Abstract

This thesis is devoted to the existence and uniqueness of mild solutions for Semilinear functional differential equations with fractional order involving the Riemann-Liouville fractional derivative. Our approach is based on resolvent operators, the Banach contraction principle, the nonlinear alternative and Schaefer's fixed point theorem type, the technique of measures of noncompactness with Mönch's fixed point theorem , Burton and Kirk's fixed point theorem. Our works will be considered on Banach spaces.

Key words and phrases

Semilinear functional differential equation, fractional derivative, fractional integral, fixed point, mild solutions, resolvent operator, finite delay, infinite delay, state-dependent delay, measures of noncompactness, phase space, perturbed differential equations

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Kheira Mekhalfi.

Contents

Introduction

The traditional integral and derivative are, to say the least, a staple for the technology professional, essential as a means of understanding and working with natural and artificial systems. Fractional Calculus is a field of mathematic study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Consider the physical meaning of the exponent. According to our primary school teachers exponents provide a short notation for what is essentially a repeated multiplication of a numerical value. This concept in itself is easy to grasp and straight forward. However, this physical definition can clearly become confused when considering exponents of non integer value. While almost anyone can verify that $x^3 = x.x.x$, how might one describe the physical meaning of $x^{3.4}$, or moreover the transcendental exponent x^{π}. One cannot conceive what it might be like to multiply a number or quantity by itself 3.4 times, or π times, and yet these expressions have a definite value for any value x, verifiable by infinite series expansion, or more practically, by calculator. Now, in the same way consider the integral and derivative. Although they are indeed concepts of a higher complexity by nature, it is still fairly easy to physically represent their meaning. Once mastered, the idea of completing numerous of these operations, integrations or differentiations follows naturally. Given the satisfaction of a very few restrictions (e.g. function continuity), completing n integrations can become as methodical as multiplication. But the curious mind can not be restrained from asking the question what if n were not restricted to an integer value? Again, at first glance, the physical meaning can become convoluted (pun intended), but as this report will show, fractional calculus fowls quite naturally from our traditional definitions. And just as fractional exponents such as the square root may find their way into innumerable equations and applications, it will become apparent that integrations of order 1/2 and beyond can find practical use in many modern problems.

Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the 20th century. However it is in the past 100 years that the most intriguing leaps in engineering and scientific application have been found. The mathematics has in some cases had to change to meet the requirements of physical reality. Caputo reformulated the more 'classic' definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations. As recently as 1996, Kolowankar reformulated again, the Riemann-Liouville fractional derivative in order to differentiate no-where differentiable fractal functions. Leibniz's response, based on studies over the intervening 300 years, has proven at least half right. It is clear that within the 20th century especially numerous applications and physical manifestations of fractional calculus have been found. However, these applications and the mathematical background surrounding fractional calculus are far from paradoxical. While the physical meaning is difficult (arguably impossible) to grasp, the definitions themselves are no more rigorous than those of their integer order counterparts.

In the last few decades, the subject of fractional differential equations has become a hot topic for the researchers due to its intensive development and applications in the field of physics, mechanics, chemistry, engineering, etc. For a reader interested in the systematic development of the topic, we refer the books Kilbas et al. [27, 40], Miller and Ross [33], Podlubny [37], Oldham et al. [36], Lakshmikantham et al [28] . Differential equations with fractional order have recently proved to be valuable tools for the description of hereditary properties of various materials and systems. Many phenomena in engineering, physics,continuum mechanics, signal processing, electro-magnetics, viscoelasticity, electrochemistry, electromagnetism and science describes efficiently by fractional order differential equations. For more details, see [29]. For some recent developments on the subject, see for instance [1, 2, 5, 6, 26, 35] and references cited therein.

It is well known that one important way to introduce the concept of mild solutions for fractional evolution equations is based on some probability densities and Laplace transform. This method was initialed by El-Borai [20]. For some recent developments see the paper [42], [19], [4]. Another approach to treat abstract equations with fractional derivatives based on the well developed theory of resolvent operators for integral equations [22]. Motivated by the approach in [22], Ye *et al.* [41] studied the existence, uniqueness and continuous dependence of the mild solutions for a class of fractional neutral functional differential equations with infinite delay, by using the Krasnoselskki fixed point theorem and the theory of resolvent operators. The fractional derivative in [41] is understood in the Caputo sense.

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years. For the theory of differential equations with state dependent delay and their applications, we reefer the reader to the papers [9, 17].

Recently in [11, 12], motivated by the approach in [22], we studied fractional order semilinear functional differential equations defined on a compact real interval with finite delay and infinite delay. Existence and uniqueness of solutions are proved, based on the theory of resolvent operators and Banach's contraction principle and Leray-Schauder nonlinear alternative. We emphasize that in [11, 12] the fractional derivative is understood in the Riemann-Liouville sense.

This thesis is devoted to the existence of mild solutions for various types of fractional differential equations, our main tool is resolvent operators, the Banach contraction principle, the nonlinear alternative and Schaefer's fixed point theorem type, Burton and Kirk's fixed point theorem, Mönch's fixed point theorem combined with the technique of measure of noncompactness. We have organized this thesis as follows

In Chapter 1, we introduce the theory of fractional calculus, In Section 1, we give the birth of fractional calculus. In Section 2, we give definition and Proprieties of Gamma function. In Section 3, we give definitions and proprieties of derivations of Riemann-Liouville.

In Chapter 2, we introduce notations and some preliminary notions. In Section 1, we give some notations and definitions from the theory of metric and Banach spaces. In Section 2, we give the theory of resolvent operator. In Section 3, we give some definitions and properties of measure of noncompactness. In Section 4, we give some fixed point theorems

In Chapter 3, we study the existence of mild solutions of semilinear fractional differential equations with finite delay of the form

$$D^{\alpha}y(t) = Ay(t) + f(t, y_t), \quad t \in J := [0, b], \ 0 < \alpha < 1, \tag{1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{2}$$

where D^{α} is the standard Riemann-Liouville fractional derivative, $f : J \times C([-r, 0], E) \to E$ is a continuous function, $A : D(A) \subset E \to E$ is a densely defined closed linear operator on E, $\phi : [-r, 0] \to E$ a given continuous function with $\phi(0) = 0$ and $(E, |\cdot|)$ a real Banach space. In Section 2, we give our main existence results with the Banach contraction principle. In Section 3, we give our main existence results with the nonlinear alternative. An example will be presented in the last section illustrating the abstract theory.

In Chapter 4, we study the existence of mild solutions of semilinear fractional differential equations with infinite delay of the form

$$D^{\alpha}y(t) = Ay(t) + f(t, y_t), \quad t \in J := [0, b], \ 0 < \alpha < 1 \tag{3}$$

$$y_0 = \phi \in \mathcal{B}, \tag{4}$$

where $f : J \times \mathcal{B} \to E$ is a continuous function, $A : D(A) \subset E \to E$ is a densely defined closed linear operator on E, $\phi : \mathcal{B} \to E$ a given continuous function with $\phi(0) = 0$, and $\mathcal{B}$ is called a phase space.

In Section 2, we give definition and Proprieties of phase space. In Section 3, we give our main existence results with the Banach contraction principle. In Section 4, we give our main existence results with the nonlinear alternative and Schaefer's fixed point theorem. An example will be presented in the last section illustrating the abstract theory.

In Chapter 5, we extend such results to the case of state dependent delay by virtue

of resolvent operator and to initiate the application of the technique of measures of non-compactness to investigate the problem of the existence of mild solutions. Especially that technique combined with an appropriate fixed point theorem has proved to be a very useful tool in the study of the existence of solutions for several types of integral and differential equations; see for example [14, 7, 25, 34, 44].
In Section 2, we give our main existence results. An example will be presented in the last section illustrating the abstract theory.

In Chapter 6, we give existence results for various classes of initial value problems for fractional semilinear perturbed functional differential equations. Our results is based upon an application of Burton and Kirk's fixed point theorem.
In Section 2, we will be concerned with semilinear perturbed fractional differential equation with finite delay of the form

$$D^{\alpha}y(t) - Ay(t) = f(t, y_t) + g(t, y_t), \quad t \in J := [0, b],\ 0 < \alpha < 1, \tag{5}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{6}$$

In Section 3, we consider semilinear perturbed fractional differential equation with infinite delay of the form

$$D^{\alpha}y(t) - Ay(t) = f(t, y_t) + g(t, y_t), \quad t \in J := [0, b],\ 0 < \alpha < 1, \tag{7}$$

$$y(t) = \phi \quad \in \mathcal{B}, \tag{8}$$

An example will be presented in the last section illustrating the abstract theory.

Chapter 1

The fractional calculus theory

1.1 Birth of fractional calculus

In a letter dated 30th September 1695, L'Hopital wrote to Leibniz asking him a particular notation that he had used in his publication for the nth-derivative of a function

$$\frac{D^n f(x)}{Dx^n}$$

L'Hopital's posed the question to Leibniz, what would the result be if $n = 1/2$. Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn." In these words fractional calculus was born. Studies over the intervening 300 years have proved at least half right. It is clear that within the twentieth century especially numerous applications have been found. However, these applications and mathematical background surrounding fractional calculus are far from paradoxical.While the physical meaning is difficult to grasp, the definitions are no more rigorous than integer order counterpart.

Following L'Hopital's and Liebniz's first inquisition, fractional calculus was primarily a study reserved for the best minds in mathematics. Fourier, Euler, Laplace are among the many that dabbled with fractional calculus and the mathematical consequences . Many found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most famous of these definitions that have been popularized in the world of fractional calculus (not yet the world as a whole) are the

Riemann-Liouville and Grunwald-Letnikov definition. While the shear number of actual definitions are no doubt as numerous as the men and women that study this field.

Fractional calculus is a generalization of integration and differentiation to non-integer order operator ${}_a\mathrm{D}_t^{\alpha}$, where a and t denote the limits of the operation and α denotes the fractional order such that

$${}_a\mathrm{D}_t^{\alpha} = \begin{cases} \frac{d^{\alpha}}{dt^{\alpha}}, & \Re(\alpha) > 0; \\ 1, & \Re(\alpha) = 0; \\ \int_a^t (dt)^{-\alpha}, & \Re(\alpha) < 0. \end{cases}$$

where generally it is assumed that $\alpha \in \mathbb{R}$, but it may also be a complex number [18]. One of the reasons why fractional calculus is not yet found in elementary texts is a certain degree of controversy found in the theory [33]. This is why there is not a single definition for a fractional-order differintegral operator. Rather there are multiple definitions which may be useful in a specific situation. Further several commonly used definitions of fractional-order operators are presented.

1.2 Gamma function

One of the basic functions of the fractional calculus is Euler's Gamma function. This function generalizes the factorial $n!$ and allows n to take non-integer values.

Definition 1.2.1 *The definition of the gamma function is given by*

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \quad z \in \mathbb{C}.$$

when the real part of z is positive $\Re e(z) > 0$

From this definition it is clear that $\Gamma(z)$ is analytic for $\Re e(z) > 0$. By using integration by parts we find that

$$\Gamma(z+1) = \int_0^{\infty} e^{-t} t^{(z+1)-1} dt = \int_0^{\infty} e^{-t} t^{z} dt \tag{1.1}$$

$$= [-e^{-t} t^{z}]_{t=0}^{t=\infty} + z \int_0^{\infty} e^{-t} t^{z-1} dt \tag{1.2}$$

$$= z\Gamma(z), \quad \Re e(z) > 0. \tag{1.3}$$

Hence we have

$$\Gamma(z+1) = z\Gamma(z), \quad \mathfrak{Re}(z) > 0. \tag{1.4}$$

Further we have

$$\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t}|_0^\infty = 1. \tag{1.5}$$

Combining (1.4) and (1.5), this leads to

$$\Gamma(n+1) = n\Gamma(n) = n!\ , \quad (n = 0, 1, 2, \ldots).$$

The above property is valid for positive values of z. Another important property of the Gamma function is that it has simple poles at $z = 0, -1, -2, -3, ...$

1.3 Riemann-Liouville fractional derivative

In this section we give the definitions of the Riemann-Liouville fractional integrals, fractional derivatives and present some of their properties. More detailed information may be found in this books [33, 36, 37, 27].

Definition 1.3.1 *The fractional (arbitrary) order integral of the function $f \in L^1([a,b])$ of order $\alpha \in \mathbb{R}^+$ is defined by*

$$({}_aI^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

where $\Gamma(.)$ is the Gamma function.

We first define the fractional differintegral operator according to Riemann-Liouville.

Definition 1.3.2

$$({}_aD^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \qquad n-1 \leq \alpha < n$$

where n is integer, α is real number and $\Gamma(.)$ is the Gamma function.

Example 1.3.1 *Calculate the fractional order derivative of the puissance function*

$$\begin{aligned} f:[a,b] &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x)=(x-a)^{\beta}, \quad \beta>0. \end{aligned}$$

In effect after the definition of ${}_aD^{\alpha}$ *we have*

$$({}_aD^{\alpha}f)(x)=\left(\frac{d}{dx}\right)^n({}_aI^{n-\alpha}f)(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_a^x\frac{f(t)}{(x-t)^{\alpha-n+1}}dt$$

$$({}_aI^{n-\alpha}f)(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x(t-a)^{\beta}(x-t)^{-\alpha+n-1}dt$$

By change of variable $t=x-r(x-a)$*, we have* $dt=-(x-a)dr$ *then*

$$\begin{aligned} ({}_aI^{n-\alpha}f)(x) &= -\frac{(x-a)^{-\alpha+n-1}}{\Gamma(n-\alpha)}\int_1^0 r^{-\alpha+n-1}(1-r)^{\beta}(x-a)^{\beta}(x-a)dr \\ &= \frac{(x-a)^{n+\beta-\alpha}}{\Gamma(n-\alpha)}\int_1^0(1-r)^{\beta}r^{-\alpha+n-1}dr \\ &= \frac{(x-a)^{n+\beta-\alpha}B(\beta+1,n-\alpha)}{\Gamma(n-\alpha)} \\ &= \frac{(x-a)^{n+\beta-\alpha}\Gamma(\beta+1)\Gamma(n-\alpha)}{\Gamma(\beta+1-\alpha+n)\Gamma(n-\alpha)} \\ &= \frac{(x-a)^{n+\beta-\alpha}\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha+n)} \\ \left(\frac{d}{dx}\right)^n({}_aI^{n-\alpha}f)(x) &= \frac{\Gamma(\beta+1)(n+\beta-\alpha)(n+\beta-\alpha-1)...(\beta-\alpha+1)}{\Gamma(\beta+1-\alpha+n)}(x-a)^{\beta-\alpha}. \end{aligned}$$

Where $B(p,q)$ *is the Beta function defined by*

$$\begin{aligned} B(p,q) &= \int_0^1 t^{p-1}(1-t)^{q-1}dt \\ &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad (p,q)\in\mathbb{C}^2, \text{ and } \Re e(p)>0, \ \Re e(q)>0. \end{aligned}$$

Since

$$\Gamma(\beta-\alpha+n)=(\beta-\alpha)(\beta-\alpha+1)...(\beta-\alpha+n-1)\Gamma(\beta-\alpha).$$

Obtained

$$\begin{aligned} \left(\frac{d}{dx}\right)^n({}_aI^{n-\alpha}f)(x) &= \frac{\Gamma(\beta+1)\Gamma(\beta-\alpha+n)(n+\beta-\alpha)}{(\beta-\alpha)\Gamma(\beta-\alpha)\Gamma(n+\beta-\alpha+1)}(x-a)^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}. \end{aligned}$$

Hence

$$({}_aD^{\alpha}f)(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}.$$

Remark 1.3.1 *If we take* $\beta = 0$*, we get the following result:*

$$({}_aD^{\alpha}1)(x) = \frac{1}{\Gamma(1-\alpha)}(x-a)^{-\alpha},$$

that is to say that the derivative of Riemann-Liouville of a constant function is no longer zero.

Lemma 1.3.1 *[27] Let* $\alpha \in \mathbb{R}^+$ *and* $n \in \mathbb{N}$ *such that* $n-1 \leq \alpha < n$ *and* $f[a,b] \longrightarrow \mathbb{R}$ *a given function. Suppose that* ${}_aD^{\alpha}f = 0$*. Then*

$$f(x) = \sum_{k=0}^{n-1} c_k \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha-n)}(x-a)^{k+\alpha-n},$$

or c_k *are any constants.*

The fractional-order differentiation has the following properties:

1. If $f(t)$ is an analytic function, then the fractional-order differentiation $({}_0\mathrm{D}^{\alpha}f)(t)$ is also analytic with respect to t.

2. If $\alpha = n$ and $n \in \mathbb{Z}^+$, then the operator ${}_0\mathrm{D}^{\alpha}$ can be understood as the usual operator d^n/dt^n.

3. Operator of order $\alpha = 0$ is the identity operator: $({}_0\mathrm{D}^{\alpha}f)(t) = f(t)$.

4. Fractional-order differentiation is linear; if a, b are constants, then

$${}_0\mathrm{D}^{\alpha}[af(t) + bg(t)] = a({}_0\mathrm{D}^{\alpha}f)(t) + b({}_0\mathrm{D}^{\alpha}g)(t).$$

5. For the fractional-order operators with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and under reasonable constraints on the function $f(t)$ it holds the additive law of exponents:

$${}_0\mathrm{D}^{\alpha}[({}_0\mathrm{D}^{\beta}f)(t)] =_0 \mathrm{D}^{\beta}[({}_0\mathrm{D}^{\alpha}f)(t)] = ({}_0\mathrm{D}^{\alpha+\beta}f)(t)$$

6. The fractional-order derivative commutes with integer-order derivative

$$\frac{d^n}{dt^n}({}_a\mathrm{D}^{\alpha}f)(t) =_a \mathrm{D}^{\alpha}\Big(\frac{d^n f(t)}{dt^n}\Big) = ({}_a\mathrm{D}^{\alpha+n}f)(t),$$

under the condition $t = a$ we have $f^{(k)}(a) = 0, (k = 0, 1, 2, ..., n-1)$.

Chapter 2

Preliminaries

In this Chapter, we introduce notations, definitions, fractional calculus, resolvent operator, measure of noncompactness, lemmas and fixed point theorems which are used throughout this thesis.

