A study of the problem of the Cauchy-Riemann operator

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Contents

INTRODUCTION.

The theory of several complex variables, namely the theory of holomorphic functions of several variables and the problem of the $\overline{\partial}$−operator have been a subject of intensive studies during the twentieth century. This story began in 1906 with H. Poincaré which observes that the bi-disc $D \times D \subset \mathbb{C}^2$ and the unit ball $B \subset \mathbb{C}^2$ are not analytically isomorph, and F. Hartogs which observes that the Riemann's theorem does not work in $\mathbb{C}^2$. The theory of several complex variables seams then to be radically different and not a simple generalization of the theory in $\mathbb{C}$. Till the early fifties this theory was developed by constructive methods, that is by integral formulas. We emphasize the work of A. Weil in 1935 [22], and of K. Oka in the period 1936 till 1951 [22].
In the fifties H. Cartan, and H. Grauert [7] discovered by means of the theory of sheaves introduced in 1945 by J. Leray, that the theory of integral formulas can be reduced to a minimum and, moreover, that the theory of Oka admits far-reaching generalizations.
In the sixties L. Hörmander [13], J.J. Kohn [7] deduced the results of Oka with the use of methods of partial differential equations, that is by $\mathbf{L}^2$−estimates for the $\overline{\partial}$−operator.

However, in the seventies integral representation formulas turned out to be the natural method for solving several problems related to the $\overline{\partial}-$operator which are connected with the behavior at the boundary. The basic tool is an integral formula for holomorphic functions discovered in 1955 by J. Leray [16], which contains the Weil formula as a special case.
We observe however, that all the theory of several complex variables mentioned above, namely the theory of the Cauchy-riemann operator $\overline{\partial}$ is build on the commutative group $(\mathbb{C}^n, +)$. We refer to this theory as the commutative theory of the $\overline{\partial}-$operator. The problem turned out to be different, and far-rich, when the the space $\mathbb{C}^n$ is endowed with a structure of non commutative group. Let $\mathbb{H} = (\mathbb{C}^n, *)$ be a simply connected $2-$step nilpotent Lie group, our aim in this thesis is to solve the following two problems:

1. **Problem:** How to construct for the group $\mathbb{H}$ the analogous $\overline{\partial}_{_L}$ of the classical Cauchy-Riemann operator $\overline{\partial}$ of the commutative group $(\mathbb{C}^n, +)$?

2. **Problem:** Can one solve the equation $\overline{\partial}_{_L} u = f$, with Hölderian estimates?

This thesis is divided into two chapters and an appendix:
In chapter 1, we solve the first problem mentioned above when the group $\mathbb{H} = (\mathbb{C}^n, *)$ is 2-step nilpotent. That is:
Let $\mathbb{H} = (\mathbb{C}^n, *)$ be a 2-step nilpotent Lie group, and $\mathcal{H}$ its Lie algebra. We attach to each Lie subalgebra $L \lhd \mathcal{H}$ of $\mathcal{H}$ containing the center $Z(\mathcal{H})$ of $\mathcal{H}$ a new Lie algebra denoted $\mathcal{H}_L$, in such a way that the family

$$\mathcal{H}_\bullet = \Big\{ \mathcal{H}_L \Big\}_{L \lhd \mathcal{H}}$$

forms a category of Lie algebras, and for each open set $\Omega \subset \mathbb{H}$, and each integers $0 \leq p_1 \leq m$, and $0 \leq p_2 \leq n - m$, $l \in \mathbb{N} \cup \{+\infty\}$, and each $0 \leq \gamma < 1$, we attach to $L \lhd \mathcal{H}$ a module $\mathcal{C}^{\gamma+l}_{((p_1,p_2),q)_L}(\Omega)$ of differential forms with $(l+\gamma)-$Hölder coefficients, in such a way that for fixed L, the family of modules

$$\mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_L}(\Omega) = \Big\{ \mathcal{C}^{\gamma+l}_{((p_1,p_2),q)_L}(\Omega) \Big\}_{l,q}$$

forms a complex of graded modules, and for running $L \lhd \mathcal{H}$, the family of complexes

$$\mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_\bullet}(\Omega) = \left\{ \mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_L}(\Omega) \right\}_L .$$

forms a category of complexes. Once defined the first and the second forms of structure ϕ and σ^L of the group $\mathbb{H}$, and the the left invariant vector fields $\mathcal{Z}_j$, $\overline{\mathcal{Z}}_j$, we attach to each $L \lhd \mathcal{H}$ a differential operator denoted $\overline{\partial}_{_L}$ generalizing the classical $\overline{\partial}$, then we study their properties. The fundamental result of chapter 1 is described by the following theorem.

Theorem

Let $\Omega \subset \mathbb{H}$ be an open set of the group $\mathbb{H}$, and let $0 \leq \gamma < 1$, $l \in \mathbb{N} \cup \{+\infty\}$. Then for each subalgebra $L \lhd \mathcal{H}$, and each integers p_1, p_2, q with $0 \leq p_1 \leq m$, $0 \leq p_2 \leq n-m$, $0 \leq q \leq n$, there exists one and only one first order linear differential operator:

$$\overline{\partial}_{_L} : \mathcal{C}^{\gamma+l}_{((p_1,p_2),q)_L}(\Omega) \longrightarrow \mathcal{C}^{\gamma+l-1}_{((p_1,p_2),q+1)_L}(\Omega)$$

such that:

1. *$\overline{\partial}_{_L}$ is left $\mathbb{H}-$invariant.*

2. *If $\langle\ ,\ \rangle$ denotes the pairing between vector fields and 1-differential forms, then for every $\mathcal{C}^\infty$ function f,*

$$\left\langle \overline{\mathcal{Z}}_j, \overline{\partial}_{_L} f \right\rangle = \overline{\mathcal{Z}}_j(f) \qquad \text{for all} \quad 1 \leq j \leq n.$$

3. *The 1-forms of structure ϕ and $\overline{\phi}$ satisfy the following " $L-$equations of structure":*

$$\begin{cases} \overline{\partial}_{_L}\phi = \quad \sigma^L \\ \overline{\partial}_{_L}\overline{\phi} = -\sigma^L. \end{cases}$$

4. *For all $f, g \in \mathcal{C}^\infty_{(\bullet)}(\Omega)$, $\overline{\partial}_{_L}$ satisfies Leibnitz'rule, that is*

$$\overline{\partial}_{_L}(f \wedge g) = \overline{\partial}_{_L} f \wedge g + (-1)^\nu f \wedge \overline{\partial}_{_L} g, \qquad \nu = deg(f).$$

When $L \lhd \mathcal{H}$ is fixed, we refer to the differential operator with variable coefficients $\overline{\partial}_{_L}$ as the left Cauchy-Riemann operator of the group $\mathbb{H}$ attached to $L \lhd \mathcal{H}$, and when L runs over all subalgebras of $\mathcal{H}$ containing the center $Z(\mathcal{H})$, we obtain a functor of categories:

$$\overline{\partial}_\bullet : \mathcal{H}_\bullet \longrightarrow \mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_\bullet}(\Omega)$$

called the left functor of Cauchy-Riemann.
In chapter 2, we study for each $L \lhd \mathcal{L}$, the differential equation

$$\overline{\partial_{_L}} u = f.$$

We begin by considering the case $f = 0$, whose solutions are nothing but $h-$holomorphic functions. We characterize on Ω the space of left $h-$holomorphic functions, that is the space $ker(\overline{\partial}_{_L})$ of solutions of $\overline{\partial}_{_L} u = 0$, and introduce the corresponding domains of left $h-$pseudoconvexity of bounded deviation. We prove for local solvability, the following result, called Dolbeault-Grothendieck lemma.

Theorem
Let $\Omega = D_1 \times ... \times D_n$ be an open polydisc of $\mathbb{H}$ and let $f \in \mathcal{C}^\infty_{(p,q+1)_L}(\Omega)$ satisfy the condition $\overline{\partial}_{_L} f \in \mathcal{J}_{(s)}(\Omega)$. If $\Omega' \subset\subset \Omega$ (that is Ω' is relatively compact in Ω), we can find $u \in \mathcal{C}^\infty_{(p,q)_\mathcal{H}}(\Omega')$ such that $f - \overline{\partial}_{_L} u \in \mathcal{J}_{(s)}(\Omega')$.
Then, we construct for $\overline{\partial}_{_L}$ an integral formula of Leray Koppelman type. This generalizes to the $\overline{\partial}_{_L}-$operator, the Leray Koppelman formula for the classical Cauchy-Riemann operator $\overline{\partial}$. Then, we prove for the $\overline{\partial}_{_L}-$operator, by means of this formula the following existence theorem with Hölderien estimates.

Theorem
Let $\Omega \subset\subset \mathbb{H}$ be a $h-$pseudoconvex open set of deviation $\mathcal{D}ev(\Omega) = r$, with $\mathcal{C}^\infty$ boundary, and f a continuous differential form up to the boundary, that is $f \in \mathcal{C}^0_{((p_1,p_2),q+1)_L}(\overline{\Omega})$ satisfying in Ω the compatibility condition $\partial_{_L} f = 0$. Then there exists a $\frac{1}{r}-$Hölder differential form $u \in \mathcal{C}^{\frac{1}{r}}_{((p_1,p_2),q)_L}(\Omega)$ such that $\overline{\partial}_{_L} u = f$.

Remark.

1. For the commodity of the reader, all the basic tools (namely: definitions) that we are led to constantly use, are recalled in a background in the begining of the thesis.

2. We give in appendix II, a list of some interesting differential operators related to $\overline{\partial}_{_L}$, namely some Laplacians.

Background

For the commodity of the reader, we recall in what follows, the main definitions and properties which we shall constantly use in this thesis.

0.1 Lie groups and Lie algebras

0.1.1 Lie groups

Definition 0.1.1. A Lie group is a differentiable([1]) manifold $\mathbb{H}$ endowed with a group law

$$* : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$$

$$(z, \xi) \longmapsto z * \xi$$

such that the map $(z, \xi) \longmapsto z * \xi^{-1}$ is differentiable. That is if the two following maps

1. The group law $(z, \xi) \longmapsto z * \xi$
2. The inverse map $\xi \longmapsto \xi^{-1}$

are both differentiable. The map

$$\tau_z : \mathbb{H} \longrightarrow \mathbb{H}$$

$$\xi \longmapsto \tau_z(\xi) = z * \xi$$

is called the left translation defined by the element $z \in \mathbb{H}$.

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[1] In this thesis, differentiable means always $\mathcal{C}^\infty$.

Definition 0.1.2. Let $\mathbb{H}$ be a Lie group, and denote by 0 the neutral element of $\mathbb{H}$, and by $-z$ the symmetric element of z. The differential of the left translation

$$\tau_{-z} : \xi \longmapsto (-z) * \xi$$

at $\xi = z$ is a vectorial 1-differential form ϕ called the first form of structure of the group $\mathbb{H}$.

Properties
The first form of structure ϕ of the group $\mathbb{H}$, is characterized by the conditions:

1. ϕ satisfies $\phi(0) = Id$.
2. ϕ is left invariant. That is, for all $z \in \mathbb{H}$

$$\tau_z^*(\phi) = \phi.$$

Definition 0.1.3. With the notation above, the 2-differential form

$$\sigma := d\phi$$

is called the second form of structure of the group $\mathbb{H}$.

Remark.
1) The second form of structure σ of the group $\mathbb{H}$, is left invariant.
2) The group $\mathbb{H}$ is commutative if and only if $\sigma = 0$.
Example. Let $\mathbb{H} = \mathbb{C}^n$ endowed with the usual addition

$$z * \xi = z + \xi$$

Then $\mathbb{H} = (\mathbb{C}^n, +)$ is a Lie group. The left translation is $\tau_z(\xi) = z + \xi$, the first form of structure is $\phi(z) = dz$, and the second form of structure is $\sigma(z) = 0$.

0.1.2 Lie algebras

Definition 0.1.4. An abstract Lie algebra $\mathcal{H}$ is a complex linear space endowed with a skew bilinear map denoted [,]:

$$[\ ,\ ] : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$$

$$(X,Y) \longmapsto [X,Y]$$

such that the following condition (called identity of Jacobi) is satisfied

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0.$$

The Lie algebra $\mathcal{H}$ is said to be 2-step nilpotent if for all $X, Y, Z \in \mathcal{H}$, we have

$$[[X,Y],Z] = 0.$$

Definition 0.1.5. Let $\mathbb{H}$ be a Lie group, and let 0 be its neutral element. A left invariant vector fields $\mathcal{Z}(z)$ over $\mathbb{H}$ is completely determined by its value at 0, that is

$$\mathcal{Z}(z) = (\tau_z)^* \, \mathcal{Z}(0).$$

This means that the linear space of left invariant vector field is isomorphic to the tangent space $T_0\mathbb{H}$. The space of left invariant vector fields endowed with the usual commutator

$$[X,Y] = X \circ Y - Y \circ X \qquad (0.1.1)$$

is a Lie algebra, called the Lie algebra of the group $\mathbb{H}$.
We observe then, that the Lie algebra $\mathcal{H}$ of the group $\mathbb{H}$ endowed with the commutator (0.1.1) is nothing but $\mathcal{H} = T_0\mathbb{H}$.

0.2 Several complex variables

For all these notions, see Hörmander [13]

0.2.1 The Cauchy-Riemann operator

Let D be an open set of $\mathbb{C}^n$, $z = (z_1, ..., z_n) \in D$, and let

$$f : D \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}$$

be a $\mathcal{C}^\infty$ complex valued function. We define for all $1 \leq j \leq n$, the differential forms

$$dz_j = dx_j + idy_j$$

$$d\overline{z}_j = dx_j - idy_j$$

and the differential operators:

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right)$$
$$\frac{\partial f}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right)$$
$$\partial f = \sum_j \frac{\partial f}{\partial z_j} dz_j$$
$$\overline{\partial} f = \sum_j + \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j$$

We observe the following

$$df = \partial f + \overline{\partial} f.$$

Definition 0.2.1.

1. The differential operator $\overline{\partial}$ is called the Cauchy-Riemann operator. The differential equation $\overline{\partial} u = f$, is called the Cauchy-Riemann equation. For functions, this is equivalent to the system
$$\frac{\partial u}{\partial \overline{z}_j} = f_j.$$
2. The $\mathcal{C}^\infty$ function f is sa id to be holomorphic if $\overline{\partial} f = 0$, that is if
$$\frac{\partial f}{\partial \overline{z}_j} = 0.$$

Proposition 0.2.2.

1. *The function f is holomporphic if and only if f is analytic.*
2. *We have for all f the identities:*
$$d^2 f = \partial^2 f = \overline{\partial}^2 f = \partial \circ \overline{\partial} f + \overline{\partial} \circ \partial f = 0.$$
3. *The differential operators $\frac{\partial}{\partial z_j}$, $\frac{\partial}{\partial \overline{z}_j}$, viewed as vector fields , and the differential operator $\overline{\partial}$, and the first form of structure $\phi(z) = dz$ are all left invariant by the group $\mathbb{H} = (\mathbb{C}^n, +)$.*

0.2.2 Pseudoconvex domains in $\mathbb{C}^n$

Let $D \subset \mathbb{C}^n$ be an open set defined by the real valued function $\varphi : V_{\overline{D}} \longrightarrow \mathbb{R}$, that is

$$D := \{z \in V_{\overline{D}}, \quad \varphi(z) < 0\}.$$

and

$$\partial D := \{z \in V_{\overline{D}}, \quad \varphi(z) = 0\}.$$

Definition 0.2.3. 1. The quadratic form

$$L_z[\varphi](\xi) = \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}(z)\xi_j\overline{\xi}_k$$

is called the Levi form of φ at z.

2. The domain D is said to be pseudoconvex if $L_z[\varphi](\xi)$ is positive at all $z \in \partial D$, and for all $\xi \in T_z\partial D$.

3. The domain D is said to be strictly pseudoconvex if $L_z[\varphi](\xi)$ is positive defined at all $z \in \partial D$, and for all $\xi \in T_z\partial D$.

0.3 Categories

0.3.1 Definition of a category

Definition 0.3.1. A category is defined by three things:

1. A collection $\mathcal{C}$ of objects : X, Y, Z, T, ..., (in general these objects are sets endowed with structures), that is

$$\mathcal{A} = \Big\{X, Y, Z, T, ...\Big\}.$$

2. For all pair of objects (X, Y), there exists a set of morphisms

$$\mathcal{M}or(X, Y) = \big\{f : X \longrightarrow Y\big\}$$

3. For all triplet of objects (X, Y, Z), there exists a composition law $\circ$

$$\mathcal{M}or(X, Y) \times \mathcal{M}or(Y, Z) \longrightarrow \mathcal{M}or(X, Z)$$

$$(f, g) \longmapsto g \circ f$$

such that the following two conditions are fulfilled:

- For all object X there exits a morphism $Id_X : X \longrightarrow X$ called the morphism identity.
- If $f \in \mathcal{M}or(X,Y)$, $g \in \mathcal{M}or(Y,Z)$, and $h \in \mathcal{M}or(Z,T)$, the law $\circ$ is associative, that is:

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

0.3.2 Functors of Categories

Definition 0.3.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. The correspondence

$$F : \mathcal{A} \longrightarrow \mathcal{B}$$

is called a functor of categories, if F associates to each object X of $\mathcal{A}$, one and only one object $F(X)$ of $\mathcal{B}$, and to each morphism $f \in \mathcal{M}or(X,Y)$ one and only one morphism $F(f) \in \mathcal{M}or\big(F(X), F(Y)\big)$ such that the following conditions are fulfilled:

1. For all $X \in \mathcal{A}$ we have

$$F\big(Id_X\big) = Id_{F(X)}$$

2. If $f \in \mathcal{M}or(X,Y)$, and $g \in \mathcal{M}or(Y,Z)$,
   then $F(f) \in \mathcal{M}or\big(F(X), F(Y)\big)$, and $F(g) \in \mathcal{M}or\big(F(Y), F(Z)\big)$ and furthermore, we have

$$F(g \circ f) = F(g) \circ F(f).$$

The second condition means that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & F(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F(f)} \\ Y & \xrightarrow{F} & F(Y) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle F(g)} \\ Z & \xrightarrow{F} & F(Z) \end{array}$$

is commutative.

Chapter 1

The functor $\overline{\partial}_\bullet$

1.1 The category of Lie algebras $\mathcal{H}_\bullet$

1.1.1 The 2-step nilpotent group $\mathbb{H} = (\mathbb{C}^n, *)$

We organize $\mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{n-m}$ as a Lie group $\mathbb{H} = (\mathbb{C}^n, *)$ with a group law $*$ defined for $z = (z_1, ..., z_n) \in \mathbb{C}^n$, and $\xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^n$ by

$$z * \xi = z + \xi + \frac{1}{2}\Big(A(z, \overline{\xi}) - A(\xi, \overline{z})\Big) \tag{1.1.1}$$

where $A = (A_1, ..., A_n) : \mathbb{C}^m \times \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is a bilinear map

$$\begin{cases} A_k(z, \xi) = 0 & \text{for} \quad 1 \leq k \leq m \\ A_k(z, \xi) = \displaystyle\sum_{i,j=1}^{m} a_{i,j}^k z_i \xi_j, & \text{for} \quad m+1 \leq k \leq n \end{cases} \tag{1.1.2}$$

with complex coefficients $a_{i,j}^k$ satisfying

$$\begin{cases} a_{i,j}^k = 0 & \text{for} \quad 1 \leq k \leq m \\ \overline{a_{i,j}^k} = -a_{j,i}^k, & \text{for} \quad m+1 \leq k \leq n. \end{cases} \tag{1.1.3}$$

The Lie group $\mathbb{H}$ is clearly 2-step nilpotent with 0 as unit element, and $-z$ as inverse element of $z \in \mathbb{H}$. We denote by $Z(\mathbb{H})$ the center of $\mathbb{H}$, that is

$$Z(\mathbb{H}) := \Big\{ z \in \mathbb{H}, \quad z * \xi = \xi * z \quad \text{for all } \xi \in \mathbb{H} \Big\}.$$

1.1.2 The forms of structure ϕ and σ of the group $\mathbb{H}$

Let $\phi = (\phi_1, ..., \phi_n)$ be the differential at $\xi = z$ of the left translation

$$\tau_{-z} : \xi \longmapsto (-z) * \xi.$$

The 1-form ϕ which defines the parallelism of the group $\mathbb{H}$ is then given by

$$\begin{cases} \phi_k = dz_k & \text{for } 1 \leq k \leq m \\ \phi_k = dz_k + \dfrac{1}{2}\displaystyle\sum_{j=1}^{m}\left(\frac{\partial A_k}{\partial z_j}dz_j - \frac{\partial A_k}{\partial \overline{z}_j}d\overline{z}_j\right) & \text{for } m+1 \leq k \leq n, \end{cases} \tag{1.1.4}$$

and its conjugate $\overline{\phi} = (\overline{\phi}_1, ..., \overline{\phi}_n)$ by

$$\begin{cases} \overline{\phi}_k = d\overline{z}_k & \text{for } 1 \leq k \leq m \\ \overline{\phi}_k = d\overline{z}_k - \dfrac{1}{2}\displaystyle\sum_{j=1}^{m}\left(\frac{\partial \overline{A}_k}{\partial z_j}dz_j - \frac{\partial \overline{A}_k}{\partial \overline{z}_j}d\overline{z}_j\right) & \text{for } m+1 \leq k \leq n. \end{cases} \tag{1.1.5}$$

By differentiating (1.1.4), ϕ satisfies the following equations of structure

$$\begin{cases} d\phi_k = 0 & \text{for } \quad 1 \leq k \leq m \\ d\phi_k = \displaystyle\sum_{i,j=1}^{m} a_{i,j}^k \phi_i \wedge \overline{\phi}_j & \text{for } \quad m+1 \leq k \leq n \end{cases} \tag{1.1.6}$$

where $a_{i,j}^k$ are the constants (1.1.3).
Let $\sigma = (\sigma_1, ..., \sigma_n)$ be the differential form defined by:

$$\sigma := d\phi$$

that is

$$\begin{cases} \sigma_k = 0 & \text{for } \quad 1 \leq k \leq m \\ \sigma_k = \displaystyle\sum_{i,j=1}^{m} a_{i,j}^k \phi_i \wedge \overline{\phi}_j & \text{for } \quad m+1 \leq k \leq n. \end{cases} \tag{1.1.7}$$

We refer to ϕ and $\overline{\phi}$ as the first forms of structure and to σ as the second form of structure of the group $\mathbb{H}$.

1.1.3 The Lie algebra $\mathcal{H}$ of the group $\mathbb{H}$

By duality with (1.1.4) and (1.1.5), we define the following vector fields:

$$\begin{cases} \mathcal{Z}_j = \dfrac{\partial}{\partial z_j} - \dfrac{1}{2}\displaystyle\sum_{k=m+1}^{n}\left(\frac{\partial A_k}{\partial z_j}\frac{\partial}{\partial z_k} + \frac{\partial \overline{A}_k}{\partial z_j}\frac{\partial}{\partial \overline{z}_k}\right) & \text{for } 1 \le j \le m \\ \\ \mathcal{Z}_k = \dfrac{\partial}{\partial z_k} \qquad\qquad \text{for } m+1 \le k \le n \end{cases} \tag{1.1.8}$$

and

$$\begin{cases} \overline{\mathcal{Z}}_j = \dfrac{\partial}{\partial \overline{z}_j} + \dfrac{1}{2}\displaystyle\sum_{k=m+1}^{n}\left(\frac{\partial A_k}{\partial \overline{z}_j}\frac{\partial}{\partial z_k} + \frac{\partial \overline{A}_k}{\partial \overline{z}_j}\frac{\partial}{\partial \overline{z}_k}\right) & \text{for } 1 \le j \le m \\ \\ \overline{\mathcal{Z}}_k = \dfrac{\partial}{\partial \overline{z}_k}, \qquad\qquad \text{for } m+1 \le k \le n \end{cases} \tag{1.1.9}$$

where A_k, $\Bigg(\text{respectively, } \overline{A}_k\Bigg)$ is short for $A_k(z,\overline{z})$, $\Bigg(\text{respectively, } \overline{A}_k(z,\overline{z})\Bigg)$, and then, the Lie algebra $\mathcal{H}$ of the group $\mathbb{H}$ is the $\mathbb{R}-$linear space spanned by the vector fields $\left\{\mathcal{Z}_k, \overline{\mathcal{Z}}_k\right\}_{1\le k\le n}$, and endowed with the commutators

$$\begin{cases} \left[\mathcal{Z}_i, \overline{\mathcal{Z}}_j\right] = \displaystyle\sum_{k=m+1}^{n} a_{i,j}^k \Bigg(\mathcal{Z}_k - \overline{\mathcal{Z}}_k\Bigg) & \text{for } 1 \le i,j \le m, \\ \text{the other brackets are zero.} \end{cases} \tag{1.1.10}$$

1.1.4 The metric group $(\mathbb{H}, g_{\mathcal{H}})$

We need in that follows, to endow the Lie algebra $\mathcal{H}$ with the Hermitian inner product $\langle\ ,\ \rangle_{\mathcal{H}}$ which makes the basis $\mathcal{B} = \left\{\mathcal{Z}_k, \overline{\mathcal{Z}}_k\right\}_{1\le k\le n}$ orthonormal, that is

$$\langle \mathcal{Z}_j, \overline{\mathcal{Z}}_k\rangle_{\mathcal{H}} = \delta_{j,k}.$$

Consequently, the group $\mathbb{H}$ is endowed with the associated left invariant([1]) metric

$$g_{\mathcal{H}} := \langle \phi, \phi \rangle_{\mathcal{H}} = \sum_{j=1}^{n} \phi_j \overline{\phi}_j. \tag{1.1.11}$$

1.1.5 Construction of the category $\mathcal{H}_\bullet = \{\mathcal{H}_L\}_{L \lhd \mathcal{H}}$

Let $Z(\mathcal{H})$ denotes the center of the Lie algebra $\mathcal{H}$, that is

$$Z(\mathcal{H}) := \Big\{ X \in \mathcal{H}, \quad [X,Y] = 0 \quad \text{for all} \quad Y \in \mathcal{H} \Big\}.$$

and let $L \lhd \mathcal{H}$ denotes a subalgebra L of $\mathcal{H}$ containing the center $Z(\mathcal{H})$. The Lie algebra $\mathcal{H}$ can be decomposed as a direct sum

$$\mathcal{H} = L \oplus L^{\perp}.$$

With this notation, we define in $\mathbb{C}^n$ via the following bracket

$$\begin{cases} [X,Y]_{L} := [X,Y] & \text{if } X \in L^{\perp}, \text{ and } Y \in L^{\perp} \\ \\ [X,Y]_{L} := 0 & \text{otherwise} \end{cases} \tag{1.1.12}$$

a new structure of Lie algebra denoted $\mathcal{H}_L = \big(\mathbb{C}^n, [\ ,\ ]_L\big)$. We observe that $\mathcal{H}_L$ is simply obtained from $\mathcal{H}$ by extension of the center, that is

$$Z(\mathcal{H}) \subseteq L \subseteq Z(\mathcal{H}_L).$$

$\mathcal{B} = \big\{\mathcal{Z}_k, \overline{\mathcal{Z}}_k\big\}_{1 \le k \le n}$ will always be regarded as constituting simultaneously a basis of $\mathcal{H}$ and a basis of $\mathcal{H}_L$, and then the decomposition of the bracket $\big[\mathcal{Z}_i, \overline{\mathcal{Z}}_j\big]_L$ as linear combination of the vector fields $\mathcal{Z}_k, \overline{\mathcal{Z}}_k$

$$\big[\mathcal{Z}_i, \overline{\mathcal{Z}}_j\big]_L = \sum_{k=1}^{n} \lambda_{i,j}^k \Big(\mathcal{Z}_k - \overline{\mathcal{Z}}_k \Big), \tag{1.1.13}$$

gives with respect to the Lie algebra $\mathcal{H}_L$, the constants of structure $\lambda_{i,j}^k$, with

$$\begin{cases} \lambda_{i,j}^k = a_{i,j}^k & \text{if } \mathcal{Z}_i \in L^{\perp}, \text{ and } \overline{\mathcal{Z}}_j \in L^{\perp} \\ \lambda_{i,j}^k = 0 & \text{otherwise.} \end{cases} \tag{1.1.14}$$

[1]We shall sketch that the metric $g_{\mathcal{H}}$ is invariant by the group $\mathbb{H}$

Now, let $K \lhd \mathcal{H}$ and $L \lhd \mathcal{H}$, then the linear mapping $g_{K,L} : \mathcal{H}_K \longrightarrow \mathcal{H}_L$ evaluated on a vector $X \in \mathcal{B}$ by

$$f_{K,L}(X) := \begin{cases} X & \text{if } X \notin K^\perp \cup L^\perp \\ 0 & \text{otherwise} \end{cases} \tag{1.1.15}$$

is a morphism of Lie algebras. This leads to consider the following category of Lie algebras attached to the metric group $\mathbb{H}$.

