Contents

1 Introduction

1.1 About wave models with and without damping terms

Let us briefly repeat some statements about energy estimates and $L^p - L^q$ decay estimates on the conjugate line for some wave models with and without dissipation.

1.1.1 Free wave equation

The Cauchy problem to the homogeneous linear wave equation

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1.1}$$

where $u_0, u_1 \in \mathcal{C}_0^\infty$ was well studied by many authors. One of the most cited results is the a-priori estimate along the conjugate line

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} \le C(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}\big(\|u_0\|_{L^{p,N_p+1}} + \|u_1\|_{L^{p,N_p}}\big) \tag{1.1.2}$$

for $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p \le 2$ and $N_p = n(\frac{1}{p} - \frac{1}{q})$. There exist different approaches in order to prove the estimate (1.1.2). In a paper by von Wahl one can find calculations with exact solution representations for the free wave equation to the space dimension $n = 3$, [vW71]. Another ansatz is achieved for instance by Strichartz, Brenner and Pecher in [[38], [5], [29]]. They applied Fourier integral operators and the method of stationary phase. The classical definition of energy for the linear wave equation is stated by

$$E_W[u](t) := \frac{1}{2}\Big(\|u_t(t,\cdot)\|_{L^2}^2 + \|\nabla u(t,\cdot)\|_{L^2}^2\Big). \tag{1.1.3}$$

Deductively, the inequality (1.1.2) comprises the estimate $E_W[u](t) \le C\big(\|u_0\|_{H^1} + \|u_1\|_{L^2}\big)$ for all times t. However, this estimate is weaker than the energy conservation $E_W[u](t) \equiv E_W[u](0)$ which is a consequence of $E_W[u]'(t) = 0$ for all times t.

1.1.2 Wave models with external damping

A further problem of interest is the Cauchy problem for the dissipative wave equation with external damping

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1.4}$$

where $u_0, u_1 \in \mathcal{C}_0^\infty$. Matsumura proved in [21] that the dissipation has an improving influence on the $L^p - L^q$ decay estimate, that is,

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^2} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\big(\|u_0\|_{L^{p,N_p+1}} + \|u_1\|_{L^{p,N_p}}\big) \tag{1.1.5}$$

for space dimension $n \geq 2$ with parameters p, q lying on the conjugate line, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p \leq 2$ together with $N_p = n(\frac{1}{p} - \frac{1}{q})$.

1.1.3 Structural damped wave models

Let us consider the Cauchy problem for the dissipative wave equation with structural damping

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + \mu(-\Delta)^\delta u_t(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \delta \in (0,1], & x \in \mathbb{R}^n, \end{cases} \tag{1.1.6}$$

where $u_0 \in H^1$ and $u_1 \in L^2$. Lu-Reissig proved in [20] for space dimensions $n \geq 2$ the $L^2 - L^2$ estimates

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^2} \leq C\big((1+t)^{-\frac{1}{2\delta}}\|u_0\|_{H^1} + \|u_1\|_{L^2}\big) \text{ for } \delta \in (0, 1/2], \tag{1.1.7}$$

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^2} \leq C\big((1+t)^{-\frac{1}{2(1-\delta)}}\|u_0\|_{H^1} + \|u_1\|_{L^2}\big) \text{ for } \delta \in [1/2, 1]. \tag{1.1.8}$$

Remark 1.1.1. *The last estimates show that in general we have no decay estimate for the classical wave type energy. If $u_1 \equiv 0$, then we have energy decay.*

1.2 Cauchy problem for wave models with time-dependent coefficients

1.2.1 Wave models with time-dependent propagation speed

The aim of many papers is to study the Cauchy problem of the linear wave equation

$$\begin{cases} u_{tt}(t,x) - a^2(t)\Delta u(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2.1}$$

where $u_0, u_1 \in \mathcal{C}_0^\infty$. Then, using suitable assumptions about the smooth, positive and bounded coefficient $a = a(t)$, the a-priori estimate

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} \leq C(1+t)^{s_0 - \frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}\big(\|u_0\|_{L^{p,N_p+1}} + \|u_1\|_{L^{p,N_p}}\big) \tag{1.2.2}$$

holds with real value $s_0 \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p \leq 2$ and $N_p = n(\frac{1}{p} - \frac{1}{q})$. There is a connection between the oscillating behavior of $a(t)$ and the loss of decay, i.e. the size of $s_0 \geq 0$ describes how the decay rate differs from the classical one stated in (1.1.2). This was compiled in a series of papers by Reissig/Yagdjian in 1999/2000 [31], [32], [33], [34], [35], and continued by Reissig/Smith in 2005 [30]. The latter also proved that for some coefficients

$a(t)$ there does not exist a real number s_0 such that (1.2.2) is satisfied, that is, an infinite loss of decay appears. Furthermore, we are interested in estimates for an appropriate energy $E_W[u]$ to the solutions of (1.2.1) which is defined as

$$E_W[u](t) := \frac{1}{2}\Big(\|u_t(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + a(t)^2\|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\Big). \tag{1.2.3}$$

For reasons of the time-dependent speed of propagation we cannot expect energy conservation. However, there are results about generalized energy conservation. In case of the wave equation we can speak about generalized energy conservation if the estimates $E_W[u](t) \cong E_W[u](0)$ hold true for all times $t \geq 0$. Initial work has been done by Reissig/Smith in 2005, [30] for oscillating coefficients belonging to C^2. Later, the benefit of higher regularity or even Gevrey regularity were discussed in papers of Hirosawa in order to prove generalized energy conservation, [14], [Hir10].

1.2.2 Wave models with time-dependent external dissipation

Let us provide an insight into the field of study of wave equations with time-dependent external dissipation

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + b(t)u_t(t,x) = 0, & (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2.4}$$

where $u_0, u_1 \in \mathcal{C}_0^\infty$. We mention some statements about special classes of equations (1.2.4) and explanations about the influence of the dissipation term $b(t)u_t$ on asymptotic properties for the solutions. If we choose $b(t) = \mu(1+t)^{-1}$ with a constant $\mu > 0$, that is, we are interested in scale-invariant cases, we derive the $L^p - L^q$ decay estimates as follows:

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^p} \leq C(1+t)^{\max\{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\mu}{2}, -n(\frac{1}{p}-\frac{1}{q})-1\}}\big(\|u_0\|_{L^{p,N_p+1}} + \|u_1\|_{L^{p,N_p}}\big), \tag{1.2.5}$$

where we assume again $\frac{1}{p}+\frac{1}{q} = 1$ and $1 < p \leq 2$ together with an integer $N_p = n(\frac{1}{p}-\frac{1}{q})$. This result has been proven by Wirth by applying the theory of special functions and exact solution representations in the extended phase space, [41]. Thus, it appears that the parameter μ influences the decay rate wherefore we call this dissipation term critical. In subsequent papers Wirth differentiated between effective and non-effective terms of dissipation.

The results basically were that the coefficient $b(t)$ in non-effective dissipation terms decays faster than this critical term $b(t) = \mu(1+t)^{-1}$. This implies a decay in the $L^p - L^q$ estimate close to that of the free wave equation (1.1.2), [42]. Moreover, if $b = b(t)$ belongs to L^1 we can even prove that the solutions behave asymptotically like the solutions to a related Cauchy problem of the free wave equation with respect to their energies, [43]. This is a result about scattering behavior.

In addition, damped wave equations with effective dissipation were studied in [44]. They are mainly characterized by $tb(t) \to \infty$ as t tends to infinity and $1/b \notin L^1$. Under suitable conditions for $b = b(t)$ the effective dissipation terms imply decay rates close to the one for the free damped wave equation stated in (1.1.5).

1.2.3 Structural damped wave models with time-dependent dissipation

Let us consider the Cauchy problem for dissipative wave equations with a time-dependent structural damping

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + \mu(1+t)^{-\gamma}(-\Delta)^{\delta} u_t(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \delta \in (0,1], \ \gamma > 0, \quad x \in \mathbb{R}^n, \end{cases} \tag{1.2.6}$$

where $u_0 \in H^1$ and $u_1 \in L^2$. Lu-Reissig proved in [20] $L^2 - L^2$ estimates which read as follows:

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^2} \le C\big((1+t)^{-\frac{1+\gamma}{2(1-\delta)}}\|u_0\|_{H^1} + \|u_1\|_{L^2}\big) \text{ for } \delta \in (0,1/2],\ \gamma \in (0, 1-2\delta),$$
$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^2} \le C\big((1+t)^{-\frac{1-\gamma}{2\delta}}\|u_0\|_{H^1} + \|u_1\|_{L^2}\big) \text{ for } \delta \in (0,1/2],\ \gamma \in [1-2\delta, 1),$$
$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^2} \le C\big((\log(e+t))^{-\frac{1}{2\delta}}\|u_0\|_{H^1} + \|u_1\|_{L^2}\big) \text{ for } \delta \in (0,1/2],\ \gamma = 1,$$
$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^2} \le C\big((1+t)^{-\frac{1-\gamma}{2\delta}}\|u_0\|_{H^1} + \|u_1\|_{L^2}\big) \text{ for } \delta \in [1/2,1],\ \gamma \in (0,1),$$
$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^2} \le C\big((\log(e+t))^{-\frac{1}{2\delta}}\|u_0\|_{H^1} + \|u_1\|_{L^2}\big) \text{ for } \delta \in [1/2,1],\ \gamma = 1.$$

for space dimension $n \ge 2$.

1.3 Some more information about the thesis

1.3.1 Some more explanations about the historical background

We consider the following Cauchy problem for structural damped σ-evolution models with $\sigma > 1$ and with a monotonous and positive function $b = b(t)$ in the dissipation term :

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^{\sigma} u(t,x) + b(t)(-\Delta)^{\delta} u_t(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \delta \in (0,\sigma], & x \in \mathbb{R}^n, \end{cases} \tag{1.3.1}$$

where (u_0, u_1) are initial data taken from the energy space $u_0 \in H^{\sigma}$ and $u_1 \in L^2$. We use the notations

$$u_t = \frac{\partial u}{\partial t}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}, \quad -\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}, \qquad x = (x_1, x_2, \cdots, x_n).$$

In this thesis we are interested among other things in estimates for higher order energies of solutions to the Cauchy problem for structural damped σ-evolution models. The damping term is a special time-dependent case of a family of damping operators which are introduced in Chen and Russell (1982)[6]. Theoretical arguments and empirical studies motivated them to consider such damping operators describing strong or structural damping effects. $L^p - L^q$ decay estimates are studied in the case $\delta = 0$ and $\sigma = 1$ (classical damped wave model with time-dependent external dissipation) in Wirth (2006, 2007)[42], [43] and [44]. In these papers the cases of non-effective or effective dissipation are discussed. $L^p - L^q$ decay estimates for the case $b(t) \equiv 1$ and $\delta = \sigma = 1$ which corresponds to a visco-elastic model in Shibata (2000)[37] or to a wave model with internal damping in Guenther and Lee (1996) are studied in Shibata (2000) [37]. The case $\delta \in (0,1]$ of a mixed problem for a bounded domain in $\mathbb{R}^n$ has been studied in Chen and Trigiani (1989,1990) [7], [8]. Here properties of the solutions are obtained from the properties of the corresponding semigroup to the abstract differential equation with structural damping.