2.1 Notations and some definitions

Let $(E, |.|)$ be a Banach space and $J = [0, b]$ be an interval of $\mathbb{R}$. Denote by $C(J, E)$ is the Banach space of all continuous functions from J into E equipped with the norm

$$\|y\|_\infty = \sup\{|y(t)|, t \in [0, b]\}.$$

For $\phi \in C([-r, b], E)$ the norm of ϕ is defined by

$$\|\phi\|_{\mathfrak{D}} = \sup\{|\phi(\theta)| : \theta \in [-r, b]\}.$$

$C([-r, 0], E)$ is endowed with norm defined by

$$\|\psi\|_C = \sup\{|\psi(\theta)| : \theta \in [-r, 0]\}.$$

$B(E)$ denotes the space of bounded linear operators from E into E, with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

$L^1(J,E)$ denotes the Banach space of functions $y : J \to E$ which are Bochner integrable normed by:

$$\|y\|_{L^1} = \int_0^b |y(t)|dt.$$

For properties of the Bochner integral, see for instance, Yosida [45].

Let $L^\infty(J,E)$ be the Banach space of measurable functions $y : J \to E$ which are bounded, equipped with the norm

$$\|y\|_{L^\infty} = \inf\{c > 0 : |y(t)| < c, \ a.e. \ t \in J\}.$$

L^1_{loc} denotes the space of all measurable scalar-valued functions which are integrable over each compact interval.

For a given set V of functions $v : [-r,b] \longrightarrow E$, let us denote by

$$V(t) = \{v(t) : v \in V\}, t \in [-r,b],$$

and

$$V(J) = \{v(t) : v \in V, t \in [-r,b]\}.$$

Definition 2.1.1 *Let E, F two Banach spaces, $f : J \times E \longrightarrow F$ is said to be Carathéodory if*

i) *$t \longmapsto f(t,u)$ is measurable for each $u \in E$;*

ii) *$u \longmapsto F(t,u)$ is continuous for almost each $t \in J$.*

Theorem 2.1.1 *[30] [Arzelà-Ascoli] Let Ω be a closed and bounded (i.e.,compact) domain in E. A set M of continuous functions on Ω is precompact in $C(\Omega)$ if and only if M satisfies the following pair of conditions:*

(i) *M is bounded uniformly . There is a constant c such that for every $f \in M$,*

$$|f(x)| \leq c, \qquad for \ \ all \ \ x \in \Omega$$

(ii) *M is equicontinuous. For any $\epsilon > 0$ there exists $\delta > 0$ dependent on ϵ, such that whenever $|x - y| < \delta,\ x, y \in \Omega$, then*

$$|f(x) - f(y)| < \epsilon \qquad holds\ for\ every\ f \in M.$$

(iii) *$M(x) = \{f(x) : f \in M\}$ precompact in E.*

Definition 2.1.2 *Let E, F two banach spaces, $f : J \times E \longrightarrow F$ is said compact if the image is relatively compact.*
f is said completely continuous if is continuous and the image of every bounded is relatively compact.

2.2 Resolvent operator

Consider the fractional differential equation

$$D^{\alpha}y(t) = Ay(t) + f(t), \quad t \in J,\ 0 < \alpha < 1,\ \ y(0) = 0, \tag{2.1}$$

where D^{α} is the standard Riemann-Liouville fractional derivative and A is a closed linear unbounded operator with domain $D(A)$ defined on a Banach space E and $f \in C(J, E)$. If A is closed then $D(A)$ equipped with the graph norm of A

$$\|x\|_{[D(A)]} = \|x\| + \|Ax\|$$

Equation (2.1) is equivalent to the following integral equation [27]

$$y(t) = \frac{1}{\Gamma(\alpha)}A\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,\ \ t \in J. \tag{2.2}$$

This equation can be written in the following form of integral equation

$$y(t) = h(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Ay(s)ds,\ \ t \geq 0, \tag{2.3}$$

where $h \in C(J, E)$ and

$$h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds. \tag{2.4}$$

Examples where the exact solution of (2.1) and the integral equation (2.2) are the same, are given in [6]. Let us assume that the integral equation (2.3) has an associated resolvent operator $(S(t))_{t\geq 0}$ on E.

Next we define the resolvent operator of the integral equation (2.3).

Definition 2.2.1 *[38, Chapter 1, Definition 1.3] A one parameter family of bounded linear operators $(S(t))_{t\geq 0}$ on E is called a resolvent operator for (2.2) [or solution operator for (2.2)] if the following conditions are satisfied :*

(A) *$S(t)$ is strongly continuous on $\mathbb{R}_+$ (i.e $S(\cdot)x \in C([0,\infty), E)$) and $S(0)x = x$ for all $x \in E$;*

(B) *$S(t)$ commutes with A, which means that $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and every $t \geq 0$;*

(C) *the resolvent equation holds,*

$$S(t)x = x + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} AS(s)x ds, \quad \textit{for every } \ x \in D(A) \ \textit{ and } \ t \geq 0. \tag{2.5}$$

Definition 2.2.2 *[38, Chapter 1, Definition 1.4] A resolvent $(S(t))_{t\geq 0}$ for (2.2) is called differentiable, if $S(.)x \in W^{1,1}_{loc}(\mathbb{R}_+, E)$ for each $x \in D(A)$ and there is $\varphi_A \in L^1_{loc}(\mathbb{R}^+)$ such that*

$$\|S'(t)x\| \leq \varphi_A(t)\|x\|_{[D(A)]} \ \textit{ for all } \ t > 0 \ \textit{ and each } \ x \in D(A).$$

Where $W^{1,1}(\mathbb{R}_+, E)$ is the space of all functions $f : \mathbb{R}_+ \longrightarrow E$ having distributional derivatives.

In the following, we denote by $\Sigma(\omega, \theta)$ the open sector with vertex $\omega \in \mathbb{R}$ and opening angle 2θ in the complex plane which is symmetric w.r.t. the real positive axis, i.e.

$$\Sigma(\omega, \theta) = \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| < \theta\}.$$

Definition 2.2.3 *[38, Chapter 2, Definition 2.1] A resolvent* $(S(t))_{t\geq 0}$ *for* (2.2) *is called analytic, if the function* $S(.) : \mathbb{R}_+ \longrightarrow B(X)$ *admits analytic extension to a sector* $\Sigma(0,\theta_0)$ *for some* $0 < \theta_0 < \pi/2$.

The next result follows from [38, Proposition I.1.2, Theorem II.2.4, Corollary II.2.6].

Lemma 2.2.1 *Under the above conditions the following properties are valid.*

(i) If $u(\cdot)$ *is a mild solution of (2.3) on* J*, then the function* $t \to \int_0^t S(t-s)h(s)ds$ *is continuously differentiable on* J*, and*

$$u(t) = \frac{d}{dt}\int_0^t S(t-s)h(s)ds, \quad \forall t \in J.$$

(ii) If $h \in C^\beta(J,E)$ *for some* $\beta \in (0,1)$*, then the function defined by*

$$u(t) = S(t)(h(t)-h(0)) + \int_0^t S'(t-s)[h(s)-h(t)]ds + S(t)h(0), \quad t \in J,$$

is a mild solution of (2.3) on J*, where* $C^\beta(J,E)$ *represents the space of all* β*-Hölder* E*-valued continuous functions from* J *into* E.

(iii) If $h \in C(J,[D(A)])$ *then the function* $u : J \to E$ *defined by*

$$u(t) = \int_0^t S'(t-s)h(s)ds + h(t), \quad t \in J,$$

is a mild solution of (2.3) on J.

2.3 Measure of noncompactness

The theory of measures of noncompactness has many applications in Topology, Functional analysis and Operator theory (see [7, 32, 39]).

We introduce the following definition:

Definition 2.3.1 *[31] Let* X *be a complete metric space and* Ω_X *denote the class of all bounded subsets of a metric space* X*. A map* $\gamma : \Omega_X \longrightarrow \mathbb{R}_+$ *will be called a measure of noncompactness in* X*, MNC for short, if it satisfies the following conditions for all* $Q, Q_1, Q_2 \in \Omega_X$

1. $\gamma(Q) = \gamma(\overline{Q})$ *(Invariance under closure)*

2. $\gamma(Q) = 0$ *if and only if* Q *is precompact* *(Regularity)*

3. $\gamma(Q_1 \cup Q_2) = \max\{\gamma(Q_1), \gamma(Q_2)\}$ *(Semi.additivity).*

It is easy to see that the following basic results hold for any measure of noncompactness.

Proposition 2.3.1 *[31] Any measure of noncompactness* γ *satisfies following conditions for all* $Q, Q_1, Q_2 \in \Omega_X$

$$Q_1 \subset Q_2 \; implies \; \gamma(Q_1) \leq \gamma(Q_2) \qquad (Monotonicity) \tag{2.6}$$

$$\gamma(Q) = 0 \; for \; every \; finite \; set \; Q \qquad (Non-singularity). \tag{2.7}$$

$$\gamma(Q_1 \cap Q_2) \leq \min\{\gamma(Q_1), \gamma(Q_2)\}. \tag{2.8}$$

If (Q_n) *is a decreasing sequence of nonempty, closed sets in* Ω_X *and*

$$\lim_{n\to\infty} \gamma(Q_n) = 0, \; then \; Q = \bigcap_{n=1}^{\infty} Q_n \neq \emptyset \; is \; compact \tag{2.9}$$

Now we are going to give the definitions of the Kuratowski, Hausdorff and separation measures of noncompactness. We recall the following notations. If S is a subset of a metric space (X, d) then $diam(S) = \sup\{d(s, s') : s, s' \in S\}$ is called the diameter of S, and $B_r(x_0) = \{x \in X : d(x, x_0) < r\}$ denotes the open ball of radius $r > 0$ with centre at $x_0 \in X$. A set B in a metric space (X, d) is said to be r-separated if $d(x, y) \geq r$ for all distinct $x, y \in B$, and the set B is called an r-separation of X.

Definition 2.3.2 *[31] Let* (X, d) *be a complete metric space.*

(a) *The function* $\alpha : \Omega_X \longrightarrow [0, \infty)$ *with*

$$\alpha(Q) = \inf\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} S_i, \; S_i \subset X, \; diam(S_i) \leq \epsilon \;\; for \;\; (i = 1, ..., n \in \mathbb{N})\}.$$

is called the Kuratowski measure of noncompactness.

(b) *The function* $\chi : \Omega_X \longrightarrow [0, \infty)$ *with*

$$\chi(Q) = \inf\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B_{r_k}(x_k),\ x_k \in X,\ r_k < \epsilon\ \ for\ \ (i = 1, ..., n \in \mathbb{N})\}.$$

is called the Hausdorff or ball measure of noncompactness.

(c) *The function* $\beta : \Omega_X \longrightarrow [0, \infty)$ *with*

$$\begin{aligned} \beta(Q) &= \sup\{r > 0 : Q\ \ has\ an\ infinite\ r - separation\ \} \\ &= \inf\{r > 0 : Q\ \ does\ not\ have\ an\ infinite\ r - separation\ \} \end{aligned}$$

is called separation measure of noncompactness.

The functions α, χ and β are measures of noncompactness in the sense of Definition (2.3.1) and so also satisfy (2.6)-(2.9).

If X is a Banach space then the functions α and χ have some additional properties connected with the linear structure of a normed space.

Proposition 2.3.2 *Let* X *be a Banach space,* $Q, Q_1, Q_2 \in \Omega_X$ *and* ψ *be any of the functions* α *or* χ*. Then we have*

1. $\psi(Q_1 + Q_2) \leq \psi(Q_1) + \psi(Q_2)$ *(algebraic semi.additivity),*
2. $\psi(Q + x) = \psi(Q)$ *for each* $x \in X$ *(translation invariance),*
3. $\psi(\lambda Q) = |\lambda|\psi(Q)$ *for each scalar* λ *(semi.homogeneity),*
   and, if $co(Q)$ *denotes the convex hull of* Q*,*
4. $\psi(Q) = \psi(co(Q))$ *(invariance under the passage to the convex hull).*

[31] In the following subsections we give a number of examples of measures of noncompactness in concrete spaces.

2.3.1 Measure of noncompctness in $C([a,b];E)$

$C([a,b];E)$ is the space of continuous functions on the interval $[a,b]$ with values in Banach space E with norm $\|.\|$

Example 2.3.1 *Consider that the Banach space E is separable equipped with the usual sup-norm. In this space the formula of the modulus of equicontinuity of the set of functions $\Omega \subset C([a,b];E)$ has the following form*

$$mod_C(\Omega) = \lim_{\delta \to 0} \sup_{x \in \Omega} \max_{|t_1 - t_2| < \delta} \|x(t_1) - x(t_2)\|$$

$mod_C(\Omega)$ defines an MNC in $C([a,b];E)$.

Example 2.3.2 *Let γ be a monotone MNC on E, and $\|x\| = \max_{t \in [a,b]} \|x(t)\|$. Define a scalar function γ_c on the bounded subsets of $C([a,b];E)$ by the formula*

$$\gamma_c(\Omega) = \gamma(\Omega[a,b])$$

where $\Omega[a,b] = \{x(t) : x \in \Omega,\ t \in [a,b]\}$. γ_c is a measure of noncompactness.

2.3.2 Measure of noncompctness in $C^1([a,b];E)$

Example 2.3.3 *Let $C^1([a,b];E)$ denote the Banach space of the continuously differentiable functions $x : [a,b] \longrightarrow E$, equipped with the norm $\|x\|_{C^1} = \|x\|_C + \|x'\|_C$. The $\mathfrak{M}[a,b]$-valued function C^1 , defined on the bounded subsets of $C^1([a,b];E)$ by the formula*

$$[\gamma_{C^1}(\Omega)](t) = \gamma[\Omega'(t)]$$

where $\Omega'(t) = \{x'(t);\ x \in \Omega\}$, is an MNC.

2.3.3 Measure of noncompctness in $C^n([a,b];E)$

Example 2.3.4 *Let $C^n([a,b];E)$ denote the Banach space of the n-times continuously differentiable functions $x : [a,b] \longrightarrow E$, endowed with the norm $\|x\|_{C^n} = \sum_{i=0}^{n} \|x'\|_{C^{(i)}}$ Then* *each measure of noncompactness* γ *on* $C([a,b];E)$ *generates a measure of noncompactness* γ_{C^n} *on* $C^n([a,b];E)$ *by the rule*

$$\gamma_{C^n}(\Omega) = \gamma(\Omega^{(n)})$$

where $\Omega^{(n)} = \{x^{(n)};\ x \in \Omega\}$.

2.4 Some fixed point theorems

In this section we give some fixed point theorems that will be used in the sequel.

Theorem 2.4.1 *(Banach contraction theorem)[24] Let* X *a Banach space and* $f : X \to X$ *a contraction. Then,* f *has a unique fixed point.*

Theorem 2.4.2 *(Schaefer's theorem)[43, p.29][24] Let* E *a Banach space, and* $U \subset E$ *convex with* $0 \in U$. *Let* $F : U \to U$ *is a completely continuous operator. Then either*

(i) F *has a fixed point, or*

(ii) *The set* $\mathcal{E} = \{x \in U : x = \lambda F(x),\ 0 < \lambda < 1\}$ *is unbounded.*

Theorem 2.4.3 *(Nonlinear alternative of Leray-Schauder) [24, p.135]. Let* E *be a Banach space,* C *a closed, convex subset of* E, U *an open subset of* C *and* $0 \in U$. *Suppose that* $F : \overline{U} \to C$ *is a continuous, compact (that is,* $F(\overline{U})$ *is a relatively compact subset of* C*) map. Then either*

(i) F *has a fixed point in* $\overline{U}$, *or*

(ii) *there is a* $u \in \partial U$ *(the boundary of* U *in* C*) and* $\lambda \in (0,1)$ *with* $u = \lambda F(u)$.

Theorem 2.4.4 *(Burton and Kirk theorem) [16] Let* X *be a Banach space, and A,B two operators satisfying:*

(i) A *is a contraction,*

(ii) B *is completely continuous.*

Then either

(a) *the operator equation* $y = A(y) + B(y)$ *has a solution, or*

(b) *the set* $\mathcal{E} = \{u \in X : u = \lambda A(u/\lambda) + \lambda B(u)\}$ *is unbounded for* $\lambda \in (0, 1)$.

Theorem 2.4.5 *(Mönch's theorem) [3, 34] Let* D *be a bounded, closed and convex subset of a Banach space such that* $0 \in D$*, and let* N *be a continuous mapping of* D *into itself. If the implication*

$$V = \overline{conv} N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V *of* D*, then* N *has a fixed point.*

Chapter 3

Semilinear fractional differential equations with finite delay

3.1 Introduction

This chapter is concerned with existence of mild solutions defined on a compact real interval for fractional order semilinear functional differential equations with finite delay.

In this chapter we consider the following class of semilinear differential equations:

$$D^{\alpha}y(t) = Ay(t) + f(t, y_t), \quad t \in J,\ 0 < \alpha < 1, \tag{3.1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{3.2}$$

where D^{α} is the standard Riemann-Liouville fractional derivative, $f : J \times C([-r, 0], E) \to E$ is a continuous function, $A : D(A) \subset E \to E$ is a densely defined closed linear operator on E, $\phi : [-r, 0] \to E$ a given continuous function with $\phi(0) = 0$ and $(E, |\cdot|)$ a real Banach space.

For any function y defined on $[-r, b]$ and any $t \in J$ we denote by y_t the element of $C([-r, 0], E)$ defined by :

$$y_t(\theta) = y(t + \theta), \qquad \theta \in [-r, 0].$$

Here $y_t(\theta)$ represents the history of the state from time $t - r$, up to the present time t.

In this chapter, we shall prove existence of mild solutions of the problems (3.1)-(3.2). Our approach is based on resolvent operators, the Banach contraction principle, and the nonlinear alternative of Leray-Schauder type. An example will be presented in the last section illustrating the abstract theory.

3.2 Existence of solutions

In this section we give our main existence result for problem (3.1)-(3.2). Before starting and proving this result, we give the definition of its mild solution.