Definition 1.1.1.
The category $\mathcal{H}_\bullet$ of Lie algebras attached to the metric group $\mathbb{H}$ is defined as follows:

- The objects of $\mathcal{H}_\bullet$ are the Lie algebras $\mathcal{H}_L$, where L runs over all $L \lhd \mathcal{H}$.
- For all $K \lhd \mathcal{H}$ and $L \lhd \mathcal{H}$, the set $\mathcal{M}or(\mathcal{H}_K, \mathcal{H}_L)$ of morphisms from $\mathcal{H}_K$ to $\mathcal{H}_L$ is reduced to one element, that is the mapping $f_{K,L}$ defined by (1.1.15),
$$\mathcal{M}or(\mathcal{H}_K, \mathcal{H}_L) := \left\{ f_{K,L} \right\}.$$
- the composition law $\mathcal{M}or(\mathcal{H}_G, \mathcal{H}_K) \times \mathcal{M}or(\mathcal{H}_K, \mathcal{H}_L) \longrightarrow \mathcal{M}or(\mathcal{H}_G, \mathcal{H}_L)$ is the usual composition of maps.

1.2 Modules of Holderian differential forms on $\mathbb{H}$.

1.2.1 Hölderian functions

Let Ω be a measurable subset of the group $\mathbb{H}$, and let $l \in \mathbb{N} \cup \{+\infty\}$ and $0 < \gamma < 1$. Then for every $\mathcal{C}^l-$complex-valued function f on Ω, we define

$$\|f\|_{0,\Omega} := \sup_{\xi \in \Omega} |f(\xi)|,$$

and the $\gamma-$Hölder norm $\|f\|_{\alpha,\Omega}$ by

$$\|f\|_{\gamma,\Omega} := \|f\|_{0,\Omega} + \sup_{z,\xi \in \Omega} \frac{|f(z) - f(\xi)|}{|z - \xi|^\gamma}.$$

We note the Hölder spaces:

$$\mathcal{C}^\gamma(\Omega) := \Big\{ f \in \mathcal{C}^0(\Omega), \qquad \|f\|_{\gamma,\Omega} < +\infty \Big\}$$

and for $l \in \mathbb{N}$

$$\mathcal{C}^{l+\gamma}(\Omega) := \Big\{ f \in \mathcal{C}^l(\Omega), \qquad \text{for all} \quad |\alpha| \leq l, \quad \|\partial^\alpha f\|_{\gamma,\Omega} < +\infty \Big\}.$$

1.2.2 Graded modules of differential forms on $\mathbb{H}$.

A) Hölderian differential forms of $\mathcal{H}-$type (p_1, p_2, q_2, q_2)

Let $\mathcal{C}^\infty(\Omega)$ denote the space of $\mathcal{C}^\infty$ complex-valued functions on Ω. Since the group $\mathbb{H}$ is by definition decomposed as $\mathbb{C}^m \times \mathbb{C}^{n-m}$, then we consider $\mathcal{C}^\infty(\Omega)-$combinations of the differential forms $\phi_{IK} \bigwedge \overline{\phi}_{JL}$ defined as follows: If $I = (i_1, ..., i_\alpha)$ and $J = (j_1, ..., j_\beta)$ are multi-indices of integers of $\{1, ..., m\}$ and $K = (k_1, ..., k_\gamma)$, and $L = (l_1, ..., l_\delta)$ are multi-indices of integers of $\{m+1, ..., n\}$ we set

$$\phi_{IK} := \phi_{i_1} \wedge ... \wedge \phi_{i_\alpha} \bigwedge \phi_{k_1} \wedge ... \wedge \phi_{k_\gamma}$$

and

$$\overline{\phi}_{JL} := \overline{\phi}_{j_1} \wedge ... \wedge \overline{\phi}_{j_\beta} \bigwedge \overline{\phi}_{l_1} \wedge ... \wedge \overline{\phi}_{l_\delta},$$

and if we conside $J = (j_1, ..., j_\beta)$ as multi-indice of integers of $\{1, ..., n\}$, we set then

$$\overline{\phi}_J := \overline{\phi}_{j_1} \wedge ... \wedge \overline{\phi}_{j_\beta}$$

and

$$\phi_{IK,J} = \phi_{IK} \bigwedge \overline{\phi}_J.$$

A differential form f is called a $(l+\gamma)-$Hölderian form of $\mathcal{H}-$type (p_1, p_2, q_1, q_2) $(0 \leq p_1, q_1 \leq m)$ and $(0 \leq p_2, q_2 \leq n-m)$, if f can be written in the form

$$f = \sum_{\substack{|I|=p_1, |J|=q_1 \\ |K|=p_2, |L|=q_2}}{}' f_{IK,JL}\, \phi_{IK} \wedge \overline{\phi}_{JL}$$

where $f_{IK,JL} \in \mathcal{C}^{l+\gamma}(\Omega)$ and $\sum'$ means that the summation is performed over all multi-indices with strictly increasing components. We denote by

$\mathcal{C}^{l+\gamma}_{(p_1,p_2,q_1,q_2)_{\mathcal{H}}}(\Omega)$ the $\mathcal{C}^{\infty}(\Omega)-$module of $(l+\gamma)-$Hölderian form of $\mathcal{H}-$type (p_1,p_2,q_1,q_2) on Ω.
A differential form f is called a $(l+\gamma)-$Hölderian form of $\mathcal{H}-$type $((p_1,p_2),q)$ with $(0\leq p_1\leq m, 0\leq p_2\leq n-m)$ and $(0\leq q\leq n)$ if f can be written in the form

$$f=\sideset{}{'}\sum_{\substack{|I|=p_1,|K|=p_2\\ |J|=q_1}} f_{_{IK,J}}\ \phi_{_{IK}}\wedge\overline{\phi}_{_J}$$

where $f_{_{IK,J}}\in\mathcal{C}^{l+\alpha}(\Omega)$. We denote by $\mathcal{C}^{l+\gamma}_{((p_1,p_2),q)_{\mathcal{H}}}(\Omega)$ the $\mathcal{C}^{\infty}(\Omega)-$module of $(l+\gamma)-$Hölderian form of $\mathcal{H}-$type $((p_1,p_2),q)$ on Ω.
We define in the same way the $(l+\gamma)-$Hölderian forms of $\mathcal{H}-$type $(p,(q_1,q_2))$.
In our spirit, the module $\mathcal{C}^{\infty}_{(p_1,p_2,q_1,q_2)_{\mathcal{H}}}(\Omega)$ is viewed as the main module of differential forms from which we define by linear combinations, the following modules:

$$\begin{aligned}
\mathcal{C}^{l+\gamma}_{((p_1,p_2),q)_{\mathcal{H}}}(\Omega) &:= \bigoplus_{q_1+q_2=q}\mathcal{C}^{l+\gamma}_{(p_1,p_2,q_1,q_2)_{\mathcal{H}}}(\Omega)\\
\mathcal{C}^{l+\gamma}_{(p,(q_1,q_2))_{\mathcal{H}}}(\Omega) &:= \bigoplus_{p_1+p_2=p}\mathcal{C}^{l+\gamma}_{(p_1,p_2,q_1,q_2)_{\mathcal{H}}}(\Omega)\\
\mathcal{C}^{l+\gamma}_{(p,q)_{\mathcal{H}}}(\Omega) &:= \bigoplus_{\substack{p_1+p_2=p\\ q_1+q_2=q}}\mathcal{C}^{l+\gamma}_{(p_1,p_2,q_1,q_2)_{\mathcal{H}}}(\Omega)\\
&= \bigoplus_{q_1+q_2=q}\mathcal{C}^{l+\gamma}_{(p,(q_1,q_2))_{\mathcal{H}}}(\Omega)\\
&= \bigoplus_{p_1+p_2=p}\mathcal{C}^{l+\gamma}_{((p_1,p_2),q)_{\mathcal{H}}}(\Omega).
\end{aligned}$$

If $\mathcal{C}^{l+\gamma}_{(s)}(\Omega)$ denotes the $\mathcal{C}^{l+\gamma}(\Omega)-$module of $s-$differential forms on the open set $\Omega\subseteq\mathbb{H}$ with coefficients in $\mathcal{C}^{l+\gamma}(\Omega)$, we set then

$$\begin{aligned}
\mathcal{C}^{l+\gamma}_{(s)}(\Omega) &= \bigoplus_{p+q=s}\mathcal{C}^{l+\gamma}_{(p,q)_{\mathcal{H}}}(\Omega)\\
\mathcal{C}^{l+\gamma}_{(\bullet)}(\Omega) &= \bigoplus_{s=0}^{2n}\mathcal{C}^{l+\gamma}_{(s)}(\Omega).
\end{aligned}$$

Characterization of left invariant differential forms
Let $\psi=T_z:\xi\longmapsto\psi(z)=z*\xi$ be a left translation of the group $\mathbb{H}$, and let

$\psi^* : f \longmapsto \psi^*[f]$ be the isomorphism of the module $\mathcal{C}^\infty_{(s)}(\mathbb{H})$ defined by

$$\psi^*[f](\xi) = f(\psi^{-1}(\xi))$$

The differential form $f \in \mathcal{C}^\infty_{(s)}(\mathbb{H})$ is said to be left invariant if

$$\psi^*[f] = f \qquad \text{for all} \quad \psi.$$

Proposition 1.2.1.

1. *A $\mathcal{C}^\infty$−function f is left invariant if and only if f is constant.*
2. *The 1-differential forms of structure ϕ_j and $\overline{\phi}_j$ are left invariant.*
3. *The differential form*

$$f = \sum_{|I|+|J|=s}{}' f_{I,J}\, \phi_I \wedge \overline{\phi}_J \qquad \in \ \mathcal{C}^\infty_{(s)}(\mathbb{H})$$

is left invariant if and only if the functions $f_{I,J}$ are constant.

Proof.
1) The first assertion is obvious.
2) The fact that the 1-differential forms of structure ϕ_j and $\overline{\phi}_j$ are left invariant follows from the definition of the form ϕ as differential of a left translation.
3) The third assertion is a consequence of the the first and the second assertions. □

Left invariant operators

A linear operator

$$A : \mathcal{C}^\infty_{(\bullet)}(\mathbb{H}) \longmapsto \mathcal{C}^\infty_{(\bullet)}(\mathbb{H})$$

is said to be left invariant if

$$(\psi^*)^{-1} \circ A \circ \psi^* = A \qquad \text{for all left translation} \quad \psi$$

which means in terms of commutators, that

$$[A, \psi^*] = 0. \qquad \text{for all left translation} \quad \psi.$$

Proposition 1.2.2. *Let* $A : \mathcal{C}^\infty_\bullet(\mathbb{H}) \longmapsto \mathcal{C}^\infty_{(\bullet)}(\mathbb{H})$ *be a linear operator such that:*

1. $A\left(\mathcal{C}^\infty_{(s)}(\mathbb{H})\right) \subseteq \mathcal{C}^\infty_{(s)}(\mathbb{H})$

2. *A satisfies the Leibniz's rule:*

$$A(f \wedge g) = (-1)^{deg(g)} A(f) \wedge g + (-1)^{deg(f)} f \wedge A(g).$$

Then A is left invariant if and only if

1. $A : \mathcal{C}^\infty(\mathbb{H}) \longmapsto \mathcal{C}^\infty(\mathbb{H})$ *is left invariant,*

2. *For all* $1 \leq j \leq n$*, the differential forms* $A(\phi_j)$ *and* $A(\overline{\phi}_j)$ *are left invariant.*

Proof. The necessarily condition is obvious. We prove the sufficient condition by induction on the integer $0 \leq s \leq 2n$.
By the hypothesis 1), the assertion is true for $s = 0$ and $s = 1$. Assume that this assertion is true for $s \geq 1$, that is $A : \mathcal{C}^\infty_{(s)}(\mathbb{H}) \longrightarrow \mathcal{C}^\infty_{(s)}(\mathbb{H})$ is left invariant, and prove it for $s+1$. For this let $f = g \wedge \phi_j \in \mathcal{C}^\infty_{(s+1)}(\mathbb{H})$. Starting from the following obvious identity

$$\psi^*(g \wedge \phi_j) = \psi^*(g) \wedge \psi^*(\phi_j)$$

and using Leibniz's rule, we obtain:

$$\begin{aligned} A\left(\psi^*(f)\right) &= A\left(\psi^*(g \wedge \phi_j)\right) \\ &= A\left(\psi^*(g) \wedge \psi^*(\phi_j)\right) \\ &= -A\left(\psi^*(g)\right) \wedge \psi^*(\phi_j) + (-1)^{deg(g)} \psi^*(f) \wedge A\left(\psi^*(\phi_j)\right) \\ &= -\psi^*\left(A(g)\right) \wedge \psi^*(\phi_j) + (-1)^{deg(g)} \psi^*(g) \wedge \psi^*\left(A(\phi_j)\right) \\ &= -\psi^*\left(A(g) \wedge \phi_j\right) + (-1)^{deg(g)} \psi^*\left(g \wedge A(\phi_j)\right) \\ &= \psi^*\left(-A(g) \wedge \phi_j + (-1)^{deg(g)} g \wedge A(\phi_j)\right) \\ &= \psi^*\left(A(g \wedge \phi_j)\right) \end{aligned}$$

which proves that $[A, \psi^*]\,(g \wedge \phi_j) = 0$. We prove in the same way that $[A, \psi^*]\,(g \wedge \overline{\phi}_j) = 0$. Then $A : \mathcal{C}^\infty_{(s+1)}(\mathbb{H}) \longrightarrow \mathcal{C}^\infty_{(s+1)}(\mathbb{H})$ is left invariant, which completes the proof. □

1.2.3 Modules of differential classes on $\mathbb{H}$ attached to $L \lhd \mathcal{H}$

A) The 2-form of structure σ^L attached to $L \lhd \mathcal{H}$
By analogy with (1.1.7), we define for any $L \lhd \mathcal{H}$, the vectorial 2-form of structure $\sigma^L = (\sigma_1^L, ..., \sigma_n^L)$ as follows:

$$\sigma_k^L := \sum_{i,j=1}^{m} \lambda_{i,j}^k \phi_i \wedge \overline{\phi}_j \tag{1.2.1}$$

where $\lambda_{i,j}^k$ are the constants of structure of the Lie algebra $\mathcal{H}_L$ defined in (1.1.14). ***B) The submodule $\mathcal{J}_{(s)}^L(\Omega)$ of $\mathcal{C}_{(s)}^\infty(\Omega)$ attached to $L \lhd \mathcal{H}$***
For $0 \leq s \leq 2n$, let $\mathcal{J}_{(s)}^L(\Omega)$ denote the $\mathcal{C}^\infty(\Omega)-$submodule of $\mathcal{C}_{(s)}^\infty(\Omega)$ attached to the 2-form σ^L, and defined as follows:

- For $2 \leq s \leq 2n$, the submodule $\mathcal{J}_{(s)}^L(\Omega)$ is generated by the scalar 2-forms σ_k^L (see the expression of σ_k^L in (1.2.1) above), that is

$$\mathcal{J}_{(s)}^L(\Omega) := \left\{ \sum_{k=1}^{k=n} f_k \wedge \sigma_k^L, \quad f_k \in \mathcal{C}_{(s-2)}^\infty(\Omega) \right\}$$

- For $s = 0$ and $s = 1$, we set

$$\mathcal{J}_{(0)}^L(\Omega) = \mathcal{J}_{(1)}^L(\Omega) = \{0\}\,.$$

C) The differential ideal $\mathcal{J}_\bullet^L(\Omega)$ attached to $L \lhd \mathcal{H}$
With the above notations, the submodule

$$\mathcal{J}_{(\bullet)}^L(\Omega) = \bigoplus_{s=0}^{2n} \mathcal{J}_{(s)}^L(\Omega)$$

is a graded differential([2]) ideal of $\mathcal{C}_{(\bullet)}^\infty(\Omega)$.
D) The modules of differential classes attached to $L \lhd \mathcal{H}$
Let $1 \leq p_1, p_2 \leq m$ and $1 \leq q_1, q_2 \leq n - m$ be integers with $p_1 + p_2 = p$

[2] In the sense that $df \in \mathcal{J}_{(\bullet)}^L(\Omega)$ for all $f \in \mathcal{J}_{(\bullet)}^L(\Omega)$.

and $q_1+q_2=q$, and Ω an open set of $\mathbb{H}$. We first attach to the subalgebra $L \lhd \mathcal{H}$, the following submodules([3]) of $\mathcal{C}^\infty_{(p+q)}(\Omega)$:

$$\begin{aligned}
\mathcal{C}^\infty_{((p_1,p_2),q)_L}(\Omega) &:= \mathcal{C}^\infty_{((p_1,p_2),q)_\mathcal{H}}(\Omega) + \mathcal{J}^L_{(p+q)}(\Omega) \qquad \text{with } (p_1+p_2=p)\\
\mathcal{C}^\infty_{(p,(q_1,q_2))_L}(\Omega) &:= \mathcal{C}^\infty_{(p,(q_1,q_2))_\mathcal{H}}(\Omega) + \mathcal{J}^L_{(p+q)}(\Omega) \qquad \text{with } (q_1+q_2=q)\\
\mathcal{C}^\infty_{(p,q)_L}(\Omega) &:= \mathcal{C}^\infty_{(p,q)_\mathcal{H}}(\Omega) + \mathcal{J}^L_{(p+q)}(\Omega)\\
&= \bigoplus_{p_1+p_2=p} \mathcal{C}^\infty_{((p_1,p_2),q)_L}(\Omega)\\
&= \bigoplus_{q_1+q_2=q} \mathcal{C}^\infty_{(p,(q_1,q_2))_L}(\Omega).
\end{aligned}$$

Now we define on the module $\mathcal{C}^\infty_{(s)}(\Omega)$ the relation $\sim$ as follows:

$$f \sim g \Longleftrightarrow f-g \in \mathcal{J}^L_{(s)}(\Omega).$$

The fact that $\sim$ is obviously an equivalence relation leads to the following:

Definition 1.2.3. The quotient module

$$\widetilde{\mathcal{C}}^\infty_{((p_1,p_2),q)_L}(\Omega) := \mathcal{C}^\infty_{((p_1,p_2),q)_L}(\Omega)/\sim$$

is called the module of $((p_1,p_2),q)_L-$differential classes, or differential classes of $\mathcal{H}-$type $((p_1,p_2),q)_L$, and The quotient module

$$\widetilde{\mathcal{C}}^\infty_{(p,(q_1,q_2))_L}(\Omega) := \mathcal{C}^\infty_{(p,(q_1,q_2))_L}(\Omega)/\sim$$

is called the module of $(p,(q_1,q_2))_L-$differential classes or differential classes of $\mathcal{H}-$type $(p,(q_1,q_2))_L$.
We set

$$\begin{aligned}
\widetilde{\mathcal{C}}^\infty_{(p,q)_L}(\Omega) := &\bigoplus_{p_1+p_2=p} \widetilde{\mathcal{C}}^\infty_{((p_1,p_2),q)_L}(\Omega)\\
&\bigoplus_{q_1+q_2=q} \widetilde{\mathcal{C}}^\infty_{(q,(q_1,q_2))_L}(\Omega).
\end{aligned}$$

E) A metric interpretation of differential classes attached to $L \lhd \mathcal{H}$

[3]The sums are not direct.

Since the group $\mathbb{H}$ is assumed to be metric, we can give a simple interpretation of the differential classes in terms of the metric $g_{\mathcal{H}}$.
Indeed, if we write $f, g \in \mathcal{C}^{\infty}_{(p+q)}(\Omega)$ as follows

$$f = \sum_{|I|+|J|=p+q}' f_{I,J}\ \phi_I \wedge \overline{\phi}_J$$

and

$$g = \sum_{|I|+|J|=p+q}' g_{I,J}\ \phi_I \wedge \overline{\phi}_J,$$

then the metric $g_{\mathcal{H}}$ induces on $\mathcal{C}^{\infty}_{(p+q)}(\Omega)$ the inner product

$$\langle f, g\rangle_{\mathcal{H}} := \sum_{|I|+|J|=p+q}' \int_{\Omega} f_{I,J} . \overline{g}_{I,J}\ dV_{\mathcal{H}} \tag{1.2.2}$$

where $dV_{\mathcal{H}} := \left(\frac{-i}{2}\right)^n \overline{\phi}_1 \wedge \phi_1 \wedge ... \wedge \overline{\phi}_n \wedge \phi_n$ is the $2n$-form volume on Ω with respect to the metric $g_{\mathcal{H}}$. Let $\mathcal{B}^{\infty}_{((p_1,p_2),q)_L}$ be the orthogonal of $\mathcal{J}^L_{(p+q)}$ with respect to the inner product (1.2.2), that is

$$\mathcal{B}^{\infty}_{((p_1,p_2),q)_L} := \left\{ f^{\perp} \in \mathcal{C}^{\infty}_{((p_1,p_2),q)_L}, \quad \left\langle f^{\perp}, g\right\rangle_{\mathcal{H}} = 0 \quad \text{for all} \quad g \in \mathcal{J}^L_{(p+q)} \right\}.$$

We check easily the following proposition.

Proposition 1.2.4. *The following map*

$$\mathcal{B}^{\infty}_{((p_1,p_2),q)_L} \longrightarrow \widetilde{\mathcal{C}}^{\infty}_{((p_1,p_2),q)_L}$$

$$f^{\perp} \longmapsto \widetilde{f}$$

is a linear isomorphism.

This proposition means that we can identify every differential classes $\widetilde{f} \in \widetilde{\mathcal{C}}_{((p_1,p_2),q)_L}$ to a differential form $f^{\perp} \in \mathcal{C}^{\infty}_{((p_1,p_2),q)_L}$ orthogonal to the ideal $\mathcal{J}^L_{(p+q)}$.

1.3 The differential operator $\overline{\partial}_{_L}$ defined by $L \lhd \mathcal{H}$

Let $1 \leq p_1, p_2 \leq m$ and $1 \leq q_1, q_2 \leq n-m$ be integers with $p_1 + p_2 = p$ and $q_1 + q_2 = q$, and Ω an open set of $\mathbb{H}$.
Our aim now is to prove We the following theorem.

Theorem 1.3.1.
There exists for every $L \lhd \mathcal{H}$, one and only one pair of first order linear differential operators $(\partial_{_L}, \overline{\partial}_{_L})$:

$$\partial_{_L} : \mathcal{C}^{\infty}_{(p,(q_1,q_2))_L}(\Omega) \longrightarrow \mathcal{C}^{\infty}_{(p+1,(q_1,q_2))_L}(\Omega)$$

$$\overline{\partial}_{_L} : \mathcal{C}^{\infty}_{((p_1,p_2),q)_L}(\Omega) \longrightarrow \mathcal{C}^{\infty}_{((p_1,p_2),q+1)_L}(\Omega)$$

such that:

1. $\overline{\partial}_{_L}$ *is left $\mathbb{H}-$invariant.*
2. *If $\langle\ ,\ \rangle$ denotes the pairing between vector fields and 1-differential forms, then for every $\mathcal{C}^{\infty}$ function f,*

$$\left\langle \overline{\mathcal{Z}}_j, \overline{\partial}_{_L} f \right\rangle = \overline{\mathcal{Z}}_j(f) \qquad \text{for all} \quad 1 \leq j \leq n. \tag{1.3.1}$$

3. *The 1-forms of structure ϕ and $\overline{\phi}$ satisfy the following " $L-$equations of structure":*

$$\begin{cases} \overline{\partial}_{_L}\phi = \ \ \sigma^L \\ \overline{\partial}_{_L}\overline{\phi} = -\sigma^L. \end{cases} \tag{1.3.2}$$

4. *For all $f, g \in \mathcal{C}^{\infty}_{(\bullet)}(\Omega)$, $\overline{\partial}_{_L}$ satisfies Leibnitz'rule, that is*

$$\overline{\partial}_{_L}(f \wedge g) = \overline{\partial}_{_L} f \wedge g + (-1)^{\nu} f \wedge \overline{\partial}_{_L} g, \qquad \nu = deg(f). \tag{1.3.3}$$

5. *The differential operator $\partial_{_L}$ is related to $\overline{\partial}_{_L}$ by the identity:*

$$\partial_{_L} f = \overline{(\overline{\partial}_{_L}\overline{f})} \qquad \text{for all } f \in \mathcal{C}^{\infty}_{(\bullet)}(\Omega). \tag{1.3.4}$$

Remark 1.3.2. It suffice from identities (1.3.4) above, to prove the existence and uniqueness of the $\overline{\partial}_{L}-$operator only.
The proof of theorem 1.3.1, will be done in two steps, first for $\mathcal{C}^{\infty}$ functions, then for differential forms.