1.4 Objectives of this thesis

The main aim of this thesis is to give contributions to the qualitative behavior of solutions to structural damped σ-evolution models. Of special interest is the influence on the damping term itself on qualitative properties as parabolic effect, decay estimates for energies of higher order, respectively, smoothing effect and finally on $L^p - L^q$ decay estimates on the conjugate line and away from this line. To be more concrete, we investigate for $\sigma > 1$ the Cauchy problem for the model

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\sigma u(t,x) + b(t)(-\Delta)^\delta u_t(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \delta \in (0,\sigma], & x \in \mathbb{R}^n, \end{cases} \tag{1.4.1}$$

in the case of Schwartz or Sobolev data. The main tasks are the followings:

- to develop a WKB analysis to derive representation formulas basing on Fourier multipliers for the solutions,
- to propose a classification of time-dependent dissipations to describe different properties of solutions,
- to explain the parabolic effect and smoothing properties by using the obtained representation formulas,
- to discuss the optimality of obtained results,
- to derive $L^p - L^q$ estimates on the conjugate line,
- to derive $L^p - L^q$ estimates away from the conjugate line, in particular, $L^1 - L^1$ estimates are of interest.

Energy decay estimates for variable coefficient damped dissipation terms are available from the literature under several assumptions. We refer only to some of the most cited references, the concrete relation to our results will be given later throughout the main part of the thesis. The main question is under which conditions the energy decays, that is, it tends to zero. So the main objective of this thesis is the derivation of the more general $L^2 - L^2$ estimates for solutions. These estimates rely on more structural properties of representations of solutions than estimates in the L^2-scale and can not be deduced by the same methods as the above mentioned results. Our approach is based on the one hand on explicit representations in a special case and on the other hand on asymptotic representations combined with an extensive phase plane analysis under more general assumptions, mostly adapted from the treatment of problems in WKB analysis. For completeness we mention the book of K. Yagdjian, [45], and the consideration of wave equations with increasing speed of propagation by M. Reissig and K. Yagdjian, [33], [34] and [35], for the combination with dissipation and mass terms [Rei01] and [15]. The method we used is based on the Fourier transform and Fourier multiplier representations (also called WKB representations) of solutions, therefore we consider only purely time-dependent dissipation terms. The consideration of time and spatial dependencies in the coefficient in the language of pseudo-differential and Fourier integral operators yields essential difficulties in connection with the time asymptotics and is therefore not considered in these thesis.

1.4.1 Content of this thesis

The schedule is as follows: In Chapter 2 we study the simplest case of time-independent dissipation. It turns out that even this example provides us with a lot of ideas and gives some feeling for the more general results proven later. In Chapters 3 and 4 we study the case of strictly decreasing or strictly increasing dissipations, respectively. In different cases we propose a different WKB-analysis. The main tools to develop a suitable WKB-analysis are definition of zones, symbol classes, to carry out diagonalization procedure, to estimate the fundamental solution and to glue the representations in different zones together. This will give us estimates for energies of higher order. In Chapter 5 we will discuss another property of solutions to structural damped σ-evolution models, the smoothing effect. The optimality of our results will be shown in Chapter 6 by studying scale invariant models. This allows to transform the equations to ordinary differential equations. By discussing these equations the optimality of our results can be shown. In Chapter 7 we derive $L^1 - L^\infty$ estimates and interpolation techniques gives immediately the desired $L^p - L^q$ estimates on the conjugate line. For applications to nonlinear problems such estimates away from the conjugate line are important. Finally, in Chapter 8 we discuss $L^1 - L^1$ estimates. Some concluding remarks and open problems complete the thesis.

2 Time-independent dissipation

In this Chapter we study the decay rate of the energies of higher order for solution to the Cauchy problem for structural damped σ-evolution models. We consider the special case of (1.3.1) for $\sigma > 1$ and $b = b(t)$ is a constant positive function. We set $b \equiv \mu > 0$ and devote to the model

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\sigma u(t,x) + \mu(-\Delta)^\delta u_t(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \sigma > 1, \quad \text{and} \quad \delta \in (0,\sigma]. \end{cases} \tag{2.0.1}$$

Introducing the partial Fourier transform $F_{x\to\xi}(u)(t,\xi) =: \hat{u}(t,\xi)$ after partial Fourier transformation of (2.0.1) we get

$$\begin{cases} \hat{u}_{tt}(t,\xi) + |\xi|^{2\sigma}\hat{u}(t,\xi) + \mu|\xi|^{2\delta}\hat{u}_t(t,\xi) = 0, \\ \hat{u}(0,\xi) =: \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) =: \hat{u}_1(\xi), \quad \sigma > 1, \quad \text{and} \quad \delta \in (0,\sigma]. \end{cases} \tag{2.0.2}$$

Here the characteristic equation is

$$(E_q): \qquad \lambda^2 + \mu|\xi|^{2\delta}\lambda + |\xi|^{2\sigma} = 0. \tag{2.0.3}$$

Hence the characteristic roots are

$$\lambda_1(\xi) = \frac{-\mu|\xi|^{2\delta} + \sqrt{\mu^2|\xi|^{4\delta} - 4|\xi|^{2\sigma}}}{2}; \quad \lambda_2(\xi) = \frac{-\mu|\xi|^{2\delta} - \sqrt{\mu^2|\xi|^{4\delta} - 4|\xi|^{2\sigma}}}{2}. \tag{2.0.4}$$

We have the following explicit representation of the solution:

$$\hat{u}(t,\xi) = \frac{\lambda_1 \exp(\lambda_2 t) - \lambda_2 \exp(\lambda_1 t)}{\lambda_1 - \lambda_2}\hat{u}_0(\xi) + \frac{\exp(\lambda_1 t) - \exp(\lambda_2 t)}{\lambda_1 - \lambda_2}\hat{u}_1(\xi) \text{ if } \lambda_1 \neq \lambda_2,$$
$$\hat{u}(t,\xi) = \exp(\lambda_1 t)\Big((1 - \lambda_1 t)\hat{u}_0(\xi) + t\hat{u}_1(\xi)\Big) \text{ if } \lambda_1 = \lambda_2.$$

Division of the extended phase space into zones

In this case we have a very simple division of the extended phase space. This we feel in the division of the extended phase space $\{(t,\xi) \in [0,\infty) \times \mathbb{R}^n\}$ into the following zones, where ε is here and in the following a small positive constant:

hyperbolic zone $\quad Z_{hyp}(\varepsilon) := \{(t,\xi) \; : \; \frac{\mu}{2}|\xi|^{2\delta-\sigma} \leq 1 - \varepsilon\},$

reduced zone $\quad Z_{red}(\varepsilon) := \{(t,\xi) \quad 1 - \varepsilon \leq \frac{\mu}{2}|\xi|^{2\delta-\sigma} \leq 1 + \varepsilon\},$

elliptic zone $\quad Z_{ell}(\varepsilon) := \{(t,\xi) \quad \frac{\mu}{2}|\xi|^{2\delta-\sigma} \geq 1 + \varepsilon\}.$

We introduce separation lines. By $|\xi|_k = \big(\frac{2(1+(-1)^{k+1}\varepsilon)}{\mu}\big)^{\frac{1}{2\delta-\sigma}}, k = 1, 2$, we denote the separation line between the elliptic zone and the reduced zone ($k = 1$), and between the reduced zone and the hyperbolic zone ($k = 2$). For the further calculations we use the following properties of λ_1 and λ_2:

Lemma 2.0.1. *It holds*
in the hyperbolic zone :

- $\frac{1}{|\lambda_1(\xi)-\lambda_2(\xi)|} \leq |\xi|^{-\sigma}, \qquad \lambda_1(\xi)\lambda_2(\xi) = |\xi|^{2\sigma},$

- $\lambda_1(\xi) = \overline{\lambda_2(\xi)}, \qquad |\lambda_1(\xi)| = |\lambda_2(\xi)| = |\xi|^{\sigma},$

in the elliptic zone :

- $\frac{1}{|\lambda_1(\xi)-\lambda_2(\xi)|} \leq |\xi|^{-2\delta}, \qquad \lambda_1(\xi)\lambda_2(\xi) = |\xi|^{2\sigma},$

- $\lambda_2(\xi) \leq \lambda_1(\xi) \leq 0, \quad |\lambda_1(\xi)| \leq |\lambda_2(\xi)|,$

- $-\mu|\xi|^{2\delta} \leq \lambda_2(\xi) \leq -\frac{\mu}{2}|\xi|^{2\delta}, \quad -\frac{2}{\mu}|\xi|^{2\sigma-2\delta} \leq \lambda_1(\xi) \leq -\frac{1}{\mu}|\xi|^{2\sigma-2\delta},$

in the reduced zone :

- $C_1 \leq \Re\lambda_2(\xi) \leq \Re\lambda_1(\xi) \leq C_2 < 0,$

- $C_3 \leq |\lambda_1(\xi)| \leq |\lambda_2(\xi)| \leq C_4.$

2.1 Time-independent dissipation – $\delta = \sigma/2$

In this section we study the special case of (2.0.1) for $\delta = \sigma/2$

$$\begin{cases} u_{tt} + (-\Delta)^{\sigma} u + \mu(-\Delta)^{\sigma/2} u_t = 0, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \sigma > 1. \end{cases} \tag{2.1.1}$$

After partial Fourier transformation we have

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + \mu|\xi|^{\sigma}\hat{u}_t = 0, \quad \hat{u}(0,\xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi), \quad \sigma > 1. \tag{2.1.2}$$

We have

$$(E_q): \qquad \lambda^2 + \mu|\xi|^{\sigma}\lambda + |\xi|^{2\sigma} = 0.$$

Then the discriminant $\mu^2|\xi|^{2\sigma} - 4|\xi|^{2\sigma} = |\xi|^{2\sigma}(\mu^2 - 4)$.