Definition 3.2.1 *We say that a continuous function $y : [-r, b] \to E$ is a mild solution of problem (3.1)-(3.2) if:*

1. $\displaystyle\int_0^t (t-s)^{\alpha-1} y(s)ds \in D(A) \quad \text{for } t \in J,$
2. $y(t) = \phi(t),\ t \in [-r, 0],$ *and*
3. $\displaystyle y(t) = \frac{A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s)ds,\ t \in J.$

Suppose that there exists a resolvent $(S(t))_{t\geq 0}$ which is differentiable and the function f is continuous. Then by Lemma 2.2.1 (iii), if $y : [-r, b] \to E$ is a mild solution of (3.1)-(3.2), then

$$y(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, y_s)ds & \\ + \displaystyle\int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, y_\tau)d\tau \right) ds, & t \in J, \\ \phi(t), & t \in [-r, 0]. \end{cases}$$

Our first existence result for problem (3.1)-(3.2) is based on the Banach's contraction principle.

Theorem 3.2.1 *Let $f : J \times C([-r, 0], E) \to E$ be continuous and there exists a constant $L > 0$ such that*

$$|f(t, u) - f(t, v)| \leq L\|u - v\|_C, \quad \textit{for} \quad t \in J \quad \textit{and} \ \ u, v \in C([-r, 0], E).$$

If

$$\frac{Lb^{\alpha}}{\Gamma(\alpha+1)}\left(1+\|\varphi_A\|_{L^1}\right)<1, \tag{3.3}$$

then the problem (3.1)-(3.2) has a unique mild solution on $[-r,b]$.

Proof. Transform the problem (3.1)-(3.2) into a fixed point problem. Consider the operator $F : C([-r,b],E) \to C([-r,b],E)$ defined by:

$$F(y)(t)=\begin{cases}\phi(t), & t\in[-r,0],\\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f(s,y_s)ds & \\ +\displaystyle\int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}f(\tau,y_\tau)d\tau\right)ds, & t\in[0,b].\end{cases}$$

We need to prove that F has a fixed point, which is a unique mild solution of (3.1)-(3.2) on $[-r,b]$. We shall show that F is a contraction. Let $y,z\in C([-r,b],E)$. For $t\in[0,b]$, we have

$$\begin{aligned}
&|F(y)(t)-F(z)(t)|\\
=\ & \left|\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[f(s,y_s)-f(s,z_s)]ds\right.\\
&\left.+\int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}[f(\tau,y_\tau)-f(\tau,z_\tau]d\tau\right)ds\right|\\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,y_s)-f(s,z_s)|ds\\
&+\int_0^t\varphi_A(t-s)\frac{1}{\Gamma(\alpha)}\int_0^\tau(s-\tau)^{\alpha-1}|f(\tau,y_\tau)-f(\tau,z_\tau)|d\tau ds\\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}L\|y_\tau-z_\tau\|_C ds+\frac{1}{\Gamma(\alpha)}\int_0^t\varphi_A(t-s)\int_0^s(s-\tau)^{\alpha-1}L\|y_\tau-z_\tau\|_C d\tau ds\\
\leq\ & \frac{L}{\Gamma(\alpha)}\|y-z\|_\infty\int_0^t(t-s)^{\alpha-1}ds+\frac{L}{\Gamma(\alpha)}\|y-z\|_\infty\int_0^t\varphi_A(t-s)\int_0^s(s-\tau)^{\alpha-1}d\tau ds\\
\leq\ & \frac{Lb^\alpha}{\Gamma(\alpha+1)}\|y-z\|_\infty+\frac{\|\varphi_A\|_{L^1}Lb^\alpha}{\Gamma(\alpha+1)}\|y-z\|_\infty.
\end{aligned}$$

Taking the supremum over $t\in[-r,b]$, we get

$$\|F(y)-F(z)\|_{\mathfrak{D}}\leq\frac{Lb^\alpha}{\Gamma(\alpha+1)}\left(1+\|\varphi_A\|_{L^1}\right)\|y-z\|_{\mathfrak{D}}.$$

By (3.3) F is a contraction and thus, by the contraction mapping theorem, we deduce that F has a unique fixed point. This fixed point is the mild solution of (3.1)-(3.2). $\square$

Next, we give an existence result based on the Schaefer's Theorem [24].

Theorem 3.2.2 *Let $f : J \times C([-r,0],E) \to E$ be continuous. Assume that:*

(H1) *$S(t)$ is compact for all $t > 0$;*

(H2) *there exist functions $p, q \in C(J, \mathbb{R}_+)$ such that*

$$|f(t,u)| \leq p(t) + q(t)\|u\|_C, \quad t \in J \text{ and } u \in C([-r,0],E).$$

Then, the problem (3.1)-(3.2) has at least one mild solution on $[-r,b]$, provident that

$$\frac{b^\alpha \|q\|_\infty}{\Gamma(\alpha+1)}(1 + \|\varphi_A\|_{L^1}) < 1.$$

Proof. Transform the problem (3.1)-(3.2) into a fixed point problem. Consider the operator $F : C([-r,b],E) \to C([-r,b],E)$ defined in Theorem 3.2.1, namely,

$$F(y)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s,y_s)ds & \\ + \displaystyle\int_0^t S'(t-s)\left(\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^s (s-\tau)^{\alpha-1} f(\tau,y_\tau)d\tau\right)ds, & t \in [0,b]. \end{cases}$$

In order to prove that F is completely continuous, we divide the operator F into two operators:

$$F_1(y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y_s)ds,$$

and

$$F_2(y)(t) = \int_0^t S'(t-s)F_1(y)(s)ds.$$

We prove that F_1 and F_2 are completely continuous. We note that the condition (H1) implies that $S'(t)$ is compact for all $t > 0$ (see [22, Lemma 2.2]).

Step 1: F_1 is completely continuous.

A) F_1 is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ as $n \to \infty$ in $C([-r,b],E)$. Then for $t \in [0,b]$ we have

$$\begin{aligned} |F_1(y_n)(t) - F_1(y)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| f(s, y_{ns}) - f(s, y_s) \Big| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \|f(\cdot, y_{n\cdot}) - f(\cdot, y_{\cdot})\|_\infty \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \frac{b^\alpha}{\Gamma(\alpha+1)} \|f(\cdot, y_{n\cdot}) - f(\cdot, y_{\cdot})\|_\infty. \end{aligned}$$

Since f is a continuous function, we have

$$\|F_1(y_n) - F_1(y)\|_{\mathfrak{D}} \to 0 \ \ as \ \ n \to \infty.$$

Thus F_1 is continuous.

B) F_1 maps bounded sets into bounded sets in $C([-r,b],E)$.

Indeed, it is enough to show that for any $\rho > 0$, there exists a positive constant δ such that for each $y \in B_\rho = \{y \in C([-r,b],E) : \|y\|_{\mathfrak{D}} \leq \rho\}$ one has $F_1(y) \in B_\delta$. Let $y \in B_\rho$. Since f is a continuous function, we have for each $t \in [0,b]$

$$\begin{aligned} |F_1(y)(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big| f(s, y_s) \big| ds \\ &\leq \frac{b^\alpha}{\Gamma(\alpha+1)} \big(\|p\|_\infty + \rho \|q\|_\infty \big) = \delta^* < \infty. \end{aligned}$$

Then, $\|F_1(y)\|_{\mathfrak{D}} = \max\{\|\phi\|_C, \delta^*\} = \delta$, and hence $F_1(y) \in B_\delta$.

C) F_1 maps bounded sets into equicontinuous sets of $C([-r,b],E)$.

Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$ and let $y \in B_\rho$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have

$$\begin{aligned}
& |F_1(y)(\tau_2) - F_1(y)(\tau_1)| \\
= & \left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}(\tau_2 - s)^{\alpha-1}f(s, y_s)ds - \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_1 - s)^{\alpha-1}f(s, y_s)ds\right| \\
\leq & \left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1-\epsilon}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]f(s, y_s)ds\right| \\
& + \left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1-\epsilon}^{\tau_1}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]f(s, y_s)ds\right| \\
& + \left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha-1}f(s, y_s)ds\right| \\
\leq & \frac{\|p\|_\infty + \rho\|q\|_\infty}{\Gamma(\alpha)}\left(\int_0^{\tau_1-\epsilon}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]ds\right. \\
& \left. + \int_{\tau_1-\epsilon}^{\tau_1}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha-1}ds\right).
\end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_1 maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_\rho$ we define

$$F_{1\epsilon}(y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t - s - \epsilon)^{\alpha-1}f(s, y_s)ds.$$

Note that the set

$$\left\{\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t - s - \epsilon)^{\alpha-1}f(s, y_s)ds : y \in B_\rho\right\}$$

is bounded since

$$\begin{aligned}
\left|\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t - s - \epsilon)^{\alpha-1}f(s, y_s)ds\right| & \leq (\|p\|_\infty + \rho\|q\|_\infty)\left|\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t - s - \epsilon)^{\alpha-1}ds\right| \\
& \leq \frac{\|p\|_\infty + \rho\|q\|_\infty}{\Gamma(\alpha+1)}(t - \epsilon)^\alpha.
\end{aligned}$$

Then for $t > 0$, the set

$$Y_\epsilon(t) = \{F_{1\epsilon}(y)(t) : y \in B_\rho\}$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\begin{aligned}\Big|F_1(y)(t) - F_{1\epsilon}(y)(t)\Big| &\le \frac{\|p\|_\infty + \rho\|q\|_\infty}{\Gamma(\alpha)}\Big(\int_0^{t-\epsilon}[(t-s)^{\alpha-1} - (t-s-\epsilon)^{\alpha-1}]ds \\ &\quad + \int_{t-\epsilon}^{t}(t-s)^{\alpha-1}ds\Big) \\ &\le \frac{\|p\|_\infty + \rho\|q\|_\infty}{\Gamma(\alpha+1)}(t^\alpha - (t-\epsilon)^\alpha).\end{aligned}$$

Therefore, the set $Y(t) = \{F_1(y)(t) : y \in B_\rho\}$ is precompact in E. Hence the operator F_1 is completely continuous.

Step 2: F_2 is completely continuous.

A) F_2 is continuous

The operator F_2 is continuous, since $S'(\cdot) \in C([0,b], B(E))$ and F_1 is continuous as proved in Step 1.

B) F_2 maps bounded sets into bounded sets in $C([-r,b],E)$.

let B_ρ be a bounded set as in Step 1. For $y \in B_\rho$ we have

$$\begin{aligned}|F_2(y)(t)| &= \Big|\int_0^t S'(t-s)F_1(y)(s)ds\Big| \\ &\le \int_0^t |S'(t-s)||F_1(y)(s)|ds \\ &\le \int_0^t \varphi_A(t-s)\|F_1(y)(s)\|_{[D(A)]}ds \\ &\le \frac{\|\varphi\|_{L^1} b^\alpha(\|p\|_\infty + \rho\|q\|_\infty)}{\Gamma(\alpha+1)} = \delta'.\end{aligned}$$

Thus, there exists a positive number δ' such that $\|F_2(y)\|_{\mathfrak{D}} \le \delta'$. This means that $F_2(y) \in B_{\delta'}$.

C) F_2 maps bounded sets into equicontinuous sets in $C([-r,b],E)$.

Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$ and let B_ρ be a bounded set as in Step 1. Let $y \in B_\rho$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have

$$\begin{aligned}
& |F_2(y)(\tau_2) - F_2(y)(\tau_1)| \\
= & \left| \int_0^{\tau_2} S'(\tau_2 - s)F_1(y)(\tau_2)ds - \int_0^{\tau_1} S'(\tau_1 - s)F_1(y)(\tau_1)ds \right| \\
\leq & \frac{b^\alpha\big(\|p\|_\infty + \rho\|q\|_\infty\big)}{\Gamma(\alpha+1)}\bigg(\int_0^{\tau_1-\epsilon} |S'(\tau_2 - s) - S'(\tau_1 - s)|\, ds \\
& + \int_{\tau_1-\epsilon}^{\tau_1} |S'(\tau_2 - s) - S'(\tau_1 - s)|\, ds + \int_{\tau_1}^{\tau_2} |S'(\tau_2 - s)| ds\bigg).
\end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_2 maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_\rho$ we define

$$F_{2\epsilon}(y)(t) = S'(\epsilon)\int_0^{t-\epsilon} S'(t - s - \epsilon)F_1(y)(s)ds.$$

Since $S'(t)$ is a compact operator for $t > 0$, the set

$$Y_\epsilon(t) = \{F_{2\epsilon}(y)(t) : y \in B_\rho\}$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\Big|F_2(y)(t) - F_{2\epsilon}(y)(t)\Big| \leq \frac{\|\varphi_A\|_{L^1}\big(\|p\|_\infty + \rho\|q\|_\infty\big)}{\Gamma(\alpha+1)}\big(t^\alpha - (t-\epsilon)^\alpha\big).$$

Then $Y(t) = \{F_2(y)(t) : y \in B_\rho\}$ is precompact in E. Hence the operator F_2 is completely continuous.

Step 3: A priori bound on solutions.

Now, it remains to show that the set

$$\mathcal{E} = \{y \in C([-r, b], E) : y = \lambda F(y), \quad 0 < \lambda < 1\}$$

is bounded.

Let $y \in \mathcal{E}$ be any element. Then, for each $t \in [0, b]$,

$$\begin{aligned}
y(t) = \lambda F(y)(t) \quad = \quad & \lambda\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_s)ds \\
& + \lambda\int_0^t S'(t-s)\bigg(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} f(\tau, y_\tau)d\tau\bigg)ds.
\end{aligned}$$

Then

$$\begin{aligned}
|y(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y_s)ds + \int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} f(\tau,y_\tau)d\tau\right)ds\right| \\
&\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y_s)ds\right| + \left|\int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} f(\tau,y_\tau)d\tau\right)ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} |f(s,y_s)|ds + \int_0^t \varphi_A(t-s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} |f(\tau,y_\tau)|d\tau ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\|p\|_\infty + \|q\|_\infty\|y_s\|_C]ds \\
&\quad + \int_0^t \varphi_A(t-s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\|p\|_\infty + \|q\|_\infty\|y_s\|_C]d\tau ds \\
&\leq \frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{b^\alpha\|q\|_\infty}{\Gamma(\alpha+1)}\|y_s\|_C + \frac{\|\varphi_A\|_{L^1} b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{\|\varphi_A\|_{L^1} b^\alpha\|q\|_\infty}{\Gamma(\alpha+1)}\|y_s\|_C \\
&\leq \frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)}\left(1+\|\varphi_A\|_{L^1}\right) + \frac{b^\alpha\|q\|_\infty}{\Gamma(\alpha+1)}\left(1+\|\varphi_A\|_{L^1}\right)\|y\|_{\mathfrak{D}},
\end{aligned}$$

and consequently

$$\|y\|_{\mathfrak{D}} \leq \frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)}\left(1+\|\varphi_A\|_{L^1}\right)\left\{1 - \frac{b^\alpha\|q\|_\infty}{\Gamma(\alpha+1)}\left(1+\|\varphi_A\|_{L^1}\right)\right\}^{-1}.$$

Hence the set $\mathcal{E}$ is bounded. As a consequence of Theorem 2.4.2 we deduce that F has at least a fixed point which gives rise to a mild solution of problem (3.1)-(3.2) on $[-r,b]$. □

3.3 An example

As an application of our results we consider the following fractional time partial functional differential equation of the form

$$\begin{aligned}
\frac{\partial^\alpha}{\partial t^\alpha}u(t,x) &= \frac{\partial^2}{\partial x^2}u(t,x) + Q(t,u(t-r,x)),\ x\in[0,\pi], t\in[0,b], \alpha\in(0,1), && (3.4)\\
u(t,0) &= u(t,\pi) = 0,\ t\in[0,b], && (3.5)\\
u(t,x) &= \phi(t,x),\quad x\in[0,\pi],\quad t\in[-r,0], && (3.6)
\end{aligned}$$

where $r>0$, $\phi : [-r,0]\times[0,\pi] \to \mathbb{R}$ is continuous and $Q : [0,b]\times\mathbb{R} \to \mathbb{R}$ is a given function.

To study this system, we take $E = L^2[0,\pi]$ and let A be the operator given by $Aw = w''$ with domain $D(A) = \{w \in E, w, w' \text{ are absolutely continuous}, w'' \in E, w(0) = w(\pi) = 0\}$.

Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \qquad w \in D(A),$$

where $(\cdot,\cdot)$ is the inner product in L^2 and $w_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(nx)$, $n = 1, 2, \dots$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on E and is given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} (w, w_n) w_n, \qquad w \in E.$$

From these expressions it follows that $(T(t))_{t\geq 0}$ is uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda - A)^{-1}$ is compact operator for all $\lambda \in \rho(A)$.

From [38, Example 2.2.1] we know that the integral equation

$$u(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s)ds, \ s \geq 0,$$

has an associated analytic resolvent operator $(S(t))_{t\geq 0}$ on E given by

$$S(t) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_{r,\theta}} e^{\lambda t} (\lambda^\alpha - A)^{-1} d\lambda, & t > 0, \\ I, & t = 0, \end{cases}$$

where $\Gamma_{r,\theta}$ denotes a contour consisting of the rays $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ for some $\theta \in (\pi, \frac{\pi}{2})$. $S(t)$ is differentiable (Proposition 2.15 in [5], Theorem 2.2 in [38]) and there exists a constant $M > 0$ such that $\|S'(t)x\| \leq M\|x\|$, for $x \in D(A)$, $t > 0$.

To represent the differential system $(3.4) - (3.6)$ in the abstract form (3.1)-(3.2), let

$$\begin{aligned} y(t)(x) &= u(t,x), \quad t \in [0,b], \ x \in [0,\pi] \\ \phi(\theta)(x) &= \phi(\theta, x), \quad \theta \in [-r, 0], \ x \in [0,\pi] \\ f(t,\phi)(x) &= Q(t, \phi(\theta, x)), \quad \theta \in [-r, 0], \ x \in [0,\pi] \end{aligned}$$

Choose b such that

$$\frac{Lb^\alpha}{\Gamma(\alpha+1)}(1+M) < 1.$$

Since the conditions of Theorem 3.2.1 are satisfied, there is a function $u \in C([-r, b], L^2[0, \pi])$ which is a mild solution of (3.4)-(3.6).