1.3.1 The differential operator $\overline{\partial}_{L}$ for functions.

Proof. **(of theorem 1.3.1 for functions.)**
Let's first prove that if the operator $\overline{\partial}_{L}$ exists for $\mathcal{C}^{\infty}$ functions, then it will be unique. Indeed, since by definition of the modules $\mathcal{J}^{L}_{(s)}(\Omega$, we have for $s=0$ and $s=1$

$$\mathcal{J}^{L}_{(0)}(\Omega)=\mathcal{J}^{L}_{(1)}(\Omega)=\{0\}$$

then

$$\mathcal{C}^{\infty}_{((0,0),0)_L}(\Omega)=\mathcal{C}^{\infty}(\Omega)$$

and

$$\mathcal{C}^{\infty}_{((0,0),1)_L}(\Omega)=\mathcal{C}^{\infty}_{(0,1)_{\mathcal{H}}}(\Omega).$$

Now let $f\in\mathcal{C}^{\infty}(\Omega)$, and write $\overline{\partial}_{L}f\in\mathcal{C}^{\infty}_{(0,1)_{\mathcal{H}}}(\Omega)$ as linear combination of $\overline{\phi}_k$, $1\leq k\leq n$, with $\mathcal{C}^{\infty}$ coefficients $P_k(f)$

$$\overline{\partial}_{L}f=\sum_{k=1}^{n}P_k(f)\overline{\phi}_k.$$

Since $\overline{\partial}_{L}$ is a linear differential operator, then $supp\left(\overline{\partial}_{L}f\right)\subseteq supp(f)$, which implies that $supp\left(P_k(f)\right)\subseteq supp(f)$ for each $1\leq k\leq n$. By Peeter's theorem, P_k is then a linear differential operator, that is

$$P_k=\sum_{j=1}^{n}a_{j,k}(z)\mathcal{Z}_j+b_{j,k}(z)\overline{\mathcal{Z}}_j,\qquad z\in\Omega.$$

where $a_{j,k},b_{j,k}\in\mathcal{C}^{\infty}(\Omega)$ are $\mathcal{C}^{\infty}$ coefficients. Hence

$$\overline{\partial}_{L}=\sum_{j,k=1}^{n}\left(a_{j,k}(z)\mathcal{Z}_j+b_{j,k}(z)\overline{\mathcal{Z}}_j\right)\overline{\phi}_k.$$

Since by condition (1), $\overline{\partial}_{L}$ is left $\mathbb{H}-$invariant, then the coefficients $a_{j,k},b_{j,k}\in\mathcal{C}^{\infty}(\Omega)$ are constant functions, and from condition (2) we obtain

$$a_{j,k}=0\quad\text{and}\quad b_{j,k}=\delta_{j,k}$$

where $\delta_{j,k}$ is the Kronecker symbol. This means that the $\overline{\partial}_{L}-$operator must be defined for $\mathcal{C}^\infty$ functions by

$$\overline{\partial}_{L} f = \sum_{j=1}^{n} \overline{\mathcal{Z}}_j(f)\overline{\phi}_j. \tag{1.3.5}$$

For the existence, it suffices to observe that the differential operator $\overline{\partial}_{L}$ defined by (1.3.5) satisfies in fact the conditions (1), (2), which proves its existence for $\mathcal{C}^\infty$ functions.
Since by condition (5), we have for all $f \in \mathcal{C}^\infty(\Omega)$, $\partial_{L} f = \overline{(\overline{\partial}_{L}\overline{f})}$, then the $\partial_{L}-$operator must be defined for $\mathcal{C}^\infty$ functions by

$$\partial_{L} f = \sum_{j=1}^{n} \mathcal{Z}_j(f)\phi_j. \tag{1.3.6}$$

and then conditions (1), (2), (5) are all satisfied. □

Remark 1.3.3. From formulas (1.3.5), (1.3.6), we observe that the differential operators ∂_{L} and $\overline{\partial}_{L}$ acting on functions are independent of the choice of the subalgebra $L \lhd \mathcal{H}$. For this raison, we denote them when acting on functions, indifferently by ∂_{L} , $\overline{\partial}_{L}$ or by $\partial_{\mathbb{H}}$, $\overline{\partial}_{\mathbb{H}}$, and we write for $\mathcal{C}^\infty$ functions

$$\partial_{L} f = \partial_{\mathbb{H}} f = \sum_{j=1}^{n} \mathcal{Z}_j(f)\phi_j$$

$$\overline{\partial}_{L} f = \overline{\partial}_{\mathbb{H}} f = \sum_{j=1}^{n} \overline{\mathcal{Z}}_j(f)\overline{\phi}_j.$$

1.3.2 The differential operators $\overline{\partial}_{L}$ for differential forms.

A) Extension of the vector fields $\mathcal{Z}_j$ ***and*** $\overline{\mathcal{Z}}_j$ ***to differential forms***
Let $L \lhd \mathcal{H}$. To define the differential operators $\overline{\partial}_{L}$ and ∂_{L} for differential forms, Formulas (1.3.5) and (1.3.6) suggest to extend the action of the left vector fields $\overline{\mathcal{Z}}_j \in \mathcal{H}$ and $\mathcal{Z}_j \in \mathcal{H}$ to linear operators $\overline{\mathcal{Z}}_j^L$ and $\mathcal{Z}_j^L$ acting on differential forms.
Indeed, the vector fields $\mathcal{Z}_j$, and $\overline{\mathcal{Z}}_j$ can be viewed simultaneously as vectors of the Lie algebra $\mathcal{H}$, that is, as linear differential operators acting on

$\mathcal{C}^\infty-$functions, by formulas (1.1.8) , and (1.1.9), and as vectors of the Lie algebra $\mathcal{H}_L$, which means that $\mathcal{Z}_j$ and $\overline{\mathcal{Z}}_j$ act on the vectors $X \in \mathcal{H}_L$ by the ad-endomorphisms $ad_{_L}\mathcal{Z}_j$ and $ad_{_L}\overline{\mathcal{Z}}_j$ as follows

$$ad_{_L}\mathcal{Z}_j : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

$$X \longrightarrow ad_{_L}\mathcal{Z}_j(X) := \left[\mathcal{Z}_j, X\right]_{_L}, \tag{1.3.7}$$

$$ad_{_L}\overline{\mathcal{Z}}_j : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

$$X \longrightarrow ad_{_L}\overline{\mathcal{Z}}_j(X) := \left[\overline{\mathcal{Z}}_j, X\right]_{_L}. \tag{1.3.8}$$

Then, using the brackets (1.1.13), we deduce by duality with (1.3.7) and (1.3.8), that $\mathcal{Z}_j$ and $\overline{\mathcal{Z}}_j$ act on the 1-differential forms ϕ_k and $\overline{\phi}_k$ by:

$$\phi_k \longmapsto \left(ad_{_L}\mathcal{Z}_j\right)^* (\phi_k) = \sum_{i=1}^{n} \overline{\lambda_{i,j}^k}\,\overline{\phi}_i \tag{1.3.9}$$

$$\overline{\phi}_k \longmapsto \left(ad_{_L}\mathcal{Z}_j\right)^* \left(\overline{\phi}_k\right) = \sum_{i=1}^{n} \lambda_{i,j}^k\,\overline{\phi}_i \tag{1.3.10}$$

$$\phi_k \longmapsto \left(ad_{_L}\overline{\mathcal{Z}}_j\right)^* (\phi_k) = \sum_{i=1}^{n} \overline{\lambda_{i,j}^k}\,\phi_i \tag{1.3.11}$$

$$\overline{\phi}_k \longmapsto \left(ad_{_L}\overline{\mathcal{Z}}_j\right)^* \left(\overline{\phi}_k\right) = \sum_{i=1}^{n} \lambda_{i,j}^k\,\phi_i. \tag{1.3.12}$$

This leads to define the linear operators $\mathcal{Z}_j$, and $\overline{\mathcal{Z}}_j$.

Definition 1.3.4. Let $L \lhd \mathcal{H}$, and $1 \leq j \leq n$. We consider the following linear operators $\mathcal{Z}_j^L$, $\overline{\mathcal{Z}}_j^L$:

1. $\overline{\mathcal{Z}}_j^L : \mathcal{C}^\infty_{(s)}(\Omega) \longrightarrow \mathcal{C}^\infty_{(s)}(\Omega)$ is defined by the conditions:

   (a) on a $\mathcal{C}^\infty$ function f, $\quad \overline{\mathcal{Z}}_j^{\mathbb{L}}(f) := \overline{\mathcal{Z}}_j(f)$.

   (b) On the first 1-forms of structure ϕ_k and $\overline{\phi}_k$, $\;\overline{\mathcal{Z}}_j^L$ acts as $\left(ad_{_L}\overline{\mathcal{Z}}_j\right)^*$

$$\begin{cases} \overline{\mathcal{Z}}_j^L(\phi_k) := \left(ad_{_L}\overline{\mathcal{Z}}_j\right)^* (\phi_k) \\ \overline{\mathcal{Z}}_j^L(\overline{\phi}_k) := \left(ad_{_L}\overline{\mathcal{Z}}_j\right)^* (\overline{\phi}_k). \end{cases}$$

(c) On arbitrary $\mathcal{C}^\infty$ differential forms, $\overline{\mathcal{Z}}_j^L$ acts by Leibnitz' rule:

$$\overline{\mathcal{Z}}_j^L(f \wedge g) = \overline{\mathcal{Z}}_j^L(f) \wedge g + f \wedge \overline{\mathcal{Z}}_j^L(g) \qquad \nu = deg(g).$$

.

2. $\mathcal{Z}_j^L : \mathcal{C}^\infty_{(s)}(\Omega) \longrightarrow \mathcal{C}^\infty_{(s)}(\Omega)$ is defined for all $f \in \mathcal{C}^\infty_{(\bullet)}(\Omega)$, by

$$\mathcal{Z}_j^L(f) := \overline{\overline{\mathcal{Z}}_j^L(\overline{f})}.$$

Proposition 1.3.5.
We have for all $L \lhd \mathcal{H}$ *and* $1 \leq i, j, k \leq n$, *the following properties:*
1) $\overline{\mathcal{Z}}_j^L$ *is left* $\mathbb{H}-$*invariant.*
2) the following compositions hold

$$\begin{cases} \overline{\mathcal{Z}}_i^L \circ \overline{\mathcal{Z}}_j^L \left(\phi_{k.}\right) = 0 \\ \overline{\mathcal{Z}}_i^L \circ \overline{\mathcal{Z}}_j^L \left(\overline{\phi}_k\right) = 0. \end{cases} \tag{1.3.13}$$

3) The following commutators hold

$$\left[\mathcal{Z}_i^L, \overline{\mathcal{Z}}_j^L\right] = \sum_{k=1}^n \left(\lambda_{i,j}^k \mathcal{Z}_k^L + \overline{\lambda_{i,j}^k} \overline{\mathcal{Z}}_k^L\right). \tag{1.3.14}$$

Proof.
1) The operator $\overline{\mathcal{Z}}_j^L$ is left invariant by proposition 1.2.2.
2) Since $\overline{\mathcal{Z}}_j^L$ is defined by $\left(ad_L \overline{\mathcal{Z}}_j\right)^*$, then the compositions (1.3.13) follow from the fact that the group $\mathbb{H}$ is $2-$step nilpotent.
3) Since $\mathcal{Z}_i^L$ and $\overline{\mathcal{Z}}_j^L$ satisfy the Leibniz's rule, then $\left[\mathcal{Z}_i^L, \overline{\mathcal{Z}}_j^L\right]$ satisfies the same rule, that is

$$\left[\mathcal{Z}_i^L, \overline{\mathcal{Z}}_j^L\right](f \wedge g) = \left[\mathcal{Z}_i^L, \overline{\mathcal{Z}}_j^L\right](f) \wedge g + f \wedge \left[\mathcal{Z}_i^L, \overline{\mathcal{Z}}_j^L\right](g)$$

and then, to determine completely $\left[\mathcal{Z}_i^L, \overline{\mathcal{Z}}_j^L\right]$, it suffices to evaluate it at $\mathcal{C}^\infty-$functions and at the forms ϕ_k and $\overline{\phi}_k$. For this, we have on functions,

$$\left[\mathcal{Z}_i^L, \overline{\mathcal{Z}}_j^L\right] = \left[\mathcal{Z}_i, \overline{\mathcal{Z}}_j\right]_L = \sum_{k=1}^n \lambda_{i,j}^k \mathcal{Z}_k + \overline{\lambda_{i,j}^k} \overline{\mathcal{Z}}_k$$

and on the forms ϕ_k and $\overline{\phi}_k$:

$$\begin{aligned}
\left[\mathcal{Z}_i^L, \overline{\mathcal{Z}}_j^L\right] &= \left[(ad_L\mathcal{Z}_i)^*, \left(ad_L\overline{\mathcal{Z}}_j\right)^*\right] \\
&= (ad_L\mathcal{Z}_i)^* \circ \left(ad_L\overline{\mathcal{Z}}_j\right)^* - (ad_L\mathcal{Z}_i)^* \circ \left(ad_L\overline{\mathcal{Z}}_j\right)^* \\
&= \left(ad_L\overline{\mathcal{Z}}_j \circ ad_L\mathcal{Z}_i\right)^* - \left(ad_L\mathcal{Z}_i \circ ad_L\overrightarrow{\mathcal{Z}}_j\right)^* \\
&= \left(ad_L\overline{\mathcal{Z}}_j \circ ad_L\mathcal{Z}_i - ad_L\mathcal{Z}_i \circ ad_L\overline{\mathcal{Z}}_j\right)^* \\
&= \left(ad_L\left[\mathcal{Z}_i, \overline{\mathcal{Z}}_j\right]\right)^* \\
&= \sum_{k=1}^{n} \lambda_{i,j}^k (ad_L\mathcal{Z}_k)^* + \overline{\lambda_{i,j}^k}\left(ad_L\overline{\mathcal{Z}}_k\right)^*
\end{aligned}$$

We obtain then

$$\left[\mathcal{Z}_i^L, \overline{\mathcal{Z}}_j^L\right] = \sum_{k=1}^{n} \lambda_{i,j}^k \mathcal{Z}_k^L + \overline{\lambda_{i,j}^k}\,\overline{\mathcal{Z}}_k^L$$

which proves (1.3.14). □

B) Extension of the operators ∂_L and $\overline{\partial}_L$ to differential forms

Proof. **(of theorem 1.3.1 for differential forms.)** To complete the proof of theorem 1.3.1, it remains now to extend the linear differential operator $\overline{\partial}_L$ defined in (1.3.5) to differential forms.
For this, let f be a $\mathcal{C}^\infty-$differential form, and define

$$\overline{\partial}_L f = \sum_{j=1}^{n} \overline{\phi}_j \wedge \overline{\mathcal{Z}}_j^L(f). \tag{1.3.15}$$

The first order linear differential operator $\overline{\partial}_L$ defined by (1.3.15) satisfies the conditions (1), (2), (3), (4) of theorem 1.3.1. Indeed,
1) Since $\overline{\mathcal{Z}}_j^L$ and $\overline{\phi}_j$ are left invariant, then $\overline{\partial}_L$ is left invariant.
2) The condition (2) is already satisfied in the construction of $\overline{\partial}_L$ for functions.

3) Let us check for each $1 \leq k \leq n$, that $\overline{\partial}_L \phi = \sigma^L$. Indeed, we have:

$$\begin{aligned}
\overline{\partial}_L(\phi_k) &= \sum_{j=1}^{n} \overline{\phi}_j \wedge \overline{\mathcal{Z}}_j^L(\phi_k) \\
&= \sum_{j=1}^{n} \overline{\phi}_j \wedge \left(ad_L \overline{\mathcal{Z}}_j\right)^* (\phi_k) \\
&= \sum_{i,j=1}^{n} \lambda_{i,j}^k \phi_j \wedge \overline{\phi}_i \qquad \Big(\text{from } (1.3.11)\Big) \\
&= -\sigma_k^L.
\end{aligned}$$

The identity $\overline{\partial}_L \overline{\phi} = -\sigma^L$ can be proved by a similar method.
4) Since by definition, $\overline{\mathcal{Z}}_j^L$ satisfies Leibnitz'rule , then $\overline{\partial}_L$ observes this rule.
5) If we define the linear operators ∂_L for every $\mathcal{C}^\infty-$differential form f, as follows:

$$\partial_L f := \sum_{j=1}^{n} \phi_j \wedge \mathcal{Z}_j^L(f) \tag{1.3.16}$$

then the pair of linear operators $\left(\partial_L, \overline{\partial}_L\right)$ satisfies obviously the conditions (5) of theorem 1.3.1. The proof is then complete. □

Proposition 1.3.6.
The $\overline{\partial}_L-$operator is left invariant, and satisfies furthermore the following properties:

$$\begin{cases} \overline{\partial}_L(\phi_k) = \sigma_k^L \\ \overline{\partial}_L(\overline{\phi}_k) = -\sigma_k^L \end{cases} \tag{1.3.17}$$

and then

$$\overline{\partial}_L\left(\mathcal{J}_{(\bullet)}^L(\Omega)\right) \subseteq \mathcal{J}_{(\bullet)}^L(\Omega) \tag{1.3.18}$$

$$\overline{\partial}_L\left(\mathcal{C}^\infty_{((p_1,p_2),q)_L}(\Omega)\right) \subseteq \mathcal{C}^\infty_{((p_1,p_2),q+1)_L}(\Omega) \tag{1.3.19}$$

$$\overline{\partial}_L^2 f = (-1)^{deg(f)+1} \sum_{k=1}^{k=n} \overline{\mathcal{Z}}_k^L(f) \wedge \sigma_k^L \qquad \in \mathcal{J}_{(\bullet)}^L(\Omega). \tag{1.3.20}$$

Proof.
1) The identities (1.3.17) are obvious. From (1.3.17) we observe that $\overline{\partial}_L \sigma_k^L \in \mathcal{J}^L_{(\bullet)}(\Omega)$, which implies the inclusion (1.3.18) by leibniz formula.
2) Let $g = f + f_0 \in \mathcal{C}^\infty_{((p_1,p_2),q)_L}(\Omega)$, with $f_0 \in \mathcal{J}^L_{(p+q)}(\Omega)$, and

$$f = \sum\nolimits'_{\substack{|I|=p_1,|K|=p_2\\|J|=q}} f_{IK,J}\ \phi_{IK} \wedge \overline{\phi}_J \qquad \in \mathcal{C}^\infty_{((p_1,p_2),q)_{\mathcal{H}}}(\Omega).$$

If we compute $\overline{\partial}_L f$ by Leibniz formula, we obtain:

$$\overline{\partial}_L f = \sum_{\substack{|I|=p_1,|K|=p_2\\|J|=q}}{}' \overline{\partial}_L f_{IK,J} \wedge \phi_{IK} \wedge \overline{\phi}_J + f_{IK,J}\ \overline{\partial}_L(\phi_{IK} \wedge \overline{\phi}_J)$$

Since by condition (4) of theorem 1.3.1, we have $\overline{\partial}_L(\phi_{IK} \wedge \overline{\phi}_J) \in \mathcal{J}^L_{(\bullet)}(\Omega)$, then $\overline{\partial}_L f_{IK,J} \wedge \phi_{IK} \wedge \overline{\phi}_J \in \mathcal{C}^\infty_{((p_1,p_2),q+1)_{\mathcal{H}}}(\Omega)$, and $f_{IK,J}\ \overline{\partial}_L(\phi_{IK} \wedge \overline{\phi}_J) + \overline{\partial}_L f_0 \in \mathcal{J}^L_{(p+q)}(\Omega)$, which proves inclusion (1.3.19).
3) Let $\nu = deg(f)$. We have:

$$\begin{aligned}
\overline{\partial}^2_L f &= \overline{\partial}_L(\overline{\partial}_L f)\\
&= \overline{\partial}_L\left(\sum_{j=1}^n \overline{\mathcal{Z}}^L_j(f) \wedge \overline{\phi}_j\right)\\
&= \sum_{k=1}^n \overline{\mathcal{Z}}^L_k\left(\sum_{j=1}^n \overline{\mathcal{Z}}^L_j(f) \wedge \overline{\phi}_j\right) \wedge \overline{\phi}_k\\
&= \sum_{j,k=1}^n \overline{\mathcal{Z}}^L_k \overline{\mathcal{Z}}^L_j(f) \wedge \overline{\phi}_j \wedge \overline{\phi}_k \ + \ (-1)^\nu \sum_{k=1}^n f \wedge \overline{\mathcal{Z}}^L_k(\overline{\phi}_j) \wedge \overline{\phi}_k\\
&= \sum_{k<j} \left[\overline{\mathcal{Z}}^L_j, \overline{\mathcal{Z}}^L_k\right](f) \wedge \overline{\phi}_j \wedge \overline{\phi}_k \ + \ (-1)^\nu \sum_{j=1}^n \overline{\mathcal{Z}}^L_j(f) \wedge \left(\sum_{k=1}^n \overline{\mathcal{Z}}^L_k(\overline{\phi}_j)\right) \wedge \overline{\phi}_k\\
&= (-1)^\nu \sum_{j=1}^n \overline{\mathcal{Z}}^L_j(f) \wedge \left(\sum_{k=1}^n \sum_{i=1}^n \overline{\lambda^j_{i,k}}\phi_i\right) \wedge \overline{\phi}_k\\
&= (-1)^\nu \sum_{j=1}^n \overline{\mathcal{Z}}^L_j(f) \wedge \left(\sum_{i,k=1}^n \overline{\lambda^j_{i,k}}\phi_i) \wedge \overline{\phi}_k\right)\\
&= (-1)^{\nu+1} \sum_{j=1}^n \overline{\mathcal{Z}}^L_j(f) \wedge \sigma^L_j.
\end{aligned}$$

□

Corollary 1.3.7.
For $L = \mathcal{H}$, the $\overline{\partial}_{\mathbb{H}}-$operator satisfies the following particular properties:

$$\overline{\partial}_{\mathbb{H}}(\phi) = \overline{\partial}_{\mathbb{H}}(\overline{\phi}) = 0. \tag{1.3.21}$$

$$\overline{\partial}_{\mathbb{H}}^{2} = 0. \tag{1.3.22}$$

If $f = \sum'_{|I|=p,|J|=q} f_{I,J}\ \phi_I \wedge \overline{\phi}_J\ \in\ \mathcal{C}^{\infty}_{(p,q)_{\mathcal{H}}}(\Omega)$, then

$$\overline{\partial}_{\mathbb{H}} f = \sum'_{|I|=p,|J|=q} \overline{\partial}_{\mathbb{H}} f_{I,J} \wedge \phi_I \wedge \overline{\phi}_J \in\ \mathcal{C}^{\infty}_{(p,q+1)_{\mathcal{H}}}(\Omega). \tag{1.3.23}$$

Proof.
This follows from the fact that for $L = \mathcal{H}$, we have $\sigma^{\mathcal{H}} = 0$, and then $\mathcal{J}^{\mathcal{H}}_{(\bullet)} = \{0\}$. □

1.3.3 The $\overline{\partial}_{L}-$operator for differential classes.

To define the $\overline{\partial}_{L}-$operator for differential classes, we may make use of the following proposition.

Proposition 1.3.8. *Let $f, g \in \mathcal{C}^{\infty}_{((p_1,p_2),q)_L}(\Omega)$. If $f \sim g$, then $\overline{\partial}_{L} f \sim \overline{\partial}_{L} g$.*

Proof. Since $f \sim g$, then there exists $h \in \mathcal{J}^{L}_{(p+q)}(\Omega)$ such that $f - g = h$. But from (1.3.18) we have $\overline{\partial}_{L}\left(\mathcal{J}^{L}_{(p+q)}(\Omega)\right) \subseteq \mathcal{J}^{L}_{(p+q+1)}(\Omega)$, then $\overline{\partial}_{L} f \sim \overline{\partial}_{L} g$. □

Definition 1.3.9. The $\overline{\partial}_{L}-$operator for differential classes is defined as follows:

$$\overline{\partial}_{L} : \widetilde{\mathcal{C}}^{\infty}_{((p_1,p_2),q)_L}(\Omega) \longrightarrow \widetilde{\mathcal{C}}^{\infty}_{((p_1,p_2),q+1)_L}(\Omega)$$

with

$$\overline{\partial}_{L} \widetilde{f} := \widetilde{\overline{\partial}_{L} f}. \tag{1.3.24}$$

Remark 1.3.10. Form proposition 1.3.8, The $\overline{\partial}_{L}-$operator for differential classes is well defined.

Proposition 1.3.11. *For every differential class $\widetilde{f}$, we have $\overline{\partial}_L^2 \widetilde{f} = 0$.*

Proof. This follows from identity (1.3.20). □

Definition 1.3.12. *The first order linear differential operator*

$$\overline{\partial}_L : \mathcal{C}^\infty_{((p_1,p_2),q)_L}(\Omega) \longrightarrow \mathcal{C}^\infty_{((p_1,p_2),q+1)_L}(\Omega)$$

defined by

$$\overline{\partial}_L f = \sum_{j=1}^{n} \overline{\mathcal{Z}}_j^L(f) \wedge \overline{\phi}_j$$

is called the left Cauchy-Riemann operator of the group $\mathbb{H}$ attached to $L \lhd \mathcal{H}$.

1.3.4 The linear connexion d_L

Definition 1.3.13. Let $L \lhd \mathcal{H}$, and define the following linear connexions:

1. $\Gamma^L : \mathcal{C}^\infty_{(s)}(\Omega) \longrightarrow \mathcal{C}^\infty_{(s+1)}(\Omega)$ is defined by the conditions:
   (a) On a $\mathcal{C}^\infty$ function f, $\quad \Gamma^L(f) := 0.$
   (b) On the first 1-forms of structure ϕ_k and $\overline{\phi}_k$, Γ^L acts as follows:
   $$\begin{cases} \Gamma^L(\phi_k) := \Gamma^L(\overline{\phi}_k) = 0 & \text{for} \quad 1 \leq k \leq m \\ \Gamma^L(\phi_k) := -\Gamma^L(\overline{\phi}_k) = 2(\sigma_k^L - \sigma_k) & \text{for} \quad m+1 \leq k \leq n. \end{cases}$$
   (c) On arbitrary $\mathcal{C}^\infty$ differential forms, Γ^L acts by Leibnitz' rule:
   $$\Gamma^L(f \wedge g) = \Gamma^L(f) \wedge g + (-1)^\nu . f \wedge \Gamma^L(g), \qquad \nu = deg(f).$$
2. $d_L : \mathcal{C}^\infty_{(s)}(\Omega) \longrightarrow \mathcal{C}^\infty_{(s+1)}(\Omega)$ is defined by:
   $$d_L := d + \Gamma^L. \tag{1.3.25}$$

Lemma 1.3.14. *For every $\mathcal{C}^\infty-$function f*

$$\left[\mathcal{Z}_j^L, d\right](f) = \sum_{i,k=1}^{n} \left(\left(a_{i,j}^k - \lambda_{i,j}^k\right) \mathcal{Z}_k(f) + \left(\overline{a_{i,j}^k} - \overline{\lambda_{i,j}^k}\right) \overline{\mathcal{Z}}_k(f) \right) \overline{\phi}_i. \tag{1.3.26}$$

Proof. Let f be a $\mathcal{C}^\infty-$function. Then

$$\begin{aligned}
\overline{\mathcal{Z}}_j(df) &= \overline{\mathcal{Z}}_j\left(\sum_{k=1}^{n}\mathcal{Z}_k(f)\phi_k + \overline{\mathcal{Z}}_k(f)\overline{\phi}_k\right)\\
&= \sum_{k=1}^{n}\overline{\mathcal{Z}}_j\circ\mathcal{Z}_k(f)\phi_k + \sum_{k=1}^{n}\mathcal{Z}_k(f)\overline{\mathcal{Z}}_j\left(\phi_k\right)\\
&+ \sum_{k=1}^{n}\overline{\mathcal{Z}}_j\circ\overline{\mathcal{Z}}_k(f)\overline{\phi}_k + \sum_{k=1}^{n}\mathcal{Z}_k(f)\overline{\mathcal{Z}}_j\left(\overline{\phi}_k\right)\\
&= \sum_{k=1}^{n}\mathcal{Z}_k\circ\overline{\mathcal{Z}}_j(f)\phi_k + \sum_{k=1}^{n}\overline{\mathcal{Z}}_k\circ\overline{\mathcal{Z}}_j(f)\overline{\phi}_k\\
&+ \sum_{k=1}^{n}\left[\overline{\mathcal{Z}}_k,\overline{\mathcal{Z}}_j\right](f)\overline{\phi}_k\\
&+ \sum_{k=1}^{n}\mathcal{Z}_k(f)\overline{\mathcal{Z}}_j\left(\phi_k\right) + \sum_{k=1}^{n}\mathcal{Z}_k(f)\overline{\mathcal{Z}}_j\left(\overline{\phi}_k\right)\\
&= d\left(\overline{\mathcal{Z}}_j\right)(f)\\
&+ \sum_{i,k=1}^{n}\left(a^i_{j,k}\mathcal{Z}_i(f) + \overline{a^i_{j,k}}\overline{\mathcal{Z}}_i(f)\right)\overline{\phi}_k\\
&- \sum_{i,k=1}^{n}\lambda^i_{j,k}\mathcal{Z}_k(f)\overline{\phi}_i - \sum_{i,k=1}^{n}\overline{\lambda^i_{j,k}}\overline{\mathcal{Z}}_k(f)\overline{\phi}_i\\
&= d\left(\overline{\mathcal{Z}}_j\right)(f) + \sum_{i,k=1}^{n}\left(\left(a^i_{j,k}-\lambda^i_{j,k}\right)\mathcal{Z}_i(f) + \left(\overline{a^i_{j,k}}-\overline{\lambda^i_{j,k}}\overline{\mathcal{Z}}_i(f)\right)\right)\overline{\phi}_k
\end{aligned}$$

which implies that

$$\left[\overline{\mathcal{Z}}_j, d\right](f) = \sum_{i,k=1}^{n}\left(\left(a^i_{j,k}-\lambda^i_{j,k}\right)\mathcal{Z}_i(f) + \left(\overline{a^i_{j,k}}-\overline{\lambda^i_{j,k}}\overline{\mathcal{Z}}_i(f)\right)\right)\overline{\phi}_k$$

and completes the proof of (1.3.26). □

Corollary 1.3.15. $\overline{\mathcal{Z}}^L_j$ is a Lie derivative if and only if $L = Z(\mathcal{H})$, that is:

$$\left[d, \overline{\mathcal{Z}}^L_j\right] = 0 \Longleftrightarrow L = Z(\mathcal{H}). \tag{1.3.27}$$

Proof. This follows from identity (1.3.26). □

Proposition 1.3.16. *For every $f \in \mathcal{C}^{\infty}_{(\bullet)}(\Omega)$, we have the decomposition*

$$d_{_L} f = \partial_{_L} f + \overline{\partial}_{_L} f. \tag{1.3.28}$$

Proof. By observing that $d_{_L}$ satisfies Leibniz's rule, it suffices then to prove formula (1.3.28) only for functions and for the 1-differential forms of structure ϕ_k and $\overline{\phi}_k$.
1) Using formulas (1.1.4), (1.1.5), and (1.1.8), (1.1.9), we deduce immediately for every $\mathcal{C}^{\infty}$ function f

$$df = \partial_{_L} f + \overline{\partial}_{_L} f.$$

2) we have from (1.1.6)

$$d\phi_k = \sigma_k$$

$$d\overline{\phi}_k = -\sigma_k.$$

and from (1.3.17)

$$\partial_{_L}(\phi_k) = -\partial_{_L}(\overline{\phi}_k) = \sigma^L_k$$

$$\overline{\partial}_{_L}(\phi_k) = -\overline{\partial}_{_L}(\overline{\phi}_k) = \sigma^L_k.$$

Then $d_{_L} = \partial_{_L} f + \overline{\partial}_{_L} f$, which competes the proof. □

1.4 The category of complexes $\mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)\bullet}$

1.4.1 The $\overline{\partial}_{_L}-$complex defined by $L \lhd \mathcal{H}$.

To construct a good $\overline{\partial}_{\mathbb{L}}-$cohomology of differential classes $\big($ie, differential forms modulo the ideal $\mathcal{J}^L_{(\bullet)}(\Omega)\big)$, we are led to define the following notions.

