,\ \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \qquad t \ge 0, \\ v(\theta,\xi) = v_0(\theta,\xi), \qquad -\infty < \theta \le 0,\ \xi \in [0,\pi], \end{cases} \tag{3.3.7}$$

To represent this system in the abstract form (3.1.1)-(3.1.2), *we choose the space*
$E = L^2([0,\pi],\mathbb{R})$ *and the operator $A : D(A) \subset E \to E$ is given by $A\omega = \omega''$ with domain*

$$D(A) := H^2(0,t) \cap H_0^1(0,t).$$

It is well known that A is an infinitesimal generator of an α-resolvent family $(T_\alpha(t))_{t\ge 0}$ on E. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, *with $n \in \mathbb{N}$ and corresponding normalized eigenfunctions given by*

$$z_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(n\xi).$$

In addition, $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E. and

$$T_\alpha(t)x = \sum_{n=1}^{\infty} \exp^{-n^2 t}(x, z_n)z_n \ for\ x \in E \ and\ t \geq 0.$$

Theorem 3.3.2 *Let $\mathcal{B} = BUC(\mathbb{R}, E)$ and $\varphi \in \mathcal{B}$. Assume that condition (H_φ) holds, $\rho_i : [0, \infty) \to [0, \infty), i = 1, 2$, are continuous and the functions $a_i : \mathbb{R} \to \mathbb{R}$ are continuous for $i = 1, 2, 3$. Then there exists a unique mild solution of* (3.3.7).

Proof 3.3.3 *By the assumptions of the above theorem, we have that*

$$f(t, \psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s, \xi)ds,$$

$$g(t, \psi)(\xi) = \int_{-\infty}^{0} a_3(s)\psi(s, \xi)ds,$$

$$\rho(s, \psi) = s - \rho_1(s)\rho_2 \left(\int_0^{\pi} a_2(\theta)|\psi(0, \xi)|^2 d\theta \right),$$

are well defined functions, which permit to transform system (3.3.7) *into the abstract system* (3.1.1)-(3.1.2). *Moreover, the function f is bounded linear operator. Now, the existence of a mild solution can be deduced from a direct application of theorem* (3.3.2). *We have the following result.*

Chapter 4

Fractional Functional Differential Inclusions with State Dependent Delay

4.1 Introduction

Our interest in this chapter is to get existence and uniqueness of mild solutions for fractional functional differential inclusions with state-dependent delay in the infinite case. The problem studied here is the following fractional functional differential inclusion of the forme

$$D^{\alpha}x(t) \in Ax(t) + F(t, x_{\rho(t,x_t)}), t \in [0, +\infty), \tag{4.1.1}$$

$$x(t) = \varphi(t), t \in (-\infty, 0], \tag{4.1.2}$$

where $\alpha \in (0,1)$, $A : D(A) \subset E \to E$ is the infinitesimal generator of an α-resolvent family $(T_\alpha(t))_{t\geq 0}$ defined on a separable real Banach space $(E, \|.\|_E)$, D^α is the fractional Riemann-Liouville derivative of order α, $F : J\times\mathcal{B} \to \mathcal{P}(E)$ is a multi-valued map with nonempty compact values, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $\rho : J\times\mathcal{B} \to \mathbb{R}$ and $\varphi : (-\infty, 0] \to E$ are appropriate given continuous functions, φ belongs to an abstract space denoted $\mathcal{B}$.

4.2 Existence results

Definition 4.2.1 *A function x is said to be a mild solution of* (4.1.1)−(4.1.2) *if x satisfies*

$$x(t) = \begin{cases} \varphi(t), t \in (-\infty, 0], \\ T_\alpha(t)\varphi(0) + \int_0^t T_\alpha(t-s)F(s, x_{\rho(s,x_s)})ds, t \in J. \end{cases} \tag{4.2.3}$$

Set $\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \leq 0\}$. We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce following hypothesis:

- (H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\varphi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that
$$\|\varphi_t\|_{\mathcal{B}} \leq L^\varphi(t)\|\varphi\|_{\mathcal{B}} \, forevery \, t \in \mathcal{R}(\rho^-).$$

  Remark 4.2.2 *The hypothesis (H_φ), is frequently verified by continuous and bounded functions. For more details, see for instance([61], Proposition 7.1.1).*

- (H_1) There exists a constant $M \geq 1$ such that $\|T_\alpha(t)\|_{B(E)} \leq M$, for all $t \in J$.
- (H_2) The multi-function $F : J \times \mathcal{B} \to \mathcal{P}_{cp,cv}(E)$ is Carathéodory.
- (H_3) For every $n \in \mathbb{N}$, there exists a positive function $l_n \in L^\infty(J, \mathbb{R}^+)$ such that:
$$H_d(F(t,u), F(t,v)) \leq l_n(t)\|u - v\|_{\mathcal{B}} \text{ for } t \in [0,n], \text{ and } u, v \in \mathcal{B},$$
  with $d(0, F(t,0)) \leq l_n(t)$, for $t \in [0,n]$.

Remark 4.2.3 *By (H_2) we can see that*

$$\|F(t,x)\|_{\mathcal{P}} \leq l_n(1 + \|x\|_{\mathcal{B}_\infty}), for\, allt \in J and x \in \mathcal{B}_\infty$$

For every $n \in \mathbb{N}^*$, we define in $\mathcal{B}_\infty = \mathcal{C}(\mathbb{R}, E)$ the semi-norms by:$\|x\|_n := \sup_{t\in[0,n]} \|x(t)\|_E$. Then $(\mathcal{B}_\infty, \|x\|_n)$ is a Fréchet space.

Lemma 4.2.4 *[59]. Let $v : J \to [0, \infty)$ be a function and ω be a non-negative locally integrable function on J. If there are constants $c > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq \omega(t) + c\int_0^t \frac{v(s)}{(t-s)^\alpha}ds.$$

then there exists a constant $K = K(\alpha)$ such that, for every $t \in J$, we have

$$v(t) \leq \omega(t) + Kc\int_0^t \frac{\omega(s)}{(t-s)^\alpha}ds.$$

Theorem 4.2.5 *Suppose that the hypothesis (H_φ) and $(H_1) - (H_3)$ are satisfied. Moreover assume that the following condition holds:*

$$l \; := \; \frac{Mn^\alpha l_n^*}{\Gamma(1+\alpha)} < 1, \textit{for each} n \in \mathbb{N}* \tag{4.2.4}$$

where $l_n^* = \|l_n\|_{L^\infty}$. Then the problem (4.1.1)-(4.1.2) has a mild solution.

Proof 4.2.6 *We transform the problem (4.1.1)-(4.1.2) into a fixed point theorem. Define the multi-valued operator $N : \mathcal{B}_{+\infty} \to \mathcal{P}(\mathcal{B}_\infty)$ by*

$$N(x)(t) = \begin{cases} \varphi(t), t \in (-\infty, 0], \\ T_\alpha(t)\varphi(0) + \int\limits_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} T_\alpha(t-s)f(s)ds, t \in J. \end{cases} \tag{4.2.5}$$