2.1.1 Treatment in the case $\mu < 2$

Proposition 2.1.1. *The following estimates hold for* $\mu \in (0, 2)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C|\xi|^{\sigma}t\big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq \sigma,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\big(-C|\xi|^{\sigma}t\big)\Big(\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big)\Big) \quad \text{for} \quad |\beta| \geq 0.$$

Proof. In this case we have $|\xi|^{2\sigma}(\mu^2-4)<0$ hence, $\lambda_1, \lambda_2 \in \mathbb{C}$. By using the properties of λ_1 and λ_2 from Lemma 2.0.1 and using the representation of solution we have

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}(t,\xi)| &\lesssim \max\Big\{|\lambda_1 \exp\big(\lambda_2 t\big)|, |\lambda_2 \exp\big(\lambda_1 t\big)|\Big\}\frac{|\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_0(\xi)| \\
&+ \max\Big\{|\exp\big(\lambda_1 t\big)|, |\exp\big(\lambda_2 t\big)|\Big\}\frac{|\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_1(\xi)| \\
&\lesssim \frac{|\lambda_2|}{|\lambda_1-\lambda_2|}|\xi|^{|\beta|}|\exp\big(\lambda_1 t\big)||\hat{u}_0(\xi)| + \frac{1}{|\lambda_1-\lambda_2|}|\xi|^{|\beta|}|\exp\big(\lambda_1 t\big)|\hat{u}_1(\xi)|.
\end{aligned}
$$

Consequently, we derived for $|\beta| \geq \sigma$ the a-priori estimates

$$
|\xi|^{|\beta|}|\hat{u}(t,\xi)| \exp\big(-C|\xi|^{\sigma}t\big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big).
$$

In the same way we conclude for $|\beta| \geq 0$ the a-priori estimates

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \quad &\lesssim \exp\big(-\frac{\mu}{2}|\xi|^{\sigma}t\big)\Big(\frac{|\lambda_1\lambda_2|}{|\lambda_1-\lambda_2|}|\xi|^{|\beta|}|\hat{u}_0(\xi)| + \frac{|\lambda_1|}{|\lambda_1-\lambda_2|}|\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) \\
&\lesssim \exp\big(-C|\xi|^{\sigma}t\big)\Big(\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big)\Big).
\end{aligned}
$$

All desired estimates are proved. □

2.1.2 Treatment in the case $\mu = 2$

Proposition 2.1.2. *The following estimates hold for $\mu = 2$:*

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}(t,\xi)| &\lesssim \exp\big(-C|\xi|^{\sigma}t\big)\big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\big) \quad \textit{for} \quad |\beta| \geq \sigma, \\
|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| &\lesssim \exp\big(-C|\xi|^{\sigma}t\big)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big) \quad \textit{for} \quad |\beta| \geq 0.
\end{aligned}
$$

Proof. In this case we have $(\mu^2-4)|\xi|^{2\sigma} = 0$. Therefore, let us use the representation of solution

$$
\begin{aligned}
\hat{u}(t,\xi) &= \exp(\lambda_1 t)\big((1-\lambda_1 t)\hat{u}_0(\xi) + t\hat{u}_1(\xi)\big), \\
\hat{u}_t(t,\xi) &= \exp(\lambda_1 t)\big(-\lambda_1^2 t\hat{u}_0(\xi) + (1+\lambda_1 t)\hat{u}_1(\xi)\big).
\end{aligned}
$$

By using the properties of λ_1 and λ_2 from Lemma 2.0.1 we have

$$
|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C|\xi|^{\sigma}t\big)\big((1+|\lambda_1|t)|\xi|^{|\beta|}|\hat{u}_0(\xi)| + t|\xi|^{|\beta|}|\hat{u}_1(\xi)|\big).
$$

Consequently, we derived for $|\beta| \geq \sigma$ the a-priori estimates

$$
|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C|\xi|^{\sigma}t\big)\big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\big).
$$

In the same way we conclude for $|\beta| \geq 0$ the a-priori estimates

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| &\lesssim \exp\big(-C|\xi|^{\sigma}t\big)\big(|\lambda_1|^2 t|\xi|^{|\beta|}|\hat{u}_0(\xi)| + (1+|\lambda_1|t)|\xi|^{|\beta||}\hat{u}_1(\xi)|\big) \\
&\lesssim \exp\big(-C|\xi|^{\sigma}t\big)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big),
\end{aligned}
$$

here C is used as a universal constant. All desired estimates are proved. □

2.1.3 Treatment in the case $\mu > 2$

Proposition 2.1.3. *The following estimates hold for* $\mu \in (2,\infty)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C|\xi|^{\sigma}t\big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\big(-C|\xi|^{\sigma}t\big)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big) \quad \text{for} \quad |\beta| \geq 0.$$

Proof. In this case we have $|\xi|^{2\sigma}(\mu^2-4) > 0$. We arrive at $\lambda_1,\lambda_2 \in \mathbb{R}$. By using the properties of λ_1 and λ_2 from Lemma 2.0.1 and using the representation of solution we have

$$\begin{aligned}|\xi|^{|\beta|}|\hat{u}(t,\xi)| &\lesssim \max\Big\{|\lambda_1|\exp\big(\lambda_2 t\big), |\lambda_2|\exp\big(\lambda_1 t\big)\Big\}\frac{|\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_0(\xi)| \\ &\quad + \max\Big\{\exp\big(\lambda_1 t\big), \exp\big(\lambda_2 t\big)\Big\}\frac{|\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_1(\xi)| \\ &\lesssim \frac{|\lambda_2|}{|\lambda_1-\lambda_2|}|\xi|^{|\beta|}\exp\big(\lambda_1 t\big)|\hat{u}_0(\xi)| + \frac{1}{|\lambda_1-\lambda_2|}|\xi|^{|\beta|}\exp\big(\lambda_1 t\big)|\hat{u}_1(\xi)|.\end{aligned}$$

Consequently, we derived for $|\beta| \geq \sigma$ the a-priori estimates

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C|\xi|^{\sigma}t\big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big).$$

In the same way we conclude for $|\beta| \geq 0$ the a-priori estimates

$$\begin{aligned}|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| &\lesssim \frac{|\lambda_1\lambda_2|}{|\lambda_1-\lambda_2|}|\xi|^{|\beta|}\exp\big(\lambda_1 t\big)|\hat{u}_0(\xi)| + \frac{|\lambda_2|}{|\lambda_1-\lambda_2|}|\xi|^{|\beta|}\exp\big(\lambda_1 t\big)|\hat{u}_1(\xi)| \\ &\lesssim \exp\big(-C|\xi|^{\sigma}t\big)|\xi|^{|\beta|+\sigma}\exp\big(-C|\xi|^{\sigma}t\big)|\hat{u}_0(\xi)| + |\xi|^{|\beta|}\exp\big(-C|\xi|^{\sigma}t\big)|\hat{u}_1(\xi)|.\end{aligned}$$

All desired estimates are proved. □

2.1.4 Energy estimates

The statement for $\mu \in (0,2)$, $\mu = 2$ and $\mu \in (2,\infty)$ are same. We distinguish between *small frequencies* $\{0 < |\xi| \leq 1\}$ and *large frequencies* $\{|\xi| \geq 1\}$.
For small frequencies we may use the estimates as before. They imply a potential type decay. This we will show in the following statement.

Corollary 2.1.4. *The following estimates hold for the small frequencies* $0 < |\xi| \leq 1$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim (1+t)^{-\frac{|\beta|}{\sigma}}|\hat{u}_0(\xi)| + (1+t)^{-\frac{|\beta|-\sigma}{\sigma}}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim (1+t)^{-\frac{|\beta|+\sigma}{\sigma}}|\hat{u}_0(\xi)| + (1+t)^{-\frac{|\beta|}{\sigma}}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq 0.$$

For large frequencies we may use the estimates as before, too. They imply an exponential type decay. This we will show in the following statement.

Corollary 2.1.5. *The following estimates hold for all large frequencies* $|\xi| \geq 1$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp(-Ct)\big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\big) \quad \text{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp(-Ct)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big) \quad \text{for} \quad |\beta| \geq 0.$$

2.1.5 Conclusion

Taking into consideration all the estimates from this section we arrive at the estimates for the energies of higher order $\|\nabla^\beta u(t,\cdot)\|_{L^2}$ and for $\|\nabla^\beta u_t(t,\cdot)\|_{L^2}$ in the case $\delta = \sigma/2$.

Theorem 2.1.6. *The solution $u = u(t,x)$ to (2.1.1) with $\delta = \frac{\sigma}{2}$ satisfies the following estimates :*

$$\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 \lesssim (1+t)^{-2\frac{|\beta|}{\sigma}} \|u_0\|_{H^{|\beta|}}^2 + (1+t)^{-2\frac{|\beta|-\sigma}{\sigma}} \|u_1\|_{H^{|\beta|-\sigma}}^2 \quad \text{for} \quad |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 \lesssim (1+t)^{-2\frac{|\beta|+\sigma}{\sigma}} \|u_0\|_{H^{|\beta|+\sigma}}^2 + (1+t)^{-2\frac{|\beta|}{\sigma}} \|u_1\|_{H^{|\beta|}}^2 \quad \text{for} \quad |\beta| \geq 0.$$

2.2 Time-independent dissipation – $\delta \in (0, \sigma/2)$

In this section we study the other cases $\delta \neq \frac{\sigma}{2}$. Firstly we study the case $\delta \in (0, \sigma/2)$, $\sigma > 1$ that is, the model :

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = 0, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \delta \in (0, \sigma/2). \end{cases} \tag{2.2.1}$$

After partial Fourier transformation we have

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + \mu|\xi|^{2\delta}\hat{u}_t = 0, \quad \hat{u}(0,\xi) =: \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) =: \hat{u}_1(\xi), \quad \delta \in (0, \sigma/2). \tag{2.2.2}$$

2.2.1 Division of the extended phase space

If we consider the set $M_{p_0} := \{(t,\xi) : |\xi| \leq p_0\}$, where p_0 is sufficiently small, then this part is completely contained in the elliptic zone. The characteristic roots are real, so from the point of WKB analysis the model is elliptic, a property which determines the notation of this zone. If we consider the set $M_{p_1} := \{(t,\xi) : |\xi| \geq p_1\}$, where p_1 is sufficiently large, then this part is completely contained in the hyperbolic zone. The characteristic roots are complex, so from the point of WKB analysis the model is hyperbolic, a property which determines the notation of this zone. In the reduced zone around the separation line between elliptic and hyperbolic region the model has no special type.

2.2.2 Treatment in the hyperbolic zone

Proposition 2.2.1. *The following estimates hold for the large frequencies $|\xi| \geq p_1$, where p_1 is sufficiently large :*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C|\xi|^{2\delta}t\big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq 0,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\big(-C|\xi|^{2\delta}t\big)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big) \quad \text{for} \quad |\beta| \geq 0.$$

Proof. Applying the definition of the hyperbolic zone we have $\mu^2|\xi|^{4\delta} - 4|\xi|^{2\sigma} \leq C(\varepsilon) < 0$. We arrive at $\lambda_1, \lambda_2 \in \mathbb{C}$. By using the properties of λ_1 and λ_2 from Lemma 2.0.1 and the representation of the solution we get

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}(t,\xi)| &\lesssim \max\Big\{|\lambda_1||\exp\big(\lambda_2 t\big)|, |\lambda_2||\exp\big(\lambda_1 t\big)|\Big\}\frac{|\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_0(\xi)| \\
&\quad + \max\Big\{|\exp\big(\lambda_1 t\big)|, |\exp\big(\lambda_2 t\big)|\Big\}\frac{|\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_1(\xi)| \\
&\lesssim \frac{|\lambda_2|}{|\lambda_1-\lambda_2|}|\xi|^{|\beta|}|\exp\big(\lambda_1 t\big)||\hat{u}_0(\xi)| + \frac{1}{|\lambda_1-\lambda_2|}|\xi|^{|\beta|}|\exp\big(\lambda_1 t\big)||\hat{u}_1(\xi)|.
\end{aligned}
$$

Consequently, we derived for $|\beta| \geq 0$ the a-priori estimates

$$
|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C|\xi|^{2\delta}t\big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big).
$$

In the same way we conclude for $|\beta| \geq 0$ the a-priori estimates

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \quad &\lesssim \max\Big\{|\exp\big(\lambda_1 t\big)|, |\exp\big(\lambda_2 t\big)|\Big\}\frac{|\lambda_1\lambda_2||\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_0(\xi)| \\
&\quad + \max\Big\{|\lambda_1||\exp\big(\lambda_1 t\big)|, |\lambda_2||\exp\big(\lambda_2 t\big)|\Big\}\frac{|\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_1(\xi)| \\
&\lesssim |\exp\big(\lambda_1 t\big)|\frac{|\lambda_1\lambda_2||\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_0(\xi)| + |\exp\big(\lambda_1 t\big)|\frac{|\lambda_1||\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_1(\xi)| \\
&\lesssim \exp\Big(-C|\xi|^{2\delta}t\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big).
\end{aligned}
$$