Definition 1.4.1.

1. A differential form $f \in \mathcal{C}^{\infty}_{((p_1,p_2),q)_L}(\Omega)$ is said to be $\overline{\partial}_{_L}-$closed, if

$$\overline{\partial}_{_L} f \in \mathcal{J}^L_{(p_1+p_2+q)}(\Omega).$$

2. A differential form $f \in \mathcal{C}^{\infty}_{((p_1,p_2),q)_\mathcal{L}}(\Omega)$ is said to be $\overline{\partial}_{L}-$exact, if there exists a differential form $g \in \mathcal{C}^{\infty}_{((p_1,p_2),q-1)_L}(\Omega)$ such that

$$f - \overline{\partial}_{L} g \in \mathcal{J}^{L}_{(p_1+p_2+q)}(\Omega).$$

Now fix $L \lhd \mathcal{H}$, and Ω . According to the above definition, we can consider the complex

$$\mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_L}(\Omega) := \left\{\mathcal{C}^{\gamma+l}_{((p_1,p_2),\circ)_L}(\Omega), \overline{\partial}_{L}\right\}_{l,q}$$

defined as follows:

$$0 \longrightarrow \cdots \mathcal{C}^{\gamma+l+1}_{((p_1,p_2),q-1)_L}(\Omega) \xrightarrow{\overline{\partial}_{L}} \mathcal{C}^{\gamma+l}_{((p_1,p_2),q)_L}(\Omega) \xrightarrow{\overline{\partial}_{L}} \mathcal{C}^{\gamma+l-1}_{((p_1,p_2),q+1)_L}(\Omega) \cdots \longrightarrow 0.$$

Hence we obtain a space of cohomology

$$\mathbf{H}_{((p_1,p_2),q)_L}(\Omega) := \frac{\left\{f \in \mathcal{C}^{\infty}_{((p_1,p_2),q)_L}(\Omega), \quad \overline{\partial}_{L} f \in \mathcal{J}^{L}_{(p_1+p_2+q+1)}(\Omega)\right\}}{\mathcal{J}^{L}_{(p_1+p_2+q)}(\Omega) + Im\left\{\overline{\partial}_{L} : \mathcal{C}^{\infty}_{((p_1,p_2),q-1)_L}(\Omega) \longrightarrow \mathcal{C}^{\infty}_{((p_1,p_2),q)_L}(\Omega)\right\}}.$$

We call $\mathbf{H}_{((p_1,p_2),q)_L}(\Omega)$ the $((p_1,p_2),q)_{L}-$group of cohomology of the $\overline{\partial}_{L}-$operator over the open set Ω.

Remark 1.4.2. In the case where $L = \mathcal{H}$, the ideal $\mathcal{J}^{\mathcal{H}}_{(\bullet)}(\Omega)$ is reduced to $\{0\}$, and the $\overline{\partial}_{\mathbb{H}}-$cohomology is in fact a cohomology of differential forms. The corresponding complex in this case, is

$$0 \longrightarrow \cdots \longrightarrow \mathcal{C}^{\infty}_{((p_1,p_2),q-1)_\mathcal{H}}(\Omega) \xrightarrow{\overline{\partial}_{\mathbb{H}}} \mathcal{C}^{\infty}_{((p_1,p_2),q)_\mathcal{H}}(\Omega) \xrightarrow{\overline{\partial}_{\mathbb{H}}} \mathcal{C}^{\infty}_{((p_1,p_2),q+1)_\mathcal{H}}(\Omega) \longrightarrow \cdots \longrightarrow 0$$

and the $((p_1,p_2),q)_\mathcal{H}-$group of cohomology is the space

$$\mathbf{H}_{((p_1,p_2),q)_\mathcal{H}}(\Omega) := \frac{ker\left\{\overline{\partial}_{\mathbb{H}} : \mathcal{C}^{\infty}_{((p_1,p_2),q)_\mathcal{H}}(\Omega) \longrightarrow \mathcal{C}^{\infty}_{((p_1,p_2),q+1)_\mathcal{H}}(\Omega)\right\}}{Im\left\{\overline{\partial}_{\mathbb{H}} : \mathcal{C}^{\infty}_{((p_1,p_2),q-1)_\mathcal{H}}(\Omega) \longrightarrow \mathcal{C}^{\infty}_{((p_1,p_2),q)_\mathcal{H}}(\Omega)\right\}}.$$

1.4.2 Construction of the functor $\overline{\partial}_\bullet$

To construct the functor $\overline{\partial}_\bullet$, we are led to consider a category of complexes. Indeed, Let $K \lhd \mathcal{H}$, and $L \lhd \mathcal{H}$, and let the modules of differential forms

$$\mathcal{C}^{\gamma+l}_{((p_1,p_2),q)_K}(\Omega) = \mathcal{C}^{\gamma+l}_{((p_1,p_2),q)_\mathcal{H}}(\Omega) + \mathcal{J}^K_{p+q}(\Omega)$$

and

$$\mathcal{C}^{\gamma+l}_{((p_1,p_2),q)_L}(\Omega) = \mathcal{C}^{\gamma+l}_{((p_1,p_2),q)_\mathcal{H}}(\Omega) + \mathcal{J}^L_{p+q}(\Omega).$$

Since the group $\mathbb{H}$ is assumed to be metric, then we can decompose the following modules as direct sums

$$\mathcal{C}^{\gamma+l}_{((p_1,p_2),q)_K}(\Omega) = \mathcal{J}^K_{p+q}(\Omega) \oplus \left(\mathcal{J}^K_{p+q}(\Omega)\right)^\perp$$

$$\mathcal{C}^{\gamma+l}_{((p_1,p_2),q)_L}(\Omega) = \mathcal{J}^L_{p+q}(\Omega) \oplus \left(\mathcal{J}^L_{p+q}(\Omega)\right)^\perp$$

Consider the orthogonal projections

$$p_{K,L} \mathcal{J}^K_{p+q}(\Omega) \longrightarrow \mathcal{J}^{K\cap L}_{p+q}(\Omega)$$

$$\mathcal{J}^L_{p+q}(\Omega) \longrightarrow \mathcal{J}^{K\cap L}_{p+q}(\Omega).$$

since $\mathcal{J}^{K\cap L}_{p+q}(\Omega) \subseteq \mathcal{J}^L_{p+q}(\Omega)$, we can define the map

$$g_{K,L} : \mathcal{C}^{\gamma+l}_{((p_1,p_2),q)_K}(\Omega) \longrightarrow \mathcal{C}^{\gamma+l}_{((p_1,p_2),q)_L}(\Omega)$$

$$g_{K,L}(u) = \begin{cases} u & \text{if} \quad u \in \mathcal{C}^{\gamma+l}_{((p_1,p_2),q)_\mathcal{H}}(\Omega) \cap \left(\mathcal{J}^K_{p+q}(\Omega)\right)^\perp \\ p_{K,L}(u) & \text{if} \quad u \in \mathcal{J}^K_{p+q}(\Omega) \end{cases} \tag{1.4.1}$$

Definition 1.4.3.
The category $\mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_\bullet}(\Omega)$ of complexes attached to the metric group $\mathbb{H}$ is defined as follows:

- The objects of $\mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_\bullet}(\Omega)$ are the complexes of modules $\mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_L}(\Omega)$, where L runs over all $L \lhd \mathcal{H}$.
- For all $K \lhd \mathcal{H}$ and $L \lhd \mathcal{H}$, the set $\mathcal{M}or\left(\mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_K}(\Omega), \mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_L}(\Omega)\right)$ of morphisms from $\mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_K}(\Omega)$ to $\mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_L}(\Omega)$ is reduced to one element, that is the mapping $g_{K,L}$ defined by (1.4.1),

$$\mathcal{M}or\left(\mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_K}(\Omega), \mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_L}(\Omega)\right) := \left\{g_{K,L}\right\}.$$

- the composition law is the usual composition of maps.

Proposition 1.4.4. *The correspondence*

$$\overline{\partial_\bullet} : \mathcal{H}_\bullet \longrightarrow \mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_\bullet}(\Omega),$$

$$\mathcal{H}_L \longmapsto \mathcal{C}^{\gamma+\circ}_{((p_1,p_2),\circ)_K}(\Omega)$$

is a functor of category.

Proof. This follows from the fact that for all $L \lhd \mathcal{H}$, we have $\overline{\partial_L}\left(\mathcal{J}^L_{p+q}\right)$. □

1.5 The $\mathcal{C}^\infty$ independence of $\overline{\partial}_L$ and $\overline{\partial}$

Let $L \lhd \mathcal{H}$, and let $\overline{\partial}_L$ be the left Cauchy-Riemann defined by $L \lhd \mathcal{H}$. After the construction of $\overline{\partial}_L$, it is legitimate to ask the following:
Question. *Is the differential operator $\overline{\partial}_L$ really $\mathcal{C}^\infty$ independent of the classical Cauchy-Riemann operator $\overline{\partial}$?*
To precise the sense of this question, let $\psi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, be a diffeomorphism, and define for an open set $\Omega \subseteq \mathbb{C}^n$, the corresponding pullback isomorphism, that is:

$$\psi^* : \mathcal{C}^\infty_{(s)}(\Omega) \longrightarrow \mathcal{C}^\infty_{(s)}(\psi(\Omega))$$
$$f \longmapsto \psi^*(f) := fo\psi^{-1}.$$

Definition 1.5.1. The differential operators $\overline{\partial}_L$ and $\overline{\partial}$ are said to be $\mathcal{C}^\infty$ dependent, if there exists a diffeomorphism $\psi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, such that

$$\overline{\partial}_L = \psi^* \circ \overline{\partial} \circ (\psi^*)^{-1}$$

that is, such that for all s, the diagram

$$\begin{array}{ccc} \mathcal{C}^\infty_{(s)}(\Omega) & \xrightarrow{\overline{\partial}_L} & \mathcal{C}^\infty_{(s+1)}(\Omega) \\ \psi^* \downarrow & & \downarrow \psi^* \\ \mathcal{C}^\infty_{(s)}(\Omega) & \xrightarrow[\overline{\partial}]{} & \mathcal{C}^\infty_{(s+1)}(\Omega) \end{array}$$

is commutative.

The following theorem precise the answer to the above question.

Theorem 1.5.2.
Let $\mathbb{H} = (\mathbb{C}^n, *)$ be a the 2-step nilpotent Lie group, and let $L \lhd \mathcal{H}$.
Then $\overline{\partial}_{L}$ and $\overline{\partial}$ are $\mathcal{C}^\infty$ dependent if and only if $\mathbb{H}$ is isomorphic to $(\mathbb{C}^n, +)$.

Proof. The sufficient condition is trivial. Let us prove the necessarily condition only. First, observe the following fact: If $\overline{\partial}_{L}$ and $\overline{\partial}$ are $\mathcal{C}^\infty$ dependent, then for some diffeomorphism ψ, we have:

$$\overline{\partial}_{L} = \psi^* \circ \overline{\partial} \circ (\psi^*)^{-1}.$$

Hence

$$\overline{\partial}^2_{L} = \psi^* \circ \overline{\partial}^2 \circ (\psi^*)^{-1} = 0.$$

But this is impossible when $L \neq \mathcal{H}$. It sufficient then to prove the theorem only in the case $L = \mathcal{H}$, and only for $s = 0$, that is to prove that $\overline{\partial}_{\mathbb{H}}$ is $\mathcal{C}^\infty$ independent of $\overline{\partial}$.
Assume that the group $\mathbb{H}$ is not commutative, and that there exists a diffeomorphism $\psi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that

$$\overline{\partial}_{\mathbb{H}} = \psi^* \circ \overline{\partial} \circ (\psi^*)^{-1}.$$

Consider the group $\widetilde{\mathbb{H}} = (\mathbb{C}^n, \widetilde{*}) = \psi(\mathbb{H})$, and define its law $\widetilde{*}$ by the map $F = (F_1, ..., F_n) : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$, and consider for $\rho \in \widetilde{\mathbb{H}}$, the inverse left translations:

$$\tau_\rho : z \longrightarrow \xi = F(\rho, z)$$

and

$$\tau_{-\rho} : \xi \longrightarrow z = G(\rho, \xi) = F(-\rho, \xi).$$

Write the classical $\overline{\partial}$ in the coordinates z, that is:

$$\overline{\partial} = \left\langle \frac{\partial}{\partial \overline{z}}, d\overline{z} \right\rangle = \sum_{k=1}^{n} \frac{\partial}{\partial \overline{z}_k} d\overline{z}_k.$$

Then by the change of the coordinates z into ξ, we obtain:

$$\begin{aligned}\overline{\partial} =& \sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\frac{\partial F_j}{\partial z_k}\frac{\partial}{\partial \xi_j}+\frac{\partial \overline{F}_j}{\partial \overline{z}_k}\frac{\partial}{\partial \overline{\xi}_j}\right)\right)\left(\sum_{l=1}^{n}\left(\frac{\partial \overline{G}_k}{\partial \xi_l}d\xi_l+\frac{\partial \overline{G}_k}{\partial \overline{\xi}_l}d\overline{\xi}_l\right)\right)\\
=& \sum_{j,l=1}^{n}\left(\sum_{k=1}^{n}\frac{\partial F_j}{\partial \overline{z}_k}\frac{\partial \overline{G}_k}{\partial \xi_l}\right)\frac{\partial}{\partial \xi_j}d\xi_l+\sum_{j,l=1}^{n}\left(\sum_{k=1}^{n}\frac{\partial F_j}{\partial \overline{z}_k}\frac{\partial \overline{G}_k}{\partial \overline{\xi}_l}\right)\frac{\partial}{\partial \xi_j}d\overline{\xi}_l\\
&+\sum_{j,l=1}^{n}\left(\sum_{k=1}^{n}\frac{\partial \overline{F}_j}{\partial \overline{z}_k}\frac{\partial \overline{G}_k}{\partial \xi_l}\right)\frac{\partial}{\partial \overline{\xi}_j}d\xi_l+\sum_{j,l=1}^{n}\left(\sum_{k=1}^{n}\frac{\partial \overline{F}_j}{\partial \overline{z}_k}\frac{\partial \overline{G}_k}{\partial \overline{\xi}_l}\right)\frac{\partial}{\partial \overline{\xi}_j}d\overline{\xi}_l.\end{aligned}$$

Since $\overline{\partial}_{_\mathbb{H}}$ is left invariant by $\mathbb{H}$, then $\overline{\partial}$ is left invariant by $\widetilde{\mathbb{H}}$, and then we must have by identification:

$$\begin{cases}\displaystyle\sum_{j,l=1}^{n}\frac{\partial F_j}{\partial \overline{z}_k}\frac{\partial \overline{G}_k}{\partial \xi_l}=0\\ \displaystyle\sum_{j,l=1}^{n}\frac{\partial F_j}{\partial \overline{z}_k}\frac{\partial \overline{G}_k}{\partial \overline{\xi}_l}=0\\ \displaystyle\sum_{j,l=1}^{n}\frac{\partial \overline{F}_j}{\partial \overline{z}_k}\frac{\partial \overline{G}_k}{\partial \xi_l}=0\\ \displaystyle\sum_{j,l=1}^{n}\frac{\partial \overline{F}_j}{\partial \overline{z}_k}\frac{\partial \overline{G}_k}{\partial \overline{\xi}_l}=\delta_{j,l}\end{cases}\tag{1.5.1}$$

It follows from the system (1.5.1) that for all $\rho\in\widetilde{\mathbb{H}}$, the partial map

$$z\longmapsto F(\rho,.)$$

is holomorphic with respect to the variable z. Since furthermore, the group $\mathbb{H}=(\mathbb{C}^n,*)$ is 2-step nilpotent, then $\widetilde{\mathbb{H}}=(\mathbb{C}^n,\widetilde{*})$ is 2-step nilpotent, and hence the Taylor expansion of the map F near the origin 0 can be written by Campbell-Hausdorff formula as a second order polynomial map, that is:

$$F(\rho,z)=\rho+z+\frac{1}{2}\big[\rho,z\big]$$

where $\big[\rho,z\big]$ denotes the Lie-bracket of ρ and z.
Now decompose $\big[\rho,z\big]$ as follows

$$\big[\rho,z\big]=A(\rho,z)+B(\rho,\overline{z})+C(\overline{\rho},z)+D(\overline{\rho},\overline{z})$$

where A, B, C, D are bilinear maps $\mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$.
Since the partial map

$$z \longmapsto F(\rho, \ z \ )$$

is holomorphic with respect to the variable z, then $B = D = 0$, and since the Lie-bracket $[\ ,\ ]$ is a skew bilinear map then $C = 0$. It follows then, that

$$F(\rho, z) = \rho + z + \frac{1}{2}A(\rho, z)$$

where $A = (A_1, ..., A_n) : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is bilinear and then holomorphic.
Now let

$$\widetilde{\phi}_j = dz_j - \frac{1}{2}\sum_{k=1}^{n} \frac{\partial A_j}{\partial z_k} dz_k \qquad 1 \leq j \leq n \tag{1.5.2}$$

with A_j short for $A_j(z, z)$, be the holomorphic 1-forms of structure of the group $\widetilde{\mathbb{H}}$, and let by duality with (1.5.2),

$$\widetilde{\mathcal{Z}}_j = \frac{\partial}{\partial z_j} + \frac{1}{2}\sum_{k=1}^{n} \frac{\partial A_k}{\partial z_j}\frac{\partial}{\partial z_k} \qquad 1 \leq j \leq n \tag{1.5.3}$$

be the dual left invariant vector fields. If we write for a $\mathcal{C}^\infty$ function f, the 1-differential form $\overline{\partial} f$ as linear combination of $\widetilde{\phi}_j$ and $\overline{\widetilde{\phi}}_j$, we obtain:

$$\overline{\partial} f = \sum_{j=1}^{n} P_j(f)\widetilde{\phi}_j + \sum_{j=1}^{n} Q_j(f)\overline{\widetilde{\phi}}_j$$

where P_j and Q_j are first order linear differential operators. Since $\overline{\partial}$ is left $\widetilde{\mathbb{H}}-$invariant and does not contain the terms $\dfrac{\partial}{\partial z_j}$ and dz_j, then $P_j = 0$ and Q_j is left $\widetilde{\mathbb{H}}-$invariant. We have then with suitable constants $b_{j,k} \in \mathbb{C}$

$$Q_j = \sum_{k=1}^{n} b_{j,k}\overline{\widetilde{\mathcal{Z}}}_k$$

and then

$$\overline{\partial} = \sum_{j,k=1}^{n} b_{j,k}\overline{\widetilde{\mathcal{Z}}}_k.\overline{\widetilde{\phi}}_j. \tag{1.5.4}$$

Let the matrix $B = (b_{j,k})$. The identity (1.5.4) can be expressed using (1.5.2) and (1.5.3) as follows:

$$\overline{\partial} = \left\langle \left(I - \overline{C}(\overline{z})\right) B \frac{\partial}{\partial \overline{z}}, \left(I - \left(\overline{C}\right)^* (\overline{z})\right) d\overline{z} \right\rangle. \tag{1.5.5}$$

If we denote by $\langle\ ,\ \rangle$ the pairing between vector fields and 1-differential forms, we can rewrite $\overline{\partial}$ using (1.5.4) and (1.5.3) as follows

$$\begin{aligned}\overline{\partial} = \left\langle \frac{\partial}{\partial \overline{z}}, d\overline{z} \right\rangle &= \left\langle \left(I - \overline{C}(\overline{z})\right) B \frac{\partial}{\partial \overline{z}}, \left(I - \left(\overline{C}\right)^* (\overline{z})\right) d\overline{z} \right\rangle \\ &= \left\langle \left(I - \overline{C}(\overline{z})\right)^2 B \frac{\partial}{\partial \overline{z}}, d\overline{z} \right\rangle.\end{aligned}$$

By identification, we obtain for all $z \in \mathbb{C}^n$,

$$I = \left(I - \overline{C}(\overline{z})\right)^2 B \tag{1.5.6}$$

Since $\overline{C}(\overline{z})$ is either 1-order polynomial or 0, then (1.5.6) implies $\overline{C}(\overline{z}) = 0$. The group $\widetilde{\mathbb{H}}$ is then commutative, which contradicts the hypothesis. The theorem is then proved. □

Chapter 2

The left Cauchy-Riemann equation $\overline{\partial}_L u = f$

2.1 Local solvability of the equation $\overline{\partial}_L u = f$

Let $L \lhd \mathcal{H}$, and let $\overline{\partial}_L$ be the left Cauchy-riemann operator defined by $L \lhd \mathcal{H}$. We prove in this section the local solvability of the equation $\overline{\partial}_L u = f$. More precisely, the following theorem (called in the commutative case, the Dolbeault-grothendieck lemma), means that every $\overline{\partial}_L-$closed differential form in the sense of definition 1.4.1 is locally $\overline{\partial}_L-$exact in the sense of definition 1.4.1.