Chapter 4

Semilinear fractional differential equations with infinite delay

4.1 Introduction

In this chapter, we are going to study the existence of mild solution of fractional order differential equations with infinite delay of the form

$$D^{\alpha}y(t) = Ay(t) + f(t, y_t), \quad t \in J, \ 0 < \alpha < 1 \tag{4.1}$$

$$y_0 = \phi \in \mathcal{B}, \tag{4.2}$$

where D^{α} is the standard Riemann-Liouville fractional derivative, $f : J \times \mathcal{B} \to E$ is a continuous function, $A : D(A) \subset E \to E$ is a densely defined closed linear operator on E, $\phi : \mathcal{B} \to E$ a given continuous function with $\phi(0) = 0$ and $(E, |\cdot|)$ a real Banach space. For any function y defined on $(-\infty, b]$ and any $t \in J$, we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $-\infty$ up to the present time t and $\mathcal{B}$ is called a phase space.

4.2 Phase space

In the literature devoted to equations with finite delay, the phase space is much of time the space of all continuous functions on $[-r, 0]$, $r > 0$, endowed with the uniform norm topology. When the delay is infinite, the notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory, a usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato [21]. For detailed discussion on this topic, we refer the reader to the books by Hino *et al.* [23]. For some recent developments on the subject, see for instance [2, 8, 10, 15, 35] and references cited therein.

In all this chapter, we assume that the phase space $(\mathcal{B}, |\cdot|)$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms introduced at first by Hale and Kato in [21]:

(A_1) If $y : (-\infty, b] \to E$, $b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:

- **(i)** $y_t \in \mathcal{B}$,
- **(ii)** $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$,
- **(iii)** $\|y_t\|_{\mathcal{B}} \leq K(t)\sup\{|y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}$,

where $H > 0$ is a constant, $K, M : \mathbb{R}^+ \to \mathbb{R}^+$ with K is continuous and M is locally bounded and H, K, M are independent of $y(\cdot)$.

(A_2) For the function $y(\cdot)$ in (A_1), y_t is a $\mathcal{B}$-valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

Hereafter are some examples of phase spaces. For other details we refer, for instance to the book by Hino *et al.* [23].

Example 4.2.1 *The spaces $\mathcal{BC}$, $\mathcal{BUC}$, $\mathcal{C}^\infty$ and $\mathcal{C}^0$. Let*

- *$\mathcal{BC}$ the space of bounded continuous functions defined from $(-\infty, 0]$ to E,*

- $\mathcal{BUC}$ *the space of bounded uniformly continuous functions defined from* $(-\infty, 0]$ *to* E,

- $\mathcal{C}^\infty = \{\phi \in \mathcal{BC} : \lim_{\theta\to-\infty} \phi(\theta)$ *exists in* $E\}$,

- $\mathcal{C}^0 = \{\phi \in \mathcal{BC} : \lim_{\theta\to-\infty} \phi(\theta) = 0\}$, *endowed with the uniform norm*

$$\|\phi\| = \sup\{|\phi(\theta)| : \theta \leq 0\}.$$

We have that the spaces $\mathcal{BUC}$, $\mathcal{C}^\infty$ *and* $\mathcal{C}^0$ *satisfy conditions* $(A_1) - (A_3)$. $\mathcal{BC}$ *satisfies* (A_2), (A_3) *but* (A_1) *is not satisfied.*

Example 4.2.2 *The spaces* $\mathcal{C}_g$, $\mathcal{UC}_g$, $\mathcal{C}_g^0$ *and* $\mathcal{C}_g^\infty$. *Let* g *be a positive continuous function on* $(-\infty, 0]$. *We define:*

- $\mathcal{C}_g = \{\phi \in \mathcal{C}((-\infty, 0], E) : (\phi(\theta)/g(\theta))$ *is bounded on* $(-\infty, 0]\}$,

- $\mathcal{C}_g^0 = \{\phi \in \mathcal{C}_g : \lim_{\theta\to-\infty}(\phi(\theta)/g(\theta)) = 0\}$ *endowed with the uniform norm*

$$\|\phi\| = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)} : -\infty < \theta \leq 0\right\}.$$

We consider the following condition on the function g.

$$(\mathcal{G}) : \sup_{0\leq t\leq a} \sup\left\{\frac{g(\theta + t)}{g(\theta)} : -\infty < \theta \leq -t\right\} < \infty \text{ for all } a > 0.$$

Then we have that the spaces $\mathcal{C}_g$ *and* $\mathcal{C}_g^0$ *satisfy conditions* (A_3). *They satisfy conditions* (A_1) *and* (A_2) *if* $(\mathcal{G})$ *holds.*

4.3 Existence of solutions

Consider the following space

$$\Omega = \{y : (-\infty, b] \to E : y|_J \in C(J, E) \text{ and } y_0 \in \mathcal{B}\}$$

where $y|_J$ is the restriction of y to J. Let $\|\cdot\|_b$ be the seminorm in Ω defined by:

$$\|y\|_b = \|y_0\|_{\mathcal{B}} + \sup\{|y(s)| : 0 \leq s \leq b\},\ y \in \Omega.$$

In this section we give our main existence results for problem (4.1)-(4.2). This problem is equivalent to the following integral equation

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \dfrac{A}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} y(s) ds + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, y_s) ds, & t \in J. \end{cases}$$

Motivated by Lemma 2.2.1 and the above representation, we introduce the concept of mild solution.

Definition 4.3.1 *One says that a function $y \in \Omega$ is a mild solution of problem (4.1)-(4.2) if:*

1. $\displaystyle\int_0^t (t-s)^{\alpha-1} y(s) ds \in D(A) \quad$ *for* $\quad t \in J,$
2. $y_0 = \phi \ \in \mathcal{B}$ *and*
3. $y(t) = \dfrac{A}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} y(s) ds + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, y_s) ds,\ t \in J.$

Suppose that there exists a resolvent $(S(t))_{t\geq 0}$ which is differentiable and the function f is continuous. Then by Lemma 2.2.1 (iii), if $y : \Omega \to \Omega$ is a mild solution of (4.1)-(4.2), then

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, y_s) ds & \\ + \displaystyle\int_0^t S'(t-s) \left(\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^s (s-\tau)^{\alpha-1} f(\tau, y_\tau) d\tau \right) ds, & t \in J. \end{cases}$$

Our first existence result for problem (4.1)-(4.2) is based on the Banach's contraction principle.

Theorem 4.3.1 *Let $f : J \times \mathcal{B} \to E$ be continuous and there exists a constant $L > 0$ such that*

$$|f(t,u) - f(t,v)| \leq L \|u - v\|_{\mathcal{B}}, \quad for \quad t \in J \quad and \ u, v \in \mathcal{B}.$$

If

$$\frac{LK_b b^\alpha}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1})<1, \tag{4.3}$$

where $K_b=\sup\{|K(t)|:t\in[0,b]\}$, *then the problem (4.1)-(4.2) has a unique mild solution on* $(-\infty,b]$.

Proof. Transform the problem (4.1)-(4.2) into a fixed point problem. Consider the operator $\mathcal{A}:\Omega\to\Omega$ defined by:

$$\mathcal{A}(y)(t)=\begin{cases}\phi(t), & t\in(-\infty,0],\\ \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y_s)ds & \\ +\int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}f(\tau,y_\tau)d\tau\right)ds, & t\in J.\end{cases}$$

Let $x(\cdot):(-\infty,b]\to E$ be the function defined by:

$$x(t)=\begin{cases}\phi(t), & \text{if} \quad t\in(-\infty,0];\\ 0, & \text{if} \quad t\in J.\end{cases}$$

Then $x_0=\phi$. We denote by $\overline{z}$ the function defined by

$$\bar{z}(t)=\begin{cases}0, & \text{if} \quad t\in(-\infty,0];\\ z(t), & \text{if} \quad t\in J.\end{cases}$$

If $y(\cdot)$ satisfies

$$y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y_s)ds+\int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}f(\tau,y_\tau)d\tau\right)ds$$

we can decompose it as $y(t)=\bar{z}(t)+x(t), t\in J$ which implies $y_t=\bar{z}_t+x_t, t\in J$ and the function $z(\cdot)$ satisfies $z_0=0$ and

$$\begin{aligned} z(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,\bar{z}_s+x_s)ds \\ &\quad +\int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}f(\tau,\bar{z}_\tau+x_\tau)d\tau\right)ds. \end{aligned}$$

Let

$$\Omega_0=\{z\in\Omega \ \text{ such that } \ z_0=0\},$$

and let $\|\cdot\|_b$ be the seminorm in Ω_0 defined by

$$\|z\|_b = \|z_0\|_{\mathcal{B}} + \sup\{|z(s)| : 0 \leq s \leq b\} = \sup\{|z(s)| : 0 \leq s \leq b\}, \quad z \in \Omega_0.$$

Then $(\Omega_0, \|\cdot\|_b)$ is a Banach space. Let the operator $F : \Omega_0 \to \Omega_0$ be defined by

$$F(z)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s) ds & \\ + \displaystyle\int_0^t S'(t-s)\left(\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^s (s-\tau)^{\alpha-1} f(\tau, \bar{z}_\tau + x_\tau) d\tau\right) ds, & t \in J. \end{cases}$$

We need to prove that F has a fixed point, which is a unique mild solution of (4.1)-(4.2) on $(-\infty, b]$. We shall show that F is a contraction. Let $z, z^* \in \Omega_0$. Then we have for each $t \in J$,

$$\begin{aligned} & |F(z)(t) - F(z^*)(t)| \\ = & \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f(s, \bar{z}_s + x_s) - f(s, \bar{z}^*_s + x_s)]ds \right. \\ & \left. + \int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^\tau (s-\tau)^{\alpha-1}\big[f(\tau, \bar{z}_\tau + x_\tau) - f(\tau, \bar{z}^*_\tau + x_\tau)\big]d\tau\right)ds\right| \\ \leq & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, \bar{z}_s + x_s) - f(s, \bar{z}^*_s + x_s)|ds \\ & + \int_0^t \varphi_A(t-s)\frac{1}{\Gamma(\alpha)}\int_0^\tau (s-\tau)^{\alpha-1}|f(\tau, \bar{z}_\tau + x_\tau) - f(\tau, \bar{z}^*_\tau + x_\tau)|d\tau ds \\ \leq & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}L\|z_s - z^*_s\|_{\mathcal{B}}ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t \varphi_A(t-s)\int_0^\tau (s-\tau)^{\alpha-1}L\|z_\tau - z^*_\tau\|_{\mathcal{B}}d\tau ds \\ \leq & \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K_b \sup_{s\in[0,t]}|z(s) - z^*(s)|ds \\ & + \frac{L}{\Gamma(\alpha)}\int_0^t \varphi_A(t-s)\int_0^\tau (s-\tau)^{\alpha-1}d\tau K_b \sup_{s\in[0,t]}|z(s) - z^*(s)|ds \\ \leq & \frac{LK_b t^\alpha}{\Gamma(\alpha+1)}\|z - z^*\|_b + \frac{\|\varphi_A\|_{L^1}LK_b t^\alpha}{\Gamma(\alpha+1)}\|z - z^*\|_b. \end{aligned}$$

Taking the supremum over t we get

$$\|F(z) - F(z^*)\|_b \leq \frac{LK_b b^\alpha}{\Gamma(\alpha+1)}(1 + \|\varphi_A\|_{L^1})\|z - z^*\|_b.$$

By (4.3) F is a contraction and thus, by the contraction mapping theorem, we deduce that F has a unique fixed point z. Then $y(t) = \bar{z}(t) + x(t),\ t \in (-\infty, b]$ is a fixed point of the operator $\mathcal{A}$, which gives rise to a unique mild solution of (4.1)-(4.2). ■

Our second existence result is based on Leray-Schauder nonlinear alternative [24, p.135].

Theorem 4.3.2 *Let $f : J \times \mathcal{B} \to E$ be continuous. Assume that:*

(A_1) *$S(t)$ is compact for all $t > 0$;*

(A_2) *there exist a function $p \in C(J, \mathbb{R}^+)$, and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$|f(t,x)| \le p(t)\psi(\|x\|_{\mathcal{B}}), \quad \forall (t,x) \in J \times \mathcal{B};$$

(A_3) *there exists a constant $M > 0$ such that*

$$\frac{M}{K_b\|p\|_\infty \psi(M)\dfrac{b^\alpha}{\Gamma(\alpha+1)}(1 + \|\varphi_A\|_{L^1}) + M_b\|\phi\|_{\mathcal{B}}} > 1.$$

Then, the problem (4.1)-(4.2) has at least one mild solution on $(-\infty, b]$.

Proof. Transform the problem (4.1)-(4.2) into a fixed point problem. Consider the operator $F : \Omega_0 \to \Omega_0$ defined in Theorem 4.3.1, namely,

$$F(z)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s)ds & \\ +\displaystyle\int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} f(\tau, \bar{z}_\tau + x_\tau)d\tau\right)ds, & t \in [0, b]. \end{cases}$$

In order to prove that F is completely continuous, we divide the operator F into two operators:

$$F_1(z)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s)ds,$$

and

$$F_2(z)(t) = \int_0^t S'(t-s)F_1(z)(s)ds.$$

We prove that F_1 and F_2 are completely continuous. We note that the condition (A_1) implies that $S'(t)$ is compact for all $t > 0$ (see [22, Lemma 2.2]).

Step 1: F_1 *is completely continuous.*

A) F_1 is continuous.

Let $\{z_n\}$ be a sequence such that $z_n \to z$ in Ω_0 as $n \to \infty$. Then for $t \in [0, b]$ we have

$$\begin{aligned} |F_1(z_n)(t) - F_1(z)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| f(s, \bar{z}_{ns} + x_s) - f(s, \bar{z}_s + x_s) \Big| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \| f(\cdot, \bar{z}_{n.} + x_.) - f(\cdot, \bar{z}_. + x_.) \|_\infty \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \frac{b^\alpha}{\Gamma(\alpha+1)} \| f(\cdot, \bar{z}_{n.} + x_.) - f(\cdot, \bar{z}_. + x_.) \|_\infty. \end{aligned}$$

Since f is a continuous function, we have

$$\|F_1(z_n) - F_1(z)\|_b \to 0 \ \ as \ \ n \to \infty.$$

Thus F_1 is continuous.

B) F_1 maps bounded sets into bounded sets in Ω_0.

Indeed, it is enough to show that for any $\rho > 0$, there exists a positive constant δ such that for each $z \in B_\rho = \{z \in \Omega_0 : \|z\|_b \leq \rho\}$ one has $F_1(z) \in B_\delta$. Let $z \in B_\rho$. Since f is a continuous function, we have for each $t \in [0, b]$

$$\begin{aligned} |F_1(z)(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big| f(s, \bar{z}_s + x_s) \big| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \psi(\|\bar{z}_s + x_s\|_{\mathcal{B}}) ds \\ &\leq \frac{b^\alpha \psi(\rho^*) \|p\|_\infty}{\Gamma(\alpha+1)} = \delta < \infty, \end{aligned}$$

where

$$\begin{aligned}
\|\bar{z}_s + x_s\|_{\mathcal{B}} &\leq \|\bar{z}_s\|_{\mathcal{B}} + \|x_s\|_{\mathcal{B}}\\
&\leq K(t)\sup\{|z(t)| : 0 \leq s \leq t\} + M(t)\|z_0\|_{\mathcal{B}}\\
&\quad + K(t)\sup\{|x(t)| : 0 \leq s \leq t\} + M(t)\|x_0\|_{\mathcal{B}}\\
&\leq K(t)\sup\{|z(t)| : 0 \leq s \leq t\} + M(t)\|x_0\|_{\mathcal{B}}\\
&\leq K_b\rho + M_b\|\phi\|_{\mathcal{B}} = \rho^*,
\end{aligned}$$

and $M_b = \sup\{|M(t)| : t \in [0, b]\}$.

Then, $\|F_1(z)\|_b \leq \delta$, and hence $F_1(z) \in B_\delta$.

C) F_1 maps bounded sets into equicontinuous sets of Ω_0.

Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$ and let B_ρ be a bounded set. Let $z \in B_\rho$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have

$$\begin{aligned}
&|F_1(z)(\tau_2) - F_1(z)(\tau_1)|\\
= &\left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}(\tau_2 - s)^{\alpha-1}f(s, \bar{z}_s + x_s)ds - \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_1 - s)^{\alpha-1}f(s, \bar{z}_s + x_s)ds\right|\\
\leq &\left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1-\epsilon}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]f(s, \bar{z}_s + x_s)ds\right|\\
&+\left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1-\epsilon}^{\tau_1}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]f(s, \bar{z}_s + x_s)ds\right|\\
&+\left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha-1}f(s, \bar{z}_s + x_s)ds\right|\\
\leq &\frac{\|p\|_\infty\psi(\rho^*)}{\Gamma(\alpha)}\left(\int_0^{\tau_1-\epsilon}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]ds\right.\\
&\left. + \int_{\tau_1-\epsilon}^{\tau_1}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha-1}ds\right).
\end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_1 maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_\rho$

we define

$$F_{1\epsilon}(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} f(s, \bar{z}_s + x_s) ds.$$

Note that the set

$$\left\{ \frac{1}{\Gamma(\alpha)} \int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} f(s, \bar{z}_s + x_s) ds : z \in B_\rho \right\}$$

is bounded since

$$\begin{aligned} & \left| \frac{1}{\Gamma(\alpha)} \int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} f(s, \bar{z}_s + x_s) ds \right| \\ \leq \; & \|p\|_\infty \psi(\rho^*) \left| \frac{1}{\Gamma(\alpha)} \int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} ds \right| \\ \leq \; & \frac{\|p\|_\infty \psi(\rho^*)}{\Gamma(\alpha+1)} (t-\epsilon)^\alpha. \end{aligned}$$

Then for $t > 0$, the set

$$Z_\epsilon(t) = \{F_{1\epsilon}(z)(t) : z \in B_\rho\}$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\begin{aligned} & \left| F_1(z)(t) - F_{1\epsilon}(z)(t) \right| \\ \leq \; & \frac{\|p\|_\infty \psi(\rho^*)}{\Gamma(\alpha)} \left(\int_0^{t-\epsilon} [(t-s)^{\alpha-1} - (t-s-\epsilon)^{\alpha-1}] ds + \int_{t-\epsilon}^t (t-s)^{\alpha-1} ds \right) \\ \leq \; & \frac{\|p\|_\infty \psi(\rho^*)}{\Gamma(\alpha+1)} \left(t^\alpha - (t-\epsilon)^\alpha \right). \end{aligned}$$

Therefore, the set $Z(t) = \{F_1(z)(t) : z \in B_\rho\}$ is precompact in E. Hence the operator F_1 is completely continuous.