In this way the proposition is proved. □

2.2.3 Treatment in the elliptic zone

Proposition 2.2.2. *The following estimates hold for the small frequencies $|\xi| \leq p_0$, where p_0 is sufficiently small* :

$$
\begin{aligned}
&|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C|\xi|^{2\sigma-2\delta}t\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta}|\hat{u}_1(\xi)|\Big) \quad \textit{for} \quad |\beta| \geq 2\delta, \\
&|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C|\xi|^{2\sigma-2\delta}t\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta}|\hat{u}_0(\xi)| \\
&\qquad + \max\Big\{\exp\Big(-C|\xi|^{2\sigma-2\delta}t\Big)|\xi|^{|\beta|+2\sigma-4\delta}, \exp\big(-C|\xi|^{2\delta}t\big)|\xi|^{|\beta|}\Big\}|\hat{u}_1(\xi)| \quad \textit{for} \quad |\beta| \geq 0.
\end{aligned}
$$

Proof. Applying the definition of elliptic zone we have $\mu^2|\xi|^{4\delta} - 4|\xi|^{2\sigma} \geq C(\varepsilon) > 0$. We arrive at $\lambda_1, \lambda_2 \in \mathbb{R}$. By using the properties of λ_1 and λ_2 from Lemma 2.0.1 and the representation of the solution implies

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}(t,\xi)| &\lesssim \max\Big\{|\lambda_1|\exp\big(\lambda_2 t\big), |\lambda_2|\exp\big(\lambda_1 t\big)\Big\}\frac{|\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_0(\xi)| \\
&\quad + \max\Big\{\exp\big(\lambda_1 t\big), \exp\big(\lambda_2 t\big)\Big\}\frac{|\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_1(\xi)| \\
&\lesssim \exp\big(\lambda_1 t\big)\Big(\frac{|\lambda_2||\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_0(\xi)| + \frac{|\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_1(\xi)|\Big).
\end{aligned}
$$

Consequently, we derived for $|\beta| \geq 2\delta$ the a-priori estimates

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C|\xi|^{2\sigma-2\delta}t\big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta}|\hat{u}_1(\xi)|\Big).$$

In the same way we conclude for $|\beta| \geq 0$ the a-priori estimates

$$\begin{aligned}
|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \quad &\lesssim \max\Big\{\exp\big(\lambda_1 t\big), \exp\big(\lambda_2 t\big)\Big\}\frac{|\lambda_1\lambda_2||\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_0(\xi)| \\
&\quad + \max\Big\{|\lambda_1|\exp\big(\lambda_1 t\big), |\lambda_2|\exp\big(\lambda_2 t\big)\Big\}\frac{|\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_1(\xi)| \\
&\lesssim \exp\big(\lambda_1 t\big)\frac{|\lambda_1\lambda_2||\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_0(\xi)| \\
&\quad + \max\Big\{|\lambda_1|\exp\big(\lambda_1 t\big), |\lambda_2||\exp\big(\lambda_2 t\big)|\Big\}\frac{|\xi|^{|\beta|}}{|\lambda_1-\lambda_2|}|\hat{u}_1(\xi)| \\
&\lesssim \exp\Big(-C|\xi|^{2\sigma-2\delta}t\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta}|\hat{u}_0(\xi)| \\
&\quad + \max\Big\{\exp\Big(-C|\xi|^{2\sigma-2\delta}t\Big)|\xi|^{|\beta|+2\sigma-4\delta}, \exp\big(-C|\xi|^{2\delta}t\big)|\xi|^{|\beta|}\Big\}|\hat{u}_1(\xi)|.
\end{aligned}$$

In this way the proposition is proved. ☐

2.2.4 Treatment in the reduced zone

Proposition 2.2.3. *The following estimates hold for all frequencies $p_1 \leq |\xi| \leq p_0$, where p_0 or p_1 is sufficiently small or large, respectively* :

$$\begin{aligned}
|\hat{u}(t,\xi)| &\lesssim \exp\big(-Ct\big)\big(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|\big), \\
|\hat{u}_t(t,\xi)| &\lesssim \exp\big(-Ct\big)\big(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|\big).
\end{aligned}$$

Proof. In the reduced zone we have $0 < C_1 \leq |\xi| \leq C_2$. By using the following properties of λ_1 and λ_2 from Lemma 2.0.1 and the representation of the solution we get:

$$\begin{aligned}
|\hat{u}(t,\xi)| \quad &\lesssim |\exp(\lambda_2 t) - \lambda_2 t\exp(\tilde{\lambda}_1 t)||\hat{u}_0(\xi)| + t|\exp(\tilde{\lambda}_2 t)||\hat{u}_1(\xi)| \\
&\lesssim |\lambda_2|(1+t)\exp(-Ct)|\hat{u}_0(\xi)| + t\exp(-Ct)|\hat{u}_1(\xi)|.
\end{aligned}$$

Consequently,

$$|\hat{u}(t,\xi)| \lesssim \exp\big(-Ct\big)\big(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|\big).$$

In the same way we conclude the a-priori estimates

$$\begin{aligned}
|\hat{u}_t(t,\xi)| \quad &\lesssim |\lambda_1\lambda_2 t\exp(\tilde{\lambda}_3 t)||\hat{u}_0(\xi)| + |\exp(\lambda_1 t) - \lambda_2 t\exp(\tilde{\lambda}_4 t)||\hat{u}_1(\xi)| \\
&\lesssim |\lambda_1\lambda_2|(1+t)\exp(-Ct)|\hat{u}_0(\xi)| + |\lambda_2|(1+t)\exp(-Ct)|\hat{u}_1(\xi)|.
\end{aligned}$$

Consequently,

$$|\hat{u}_t(t,\xi)| \lesssim \exp\big(-Ct\big)\big(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|\big).$$

In this way the proposition is proved. ☐

2.2.5 Energy estimates

We distinguish between *small frequencies* $\{0 < |\xi| \leq p_0\}$ and *large frequencies* $\{|\xi| \geq p_1\}$. For large frequencies we may use the estimates from Proposition 2.2.1. They imply an exponential type decay. This we will show in the following statement.

Corollary 2.2.4. *For large frequencies $|\xi| \geq p_1$ the following estimates hold for all $t \in [0,\infty)$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp(-Ct)\big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\big) \quad \text{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp(-Ct)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big) \quad \text{for} \quad |\beta| \geq 0.$$

For small frequencies we may use the estimates from Proposition 2.2.2. They imply a potential type decay. This we will show in the following statement.

Corollary 2.2.5. *For small frequencies $0 < |\xi| \leq p_0$ the following estimates hold for all $t \in [0,\infty)$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim (1+t)^{-\frac{|\beta|}{2(\sigma-\delta)}}|\hat{u}_0(\xi)| + (1+t)^{-\frac{|\beta|-2\delta}{2(\sigma-\delta)}}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq 2\delta,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim (1+t)^{-\frac{|\beta|+2\sigma-2\delta}{2(\sigma-\delta)}}|\hat{u}_0(\xi)| + (1+t)^{\frac{|\beta|+2\sigma-4\delta}{2(\sigma-\delta)}}|\hat{u}_1(\xi)| + (1+t)^{-\frac{|\beta|}{2\delta}}|\hat{u}_1(\xi)|$$
$$\text{for} \quad |\beta| \geq 0.$$

2.2.6 Conclusion

Taking into consideration all the estimates from this section we arrive at the estimates for the energies of higher order $\|\nabla^\beta u(t,\cdot)\|_{L^2}$ and for $\|\nabla^\beta u_t(t,\cdot)\|_{L^2}$ in the case $\delta \in (0,\sigma/2)$.

Theorem 2.2.6. *The solution $u = u(t,x)$ to (2.2.1) satisfies in the case $\delta \in (0,\sigma/2)$ the following estimates* :

$$\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 \lesssim (1+t)^{-\frac{|\beta|}{\sigma-\delta}}\|u_0\|_{H^{|\beta|}}^2 + (1+t)^{-\frac{|\beta|-2\delta}{\sigma-\delta}}\|u_1\|_{H^{|\beta|-\sigma}}^2 \quad \text{for } |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 \lesssim (1+t)^{-\frac{|\beta|+2\sigma-2\delta}{\sigma-\delta}}\|u_0\|_{H^{|\beta|+\sigma}}^2 + (1+t)^{\max\{-\frac{|\beta|+2\sigma-4\delta}{\sigma-\delta},-\frac{|\beta|}{\delta}\}}\|u_1\|_{H^{|\beta|}}^2 \quad \text{for } |\beta| \geq 0.$$

Remark 2.2.1. *If we set formally $\delta = \sigma/2$ in the estimates from Theorem 2.2.6, then we get the estimates from Theorem 2.1.6.*

2.3 Time-independent dissipation – $\delta \in (\sigma/2,\sigma]$

In this section we study the special case of (2.0.1) for $\delta \in (\sigma/2,\sigma]$, $\sigma > 1$ that is, the model :

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = 0, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \delta \in (\sigma/2,\sigma]. \end{cases} \tag{2.3.1}$$

After partial Fourier transformation we have

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + \mu|\xi|^{2\delta}\hat{u}_t = 0, \quad \hat{u}(0,\xi) =: \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) =: \hat{u}_1(\xi), \quad \delta \in (\sigma/2,\sigma].$$

2.3.1 Division of the extended phase space

We divide the extended phase space into the same zones as in Section 2.2.1. But there appears a big difference. If we consider the set $M_{p_0} := \{(t,\xi) : |\xi| \leq p_0\}$, where p_0 is sufficiently small, then this part is completely contained in the hyperbolic zone. And if we consider the set $M_{p_1} := \{(t,\xi) : |\xi| \geq p_1\}$, where p_1 is sufficiently large, then this part is completely contained in the elliptic zone. For this reason we have now a 'converse division' in comparison with that one from Section 2.2.1.

2.3.2 Treatment in the hyperbolic zone

Proposition 2.3.1. *The following estimates hold for small frequencies $|\xi| \leq p_0$, where p_0 is sufficiently small* :

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C|\xi|^{2\delta}t\big)\big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\big) \quad \textit{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\big(-C|\xi|^{2\delta}t\big)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big) \quad \textit{for} \quad |\beta| \geq 0.$$

Proof. The proof is the same as the proof to Proposition 2.2.1. □

2.3.3 Treatment in the elliptic zone

Proposition 2.3.2. *The following estimates hold for large frequencies $|\xi| \geq p_1$, where p_1 is sufficiently large* :

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C|\xi|^{2\sigma-2\delta}t\big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta}|\hat{u}_1(\xi)|\Big) \quad \textit{for} \quad |\beta| \geq 0,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C|\xi|^{2\sigma-2\delta}t\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta}|\hat{u}_0(\xi)|$$
$$+\max\Big\{\exp\Big(-C|\xi|^{2\sigma-2\delta}t\Big)|\xi|^{|\beta|+2\sigma-4\delta}, \exp\big(-C|\xi|^{2\delta}t\big)|\xi|^{|\beta|}\Big\}|\hat{u}_1(\xi)| \quad \textit{for} \quad |\beta| \geq 0.$$

Proof. The proof is the same as the proof to Proposition 2.2.2. □

2.3.4 Treatment in the reduced zone

Proposition 2.3.3. *The following estimates hold for all frequencies $p_1 \leq |\xi| \leq p_0$, where p_0 or p_1 is sufficiently small or large, respectively* :

$$|\hat{u}(t,\xi)| \lesssim \exp\big(-Ct\big)\big(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|\big),$$
$$|\hat{u}_t(t,\xi)| \lesssim \exp\big(-Ct\big)\big(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|\big).$$

Proof. The proof coincides with the proof of Proposition 2.2.3. □

2.3.5 Energy estimates

We distinguish between *small frequencies* $\{0 < |\xi| \leq p_0\}$ and *large frequencies* $\{|\xi| \geq p_1\}$. For large frequencies we may use the estimates from Proposition 2.3.2. They imply an exponential type decay. This we will show in the following statement.