Theorem 2.1.1. ***(Dolbeault-Grothendieck lemma)***

1. *First statement(for differential forms). Let $\Omega = D_1 \times ... \times D_n$ be an open polydisc of $\mathbb{H}$ and let $f \in \mathcal{C}^{\infty}_{(p,q+1)_L}(\Omega)$ satisfy the condition $\overline{\partial}_L f \in \mathcal{J}_{(s)}(\Omega)$. If $\Omega' \subset\subset \Omega$ (that is Ω' is relatively compact in Ω), we can find $u \in \mathcal{C}^{\infty}_{(p,q)_{\mathcal{H}}}(\Omega')$ such that $f - \overline{\partial}_L u \in \mathcal{J}_{(s)}(\Omega')$.*

2. *Second statement(for differential classes). Let $\Omega = D_1 \times ... \times D_n$ be an open polydisc of $\mathbb{H}$ and let $\widetilde{f} \in \widetilde{\mathcal{C}}^{\infty}_{(p,q+1)}(\Omega)$ be a differential class satisfying the condition $\overline{\partial}_L \widetilde{f} = 0$. If $\Omega' \subset\subset \Omega$ (that is Ω' is relatively compact in Ω) we can find a differential class $\widetilde{u} \in \widetilde{\mathcal{C}}^{\infty}_{(p,q)}(\Omega')$ such that $\overline{\partial}_L \widetilde{u} = \widetilde{f}$.*

Proof. We follow Hörmander [4].
Let $\widetilde{f}$ be the differential class of f .We prove the theorem by induction in μ such that $\widetilde{f}$ do not involve $\widetilde{\overline{\phi_{\mu+1}}}, ..., \widetilde{\overline{\phi_n}}$.
If $\widetilde{f}$ does not involve the differential classes $\widetilde{\overline{\phi_1}}, ..., \widetilde{\overline{\phi_n}}$, then the theorem is true because in this case $\widetilde{f} = 0$ since every term in $\widetilde{f}$ is of degree $q+1 > 0$ with respect to $\widetilde{\overline{\phi}}$.
Assume the theorem true for $\mu-1$ (that is for differential classes not involving $\widetilde{\overline{\phi_\mu}}, ..., \widetilde{\overline{\phi_n}}$) and prove it for μ.
Let $\widetilde{f}$ be a differential class not involving $\widetilde{\overline{\phi_{\mu+1}}}, ..., \widetilde{\overline{\phi_n}}$. We can write

$$\widetilde{f} = \widetilde{\overline{\phi_\mu}} \wedge \widetilde{g} + \widetilde{h}$$

where $g \in \mathcal{C}^\infty_{(p,q)}(\Omega)$ and $h \in \mathcal{C}^\infty_{(p,q+1)}(\Omega)$. Observe that $\widetilde{g}$ and $\widetilde{h}$ are independent of $\widetilde{\overline{\phi_\mu}}, ..., \widetilde{\overline{\phi_n}}$. Write

$$\widetilde{g} = \sum_{|IK|=p, |JL|=q}{}' g_{IK,JL} \, \widetilde{\phi_{IK}} \wedge \widetilde{\overline{\phi_{JL}}} \; ,$$

where $g_{IK,JL} \in \mathcal{C}^\infty(\Omega)$ and $\sum'$ means that the summation is performed over all multi-indices with strictly increasing components. From the hypothesis $\overline{\partial}_L \widetilde{f} = 0$, we obtain

$$\overline{\mathcal{Z}}_\nu(g_{IK,JL}) = 0 \qquad \text{for} \quad \nu > \mu, \tag{2.1.1}$$

where $\overline{\mathcal{Z}}_\nu$ is the left invariant vector fields defined by (1.1.9).
Thus :
1) if $\mu \geq m$, then $g_{IK,JL}$ is left $\mathcal{H}-$holomorphic in the variables $\zeta_1, ..., \zeta_{n-m}$
2) if $\mu < m$, then $g_{IK,JL}$ is left $\mathcal{H}-$holomorphic in the variables $z_{\mu+1}, ..., z_m$, $\zeta_1, ...\zeta_{n-m}$.
We now choose a solution $G_{IK,JL}$ of the equation

$$\overline{\mathcal{Z}}_\mu(G_{IK,JL}) = g_{IK,JL}. \tag{2.1.2}$$

For this, set for $s \in \mathbb{C}$

$$T_\mu(s) = s(\delta_{1,\mu}, ..., \delta_{n,\mu})$$

where $\delta_{j,l}$ is the symbol of Kronecker $= 1$ if $j = l$ and 0 if $j \neq l$. We have two cases to discuss :
1) If $\mu > m$, (that is $\mu = m+k$, with $1 \leq k \leq n-m$) we begin by choosing $\varphi \in \mathcal{C}_0^\infty(D_{m+k})$ such that $\varphi(\zeta_{m+k}) = 1$ in a neighborhood Ω" of $\overline{\Omega'}$, and we set

$$G_{_{IK,JL}}(z,\zeta) = \frac{1}{2\pi i}\int_{s\in D_{m+k}} \frac{\varphi(s)g_{_{IK,JL}}(z,\zeta+T_{m+k}(s-\zeta_{m+k}))d\overline{s}\wedge ds}{s-\zeta_{m+k}}$$
$$= \frac{-1}{2\pi i}\int_{s\in\widehat{D_{m+k}}} \frac{\varphi(\zeta_{m+k}-s)g_{_{IK,JL}}(z,\zeta-T_{m+k}(s))d\overline{s}\wedge ds}{s},$$

where $\widehat{D_{m+k}} = \{\zeta_{m+k} - s : s \in D_{m+k}\}$. This expression shows first that $G_{_{IK,JL}} \in \mathcal{C}^\infty(\Omega)$, and by the Cauchy-Green formula, the equation (2.1.2) holds in Ω''. in view of (2.1.1) a differentiation under the sign of integration gives for $\nu = m + k'$ with $k' > k$

$$\overline{\mathcal{Z}}_\nu(G_{_{IK,JL}}) = 0 \qquad \text{for} \quad \nu = m + k' > \mu.$$

2) If $\mu \leq m$, , we begin by choosing $\varphi \in \mathcal{C}_0^\infty(D_\mu)$ such that $\varphi(z_\mu) = 1$ in a neighborhood Ω" of $\overline{\Omega'}$, and we set

$$G_{_{IK,JL}}(z,\zeta) = \frac{1}{2\pi i}\int_{s\in D_\mu} \frac{\varphi(s)g_{_{IK,JL}}\big(z+T_\mu(s-z_\mu),\zeta-\frac{i}{4}B(z,\overline{z})\big)d\overline{s}\wedge ds}{s-z_\mu}$$
$$= \frac{-1}{2\pi i}\int_{s\in\widehat{D_\mu}} \frac{\varphi(z_\mu-s)g_{_{IK,JL}}\big(z-T_\mu(s),\zeta-\frac{i}{4}B(z,\overline{z})\big)d\overline{s}\wedge ds}{s},$$

where $\widehat{D_\mu} = \{s - z_\mu : s \in D_\mu\}$. As above , the last expression shows that $G_{IK,JL} \in \mathcal{C}^\infty(\Omega)$. By the Cauchy-Green formula, once again, the equation (2.1.2) holds in Ω''. in view of (2.1.1) a differentiation under the sign of integration gives

$$\overline{\mathcal{Z}}_\nu(G_{_{IK,JL}}) = 0 \qquad \text{for} \quad \nu > \mu.$$

If we set

$$G = \sum_{|IK|=p,|JL|=q}{}' G_{_{IK,JL}}\,\widetilde{\phi_{IK}}\wedge\overline{\widetilde{\phi_{JL}}}\ ,$$

it follows then that in Ω'

$$\overline{\partial}_{_L}G = \sum_{|IK|=p,|JL|=q}{}' \sum_\mu \overline{\mathcal{Z}}_\nu(G_{_{IK,JL}})\,\overline{\widetilde{\phi}_\mu}\bigwedge\widetilde{\phi_{IK}}\wedge\overline{\widetilde{\phi_{JL}}} = \overline{\widetilde{\phi}_\mu}\bigwedge\widetilde{g}+\widetilde{h}_1$$

where $\widetilde{h}_1$ is the sum of the terms of $\overline{\partial}_L G$ when j runs from 1 to $\mu - 1$ and is independent of $\overline{\widetilde{\phi}}_\mu, ..., \overline{\widetilde{\phi}}_n$. Hence $\widetilde{h} - \widetilde{h}_1 = \widetilde{f} - \overline{\partial}_L G$ does not involve $\overline{\widetilde{\phi}}_\mu, ..., \overline{\widetilde{\phi}}_n$. Since $\overline{\partial}_L \widetilde{f} - \overline{\partial}_L G) = \overline{\partial}_L \widetilde{f} = 0$, then by the induction hypothesis we can find $v \in \mathcal{C}^\infty_{(p,q)}(\Omega)$ so that $\overline{\partial}_L \widetilde{v} = \widetilde{f} - \overline{\partial}_L G$. The differential class $\widetilde{u} = \widetilde{v} + G$ satisfies the equation $\overline{\partial}_L \widetilde{u} = \widetilde{f}$, which completes the proof. □

2.2 The left $\mathcal{H}-$holomorphic functions

Definition 2.2.1. The $\mathcal{C}^\infty$ complex valued function f is said to be left $\mathcal{H}-$holomorphic if the 1-differential form $\overline{\partial}_{\mathbb{H}} f$ is of $\mathcal{H}-$type $(0,1)_\mathcal{H}$, that is if $f \in ker(\overline{\partial}_{\mathbb{H}})$, which means that $\overline{\partial}_{\mathbb{H}} f = 0$, or in other words f is a solution of the system of partial differential equations

$$\overline{\mathcal{Z}}_j(f) = 0 \qquad \text{for all} \quad 1 \leq j \leq n.$$

We denote the module of left $\mathcal{H}-$holomorphic functions on Ω by $\mathcal{O}_\mathcal{H}(\Omega)$.

Example 2.2.2. Let $z = (z', z'') \in \mathbb{H} = \mathbb{C}^m \times \mathbb{C}^{n-m}$. From the definition of the vector fields $\overline{\mathcal{Z}}_j$ (see (1.1.9)), we check easily that the functions $h_1, ..., h_n$ defined on the group $\mathbb{H}$ as follows

$$\begin{cases} h_j(z) = z_j & \text{for} \quad 1 \leq j \leq m, \\ h_k(z) = z_k - \dfrac{1}{2} A_k(z', \overline{z'}) & \text{for} \quad m+1 \leq k \leq n \end{cases} \tag{2.2.1}$$

where A_k are the bilinear maps defining the group $\mathbb{H}$, are all left $\mathcal{H}-$holomorphic.

The left $\mathcal{H}-$holomorphic coordinates.

Definition 2.2.3. Let $\Omega \subset \mathbb{H}$ be a bounded open set with, and

$$h : \mathbb{H} \longrightarrow \mathbb{C}^n$$
$$z \longmapsto \mathfrak{z} = h(z)$$

be the diffeomorphism defined by equations (2.2.1) above. (Ω, h) is called the $\mathcal{H}-$chart of the group $\mathbb{H}$ over the open set Ω, and the system $(\mathfrak{z}_1, ..., \mathfrak{z}_n) \in \mathbb{C}^n$ defined by

$$\begin{cases} \mathfrak{z}_1 = h_1(z) \\ \quad \vdots \\ \mathfrak{z}_n = h_n(z) \end{cases}$$

is called the system of left $\mathcal{H}-$holomorphic coordinates of the point $z \in \Omega \subseteq \mathbb{H}$.

Remark 2.2.4. The $\mathcal{H}-$chart $h = (h_1, ..., h_n)$ defined by (2.2.1) will be of great interest in the construction of integral formulas for solving the equation $\overline{\partial}_L u = f$.

As application, let us characterize the left $\mathcal{H}-$holomorphic functions on the group $\mathbb{H}$ in terms of the $\mathcal{H}-$coordinates.

Proposition 2.2.5. *Let Ω be an open subset of $\mathbb{H}$, and let $h = (h_1, ..., h_n)$ be the $\mathcal{H}-$chart over Ω. Then $f : \Omega \longrightarrow \mathbb{C}$ is left $\mathcal{H}-$holomorphic if and only if $f \circ h^{-1} : h(\Omega) \longrightarrow \mathbb{C}$ is holomorphic([1]).*

Proof. Let $g := f \circ h^{-1}$. We have then

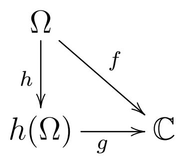
$$g(z) = f\left(z + \frac{1}{2}A(z', \overline{z'})\right). \tag{2.2.2}$$

By differentiation (2.2.2) , we find for all $1 \leq j \leq n$

$$\frac{\partial g}{\partial \overline{z}_j} = \overline{\mathcal{Z}}_j(f)$$

f is then left $\mathcal{H}-$holomorphic if and only if $\overline{\mathcal{Z}}_j(f) = 0 \Longleftrightarrow \dfrac{\partial g}{\partial \overline{z_j}} = 0$, which completes the proof. □

Remark 2.2.6. *The proposition 2.2.5 means that $f : \Omega \longrightarrow \mathbb{C}$ is left $\mathcal{H}-$holomorphic if and only if its expression $g = f \circ h^{-1} : h(\Omega) \longrightarrow \mathbb{C}$ in the h-chart (2.2.1) is holomorphic in the classical sense.*



Corollary 2.2.7. The $\mathcal{C}^\infty$ complex valued function

$$f : \Omega \subseteq \mathbb{H} \longrightarrow \mathbb{C}$$

is left $\mathcal{H}-$holomorphic if and only f is analytic with respect to the $\mathcal{H}-$holomorphic coordinates $h_1, ..., h_n$.

[1] In the classical sense.

2.2.1 Left $h-$pseudoconvexity with bounded deviation.

Notations

Let $\Omega \subset \mathbb{H}$ be a bounded open set with $\mathcal{C}^\infty-$boundary $\partial\Omega$ and

$$h : \mathbb{H} \longrightarrow \mathbb{C}^n$$

$$z \longmapsto \mathfrak{z} = h(z)$$

be the system of left h-holomorphic coordinates defined for $z = (z', z'') \in \mathbb{C}^m \times \mathbb{C}^{n-m}$ by

$$\begin{cases} \mathfrak{z}_j = z_j & \text{for} \quad 1 \leq j \leq m \\ \mathfrak{z}_k = z_k - \dfrac{1}{4} A_k(z', \overline{z'}) & \text{for} \quad m+1 \leq k \leq n. \end{cases}$$

In all that follows we note $D := h(\Omega) \subset \mathbb{C}^n$.
Now Let $V_{\overline{D}}$ be a neighborhood of $\overline{D}$, and

$$\varphi : V_{\partial D} \longrightarrow \mathbb{R}.$$

be a $\mathcal{C}^\infty$ function defined in a neighborhood $V_{\partial D}$ of $\partial D \subset \mathbb{C}^n$, then with the standard notations

$$\alpha = (\alpha_1, ..., \alpha_n)$$

$$|\alpha| = \alpha_1 + ... + \alpha_n$$

$$\alpha! = \alpha_1!...\alpha_n!$$

$$(\mathfrak{z} - \zeta)^\alpha = (\mathfrak{z}_1 - \zeta_1)^{\alpha_1}...(\mathfrak{z}_n - \zeta_n)^{\alpha_n}$$

$$\partial_\zeta^\alpha \varphi = \frac{\partial^{|\alpha|}\varphi}{\partial\zeta_1^{\alpha_1}...\partial\zeta_n^{\alpha_n}}$$

we assign to φ at each point $\zeta \in V_{\partial D}$ the following polynomials of order $2r$, $r \in \mathbb{N}^*$:

- The Levi polynomial $\left[P_\zeta^{2r}(\varphi)\right]$ of type (1,0), defined by

$$\mathfrak{z} \longmapsto \left[P_\zeta^{2r}(\varphi)\right](\mathfrak{z}) := \sum_{0 \leq |\alpha| \leq 2r} \frac{\partial_\zeta^\alpha \varphi}{\alpha!} (\mathfrak{z} - \zeta)^\alpha.$$

- The Levi polynomial $\left[\mathcal{L}^{2r}_\zeta(\varphi)\right]$ of type (1,1), defined by

$$\mathfrak{z} \longmapsto \left[\mathcal{L}^{2r}_\zeta(\varphi)\right](\mathfrak{z}) := \sum_{\substack{0<|\alpha| \\ 0<|\beta| \\ |\alpha|+|\beta|\leq 2r}} \frac{\partial^\alpha_\zeta \partial^\beta_{\overline{\zeta}} \varphi}{\alpha!\beta!} (\mathfrak{z}-\zeta)^\alpha (\overline{\mathfrak{z}}-\overline{\zeta})^\beta.$$

The particular case $\mathcal{L}^2_\zeta(\varphi)$ will be called as usual the Levi form of φ at ζ. Recall that φ is said to be plurisubharmonic in $V_{\partial D}$, if at every $\zeta \in V_{\partial D}$, the Levi form $\mathcal{L}^2_\zeta(\varphi)$ is positive.

Definition 2.2.8.
The open set $\Omega \subset \mathbb{H}$ is said to be left $h-$pseudoconvex if $D = h(\Omega) \subset \mathbb{C}^n$ is pseudoconvex in the usual sense.

We introduce in that follows for every $\mathcal{C}^\infty$ pseudoconvex open set D, a function

$$\mathcal{D}ev_{_D} : \partial D \longrightarrow \mathbb{N} \cup \{+\infty\}$$

evaluating at each $\zeta \in \partial D$, the "degree" of non strict pseudoconvexity of D. This function will play a capital role for proving existence theorems for $\overline{\partial}_{_L} u = f$ with Hölderian estimates.

Definition 2.2.9. Let D be a pseudoconvex open set of $\mathbb{C}^n$ with $\mathcal{C}^\infty-$boundary, and let $\varphi : V_{\partial D} \longrightarrow \mathbb{R}$ be a defining $\mathcal{C}^\infty$ plurisubharmonic function for D, that is:

$$D \cap V_{\partial D} = \left\{ \mathfrak{z} \in V_{\partial D} \ , \quad \varphi(\mathfrak{z}) < 0 \right\}.$$

We note the set of $\mathcal{C}^\infty$ plurisubharmonic functions on $V_{\partial D}$ defining D by $Psh(V_{\partial D})$.

- The plurisubharmonic function φ is said to be of bounded deviation at the point $\zeta \in \partial D$, if there exist a positive integer $r \in \mathbb{N}^*$, a real number $c > 0$, and a ball $B(0, R) \subset \mathbb{C}^n$ such that:

$$\left[\mathcal{L}^{2r}_\zeta(\varphi)\right](\mathfrak{z}) \geq c \left\| \mathfrak{z} - \zeta \right\|^{2r} \qquad \text{for all} \quad \mathfrak{z} \in B(0, R). \tag{2.2.3}$$

- Let the set

$$\mathbb{D}_\varphi(\zeta) := \left\{ r \in \mathbb{N}^*, \quad r \text{ satisfies } \ (2.2.3) \right\}.$$

The deviation plurisubharmonic of the function φ at the point $\zeta \in \partial D$ is then defined by:

$$\mathcal{D}ev_{\varphi}(\zeta) := \begin{cases} [\inf \mathbb{D}_{\varphi}(\zeta)] - 1 & \text{if} \quad \mathbb{D}_{\varphi}(\zeta) \neq \phi, \\ +\infty & \text{if} \quad \mathbb{D}_{\varphi}(\zeta) = \phi. \end{cases} \tag{2.2.4}$$

- We define the deviation pseudoconvex of the open set D at the point $\zeta \in \partial D$, by:

$$\mathcal{D}ev(\zeta) := \inf \Big\{ \mathcal{D}ev_{\varphi}(\zeta), \qquad \varphi \in Psh(V_{\partial D}) \Big\}, \tag{2.2.5}$$

  and we say that D is pseudoconvex with bounded deviation, if

$$\mathcal{D}ev(D) = \sup_{\zeta \in \partial D} \mathcal{D}ev(\zeta) < +\infty.$$

Remark 2.2.10.

$$\mathcal{D}ev(D) = 0 \quad \Leftrightarrow \quad D \text{ is stritly pseudoconvex.}$$

Proposition 2.2.11. *Let $D \subset \mathbb{C}^n$ be a $\mathcal{C}^{\infty}$ pseudoconvex open set. Then the deviation pseudoconvex of D*

$$\mathcal{D}ev : \partial D \longrightarrow [0, +\infty]$$
$$\zeta \longmapsto \mathcal{D}ev(\zeta)$$

is a lowersemicontinuous function.

Proposition 2.2.12. *Let $D \subset \mathbb{C}^n$ be a $\mathcal{C}^{\infty}$ pseudoconvex open set with bounded deviation. Then D is of finite type in the sense of D'Angelo.*
The converse is in general false.

2.3 Integral representation formulas for the $\overline{\partial}_L$−operator

2.3.1 The basic differential form $\mathcal{K}(u, v)$

Notations.

Let M be a $\mathcal{C}^1$−differentiable manifold, and

$$u = (u_1, \cdots, u_{n+1}) : M \longrightarrow \mathbb{C}^{n+1}$$

$$v = (v_1, \cdots, v_{n+1}) : M \longrightarrow \mathbb{C}^{n+1}$$

be $\mathcal{C}^1-$mappings. Define the differential forms

$$\omega_{n+1}(u) := \bigwedge_{j=1}^{n+1} du_j \tag{2.3.1}$$

$$\omega_n^{'}(v) := \sum_{j=1}^{n+1} (-1)^{j+1} v_j dv_1 \wedge \ldots \widehat{dv_j} \wedge \ldots dv_{n+1} \tag{2.3.2}$$

where $\widehat{dv_j}$ means that dv_j is omitted, and the scalar function

$$\langle u, v\rangle := \sum_{j=1}^{n+1} u_j v_j. \tag{2.3.3}$$

Proposition 2.3.1. *The singular differential form*

$$K_{2n+1}(u, v) := \frac{\omega_n^{'}(v) \wedge \omega_{n+1}(u)}{\langle u, v\rangle^{n+1}} \tag{2.3.4}$$

is closed (in the sense of distributions) in the open set $\Big\{x \in M; \big\langle u(x), v(x)\big\rangle \neq 0\Big\}$.

Proof. This results from a direct computation, for details, see [],[],[]. □

Proposition 2.3.2. *For every $\mathcal{C}^1-$function $g : M \longrightarrow \mathbb{C}$, we have*

$$\omega_n^{'}(g.v) = g^{n+1} \omega_n^{'}(v) \tag{2.3.5}$$

and hence

$$K_{2n+1}(u, g.v) = K_{2n+1}(u, v).$$

Proof. For the proof, it suffices to write $\omega_{n+1}^{'}(v)$ as determinant

$$\omega_n^{'}(v) = \frac{1}{n!} det\Big(v, \underbrace{dv, ..., dv}_{n}\Big)$$

that is

$$\omega_n'(v) = \frac{1}{n!} det \begin{pmatrix} v_1 & dv_1 & \cdots & dv_1 \\ \vdots & \vdots & \vdots & \vdots \\ v_{n+1} & dv_{n+1} & \cdots & dv_{n+1} \end{pmatrix}.$$

We have

$$\begin{aligned} \omega_n'(g.v) &= \frac{1}{n!} det\Big(g.v, \underbrace{d(g.v), ..., d(g.v)}_{n}\Big) \\ &= \frac{1}{n!} det\Big(g.v, \underbrace{g.dv + v\frac{dg}{g}, ..., gdv + v\frac{dg}{g}}_{n}\Big) \\ &= \frac{1}{n!} det\Big(g.v, \underbrace{g.dv, ..., g.dv}_{n}\Big) \\ &= g^{n+1}\omega_n'(v) \end{aligned}$$

as desired. □

2.3.2 An integral representation formula of Koppelman type.

The kernel $K(z, \xi)$.

Let $\Omega \subset \mathbb{H}$ be a bounded open set with $\mathcal{C}^\infty-$boundary $\partial\Omega$, $V_{\overline{\Omega}}$ a neighborhood of $\overline{\Omega}$, and let

$$\begin{aligned} h : \mathbb{H} &\longrightarrow \mathbb{C}^n \\ z &\longmapsto \mathfrak{Z} = h(z) \end{aligned}$$

be the system of left h-holomorphic coordinates defined for $z = (z', z'') \in \mathbb{C}^m \times \mathbb{C}^{n-m}$ by

$$\begin{cases} \mathfrak{Z}_j = z_j & \text{for} \quad 1 \le j \le m \\ \mathfrak{Z}_k = z_k - \dfrac{1}{4}A_k(z', \overline{z'}) & \text{for} \quad m+1 \le k \le n. \end{cases}$$

Consider the manifold $M := \Omega \times V_{\overline{\Omega}} \times \mathbb{C}$, and define the maps $u, v : M \longrightarrow \mathbb{C}^{n+1}$ by

$$\begin{cases} u(z, \xi, t) = (h(\xi) - h(z) \ , \ t) \\ v(z, \xi, t) = \left(\overline{h(\xi)} - \overline{h(z)} \ , \ \overline{t}e^{-|t|^2}\right). \end{cases} \tag{2.3.6}$$

Then the usual euclidian inner product of $u(z,\xi,t)$ and $v(z,\xi,t)$ is

$$\Big\langle u(z,\xi,t)\ ,\ v(z,\xi,t)\Big\rangle = \|h(\xi)-h(z)\|^2 + |t|^2\, e^{-|t|^2}.$$

By substituting the maps (u,v) in $\mathcal{K}_{2n+1}$, we obtain the singular differential form

$$K_{2n+1}\big(u(z,\xi,t)\ ,\ v(z,\xi,t)\big) = \frac{\omega_n'(v(z,\xi,t))\wedge\omega_{n+1}(u(z,\xi,t)}{\langle u(z,\xi,t), v(z,\xi,t)\rangle^{n+1}}.$$

Definition 2.3.3. Let the complex measure in $\mathbb{C}$

$$\mu(t) := \frac{n!}{(2\pi i)^{n+1}}.(1-|t|^2)e^{-|t|^2}d\bar{t}\wedge dt$$

and define

$$\mathcal{K}(z,\xi) = \int_{t\in\mathbb{C}} \frac{\omega_{n-1}'\Big(\overline{h(\xi)}-\overline{h(z)}\Big)\wedge\omega_n\big(h(\xi)-h(z)\big)}{\big(\|h(\xi)-h(z)\|^2+|t|^2\, e^{-|t|^2}\big)^{n+1}}\wedge\mu(t). \qquad (2.3.7)$$

The singular differential (2n-1)-form is called the kernel of Koppelman type of the generalized Heisenberg group $\mathbb{H}$.

Lemma 2.3.4. *For every bounded differential forms* $f\in\mathcal{C}^\infty_{(p,(q_1,q_2))_\mathcal{H}}(\Omega)$ *and* $\psi\in\mathcal{C}^\infty_{(p,(q_1,q_2))_\mathcal{H}}(\Omega)$*, we have:*

$$\int_{\partial\Omega}\mathcal{K}(z,\xi)\wedge f(\xi)\wedge\psi(z) = \int_{\partial\Omega\times\mathbb{C}} K_{2n+1}(z,\xi,t)\wedge f(\xi)\wedge\psi(z)$$

Since the map: $(z,\xi)\longrightarrow h(\xi)-h(z)$ is left $\mathcal{H}$−holomorphic with respect to both z and ξ, then

$$d\Big(h(\xi)-h(z)\Big) = \partial_{\mathbb{H}}(h(\xi)) - \partial_{\mathbb{H}}(h(z))$$

and

$$d\left(\overline{h(\xi)}-\overline{h(z)}\right) = \overline{\partial}_{\mathbb{H}}\left(\overline{h(\xi)}\right) - \overline{\partial}_{\mathbb{H}}\left((\overline{h(z)}\right).$$

The differential forms $\omega_n(h(\xi)-h(z))$ and $\omega_{n-1}'\left(\overline{h(\xi)}-\overline{h(z)}\right)$ may then be written as follows

$$\omega_n(h(\xi)-h(z)) = \bigwedge_{j=1}^{n}\big(\partial_{\mathbb{H}}(h_j(\xi)) - \partial_{\mathbb{H}}(h_j(z))\big)$$

and

$$\omega_{n-1}^{'}\left(\overline{h(\xi)}-\overline{h(z)}\right)=\sum_{j=1}^{n}(-1)^{j+1}\left(\overline{h_j(\xi)}-\overline{h_j(z)}\right)\bigwedge_{\substack{k=1\\k\neq j}}^{n}\left(\overline{\partial}_{\mathbb{H}}(\overline{h_k(\xi)})-\overline{\partial}_{\mathbb{H}}(\overline{h_k(z)})\right)$$

which means that $K(z,\xi)$ is of $\mathcal{H}-$bi degree $(n,n-1)_{\mathcal{H}}$ on $\Omega\times V_{\overline{\Omega}}$.

The integral operators $\mathcal{K}_\Omega$ and $\mathcal{K}_{\partial\Omega}$.

Since the kernel of Koppelman type $\mathcal{K}(z,\xi)$ is smooth outside the diagonal $\Delta=\{(z,\xi)\in\Omega^2\}$ and has integrable singularities in Δ of order $2n-1$, we can then define the following integral operators:

1. If f is a bounded differential form on Ω, we define

$$(\mathcal{K}_\Omega f)(z):=\int_{\xi\in\Omega}\mathcal{K}(z,\xi)\wedge f(\xi),\qquad z\in\Omega. \tag{2.3.8}$$

2. If f a bounded differential form on $\partial\Omega$, we define

$$(\mathcal{K}_{\partial\Omega} f)(z):=\int_{\xi\in\partial\Omega}\mathcal{K}(z,\xi)\wedge f(\xi),\qquad z\in\Omega. \tag{2.3.9}$$

Now decompose the kernel $K(z,\xi)$ as

$$\mathcal{K}(z,\xi)=\sum_{\substack{0\le p_1+p_2\le n\\0\le q\le n-1}}\mathcal{K}_{((p_1,p_2),q)}(z,\xi) \tag{2.3.10}$$

where $K_{((p_1,p_2),q)}(z,\xi)$ is a differential form of type $((p_1,p_2),q)_{\mathcal{H}}$ in z and of type $((m-p_1,n-m-p_2),n-q-1)_{\mathcal{H}}$ in ξ, then the operator $\mathcal{K}_\Omega$ can be defined for a bounded differential form f on Ω by

$$(\mathcal{K}_\Omega f)(z)=\int_{\xi\in\Omega}\mathcal{K}_{((p_1,p_2),q)}(z,\xi)\wedge f(\xi)$$

and $\mathcal{K}_{\partial\Omega}$ can be defined for founded differential form f on $\partial\Omega$ by

$$(\mathcal{K}_{\partial\Omega} f)(z)=\int_{\xi\in\partial\Omega}\mathcal{K}_{((p_1,p_2),q)}(z,\xi)\wedge f(\xi).$$

Proposition 2.3.5. *($\gamma-$Hölder estimates of $\mathcal{K}_\Omega$.)*
Let Ω be a bounded open set in $\mathbb{H}$. Then For every bounded differential form f on Ω, $\mathcal{K}_\Omega(f)$ is a $\mathcal{C}^\gamma-$form in Ω for all $0 \leq \gamma \leq 1$.