Where $f \in S_{F,x}$.
Clearly, the fixed points of the operator N are mild solutions of the problem (4.1.1)-(4.1.2).
We remark also that, for each $x \in \mathcal{B}_\infty$, the set $\mathcal{B}_\infty$ is nonempty, since by (H_2), F has a measurable selection [38].
Let x be a possible fixed point of the operator N. Given $n \in \mathbb{N}^*$ and $t \leq n$, then x should be a solution of the inclusion $x \in N(x)$ for some $\lambda \in (0, 1)$. So, by (H_1), (H_2) and (H_φ) and Lemma (4.2.4) there exists $f \in S_{F,x}$ such

that, for each $t \in J$ we have

$$\begin{aligned}\|x(t)\|_E &\leq \|T_\alpha(t)\|\|\varphi(0)\|_E + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|T_\alpha(t-s)\|_{B(E)}\|f(s)\|_E ds \\ &\leq M\|\varphi\| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|T_\alpha(t-s)\|_{B(E)}\|f(s, x_{\rho(s,x_s)})\|_E ds \\ &\leq M\|\varphi\| + M\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} l_n(s)(1+\|x_{\rho(s,x_s)}\|)ds \\ &\leq M\|\varphi\| + M\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} l_n(s)ds + M\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} l_n(s)||x_{\rho(s,x_s)}||ds \\ &\leq M\|\varphi\| + Ml_n^*\frac{n^\alpha}{\Gamma(1+\alpha)} + Ml_n^*\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|x_{\rho(s,x_s)}\|ds\end{aligned}$$

We define the function v by

$$v(t) = \sup_{s\in[0,t]} \|x(s)\|_E fort \in J.$$

Let $t^* \in [0,t]$ be such that $v(t) = \|x(t^*)\|$. If $t^* \in [0,t]$, then $v(t) = \|\varphi\|$ and if $t^* \in [0,n]$, then by the previous inequality, we have

$$v(t) \leq M\|\varphi\| + Ml_n^*\frac{n^\alpha}{\Gamma(1+\alpha)} + Ml_n^*\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds.$$

Set $\omega_n := M\|\varphi\| + Ml_n^*\frac{n^\alpha}{\Gamma(1+\alpha)}$ and $c_n := \frac{Ml_n^*}{\Gamma(\alpha)}$. Then

$$v(t) \leq \omega_n + c_n\int_0^t (t-s)^{\alpha-1} v(s)ds.$$

By Lemma (4.2.4), there exists a constant $K = K_\alpha$ such that

$$v(t) \leq \omega_n(1 + \frac{Kc_n n^\alpha}{\Gamma(\alpha+1)}) := D_n.$$

Thus, for every $t \in [0,n]$ $v(t) \leq D_n$. Since $\|x_t\| \leq v(t)$,then

$$\|x\|_n \leq \max\{\|\varphi\|, D_n\} := \Delta_n.$$

Set

$$\mathcal{U} = \{x \in \mathcal{B}_\infty : \|x\|_n < 1 + \Delta_n, n \in \mathbb{N}^*\} isopen.$$

Clearly, $\mathcal{U}$ is an open subset of $\mathcal{B}_\infty$.

We show that

$$N : \overline{\mathcal{U}} \to \mathcal{P}(\mathcal{B}_\infty),$$

is a contraction and admissible operator.
First, we prove that N is a contraction. Let $x, \overline{x} \in \mathcal{B}$ and $h \in N(x)$. Then there exists $f \in S_{F,x}$ such that for each $t \in [0,n]$, we have

$$h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} T_\alpha(t-s) f(s) ds.$$

By (H_3)it follows that

$$H_d(F(t, x_{\rho(t,x_t)}), F(t, \overline{x}_{\rho(t,x_t)})) \le l_n(t) \|x_{\rho(t,x_t)} - \overline{x}_{\rho(t,x_t)}\|.$$

Hence there is $\xi \in S_{F,\overline{x}_{\rho(t,x_t)}}$ such that

$$\|f(t) - \xi\| \le l_n(t) \|x_{\rho(t,x_t)} - \overline{x}_{\rho(t,x_t)},\ t \in [0,n].$$

Define

$$\overline{\mathcal{U}^*} \to \mathcal{P}(\mathcal{B}_\infty)$$

by

$$\overline{\mathcal{U}^*} = \{\xi \in E : \|f(t) - \xi\| \le l_n(t) \|x_{\rho(t,x_t)} - \overline{x}_{\rho(t,x_t)}\|\}.$$

Since the multi-valued operator $\mathcal{V} = \mathcal{U}^*(t) \cap \overline{x}_{\rho(t,x_t)}$ is measurable [38], there exists a function $\overline{f}(t)$, which is a measurable selection for $\mathcal{V}$.
So $\overline{f}(t) \in S_{F,\overline{x}_{\rho(t,x_t)}}$ and for each $t \in [0,n]$, we obtain

$$\|f(t) - \overline{f}(t)\| \le l_n(t) \|x_{\rho(t,x_t)} - \overline{x}_{\rho(t,x_t)}\|.$$

Let us define for each $t \in [0,n]$,

$$\overline{h}(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} T_\alpha(t-s) \overline{f}(s) ds.$$

Then for each $t \in [0,n]$ we have

$$\begin{aligned} \|h(t) - \overline{h}(t)\|_E &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|T_\alpha(t-s)\|_{B(E)} \|f(s) - \overline{f}(s)\|.ds \\ &\le \frac{M n^\alpha l_n^*}{\Gamma(1+\alpha)} \|x - \overline{x}\|_n. \end{aligned}$$

Hence

$$\|h-\overline{h}\|_n \leq l\|x-\overline{x}\|_n.$$

By an analogous relation, we obtain by interchanging the roles of x *and* $\overline{x}$, *it follows that*

$$H_d(N(x), N(\overline{x})) \leq l\|x-\overline{x}\|_n.$$

By the condition (4.2.4), N *is a contraction for all* $n \in \mathbb{N}^*$.
It remains to show that N *is an admissible operator.*
Let $x \in \mathcal{C}((-\infty,+\infty),E)$. *Define* $N : \mathcal{C}((-\infty,n],E) \to \mathcal{P}(\mathcal{C}((-\infty,n],E))$, *by*

$$N(x)(t) = \begin{cases} \varphi(t), t \in (-\infty, 0], \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} T_\alpha(t-s) f(s) ds, t \in [0,n]. \end{cases} \tag{4.2.6}$$

By $(H_1)-(H_3)$*and the fact that* F *is a multi-valued map with compact values, we can prove that for every* $x \in \mathcal{C}((-\infty,n],E)$, $N(x) \in \mathcal{P}_{cp}(\mathcal{C}((-\infty,n],E))$ *and there exists* $x_* \in \mathcal{C}((-\infty,n],E)$ *such that* $x_* \in N(x_*)$.
Let $h \in \mathcal{C}((-\infty,n],E))$,$\overline{x} \in \overline{\mathcal{U}}$ *and* $\epsilon > 0$.
Assume that $x_* \in N(\overline{x})$, *then*

$$\begin{aligned} \|\overline{x}(t) - x_*(t)\| &\leq \|\overline{x}(t) - h(t)\| + \|x_*(t) - h(t)\| \\ &\leq \|\overline{x} - N(\overline{x})\|_n + \|x_*(t) - h(t)\|. \end{aligned}$$

Since h *is arbitrary, we may suppose that*

$$h \in B(x_*, \epsilon) = \{h \in \mathcal{C}((-\infty,n],E) : \|h - x_\star\|_n \leq \epsilon\}.$$

Therefore,

$$\|\overline{x} - x_*\| \leq \|\overline{x} - N(\overline{x})\|_n + \epsilon.$$

If s *is not in* $N(\overline{x})$, *then* $\|x_* - N(\overline{x})\| \neq 0$. *Since* $N(\overline{x}\|$ *is compact, there exists* $y \in N(\overline{x}$ *such that* $\|x_* - N(\overline{x})\| = \|x_* - y\|$.
Then we have

$$\begin{aligned} \|\overline{x}(t) - y(t)\| &\leq \|\overline{x}(t) - h(t)\| + \|y(t) - h(t)\| \\ &\leq \|\overline{x} - N(\overline{x})\|_n + \|y(t) - h(t)\|. \end{aligned}$$

Thus,

$$\|\overline{x} - y\|_n \leq \|\overline{x} - N(\overline{x})\|_n + \epsilon.$$

So, N *is an admissible operator contraction. By the choice of* $\mathcal{U}$, *there is no* $x \in \partial\mathcal{U}$ *such that* $x \in \lambda N(x)$ *for some* $\lambda \in (0,1)$. *Then by applying of* (1.6.3) *we deduce that the operator* N *has a fixed point* x^* *which is a mild solution of the problem* (4.1.1)-(4.1.2).