Corollary 2.3.4. *For large frequencies $|\xi| \geq p$ the following estimates hold for all $t \in [0, \infty)$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)|^2 d\xi \lesssim \exp(-Ct)\big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta}|\hat{u}_1(\xi)|\big) \quad \textit{for} \quad |\beta| \geq 2\delta,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp(-Ct)\big(|\xi|^{|\beta|+2\sigma-2\delta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big) \quad \textit{for} \quad |\beta| \geq 0.$$

For small frequencies we may use the estimates from Proposition 2.3.1. They imply a potential type decay. This we will show in the following statement.

Corollary 2.3.5. *For small frequencies $0 < |\xi| \leq p_0$ the following estimates hold for all $t \in [0, \infty)$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim (1+t)^{-\frac{|\beta|}{2\delta}}|\hat{u}_0(\xi)| + (1+t)^{-\frac{|\beta|-\sigma}{2\delta}}|\hat{u}_1(\xi)| \quad \textit{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim (1+t)^{-\frac{|\beta|+\sigma}{2\delta}}|\hat{u}_0(\xi)| + (1+t)^{-\frac{|\beta|}{2\delta}}|\hat{u}_1(\xi)| \quad \textit{for} \quad |\beta| \geq 0.$$

2.3.6 Conclusion

Taking into consideration all the estimates from the previous sections we arrive at the estimates for the energies of higher order $\|\nabla^\beta u(t,\cdot)\|_{L^2}$ and $\|\nabla^\beta u_t(t,\cdot)\|_{L^2}$ in the case $\delta \in (\sigma/2, \sigma]$.

Theorem 2.3.6. *The solution $u = u(t,x)$ to (2.3.1) satisfies in the case $\delta \in \big(\sigma/2, \sigma\big]$ the following estimates :*

$$\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 \lesssim (1+t)^{-\frac{|\beta|}{\delta}}\|u_0\|_{H^{|\beta|}}^2 + (1+t)^{-\frac{|\beta|-\sigma}{\delta}}\|u_1\|_{H^{|\beta|-2\delta}}^2 \quad \textit{for} \quad |\beta| \geq 2\delta,$$
$$\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 \lesssim (1+t)^{-\frac{|\beta|}{\delta}}\|u_0\|_{H^{|\beta|}}^2 + (1+t)^{-\frac{|\beta|-\sigma}{\delta}}\|u_1\|_{L^2}^2 \quad \textit{for} \quad |\beta| \in [\sigma, 2\delta],$$
$$\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 \lesssim (1+t)^{-\frac{|\beta|+\sigma}{\delta}}\|u_0\|_{H^{|\beta|+2\sigma-2\delta}}^2 + (1+t)^{-\frac{|\beta|}{\delta}}\|u_1\|_{H^{|\beta|}}^2 \quad \textit{for} \ |\beta| \geq 0.$$

Remark 2.3.1. *The statements of the Theorems 2.1.6, 2.2.6 and 2.3.6 tell us that we have the parabolic effect for $\delta \in (0, \sigma]$. In general the classical "elastic energy" and classical "kinetic energy " which are related to the Cauchy problem (2.0.1) have no decay (only if $u_1 \equiv 0$), but higher order energies have a decay. The decay rate increases with the order of energy. Moreover, the decay comes from the small frequencies.*

Remark 2.3.2. *If we set formally $\delta = \sigma/2$ in the estimates from Theorem 2.3.6, then we get the estimates from Theorem 2.1.6.*

Remark 2.3.3. *If we set formally $\sigma = 1$ in the estimates for Theorems 2.1.6, 2.2.6 and 2.3.6, then we get the estimates from Theorem 2.1 in [20].*

3 Time-dependent strictly decreasing dissipation

After the study of the time-independent case $b(t) \equiv \mu$ in model (1.3.1) we will study in Sections 3 and 4 time-dependent structural dissipations $b(t)(-\Delta)^\delta u_t$. We divide our considerations into the cases

1. in Chapter 3: $b(t)$ is strictly decreasing, that is, $b'(t) < 0$ for $t > 0$,
2. in Chapter 4: $b(t)$ is strictly increasing, that is, $b'(t) > 0$ for $t > 0$.

In this section we study the decay rate of the energies of higher order for solution to the Cauchy problem for structural damped σ-evolution models. We consider the special case of (1.3.1), that is, the model :

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\sigma u(t,x) + b(t)(-\Delta)^\delta u_t(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \sigma > 1, \quad \text{and} \quad \delta \in (0,\sigma]. \end{cases} \tag{3.0.1}$$

where $b = b(t)$ is a positive decreasing function

3.1 Objectives and strategies

We will study the $L^2 - L^2$ decay estimates for the energy of the solution $u(t,\cdot)$ for structural damped $\sigma-$evolution models (3.0.1). As in the previous chapters we assume for the Cauchy data $u_0 \in H^\sigma(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. We are interested to understand the decay of the energy of the solution $u(t,\cdot)$. Let us explain our strategy. It is divided into the following steps:

- In the first step we use the partial Fourier transformation to reduce the partial differential equation to an ordinary differential equation for $\hat{u}(t,\xi)$ parameterized by ξ.
- At first we want to consider the case $\delta = \sigma/2$ to understand the difficulties and we will cope with in the cases $\delta \in (0,\sigma/2)$ and $\delta \in (\sigma/2,\sigma)$.
- We will divide the extended phase space into important zones, the hyperbolic, elliptic zone, reduced and pseudo-differential zone.
- Firstly we restrict our considerations to the hyperbolic or reduced zone, we use the "dissipative transformation". Then we will introduce an appropriate micro-energy to get for it a system of first order, in the hyperbolic zone after one step of diagonalization. The remainder becomes integrable. Here the remainder can be studied by the matrizant representation. Then we derive a representation of the fundamental solution.

- We restrict our considerations to the elliptic zone. We use another micro-energy there to get again a system of first order. This system should be diagonalized twice. We explain the matrix representation of the fundamental solution which entries can be estimated in a very effective (two steps) way.

- By using the gluing procedure we get the estimates for the elastic and the kinetic energy for small and large frequencies.

- By using the Plancherel theorem we obtain $L^2 - L^2$ estimates. Here we get two types of decay estimates, a "potential type decay" for small frequencies and an "exponential type decay" for large frequencies under additional regularity assumptions for the data.

3.2 Time-dependent strictly decreasing dissipation – $\delta = \frac{\sigma}{2}$

In this section we study the special case of (3.0.1) for $\delta = \sigma/2$, that is, the model

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + b(t)(-\Delta)^{\frac{\sigma}{2}} u_t = 0, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \sigma > 1. \end{cases} \tag{3.2.1}$$

This model corresponds to the non-constant distributed resistance of typical semiconductors [?]. In this section we assume the following conditions for $b = b(t)$:

(A1) *positivity* : $b(t) > 0$ and there exists a positive constant T_c such that $b(t) < 2$ for all $t \geq T_c$,

(A2) *decreasing behavior* : $b'(t) < 0$ for all $t \geq 0$,

(A3) *non-integrability* : $\int_0^\infty b(\tau)d\tau = \infty$.

3.2.1 Division of the extended phase space into zones

After partial Fourier transformation $\hat{u}(t,\xi) = F_{x\to\xi}(u)(t,\xi)$ in (3.2.1) we obtain the Cauchy problem

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + b(t)|\xi|^\sigma \hat{u}_t = 0, \quad \hat{u}(0,\xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi), \quad \sigma > 1.$$

We define in the extended phase space $[0,\infty) \times \mathbb{R}^n_\xi$ by the aid of the solution to this equation the elastic type energy $|\xi|^{|\beta|}|\hat{u}(t,\xi)|$ of order $|\beta| \geq \sigma$ and the kinetic type energy $|\xi|^{|\beta|}|\hat{u}_t(t,\xi)|$ of order $|\beta| \geq 0$.
In this case we have a very simple division of the extended phase space $\{(t,\xi) \in [0,\infty) \times \mathbb{R}^n_\xi\}$. We divide it into the following zones, where ε is sufficiently small:

$$\text{hyperbolic zone} \quad Z_{hyp}(\varepsilon) = \Big\{(t,\xi) : \frac{b(t)}{2} \leq 1 - \varepsilon\Big\},$$

$$\text{remaining zone} \quad Z_{rem}(\varepsilon) = \Big\{(t,\xi) : 1 - \varepsilon \leq \frac{b(t)}{2}\Big\}.$$

We denote the separating line $t = t_0$ between the hyperbolic zone and the remaining zone. We mention, that the remaining zone $Z_{rem}(\varepsilon) = [0, t_0] \times \mathbb{R}^n$ appears if there exists a positive constant t_0 satisfying $1 - \varepsilon = \frac{b(t_0)}{2}$ (see Figure 10.1 in the Appendix).

3.2.2 Treatment in the remaining zone

Proposition 3.2.1. *The Cauchy problem (3.2.1) is L^2 well-posed in the following sense : To every data $u_0 \in H^{|\beta|}, |\beta| \geq \sigma$ and $u_1 \in H^{|\beta|-\sigma}$ there exists a unique solution $u \in C([0,\infty), H^{|\beta|}) \cap C^1([0,\infty), H^{|\beta|-\sigma})$. The solution depends continuously on the data. For the energy of higher order*

$$E^{|\beta|}[u](t) := \|u(t,\cdot)\|^2_{H^{|\beta|}} + \|u_t(t,\cdot)\|^2_{H^{|\beta|-\sigma}}$$

with $|\beta| \geq \sigma$ we have the estimate

$$E^{|\beta|}[u](t) \lesssim C(t)\big(\|u_0\|^2_{H^{|\beta|}} + \|u_1\|^2_{H^{|\beta|-\sigma}}\big), \tag{3.2.2}$$

where the function $C = C(t)$ is bounded on every compact interval $[0,T]$.