Proof.
It follows from the definition of $\mathcal{K}_\Omega(f)$ that , for some constant $C > 0$, and for all $z, \xi \in \Omega$

$$\|\mathcal{K}_\Omega(f)(z) - \mathcal{K}_\Omega(f)(\zeta)\| \leq C \, \|f\|_{0,\Omega} \sum_{j=1}^{n} \int_{\xi\in\Omega} \left| \frac{\overline{h_j(\xi)} - \overline{h_j(z)}}{|h(\xi) - h(z)|} - \frac{\overline{h_j(\xi)} - \overline{h_j(\zeta)}}{|h(\xi) - h(\zeta)|} \right| dV$$

where dV is the Haar measure in $\mathbb{H}$. In view of proposition .0.7, Appendix 1, it follows that for some $C_1 > 0$

$$\|\mathcal{K}_\Omega(f)(z) - \mathcal{K}_\Omega(f)(\zeta)\| \leq C \, \|f\|_{0,\Omega} |h(z) - h(\zeta)| \left| ln \, |h(z) - h(\zeta)| \right|.$$

Since, for some $A > 0$, and $B > 0$

$$A \leq \frac{|h(z) - h(\zeta)|}{|z - \zeta|} \leq B$$

and for all $0 < \gamma < 1$, we have

$$\sup_{z,\zeta\in\Omega} |h(z) - h(\zeta)|^{1-\gamma} \left| ln \, |h(z) - h(\zeta)| \right| < +\infty$$

we obtain then the assertion of proposition 2.3.5 as required. □

Theorem 2.3.6. ***(Integral formula of Koppelman type).*** *Let $\Omega \subset \mathbb{H}$ be a bounded open set with piecewise $\mathcal{C}^1$ boundary $\partial\Omega$. Then for every $((p_1, p_2), q)_\mathcal{H}-$differential form f on $\overline{\Omega}$, we have for every $L \lhd \mathcal{H}$, the integral formula*

$$f = \mathcal{K}_{\partial\Omega} f + \overline{\partial}_L \left(\mathcal{K}_\Omega f\right) + \mathcal{K}_\Omega \left(\overline{\partial}_L f\right). \tag{2.3.11}$$

Proof. Let $\psi(z) \in \mathcal{D}_{((m-p_1, n-m-p_2), n-q)_\mathcal{H}}(\Omega)$ be a differential form with compact support of type $((m-p_1, n-m-p_2), n-q)_\mathcal{H}$, and consider the following integral:

$$I(f, \psi) := \int_{\Omega\times\partial\Omega} \mathcal{K}(z, \xi) \wedge f(\xi) \wedge \psi(z).$$

Taking into account the definition of the kernel $\mathcal{K}(z,\xi)$ (see (2.3.7)), we have:

$$\begin{aligned} I(f,\psi) &= \int_{\Omega\times\partial\Omega} \mathcal{K}(z,\xi)\wedge f(\xi)\wedge\psi(z) \\ &= \int_{\Omega\times\partial\Omega\times\mathbb{C}} K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)\wedge f(\xi)\wedge\psi(z) \\ &= \int_{\Omega\times\partial(\Omega\times\mathbb{C})} K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)\wedge f(\xi)\wedge\psi(z). \end{aligned}$$

Since $K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)$ has no singularities on $Supp\psi\times\partial(\Omega\times\mathbb{C})\subset\subset\Omega\times\partial(\Omega\times\mathbb{C})$, and ψ vanishes on $\partial\Omega$, then

$$I(f,\psi) = \int_{\partial(\Omega\times\Omega\times\mathbb{C})} K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)\wedge f(\xi)\wedge\psi(z).$$

Now let $L\lhd\mathcal{H}$, and write in the product $\mathbb{H}\times\mathbb{H}$, the exterior differential operator $d_{z,\xi}$ in terms of the connexion $d_{L\times L}$ (see(1.3.25))

$$d_{z,\xi} = d_{L\times L} - \Gamma^{L\times L}.$$

From the identity (1.3.28), we obtain in $\mathbb{H}\times\mathbb{H}\times\mathbb{C}$ the following decomposition:

$$\begin{aligned} d_{z,\xi,t} &= d_{z,\xi} + d_t \\ &= d_{L\times L} - \Gamma^{L\times L} + d_t \\ &= \partial_{L\times L} + \overline{\partial}_{L\times L} - \Gamma^{L\times L} + \partial_t + \overline{\partial}_t. \end{aligned}$$

In view of the decomposition (2.3.10), the differential form

$$K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)\wedge f(\xi)\wedge\psi(z)$$

is of total $\mathcal{H}-$bidegree $(2n+2,2n+1)_{\mathcal{H}}$. Then from the decomposition of $d_{z,\xi,t}$, and the definition of $\Gamma^{L\times L}$, we obtain:

$$\begin{cases} \Big(\partial_{L\times L}+\partial_t\Big)\Big[K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)\wedge f(\xi)\wedge\psi(z)\Big] = 0 \\ \Gamma^{L\times L}\Big[K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)\wedge f(\xi)\wedge\psi(z)\Big] = 0 \\ d_{z,\xi,t}\Big[K_{2n+1}(z,\xi)\wedge f(\xi)\wedge\psi(z)\Big] = (\overline{\partial}_{L\times L}+\overline{\partial}_t)\Big[K_{2n+1}(z,\xi)\wedge f(\xi)\wedge\psi(z)\Big]. \end{cases}$$

Stokes' formula gives on $\Omega \times \Omega \times \mathbb{C}$:

$$\begin{aligned} I(f,\psi) &= \int_{\Omega\times\Omega\times\mathbb{C}} d_{z,\xi,t}\left[K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)\wedge f(\xi)\wedge\psi(z)\right] \\ &= \int_{\Omega\times\Omega\times\mathbb{C}} (\overline{\partial}_{L\times L}+\overline{\partial}_t)\left[K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)\wedge f(\xi)\wedge\psi(z)\right] \end{aligned}$$

Then

$$\begin{aligned} &I(f,\psi) = \\ &\int_{\Omega\times\Omega\times\mathbb{C}}\left[(\overline{\partial}_{L\times L}+\overline{\partial}_t)K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)\right]\wedge f(\xi)\wedge\psi(z) \\ &\quad - \int_{\Omega\times\Omega\times\mathbb{C}} K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)\wedge\left[\overline{\partial}_L f(\xi)\right]\wedge\psi(z) \\ &\quad - (-1)^{p_1+p_2+q}\int_{\Omega\times\Omega\times\mathbb{C}} K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)\wedge f(\xi)\wedge\left[\overline{\partial}_L\psi(z)\right]. \end{aligned} \tag{2.3.12}$$

Since $(\overline{\partial}_{L\times L}+\overline{\partial}_t)\left[K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)\right] = [\Delta]\otimes\delta_{(t=0)}$ where $[\Delta]$ is the current of integration on the diagonal $\Delta\subset\mathbb{H}\times\mathbb{H}$, and $\delta_{(t=0)}$ is the Dirac measure at $t=0$, then

$$\begin{aligned} &\int_{\Omega\times\Omega\times\mathbb{C}}\left[(\overline{\partial}_{L\times L}+\overline{\partial}_t)K_{2n+1}\Big(u(z,\xi,t),v(z,\xi,t)\Big)\right]\wedge f(\xi)\wedge\psi(z) \\ &= \int_{\Omega\times\Omega\times\mathbb{C}}\big([\Delta]\otimes\delta_{(t=0)}\big)\wedge f(\xi)\wedge\psi(z) \\ &= \int_\Omega f(z)\wedge\psi(z). \end{aligned}$$

If $\langle\ ,\ \rangle$ denotes the pairing between currents and test forms on Ω, then after integrating $\overline{\partial}_L\psi$ by parts, equality (2.3.12) is equivalent to the integral representation formula (2.3.11). The proof is then complete. □

2.3.3 An integral representation formula of Leray-Koppelman type.

The Leray section $\big(w(z,\xi),g(z,\xi)\big)\in\mathbb{C}^{n+1}$.

Notations.

Let $\Omega \subset \mathbb{H}$ be a bounded open set with $\mathcal{C}^1-$boundary, $V_{\partial\Omega}$ a neighborhood of $\partial\Omega$, and u v the maps defined in (2.3.6), that is

$$u(z,\xi,t) = \big(h(\xi) - h(z) \ , \ t\big) \ \in \mathbb{C}^{n+1}$$
$$v(z,\xi,t) = \Big(\overline{h(\xi)} - \overline{h(z)} \ , \ \bar{t}e^{-|t|^2}\Big) \ \in \mathbb{C}^{n+1}.$$

Now consider a map $w : \Omega \times V_{\partial\Omega} \longrightarrow \mathbb{C}^n$:

$$w(z,\xi) = \Big(w_1(z,\xi), ..., w_n(z,\xi)\Big) \ \in \mathbb{C}^{n+1},$$

and a complex valued function $g : \Omega \times V_{\partial\Omega} \longrightarrow \mathbb{C}$, and set:

$$\widetilde{w}(z,\xi,t) := \Big(w(z,\xi) \ , \ \bar{t}e^{-|t|^2}.g(z,\xi)\Big) \in \mathbb{C}^{n+1} \tag{2.3.13}$$

$$\begin{aligned} N_0(z,\xi,t) &:= \Big\langle u(z,\xi,t), v(z,\xi,t)\Big\rangle \\ &= \sum_{j=1}^{n} |h_j(\xi) - h_j(z)|^2 + |t|^2 e^{-|t|^2} \end{aligned} \tag{2.3.14}$$

$$\begin{aligned} N(z,\xi,t) &:= \Big\langle u(z,\xi,t), \widetilde{w}(z,\xi,t)\Big\rangle \\ &= \sum_{j=1}^{n} w_j\big(z,\xi).(h_j(\xi) - h_j(z)\big) + |t|^2 e^{-|t|^2}.g(z,\xi) \end{aligned} \tag{2.3.15}$$

and denote by $F^w_{(z,t)}$ the following subset of $\partial\Omega$:

$$F^w_{(z,t)} := \Big\{\xi \in \partial\Omega, \quad N(z,\xi,t) = 0\Big\}$$

and by $\mu_{\partial\Omega}$ the Lebeagues measure of the boundary $\partial\Omega$. We are lead to the following definition.

Definition 2.3.7. With the above notations, we say that the map

$$(w,g) : \Omega \times V_{\partial\Omega} \longrightarrow \mathbb{C}^{n+1}$$
$$(z,\xi) \longmapsto \big(w(z,\xi), \ g(z,\xi) \ \big)$$

is a Leray section for Ω, if the following two conditions are fulfilled:

1. For all $(z,t)\in\Omega\times\mathbb{C}$, $\qquad \mu_{\partial\Omega}\left(F^{w}_{(z,t)}\right)=0.$

2. for all $z\in\Omega$, $\qquad \left|\int_{t\in\mathbb{C}}\int_{\xi\in\partial\Omega}\frac{g(z,\xi)}{[N(z,\xi,t)]^{n+1}}\,\mu_{\partial\Omega}(\xi)\otimes\mu(t)\right|<+\infty.$

Now let (w,g) be a Leray section for Ω and consider on the manifold

$$M:=\Omega\times V_{\partial\Omega}\times\mathbb{C}$$

the homotopy $\rho=(\rho_1,...,\rho_{n+1})$ defined for $s\in[0,1]$, by

$$\rho(z,\xi,t,s):=\frac{(1-s).v(z,\xi)}{N_0(z,\xi,t)}+\frac{s.\widetilde{w}(z,\xi,t)}{N(z,\xi,t)}. \tag{2.3.16}$$

It is clear by (2.3.16) that for all $(z,\xi,t,s)\in M\times[0,1]$,

$$\Big\langle u(z,\xi,t),\rho(z,\xi,t,s)\Big\rangle=1. \tag{2.3.17}$$

By substituting v by the maps w and ρ respectively in the forms $\omega^{'}_{n-1}(v)$ and $\omega^{'}_{n}(v)$ (see (2.3.2)), we obtain :

$$\omega^{'}_{n-1}(w(z,\xi))=\sum_{j=1}^{n}(-1)^{j+1}w_j(z,\xi)\bigwedge_{\substack{k=1\\k\neq j}}^{n}\overline{\partial}_{\mathbb{H}\times\mathbb{H}}(w_k(z,\xi))$$

and

$$\omega^{'}_{n}(\rho(z,\xi,t,s))=\sum_{j=1}^{n}(-1)^{j+1}\rho_j(z,\xi,t,s)\bigwedge_{\substack{k=1\\k\neq j}}^{n+1}(\overline{\partial}_{\mathbb{H}\times\mathbb{H}}+d_t+d_s)(\rho_k(z,\xi,t,s)).$$

Definition 2.3.8. Let the complex measure in $\mathbb{C}$

$$\mu(t):=\frac{n!}{(2\pi i)^{n+1}}.(1-|t|^2)e^{-|t|^2}d\overline{t}\wedge dt$$

and define

$$L_{2n+1}(z,\xi,t):=\frac{n!}{(2i\pi)^{2n+1}}K_{2n+1}\left(u(z,\xi,t),\widetilde{w}(z,\xi,t)\right)$$

$$R_{2n+1}(z,\xi,t,s) := \frac{n!}{(2i\pi)^{2n+1}} K_{2n+1}(u(z,\xi,t), \rho(z,\xi,t,s))$$

and

$$\mathcal{L}(z,\xi) = \int_{t\in\mathbb{C}} g(z,\xi). \frac{\omega'_{n-1}(w(z,\xi)) \wedge \omega_n(h(\xi)-h(z))}{\langle u(z,\xi,t), \widetilde{w}(z,\xi,t)\rangle^{n+1}} \wedge \mu(t)$$

$$\mathcal{R}(z,\xi,s) = \int_{t\in\mathbb{C}} g(z,\xi).\omega'_n(\rho(z,\xi,t,s)) \wedge \omega_n(h(\xi)-h(z)) \wedge \mu(t).$$

The differential forms $\mathcal{L}(z,\xi)$ and $\mathcal{R}(z,\xi,s)$ are called the Leray kernels of the generalized Heisenberg group $\mathbb{H}$.

Lemma 2.3.9. *For every bounded differential forms $f \in \mathcal{C}^\infty_{(p,(q_1,q_2))_\mathcal{H}}(\Omega)$ and $\psi \in \mathcal{C}^\infty_{(p,(q_1,q_2))_\mathcal{H}}(\Omega)$, we have:*

$$\int_{\partial\Omega} \mathcal{L}(z,\xi) \wedge f(\xi) \wedge \psi(z) = \int_{\partial\Omega\times\mathbb{C}} L_{2n+1}(z,\xi,t) \wedge f(\xi) \wedge \psi(z)$$

$$\int_{\partial\Omega} \mathcal{R}(z,\xi,s) \wedge f(\xi) \wedge \psi(z) = \int_{\partial\Omega\times\mathbb{C}} R_{2n+1}(z,\xi,t,s) \wedge f(\xi) \wedge \psi(z).$$

The integral operators $\mathcal{L}_{\partial\Omega}$ and $\mathcal{R}_{\partial\Omega}$

Let f be a bounded differential form on $\partial\Omega$, and (w,g) a Leray section for Ω. Since by (2.3.17) and the conditions (1) and (2) of definition 2.3.7, the differential forms $L(z,\xi) \wedge f(\xi)$ and $R(z,\xi,s) \wedge f(\xi)$ are integrable on $V_{\partial\Omega}$ and on $V_{\partial\Omega} \times [0,1]$ respectively, we can then define

$$\big(\mathcal{L}_{\partial\Omega} f\big)(z) := \int_{\xi\in\partial\Omega} \mathcal{L}\big(z,\xi\big) \wedge f(\xi) \tag{2.3.18}$$

and

$$\big(\mathcal{R}_{\partial\Omega} f\big)(z) := \int_{\substack{\xi\in\partial\Omega \\ 0\leq s\leq 1}} \mathcal{R}\big(z,\xi,s\big) \wedge f(\xi). \tag{2.3.19}$$

If we consider the unique decompositions

$$\mathcal{L}\big(z,\xi\big) = \sum_{\substack{0\leq p_1+p_2\leq n \\ 0\leq q\leq n-1}} \mathcal{L}_{((p_1,p_2),q)}(z,\xi)$$

$$\mathcal{R}\big(z,\xi,s\big) = \sum_{\substack{0\le p_1+p_2\le n\\ 0\le q\le n-1}} \mathcal{R}_{((p_1,p_2),q)}(z,\xi,s)$$

where $\mathcal{L}_{((p_1,p_2),q)}(z,\xi)$ is of type $((p_1,p_2),q)_{\mathcal{H}}$ in z and $((m-p_1,n-m-p_2),q)_{\mathcal{H}}$ in ξ and $R_{((p_1,p_2),q)}(z,\xi,s)$ is of type $((p_1,p_2),q)_{\mathcal{H}}$ in z and $((m-p_1,n-m-p_2),q-1)_{\mathcal{H}}$ in (ξ,s), then the integral operators $\mathcal{L}_{\partial\Omega}$ and $\mathcal{R}_{\partial\Omega}$ may be defined for $f\in\mathcal{C}^\infty_{((p_1,p_2),q)_{\mathcal{H}}}(\Omega)$ as follows:

$$\big(\mathcal{L}_{\partial\Omega}f\big)(z) := \int_{\xi\in\partial\Omega} \mathcal{L}_{((p_1,p_2),q)}(z,\xi)\wedge f(\xi)$$

and

$$\big(\mathcal{R}_{\partial\Omega}f\big)(z) := \int_{\substack{\xi\in\partial\Omega\\ 0\le t\le 1}} \mathcal{R}_{((p_1,p_2),q)}(z,\xi,s)\wedge f(\xi).$$

Theorem 2.3.10. ***(Integral formula of Leray-Koppelman type).*** *Let $\Omega\subset\mathbb{H}$ be a bounded open set with piecewise $\mathcal{C}^1$ boundary $\partial\Omega$, $V_{\partial\Omega}$ a bounded neighborhood of $\partial\Omega$ and (w,g) a Leray section for Ω such that the derivatives of (w,g) of order ≤ 2 in z and the derivatives of (w,g) of order ≤ 1 in ξ are continuous on $\Omega\times V_{\partial\Omega}$. Then for every $((p_1,p_2),q)_{\mathcal{H}}-$differential form f of class $\mathcal{C}^1$ on $\overline{\Omega}$ we have*

$$f = \mathcal{L}_{\partial\Omega}f + \overline{\partial}_L\left(\mathcal{R}_{\partial\Omega}+\mathcal{K}_\Omega\right)f + \left(\mathcal{R}_{\partial\Omega}+\mathcal{K}_\Omega\right)\overline{\partial}_L f. \tag{2.3.20}$$

Proof. To prove (2.3.20), we have only by Koppelman formula to prove in the sense of distributions the following identity:

$$\overline{\partial}_L\mathcal{R}_{\partial\Omega}f = \mathcal{K}_{\partial\Omega}f - \mathcal{L}_{\partial\Omega}f + \mathcal{R}_{\partial\Omega}\overline{\partial}_L f \qquad \text{in}\quad \Omega. \tag{2.3.21}$$

Indeed, let $\psi\in\mathcal{D}_{(m-p_1,n-m-p_2,n-q)_{\mathcal{H}}}(\Omega)$. With the notation:

$$R_{2n+1}(z,\xi,t,s) = \frac{n!}{(2i\pi)^{2n+1}}K_{2n+1}\big(u(z,\xi,t),\rho(z,\xi,t,s)\big),$$

consider the integral

$$J(f,\psi) := \int_{\Omega\times\partial\Omega\times[0,1]} d\left[R_{2n+1}(z,\xi,s)\wedge f(\xi)\wedge\psi(z)\right].$$

Since $\big(w(z,\xi),g(z,\xi)\big)$ is a Leray section for Ω, then $R_{2n+1}(z,\xi,t,s)\wedge f(\xi)\wedge\psi(z)$ has integrable singularities on $\Omega\times Supp\psi\times\mathbb{C}\times[0,1]$, and since $\psi(z)$ vanishes on $\partial\Omega$, then the integral $J(f,\psi)$ can be written as follows

$$J(f,\psi) := \int_{\partial(\Omega\times\Omega\times\mathbb{C})\times[0,1]} d\left[R_{2n+1}(z,\xi,t,s)\wedge f(\xi)\wedge\psi(z)\right].$$

Let $L \lhd \mathcal{H}$, and write in the product $\mathbb{H} \times \mathbb{H}$, the exterior differential operator $d_{z,\xi}$ in terms of the connexion $d_{L\times L}$ (see(1.3.25))

$$d_{z,\xi} = d_{L\times L} - \Gamma^{L\times L}.$$

From the identity (1.3.28), we obtain in $\mathbb{H} \times \mathbb{H} \times \mathbb{C}$ the following decomposition:

$$\begin{aligned} d &= d_{z,\xi} + d_t + d_s \\ &= d_{L\times L} - \Gamma^{L\times L} + d_t + d_s \\ &= \partial_{L\times L} + \overline{\partial}_{L\times L} - \Gamma^{L\times L} + \partial_t + \overline{\partial}_t + d_s. \end{aligned}$$

Since in view of the decomposition (2.3.19), the differential form

$$R_{2n+1}(z, \xi, t, s) \wedge f(\xi) \wedge \psi(z)$$

is of total $\mathcal{H}-$bidegree $(2n+2, 2n+1)_{\mathcal{H}}$, then:

$$\begin{cases} \left(\partial_{L\times L} + \partial_t\right) [R_{2n+1}(z, \xi, t, s) \wedge f(\xi) \wedge \psi(z)] = 0 \\ \Gamma^{L\times L} [R_{2n+1}(z, \xi, t, s) \wedge f(\xi) \wedge \psi(z)] = 0 \\ d [R_{2n+1}(z, \xi, t, s) \wedge f(\xi) \wedge \psi(z)] = \\ \left(\overline{\partial}_{L\times L} + \overline{\partial}_t + d_s\right) [R_{2n+1}(z, \xi, t, s) \wedge f(\xi) \wedge \psi(z)] \end{cases}$$

Hence

$$\begin{aligned} J(f, \psi) = &- \int_{\Omega\times\partial\Omega\times\mathbb{C}\times[0,1]} R_{2n+1}(z, \xi, t, s) \wedge \overline{\partial}_L [f(\xi)] \wedge \psi(z) \\ &- (-1)^{p_1+p_2+q} \int_{\Omega\times\partial\Omega\times\mathbb{C}\times[0,1]} R_{2n+1}(z, \xi, t, s) \wedge f(\xi) \wedge \overline{\partial}_L [\psi(z)] \end{aligned}$$

and by Stokes' formula:

$$\begin{aligned} J(f, \psi) = &\int_{\Omega\times\partial\Omega\times\mathbb{C}} R_{2n+1}(z, \xi, t, 1) \wedge f(\xi) \wedge \psi(z) \\ &- \int_{\Omega\times\partial\Omega\times\mathbb{C}} R_{2n+1}(z, \xi, t, 0) \wedge f(\xi) \wedge \psi(z). \end{aligned}$$

From identity (2.3.5), we deduce:

$$\begin{cases} R_{2n+1}(z, \xi, t, 1) \wedge f(\xi) \wedge \psi(z) = L_{2n+1}(z, \xi, t) \wedge f(\xi) \wedge \psi(z) \\ R_{2n+1}(z, \xi, t, 0) \wedge f(\xi) \wedge \psi(z) = K_{2n+1}(z, \xi, t) \wedge f(\xi) \wedge \psi(z), \end{cases}$$

According to lemma 2.3.9, we obtain in one hand

$$J(f,\psi) = -\int_{\Omega\times\partial\Omega\times[0,1]} \mathcal{R}(z,\xi,s)\wedge\overline{\partial}_L\left[f(\xi)\right]\wedge\psi(z)$$
$$-(-1)^{p_1+p_2+q}\int_{\Omega\times\partial\Omega\times[0,1]} \mathcal{R}(z,\xi,s)\wedge f(\xi)\wedge\overline{\partial}_L\left[\psi(z)\right]$$

and in the other hand

$$J(f,\psi) = \int_{\Omega\times\partial\Omega} \mathcal{L}(z,\xi)\wedge f(\xi)\wedge\psi(z)$$
$$-\int_{\Omega\times\partial\Omega} \mathcal{K}(z,\xi)\wedge f(\xi)\wedge\psi(z).$$

Finally, by integrating by parts $\overline{\partial}_L\psi$, we deduce (2.3.21). This completes the proof. □

Theorem 2.3.11.
Let (w,g) be a Leray section for Ω. If w is left $\mathcal{H}-$holomorphic in z, then for every differential form $f\in\mathcal{C}_{(p_1,p_2,q)_{\mathcal{H}}}(\overline{\Omega})$ of with $q\geq 1$, we have:

$$f = \overline{\partial}_L\left(\mathcal{R}_{\partial\Omega}+\mathcal{K}_\Omega\right)f + \left(\mathcal{R}_{\partial\Omega}+\mathcal{K}_\Omega\right)\overline{\partial}_L f. \qquad (2.3.22)$$

Proof. Let (w,g) be a Leray section for Ω. From Leray-koppelman formula (2.3.20), we have in the sense of distributions:

$$f = \mathcal{L}_{\partial\Omega}f + \overline{\partial}_L\left(\mathcal{R}_{\partial\Omega}+\mathcal{K}_\Omega\right)f + \left(\mathcal{R}_{\partial\Omega}+\mathcal{K}_\Omega\right)\overline{\partial}_L f.$$

that is for all $f\in\mathcal{C}_{(p_1,p_2,q)_{\mathcal{H}}}(\overline{\Omega})$ and all $\psi\in\mathcal{D}_{(m-p_1,n-m-p_2,n-q)_{\mathcal{H}}}(\Omega)$:

$$\int_{\Omega\times\partial\Omega\times)} f(\xi)\wedge\psi(z) = \int_{\Omega\times\partial\Omega\times\mathbb{C}} \mathcal{L}(z,\xi)\wedge f(\xi)\wedge\psi(z)$$
$$+\int_{\Omega\times\partial\Omega\times\mathbb{C})\times[0,1]} \mathcal{R}(z,\xi,s)\wedge\left[\overline{\partial}_L f(\xi)\right]\wedge\psi(z)$$
$$-\int_{\Omega\times\partial\Omega\times\mathbb{C})\times[0,1]} \mathcal{R}(z,\xi,s)\wedge f(\xi)\wedge\overline{\partial}_L\psi(z).$$

recall that

$$\mathcal{L}(z,\xi) = \int_{t\in\mathbb{C}} g(z,\xi).\frac{\omega'_{n-1}\left(w(z,\xi)\right)\wedge\omega_n(h(\xi)-h(z))}{\left\langle u(z,\xi,t),\widetilde{w}(z,\xi,t)\right\rangle^{n+1}}\wedge\mu(t)$$

where

$$\omega'_{n-1}(w(z,\xi)) = \sum_{j=1}^{n}(-1)^{j+1}w_j(z,\xi)\bigwedge_{\substack{k=1\\k\neq j}}^{n}\overline{\partial}_{\mathbb{H}\times\mathbb{H}}(w_k(z,\xi)).$$

Since $w(z,\xi)$ is left $\mathcal{H}-$holomorphic in z, then $\omega'_{n-1}(w(z,\xi))$ does not involve $\overline{\phi}_k(z)$, which implies for reason degrees, that the differential form $\mathcal{L}(z,\xi)\wedge f(\xi)\wedge\psi(z)$ contains at least the term

$$\bigwedge_{j=1}^{n}\overline{\phi_j(\xi)}\wedge\phi_j(\xi)\bigwedge_{j=1}^{n}\overline{\phi_j(z)}\wedge\phi_j(z)$$

and since $dim_{\mathbb{R}}\partial\Omega = 2n-1$, then we must have:

$$\int_{\Omega}\left(\mathcal{L}_{\partial\Omega}f\right)(z)\wedge\psi(z) = \int_{\Omega\times\partial\Omega}\mathcal{L}(z,\xi)\wedge f(\xi)\wedge\psi(z) = 0,$$

that is in the sense of distributions

$$\mathcal{L}_{\partial\Omega}f = 0 \quad \text{for all} \quad f\in\mathcal{C}_{(p_1,p_2,q)_{\mathcal{H}}}(\overline{\Omega}).$$

This implies formula (2.3.22), as required and completes the proof. □

2.4 The solvability of $\overline{\partial}_L u = f$ with uniform estimates

Let $L\lhd\mathcal{H}$. Our aim now is to prove existence theorems with Hölderian estimates for the $\overline{\partial}_L-$ complex on a left $h-$pseudoconvex open set $\Omega\subset\mathbb{H}$ with "bounded deviation".