Step 2: F_2 *is completely continuous.*

A) F_2 is continuous

The operator F_2 is continuous, since $S'(\cdot) \in C(J, B(E))$ and F_1 is continuous as proved in Step 1.

B) F_2 maps bounded sets into bounded sets in Ω_0.

let B_ρ be a bounded set as in Step 1. For $z \in B_\rho$ we have

$$\begin{aligned}|F_2(z)(t)| &\leq \int_0^t |S'(t-s)||F_1(z)(s)|ds\\ &\leq \int_0^t \varphi_A(t-s)\|F_1(z)(s)\|_{[D(A)]}ds\\ &\leq \frac{\|\varphi_A\|_{L^1} b^\alpha \|p\|_\infty \psi(\rho^*)}{\Gamma(\alpha+1)} = \delta'.\end{aligned}$$

Thus, there exists a positive number δ' such that $\|F_2(z)\|_b \leq \delta'$. This means that $F_2(z) \in B_{\delta'}$.

C) F_2 maps bounded sets into equicontinuous sets in Ω_0.

Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$ and let B_ρ be a bounded set as in Step 1. Let $z \in B_\rho$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have

$$\begin{aligned}&|F_2(z)(\tau_2) - F_2(z)(\tau_1)|\\ =& \left| \int_0^{\tau_2} S'(\tau_2 - s)F_1(z)(\tau_2)ds - \int_0^{\tau_1} S'(\tau_1 - s)F_1(z)(\tau_1)ds \right|\\ \leq& \frac{b^\alpha \|p\|_\infty \psi(\rho^*)}{\Gamma(\alpha+1)} \bigg(\int_0^{\tau_1 - \epsilon} |S'(\tau_2 - s) - S'(\tau_1 - s)|\, ds\\ &+ \int_{\tau_1-\epsilon}^{\tau_1} |S'(\tau_2 - s) - S'(\tau_1 - s)|\, ds + \int_{\tau_1}^{\tau_2} |S'(\tau_2 - s)|ds \bigg).\end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_2 maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_\rho$ we define

$$F_{2\epsilon}(z)(t) = S'(\epsilon) \int_0^{t-\epsilon} S'(t - s - \epsilon)F_1(z)(s)ds.$$

Since $S'(t)$ is a compact operator for $t > 0$, the set

$$Z_\epsilon(t) = \{F_{2\epsilon}(z)(t) : z \in B_\rho\}$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\Big|F_2(z)(t) - F_{2\epsilon}(z)(t)\Big| \leq \frac{\|\varphi_A\|_{L^1} \|p\|_\infty \psi(\rho^*)}{\Gamma(\alpha+1)} \Big(t^\alpha - (t-\epsilon)^\alpha\Big).$$

Then $Z(t) = \{F_2(z)(t) : z \in B_\rho\}$ is precompact in E. Hence the operator F_2 is completely continuous.

Step 3: *We show there exists an open set $U \subset C(J, E)$ with $z \notin \lambda F(z)$ for $\lambda \in (0,1)$ and $z \in \partial U$.*

Let $\lambda \in (0,1)$ and

$$\begin{aligned} z(t) = \lambda F(z)(t) &= \lambda \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s) ds \\ &+ \lambda \int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, \bar{z}_\tau + x_\tau) d\tau \right) ds. \end{aligned}$$

Then

$$\begin{aligned} |z(t)| &\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s) ds \right. \\ &+ \left. \int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, \bar{z}_\tau + x_\tau) d\tau \right) ds \right| \\ &\leq \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} |f(s, \bar{z}_s + x_s)| ds \\ &+ \int_0^t \frac{\varphi_A(t-s)}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} |f(\tau, \bar{z}_\tau + x_\tau)| d\tau ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \psi(\|\bar{z}_s + x_s\|_{\mathcal{B}}) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \varphi_A(t-s) \int_0^s (s-\tau)^{\alpha-1} p(s) \psi(\|\bar{z}_s + x_s\|_{\mathcal{B}}) d\tau ds. \end{aligned} \tag{4.4}$$

But

$$\|\bar{z}_s + x_s\|_{\mathcal{B}} \leq K_b \sup\{|z(s)| : 0 \leq s \leq t\} + M_b \|\phi\|_{\mathcal{B}}$$

as proved in Step 1. If we let $w(t)$ be the right-hand side of the above inequality then we have that

$$\|\bar{z}_s + x_s\|_{\mathcal{B}} \leq w(t), \quad t \in J,$$

and therefore (4.4) becomes

$$\begin{aligned} |z(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \psi(w(s)) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \varphi_A(t-s) \int_0^s (s-\tau)^{\alpha-1} p(s) \psi(w(s)) d\tau ds. \end{aligned} \tag{4.5}$$

Using (4.5) in the definition of w, we have

$$\begin{aligned} w(t) &= K_b \sup\{|z(s)| : 0 \le s \le t\} + M_b\|\phi\|_{\mathcal{B}} \\ &\le K_b \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)\psi(w(s))ds \\ &\quad + K_b \frac{1}{\Gamma(\alpha)} \int_0^t \varphi_A(t-s) \int_0^s (s-\tau)^{\alpha-1} p(s)\psi(w(s))d\tau ds + M_b\|\phi\|_{\mathcal{B}}. \end{aligned}$$

Then

$$\|w\| \le K_b\|p\|_\infty \psi(\|w\|)\frac{b^\alpha}{\Gamma(\alpha+1)} + K_b\|p\|_\infty \psi(\|w\|)\frac{b^\alpha}{\Gamma(\alpha+1)}\|\varphi_A\|_{L^1} + M_b\|\phi\|_{\mathcal{B}}$$

and consequently

$$\frac{\|w\|}{K_b\|p\|_\infty \psi(\|w\|)\dfrac{b^\alpha}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1}) + M_b\|\phi\|_{\mathcal{B}}} \le 1.$$

Thus, by (A_3), there exists M such that $\|w\| \ne M$. Let us set

$$U = \{z \in C(J,E) : \|z\| < M\}.$$

From the choice of U, there is no $z \in \partial U$ such that $z = \lambda F(y)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 2.4.3), we deduce that F has a fixed point $z \in \overline{U}$. Then $y = \overline{z} + x$ is a solution of (4.1)-(4.2) on $(-\infty, b]$. This completes the proof. ■

Finally, we give an existence result based upon Schaefer's fixed point theorem.

Theorem 4.3.3 *Let $f : J \times \mathcal{B} \to E$ be continuous. Assume that:*

(B_1) *$S(t)$ is compact for all $t > 0$;*

(B_2) *there exist functions $p, q \in C(J, \mathbb{R}_+)$ such that*

$$|f(t,u)| \le p(t) + q(t)\|u\|_{\mathcal{B}}, \quad t \in J \text{ and } u \in \mathcal{B}.$$

Then, the problem (4.1)-(4.2) has at least one mild solution on $(-\infty, b]$, provident that

$$\frac{b^\alpha K_b\|q\|_\infty}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1}) < 1.$$

Proof. Define F as in the proof of Theorem 4.3.1. As in Theorem 4.3.2 we can prove that F is completely continuous. Here we prove that the set

$$\mathcal{E}=\{z\in\Omega_0 : z=\lambda F(z),\quad 0<\lambda<1\}$$

is bounded.

Let $z\in\mathcal{E}$ be any element. Then, for each $t\in[0,b]$,

$$\begin{aligned}
|z(t)| &\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^t p(s)(t-s)^{\alpha-1}ds+\int_0^t(t-s)^{\alpha-1}q(s)\Big[K_b\|z\|_b+M_b\|\phi\|_{\mathcal{B}}\Big]ds\right)\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t\varphi_A(t-s)\int_0^s(s-\tau)^{\alpha-1}\left(p(s)+q(s)\Big[K_b\|z\|_b+M_b\|\phi\|_{\mathcal{B}}\Big]\right)d\tau ds\\
&\leq \frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)}+\frac{b^\alpha\|q\|_\infty}{\Gamma(\alpha+1)}\Big[K_b\|z\|_b+M_b\|\phi\|_{\mathcal{B}}\Big]+\frac{b^\alpha\|\varphi_A\|_{L^1}\|p\|_\infty}{\Gamma(\alpha+1)}\\
&+\frac{b^\alpha\|\varphi_A\|_{L^1}\|q\|_\infty}{\Gamma(\alpha+1)}\Big[K_b\|z\|_b+M_b\|\phi\|_{\mathcal{B}}\Big]\\
&= \frac{b^\alpha}{\Gamma(\alpha+1)}\Bigg[\|p\|_\infty(1+\|\varphi_A\|_{L^1})+\|q\|_\infty\|\phi\|_{\mathcal{B}}M_b(1+\|\varphi_A\|_{L^1})\Bigg]\\
&+\frac{b^\alpha K_b\|q\|_\infty}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1})\|z\|_b\\
&= \frac{b^\alpha}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1})(\|p\|_\infty+\|q\|_\infty\|\phi\|_{\mathcal{B}}M_b)\\
&+\frac{b^\alpha K_b\|q\|_\infty}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1})\|z\|_b
\end{aligned}$$

and consequently

$$\|z\|_b\leq\frac{b^\alpha}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1})(\|p\|_\infty+\|q\|_\infty\|\phi\|_{\mathcal{B}}M_b)\left\{1-\frac{b^\alpha K_b\|q\|_\infty}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1})\right\}^{-1}.$$

Hence the set $\mathcal{E}$ is bounded. As a consequence of Theorem 2.4.3 we deduce that F has at least a fixed point, then the operator $\mathcal{A}$ has one, which gives rise to a mild solution of (4.1)-(4.2) on $(-\infty,b]$. ■

4.4 An example

As an application of our results we consider the following fractional time partial functional differential equation of the form

$$
\begin{aligned}
\frac{\partial^\alpha}{\partial t^\alpha}u(t,x) &= \frac{\partial^2}{\partial x^2}u(t,x) + \int_{-\infty}^{0} P(\theta)g\big(t,u(t+\theta,x)\big)d\theta, \\
&\qquad x\in[0,\pi],\ t\in[0,b],\ 0<\alpha<1, &(4.6)\\
u(t,0) &= u(t,\pi)=0,\ t\in[0,b], &(4.7)\\
u(t,x) &= u_0(t,x),\quad x\in[0,\pi],\quad t\in(-\infty,0], &(4.8)
\end{aligned}
$$

where $P:(-\infty,0]\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ and $u_0:(-\infty,0]\times[0,\pi]\to\mathbb{R}$ are continuous functions. To study this system, we take $E=L^2[0,\pi]$ and let A be the operator given by $Aw=w''$ with domain $D(A)=\{w\in E, w, w'$ are absolutely continuous, $w''\in E, w(0)=w(\pi)=0\}$.

Then

$$
Aw=\sum_{n=1}^{\infty} n^2(w,w_n)w_n, \qquad w\in D(A),
$$

where $(\cdot,\cdot)$ is the inner product in L^2 and $w_n(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin(nx)$, $n=1,2,\dots$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on E and is given by

$$
T(t)w=\sum_{n=1}^{\infty} e^{-n^2t}(w,w_n)w_n, \qquad w\in E.
$$

From these expressions it follows that $(T(t))_{t\geq 0}$ is uniformly bounded compact semigroup, so that $R(\lambda,A)=(\lambda-A)^{-1}$ is compact operator for all $\lambda\in\rho(A)$.

From [38, Example 2.2.1] we know that the integral equation

$$
u(t)=h(t)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Au(s)ds,\ s\geq 0,
$$

has an associated analytic resolvent operator $(S(t))_{t\geq 0}$ on E given by

$$
S(t)=\begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_{r,\theta}} e^{\lambda t}(\lambda^\alpha-A)^{-1}d\lambda, & t>0,\\ I, & t=0,\end{cases}
$$

where $\Gamma_{r,\theta}$ denotes a contour consisting of the rays $\{re^{i\theta}: r\geq 0\}$ and $\{re^{-i\theta}: r\geq 0\}$ for some $\theta\in(\pi,\frac{\pi}{2})$. $S(t)$ is differentiable (Proposition 2.15 in [5]) and there exists a constant $M>0$ such that $\|S'(t)x\|\leq M\|x\|$, for $x\in D(A)\,t>0$.

For the phase space $\mathcal{B}$, we choose the well-known space $\mathcal{BUC}(\mathbb{R}^-, E)$ of uniformly bounded continuous functions equipped with the following norm:

$$\|\varphi\| = \sup_{\theta \leq 0} |\varphi(\theta)| \text{ for } \varphi \in \mathcal{B}.$$

To represent the system (4.6)-(4.8) in the abstract form (4.1)-(4.2) we consider $\varphi \in \mathcal{BUC}(\mathbb{R}^-, E)$, $x \in [0, \pi]$ and introduce the functions

$$\begin{aligned} y(t)(x) &= u(t,x), \quad t \in [0,b], x \in [0,\pi], \\ \phi(\theta)(x) &= u_0(\theta,x), \quad -\infty < \theta \leq 0, x \in [0,\pi], \\ f(t,\varphi)(x) &= \int_{-\infty}^{0} P(\theta) g\big(t, \varphi(\theta)(x)\big) d\theta, \quad -\infty < \theta \leq 0, x \in [0,\pi]. \end{aligned}$$

Then the problem (4.6)-(4.8) takes the following abstract form:

$$\begin{cases} D^\alpha y(t) &= Ay(t) + f(t, y_t), \quad t \in J = [0,b], \quad 0 < \alpha < 1; \\ y_0 &= \phi \in \mathcal{B}. \end{cases} \tag{4.9}$$

We assume the following assumptions:

(i) P is integrable on $(-\infty, 0]$.

(ii) There exist a continuous increasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$|g(t,v)| \leq \psi(|v|), \quad \text{for} \quad v \in \mathbb{R}.$$

By the dominated convergence theorem of Lebesgue, we can show that f is a continuous function of $\mathcal{B}$ in E. On the other hand, we have for $\varphi \in \mathcal{B}$ and $x \in [0, \pi]$

$$|f(t,\varphi)(x)| \leq \int_{-\infty}^{0} |P(\theta)| g\big(t, |\varphi(\theta)(x)|\big) d\theta.$$

Since the function ψ is increasing, we have

$$|f(t,\varphi)| \leq \int_{-\infty}^{0} |P(\theta)| d\theta \psi\big(\|\varphi\|_{\mathcal{B}}\big) \text{ for } \varphi \in \mathcal{B}.$$

Choose b such that

$$\frac{Lb^\alpha}{\Gamma(\alpha+1)}(1+M) < 1.$$

Since the conditions of Theorem 4.3.1 are satisfied, there is a function $u \in C((-\infty, b], L^2[0,\pi])$ which is a mild solution of (4.6)-(4.8).

Chapter 5

Semilinear fractional differential equations with State-Dependent Delay

5.1 Introduction

This chapter is concerned with existence of mild solutions defined on a compact real interval for fractional order semilinear functional differential equations with state-dependent delay of the form

$$D^{\alpha}y(t) = Ay(t) + f(t, y(t - \rho(y(t)))) \quad , t \in J = [0, b], 0 < \alpha < 1 \tag{5.1}$$

$$y(t) = \phi(t) \quad , t \in [-r, 0] \tag{5.2}$$

where D^{α} is the standard Riemann-Liouville fractional derivative, $f : J \times C([-r, 0], E) \rightarrow E$ is a continuous function, $A : D(A) \subset E \rightarrow E$ is a densely defined closed linear operator on E. $\phi : [-r, 0] \rightarrow E$ a given continuous function with $\phi(0) = 0$ and $(E, |.|)$ a real Banach space. ρ is a positive bounded continuous function on $C([-r, 0], E)$. r is the maximal delay defined by

$$r = \sup_{y \in C} \rho(y).$$

5.2 Existence of solutions

In this section we give our main existence results for problem (5.1)-(5.2). This problem is equivalent to the following integral equation

$$y(t)=\begin{cases}\dfrac{A}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}y(s)ds+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f(s,y(s-\rho(y(s))))ds, & t\in J,\\ \phi(t), & t\in[-r,0].\end{cases}$$

Motivated by Lemma 2.2.1 and the above representation, we introduce the concept of mild solution.

Definition 5.2.1 *We say that a continuous function $y:[-r,b]\to E$ is a mild solution of problem (5.1)-(5.2) if:*

1. $\displaystyle\int_0^t (t-s)^{\alpha-1}y(s)ds\in D(A)$ *for* $t\in J$,
2. $y(t)=\phi(t)$, $t\in[-r,0]$, *and*
3. $y(t)=\dfrac{A}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}y(s)ds+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f(s,y(s-\rho(y(s))))ds$, $t\in J$.

Suppose that there exists a resolvent $(S(t))_{t\geq 0}$ which is differentiable and the function f is continuous. Then by Lemma 2.2.1 (iii), if $y:[-r,b]\to E$ is a mild solution of (5.1)-(5.2), then

$$y(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f(s,y(s-\rho(y(s))))ds & \\ +\displaystyle\int_0^t S'(t-s)\left(\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^s (s-\tau)^{\alpha-1}f(\tau,y(\tau-\rho(y(\tau))))d\tau\right)ds, & t\in J,\\ \phi(t), & t\in[-r,0].\end{cases}$$

Lemma 5.2.1 *[44] Let D be a bounded, closed and convex subset of the Banach space $C(J,E)$, G a continuous function on $J\times J$ and f a function from $J\times C([-r,0],E)\longrightarrow E$ which satisfies the Carathéodory conditions and there exists $p\in L^1(J,\mathbb{R}_+)$ such that for each $t\in J$ and each bounded set $B\subset C([-r,0],E)$ we have*

$$\lim_{k\to 0^+}\alpha(f(J_{t,k}\times B))\leq p(t)\alpha(B);\ \textit{here}\ J_{t,k}=[t-k,t]\cap J.$$

If V is an equicontinuous subset of D, then

$$\alpha\left(\left\{\int_J G(s,t)f(s,y_s)ds : y\in V\right\}\right) \le \int_J \|G(t,s)\|p(s)\alpha(V(s))ds.$$

To prove the main results, we assume the following conditions:

(H1) The operator $S'(t)$ is compact for all $t>0$; and

$$\|S'(t)x\| \le \varphi_A(t)\|x\|_{[D(A)]} \text{ for all } t>0 \text{ and each } x\in D(A).$$

(H2) $f : J\times C([-r,0],E) \longrightarrow E$ is of Carathéodory.