Proof. Due to our assumption we consider in the phase space the Cauchy problem

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + b(t)|\xi|^{\sigma}\hat{u}_t = 0, \;\; \hat{u}(0,\xi) = \hat{u}_0(\xi), \;\; \hat{u}_t(0,\xi) = \hat{u}_1(\xi) \tag{3.2.3}$$

with $\hat{u}_0 \in L^{2,|\beta|}, \hat{u}_1 \in L^{2,|\beta|-\sigma}$. Here we use $L^{2,|\beta|} := F(H^{|\beta|})$, that is, the Fourier image of $H^{|\beta|}$. Firstly, we restrict ourselves to the set of large frequencies $\{|\xi| \geq M\}$. We define the energy

$$2E[\hat{u}](t,\xi) = \hat{u}_t(t,\xi)^2 + |\xi|^{2\sigma}\hat{u}(t,\xi)^2.$$

Straight-forward calculations give

$$d_t E[\hat{u}](t,\xi) \leq 0, \quad \text{hence,} \quad E[\hat{u}](t,\xi) \leq E[\hat{u}](0,\xi).$$

For small frequencies $\{|\xi| \leq M\}$ we define the energy

$$2\check{E}[\hat{u}](t,\xi) = \hat{u}_t(t,\xi)^2 + \hat{u}(t,\xi)^2.$$

Straight-forward calculations give

$$d_t \check{E}[\hat{u}](t,\xi) \leq C\check{E}[\hat{u}](t,\xi), \quad \text{hence,} \quad \check{E}[\hat{u}](t,\xi) \leq C(t)\check{E}[\hat{u}](0,\xi).$$

Both estimates together yield the unique existence and continuous dependence on the data of the solution $\hat{u}$ satisfying

$$\hat{u} \in L^\infty([0,\infty), L^{2,|\beta|}) \quad \text{and} \quad \hat{u}_t \in L^\infty([0,\infty), L^{2,|\beta|-\sigma}).$$

Moreover, we prove

$$\hat{u} \in C([0,\infty), L^{2,|\beta|}) \quad \text{and} \quad \hat{u}_t \in C([0,\infty), L^{2,|\beta|-\sigma}).$$

Firstly, we prove $\hat{u} \in C([0,T], L^{2,|\beta|})$ for all $T > 0$. By the definition of $C([0,T], L^{2,|\beta|})$ we have to show for all $t_1, t_2 \in [0,T]$ the relation $\lim_{t_1 \to t_2} \|\hat{u}(t_1,\cdot) - \hat{u}(t_2,\cdot)\|_{L^{2,|\beta|}} = 0$. We have

$$\begin{aligned}\|\hat{u}(t_1,\cdot) - \hat{u}(t_2,\cdot)\|^2_{L^{2,|\beta|}} &= \int_{\mathbb{R}^n_\xi} \langle\xi\rangle^{2|\beta|}|\hat{u}(t_1,\xi) - \hat{u}(t_2,\xi)|^2 d\xi \\ &= \int_{B_R(0)} \langle\xi\rangle^{2|\beta|}|\hat{u}(t_1,\xi) - \hat{u}(t_2,\xi)|^2 d\xi + \int_{\mathbb{R}^n_\xi \setminus B_R(0)} \langle\xi\rangle^{2|\beta|}|\hat{u}(t_1,\xi) - \hat{u}(t_2,\xi)|^2 d\xi.\end{aligned}$$

By using the continuity of Lebesgue measure we conclude for the second term

$$\int_{\mathbb{R}^n_\xi \setminus B_R(0)} \langle\xi\rangle^{2|\beta|}|\hat{u}(t_1,\xi)-\hat{u}(t_2,\xi)|^2 d\xi \\ \leq \int_{\mathbb{R}^n_\xi \setminus B_R(0)} \langle\xi\rangle^{2|\beta|}|\hat{u}(t_1,\xi)|^2 d\xi + \int_{\mathbb{R}^n_\xi \setminus B_R(0)} \langle\xi\rangle^{2|\beta|}|\hat{u}(t_2,\xi)|^2 d\xi \leq \varepsilon(R,T)$$

with $\varepsilon(R,T) \to 0$ for $R \to \infty$. This follows for $t \in [0,T]$ from the energy estimate

$$E[\hat{u}](t,\xi) \leq E[\hat{u}](0,\xi), \quad \text{hence,} \quad |\xi|^{2\sigma}|\hat{u}(t,\xi)|^2 \leq |\hat{u}_1(\xi)|^2 + |\xi|^{2\sigma}|\hat{u}_0(\xi)|^2.$$

Hence,

$$\int_{\mathbb{R}^n_\xi \setminus B_R(0)} \langle\xi\rangle^{2|\beta|}|\hat{u}(t,\xi)|^2 d\xi \leq \int_{\mathbb{R}^n_\xi \setminus B_R(0)} \langle\xi\rangle^{2|\beta|}|\hat{u}_0(\xi)|^2 d\xi + \int_{\mathbb{R}^n_\xi \setminus B_R(0)} \langle\xi\rangle^{2|\beta|-2\sigma}|\hat{u}_1(\xi)|^2 d\xi \leq \varepsilon(R,T).$$

Taking into account that the coefficients in (3.2.3) are continuous we get the uniform continuity of $\langle\xi\rangle^{|\beta|}\hat{u}(t,\xi)$ on the compact set $[0,T]\times B_R(0)$. So

$$\lim_{t_1\to t_2} \int_{B_R(0)} \langle\xi\rangle^{2|\beta|}|\hat{u}(t_1,\xi)-\hat{u}(t_2,\xi)|^2 d\xi = 0.$$

Summarizing, we conclude immediately $\hat{u} \in C([0,\infty), L^{2,|\beta|})$. In same way we can prove $\hat{u}_t \in C([0,\infty), L^{2,|\beta|-\sigma})$. This gives the desired statement about the well-posedness. □

3.2.3 Treatment in the hyperbolic zone

Proposition 3.2.2. *The following estimates hold for all* $t \in [t_0,\infty)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^\sigma}{2}\int_{t_0}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_0,\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_0,\xi)|\Big) \text{ for } |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^\sigma}{2}\int_{t_0}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(t_0,\xi)| + |\xi|^{|\beta|}|\hat{u}_t(t_0,\xi)|\Big) \text{ for } |\beta| \geq 0.$$

Proof. Applying the diagonalization procedure (see in Appendix Section 10.1.1) we get after the first step of diagonalization that the entries of the second matrix $\mathcal{R}_1(t)$ are uniformly integrable over the hyperbolic zone. For this reason $\mathcal{R}_1(t)$ belongs to $L^1_{loc}\big(Z_{hyp}(\varepsilon)\big)$. We can write $V^{(1)}(t,\xi) = E_1(t,s,\xi)V^{(1)}(s,\xi)$, where $E_1(t,s,\xi)$ is the fundamental solution, that is the solution of the system

$$D_t E_1(t,s,\xi) = \big(\mathcal{D}(t,\xi)+\mathcal{R}_1(t)\big)E_1(t,s,\xi), \;\; E_1(s,s,\xi) = I_2, \;\text{ for all }\; t\geq s \;\text{ and }\; (s,\xi)\in Z_{hyp}(\varepsilon).$$

First, we get the fundamental solution $E_d = E_d(t,s,\xi)$ of the diagonal part of this system,

$$D_t E_d(t,s,\xi) = \mathcal{D}(t,\xi)E_d(t,s,\xi), \;\; E_d(s,s,\xi) = I_2, \;\text{ for all }\; t\geq s \;\text{ and }\; (s,\xi)\in Z_{hyp}(\varepsilon).$$

That is,

$$E_d(t,s,\xi) = \begin{pmatrix} \exp\Big(i\int_s^t p(\tau,\xi)d\tau\Big) & 0 \\ 0 & \exp\Big(-i\int_s^t p(\tau,\xi)d\tau\Big) \end{pmatrix}.$$

Then the following estimate holds:

$$\|E_d(t,s,\xi)\| \le C \quad \text{for all} \quad t \ge s \;\text{ and }\; (s,\xi) \in Z_{hyp}(\varepsilon).$$

Now we set $E_1(t,s,\xi) = E_d(t,s,\xi)Q(t,s,\xi)$. We have to find the solution $Q(t,s,\xi)$ to the system

$$D_t\big(E_d(t,s,\xi)Q(t,s,\xi)\big) = \big(\mathcal{D}(t,\xi) + \mathcal{R}_1(t)\big)E_d(t,s,\xi)Q(t,s,\xi), \quad Q(s,s,\xi) = I_2,$$

for all $t \ge s$ and $(s,\xi) \in Z_{hyp}(\varepsilon)$. Hence, we have to study the system

$$D_tQ(t,s,\xi) = P(t,s,\xi)Q(t,s,\xi), \quad Q(s,s,\xi) = I_2, \quad \text{for all} \quad t \ge s \;\text{ and }\; (s,\xi) \in Z_{hyp}(\varepsilon),$$

where

$$P(t,s,\xi) := E_d^{-1}(t,s,\xi)\mathcal{R}_1(t)E_d(t,s,\xi).$$

Applying the Peano-Baker formula from Proposition 10.1.10 (see Appendix) we get

$$\|Q(t,s,\xi)\| \le \exp\Big(\int_s^t \|\mathcal{R}_1(t)\|d\tau\Big) \le C \quad \text{for all} \quad t \ge s \;\text{ and }\; (s,\xi) \in Z_{hyp}(\varepsilon).$$

Hence,

$$\|E_1(t,s,\xi)\| = \|E_d(t,s,\xi)\|\|Q(t,s,\xi)\| \le C \quad \text{for all} \quad t \ge s \;\text{ and }\; (s,\xi) \in Z_{hyp}(\varepsilon).$$

Finally, we obtain the following estimate for the transformed micro-energy $V^{(1)}(t,\xi)$. In the hyperbolic zone for all $t \ge t_0$:

$$|V^{(1)}(t,\xi)| \lesssim C|V^{(1)}(t_0,\xi)|, \quad \Big|\begin{pmatrix} p(t,\xi)v(t,\xi) \\ D_tv(t,\xi)\end{pmatrix}\Big| \lesssim C\Big|\begin{pmatrix} p(t_0,\xi)v(t_0,\xi) \\ D_tv(t_0,\xi)\end{pmatrix}\Big| \quad \text{for all} \quad t \ge t_0,$$

respectively, with a constant C which is independent of $t \in [t_0,\infty)$. After backward transformation $\hat{u}(t,\xi) = \exp\Big(-\frac{|\xi|^\sigma}{2}\int_0^t b(s)ds\Big)v(t,\xi)$ we have

$$\begin{pmatrix} p(t,\xi)\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix} = \exp\Big(-\frac{|\xi|^\sigma}{2}\int_0^t b(\tau)d\tau\Big)\begin{pmatrix} 1 & 0 \\ -i\frac{b(t)|\xi|^\sigma}{2p(t,\xi)} & 1\end{pmatrix}\begin{pmatrix} p(t,\xi)v(t,\xi) \\ D_tv(t,\xi)\end{pmatrix}.$$