Proposition 2.4.1. *Let*

$$D = \left\{\mathfrak{z}\in V_{\overline{D}},\quad \varphi(\mathfrak{z})<0\right\}$$

be a $\mathcal{C}^\infty$ pseudoconvex open set with bounded deviation, and let the normal vector field over ∂D

$$\overrightarrow{N}(\zeta) = \left(Re\frac{\partial\varphi}{\partial\zeta_1},...,Re\frac{\partial\varphi}{\partial\zeta_n},Im\frac{\partial\varphi}{\partial\zeta_1},...,Im\frac{\partial\varphi}{\partial\zeta_n}\right)$$

and the function

$$\eta : V_{\overline{D}} \times \partial D \longrightarrow [0,1]$$

$$(\mathfrak{z},\zeta) \longmapsto \eta(\mathfrak{z},\zeta) := \left| cos\left(\overrightarrow{N}(\zeta), \overrightarrow{\mathfrak{z}\zeta}\right)\right|. \tag{2.4.1}$$

We denote for all $\mathfrak{z} \in V_{\overline{D}}$, by $E_{\mathfrak{z}}$ the subset $E_{\mathfrak{z}}$ of ∂D defined by:

$$E_{\mathfrak{z}} = \left\{\zeta \in \partial D, \quad \eta(\mathfrak{z},\zeta) > 0\right\}.$$

Then the exist a positive integer $r \in \mathbb{N}^*$ and real numbers $b > 0$, $c > 0$, and $1 > \varepsilon > 0$ such that:

- *If $(\mathfrak{z},\zeta) \in D \times E_{\mathfrak{z}}$ satisfies $\|\mathfrak{z} - \zeta\| \leq min\{\varepsilon \ , \ b.\eta(\mathfrak{z},\zeta)\}$, then*

$$-Re\sum_{j=1}^{n} \frac{\partial\varphi}{\partial\zeta_j}(\zeta_j - \mathfrak{z}_j) \geq c\Big[d(\mathfrak{z},\partial D) + \|\mathfrak{z}-\zeta\|^{2r}\Big]. \tag{2.4.2}$$

- *If $(\mathfrak{z},\zeta) \in D \times E_{\mathfrak{z}}$ satisfies $\|\mathfrak{z} - \zeta\| > min\{\varepsilon \ , \ b.\eta(\mathfrak{z},\zeta)\}$, then*

$$\left|Re\sum_{j=1}^{n} \frac{\partial\varphi}{\partial\zeta_j}(\zeta_j - \mathfrak{z}_j)\right| \geq c\|\mathfrak{z}-\zeta\|\Big(\eta(\mathfrak{z},\zeta)\Big)^{2r-1}. \tag{2.4.3}$$

To prove the proposition, we need the following lemmas

Lemma 2.4.2. *Let*

$$D = \left\{\mathfrak{z} \in V_{\overline{D}}, \quad \varphi(\mathfrak{z}) < 0\right\}$$

be an open set defined by the $\mathcal{C}^\infty-$function $\varphi : V_{\overline{D}} \longrightarrow \mathbb{R}$, with $d\varphi \neq 0$ on ∂D, and let $m = \inf_{\zeta\in\partial D}\left\|\overrightarrow{N}(\zeta)\right\|$. Then there exists a positive number $0 < \varepsilon < 1$, such that for all $\mathfrak{z} \in D$ satisfying $d(\mathfrak{z},\partial D) \leq \varepsilon$, we have

$$-\varphi(\mathfrak{z}) \geq m.d(\mathfrak{z},\partial D). \tag{2.4.4}$$

Proof. Let $\zeta \in \partial D$, and $\mathfrak{z} \in D$ such that

$$d(\mathfrak{z},\partial D) = \|\mathfrak{z}-\zeta\|$$

that is $\overrightarrow{N}(\zeta)//\overrightarrow{\zeta\mathfrak{z}}$, or in other words $\eta(\mathfrak{z},\zeta)=1$.
Write in the ball $B(\zeta,\varepsilon_1)=\{\mathfrak{z}\in\mathbb{C}^n, \|\mathfrak{z}-\zeta\|\leq\varepsilon_1\}$, with $0<\varepsilon_1<1$, the Taylor expansion of order 2 of $-\varphi$:

$$\begin{aligned}-\varphi(\mathfrak{z}) =& -2Re\sum_{j=1}^n \frac{\partial\varphi}{\partial\zeta_j}(\mathfrak{z}_j-\zeta_j)\\ &- \sum_{|\alpha+\beta|=2}\frac{\partial_\zeta^\alpha\partial_{\overline{\zeta}}^\beta\varphi}{2}(\mathfrak{z}-\zeta)^\alpha(\overline{\mathfrak{z}}-\overline{\zeta})^\beta + o\left(\|\mathfrak{z}-\zeta\|^2\right).\end{aligned} \tag{2.4.5}$$

Since $0<\varepsilon_1<1$, we have for $\|\mathfrak{z}-\zeta\|\leq\varepsilon_1$

$$\left|\sum_{|\alpha+\beta|=2}\frac{\partial_\zeta^\alpha\partial_{\overline{\zeta}}^\beta\varphi}{2}(\mathfrak{z}-\zeta)^\alpha(\overline{\mathfrak{z}}-\overline{\zeta})^\beta\right| \leq b_2\|\mathfrak{z}-\zeta\|^2$$

where

$$b_2 := \frac{1}{2}\sum_{|\alpha+\beta|=2}\sup_{\zeta\in\partial D}\left|\partial_\zeta^\alpha\partial_{\overline{\zeta}}^\beta\varphi(\zeta)\right|,$$

and then, if we choose $0<\varepsilon_2<\varepsilon_1$ so small that for $\|\mathfrak{z}-\zeta\|\leq\varepsilon_2$, we obtain

$$\left|\sum_{|\alpha+\beta|=2}\frac{\partial_\zeta^\alpha\partial_{\overline{\zeta}}^\beta\varphi}{2}(\mathfrak{z}-\zeta)^\alpha(\overline{\mathfrak{z}}-\overline{\zeta})^\beta + o\left(\|\mathfrak{z}-\zeta\|^2\right)\right| \leq \frac{b_2}{2}\|\mathfrak{z}-\zeta\|^2. \tag{2.4.6}$$

By the fact that $\eta(\mathfrak{z},\zeta)=1$, the following hold

$$\begin{aligned}2\left|Re\sum_{j=1}^n\frac{\partial\varphi}{\partial\zeta_j}(\mathfrak{z}_j-\zeta_j)\right| &= \left\|\overrightarrow{N}(\zeta)\right\|.\|\mathfrak{z}-\zeta\|\\ &\geq m.\|\mathfrak{z}-\zeta\|.\end{aligned} \tag{2.4.7}$$

Now let $\varepsilon := \min\left\{\varepsilon_2, \dfrac{m}{b_2}\right\}$. Then (2.4.6) and (2.4.7) imply for $\|\mathfrak{z}-\zeta\|\leq\varepsilon$

$$\left|\sum_{|\alpha+\beta|=2}\frac{\partial_\zeta^\alpha\partial_{\overline{\zeta}}^\beta\varphi}{2}(\mathfrak{z}-\zeta)^\alpha(\overline{\mathfrak{z}}-\overline{\zeta})^\beta + o\left(\|\mathfrak{z}-\zeta\|^2\right)\right| \leq \left|Re\sum_{j=1}^n\frac{\partial\varphi}{\partial\zeta_j}(\mathfrak{z}_j-\zeta_j)\right|. \tag{2.4.8}$$

Taking into account (2.4.8), we deduce first from Taylor formula (2.4.5), that for all $\|\mathfrak{z} - \zeta\| \leq \varepsilon$,

$$-Re\sum_{j=1}^{n}\frac{\partial\varphi}{\partial\zeta_j}(\mathfrak{z}_j - \zeta_j) > 0$$

and then

$$\begin{aligned}-\varphi(\mathfrak{z}) &\geq -Re\sum_{j=1}^{n}\frac{\partial\varphi}{\partial\zeta_j}(\mathfrak{z}_j - \zeta_j)\\ &= m.\,\|\mathfrak{z} - \zeta\|\\ &= m.d(\mathfrak{z}, \partial D).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.4.3. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex open set, and let $\Delta(\mathfrak{z}, \zeta)$ be the line through the points $\mathfrak{z} \in D$ and $\zeta \in \partial D$. Then for all $\varepsilon > 0$ small enough, there exists a point $\mathfrak{z}_\varepsilon \in D \cap \Delta(\mathfrak{z}, \zeta)$, such that $\|\mathfrak{z}_\varepsilon - \zeta\| \leq \varepsilon$.*

Proof.
Let $\zeta \in \partial D$, that is $\varphi(\zeta) = 0$. Since by hypothesis, the open set D is of bounded deviation, then there exist $r \in \mathbb{N}^*$, $c_1 > 0$ and $0 < \varepsilon_0 < 1$ such that for $\|\mathfrak{z} - \zeta\| \leq \varepsilon_0$,

$$\left[\mathcal{L}_\zeta^{2r}(\varphi)\right](\mathfrak{z}) \geq c_1\left\|\mathfrak{z} - \zeta\right\|^{2r}. \tag{2.4.9}$$

Let the Taylor expansion of φ of order $2r$ in a neighborhood of $\zeta \in \partial D$:

$$\varphi(\mathfrak{z}) = \varphi(\zeta) + 2Re\left[P_\zeta^{2r}(\varphi)\right](\mathfrak{z}) + \left[\mathcal{L}_\zeta^{2r}(\varphi)\right](\mathfrak{z}) + o(\|\mathfrak{z} - \zeta\|^{2r}).$$

We can choose ε_1, with $0 < \varepsilon_1 < \varepsilon_0$ such that for $\|\mathfrak{z} - \zeta\| \leq \varepsilon_2$,

$$-2Re\left[P_\zeta^{2r}(\varphi)\right](\mathfrak{z}) \geq -\varphi(\mathfrak{z}) + \frac{c_1}{2}\|\mathfrak{z} - \zeta\|^{2r}. \tag{2.4.10}$$

Decompose the Levi polynomial $\left[P_\zeta^{2r}(\varphi)\right](\mathfrak{z})$ as follows:

$$\left[P_\zeta^{2r}(\varphi)\right](\mathfrak{z}) = \sum_{j=1}^{n}\frac{\partial\varphi}{\partial\zeta_j}(\zeta_j - \mathfrak{z}_j) + \sum_{2\leq|\alpha|\leq 2r}\frac{\partial_\zeta^\alpha\varphi}{\alpha!}(\mathfrak{z} - \zeta)^\alpha. \tag{2.4.11}$$

Since $0 < \varepsilon_1 < 1$, we have for $\|\mathfrak{z} - \zeta\| \leq \varepsilon_1$,

$$\left|\sum_{2\leq|\alpha|\leq 2r}\frac{\partial_\zeta^\alpha\varphi}{\alpha!}(\mathfrak{z} - \zeta)^\alpha\right| \leq b_1\|\mathfrak{z} - \zeta\|^2 \tag{2.4.12}$$

where

$$b_1 := \sum_{2\leq|\alpha|\leq 2r} \frac{1}{\alpha!} \sup_{\zeta\in\partial D} \left|\partial_\zeta^\alpha \varphi(\zeta)\right|,$$

and for all $(\mathfrak{z},\zeta) \in D \times \partial D$, we have

$$2\left|Re\sum_{j=1}^{n} \frac{\partial\varphi}{\partial\zeta_j}(\zeta_j - \mathfrak{z}_j)\right| \geq m\,\|\mathfrak{z}-\zeta\|\,.\eta(\mathfrak{z},\zeta) \tag{2.4.13}$$

where

$$m := \inf_{\zeta\in\partial D} \left\|\overrightarrow{N}(\zeta)\right\| > 0.$$

Now let $(\mathfrak{z},\zeta) \in D \times E_3$, that is:

$$\begin{cases} -\varphi(\mathfrak{z}) > 0 \\ \eta(\mathfrak{z},\zeta) > 0, \end{cases}$$

and to simplify notations, set

$$\delta_1(\mathfrak{z},\zeta) = min\left\{\varepsilon_1 \ , \ \frac{m}{2b_1}\eta(\mathfrak{z},\zeta)\right\}. \tag{2.4.14}$$

We are led to discuss the following two cases:
First case.
Let the point $(\mathfrak{z},\zeta) \in D \times E_3$ satisfying the condition

$$\|\mathfrak{z}-\zeta\| \leq \delta_1(\mathfrak{z},\zeta). \tag{2.4.15}$$

Under this condition, (2.4.12) and (2.4.13) imply:

$$\left|\sum_{2\leq|\alpha|\leq 2r} \frac{\partial_\zeta^\alpha\varphi}{\alpha!}(\mathfrak{z}-\zeta)^\alpha\right| \leq \left|Re\sum_{j=1}^{n}\frac{\partial\varphi}{\partial\zeta_j}(\zeta_j-\mathfrak{z}_j)\right|. \tag{2.4.16}$$

Let us substitute the decomposition (2.4.11) in (2.4.10). Then by making use of (2.4.16), we deduce first from inequality (2.4.10) that

$$-Re\sum_{j=1}^{n}\frac{\partial\varphi}{\partial\zeta_j}(\zeta_j-\mathfrak{z}_j) > 0 \tag{2.4.17}$$

and then that

$$-Re\sum_{j=1}^{n}\frac{\partial\varphi}{\partial\zeta_j}(\zeta_j - \mathfrak{z}_j) \geq -\frac{1}{3}\varphi(\mathfrak{z}) + \frac{c_1}{6}\|\mathfrak{z}-\zeta\|^{2r}. \tag{2.4.18}$$

It remains now to estimate $-\varphi(\mathfrak{z})$ in terms of $d(\mathfrak{z}, \partial D)$. For this, we know by lemma 2.4.2 that there exists a positive number $0 < \varepsilon_2 < \varepsilon_1$ so small that $\|\mathfrak{z}-\zeta\| \leq \varepsilon_2$, we have

$$-\varphi(\mathfrak{z}) \geq m.d(\mathfrak{z}, \partial D).$$

With the following choice of constants:

$$\varepsilon := \min\left\{\varepsilon_2, \frac{m}{b_2}\right\}, \qquad b := \frac{m}{2b_1}, \qquad c := \min\left\{\frac{m}{3}, \frac{c_1}{6}\right\}$$

we deduce then from (2.4.18), the first part of the proposition, that is :
If the point $(\mathfrak{z}, \zeta) \in D \times E_3$ satisfies the condition

$$\|\mathfrak{z}-\zeta\| \leq min\Big\{\varepsilon \ , \ b.\eta(\mathfrak{z},\zeta)\Big\}, \tag{2.4.19}$$

then

$$-Re\sum_{j=1}^{n}\frac{\partial\varphi}{\partial\zeta_j}(\zeta_j - \mathfrak{z}_j) \geq c\Big[d(\mathfrak{z},\partial D) + \|\mathfrak{z}-\zeta\|^{2r}\Big].$$

Second case.
Let the point $(\mathfrak{z}, \zeta) \in D \times E_3$ satisfying the condition

$$\|\mathfrak{z}-\zeta\| > \delta(\mathfrak{z},\zeta). \tag{2.4.20}$$

where

$$\delta(\mathfrak{z},\zeta) := min\Big\{\varepsilon \ , \ b.\eta(\mathfrak{z},\zeta)\Big\}.$$

By lemma 2.4.3, there exists $\mathfrak{z}^\delta \in D \cap \Delta(\mathfrak{z},\zeta)$ such that $\|\mathfrak{z}^\delta - \zeta\| = \delta(\mathfrak{z},\zeta)$. The point $\mathfrak{z}^\delta$ is defined by

$$\mathfrak{z}^\delta = \left(1 - \frac{t}{\|\mathfrak{z}-\zeta\|}\right)\zeta + \frac{t}{\|\mathfrak{z}-\zeta\|}.\mathfrak{z} \qquad \text{with} \quad |t| = \delta(\mathfrak{z},\zeta).$$

Observe that

$$\mathfrak{z}^\delta - \zeta = \frac{t}{\|\mathfrak{z}-\zeta\|}(\mathfrak{z}-\zeta)$$

and

$$\left\|\mathfrak{z}^\delta - \zeta\right\| = |t| = \delta(\mathfrak{z}, \zeta)$$

which implies that the point $(\mathfrak{z}^\delta, \zeta) \in D \times E_3$ satisfies condition (2.4.19). By applying (2.4.18), we obtain

$$\begin{aligned} -Re \sum_{j=1}^{n} \frac{\partial \varphi}{\partial \zeta_j}(\zeta_j - \mathfrak{z}_j^\delta) &\geq -\frac{1}{3}\varphi(\mathfrak{z}^\delta) + \frac{c_1}{6}\|\mathfrak{z}^\delta - \zeta\|^{2r} \\ &\geq \frac{c_1}{6}\|\mathfrak{z}^\delta - \zeta\|^{2r}. \end{aligned}$$

If we write $\zeta - \mathfrak{z}^\delta = \frac{t}{\|\mathfrak{z}-\zeta\|}(\mathfrak{z} - \zeta)$, then

$$\frac{|t|}{\|\mathfrak{z} - \zeta\|}\left| Re \sum_{j=1}^{n} \frac{\partial \varphi}{\partial \zeta_j}(\zeta_j - \mathfrak{z}_j)\right| \geq \frac{c_1}{6}.|t|^{2r}$$

The above choice of the constant $c := \min\left\{\frac{m}{3}, \frac{c_1}{6}\right\}$, and the fact that $|t| = \delta(\mathfrak{z}, \zeta)$, give

$$\left| Re \sum_{j=1}^{n} \frac{\partial \varphi}{\partial \zeta_j}(\zeta_j - \mathfrak{z}_j)\right| \geq c.\|\mathfrak{z} - \zeta\|.\Big(\delta(\mathfrak{z}, \zeta)\Big)^{2r-1}$$

which proves the second part of the proposition. □

Proposition 2.4.4. *Let*

$$\Omega = \Big\{z \in \mathbb{H}, \quad \psi(z) < 0\Big\}$$

be a $\mathcal{C}^\infty$ *bounded left* $h-$*pseudoconvex open set in* $\mathbb{H}$ *with bounded deviation. With the following notations:*

$$\begin{cases} \varphi = \psi \circ h^{-1} \\ \zeta = h(\xi) \\ \mathfrak{z} = h(z), \end{cases}$$

let

$$D = h(\Omega) = \Big\{\mathfrak{z} \in \mathbb{C}^n, \quad \varphi(\mathfrak{z}) < 0\Big\}$$

and let $r = \mathcal{D}ev(D)$, and $\overrightarrow{N}(\zeta)$ be the normal vector field over ∂D, and for all $\mathfrak{z} \in V_{\overline{D}}$

$$F_{\mathfrak{z}} = \left\{ \zeta \in \partial D, \quad cos\left(\overrightarrow{N}(\zeta), \overrightarrow{\mathfrak{z}\zeta} \right) = 0 \right\}.$$

If $\mu_{\partial D}(F_{\mathfrak{z}}) = 0$, then the $\mathcal{C}^{\infty}-$map $(z,\xi) \longmapsto (w(z,\xi), g(z,\xi)) \in \mathbb{C}^{n+1}$ defined by

$$\begin{cases} w(z,\xi) = -\left(\dfrac{\partial \varphi}{\partial \zeta_1}(\zeta), \cdots, \dfrac{\partial \varphi}{\partial \zeta_n}(\zeta) \right) \\ g(z,\xi) = Re\ w(z,\xi). \left| \left\langle \overrightarrow{N}(\zeta), \overrightarrow{\mathfrak{z}\zeta} \right\rangle \right|^{2r(n+1)} \end{cases}$$

is a Leray section for Ω.

Proof. Since the condition $\mu_{\partial D}(F_{\mathfrak{z}}) = 0$ is fulfilled, then to prove that (w,g) is a Leray section for Ω, we have only to prove that for all $z \in \Omega$

$$\left| \int_{t\in\mathbb{C}} \int_{\xi\in\partial\Omega} \frac{g(z,\xi)}{[N(z,\xi,t)]^{n+1}}\ \mu_{\partial\Omega}(\xi) \otimes \mu(t) \right| < +\infty.$$

For this, let by (2.3.15),

$$\begin{aligned} N(z,\xi,t) &:= \left\langle u(z,\xi,t), \widetilde{w}(z,\xi,t) \right\rangle \\ &= \sum_{j=1}^{n} w_j\big(z,\xi).(h_j(\xi) - h_j(z)\big) + |t|^2 e^{-|t|^2}.g(z,\xi). \end{aligned}$$

where

$$\begin{aligned} u(\mathfrak{z},\zeta,t) &= \big(\zeta - \mathfrak{z}\ ,\ t\big) \ \in \mathbb{C}^{n+1} \\ w(z,\xi) &= -\left(\frac{\partial \varphi}{\partial \zeta_1}(\zeta), \cdots, \frac{\partial \varphi}{\partial \zeta_n}(\zeta) \right) \\ g(z,\xi) &= Re\ w(z,\xi). \left| \left\langle \overrightarrow{N}(\zeta), \overrightarrow{\mathfrak{z}\zeta} \right\rangle \right|^{2r(n+1)}. \end{aligned}$$

We have the following estimates:

$$\begin{aligned} |N(z,\xi,t)| &\geq |ReN(z,\xi,t)| \\ &\geq |Re\ w(\mathfrak{z},\zeta)| \left[1 + |t|^2 e^{-|t|^2} \left| \left\langle \overrightarrow{N}(\zeta), \overrightarrow{\mathfrak{z}\zeta} \right\rangle \right|^{2r(n+1)} \right] \\ &\geq |Re\ w(\mathfrak{z},\zeta)|\,. \end{aligned}$$

observe that

$$\left|\left\langle \overrightarrow{N}(\zeta), \overrightarrow{\mathfrak{z}\zeta} \right\rangle\right|^{2r(n+1)} = \left\|\overrightarrow{N}(\zeta)\right\| . \|\zeta - \mathfrak{z}\| \left|cos\left(\overrightarrow{N}(\zeta), \overrightarrow{\mathfrak{z}\zeta}\right)\right|$$

Since D is pseudoconvex of finite deviation $\mathcal{D}ev(D) = r$, then by the use of (2.4.2) or (2.4.3) of proposition 2.4.1, we obtain:

$$\left|\int_{t\in\mathbb{C}}\int_{\xi\in\partial\Omega} \frac{g(z,\xi)}{[N(z,\xi,t)]^{n+1}}\, \mu_{\partial\Omega}(\xi)\otimes\mu(t)\right|$$
$$< \left|\int_{t\in\mathbb{C}}\int_{\xi\in\partial\Omega} \frac{\left|Re\, w(z,\xi). \left|\left\langle \overrightarrow{N}(\zeta), \overrightarrow{\mathfrak{z}\zeta} \right\rangle\right|^{2r(n+1)}\right|}{[|Rew(z,\xi)|]^{n+1}}\, \mu_{\partial\Omega}(\xi)\otimes\mu(t)\right| < +\infty.$$

□

Theorem 2.4.5. *Let Ω be a left $\mathcal{H}-$pseudoconvex open set of finite deviation $r = \mathcal{D}ev(\Omega)$ of the group $\mathbb{H}$ with $\mathcal{C}^\infty-$boundary. let (w,g) be a Leray section for Ω as defined in proposition 2.4.4. Then there exists a positive number $C > 0$, such that, for any bounded $((p_1,p_2),q)_{\mathcal{H}}$ differential form f on $\partial\Omega$*

$$\|\mathcal{R}_{\partial\Omega} f\|_{\frac{1}{2r},\Omega} \leq C.\, \|f\|_{r,\Omega}\,. \tag{2.4.21}$$

Proof. Write to simplify

$$N_0^2 = \left\|u(z,\xi,t)\right\|^2$$
$$= \left|h(\xi) - h(z)\right|^2 + |t|^2\, e^{-|t|^2}$$

and

$$N = \left\langle u(z,\xi,t), \widetilde{w}(z,\xi,t)\right\rangle$$
$$= \left\langle w(z,\xi), h(\xi) - h(z)\right\rangle + |t|^2\, e^{-|t|^2} g(z,\xi).$$

By definition $\mathcal{R}_{\partial\Omega}$ can be expressed as a determinant, and then

$$\left(\mathcal{R}_{\partial\Omega} f\right)(z)$$
$$= \int_{\partial\Omega\times\mathbb{C}\times[0,1]} g(z,\xi) f(\xi) \wedge \sum_{j=0}^{n-1} p_j(s) det_{1,1,n-j-2,j}\left(\frac{w}{N}, \frac{v}{N_0^2}, \frac{\overline{\partial}_{\mathbb{H}} w}{N}, \frac{\overline{\partial}_{\mathbb{H}} v}{N_0^2}\right) \wedge ds \wedge \omega(u) \wedge \mu(t)$$

where $p_j(s)$ is a polynomial in s.
Further, by multi-linearity of the determinant,

$$\begin{aligned}&\left(\mathcal{R}_{\partial\Omega} f\right)(z)\\&= \int_{\partial\Omega\times\mathbb{C}\times[0,1]} g(z,\xi)f(\xi)\wedge\sum_{j=0}^{n-1} p_j(s)\frac{det_{1,1,n-j-2,j}\left(w,v,\overline{\partial}_{\mathbb{H}}w,\overline{\partial}_{\mathbb{H}}v\right)}{N^{n-j-1}N_0^{2j+2}}\wedge ds\wedge\omega(u)\wedge\mu(t).\end{aligned}$$