4.3 Example

To illustrate our results, consider the following system

$$\begin{cases} D^{\alpha}_{0,t}u(t,\xi) \in a(t,\xi)\frac{\partial^2 u}{\partial \xi^2}(t,\xi) \\ +\int_{-\infty}^{0} P(\theta)R(t,u(t+\rho(\theta),\xi)d\theta, t\geq 0, \xi\in[0,\pi], \\ u(t,0)=u(t,\pi)=0, t\geq 0, \\ u(\theta,\xi)=u_0(\theta,\xi), -\infty<\theta\leq 0, \xi\in[0,\pi]. \end{cases} \tag{4.3.7}$$

Where $a(t,\xi)$ is a continuous function and is uniformly Hölder continuous in t, $\rho : [0,+\infty)\times\mathcal{C} \to (-\infty,+\infty)$ is continuous, $P : (-\infty,0] \to \mathbb{R}$, $u : (-\infty,0]\times\mathbb{R} \to \mathbb{R}$ and $u_0 : (-\infty,0]\times[0,\pi] \to \mathbb{R}$ are continuous functions, $R : (-\infty,0]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map with compact convex values and $D^{\alpha}_{0,t}u(t,\xi)$ denotes the Riemann-Liouville fractional derivative of order $\alpha\in(0,1]$ of u with respect to t. It is defined by the expression

$$D^{\alpha}_{0,t} = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t (t-s)^{-\alpha}u(s,\xi)ds.$$

Proof 4.3.1 *Let $E = L^2([0,\pi],\mathbb{R})$ and define the operator $A : D(A)\subset E \to E$ by $A\omega = a(t,\xi)\omega''$ with domain*

$$D(A) := H^2(0,\pi)\cap H^1_0(0,\pi).$$

For $\xi\in[0,\pi]$, we have

$$x(t)(\xi) = u(t,\xi), t\in[0,+\infty),$$

$$\varphi(\theta)(\xi) = u_0(t,\xi), -\infty<\theta\leq 0,$$

and

$$F(t,\eta)(\xi) = \int_{-\infty}^{0} P(\theta)R(t,u(t,\eta(\theta)(\xi)d\theta, -\infty<\theta\leq 0.$$

Then the problem (4.3.7) *takes the fractional differential inclusion form* (4.1.1)-(4.1.2).
In order to show the existence of mild solutions of problem (4.3.7)*, we suppose the following assumptions:*

- *u is Lipschitz with respect to its second argument. Let $lip(u)$ denotes the Lipschitz constant of u*
- *There exists $p \in L_\infty(J, \mathbb{R}^+)$ such that*

$$|R(t,\eta)| \leq p(t) + |\eta|), \textit{for } t \in J,\ \ \eta \in \mathbb{R}.$$

- *P is integrable on $(-\infty, 0]$.*

By the dominated convergence theorem, we can show that $f \in S_{F,x}$ is a continuous function from $\mathcal{C}(-\infty,0], E)$ to E. In fact, for $\eta \in \mathbb{R}$ and $\xi \in [0,\pi]$, we have

$$|F(t,\eta)(\xi)| \leq \int_{-\infty}^{0} |p(t)P(\theta)| \, (1 + |(\eta(\theta))(\xi)|) d\theta.$$

Thus

$$\|F(t,\eta)\|_{\mathcal{P}(E)} \leq p(t) \int_{-\infty}^{0} |P(\theta)| \, d\theta (1 + |\eta|).$$

Under the above assumptions, if we assume that condition (4.2.4) *in Theorem* (4.2.5) *is true, then the problem* (4.3.7) *has a mild solution which is defined in* $(-\infty, +\infty)$.

Chapter 5

Fractional Neutral Functional Differential Inclusions with State Dependent Delay

5.1 Introduction

In this chapter we provide sufficient conditions for the existence and uniqueness of mild solutions for a class of neutral abstract fractional functional differential inclusions with state-dependent delay by using the nonlinear alternative of Frigon for admissible contractions maps in Fréchet spaces. Also an example is given to illustrate our results. Recently Baghli *et al.* [17], have proved global existence and uniqueness results for functional differential evolution inclusions with state dependent delay in the integer case. Motivated by the above paper, our interest is to get existence and uniqueness of mild solutions for the following fractional functional differential inclusion with state dependent delay of the forme

$$^{c}D_{0}^{\alpha}[x(t)-g(t,x_{\rho(t,x_t)})]\in A[x(t)-g(t,x_{\rho(t,x_t)})]+F(t,x_{\rho(t,x_t)})t\in[0,+\infty), \tag{5.1.1}$$

$$x(t)=\varphi(t), t\in(-\infty,0], \tag{5.1.2}$$

where $r>0,\ \alpha\in(0,1],\ ^{c}D_{0}^{\alpha}$ is the fractional Caputo derivative of order $\alpha\in(0,1],\ F:J\times\mathcal{B}\to\mathcal{P}(E)$ is a multi-valued map with nonempty compact values, $(E,\|\cdot\|_E)$ is a Banach space, $\mathcal{P}(E)$ is the family of all nonempty subsets of $E,\ g:J\times\mathcal{B}\to E,\ \rho:J\times\mathcal{B}\to\mathbb{R}$ and $\varphi:(-\infty,0]\to E$ are given continuous functions, A is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{T(t)\}_{t>0}$ in E and $\mathcal{B}$ is called the phase space. Our results are based into the following nonlinear alternative of Frigon for contractive multi-valued maps in Fréchet spaces. (see (4.2.5))

In the sequel we make use of the following Gronwall's lemma.

Lemma 5.1.1 *[56] Let $\upsilon : J \to [0,\infty)$ be a real function and ω be a non-negative, locally integrable function on J. If there are constants $c > 0$ and $0 < \alpha < 1$ such that*

$$\upsilon(t) \leq \omega(t) + c\int_0^t \frac{\upsilon(s)}{(t-s)^\alpha}ds,$$

then there exists a constant $\delta = \delta(\alpha)$ such that, for every $t \in J$, then

$$\upsilon(t) \leq \omega(t) + \delta c\int_0^t \frac{\omega(s)}{(t-s)^\alpha}ds$$

5.2 Existence of mild solutions

Set

$$\Omega := \{x : (-\infty, a] \to E : \ x_0 \in \mathcal{B} \text{ and} u|_J \in \mathcal{C}\}.$$

we now introduce the definition of mild solution to (5.1.1)-(5.1.2).