(H3) There exist functions $p\in L^\infty(J,\mathbb{R}_+)$ such that

$$|f(t,u)| \le p(t)(\|u\|_C+1), \text{ for a.e. } t\in J \text{ and } u\in C([-r,0],E).$$

(H4) For almost each $t\in J$ and each bounded set $B\subset C([-r,0],E)$ we have

$$\lim_{k\to 0^+}\alpha(f(J_{t,k}\times B)) \le p(t)\alpha(B); \text{ here } J_{t,k}=[t-k,t]\cap J.$$

Our main result reads as follows:

Theorem 5.2.1 *Assume that the conditions $(H1)-(H4)$ are satisfied. Then the problem (5.1)-(5.2) has at least one mild solution on $[-r,b]$, provident that*

$$\frac{b^\alpha\|p\|_{L^\infty}(1+\|\varphi_A\|_{L^1})}{\Gamma(\alpha+1)} < 1. \tag{5.3}$$

Proof. Transform the problem (5.1)-(5.2) into a fixed point problem. Consider the operator

$N : C([-r,b],E)\to C([-r,b],E)$ defined by,

$$N(y)(t)=\begin{cases}\phi(t), & t\in[-r,0],\\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f(s,y(s-\rho(y(s))))ds & \\ +\displaystyle\int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}f(\tau,y(\tau-\rho(y(\tau))))d\tau\right)ds, & t\in[0,b].\end{cases}$$

Let $\gamma > 0$ be such that

$$\gamma \geq \frac{b^{\alpha}\|p\|_{L^{\infty}}}{\Gamma(\alpha+1) - b^{\alpha}\|p\|_{L^{\infty}}}, \tag{5.4}$$

and consider the set

$$D_{\gamma} = \{y \in C([-r,b],E) : \|y\|_{\infty} \leq \gamma\}.$$

Clearly, the subset D_{γ} is closed, bounded and convex. We shall show that N satisfies the assumptions of Theorem 2.4.5.

In order to prove that N is completely continuous, we divide the operator N into two operators:

$$N_1(y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s-\rho(y(s)))ds,$$

and

$$N_2(y)(t) = \int_0^t S'(t-s)N_1(y)(s)ds.$$

We prove that N_1 and N_2 are completely continuous.

Step 1: N_1 is completely continuous.

A) N_1 is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ as $n \to \infty$ in $C([-r,b],E)$, then for $t \in [0,b]$. Note that $-r \leq s - \rho(y(s)) \leq s$ for each $s \in J$ we have,

$$|N_1(y_n)(t) - N_1(y)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|f(s, y_n(s-\rho(y_n(s))) - f(s, y(s-\rho(y(s)))|ds$$

Since f is a Carathéodory function for $t \in J$, and from the continuity of ρ, we have by the dominated convergence theorem of Lebesgue, the right member of the above inequality tends to zero as $n \to \infty$.

$$\|N_1(y_n) - N_1(y)\|_{\infty} \to 0 \ \ as \ \ n \to \infty.$$

Thus N_1 is continuous.

B) $N_1(D_\gamma) \subset D_\gamma$ is bounded.

For each $y \in D_\gamma$ by $(H3)$ and (4.1) we have for each $t \in [0, b]$

$$\begin{aligned}
|N_1(y)(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s-\rho(y(s)))ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|f(s, y(s-\rho(y(s)))\big|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} p(s)(\|y(s)\|+1)ds \\
&\leq \frac{(\gamma+1)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} p(s)ds \\
&\leq \frac{b^\alpha(\gamma+1)\|p\|_{L^\infty}}{\Gamma(\alpha+1)} \\
&\leq \gamma.
\end{aligned}$$

Then $N_1(D_\gamma) \subset D_\gamma$.

C) $N_1(D_\gamma)$ is equicontinuous.

Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have for any $y \in D_\gamma$.

$$\begin{aligned}
& |N_1(y)(\tau_2) - N_1(y)(\tau_1)| \\
= & \left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2} (\tau_2-s)^{\alpha-1} f(s, y(s-\rho(y(s)))ds - \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} (\tau_1-s)^{\alpha-1} f(s, y(s-\rho(y(s)))ds\right| \\
\leq & \left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1-\epsilon} [(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}] f(s, y(s-\rho(y(s)))ds\right| \\
& + \left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1-\epsilon}^{\tau_1} [(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}] f(s, y(s-\rho(y(s)))ds\right| \\
& + \left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1} f(s, y(s-\rho(y(s)))ds\right| \\
\leq & \frac{(\gamma+1)\|p\|_{L^\infty}}{\Gamma(\alpha)}\left(\int_0^{\tau_1-\epsilon} [(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}]ds\right. \\
& \left. + \int_{\tau_1-\epsilon}^{\tau_1} [(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}]ds + \int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1}ds\right).
\end{aligned}$$

As $\tau_1 \longrightarrow \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero.

Then $N_1(D_\gamma)$ is continuous and completely continuous

Step 2: N_2 is completely continuous.

A) N_2 is continuous

The operator N_2 is continuous, since $S'(\cdot) \in C([0,b], B(E))$ and N_1 is continuous as proved in Step 1.

B) $N_2(D_\gamma) \subset D_\gamma$ is bounded.

For $y \in D_\gamma$ we have

$$\begin{aligned}
|N_2(y)(t)| &\leq \int_0^t |S'(t-s)||N_1(y)(s)|ds \\
&\leq \int_0^t \varphi_A(t-s)\|N_1(y)(s)\|_{[D(A)]}ds \\
&\leq \frac{\|\varphi_A\|_{L^1} b^\alpha(\gamma+1)\|p\|_{L^\infty}}{\Gamma(\alpha+1)} \\
&\leq \gamma.
\end{aligned}$$

Then $N_2(D_\gamma) \subset D_\gamma$.

C) $N_2(D_\gamma)$ is equicontinuous.

Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have for any $y \in D_\gamma$;

$$\begin{aligned}
|N_2(y)(\tau_2) - N_2(y)(\tau_1)| &= \left| \int_0^{\tau_2} S'(\tau_2-s)N_1(y)(\tau_2)ds - \int_0^{\tau_1} S'(\tau_1-s)N_1(y)(\tau_1)ds \right| \\
&\leq \frac{b^\alpha(\gamma+1)\|p\|_{L^\infty}}{\Gamma(\alpha+1)} \left(\int_0^{\tau_1-\epsilon} |S'(\tau_2-s) - S'(\tau_1-s)|\, ds \right. \\
&\quad \left. + \int_{\tau_1-\epsilon}^{\tau_1} |S'(\tau_2-s) - S'(\tau_1-s)|\, ds + \int_{\tau_1}^{\tau_2} |S'(\tau_2-s)|ds \right).
\end{aligned}$$

As $\tau_1 \longrightarrow \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. Then $N_2(D_\gamma)$ is continuous and completely continuous

Now let V be a subset of D_γ such that $V \subset \overline{conv}(N(V) \cup \{0\})$.
V is bounded and equicontinuous and therefore the function $v \longrightarrow v(t) = \alpha(V(t))$ is continuous on $[-r, b]$. By (H4), Lemma 5.2.1 and the properties of the measure α we have

for each $t \in [-r, b]$,

$$\begin{aligned}
v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\
&\leq \alpha(N(V)(t)) \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \alpha(V(s)) ds \\
&\quad + \int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} p(s) \alpha(V(\tau)) d\tau \right) ds \\
&\leq \frac{\|p\|_{L^\infty}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds + \frac{\|p\|_{L^\infty} \|\varphi_A\|_{L^1}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds \\
&\leq \|v\|_\infty \frac{b^\alpha \|p\|_{L^\infty}}{\Gamma(\alpha+1)} + \|v\|_\infty \frac{b^\alpha \|p\|_{L^\infty} \|\varphi_A\|_{L^1}}{\Gamma(\alpha+1)} \\
&\leq \|v\|_\infty \frac{b^\alpha \|p\|_{L^\infty} (1 + \|\varphi_A\|_{L^1})}{\Gamma(\alpha+1)}
\end{aligned}$$

This means that

$$\|v\|_\infty \left(1 - \frac{b^\alpha \|p\|_{L^\infty} (1 + \|\varphi_A\|_{L^1})}{\Gamma(\alpha+1)} \right) \leq 0$$

By (5.3) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in [-r, b]$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D_γ. Applying now Theorem 5.2.1 we conclude that N has a fixed point which is a mild solution for the problem (5.1)-(5.2).

5.3 An example

To apply our pervious result, we consider the following partial functional differential equation with fractional order for some $p > 1$

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(t,y) = \Delta u(t,y) + \theta(t) |u(t - \tau(u(t,y)), y)|^p, \quad for\ y \in \Omega, t \in [0,T]\ and\ 0 < \alpha < 1; \\ u(t,y) = 0, \quad for\ \ y \in \partial\Omega\ and\ t \in [0,T]; \\ u(t,y) = u_0(t,y), \quad for\ \ y \in \Omega\ and\ -\tau_{max} \leq t \leq 0. \end{cases} \tag{5.5}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with regular boundary $\partial\Omega$. $u_0 \in \mathcal{C}^2([-\tau_{max}, 0] \times \Omega, \mathbb{R}^n)$, θ is a continuous function from $[0,T]$ to $\mathbb{R}$ and $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$. The delay function

τ is bounded positive continuous function in $\mathbb{R}^n$, let τ_{max} be the maximal delay which is defined by

$$\tau_{max} = \sup_{y\in\mathbb{R}} \tau(y).$$

Let $E = L^2[0,\pi]$ and let A be the operator given by $Aw = w''$ with domain $D(A) = \{w \in E, w, w'$ are absolutely continuous, $w'' \in E, w(0) = w(\pi) = 0\}$.

Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \qquad w \in D(A),$$

where $(\cdot,\cdot)$ is the inner product in L^2 and $w_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(nx)$, $n = 1, 2, \ldots$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on E and is given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} (w, w_n) w_n, \qquad w \in E.$$

From these expressions it follows that $(T(t))_{t\geq 0}$ is uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda - A)^{-1}$ is compact operator for all $\lambda \in \rho(A)$.

From [38, Example 2.2.1] we know that the integral equation

$$u(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Au(s) ds, \; s \geq 0,$$

has an associated analytic resolvent operator $(S(t))_{t\geq 0}$ on E given by

$$S(t) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_{r,\theta}} e^{\lambda t} (\lambda^\alpha - A)^{-1} d\lambda, & t > 0, \\ I, & t = 0, \end{cases}$$

where $\Gamma_{r,\theta}$ denotes a contour consisting of the rays $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ for some $\theta \in (\pi, \frac{\pi}{2})$. $S(t)$ is differentiable (Proposition 2.15 in [5]) and there exists a constant $M > 0$ such that $\|S'(t)x\| \leq M\|x\|$, for $x \in D(A), \; t > 0$.

Let f be the function defined from $[0,T] \times E$ to E by

$$f(t,\varphi)(y) = \theta(t)|\varphi(y)|^p \quad for \;\; \varphi \in E \; and \;\; y \in \Omega.$$

Let u be a solution of Equation (5.5). Then $y(t) = u(t,.)$ is a solution of the following equation

$$\begin{cases} D^{\alpha}y(t) & = Ay(t) + f(t, y(t - \tau(y(t)))) \quad for \;\; t \in [0,T], 0 < \alpha < 1; \\ y(t) & = \phi(t) \quad , t \in [-\tau_{max}, 0], \end{cases} \tag{5.6}$$

where the initial value function ϕ is given by

$$\phi(t)(y) \quad = \quad u_0(t,y) \quad for \;\; t \in [-\tau_{max}, 0] \; and \;\; y \in \Omega.$$

We can show that problem (5.5) is an abstract formulation of problem (5.6). Under suitable conditions, Theorem 5.2.1 implies that problem (5.6) has a unique solution y on $[-\tau_{max}, T] \times \Omega$.

Chapter 6

Semilinear perturbed fractional differential equations

6.1 Introduction

The purpose of this chapter is to extend such results to perturbed functional differential equations with fractional order. Our results is based upon an application of Burton and Kirk's fixed point theorem for the sum of a contraction operator and a completely continuous operator (see [13, 16] for an application of this fixed point theorem to a class of perturbed semilinear functional differential equations of neutral type with infinite delay).

In this chapter, we give existence results for various classes of initial value problems for fractional semilinear perturbed functional differential equations , both cases of finite and infinite delay are considered. More precisely this chapter is organized as follows. In the second section we will be concerned with semilinear perturbed fractional differential equation with finite delay. In the third section, we consider semilinear perturbed fractional differential equation with infinite delay. An example will be presented in the last section illustrating the abstract theory.

6.2 Existence results for finite delay problems

In the following we will extend the previous results to the case when the delay is finite. More precisely we consider the following problem

$$D^{\alpha}y(t) - Ay(t) = f(t, y_t) + g(t, y_t), \quad t \in J := [0, b], \; 0 < \alpha < 1, \tag{6.1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{6.2}$$

where D^{α} is the standard Riemann-Liouville fractional derivative, $f, g : J \times C([-r, 0], E) \to E$ is a continuous function, $A : D(A) \subset E \to E$ is a densely defined closed linear operator on E, $\phi : [-r, 0] \to E$ a given continuous function with $\phi(0) = 0$ and $(E, |\cdot|)$ a real Banach space. For any function y defined on $[-r, b]$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], E)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$ up to the present time t

Before stating our main results in this section for problem (6.1) and (6.2) we give the definition of the mild solution.

Definition 6.2.1 *We say that a continuous function $y : [-r, b] \to E$ is a mild solution of problem (6.1)-(6.2) if:*

1. $\displaystyle\int_0^t (t-s)^{\alpha-1} y(s)ds \in D(A)$ *for* $t \in J$,
2. $y(t) = \phi(t)$, $t \in [-r, 0]$, *and*
3. $\displaystyle y(t) = \frac{A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(f(s, y_s) + g(s, y_s))ds$, $t \in J$.

Suppose that there exists a resolvent $(S(t))_{t\geq 0}$ which is differentiable and the functions f, g is continuous. Then by Lemma 2.2.1 (iii), if $y : [-r, b] \to E$ is a mild solution of (6.1)-(6.2), then

$$y(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}(f(s, y_s) + g(s, y_s))ds \\ + \displaystyle\int_0^t S'(t-s) \left(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1}(f(\tau, y_\tau) + g(\tau, y_\tau)d\tau \right) ds, & t \in J, \\ \phi(t), & t \in [-r, 0]. \end{cases}$$

Our main result reads.

Theorem 6.2.1 *Assume that*

(H1) *$S(t)$ is compact for all $t>0$;*

(H2) *there exists a constant $L>0$ such that*

$$|g(t,u)-g(t,v)| \leq L\|u-v\|_C, \quad for \quad t \in J \quad and \ u,v \in C([-r,0],E)$$

with

$$\frac{(1+\|\varphi_A\|_{L^1})Lb^\alpha}{\Gamma(\alpha+1)} < 1; \tag{6.3}$$

(H3) *there exist functions $p,q \in C(J,\mathbb{R}_+)$ such that*

$$|f(t,u)| \leq p(t)+q(t)\|u\|_C, \quad t \in J \ and \ u \in C([-r,0],E).$$

Then, the problem (6.1)-(6.2) has at least one mild solution on $[-r,b]$, provident that

$$\frac{b^\alpha(\|q\|_\infty+L)}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1}) < 1. \tag{6.4}$$

Proof. Transform the problem (6.1)-(6.2) into a fixed point problem. Consider the two operators

$$F,G : C([-r,b],E) \to C([-r,b],E)$$

defined by:

$$F(y)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \dfrac{1}{\Gamma(\alpha)}\left(\displaystyle\int_0^t (t-s)^{\alpha-1} f(s,y_s)ds + \int_0^t S'(t-s)\int_0^s (s-\tau)^{\alpha-1} f(\tau,y_\tau)d\tau\right)ds, & t \in [0,b]. \end{cases}$$

$$G(y)(t) = \begin{cases} 0, & t \in [-r,0], \\ \dfrac{1}{\Gamma(\alpha)}\left(\displaystyle\int_0^t (t-s)^{\alpha-1} g(s,y_s)ds + \int_0^t S'(t-s)\int_0^s (s-\tau)^{\alpha-1} g(\tau,y_\tau)d\tau\right)ds, & t \in [0,b]. \end{cases}$$

Then the problem of finding the solution of problem (6.1)-(6.2) is reduced to finding the solution of the operator equation $F(y)(t)+G(y)(t)=y(t), t \in [-r,b]$. We will show that the operators F and G satisfy all conditions of Theorem2.4.4 . The proof will be given in several steps.

Step 1: F is completely continuous.

In order to prove that F is completely continuous, we divide the operator F into two operators:

$$F_1(y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds,$$

and

$$F_2(y)(t) = \int_0^t S'(t-s) F_1(y)(s) ds.$$

We prove that F_1 and F_2 are completely continuous. We note that the condition (H1) implies that $S'(t)$ is compact for all $t > 0$ (see [22, Lemma 2.2]).

• **F_1 is completely continuous**

At first, we prove that F_1 is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ as $n \to \infty$ in $C([-r, b], E)$. Then for $t \in [0, b]$ we have

$$\begin{aligned} |F_1(y_n)(t) - F_1(y)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| f(s, y_{ns}) - f(s, y_s) \Big| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \| f(\cdot, y_{n\cdot}) - f(\cdot, y_{\cdot}) \|_\infty \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \frac{b^\alpha}{\Gamma(\alpha+1)} \| f(\cdot, y_{n\cdot}) - f(\cdot, y_{\cdot}) \|_\infty. \end{aligned}$$

Since f is a continuous function, we have

$$\|F_1(y_n) - F_1(y)\|_{\mathfrak{D}} \to 0 \ \ as \ \ n \to \infty.$$

Thus F_1 is continuous.