Using the relation

$$\begin{pmatrix} 1 & 0 \\ i\frac{b(t)|\xi|^\sigma}{2p(t,\xi)} & 1\end{pmatrix}\begin{pmatrix} p(t,\xi)\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix} = \exp\Big(-\frac{|\xi|^\sigma}{2}\int_0^t b(\tau)d\tau\Big)\begin{pmatrix} p(t,\xi)v(t,\xi) \\ D_tv(t,\xi)\end{pmatrix}$$

we are able to estimate

$$\begin{aligned}
\Big|\begin{pmatrix} 1 & 0 \\ i\frac{b(t)|\xi|^\sigma}{2p(t,\xi)} & 1\end{pmatrix}\begin{pmatrix} p(t,\xi)\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix}\Big| &\lesssim \exp\Big(-\frac{|\xi|^\sigma}{2}\int_0^t b(\tau)d\tau\Big)\Big|\begin{pmatrix} p(t_0,\xi)v(t_0,\xi) \\ D_tv(t_0,\xi)\end{pmatrix}\Big| \\
&\lesssim \exp\Big(-\frac{|\xi|^\sigma}{2}\int_0^t b(\tau)d\tau\Big)\exp\Big(\frac{|\xi|^\sigma}{2}\int_0^{t_0} b(\tau)d\tau\Big)\Big|\begin{pmatrix} 1 & 0 \\ i\frac{b(t_0)|\xi|^\sigma}{2p(t_0,\xi)} & 1\end{pmatrix}\begin{pmatrix} p(t_0,\xi)\hat{u}(t_0,\xi) \\ D_t\hat{u}(t_0,\xi)\end{pmatrix}\Big| \\
&\lesssim \exp\Big(-\frac{|\xi|^\sigma}{2}\int_{t_0}^t b(\tau)d\tau\Big)\Big|\begin{pmatrix} 1 & 0 \\ i\frac{b(t_0)|\xi|^\sigma}{2p(t_0,\xi)} & 1\end{pmatrix}\begin{pmatrix} p(t_0,\xi)\hat{u}(t_0,\xi) \\ D_t\hat{u}(t_0,\xi)\end{pmatrix}\Big|.
\end{aligned}$$

The equivalence $p(t,\xi) \sim |\xi|^\sigma$ in $Z_{hyp}(\varepsilon)$ gives the next a-priori estimates for all $t \in [t_0,\infty)$:

$$\Big|\begin{pmatrix} |\xi|^\sigma\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix}\Big| \lesssim \exp\Big(-\frac{|\xi|^\sigma}{2}\int_{t_0}^t b(\tau)d\tau\Big)\Big|\begin{pmatrix} |\xi|^\sigma\hat{u}(t_0,\xi) \\ D_t\hat{u}(t_0,\xi)\end{pmatrix}\Big|.$$

Consequently, we derived for $|\beta| \geq \sigma$ the a-priori estimates

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^\sigma}{2}\int_{t_0}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_0,\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_0,\xi)|\Big).$$

In the same way we conclude for $|\beta| \geq 0$ the a-priori estimates

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^\sigma}{2}\int_{t_0}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(t_0,\xi)| + |\xi|^{|\beta|}|\hat{u}_t(t_0,\xi)|\Big).$$

All desired estimates are proved. □

3.2.4 Gluing procedure

Now we have to glue the estimates from the Propositions 3.2.1 and 3.2.2. If we consider the set $M_\varepsilon := \{|\xi| \geq \varepsilon > 0\}$, then it is clear that an "exponential type decay" for the higher order energies follows from Proposition 3.2.2 under the usual regularity assumptions for the data from the Cauchy problem for the σ-evolution model . Thus the interesting case is to glue all estimates for small frequencies, let us say for $\{|\xi| \leq \varepsilon\}$, where ε is sufficiently small.*case 1:* $t \leq t_0$In this case we apply Proposition 3.2.1. *case 2:* $t \in [t_0, \infty]$Now we have to glue the

estimates from Propositions 3.2.1 and 3.2.2, in fact, we also use Proposition 7.4.1. We have the following statement:

Corollary 3.2.3. *The following estimates hold for all* $t \in [t_0, \infty)$ *:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^\sigma}{2}\int_{t_0}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \text{ for } |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^\sigma}{2}\int_{t_0}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) \text{ for } |\beta| \geq 0.$$

3.2.5 Energy estimates

Using (3.2.1) we can restrict ourselves to $t \geq \max\{2t_0, 1\}$. From Proposition 3.2.2 we conclude the following result:

Corollary 3.2.4. *The following estimates hold for* $t \in [t_0, \infty]$ *:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\sigma}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{\sigma}}|\hat{u}_1(\xi)| \quad \text{for } |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma}{\sigma}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\sigma}}|\hat{u}_1(\xi)| \quad \text{for } |\beta| \geq 0.$$

3.2.6 Conclusion

We obtain the following statement from Corollary 3.2.4 :

Theorem 3.2.5. *Let us consider the Cauchy problem (3.2.1), where $b = b(t)$ satisfies the assumptions (A1) to (A3). Then the solution $u = u(t,x)$ satisfies in the case $\delta = \sigma/2$ the following estimates for the energies of higher order :*

$$\begin{aligned}
\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 &\lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-2\frac{|\beta|}{\sigma}}\|u_0\|_{H^{|\beta|}}^2 + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-2\frac{|\beta|-\sigma}{\sigma}}\|u_1\|_{H^{|\beta|-\sigma}}^2 \\
&\quad \text{for } \ |\beta| \geq \sigma, \\
\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 &\lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-2\frac{|\beta|+\sigma}{\sigma}}\|u_0\|_{H^{|\beta|+\sigma}}^2 + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-2\frac{|\beta|}{\sigma}}\|u_1\|_{H^{|\beta|}}^2 \\
&\quad \text{for } \ |\beta| \geq 0.
\end{aligned}$$

Remark 3.2.1. *If we set formally $\sigma = 1$ and $b(t) = (1+t)^{-\gamma},\ \gamma \in (0,1]$ in the estimates from Theorem 3.2.5, then we get the estimates from Theorem 3.18 in [20].*

3.2.7 Some other examples

Typical examples for coefficients $b = b(t)$ are

$b_n(t) = \mu(1+t)^{-\gamma}\big(\log(e+t)\big)^{-\gamma_1}\cdot\cdot\cdot\big(\log^{[n]}(e^{[n]}+t)\big)^{-\gamma_n},$ with nonnegative

$\mu,\ \gamma_i,\ \ i = 1,\cdot\cdot\cdot,n,$ and $\gamma \in (0,1)$. Here we use, $\log^{[0]}(x) = x,\ e^{[0]} = 1$ and

$\log^{[n+1]}(x) = \log\log^{[n]}(x),\ e^{[n+1]} = e^{e^{[n]}}.$

Example 3.2.1. *If we choose $b = b_0(t)$ in this way $b(t) = (1+t)^{-\gamma},\ \gamma \in (0,1]$. Then $b = b(t)$ satisfies the assumptions of Theorem 3.2.5. Consequently, the following estimates for the energies of higher order hold :*

$$\begin{aligned}
&\gamma \in (0,1): \\
&\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 \lesssim \big(1+t\big)^{-2|\beta|\frac{(1-\gamma)}{\sigma}}\|u_0\|_{H^{|\beta|}}^2 + \big(1+t\big)^{-2(|\beta|-\sigma)\frac{1-\gamma}{\sigma}}\|u_1\|_{H^{|\beta|-\sigma}}^2 \text{ for } |\beta| \geq \sigma, \\
&\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 \lesssim \big(1+t\big)^{-2(|\beta|+\sigma)\frac{1-\gamma}{\sigma}}\|u_0\|_{H^{|\beta|+\sigma}}^2 + \big(1+t\big)^{-2|\beta|\frac{1-\gamma}{\sigma}}\|u_1\|_{H^{|\beta|}}^2 \text{ for } |\beta| \geq 0, \\
&\gamma = 1: \\
&\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 \lesssim \big(\log(e+t)\big)^{-2\frac{|\beta|}{\sigma}}\|u_0\|_{H^{|\beta|}}^2 + \big(\log(e+t)\big)^{-2\frac{|\beta|-\sigma}{\sigma}}\|u_1\|_{H^{|\beta|-\sigma}}^2 \text{ for } |\beta| \geq \sigma, \\
&\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 \lesssim \big(\log(e+t)\big)^{-2\frac{|\beta|+\sigma}{\sigma}}\|u_0\|_{H^{|\beta|+\sigma}}^2 + \big(\log(e+t)\big)^{-2\frac{|\beta|}{\sigma}}\|u_1\|_{H^{|\beta|}}^2 \text{ for } |\beta| \geq 0.
\end{aligned}$$

Example 3.2.2. *We choose $b(t) = (1+t)^{-\gamma}(\log(e+t))^{-1}$ with $\gamma \in (0,1]$. Then $b = b(t)$ satisfies the assumptions of Theorem 3.2.5. Consequently, the following estimates for the energies of* *higher order hold* :

$$
\begin{aligned}
&\gamma \in (0,1):\\
&\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 \lesssim \Big((1+t)^{1-\gamma}(\log(e+t))^{-1}\Big)^{-2\frac{|\beta|}{\sigma}}\|u_0\|_{H^{|\beta|}}^2\\
&\qquad\qquad\qquad +\Big((1+t)^{1-\gamma}(\log(e+t))^{-1}\Big)^{-2\frac{|\beta|-\sigma}{\sigma}}\|u_1\|_{H^{|\beta|-\sigma}}^2 \quad \text{for } |\beta|\geq\sigma,\\
&\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 \lesssim \Big((1+t)^{1-\gamma}(\log(e+t))^{-1}\Big)^{-2\frac{|\beta|+\sigma}{\sigma}}\|u_0\|_{H^{|\beta|+\sigma}}^2\\
&\qquad\qquad\qquad +\Big((1+t)^{1-\gamma}(\log(e+t))^{-1}\Big)^{-2\frac{|\beta|}{\sigma}}\|u_1\|_{H^{|\beta|}}^2 \quad \text{for } |\beta|\geq 0,\\
&\gamma = 1:\\
&\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 \lesssim \Big(\log^{[2]}(e^{[2]}+t)\Big)^{-2\frac{|\beta|}{\sigma}}\|u_0\|_{H^{|\beta|}}^2 + \Big(\log^{[2]}(e^{[2]}+t)\Big)^{-2\frac{|\beta|-\sigma}{\sigma}}\|u_1\|_{H^{|\beta|-\sigma}}^2\\
&\qquad\qquad \text{for } |\beta|\geq\sigma,\\
&\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 \lesssim \Big(\log^{[2]}(e^{[2]}+t)\Big)^{-2\frac{|\beta|+\sigma}{\sigma}}\|u_0\|_{H^{|\beta|+\sigma}}^2 + \Big(\log^{[2]}(e^{[2]}+t)\Big)^{-2\frac{|\beta|}{\sigma}}\|u_1\|_{H^{|\beta|}}^2\\
&\qquad\qquad \text{for } |\beta|\geq 0.
\end{aligned}
$$