Definition 5.2.1 *[96] A function $x \in \Omega$ is said to be a mild solution of* (5.1.1)-(5.1.2) *if there exists a function $f \in S_{F,x}$ such that*

$$x(t) = \begin{cases} \varphi(t), t \in (-\infty, 0], \\ S_\alpha(t)(\varphi(0) - g(0, x_{\rho(0,x_0)})) + g(t, x_{\rho(t,x_t)}) \\ + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s)ds, t \in J \end{cases} \tag{5.2.3}$$

Remark 5.2.2 *It is not difficult to verify that for $y \in [0,1]$,*

$$\int_0^\infty \theta^y \xi_\alpha(\theta)d\theta = \int_0^\infty \theta^{-\alpha y}\overline{w}_\alpha(\theta)d\theta \tag{5.2.4}$$

$$= \frac{\Gamma(1+y)}{\Gamma(1+\alpha y)}. \tag{5.2.5}$$

Lemma 5.2.3 *[96] For any $t \geq 0$, the operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties:*

- *(a) For any fixed $t \geq 0$, S_α and T_α are linear and bounded operators, ie. for any $x \in E$,*

$$\|S_\alpha(t)x\|_E \leq M\|x\|_E,\ \|T_\alpha(t)x\|_E \leq \frac{M}{\Gamma(\alpha)}\|x\|_E.$$

- *(b) $\{S_\alpha(t) : t \geq 0\}$ and $\{T_\alpha(t);\ t \geq 0\}$ are strongly continuous.*

- *(c) For every $t \geq 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are also compact operators.*

Set $\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B},\ \rho(s,\varphi) \leq 0\}$. We always assume that $x : t \mapsto x_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$.
Let us introduce the following hypotheses:

- (H_1) The multifunction $F : J \times \mathcal{B} \to \mathcal{P}_{\rfloor_{\surd},\rfloor\sqsubseteq}(E)$ is Carathéodory,

- (H_2) For every $n \in \mathbb{N}$, there exists a positive function $l_n \in L^\infty(J,\mathbb{R})$ such that

$$H_d(F(t,x),F(t,y)) \leq l_n(t)\|x-y\|_{\mathcal{B}}, for\ t \in [0,n],\ x,y \in \mathcal{B}$$

  and $\quad d(0,F(t,0)) \leq l_n(t) \quad$ for $\quad t \in [0,n]$.

- (H_3) For every $n \in \mathbb{N}$, there exists a constant $l_g > 0$, such that

$$\|g(t,x) - g(t,y)\|_E \leq l_g\|x-y\|_{\mathcal{B}},\ and\ \|g(t,0)\|_E \leq l_g,$$

  for each $t \in [0,n]$, and $x,y \in \mathcal{B}$.

Remark 5.2.4 *By (H_2), we can see that*

$$\|F(t,x)\|_{\mathcal{P}} \leq l_n(t)(1+\|x\|_{\mathcal{B}});\ for\ all\ t \in J\ and\ x \in \mathcal{B}.$$

Also, by (H_3), we can see that

$$\|g(t,x)\|_E \leq l_g(1+\|x\|_{\mathcal{B}}),\ for\ all\ t \in J\ and\ x \in \mathcal{B}.$$

Theorem 5.2.5 *Suppose that the hypotheses* $(H_1)-(H_3)$ *are satisfied. Moreover, assume that the following condition holds:*

$$\ell := 2MK_n l_g + \frac{MK_n l_n^* n^\alpha}{\Gamma(1+\alpha)} < 1; \quad \textit{for each} \;\; n \in \mathbb{N}^*, \tag{5.2.6}$$

where $l_n^* := \|l_n\|_{L^\infty}$*. Then the problem* (5.1.1)-(5.1.2) *has a mild solution.*

Proof 5.2.6 *Transform the problem* (5.1.1)-(5.1.2) *into a fixed point problem. Consider the multi-valued operator* $N : \Omega \to \mathcal{P}(\Omega)$ *defined by:*

$$(Nx)(t) = \left\{ h \in \Omega : h(t) = \left\{ \begin{array}{l} \varphi(t), t \in (-\infty, 0], \\ S_\alpha(t)(\varphi(0) - g(0, x_{\rho(0,x_0)})) + g(t, x_{\rho(t,x_t)}) \\ + \int\limits_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s) ds, t \in J. \end{array} \right. \right\}$$

where $f \in S_{F,x}$*. Clearly, the fixed points of the operator* N *are mild solutions of the problem* (5.1.1)-(5.1.2) *We remark also that, for each* $x \in \mathcal{B}_\infty$*, the set* $S_{F,x}$ *is nonempty since, by* (H_1)*,* F *has a measurable selection [38], (Theorem III.6).*

For $\varphi \in \mathcal{B}$*, we define the function* $y : \mathbb{R} \to E$ *as follows:*

$$y(t) = \left\{ \begin{array}{l} \varphi(t), t \leq 0, \\ 0, t \in J. \end{array} \right.$$

Then $x_0 = \varphi$*. For each function* $z \in \Omega$ *with* $z(0) = 0$*, we define the function* $\bar{z}$ *by*

$$\bar{z}(t) = \left\{ \begin{array}{l} 0, t \leq 0, \\ z(t), t \in J. \end{array} \right.$$

Let $x(\cdot)$ *satisfies*

$$\begin{cases} x(t) = \varphi(t), t \in (-infty, 0] \\ x(t) = S_\alpha(t)(\varphi(0) - g(0, x_{\rho(0,x_0)})) + g(t, x_{\rho(t,x_t)}) \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s) ds, t \in J, \end{cases}$$

where $f \in S_{F,x}$, *and decompose* $x(\cdot)$ *as* $x(t) = \bar{z}(t) + y(t)$, $t \geq 0$, *which implies* $x_t = \bar{z}_t + y_t$, *for every* $t \in J$ *and the function* $z(\cdot)$ *satisfies* $z_0 = 0$ *and for* $t \in J$, *we get that*

$$\begin{aligned} z(t) &= S_\alpha(t)(\varphi(0) - g(0, \bar{z}_{\rho(0,x_0)} + y_{\rho(0,x_0)})) + g(t, \bar{z}_{\rho(t,x_t)} + y_{\rho(t,x_t)}) \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s) ds. \end{aligned}$$

For each $n \in \mathbb{N}$, *set*

$$C_0 = \{w \in C([0,n], E) : \ w(0) = 0\},$$

and let $\|.\|_n$ *be the semi-norm in* C_0 *defined by*

$$\|w\|_n = \|w_0\|_{\mathcal{B}} + \sup_{t \in [0,n]} \|w(t)\| = \sup_{t \in [0,n]} \|w(t)\|, w \in C_0.$$

Then C_0 *is a Fréchet space with these semi-norms family* $\|.\|_n$.

Define the operator $P : C_0 \to C_0$ *by:*

$$\begin{aligned} (Pz)(t) &= S_\alpha(t)(\varphi(0) - g(0, \bar{z}_{\rho(0,x_0)} + y_{\rho(0,x_0)})) + g(t, \bar{z}_{\rho(t,x_t)} + y_{\rho(t,x_t)}) \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s) ds. \end{aligned} \tag{5.2.7}$$

Obviously the operator N *has a fixed point is equivalent to say that* P *has one, so it turns to prove that* P *has a fixed point. Let* $z \in C_0$ *be such that* $z = \lambda P(z)$ *for some* $\lambda \in (0,1)$. *Then for each* $t \in [0,n]$, *there exists* $f \in S_{F,x}$

such that, for each $t \in J$, we have

$$\begin{aligned}
\|z(t)\|_E &\leq \|S_\alpha(t)(\varphi(0) - g(0, \bar{z}_{\rho(0,x_0)} + y_{\rho(0,x_0)})) + g(t, \bar{z}_{\rho(t,x_t)} + y_{\rho(t,x_t)})\| \\
&+ \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s) f(s)\|_E ds \\
&\leq M\|\varphi(0) - g(0, \bar{z}_{\rho(0,x_0)} + y_{\rho(0,x_0)})\|_E + \|g(t, \bar{z}_{\rho(t,x_t)} + y_{\rho(t,x_t)})\|_E \\
&+ \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s)\|_E ds \\
&\leq M\|\varphi\|_{\mathcal{B}} + Ml_g(1 + \|\bar{z}_{\rho(0,x_0)} + y_{\rho(0,x_0)}\|_{\mathcal{B}}) \\
&+ Ml_g(1 + \|\bar{z}_{\rho(t,x_t)} + y_{\rho(t,x_t)}\|_{\mathcal{B}}) \\
&+ \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l_n^*(1 + \|\bar{z}_{\rho(s,x_s)} + y_{\rho(s,x_s)}\|_{\mathcal{B}}) ds \\
&\leq M\|\varphi\|_{\mathcal{B}} + 2Ml_g + \frac{Ml_n^* n^\alpha}{\Gamma(1+\alpha)} + 2Ml_g\|\bar{z}_{\rho(t,x_t)} + y_{\rho(t,x_t)}\|_{\mathcal{B}} \\
&+ \frac{Ml_n^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\bar{z}_{\rho(s,x_s)} + y_{\rho(s,x_s)}\|_{\mathcal{B}} ds.
\end{aligned}$$