Next, we prove that F_1 maps bounded sets into bounded sets in $C([-r, b], E)$. Indeed, it is enough to show that for any $\rho > 0$, there exists a positive constant δ such that for each $y \in B_\rho = \{y \in C([-r, b], E) : \|y\|_{\mathfrak{D}} \leq \rho\}$ one has $F_1(y) \in B_\delta$. Let $y \in B_\rho$. Since f is a continuous function, we have for each $t \in [0, b]$

$$\begin{aligned} |F_1(y)(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_s)ds\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|f(s, y_s)\big|ds \\ &\leq \frac{(\|p\|_\infty + \rho\|q\|_\infty)b^\alpha}{\Gamma(\alpha+1)} = \delta^* < \infty. \end{aligned}$$

Then, $\|F_1(y)\|_{\mathfrak{D}} = \max\{\|\phi\|_C, \delta^*\} = \delta$, and hence $F_1(y) \in B_\delta$.

Now, we prove that F_1 maps bounded sets into equicontinuous sets of $C([-r, b], E)$. Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$ and let B_ρ be a bounded set. Let $y \in B_\rho$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have

$$\begin{aligned} & |F_1(y)(\tau_2) - F_1(y)(\tau_1)| \\ = & \left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2} (\tau_2 - s)^{\alpha-1} f(s, y_s)ds - \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} (\tau_1 - s)^{\alpha-1} f(s, y_s)ds\right| \\ \leq & \left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1-\epsilon} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}] f(s, y_s)ds\right| \\ & + \left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1-\epsilon}^{\tau_1} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}] f(s, y_s)ds\right| + \left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} f(s, y_s)ds\right| \\ \leq & \frac{(\|p\|_\infty + \rho\|q\|_\infty)}{\Gamma(\alpha)}\Bigg(\int_0^{\tau_1-\epsilon} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]ds \\ & + \int_{\tau_1-\epsilon}^{\tau_1} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]ds + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1}ds\Bigg). \end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_1 maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_\rho$ we define

$$F_{1\epsilon}(y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon} (t - s - \epsilon)^{\alpha-1} f(s, y_s)ds.$$

Note that the set

$$\left\{\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon} (t - s - \epsilon)^{\alpha-1} f(s, y_s)ds : y \in B_\rho\right\}.$$

is bounded since

$$\begin{aligned}\left|\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}f(s,y_s)ds\right| &\leq (\|p\|_\infty+\rho\|q\|_\infty)\left|\frac{1}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}ds\right|\\ &\leq \frac{(\|p\|_\infty+\rho\|q\|_\infty)(t-\epsilon)^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

Then for $t>0$, the set

$$Y_\epsilon(t)=\{F_{1\epsilon}(y)(t) : y\in B_\rho\}$$

is precompact in E for every ϵ, $0<\epsilon<t$. Moreover

$$\begin{aligned}\left|F_1(y)(t)-F_{1\epsilon}(y)(t)\right| &\leq \frac{(\|p\|_\infty+\rho\|q\|_\infty)}{\Gamma(\alpha)}\left(\int_0^{t-\epsilon}[(t-s)^{\alpha-1}-(t-s-\epsilon)^{\alpha-1}]ds+\int_{t-\epsilon}^t(t-s)^{\alpha-1}ds\right)\\ &\leq \frac{(\|p\|_\infty+\rho\|q\|_\infty)(t^\alpha-(t-\epsilon)^\alpha)}{\Gamma(\alpha+1)}.\end{aligned}$$

Therefore, the set $Y(t)=\{F_1(y)(t) : y\in B_\rho\}$ is precompact in E. Hence the operator F_1 is completely continuous.

•F_2 is completely continuous

The operator F_2 is continuous, since $S'(\cdot)\in C([0,b],B(E))$ and F_1 is continuous as proved in Step 1.

Now, let B_ρ be a bounded set as in Step 1. For $y\in B_\rho$ we have

$$\begin{aligned}|F_2(y)(t)| &\leq \int_0^t|S'(t-s)||F_1(y)(s)|ds\\ &\leq \int_0^t\varphi_A(t-s)\|F_1(y)(s)\|_{[D(A)]}ds\\ &\leq \frac{\|\varphi_A\|_{L^1}(\|p\|_\infty+\rho\|q\|_\infty)b^\alpha}{\Gamma(\alpha+1)}=\delta'.\end{aligned}$$

Thus, there exists a positive number δ' such that $\|F_2(y)\|_{\mathfrak{D}}\leq\delta'$. This means that $F_2(y)\in B_{\delta'}$.

Next, we shall show that F_2 maps bounded sets into equicontinuous sets in $C([-r,b],E)$. Let $\tau_1,\tau_2\in J$, $\tau_2>\tau_1$ and let B_ρ be a bounded set as in Step 1. Let $y\in B_\rho$.

Then if $\epsilon>0$ and $\epsilon\leq\tau_1\leq\tau_2$ we have,

$$\begin{aligned}
& |F_2(y)(\tau_2) - F_2(y)(\tau_1)| \\
= & \left| \int_0^{\tau_2} S'(\tau_2 - s)F_1(y)(\tau_2)ds - \int_0^{\tau_1} S'(\tau_1 - s)F_1(y)(\tau_1)ds \right| \\
\leq & \frac{(\|p\|_\infty + \rho\|q\|_\infty)b^\alpha}{\Gamma(\alpha+1)} \left(\int_0^{\tau_1-\epsilon} |S'(\tau_2 - s) - S'(\tau_1 - s)|\, ds \right. \\
& \left. + \int_{\tau_1-\epsilon}^{\tau_1} |S'(\tau_2 - s) - S'(\tau_1 - s)|\, ds + \int_{\tau_1}^{\tau_2} |S'(\tau_2 - s)| ds \right).
\end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_2 maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_\rho$ we define

$$F_{2\epsilon}(y)(t) = S'(\epsilon) \int_0^{t-\epsilon} S'(t - s - \epsilon)F_1(y)(s)ds.$$

Since $S'(t)$ is a compact operator for $t > 0$, the set

$$Y_\epsilon(t) = \{F_{2\epsilon}(y)(t) : y \in B_\rho\}$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\left| F_2(y)(t) - F_{2\epsilon}(y)(t) \right| \leq \frac{\|\varphi_A\|_{L^1}(\|p\|_\infty + \rho\|q\|_\infty)\big(t^\alpha - (t-\epsilon)^\alpha\big)}{\Gamma(\alpha+1)}.$$

Then $Y(t) = \{F_2(y)(t) : y \in B_\rho\}$ is precompact in E. Hence the operator F_2 is completely continuous.

Step 2: G is a contraction.

Let $y, z \in C([-r, b], E)$. For $t \in [0, b]$, we have

$$\begin{aligned}
& |G(y)(t) - G(z)(t)| \\
= & \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1}[g(s, y_s) - g(s, z_s)]ds + \int_0^t S'(t-s) \int_0^s (s-\tau)^{\alpha-1}[g(\tau, y_\tau) - g(\tau, z_\tau]d\tau ds \right| \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|g(s, y_s) - g(s, z_s)|ds + \int_0^t \varphi_A(t-s) \int_0^\tau (s-\tau)^{\alpha-1}|g(\tau, y_\tau) - g(\tau, z_\tau)|d\tau ds \\
\leq & \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1}L\|y_\tau - z_\tau\|_C ds + \int_0^t \varphi_A(t-s) \int_0^s (s-\tau)^{\alpha-1}L\|y_\tau - z_\tau\|_C d\tau ds \right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{L}{\Gamma(\alpha)}\|y-z\|_{\mathfrak{D}}\int_0^t (t-s)^{\alpha-1}ds + \frac{L}{\Gamma(\alpha)}\|y-z\|_{\mathfrak{D}}\int_0^t \varphi_A(t-s)\int_0^s (s-\tau)^{\alpha-1}d\tau ds\\
&\leq \frac{Lt^\alpha}{\Gamma(\alpha+1)}\|y-z\|_{\mathfrak{D}} + \frac{\|\varphi_A\|_{L^1}Lt^\alpha}{\Gamma(\alpha+1)}\|y-z\|_{\mathfrak{D}}.
\end{aligned}
$$

Taking the supremum over $t \in [-r,b]$, we get

$$
\|G(y)-G(z)\|_{\mathfrak{D}} \leq \frac{Lb^\alpha}{\Gamma(\alpha+1)}\left(1+\|\varphi_A\|_{L^1}\right)\|y-z\|_{\mathfrak{D}}.
$$

Which is a contraction, since $\frac{(1+\|\varphi_A\|_{L^1})Lb^\alpha}{\Gamma(\alpha+1)} < 1$, by the condition (6.3).

Step 3: A priori bound on solutions.

Now, it remains to show that the set

$$
\mathcal{E} = \{y \in C([-r,b],E) : y = \lambda F(y) + \lambda G(\frac{y}{\lambda}), \quad 0<\lambda<1\}
$$

is bounded.

Let $y \in \mathcal{E}$ be any element. Then, for each $t \in [0,b]$,

$$
\begin{aligned}
y(t) &= \lambda F(y)(t) + \lambda G(\frac{y}{\lambda})(t)\\
&= \lambda\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y_s)ds + \int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}f(\tau,y_\tau)d\tau\right)ds\right]\\
&\quad +\lambda\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,\frac{y_s}{\lambda})ds + \int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}g(\tau,\frac{y_\tau}{\lambda})d\tau\right)ds\right]
\end{aligned}
$$

Then

$$
\begin{aligned}
|y(t)| \leq\ & \frac{1}{\Gamma(\alpha)}\Bigg[\int_0^t (t-s)^{\alpha-1}[\|p\|_\infty + \|q\|_\infty\|y_s\|_C]ds\\
&+\int_0^t \varphi_A(t-s)\int_0^s (s-\tau)^{\alpha-1}[\|p\|_\infty + \|q\|_\infty\|y_s\|_C]d\tau ds\Bigg]\\
&+\frac{1}{\Gamma(\alpha)}\Bigg[\int_0^t (t-s)^{\alpha-1}\Big|g(s,\frac{y_s}{\lambda})-g(s,0)\Big|ds + \int_0^t (t-s)^{\alpha-1}|g(s,0)|ds\\
&+\int_0^t \varphi_A(t-s)\int_0^s (s-\tau)^{\alpha-1}\Big|g(\tau,\frac{y_\tau}{\lambda})-g(\tau,0)\Big|d\tau ds\\
&+\int_0^t \varphi_A(t-s)\int_0^s (s-\tau)^{\alpha-1}|g(\tau,0)|d\tau ds\Bigg]
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{b^\alpha\|q\|_\infty}{\Gamma(\alpha+1)}\|y_s\|_C + \frac{\|\varphi_A\|_{L^1} b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{\|\varphi_A\|_{L^1} b^\alpha\|q\|_\infty}{\Gamma(\alpha+1)}\|y_s\|_C \\
&\quad + \frac{b^\alpha L}{\Gamma(\alpha+1)}\|y_s\|_C + \frac{b^\alpha L\|\varphi_A\|_{L^1}}{\Gamma(\alpha+1)}\|y_s\|_C + \frac{(1+\|\varphi_A\|_{L^1})}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}|g(s,0)|ds \\
&\le \left(\frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(s,0)|ds\right)(1+\|\varphi_A\|_{L^1}) + \frac{b^\alpha(\|q\|_\infty + L)}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1})\|y\|_{\mathfrak{D}}
\end{aligned}$$

and consequently

$$\|y\|_{\mathfrak{D}} \le \left(\frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(s,0)|ds\right)(1+\|\varphi_A\|_{L^1})\left\{1 - \frac{b^\alpha(\|q\|_\infty + L)}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1})\right\}^{-1}.$$

This shows that the set $\mathcal{E}$ is bounded. As a result the conclusion (b) of Theorem 2.4.4 does not hold. Hence the conclusion (a) holds and consequently $F(y)+G(y)$ has a fixed point which is a mild solution of problem (6.1)-(6.2) on $[-r,b]$.

6.3 Existence results for infinite delay problems

In the following we will extend the previous results to the case when the delay is infinite. More precisely we consider the following problem

$$D^\alpha y(t) - Ay(t) = f(t,y_t) + g(t,y_t), \quad t \in J := [0,b],\ 0<\alpha<1, \tag{6.5}$$

$$y(t) = \phi \quad \in \mathcal{B}, \tag{6.6}$$

where D^α is the standard Riemann-Liouville fractional derivative, $f,g : J\times\mathcal{B} \to E$ is a continuous function, $A : D(A) \subset E \to E$ is a densely defined closed linear operator on E, $\phi : \mathcal{B} \to E$ a given continuous function with $\phi(0) = 0$.

Before stating our main results in this section for problem (6.5) and (6.6) we give the definition of the mild solution.

Definition 6.3.1 *We say that a function $y \in \Omega$ is a mild solution of problem (6.5)-(6.6) if:*

1. $\int_0^t (t-s)^{\alpha-1}y(s)ds \in D(A)$ *for $t \in J$,*
2. $y(t) = \phi \in \mathcal{B}$, *and*

3. $y(t) = \frac{A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (f(s, y_s) + g(s, y_s)) ds, \ t \in J.$

Suppose that there exists a resolvent $(S(t))_{t \geq 0}$ which is differentiable and the functions f, g is continuous. Then by Lemma 2.2.1 (iii), if $y : \Omega \to \Omega$ is a mild solution of (6.5)-(6.6), then

$$y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (f(s, y_s) + g(s, y_s)) ds \\ + \int_0^t S'(t-s) \Big(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} (f(\tau, y_\tau) + g(\tau, y_\tau) d\tau \Big) ds, \quad t \in J, \\ \phi \in \mathcal{B}. \end{cases}$$

Let $x(\cdot) : (-\infty, b] \to E$ be the function defined by:

$$x(t) = \begin{cases} \phi(t), & \text{if} \quad t \in (-\infty, 0]; \\ 0, & \text{if} \quad t \in J. \end{cases}$$

Then $x_0 = \phi$. We denote by $\overline{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} 0, & \text{if} \quad t \in (-\infty, 0]; \\ z(t), & \text{if} \quad t \in J. \end{cases}$$

If $y(\cdot)$ satisfies

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (f(s, y_s) + g(s, y_s)) ds + \int_0^t S'(t-s) \Big(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} (f(\tau, y_\tau) + g(\tau, y_\tau)) d\tau \Big) ds$$

we can decompose it as $y(t) = \bar{z}(t) + x(t), t \in J$ which implies $y_t = \bar{z}_t + x_t, t \in J$ and the function $z(\cdot)$ satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (f(s, \bar{z}_s + x_s) + g(s, \bar{z}_s + x_s)) ds \\ &+ \int_0^t S'(t-s) \Big(\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} (f(\tau, \bar{z}_\tau + x_\tau) + g(\tau, \bar{z}_\tau + x_\tau)) d\tau \Big) ds. \end{aligned}$$

Let

$$\Omega_0 = \{ z \in \Omega \ \text{ such that } \ z_0 = 0 \},$$

and let $\| \cdot \|_b$ be the seminorm in Ω_0 defined by

$$\|z\|_b = \|z_0\|_{\mathcal{B}} + \sup\{ |z(s)| : 0 \leq s \leq b \} = \sup\{ |z(s)| : 0 \leq s \leq b \}, \quad z \in \Omega_0.$$

Then $(\Omega_0, \|\cdot\|_b)$ is a Banach space. Let the operator $F, G : \Omega_0 \to \Omega_0$ be defined by

$$F(z)(t) = \begin{cases} \phi \in \mathcal{B}, \\ \dfrac{1}{\Gamma(\alpha)}\left(\displaystyle\int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s)ds + \int_0^t S'(t-s)\int_0^s (s-\tau)^{\alpha-1} f(\tau, \bar{z}_\tau + x_\tau)d\tau\right)ds, \\ t \in [0,b]. \end{cases}$$

$$G(z)(t) = \begin{cases} 0, \qquad t \in \mathcal{B}, \\ \dfrac{1}{\Gamma(\alpha)}\left(\displaystyle\int_0^t (t-s)^{\alpha-1} g(s, \bar{z}_s + x_s)ds + \int_0^t S'(t-s)\int_0^s (s-\tau)^{\alpha-1} g(\tau, \bar{z}_\tau + x_\tau)d\tau\right)ds, \\ t \in [0,b]. \end{cases}$$

Our main result reads.

Theorem 6.3.1 *Assume that*

(H1) $S(t)$ *is compact for all* $t > 0$;

(H2) *there exists a constant* $L > 0$ *such that*

$$|g(t,u) - g(t,v)| \leq L\|u - v\|_{\mathcal{B}}, \quad for \quad t \in J \quad and \quad u, v \in \mathcal{B}$$

with

$$\frac{LK_b b^\alpha}{\Gamma(\alpha+1)}(1 + \|\varphi_A\|_{L^1}) < 1; \tag{6.7}$$

(H3) *there exist functions* $p, q \in C(J, \mathbb{R}_+)$ *such that*

$$|f(t,u)| \leq p(t) + q(t)\|u\|_{\mathcal{B}}, \quad t \in J \text{ and } u \in \mathcal{B}.$$

Then, the problem (6.5)-(6.6) has at least one mild solution on $(-\infty, b]$, *provident that*

$$\frac{b^\alpha K_b(\|q\|_\infty + L)}{\Gamma(\alpha+1)}(1 + \|\varphi_A\|_{L^1}) < 1.$$

Proof. The problem of finding the solution of problem (6.5)-(6.6) is reduced to finding the solution of the operator equation $F(z)(t) + G(z)(t) = y(t), t \in \Omega_0$. We will show that the operators F and G satisfy all conditions of Theorem2.4.4 . The proof will be given in several steps.

Step 1: F is completely continuous.

In order to prove that F is completely continuous, we divide the operator F into two operators:

$$F_1(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s) ds,$$

and

$$F_2(z)(t) = \int_0^t S'(t-s) F_1(z)(s) ds.$$

We prove that F_1 and F_2 are completely continuous. We note that the condition (H1) implies that $S'(t)$ is compact for all $t > 0$ (see [22, Lemma 2.2]).

• **F_1 is completely continuous**

At first, we prove that F_1 is continuous. Let $\{z_n\}$ be a sequence such that $z_n \to z$ in Ω_0 as $n \to \infty$. Then for $t \in [0, b]$ we have

$$\begin{aligned} |F_1(z_n)(t) - F_1(z)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| f(s, \bar{z}_{ns} + x_s) - f(s, \bar{z}_s + x_s) \Big| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \| f(\cdot, \bar{z}_{n.} + x_.) - f(\cdot, \bar{z}_. + x_.) \|_\infty \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \frac{b^\alpha}{\Gamma(\alpha+1)} \| f(\cdot, \bar{z}_{n.} + x_.) - f(\cdot, \bar{z}_. + x_.) \|_\infty . \end{aligned}$$

Since f is a continuous function, we have

$$\|F_1(z_n) - F_1(z)\|_b \to 0 \;\; as \;\; n \to \infty.$$

Thus F_1 is continuous.