Integrating with respect to s, we obtain

$$\begin{aligned}&\left(\mathcal{R}_{\partial\Omega} f\right)(z)\\&= \sum_{j=0}^{n-1} A_j\int_{\partial\Omega\times\mathbb{C}} \frac{g(z,\xi)f(\xi)\wedge det_{1,1,n-j-2,j}\left(w,v,\overline{\partial}_{\mathbb{H}}w,\overline{\partial}_{\mathbb{H}}v\right)}{N^{n-j-1}N_0^{2j+2}}\wedge ds\wedge\omega(u)\wedge\mu(t).\end{aligned}$$

where $A_j = \int_0^1 p_j(s)ds$. Hence the coefficients of the differential form $\left(\mathcal{R}_{\partial\Omega} f\right)(z)$ are linear combinations of integrals of the type

$$F_k(z) = \int_{\partial\Omega\times\mathbb{C}} \frac{g(z,\xi)f_J(\xi)\lambda(z,\xi)}{N^{n-j-1}N_0^{2j+2}}\bigwedge_{j\neq k}\overline{\phi}_j(\xi)\wedge\omega(h(\xi)) \qquad (2.4.22)$$

where $0 \le j \le n-2$, $1 \le k \le n$, f_J is a combination of coefficients of the form f, and $\lambda(z,\xi)$ is a product the functions $w_j(z,\xi)$, $h_j(\xi) - h_j(z)$, and $\overline{\mathcal{Z}}_i(w_j)$, $1 \le i,j \le n$. Since $\lambda(z,\xi)$ contains at least one of the factors $\overline{h_j(z)} - \overline{h_j(\xi)}$, then for some constant $C_1 > 0$, we have

$$|\lambda(z,\xi)| \le C_1\left|h_j(z) - h_\xi\right|.$$

To estimate the integral (2.4.22), we apply proposition .0.8 in appendix. In view of this proposition, it is sufficient to prove for some $C > 0$, and for each $1 \le i \le n$, that

$$\left|\mathcal{Z}_j\left(F_k\right)(z)\right|, \left|\overline{\mathcal{Z}}_j\left(F_k\right)(z)\right| \le C\frac{\|f\|_{0,\Omega}}{\left[d(z,\partial\Omega\right]^{1-\frac{1}{2r}}}. \qquad (2.4.23)$$

Recalling that $N_0^2 = \left|h(\xi) - h(z)\right|^2 + |t|^2 e^{-|t|^2}$, we have

$$\mathcal{Z}_j\left[\frac{g(z,\xi)\lambda(z,\xi)}{N^{n-j-1}N_0^{2j+2}}\right] = \frac{\mathcal{Z}_j(g(z,\xi))\cdot\lambda(z,\xi)}{N^{n-j-1}N_0^{2j+2}} + \frac{g(z,\xi)\cdot\mathcal{Z}_j(\lambda(z,\xi))}{N^{n-j-1}N_0^{2j+2}} + \frac{(j+1)\left(\overline{h_j(\xi)} - \overline{h_j(z)}\right)g(z,\xi)\cdot\lambda(z,\xi)}{N^{n-j-1}N_0^{2j+4}} - \frac{(n-j-1)g(z,\xi)\cdot\lambda(z,\xi)\cdot\mathcal{Z}_j(N)}{N^{n-j}N_0^{2j+2}}\,.$$

Since $\mathcal{Z}_j(g)$, $\mathcal{Z}_j(\lambda)$, are bounded for $(z,\xi)\in\Omega\times\partial\Omega$, and $|\lambda(z,\xi)| \leq C_1\left|h_j(z) - h_\xi\right|$, this implies that for some $C_2 > 0$

$$\left|\mathcal{Z}_j\left[\frac{g(z,\xi)\lambda(z,\xi)}{N^{n-j-1}N_0^{2j+2}}\right]\right| \leq \frac{C_2}{|N|^{n-j-1}N_0^{2j+2}} + \frac{C_2}{|N|^{n-j}N_0^{2j+1}} \tag{2.4.24}$$

An estimates similar to (2.4.24) hold for the differential operator $\overline{\mathcal{Z}}_j$. Hence we can find $C_3 > 0$ such that

$$\left|\mathcal{Z}_j\left(F_k\right)(z)\right|, \left|\overline{\mathcal{Z}}_j\left(F_k\right)(z)\right| \leq C\left\|f\right\|_{0,\Omega}\left[\int_{\partial\Omega}\frac{|g(z,\xi)|\,\mu_{\partial\Omega}}{|N|^{n-j-1}N_0^{2j+2}} + \int_{\partial\Omega}\frac{|g(z,\xi)|\,\mu_{\partial\Omega}}{|N|^{n-j}N_0^{2j+1}}\right]$$

where $\mu_{\partial\Omega}$ is Lebeasgue's measure on $\partial\Omega$. Now set:

$$\begin{cases} \psi = \varphi\circ h^{-1} \\ \zeta = h(\xi) \\ \mathfrak{z} = h(z) \\ D = h(\Omega) \\ \mu_{\partial D} = \left(h^{-1}\right)^*\mu_{\partial\Omega} \\ \eta(\mathfrak{z},\zeta) = \left|cos\left(\overrightarrow{N}(\zeta), \overrightarrow{\mathfrak{z}\zeta}\right)\right| \\ E_{\mathfrak{z}} = \left\{\zeta\in\partial D, \quad \eta(\mathfrak{z},\zeta) > 0\right\} \end{cases}$$

Therefore, to show (2.4.22), it is sufficient using a partition of unity to show that for every $\zeta\in\partial D$, there exists a neighborhood U_ζ of ζ and a real number

$C_\zeta > 0$ such that:

$$\int_{E_\mathfrak{z}\cap U_\zeta} \frac{|g(\mathfrak{z},\zeta)|\,\mu_{\partial D}}{|N|^{n-j-1}\,|\zeta-\mathfrak{z}|^{2j+2}} \le \frac{C_\zeta}{\Big[d(\mathfrak{z},\partial D)\Big]^{1-\frac{1}{2r}}} \tag{2.4.25}$$

and

$$\int_{E_\mathfrak{z}\cap U_\zeta} \frac{|g(\mathfrak{z},\zeta)|\,\mu_{\partial D}}{|N|^{n-j}\,|\zeta-\mathfrak{z}|^{2j+1}} \le \frac{C_\zeta}{\Big[d(\mathfrak{z},\partial D)\Big]^{1-\frac{1}{2r}}}. \tag{2.4.26}$$

Let us prove (2.4.26). For this fix $\mathfrak{z} \in E_\mathfrak{z} \subseteq \partial D$. We know from proposition 2.4.1, that the exist real numbers $b > 0$, $c > 0$, and $1 > \varepsilon > 0$ such that:

- If $(\mathfrak{z},\zeta) \in D \times E_\mathfrak{z}$ satisfies $\big\|\mathfrak{z}-\zeta\big\| \le min\big\{\varepsilon\ ,\ b.\eta(\mathfrak{z},\zeta)\big\}$, then

$$-Re\sum_{j=1}^{n}\frac{\partial\varphi}{\partial\zeta_j}(\zeta_j-\mathfrak{z}_j) \ge c\Big[d(\mathfrak{z},\partial D)+\|\mathfrak{z}-\zeta\|^{2r}\Big].$$

- If $(\mathfrak{z},\zeta) \in D \times E_\mathfrak{z}$ satisfies $\big\|\mathfrak{z}-\zeta\big\| > min\big\{\varepsilon\ ,\ b.\eta(\mathfrak{z},\zeta)\big\}$, then

$$\left|Re\sum_{j=1}^{n}\frac{\partial\varphi}{\partial\zeta_j}(\zeta_j-\mathfrak{z}_j)\right| \ge c\big\|\mathfrak{z}-\zeta\big\|\Big(\eta(\mathfrak{z},\zeta)\Big)^{2r-1}.$$

1) If $\zeta \in E_\mathfrak{z} \cap U_\zeta \subset \partial D$ is such that $\big\|\mathfrak{z}-\zeta\big\| \le min\big\{\varepsilon\ ,\ b.\eta(\mathfrak{z},\zeta)\big\}$, we have then for some positive number $C_3 > 0$

$$\begin{aligned}
\int_{E_\mathfrak{z}\cap U_\zeta} \frac{|g(\mathfrak{z},\zeta)|\,\mu_{\partial D}}{|N|^{n-j}\,|\zeta-\mathfrak{z}|^{2j+1}} &\le C_3.\int_{E_\mathfrak{z}\cap U_\zeta}\frac{\mu_{\partial D}}{\big|d(\mathfrak{z},\partial D)+|\zeta-\mathfrak{z}|^{2r}\big|^{n-j}\,|\zeta-\mathfrak{z}|^{2j+1}}\\
&\le C_4.\int_{\substack{x\in\mathbb{R}^{2n-1}\\ |x|\le R}}\frac{dx_1\wedge\ldots\wedge dx_{2n-1}}{\Big[d(\mathfrak{z},\partial D)+x_1^{2r}\Big]^{n-j}\,x_1^{2j+1}}\\
&\le \frac{C_5}{\Big[d(\mathfrak{z},\partial D)\Big]^{1-\frac{1}{2r}}} \qquad \text{(by proposition (.0.9)).}
\end{aligned}$$

2) If $\zeta \in E_{\mathfrak{z}} \cap U_\zeta \subset \partial D$ is such that $\left\|\mathfrak{z} - \zeta\right\| > min\left\{\varepsilon\ ,\ b.\eta(\mathfrak{z},\zeta)\right\}$, the integral

$$\int_{E_{\mathfrak{z}}\cap U_\zeta} \frac{|g(\mathfrak{z},\zeta)|\,\mu_{\partial D}}{|N|^{n-j}\,|\zeta - \mathfrak{z}|^{2j+1}}$$

is finite, and then there exists C_6, such that

$$\int_{E_{\mathfrak{z}}\cap U_\zeta} \frac{|g(\mathfrak{z},\zeta)|\,\mu_{\partial D}}{|N|^{n-j}\,|\zeta - \mathfrak{z}|^{2j+1}} \leq \frac{C_6}{\left[d(\mathfrak{z},\partial D\right]^{1-\frac{1}{2r}}}.$$

The estimate (2.4.26) is then proved. □

The Hölderian exponent $\frac{1}{2r}$ is the best one possible

We construct an example similar to E.M.Stein's example which shows that the exponent $\frac{1}{r}$ is the best one in theorem 2.4.5.

Example. Following E.M.Stein (see [11]), let $\mathbb{H} = \mathbb{C}^2$ endowed with the group Law

$$(z_1,z_2) * (\xi_1,\xi_2) = \left(z_1 + \xi_1, z_2 + \xi_2 + \frac{1}{2}\big(z_1\overline{\xi}_1 - \xi_1\overline{z}_1\big)\right)$$

The conjugate complex form of structure is

$$\overline{\phi(z_1,z_2)} = \big(d\overline{z}_1, d\overline{z}_2 - \frac{1}{2}\big(\overline{z}_1 dz_1 - z_1 d\overline{z}_1\big)\big)$$

and the left $\mathcal{H}-$holomorphic coordinates of (z_1,z_2) are then

$$h(z_1,z_2) = \left(z_1, z_2 - \frac{1}{2}|z_1|^2\right).$$

Let

$$\Omega := \left\{(z_1,z_2) \in \mathbb{H},\quad |z_1|^{2r} + \left|z_2 - \frac{1}{2}|z_1|^2\right|^{2r} < 1\right\}.$$

Since $h(\Omega)$ is define in $\mathbb{C}^2$ by $|\zeta_1|^{2r} + |\zeta_2|^{2r} < 1$, we check easily that Ω is left $h-$pseudovonvex of bounded deviation, in $\mathbb{H}$, with deviation $\mathcal{D}ev(\Omega) = r$. Let $ln(z_1 - 1)$ where $z_1 \notin [1, +\infty[$, be the branch of the logarithm with

$0 < Arg\big(ln(z_1 - 1)\big) < 2\pi$, and consider in $\overline{\Omega}$ the following $(0,1)_{\mathcal{H}}-$differential form

$$f(z_1, z_2) := \begin{cases} \dfrac{d\overline{z}_2 - \frac{1}{2}\big(\overline{z}_1 dz_1 - z_1 d\overline{z}_1\big)}{ln(z_1 - 1)} & \text{if} \quad (z_1, z_2) \in \overline{\Omega}\backslash(1,0), \\ \qquad\qquad 0 & \text{if} \quad (z_1, z_2) = (1,0). \end{cases}$$

f is trivially $\mathcal{C}^\infty$ in Ω and continuous in $\overline{\Omega}$, and we check easily by the definition of the $\overline{\partial}_{\mathbb{H}}-$operator that $\overline{\partial}_{\mathbb{H}} f = 0$ in Ω.

Proposition 2.4.6. *If $\alpha > \frac{1}{2r}$, then there does not exit a function u in Ω such that $\overline{\partial}_{\mathbb{H}} u = f$ and $\|u\|_\alpha < \infty$.*

Proof. Let u be a solution of $\overline{\partial}_{\mathbb{H}} u = f$ in Ω. An elementary calculus gives $\overline{\partial}_{\mathbb{H}}\left(\frac{\overline{z}_2}{ln(z_1-1)}\right) = f$, and then the function $u(z_1, z_2) - \frac{\overline{z}_2}{ln(z_1-1)}$ is left $\mathcal{H}-$holomorphic in Ω. Let $\varepsilon > 0$ be so small that

$$\left\{(z_1, z_2) \in \mathbb{H}, z_1 = 1 - \varepsilon, \quad \left|z_2 - \frac{1}{2}|z_1|^2\right| = \varepsilon^{\frac{1}{2r}}\right\} \subset \Omega$$

and

$$\left\{(z_1, z_2) \in \mathbb{H}, z_1 = 1 - 2\varepsilon, \quad \left|z_2 - \frac{1}{2}|z_1|^2\right| = \varepsilon^{\frac{1}{2r}}\right\} \subset \Omega.$$

Since $u(z_1, z_2) - \frac{\overline{z}_2}{ln(z_1-1)}$ is left $\mathcal{H}-$holomorphic in Ω, this implies

$$\overline{\mathcal{Z}}_2\left(u(z_1, z_2) - \frac{\overline{z}_2}{ln(z_1 - 1)}\right) = \frac{\partial}{\partial \overline{z}_2}\left(u(z_1, z_2) - \frac{\overline{z}_2}{ln(z_1 - 1)}\right) = 0$$

then the classical Cauchy's formula gives

$$\int_{|z_2 - \frac{1}{2}(1-\varepsilon)^2| = \varepsilon^{\frac{1}{2r}}} u(1 - \varepsilon, z_2) dz_2 = \int_{|z_2 - \frac{1}{2}(1-\varepsilon)^2| = \varepsilon^{\frac{1}{2r}}} \frac{\overline{z}_2 dz_2}{ln(-\varepsilon)} = \frac{2i\pi\varepsilon}{ln(-\varepsilon)}$$

and

$$\int_{|z_2 - \frac{1}{2}(1-2\varepsilon)^2| = \varepsilon^{\frac{1}{2r}}} u(1 - 2\varepsilon, z_2) dz_2 = \int_{|z_2 - \frac{1}{2}(1-2\varepsilon)^2| = \varepsilon^{\frac{1}{2r}}} \frac{\overline{z}_2 dz_2}{ln(-2\varepsilon)} = \frac{2i\pi\varepsilon}{ln(-2\varepsilon)}.$$

Since $\|u\|_\alpha < \infty$ this implies that for some constant $C > 0$,

$$\left|\frac{1}{ln(-\varepsilon)} - \frac{1}{ln(-2\varepsilon)}\right| \leq C\varepsilon^{\alpha - \frac{1}{2r}}$$

which means that

$$\frac{ln2}{|ln(-\varepsilon)ln(-2\varepsilon)|} \leq C\varepsilon^{\alpha-\frac{1}{2r}}.$$

But the last inequality is impossible for $\alpha > \frac{1}{2r}$, and $\varepsilon \longrightarrow 0$. $\square$

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Appendix I. Estimation of some Integrals

We recall in this Appendix without proof (see []) some estimates of some integrals.

Proposition .0.7. *Let $B(0,R)$ be the ball of $\mathbb{R}^n$ of center 0 and radius R with $0 < R < \infty$. Then For every $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n) \in \mathbb{R}^n$, there exists a constant $C > 0$ such that*

$$\int_{B(0,R)} \left| \frac{x_1 - a_1}{||x-a||^n} - \frac{x_1 - b_1}{||x-b||^n} \right| dx_1 \wedge ... \wedge dx_n \leq C||a-b||ln||a-b||.$$

Proposition .0.8. *Let D be a bounded domain of $\mathbb{R}^n$ with $\mathcal{C}^2-$boundary. Then there exists a positive constant $C > 0$ depending only on D with the following property: If $f \in \mathcal{C}^1(D)$ is such that for some $k > 0$ and $0 < \alpha < 1$ we have*

$$||df(x)|| \leq k\,[d(x, \partial D)]^{1-\alpha} \qquad \textit{for all } x \in D,$$

then

$$|f(x) - f(y)| \leq C.k|x-y|^{\alpha} \qquad \textit{for all } x, y \in D.$$

Proposition .0.9. *Let $B(0,R)$ be the ball of $\mathbb{R}^n$ of center 0 and radius R with $0 < R < \infty$. Then there exists a positive constant $C > 0$ such that for all $\varepsilon > 0$*

$$\int_{B(0,R)} \frac{dx_1 \wedge ... \wedge dx_n}{(\varepsilon + ||x||^2)||x||^{n-1}} \leq \frac{C}{\sqrt{\varepsilon}}.$$

Appendix II some Differential operators on the group $\mathbb{H}$

Left invariant metrics

Let g be a Riemannian metric on the group $\mathbb{H}$. We say that g is $\mathcal{H}-$hermitian, if g can be written in terms of the 1-structure forms ϕ and $\overline{\phi}$ as follows

$$g(\mathfrak{Z}) = \sum_{\mu,\nu=1}^{n} g_{\mu,\nu}(\mathfrak{Z}) \;\; \phi_\mu \otimes \overline{\phi}_\nu \tag{.0.27}$$

where $(g_{\mu,\nu})$ is a positive hermitian matrix with $\mathcal{C}^\infty$ coefficients.
The fundamental $(1,1)_{\mathcal{H}}-$form associated to g is the positive $(1,1)_{\mathcal{H}}-$form

$$\omega = -Img = \frac{i}{2} \sum_{\mu,\nu=1}^{n} g_{\mu,\nu}(\mathfrak{Z}) \phi_\mu \wedge \overline{\phi}_\nu.$$

Definition .0.10. The metric g is said to be $\mathcal{H}-$Kähler on $\Omega \subseteq \mathbb{H}$, if $d_{\mathbb{H}}\omega = 0$ or in other words if $d\omega \in \mathcal{J}_3^\infty(\Omega)$.

Proposition .0.11. *Every left invariant Riemannian metric on the group $\mathbb{H}$ is $\mathcal{H}-$Kähler.*

Proof. Let ω be a Riemannian metric on $\mathbb{H}$. Write

$$\omega = \sum_{\mu,\nu=1}^{n} g_{\mu,\nu}(\mathfrak{Z}) \;\; \phi_\mu \wedge \overline{\phi}_\nu.$$

Since ω is left invariant, then the functions $g_{\mu,\nu}(\mathfrak{Z}) = g_{\mu,\nu} \in \mathbb{C}$ are constants. That is

$$\omega = \sum_{\mu,\nu=1}^{n} g_{\mu,\nu} \;\; \phi_\mu \wedge \overline{\phi}_\nu$$

which implies that $d_{\mathbb{H}}\omega = 0$. This competes the proof . □

The space $\mathbf{L}^2_{(p,q)_{\mathcal{H}}}(\Omega)$

Let us consider the open set group $\Omega \subseteq \mathbb{H}$ endowed with the particular $\mathcal{H}-$Kähler metric

$$\omega = -Img = \frac{i}{2}\sum_{\nu=1}^{n} \phi_\nu \wedge \overline{\phi}_\nu.$$

and the Haar measure $d\lambda = \omega^n$. For the forms

$$f = \sum_{|I|=p,|J|=q}^{\prime} f_{I,J}\ \phi_I \wedge \overline{\phi}_J \quad \in \mathcal{C}^\infty_{(p,q)_{\mathcal{H}}}(\Omega)$$

and

$$g = \sum_{|I|=p,|J|=q}^{\prime} g_{I,J}\ \phi_I \wedge \overline{\phi}_J \quad \in \mathcal{C}^\infty_{(p,q)_{\mathcal{H}}}(\Omega)$$

with coefficients $f_{I,J}, g_{I,J} \in \mathbf{L}^2(\Omega)$, we set

$$\langle f, g\rangle_{(p,q)_{\mathcal{H}}} = \sum_{|I|=p,|J|=q}^{\prime} \int_\Omega f_{I,J} \cdot g_{I,J} d\lambda$$

and

$$||f||_{(p,q)_{\mathcal{H}}} = \sqrt{\sum_{|I|=p,|J|=q}^{\prime} \int_\Omega |f_{I,J}|^2 d\lambda}.$$

The Laplace-Beltrami operators $\Box^{\prime}_{\mathbb{H}}$ and $\Box^{\prime\prime}_{\mathbb{H}}$

The $\overline{\partial}_{\mathbb{H}}-$operator defines a linear, closed, densely defined operator T

$$T : D_T \subset \mathbf{L}^2_{(p,q)_{\mathcal{H}}}(\Omega) \longrightarrow \mathbf{L}^2_{(p,q+1)_{\mathcal{H}}}(\Omega)$$

with a domain

$$D_T = \left\{ f \in \mathbf{L}^2_{(p,q)_{\mathcal{H}}}(\Omega); \quad \overline{\partial}_{\mathbb{H}} f \in \mathbf{L}^2_{(p,q+1)_{\mathcal{H}}}(\Omega) \right\}.$$

If $f \in D_T$, we set $T(f) := \overline{\partial}_{\mathbb{H}} f$.

Lemma .0.12. ***(The adjoint operator of $\overline{\partial}_{\mathbb{H}}$)***
If $f = \displaystyle\sum_{|I|=p,|J|=q+1}^{\prime} f_{I,J}\ \phi_I \wedge \overline{\phi}_J \quad \in \quad D(T^*)$, *then*

$$T^*(f) = (-1)^{p-1} \sum_{|I|=p,|K|=q}^{\prime} \sum_{\mu=1}^{n} \mathcal{Z}_\mu\left(f_{I,\mu K}\right)\ \phi_I \wedge \overline{\phi}_K.$$

Proof. Let

$$g = \sum_{|I|=p,|K|=q}^{\prime} g_{I,K}\ \phi_I \wedge \overline{\phi}_K \in \mathbf{D}_{(p,q)_\mathcal{H}}(\Omega).$$

where $\mathbf{D}_{(p,q)_\mathcal{H}}(\Omega)$ is the space of $(p,q)_\mathcal{H}-$differential forms with compact supports. The expression

$$\begin{aligned} \overline{\partial}_{\mathbb{H}}(g) &= \sum_{|I|=p,|K|=q}^{\prime} \overline{\partial}_{\mathbb{H}} g_{I,K} \wedge \phi_I \wedge \overline{\phi}_K \\ &= \sum_{|I|=p,|K|=q}^{\prime} \sum_{\mu=1}^{\mu=n} \overline{\mathcal{Z}}_\mu(g_{I,K})\overline{\phi}_\mu \wedge \phi_I \wedge \overline{\phi}_K \end{aligned}$$

shows that the identity $\langle T^* f, g\rangle_{(p,q)_\mathcal{H}} = \langle f, Tg\rangle_{(p,q)_\mathcal{H}}$, can be written in the form

$$\int_\Omega \sum_{|I|=p,|K|=q}^{\prime} (T^* f)_{I,K}.\overline{g_{I,K}} d\lambda = (-1)^p \int_\Omega \sum_{|I|=p,|K|=q}^{\prime} \left(\sum_{\mu=1}^{\mu=n} f_{I,\mu K}.\overline{\overline{\mathcal{Z}}_\mu(g_{I,K})}\right) d\lambda.$$

Then an integration by parts in the right hand side of the abve equality, gives the expression of T^* in the lemma. □

The operator T^* is the Hilbertian adjoint of the non bounded differential operator $\overline{\partial}_{\mathbb{H}}$ acting on the Hilbert space of square integrable $(p,q)_\mathcal{H}-$differential forms $\mathbf{L}^2_{(p,q)_\mathcal{H}}(\Omega,\varphi)$. We set

$$T^* = \overline{\partial}_{\mathbb{H}}^*$$

Definition .0.13. The self-adjoint differential operator

$$\Box''_{\mathbb{H}} := \overline{\partial}_{\mathbb{H}}\ \overline{\partial}_{\mathbb{H}}^* + \overline{\partial}_{\mathbb{H}}^*\ \overline{\partial}_{\mathbb{H}}$$

is the Laplace-Beltrami operator or the Neumann operator associated to the $\overline{\partial}_{\mathbb{H}}-$operator.
We construct by the same way the Laplace beltrami or Neumann operator associated to the $\partial_{\mathbb{H}}-$operator.

$$\square'_{\mathbb{H}} := \partial_{\mathbb{H}}\ \partial^*_{\mathbb{H}} + \partial^*_{\mathbb{H}}\ \partial_{\mathbb{H}}.$$

List of the main differential operators

Here is a list of differential operators of the hermitian geometry of the group $\mathbb{H}$ and their complex counterparts. $\star$ denotes the Hodge star operator:

$$\begin{cases}
\partial_{\mathbb{H}} & , \quad \partial^{\star}_{\mathbb{H}} = -\star\overline{\partial}_{\mathbb{H}}\star \\
\overline{\partial}_{\mathbb{H}} & , \quad \overline{\partial}^{\star}_{\mathbb{H}} = -\star\partial_{\mathbb{H}}\star \\
d = D_{\mathbb{H}} + \overline{D}_{\mathbb{H}} & , \quad \delta = D^{\star}_{\mathbb{H}} + \overline{D}^{\star}_{\mathbb{H}} \\
d_{\mathbb{H}} = \partial_{\mathbb{H}} + \overline{\partial}_{\mathbb{H}} & , \quad \delta_{\mathbb{H}} = \partial^{\star}_{\mathbb{H}} + \overline{\partial}^{\star}_{\mathbb{H}} \\
\Delta = d\delta + \delta d & , \quad \Delta^* = \Delta \\
\Delta'_{\mathbb{H}} = D_{\mathbb{H}}D^{\star}_{\mathbb{H}} + D^{\star}_{\mathbb{H}}D_{\mathbb{H}} & , \quad (\Delta'_{\mathbb{H}})^{\star} = \Delta'_{\mathbb{H}} \\
\Delta''_{\mathbb{H}} = \overline{D}_{\mathbb{H}}\overline{D}^{\star}_{\mathbb{H}} + \overline{D}^{\star}_{\mathbb{H}}\overline{D}_{\mathbb{H}} & , \quad (\Delta''_{\mathbb{H}})^{\star} = \Delta''_{\mathbb{H}} \\
\square_{\mathbb{H}} = d_{\mathbb{H}}\delta_{\mathbb{H}} + \delta_{\mathbb{H}}d_{\mathbb{H}} & , \quad \square^{\star}_{\mathbb{H}} = \square_{\mathbb{H}} \\
\square'_{\mathbb{H}} = \partial_{\mathbb{H}}\partial^{\star}_{\mathbb{H}} + \partial^{\star}_{\mathbb{H}}\partial_{\mathbb{H}} & , \quad (\square'_{\mathbb{H}})^{\star} = \square'_{\mathbb{H}} \\
\square''_{\mathbb{H}} = \overline{\partial}_{\mathbb{H}}\overline{\partial}^{\star}_{\mathbb{H}} + \overline{\partial}^{\star}_{\mathbb{H}}\overline{\partial}_{\mathbb{H}} & , \quad (\square''_{\mathbb{H}})^{\star} = \square''_{\mathbb{H}}
\end{cases} \tag{.0.28}$$