Using the assumption (A_1) we get

$$\begin{aligned}
\|\bar{z}_{\rho(t,\bar{z}_s+y_s)} + y_{\rho(t,\bar{z}_s+y_s)}\|_{\mathcal{B}} &\leq \|\bar{z}_{\rho(t,\bar{z}_t+y_t)}\|_{\mathcal{B}} + \|y_{\rho(t,\bar{z}_t+y_t)}\|_{\mathcal{B}} \\
&\leq K(t)\|\bar{z}(t)\|_E + M(t)\|\bar{z}_0\|_{\mathcal{B}} + K(t)\|y(t)\|_E + M(t)\|y_0\|_{\mathcal{B}} \\
&\leq K_n\|\bar{z}(t)\|_E + M_n M\|\varphi(0)\| + M_n\|\varphi\|_{\mathcal{B}} \\
&\leq K_n\|\bar{z}(t)\|_E + M_n M H\|\varphi\|_{\mathcal{B}} + M_n\|\varphi\|_{\mathcal{B}} \\
&\leq K_n\|\bar{z}(t)\|_E + (K_n M H + M_n)\|\varphi\|_{\mathcal{B}} \\
&\leq c_n + K_n\|\bar{z}(t)\|_E,
\end{aligned}$$

where $c_n = (K_n MH + M_n)\|\varphi\|_{\mathcal{B}}$. Then,we obtain

$$\begin{aligned}
\|z(t)\|_E &\leq M\|\varphi\|_{\mathcal{B}} + 2Ml_g + \frac{Ml_n^* n^\alpha}{\Gamma(1+\alpha)} + 2Ml_g(c_n + K_n\|z(t)\|_E) \\
&+ \frac{Ml_n^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (c_n + K_n\|z(s)\|_E) ds \\
&\leq M\|\varphi\|_{\mathcal{B}} + 2Ml_g + \frac{Ml_n^* n^\alpha}{\Gamma(1+\alpha)} + 2Ml_g c_n + 2Ml_g K_n\|z(t)\|_E \\
&+ \frac{Ml_n^* c_n n^\alpha}{\Gamma(1+\alpha)} + \frac{Ml_n^* K_n}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z(s)\|_E ds.
\end{aligned}$$

Then

$$\begin{aligned}(1-2Ml_gK_n)\|z(t)\|_E &\leq M\|\varphi\|_{\mathcal{B}} + (1+c_n)\left(2Ml_g + \frac{Ml_n^*n^\alpha}{\Gamma(1+\alpha)}\right)\\ &+ \frac{Ml_n^*K_n}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|z(s)\|_E ds.\end{aligned}$$

Set

$$w_n := \frac{1}{1-2Ml_gK_n}M\|\varphi\|_{\mathcal{B}} + (1+c_n)\left(2Ml_g + \frac{Ml_n^*n^\alpha}{\Gamma(1+\alpha)}\right),$$

and

$$c_n = \frac{Ml_n^*K_n}{(1-2Ml_gK_n)\Gamma(\alpha)}.$$

Thus

$$\|z(t)\|_E \leq w_n + c_n\int_0^t (t-s)^{\alpha-1}\|z(s)\|_E ds.$$

By Lemma (5.1.1), *there exists a constant* $\delta := \delta(\alpha)$ *such that*

$$\begin{aligned}z(t) &\leq w_n\left(1+\delta c_n\int_0^t (t-s)^{\alpha-1}ds\right)\\ &\leq w_n\left(1+\frac{\delta c_n n^\alpha}{\alpha}\right) := D_n.\end{aligned}$$

Set

$$\mathcal{U} = \{x \in C_0 : \|x\|_n < 1 + D_n, n \in \mathbb{N}*\}.$$

Clearly, $\mathcal{U}$ *is an open subset of* C_0.

We show that $P : \overline{\mathcal{U}} \to C_0$ *is a contraction and an admissible operator. Let* $z,\ z* \in C_0$, *such that for each* $t \in [0,n], n \in \mathbb{N}*$,

$$\begin{aligned}z(t) &= S_\alpha(t)(\varphi(0) - g(0,\bar{z}_{\rho(0,z_0)})) + g(t,\bar{z}_{\rho(t,z_t)})\\ &+ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s)ds,\end{aligned}$$

where $f \in S_{F,z}$ *and* $z_{\rho(t,z_t)} = \bar{z}_{\rho(t,\bar{z}_t+y_t)} + y_{\rho(t,\bar{z}_t+y_t)}$ *and*

$$\begin{aligned}z^*(t) &= S_\alpha(t)(\varphi(0) - g(0,\bar{z}^*_{\rho(0,z_0)})) + g(t,\bar{z}^*_{\rho(t,z_t)})\\ &+ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f^*(s)ds,\end{aligned}$$

where $f^* \in S_{F,z*}$ *and* $z*_{\rho(t,z_t)} = \bar{z} *_{\rho(t,\bar{z}_t+y_t)} + y_{\rho(t,\bar{z}_t+y_t)} \in C_0$. *Then,*

$$\begin{aligned}
\|(Pz)(t)-(Pz^*)(t)\|_E &\leq \|S_\alpha(t)(g(0,\bar{z}_{\rho(0,z_0)})-g(0,z^*_{\rho(0,z_0)}))\|_E \\
&+ \|g(t,z_{\rho(t,z_t)})-g(t,z^*_{\rho(t,z_t)})\|_E \\
&+ \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)(f(s)-f^*(s)\|_E ds \\
&\leq Ml_g\|z_{\rho(0,z_0)}-z^*_{\rho(0,z_0)}\|_{\mathcal{B}} + Ml_g\|z_{\rho(t,z_t)}-z^*_{\rho(t,z_t)}\|_{\mathcal{B}} \\
&+ \frac{Ml_n^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|z_{\rho(s,z_s)}-z^*_{\rho(s,x_s)}\|_{\mathcal{B}} ds \\
&\leq Ml_g\|\bar{z}_{\rho(0,\bar{z}_0+y_0)}-\bar{z}^*_{\rho(0,\bar{z}_0+y_0)}\|_{\mathcal{B}} \\
&+ Ml_g\|\bar{z}_{\rho(t,\bar{z}_t+y_t)}-\bar{z}^*_{\rho(t,\bar{z}_t+y_t)}\|_{\mathcal{B}} \\
&+ \frac{Ml_n^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\bar{z}_{\rho(s,\bar{z}_s+y_s)}-\bar{z}^*_{\rho(s,\bar{z}_s+y_s)}\|_{\mathcal{B}} ds \\
&\leq 2MK_nl_g\|z-z^*\|_n + \frac{MK_nl_n^*n^\alpha}{\Gamma(1+\alpha)}\|z-z^*\|_n \\
&\leq \left(2MK_nl_g + \frac{MK_nl_n^*n^\alpha}{\Gamma(1+\alpha)}\right)\|z-z^*\|_n.
\end{aligned}$$

Hence

$$\|P(z)-P(\overline{z})\|_n \leq \ell\|z-\overline{z}\|_n.$$

By the condition (5.2.6), P *is a contraction for all* $n \in \mathbb{N}^*$.