Next, we prove that F_1 maps bounded sets into bounded sets in Ω_0. Indeed, it is enough to show that for any $\rho > 0$, there exists a positive constant δ such that for each $z \in B_\rho = \{z \in \Omega_0 : \|z\|_b \leq \rho\}$ one has $F_1(z) \in B_\delta$. Let $z \in B_\rho$. Since f is a continuous function, we have for each $t \in [0, b]$

$$\begin{aligned} |F_1(z)(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big| f(s, \bar{z}_s + x_s) \big| ds \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\|p\|_\infty + \|q\|_\infty\|\bar{z}_s + x_s\|_{\mathcal{B}})ds \\
&\leq \frac{b^\alpha(\|p\|_\infty + \rho^*\|q\|_\infty)}{\Gamma(\alpha+1)} = \delta < \infty,
\end{aligned}
$$

where

$$
\begin{aligned}
\|\bar{z}_s + x_s\|_{\mathcal{B}} &\leq \|\bar{z}_s\|_{\mathcal{B}} + \|x_s\|_{\mathcal{B}} \\
&\leq K(t)\sup\{|z(t)| : 0 \leq s \leq t\} + M(t)\|z_0\|_{\mathcal{B}} \\
&\quad + K(t)\sup\{|x(t)| : 0 \leq s \leq t\} + M(t)\|x_0\|_{\mathcal{B}} \\
&\leq K(t)\sup\{|z(t)| : 0 \leq s \leq t\} + M(t)\|x_0\|_{\mathcal{B}} \\
&\leq K_b\rho + M_b\|\phi\|_{\mathcal{B}} = \rho^*,
\end{aligned}
$$

and $M_b = \sup\{|M(t)| : t \in [0,b]\}$.

Then, $\|F_1(z)\|_b \leq \delta$, and hence $F_1(z) \in B_\delta$.

Now, we prove that F_1 maps bounded sets into equicontinuous sets of Ω_0. Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$ and let B_ρ be a bounded set. Let $z \in B_\rho$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have

$$
\begin{aligned}
&|F_1(z)(\tau_2) - F_1(z)(\tau_1)| \\
= &\left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}(\tau_2 - s)^{\alpha-1}f(s, \bar{z}_s + x_s)ds - \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_1 - s)^{\alpha-1}f(s, \bar{z}_s + x_s)ds\right| \\
\leq &\left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1-\epsilon}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]f(s, \bar{z}_s + x_s)ds\right| \\
&+\left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1-\epsilon}^{\tau_1}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]f(s, \bar{z}_s + x_s)ds\right| \\
&+\left|\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha-1}f(s, \bar{z}_s + x_s)ds\right| \\
\leq &\frac{(\|p\|_\infty + \rho^*\|q\|_\infty)}{\Gamma(\alpha)}\left(\int_0^{\tau_1-\epsilon}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]ds\right. \\
&\left.+\int_{\tau_1-\epsilon}^{\tau_1}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha-1}ds\right).
\end{aligned}
$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_1 maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_\rho$ we define

$$F_{1\epsilon}(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} f(s, \bar{z}_s + x_s) ds.$$

Note that the set

$$\left\{ \frac{1}{\Gamma(\alpha)} \int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} f(s, \bar{z}_s + x_s) ds : z \in B_\rho \right\}$$

is bounded since

$$\begin{aligned} & \left| \frac{1}{\Gamma(\alpha)} \int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} f(s, \bar{z}_s + x_s) ds \right| \\ \leq \; & (\|p\|_\infty + \rho^* \|q\|_\infty) \left| \frac{1}{\Gamma(\alpha)} \int_0^{t-\epsilon} (t-s-\epsilon)^{\alpha-1} ds \right| \\ \leq \; & \frac{(\|p\|_\infty + \rho^* \|q\|_\infty)}{\Gamma(\alpha+1)} (t-\epsilon)^\alpha. \end{aligned}$$

Then for $t > 0$, the set

$$Z_\epsilon(t) = \{F_{1\epsilon}(z)(t) : z \in B_\rho\}$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\begin{aligned} & \left| F_1(z)(t) - F_{1\epsilon}(z)(t) \right| \\ \leq \; & \frac{(\|p\|_\infty + \rho^* \|q\|_\infty)}{\Gamma(\alpha)} \left(\int_0^{t-\epsilon} [(t-s)^{\alpha-1} - (t-s-\epsilon)^{\alpha-1}] ds + \int_{t-\epsilon}^t (t-s)^{\alpha-1} ds \right) \\ \leq \; & \frac{(\|p\|_\infty + \rho^* \|q\|_\infty)}{\Gamma(\alpha+1)} \left(t^\alpha - (t-\epsilon)^\alpha \right). \end{aligned}$$

Therefore, the set $Z(t) = \{F_1(z)(t) : z \in B_\rho\}$ is precompact in E. Hence the operator F_1 is completely continuous.

- **F_2 is completely continuous**

The operator F_2 is continuous, since $S'(\cdot) \in C(J, B(E))$ and F_1 is continuous as proved in Step 1.

Now, let B_ρ be a bounded set as in Step 1. For $z \in B_\rho$ we have

$$\begin{aligned} |F_2(z)(t)| &\leq \int_0^t |S'(t-s)||F_1(z)(s)|ds \\ &\leq \int_0^t \varphi_A(t-s)\|F_1(z)(s)\|_{[D(A)]}ds \\ &\leq \frac{\|\varphi_A\|_{L^1} b^\alpha(\|p\|_\infty + \rho^*\|q\|_\infty)}{\Gamma(\alpha+1)} = \delta'. \end{aligned}$$

Thus, there exists a positive number δ' such that $\|F_2(z)\|_b \leq \delta'$. This means that $F_2(z) \in B_{\delta'}$.

Next, we shall show that F_2 maps bounded sets into equicontinuous sets in Ω_0. Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$ and let B_ρ be a bounded set as in Step 1. Let $z \in B_\rho$. Then if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have

$$\begin{aligned} & |F_2(z)(\tau_2) - F_2(z)(\tau_1)| \\ = & \left| \int_0^{\tau_2} S'(\tau_2 - s)F_1(z)(\tau_2)ds - \int_0^{\tau_1} S'(\tau_1 - s)F_1(z)(\tau_1)ds \right| \\ \leq & \frac{b^\alpha(\|p\|_\infty + \rho^*\|q\|_\infty)}{\Gamma(\alpha+1)} \Bigg(\int_0^{\tau_1-\epsilon} |S'(\tau_2 - s) - S'(\tau_1 - s)|\, ds \\ & + \int_{\tau_1-\epsilon}^{\tau_1} |S'(\tau_2 - s) - S'(\tau_1 - s)|\, ds + \int_{\tau_1}^{\tau_2} |S'(\tau_2 - s)|ds \Bigg). \end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero. By Arzelá-Ascoli theorem it suffices to show that F_2 maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_\rho$ we define

$$F_{2\epsilon}(z)(t) = S'(\epsilon) \int_0^{t-\epsilon} S'(t-s-\epsilon)F_1(z)(s)ds.$$

Since $S'(t)$ is a compact operator for $t > 0$, the set

$$Z_\epsilon(t) = \{F_{2\epsilon}(z)(t) : z \in B_\rho\}$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\Big|F_2(z)(t) - F_{2\epsilon}(z)(t)\Big| \leq \frac{\|\varphi_A\|_{L^1}(\|p\|_\infty + \rho^*\|q\|_\infty)}{\Gamma(\alpha+1)}\Big(t^\alpha - (t-\epsilon)^\alpha\Big).$$

Then $Z(t) = \{F_2(z)(t) : z \in B_\rho\}$ is precompact in E. Hence the operator F_2 is completely continuous.

Step 2: G is a contraction.

Let $z, z^* \in \Omega_0$. Then we have for each $t \in J$

$$\begin{aligned}
|G(z)(t) - G(z^*)(t)| &= \Bigg|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[g(s,\bar{z}_s + x_s) - g(s,\bar{z}^*_s + x_s)]ds \\
&\quad + \int_0^t S'(t-s)\left(\frac{1}{\Gamma(\alpha)}\int_0^\tau (s-\tau)^{\alpha-1}\big[g(\tau,\bar{z}_\tau + x_\tau) - g(\tau,\bar{z}^*_\tau + x_\tau)\big]d\tau\right)ds\Bigg| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(s,\bar{z}_s + x_s) - g(s,\bar{z}^*_s + x_s)|ds \\
&\quad + \int_0^t \varphi_A(t-s)\frac{1}{\Gamma(\alpha)}\int_0^\tau (s-\tau)^{\alpha-1}|g(\tau,\bar{z}_\tau + x_\tau) - g(\tau,\bar{z}^*_\tau + x_\tau)|d\tau ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}L\|z_s - z^*_s\|_{\mathcal{B}}ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \varphi_A(t-s)\int_0^\tau (s-\tau)^{\alpha-1}L\|z_\tau - z^*_\tau\|_{\mathcal{B}}d\tau ds \\
&\le \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K_b \sup_{s\in[0,t]}|z(s) - z^*(s)|ds \\
&\quad + \frac{L}{\Gamma(\alpha)}\int_0^t \varphi_A(t-s)\int_0^\tau (s-\tau)^{\alpha-1}d\tau K_b \sup_{s\in[0,t]}|z(s) - z^*(s)|ds \\
&\le \frac{LK_b t^\alpha}{\Gamma(\alpha+1)}\|z - z^*\|_b + \frac{\|\varphi_A\|_{L^1}LK_b t^\alpha}{\Gamma(\alpha+1)}\|z - z^*\|_b.
\end{aligned}$$

Taking the supremum over t we get

$$\|G(z) - G(z^*)\|_b \le \frac{LK_b b^\alpha}{\Gamma(\alpha+1)}(1 + \|\varphi_A\|_{L^1})\|z - z^*\|_b.$$

Which is a contraction, since $\frac{LK_b b^\alpha}{\Gamma(\alpha+1)}(1 + \|\varphi_A\|_{L^1}) < 1$, by the condition (6.7).

Step 3: A priori bound on solutions.

Now, it remains to show that the set

$$\mathcal{E} = \{z \in \Omega_0 : z = \lambda F(z) + \lambda G(\frac{z}{\lambda}), \quad 0 < \lambda < 1\}$$

is bounded.

Let $z \in \mathcal{E}$ be any element. Then, for each $t \in [0,b]$,

$$\begin{aligned}
z(t) &= \lambda F(z)(t) + \lambda G(\frac{z}{\lambda})(t)\\
&\leq \frac{1}{\Gamma(\alpha)}\Big[\int_0^t (t-s)^{\alpha-1}(p(s)+q(s)\|\bar{z}_s+x_s\|_{\mathcal{B}})ds\\
&\quad + \int_0^t \varphi_A(t-s)\int_0^s (s-\tau)^{\alpha-1}(p(s)+q(s)\|\bar{z}_s+x_s\|_{\mathcal{B}})d\tau ds\Big]\\
&\quad + \frac{1}{\Gamma(\alpha)}\Big[\int_0^t (t-s)^{\alpha-1}\big|g(s,\frac{\overline{z}_s+x_s}{\lambda})-g(s,0)\big|ds + \int_0^t (t-s)^{\alpha-1}|g(s,0)|ds\\
&\quad + \int_0^t \varphi_A(t-s)\int_0^s (s-\tau)^{\alpha-1}\big|g(\tau,\frac{\overline{z}_\tau+x_\tau}{\lambda})-g(\tau,0)\big|d\tau ds\\
&\quad + \int_0^t \phi_A(t-s)\int_0^s (s-\tau)^{\alpha-1}|g(\tau,0)|d\tau ds\Big]\\
&\leq \frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1}) + \frac{b^\alpha\|q\|_\infty}{\Gamma(\alpha+1)}(K_b\|z\|_b + M_b\|\phi\|_{\mathcal{B}})(1+\|\varphi_A\|_{L^1})\\
&\quad + \frac{b^\alpha L}{\Gamma(\alpha+1)}(K_b\|z\|_b + M_b\|\phi\|_{\mathcal{B}})(1+\|\varphi_A\|_{L^1}) + \frac{(1+\|\varphi_A\|_{L^1})}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(s,0)|ds\\
&= \frac{b^\alpha(\|p\|_\infty + M_b\|q\|_\infty\|\phi\|_{\mathcal{B}})}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1}) + \frac{(1+\|\varphi_A\|_{L^1})}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(s,0)|ds\\
&\quad + \frac{b^\alpha K_b(\|q\|_\infty + L)}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1})\|z\|_b
\end{aligned}$$

and consequently

$$\begin{aligned}
\|z\|_b &\leq \left(\frac{b^\alpha(\|p\|_\infty + \|q\|_\infty\|\phi\|_{\mathcal{B}}M_b)}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(s,0)|ds\right)(1+\|\varphi_A\|_{L^1})\\
&\quad \left\{1 - \frac{b^\alpha K_b(\|q\|_\infty + L)}{\Gamma(\alpha+1)}(1+\|\varphi_A\|_{L^1})\right\}^{-1}.
\end{aligned}$$

This shows that the set $\mathcal{E}$ is bounded. As a result the conclusion (b) of Theorem 2.4.4 does not hold. Hence the conclusion (a) holds and consequently $F(z)+G(z)$ has a fixed point which is a mild solution of problem (6.5)-(6.6) on $(-\infty,b]$.

6.4 An example

As an application of our results we consider the following partial perturbed functional differential equations of the form

$$\begin{aligned}
\frac{\partial^\alpha}{\partial t^\alpha}u(t,x) &= \frac{\partial^2}{\partial x^2}u(t,x)+\frac{tu(t)}{7+t^2}+Q(t,u(t-r,x)), && (6.8)\\
& \quad x\in[0,\pi], t\in[0,1], \alpha\in(0,1), && (6.9)\\
u(t,0) &= u(t,\pi)=0,\ t\in[0,1], && (6.10)\\
u(t,x) &= \varphi(t,x),\quad x\in[0,\pi],\quad t\in[-r,0], && (6.11)
\end{aligned}$$

where $r>0$, $\varphi:[-r,0]\times[0,\pi]\to\mathbb{R}$ is continuous and $Q:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given function.

To study this system, we take $E=L^2[0,\pi]$ and let A be the operator given by $Aw=w''$ with domain $D(A)=\{w\in E, w, w'$ are absolutely continuous, $w''\in E, w(0)=w(\pi)=0\}$.

Then

$$Aw=\sum_{n=1}^{\infty}n^2(w,w_n)w_n, \qquad w\in D(A),$$

where $(\cdot,\cdot)$ is the inner product in L^2 and $w_n(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin(nx)$, $n=1,2,\ldots$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on E and is given by

$$T(t)w=\sum_{n=1}^{\infty}e^{-n^2t}(w,w_n)w_n, \qquad w\in E.$$

From these expressions it follows that $(T(t))_{t\geq 0}$ is uniformly bounded compact semigroup, so that $R(\lambda,A)=(\lambda-A)^{-1}$ is compact operator for all $\lambda\in\rho(A)$.

From [38, Example 2.2.1] we know that the integral equation

$$u(t)=h(t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Au(s)ds,\ s\geq 0,$$

has an associated analytic resolvent operator $(S(t))_{t\geq 0}$ on E given by

$$S(t)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_{r,\theta}}e^{\lambda t}(\lambda^\alpha-A)^{-1}d\lambda, & t>0,\\ I, & t=0,\end{cases}$$

where $\Gamma_{r,\theta}$ denotes a contour consisting of the rays $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ for some $\theta \in (\pi, \frac{\pi}{2})$. $S(t)$ is differentiable (Proposition 2.15 in [5]) and there exists a constant $M > 0$ such that $\|S'(t)x\| \leq M\|x\|$, for $x \in D(A), t > 0$.

To represent the differential system (6.8) – (6.11) in the abstract form (6.1)-(6.2), let

$$\begin{aligned} y(t)(x) &= u(t,x), \quad t \in J,\ x \in [0,\pi] \\ \phi(\theta)(x) &= \varphi(\theta,x), \quad \theta \in [-r,0],\ x \in [0,\pi] \\ g(t,u) &= \frac{tu(t)}{7+t^2}, \quad t \in J \\ f(t,\phi)(x) &= Q(t,\varphi(\theta,x)), \quad \theta \in [-r,0],\ x \in [0,\pi] \end{aligned}$$

Assume that the function Q satisfies the following condition

(i) $|Q(t,u)| \leq \frac{1}{e^{t+2}} + \frac{1}{t+8}\|u\|_C$ for each $(t,u) \in J \times \mathbb{R}$.

and for each $u, \overline{u} \in \mathbb{R}$ and $t \in J$ we have

$$|g(t,u) - g(t,\overline{u})| \leq \frac{1}{8}\|u - \overline{u}\|_C$$

Hence condition (H2) is satisfied with $L = \frac{1}{8}$

It is clear that conditions (H1)-(H3) are satisfied. We shall show that (6.4) holds with

$$q(t) = \frac{1}{t+8}, t \in [0,1]$$

$$M = 1, b = 1, \|q\|_\infty = \frac{1}{8}$$

Indeed, we have

$$\frac{b^\alpha(\|q\|_\infty + L)}{\Gamma(\alpha+1)}(1+M) \leq \frac{1}{2\Gamma(\alpha+1)} < 1,\ for\ each\ \alpha \in (0,1].$$

Hence, Theorem 6.2.1 implies that problem (6.8)-(6.11) has a mild solution u on $[-r,1] \times \in [0,\pi]$.

Conclusion

In this thesis we consider some functional semi-linear differential equations of fractional order. By using resolvent operator theory combined with suitable fixed points theorems, we establish some existence and uniqueness results for problems with various delays, more precisely for finite delay, infinite delay and state-dependent delay.

Our main direction in the future is to consider and extend our results to the multivalued case by considering differential inclusions for both cases, convex and non-convex right-hand side.

We will also be interested by considering differential equations and inclusions with impulses.

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