It remains to show that N *is an admissible operator.*
By hypothesis $(H_1)-(H_3)$ *and in fact that* F *is a multi-valued map with compact values, we can prove that for every* $x \in C_0$, $N(x) \in \mathcal{P}_{\rfloor\surd}(C_0)$ *and there exists* $x_\star \in C_0$ *such that* $x_\star \in N(x_\star)$. *Let* $h \in C_0$, $\overline{x} \in \overline{\mathcal{U}}$ *and* $\epsilon > 0$. *Assume that* $x_\star \in P(\overline{x})$. *Then we have*

$$\begin{aligned}
\|\overline{x}(t)-x_\star(t)\| &\leq \|\overline{x}(t)-h(t)\|_E + \|x_\star(t)-h(t)\|_E \\
&\leq \|\overline{x}-N(\overline{x})\|_n + \|x_\star(t)-h(t)\|_E.
\end{aligned}$$

In fact that h is arbitrary, we may suppose that

$$h \in B(x_\star,\epsilon) = \{h \in C_0 : \|h-x_\star\|_n \leq \epsilon\}.$$

Therefore,

$$\|\overline{x}-x_\star\|_n \leq \|\overline{x}-N(\overline{x})\|_n + \epsilon.$$

If x is not in $P(\overline{x})$, then $\|x_\star - N(\overline{x})\| \neq 0$. In fact that $N(\overline{x})$ is compact, there exists $y \in N(\overline{x})$ such that $\|x_\star - N(\overline{x})\|_E = \|x_\star - y\|_E$. Then we have that

$$\begin{aligned}\|\overline{x}(t) - y(t)\|_E &\leq \|\overline{x}(t) - h(t)\| + \|y(t) - h(t)\|_E \\ &\leq \|\overline{x} - N(\overline{x})\|_n + \|y(t) - h(t)\|_E.\end{aligned}$$

Thus,

$$\|\overline{x} - y\|_n \leq \|\overline{x} - N(\overline{x})\|_n + \epsilon.$$

So, N is an admissible operator contraction.

BY the choice of $\mathcal{U}$ there is no $x \in \partial\mathcal{U}$ such that $x = \lambda P(x)$ for some $\lambda \in (0,1)$. A consequence of Theorem (1.6.3), the operator P has a fixed point $z^\star$. Then $x^\star(t) = \bar{z}^\star(t) + y^\star(t)$; $t \in (-\infty, +\infty)$ is a fixed point of the operator N, which is a mild solution of the problem (5.1.1)-(5.1.2).

5.3 Example

As an application of our results, we present the following model

$$\begin{cases} D^\alpha_{0,t}\left[v(t,\xi) - \int_{-\infty}^0 T(\theta)u(t, v(t+\rho(\theta),\xi))d\theta\right] \\ \in \frac{\partial^2}{\partial \xi^2}\left[v(t,\xi) - \int_{-\infty}^0 T(\theta)u(t, v(t+\rho(\theta),\xi))d\theta\right] \\ + \int_{-\infty}^0 P(\theta)R(t, v(t+\rho(\theta),\xi))d\theta, t \in [0,+\infty) \text{ and } \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0; \qquad t \in [0,+\infty), \\ v(\theta,\xi) = v_0(\theta,\xi)\ \theta \leq 0,\ \xi \in [0,\pi], \end{cases} \tag{5.3.8}$$

$T, P : \mathbb{R}^- \to \mathbb{R}$, $u : (-\infty, 0] \times \mathbb{R} \to \mathbb{R}$ and $v_0 : (-\infty, 0] \times [0,\pi] \to \mathbb{R}$ are continuous functions, $R : [0,+\infty) \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map with compact convex values, and ${}^cD^\alpha_{0,t}v(t,\xi)$ denotes the Caputo fractional derivative of order $\alpha \in (0,1]$ of v with respect to t. It is defined by the expression

$$D^\alpha_{0,t}v(t,\xi) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial}{\partial s}v(s,\xi)ds.$$

Set $E = L^2([0,\pi], \mathbf{R})$ and define A by $Aw = w''$ with domain

$$D(A) = H^2(0,\pi) \cap H_0^1(0,\pi).$$

For $\xi \in [0,\pi]$, we have

$$x(t)(\xi) = v(t,\xi), t \in [0,+\infty),$$

$$\varphi(\theta)(\xi) = v_0(\theta,\xi),\ \theta \leq 0,$$

$$g(t,\eta)(\xi) = \int_{-\infty}^{0} T(\theta)u(t,\eta(\theta)(\xi))d\theta, \theta \leq 0,$$

and

$$F(t,\eta)(\xi) = \int_{-\infty}^{0} P(\theta)R(t,\eta(\theta)(\xi))d\theta, \theta \leq 0.$$

Then, the problem (5.3.8) takes the neutral fractional differential inclusion form (5.1.1)-(5.1.2). In order to show the existence of mild solutions of problem (5.3.8), we suppose the following assumptions:

- u is Lipschitz with respect to its second argument. Let $lip(u)$ denotes the Lipschitz constant of u.
- There exist $p \in L_\infty(J, \mathbb{R}^+)$ such that

$$|R(t,\eta)| \leq p(t)(1 + |\eta|), \text{for} \in J, \text{and} \eta \in \mathbb{R}.$$

- T, P are integrable on $(-\infty, 0]$.

By the dominated convergence theorem, we show that $f \in S_{F,x}$ is a continuous function from $\mathcal{B}_\gamma$ to E, where $\mathcal{B}_\gamma$ is the phase space defined by

$$\mathcal{B}_\gamma := \left\{\phi \in C((-\infty,0];E) : \lim_{\theta\ to -\infty} e^{\gamma\theta}\phi(\theta) \text{exists in } E\right\}$$

endowed with the norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \theta \leq 0\}.$$

Moreover the map g is Lipschitz continuous in its second argument, in fact, we have that

$$|g(t,\eta_1) - g(t,\eta_2)| \leq \overline{M}_0 L_\star lip(u) \int_{-\infty}^{0} |T(\theta)|\, d\theta\, |\eta_1 - \eta_2|\,, \text{ for } \eta_1, \eta_2 \in \mathbb{R}.$$

On the other hand, we have for $\eta \in \mathbb{R}$ and $\xi \in [0, \pi]$,

$$|F(t, \eta)(\xi)| \leq \int_{-\infty}^{0} [|p(t)P(\theta)|](1 + |(\eta(\theta))(\xi)|)d\theta.$$

Thus

$$\|F(t, \eta)\|_{\mathcal{P}(\mathcal{E})} \leq p(t) \int_{-\infty}^{0} |P(\theta)| \, d\theta(1 + |\eta|).$$

Under the above assumptions, if we assume that the condition (5.2.6) in Theorem (5.2.5) is true, then the problem (5.3.8) has a mild solution which is defined in $(-\infty, +\infty)$.

Chapter 6

Conclusion

In this thesis, some interesting results are obtained concerning the existence and uniqueness of mild solutions for some classes of semi-linear fractional functional and neutral fractional functional differential equations and inclusions on infinite intervals with state dependent delay in Fréchet spaces. The results are based on the $\alpha-$ resolvent families theory and the argument of fixed points. Some appropriate fixed point theorems have been used: In particular we have used Frigon-Granas theorem and Frigon theorem.
For the perspective, it would be interesting to look for qualitative properties instead of quantitative ones considered in the present thesis. Another goal in the future is to look for automorphic and almost automorphic solutions of fractional functional differential equations and inclusions with state dependent delay. We can also think to apply our results in control theory.

Chapter 7

Bibliography

Bibliography

[1] S. Abbas, R. Agarwal and Benchohra, Existence results for nondensely defined impulsive semilnear functional differential equations with state-dependent delay, *Asian-Eur. J. Math*, **1**, (4), (2008), 449-468.

[2] S. Abbas and M. Benchohra, On the Set of Solutions for The Darboux problem for fractional order partial hyperbolic functional differential inclusions, *Fixed Points Theory*, **14**, (2), (2013), 253-262.

[3] S. Abbas and M. Benchohra, *Advanced Fractional Differential and Integral Equations*, Nova. Sci. Publs, New York, 2015.

[4] S. Abbas, M. Benchohra and S. A. Hammoudi, Upper, lower Solutions method and extremal solutions for impulsive discontinuous partial fractional differential inclusions, *Pan American. J. Math*, **24**, (1), (2014), 31-52.

[5] S. Abbas, M. Benchohra and G. M. Nguérékata, *Topics in fractional differential equations*, Springer, New York, 2012.

[6] S. Abbas, M. Benchohra and A N. Vityuk, On fractional order derivatives and Darboux problem for implicit differential equations, *Frac. Calc. Appl. Anal*, **15**, (2012), 168-182.

[7] S. Abbas, M. Benchohra and Y. Zhou, Fractional order partial Hyperbolic Functional Differential Equations with State Dependent Delay, *Int. J. Dyn. Sys. Diff. Eqs*, **3** (4), (2011), 459-490.

[8] M. Adimy and K. Ezzinbi, The basic theory of abstract semilnear Functional Differntial Equations with nondense domain, in " Delay Differential Equations with Applications", ed by O. Arino, M. H. Hbid and

E. Ait Dads, *NATO. Sci. Ser. Math, Phys and Chem*, Vo.**205**, (2006), Springer, Berlin, pp. 347-407.

[9] R. P. Agarwal, S. Arshad, D. O'Regan and V. Lupulescu, Fuzzy fractional integral equations under compactness type condition,*Fract. Calc. Appl. Anal*, **15**, (2012), 572-590.

[10] R. P. Agarwal, M. Belmekki and M. Benchohra, A Survey on semilinear differential Equations and inclusions involving Riemann-Liouville fractional derivative, *Adv. Diff. Equ*, **9**, 46 pages. ID 981728.

[11] N.U. Ahmed, *Semigroup Theory with applications to systems and control*, Harlow John Wiley and Sons, Inc, New York, 1991.

[12] N.U. Ahmed,*Dynamical systems, control and applications*, World. Sci. Publi. Co. Pte. Ltd, Hackensack, New York, 2006.

[13] E. Ait Dads and K. Ezzinbi, Boundedness and almost periodicity for some state-dependent delay differential equations, *Elec. J. Diff. Eqs* **67**, (2002), 13 pp.

[14] A. Arara, M. Benchohra, L. Gorniewicz and A. Ouahab, Controllability results for semilinear functional differential inclusions with unbounded delay, *Math. Bul*, **3**, (2006), 157-183.

[15] W. Arendt, C. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Monographs in Mathematics, **96**, Birkhauser Basel, 2001. r

[16] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional calculus models and numerical methods*, World Sci, New York, 2012.

[17] S. Baghli, M. Benchohra and J. J. Nieto, Global Uniqueness results for partial functional and neutral functional evolution equations with State-Dependent Delay. *J. Adv. Reas. Diff. Eqs*, **2**, (3), (2010), 35-52.

[18] D. Baleanu, Z.B. Guvenc and J.A.T. Machado, *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, New York, 2010.

[19] D. Baleanu, J.A.T. Machado and A. C. J. Luo *Fractional Dynamics and Control*, Springer, 2012.

[20] S. Baghli and M. Benchohra, Uniqueness results for partial functional differential equations in Fréchet spaces, *Fixed Points Theory*, **9**, (2) (2008), 395-406.

[21] S. Baghli and M. Benchohra, Existence results for semi linear functional differential equations involving evolution operators in Fréchet Spaces, *Georgian. Math. J.* **27**, (3), (2010), 423-436.

[22] S. Baghli and M. Benchohra, Global uniqueness results for partial functional and neutral functional evolution equations with infinite delay, *Diff. Eqs and integrals* ,(2010).

[23] S. Baghli and M. Benchohra, Perturbed functional and neutral functional evolution equations with infinite delay in Fréchet spaces, *Elec. J. Diff. Eqs*, **69**, (2008), 19 pp.

[24] K. Balachandran and J. P. Dauer, Controllability of nonlinear systems in Banach spaces: A survey dedicated to Professor Wolfram Stadler, *J. Optim. Theo. Appli*, **115**, (2002), 7-28.

[25] J. Belair and M.C. Mackey, Consumer memory and price fluctuations on commodity markets: An Integro-differential model, *J. Dynam. Diff. Eqs*, bf 1, (1989), 299-325.

[26] M. Belmekki and M. Benchohra, Existence results for fractional order semilinear functional differential equations, *Proc. A. Razmadze Math. Inst*, **146**, (2008), 9-20.

[27] M. Belmekki, M. Benchohra and K. Ezzinbi, Existence Results for some partial functonal differential equations with state dependent Delay. *J. Appli. Maths. Lett*, **24**, (2011), 1810-1816.

[28] M. Belmekki and S. K. Ntouyas, Existence results for boundary value problem for multivalued fractional differential equations, *Dyn. Sys. Appli*, **24**, (2015), 35-50.

[29] M. Benchohra, O. Bennihi and K. Ezzinbi, Semilinear functional differential equation of fractional order with state-dependent delay, *Comment. Math.* **53**, (1) (2013), 47-59.

[30] M. Benchohra, O. Bennihi and K. Ezzinbi, Fractional neutral functional differential equations with state-dependent delay, *CUBO. A Math. J*, **16**, (3) (2014), 37-53.

[31] M. Benchohra, O. Bennihi and K. Ezzinbi, Fractional functional qifferential inclusions with state-dependent delay. (Submitted).

[32] M. Benchohra, O. Bennihi and K. Ezzinbi, Fractional neutral functional differential inclusions with state-dependent delay. (Submitted).

[33] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab, Existence results for functional differential equations of fractional order with infinite delay, *J. Math. Anal. Appli*, **338**, (2008), 1340-1350.

[34] M. Benchohra and J. E. Lazreg, Nonlinear fractional implicit differential equations, *Commun. Appl Anal*, **17**, (2013), 471-482.

[35] M. Benchohra and J. E. Lazreg, Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions, *Romanian J. Math. Comput. Sc*, **4**, (1), (2014), 60-72.

[36] M. Benchohra and J. E. Lazreg, Existence results for nonlinear implicit fractional differential equations, *Surv. Math. Appl*, **9**, (2014), 79-92.

[37] M. Benchohra and S. K. Ntouyas, Controllability of second order differential inclusions in Banach spaces with nonlocal conditions, *J. Optim. Theory. Appli*, **107**, (2000), 559-505.

[38] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions, Lecture notes in Mathematics*, Vo.**580**, Springer-Verlag, New York, 1977.

[39] C. Corduneanu and V. Lakshmikantham, Equations with unbounded Delay, *A Survey. Nonli. Anal*, **TMA 4**, (5), (1980), 831-877.

[40] M. A. Darwish and S.K. Ntouyas, Semilinear functional differential equations of fractional order with state dependent delay, *Elec. J. of Diff. Eqs*, (38), (2009), 1-10.

[41] K. Deimling, *Multivalued differential equations*, Walter De Gruyter, Berlin and New York, 1992.

[42] J. V. Devi and V. Lakshmikantham, Nonsmooth and fractional differential equations, *Nonl. Anal.* (70), (2009), 4151-4157.

[43] A. Granas and J. Dugundji, *Fixed points theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, USA, 2003.

[44] R.D. Driver, and M.J. Norris, Note on uniqueness for a one-dimensional two-body problems of classical electrodynamics, *Ann. Phys,* **42**, (1967), 347-351.

[45] K. J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathemaics, (194), Springer-Verlag, New York, 2000.

[46] K. Ezzinbi, Existence and stability for some partial functional differential equations with infinite delay, *Elec. J. Diff. Eqs*, **116**, (2003), 1-13.

[47] M. Frigon, Fixed point results for multi-valued contractions on Gauge Spaces. Set Valued Mappings with Applications in Nonlinear Analysis, *Ser. Math. Anal. Appl*, Taylor § Francis, London,(**4**), (2002), 175-181.

[48] M. Frigon, Fixed point and continuation results for contractions in Metric and Gauge Spaces. Fixed Point theory and its applications, *Banach Center Publ, Polish. Acad. Sci*, Warsaw, (77), (2007), 89-114.

[49] M. Frigon and A. Granas, Résultats de type Leray-Schauder pour des contractions sur des espaces de Fréchet, *Anal. Sci. Math. Québec*, **22**, (1998), 161-168.

[50] E. P. Gastori, Controllability results for nondensely defined evolution differential inclusions with nonlocal conditions, *J. Math. Anal. Appl*, **297**, (2004), 194-211.

[51] W. G. Glockle and T. F. Nonnenmacher, A fractional calculus approach of Selfsimilar Protein Dynamics, *Biophys. J*, **68**, (1995), 46-53.

[52] L. Gorniewicz, *Topological fixed points theory of multi-valued mappings, mathematics and its applications*, Kluiver Academic Publishers, Dordrecht, **495**, (1999).

[53] J. Hale, *Theory of functional differential equations*, Springer, New York, Heidelberg, Berlin, (1977).

[54] J. Halea and J. Kato, Phase Space for Retarded Equations with Infinite Delay, *Funkcial. Ekvac*, **21**, (1978), 11-41.

[55] J. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Appli. Math. Sci, Springer-Verlag, New York, 1993.

[56] E. Hernandez, A. Prokopczyk and L. Ladeira, A Note on Partial Functional Differential Equations with State-Dependent Delay, *Nonli. Anal. Real. World. Appl*, **7**,(4), (2006), 510-519.

[57] E. Hernández, M. Pierri and G. Goncalves, Existence Results for An Impulsive Abstract Partial Differential Equation with State-Dependent Delay, *Comput. Math. Appl*, **52**, (34), (2006), 411-420.

[58] E. Hernandez, R. Satkhivel and S. Tanaka Aki, Existence Results for Impulsive Evolution Differential Equations with State-Dependent Delay, *Elec. J. Diff. Eqs*, **28**, (2008), 1-11.

[59] H. R. Henriquez, Existence of Periodic Solutions of Neutral Functional Differential Equations with Unbounded Delay, *Proyecciones*, **19**, (3), (2000), 305-329.

[60] R. Hermann, *Fractional Calculus: An Introduction For Physicists*, World Scientifc Publishing Co. Pte. Ltd, 2011.

[61] Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Unbounded Delay*, Springer-Verlag, Berlin, 1991.

[62] R. Hilfer, *Applications of Fractional Calculus in Physics*, World. Sci, Singapore, 2000.

[63] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland, Maths Studies, Elsevier Science B.V. Amsterdam, **204**, (2006).

[64] F. Kapper and W. Schappacher, Some Considerations to the Fundamental Theory of Infinite Delay Equations, *J. Diff. Eqs*, **37**, (1980), 141-183.

[65] V. Kolmanovskii, and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*. Kluwer Academic Publishers, Dordrecht, 1999.

[66] V. Lakshmikantham, Theory of Fractional Functional Differential Equations, *Nonl. Anal*, **69**, (2008), 3337-3343.

[67] V. Lakshmikantham and J. V. Devi, Theory of Fractional Differential Equations in A Banach Space, *Eur. J. Pure and Appl. Maths*, **1**, (2008), 38-45.

[68] V. Lakshmikantham, S. Leela and J. Vasundhara, *Theory of Dynamical Systems*. Cambridge. Acad. Publis, Cambridge, 2009.

[69] V. Lakshmikantham and A. S. Vatsala, Basic Theory of Fractional Differential Inequalities and Applications, *Commun. Appl. Anal*, **11**, No.(34), (2007), 395-402.

[70] V. Lakshmikantham and A. S. Vatsala, Basic Theory of Fractional Differential Equations, *Nonli. Anal*, **69**, (2008), 2677-2682.

[71] C. Lizama, Regularized Solutions for Abstract Volterra Equations, *J. Math. Anal. Appl*, **243**, (2000), 278-292.

[72] W. S. Li, Y. K. Chang and J. J. Nieto, Solvability of Impulsive Neutral Evolution Differential Inclusions with State-Dependent Delay, *Math. Comput. Model,* **49**, (2009), 1920-1927.

[73] K. Li and J. Peng, Fractional Resolvents and Fractional Evolution Equations, *Appl. Math. Lett,* **25**, (2012), 808-812.

[74] A. J. Luo and V. Afraimovich, *Long-range Interactions, Stochasticity and Fractional Dynamics*, Springer, New York, Dordrecht, Heidelberg, London, 2010.

[75] M.C. Mackey, and J. Milton, Feedback Delays and The Origin of Blood Cells dynamic, *Comm. Theor. Biol*, **1**, (1990), 299-327.

[76] M.C. Mackey, Commodity Price Fluctuations, Price Dependent Delays and Nonlinearities as Explanatory Factors. *J. Econ. Theory*, bf 48, (1989), 497-509.

[77] F. Mainardi, Fractional Calculus, Some Basic Problems in Continuum and statistical Mechanics, in Fractals and Fractional Calculus in Continuum Mechanics, *A. Carpinteri and F. Mainardi, Eds, Springer-Verlag, Wien*, (1997), pp 291-348.

[78] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, *J. Chem. Phys*, **103**, (1995), 7180-7186.

[79] J. Mikusiński, *The Bochner integral*, Birkhäuser, Basel, 1978.

[80] G. M. Mophou and G. M. Nguérékata, Existence of mild solutions for some semi-linear neutral fractional functional evolution equations With Infinite Delay, *Appl. Math. Comput*, **216**, (2010), 61-69.

[81] K. B. Oldham and J. Spanier, *The Fractional Calculus: theory and application of differentiation and integration to arbitrary order*, Academic Press, New York, London, 1974.

[82] M. D. Ortigueira, *Fractional calculus for Scientists and Engineers*. Lect. Not. Elect. Engi, **84**, Springer, Dordrecht, 2011.

[83] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[84] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.

[85] J. Prüss, *Evolutionary integral equations and applications*, Monographs Math, **87**, Birkhauser Verlag, 1993.

[86] B. Ross, *Fractional calculus and its applications, proceedings of the International Conference*, New Haven, Springer-Verlag, New York, 1974

[87] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives, theory and applications*, Gordon and Breach, Yverdon, 1993.

[88] P. Sahoo, T. Barman and J. P. Davim, Fractal analysis in machining, Springer, New York, Dordrecht, Heidelberg, London, 2011.

[89] P. Saveliev, Fixed points and selections of set-valued maps on spaces with convexity. *Int. J. Math.Sci,* **24**, (2000), 595-612.

[90] K. Schumacher, Existence and continuous dependence for differential equations with unbounded delay, *Arch. Rational Mech. Anal*, **64**, (1978), 315-335.

[91] K. Schumacher, Existence and continuous dependence for fractional neutral evolution equations, *comput. Math. Appl*, **59**, (2010), 1063-1077.

[92] V. E. Tarasov, *Fractional dynamics: Applications of cractional calculus to dynamics of particles, fields and media.* Springer, Heidelberg, 2010.

[93] K. Velusamy, R. Murugesu and P. Z. Wang, Existence results for neutral functional fractional differential equations with state dependendent delay, *Malaya. Math. J*, **11**, (2012), 50-61.

[94] J. Wu, *Theory and applications of partial functional differential equations*, Springer-Verlag, New York, 1996.

[95] Y. Zhou and F. Jiao, Existence and continuous dependance for differential equations with unbounded delay, *Math.Appl,* **59**, (2010), 1063-1077.

[96] Y. Zhou and F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, *Nonli. Anal. Real. World. Appl*, **11**, (2010), 4465-4475.