Example 3.2.3. *We choose* $b(t) = (1+t)^{-\gamma}(\log(e+t))^{-1}(\log^{[2]}(e^{[2]}+t))^{-\gamma_2}$ *with* $\gamma \in (0,1]$ *and* $\gamma_2 \in (0,1)$. *Then* $b = b(t)$ *satisfies the assumptions of Theorem 3.2.5. Consequently, the following estimates for the energies of higher order hold* :

$$
\begin{aligned}
&\gamma \in (0,1):\\
&\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 \lesssim \Big((1+t)^{1-\gamma}(\log(e+t))^{-1}(\log^{[2]}(e^{[2]}+t))^{-\gamma_2}\Big)^{-2\frac{|\beta|}{\sigma}}\|u_0\|_{H^{|\beta|}}^2\\
&\qquad +\Big((1+t)^{1-\gamma}(\log(e+t))^{-1}(\log^{[2]}(e^{[2]}+t))^{-\gamma_2}\Big)^{-2\frac{|\beta|-\sigma}{\sigma}}\|u_1\|_{H^{|\beta|-\sigma}}^2 \text{ for } |\beta|\geq\sigma,\\
&\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 \lesssim \Big((1+t)^{1-\gamma}(\log(e+t))^{-1}(\log^{[2]}(e^{[2]}+t))^{-\gamma_2}\Big)^{-2\frac{|\beta|+\sigma}{\sigma}}\|u_0\|_{H^{|\beta|+\sigma}}^2\\
&\qquad +\Big((1+t)^{1-\gamma}(\log(e+t))^{-1}(\log^{[2]}(e^{[2]}+t))^{-\gamma_2}\Big)^{-2\frac{|\beta|}{\sigma}}\|u_1\|_{H^{|\beta|}}^2 \text{ for } |\beta|\geq 0.\\
&\gamma = 1:\\
&\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 \lesssim \Big(\log^{[2]}(e^{[2]}+t)\Big)^{-2\frac{|\beta|(1-\gamma_2)}{\sigma}}\|u_0\|_{H^{|\beta|}}^2\\
&\qquad\qquad\qquad +\Big(\log^{[2]}(e^{[2]}+t)\Big)^{-2\frac{(|\beta|-\sigma)(1-\gamma_2)}{\sigma}}\|u_1\|_{H^{|\beta|-\sigma}}^2 \text{ for } |\beta|\geq\sigma,\\
&\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 \lesssim \Big(\log^{[2]}(e^{[2]}+t)\Big)^{-2\frac{(|\beta|+\sigma)(1-\gamma_2)}{\sigma}}\|u_0\|_{H^{|\beta|+\sigma}}^2\\
&\qquad\qquad\qquad +\Big(\log^{[2]}(e^{[2]}+t)\Big)^{-2\frac{|\beta|(1-\gamma_2)}{\sigma}}\|u_1\|_{H^{|\beta|}}^2 \text{ for } |\beta|\geq 0.
\end{aligned}
$$

3.3 Time-dependent strictly decreasing dissipation – $\delta \in (0,\sigma/2)$

In this section we study cases $\delta \neq \sigma/2$. We consider the special case of (3.0.1) for $\delta \in (0,\sigma/2)$, that is, the model

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + b(t)(-\Delta)^\delta u_t = 0, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \sigma > 1, \quad \delta \in (0,\sigma/2). \end{cases} \tag{3.3.1}$$

After partial Fourier transformation we obtain the Cauchy problem

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + b(t)|\xi|^{2\delta}\hat{u}_t = 0, \quad \hat{u}(0,\xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi), \quad \delta \in (0,\sigma/2), \quad \sigma > 1.$$

We pose the following conditions to the coefficient function $b = b(t)$:

(A1) *positivity* : $b(t) > 0$ for all $t \geq 0$,

(A2) *decreasing behavior* : $b'(t) < 0$ for all $t \geq 0$,

(A3) *non-integrability* : $\int_0^\infty b(\tau)d\tau = \infty$,

(A4) *higher order derivatives* : it holds $|d_t^k b(t)| \leq C_k b(t)\big(\frac{1}{1+t}\big)^k$ for $k = 1, 2$,

(A5) *useful inequalities* : there exist positive constants C_0, C_1, C_2 which are independent of t such that

$$C_0\frac{b(t)}{\Lambda(t)} \leq -\frac{b'(t)}{b(t)} \leq C_1\frac{1}{1+t} \leq C_2\frac{b(t)}{\Lambda(t)} \quad \text{with} \quad \Lambda(t) = 1 + \int_0^t b(\tau)d\tau,$$

(A6) *additional classification* : $b \in S_\alpha$ for $\alpha \in \big(1, \frac{\sigma}{2\delta}\big]$, where we introduce the family $\{S_\alpha\}_\alpha$ of classes

$$S_\alpha := \Big\{b = b(t) : \limsup_{t\to\infty}(1+t)\Lambda_1(t)^{-\alpha} < \infty, \quad \lim_{t\to\infty}(1+t)\Lambda_1(t)^{-\beta} = \infty \quad \text{for all } \beta < \alpha\Big\}.$$

3.3.1 Division of the extended phase space into zones

From the point of WKB analysis the model (3.2.1) is hyperbolic like. But now the model is hyperbolic-elliptic like. We introduce the following regions :

$$\text{the hyperbolic region:} \qquad R_{hyp} = \Big\{(t,\xi) : \frac{b(t)}{2}|\xi|^{2\delta-\sigma} < 1\Big\},$$

$$\text{the elliptic region:} \qquad R_{ell} = \Big\{(t,\xi) : \frac{b(t)}{2}|\xi|^{2\delta-\sigma} > 1\Big\}.$$

Moreover, we introduce zones. This we feel in the division of the extended phase space $\{(t,\xi) \in [0,\infty) \times \mathbb{R}^n_\xi\}$ into the following zones with $\Lambda(t) = 1 + \int_0^t b(s)ds$, where ε is sufficiently small and N is sufficiently large:

hyperbolic zone $Z_{hyp}(\varepsilon) = \Big\{(t,\xi) : \dfrac{b(t)}{2}|\xi|^{2\delta-\sigma} \le 1-\varepsilon\Big\}$,

reduced zone $Z_{red}(\varepsilon) = \Big\{(t,\xi) : 1-\varepsilon \le \dfrac{b(t)}{2}|\xi|^{2\delta-\sigma} \le 1+\varepsilon\Big\}$,

elliptic zone $Z_{ell}(\varepsilon,N) = \Big\{(t,\xi) : \dfrac{b(t)}{2}|\xi|^{2\delta-\sigma} \ge 1+\varepsilon \quad \text{and} \quad \Lambda(t)|\xi|^{2\delta} \ge N\Big\}$,

pseudo-differential zone $Z_{pd}(\varepsilon,N) = \Big\{(t,\xi) : \dfrac{b(t)}{2}|\xi|^{2\delta-\sigma} \ge 1+\varepsilon \quad \text{and} \quad \Lambda(t)|\xi|^{2\delta} \le N\Big\}$.

We introduce separation lines. By $t_k = t_k(|\xi|), k = 0,1,2$, we denote the separation line between the pseudo-differential zone and the elliptic zone ($k=0$), between the elliptic zone and the reduced zone ($k=1$), and between the reduced zone and the hyperbolic zone ($k=2$) (see Figure 10.2 in the Appendix).

Lemma 3.3.1. *If* $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$, *then the elliptic zone is a compact set. If* $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = 0$, *then the elliptic zone is not a compact set. Here* $\delta \in (0, \frac{\sigma}{2})$.

Proof. By using the definition of the separating line between the pseudo-differential and elliptic zone we have

$$\Lambda(t)|\xi|^{2\delta} = N, \qquad \text{hence,} \qquad |\xi| = \Big(\frac{N}{\Lambda(t)}\Big)^{\frac{1}{2\delta}}.$$

Analogously, from the separating line between the elliptic and reduced zone we have

$$\frac{b(t)}{2}|\xi|^{2\delta-\sigma} = 1+\varepsilon, \qquad \text{hence,} \qquad |\xi| = \Big(\frac{b(t)}{2(1+\varepsilon)}\Big)^{\frac{1}{\sigma-2\delta}}.$$

In order that the elliptic zone has really its own meaning one should guarantee

$$\Big(\frac{N}{\Lambda(t)}\Big)^{\frac{1}{2\delta}} \le \Big(\frac{b(t)}{2(1+\varepsilon)}\Big)^{\frac{1}{\sigma-2\delta}}, \qquad b(t)\Lambda(t)^{\frac{\sigma-2\delta}{2\delta}} \ge \frac{2(1+\varepsilon)}{N^{\frac{2\delta-\sigma}{2\delta}}}.$$

After integration we conclude

$$\Lambda(t)^{\frac{\sigma}{2\delta}} \ge 1 + \frac{\sigma(1+\varepsilon)}{\delta N^{\frac{2\delta-\sigma}{2\delta}}}t, \quad \text{hence,} \quad \Lambda(t)^{\frac{\sigma}{2\delta}} \ge \frac{\sigma(1+\varepsilon)}{\delta N^{\frac{2\delta-\sigma}{\delta}}}(1+t).$$

Then we get for large t the inequality

$$(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} \le \frac{\delta N^{\frac{2\delta-\sigma}{2\delta}}}{\sigma(1+\varepsilon)}.$$

In this way the lemma is proved. □

We divide the further considerations into the following cases:

case 1: $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = 0$,
case 2: $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$. In the case 1 we divide the extended phase space into the hyperbolic zone $Z_{hyp}(\varepsilon)$, the reduced zone $Z_{red}(\varepsilon)$, the elliptic zone $Z_{ell}(\varepsilon,N)$, and the pseudo-differential zone $Z_{pd}(\varepsilon,N)$. In the case 2 we have only the hyperbolic zone $Z_{hyp}(\varepsilon)$, the reduced zone $Z_{red}(\varepsilon)$, and the pseudo-differential zone $Z_{pd}(\varepsilon,N)$. The elliptic zone is a compact set in case 2.

Remark 3.3.1. *In our considerations we omit the "critical case"*

$$\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} \in (0,\infty).$$

Let us devote to the first case.

3.3.2 Treatment in the hyperbolic zone

Proposition 3.3.2. *The following estimates hold for all $t \in [t_2(|\xi|),\infty)$, where $t_2(|\xi|) = 0$ for large frequencies :*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{2}\int_{t_2(|\xi|)}^{t} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_2(|\xi|),\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_2(|\xi|),\xi)|\Big)$$
for $|\beta| \geq \sigma$,
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{2}\int_{t_2(|\xi|)}^{t} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(t_2(|\xi|),\xi)| + |\xi|^{|\beta|}|\hat{u}_t(t_2(|\xi|),\xi)|\Big)$$
for $|\beta| \geq 0$.

Proof. Applying the diagonalization procedure (see in Appendix Section 10.1.1) we get after the first step of diagonalization that the entries of the second matrix $\mathcal{R}_1(t,\xi)$ are uniformly integrable over the hyperbolic zone. For this reason they belong to $L^1_{loc}\big(Z_{hyp}(\varepsilon)\big)$. We can write $V^{(1)}(t,\xi) = E_1(t,s,\xi)V^{(1)}(s,\xi)$, where $E_1(t,s,\xi)$ is the fundamental solution, that is the solution of the system

$$D_tE_1(t,s,\xi) = \big(D(t,\xi)+\mathcal{R}_1(t,\xi)\big)E_1(t,s,\xi),\ E_1(s,s,\xi) = I_2 \text{ for all } t \geq s \text{ and } (s,\xi) \in Z_{hyp}(\varepsilon).$$

First we take the fundamental solution $E_d = E_d(t,s,\xi)$ of the diagonal part of this system, that is,

$$D_tE_d(t,s,\xi) = \mathcal{D}(t,\xi)E_d(t,s,\xi), \quad E_d(s,s,\xi) = I_2, \quad \text{for all} \quad t \geq s \quad \text{and} \quad (s,\xi) \in Z_{hyp}(\varepsilon).$$

We have

$$E_d(t,s,\xi) = \begin{pmatrix} \exp\big(i\int_s^t p(\tau,\xi)d\tau\big) & 0 \\ 0 & \exp\big(-i\int_s^t p(\tau,\xi)d\tau\big) \end{pmatrix}.$$

Then the following estimate holds:

$$\|E_d(t,s,\xi)\| \leq C \quad \text{for all} \quad t \geq s \ \text{ and } \ (s,\xi) \in Z_{hyp}(\varepsilon).$$

Now we introduce $E_1(t,s,\xi) = E_d(t,s,\xi)Q(t,s,\xi)$. We have to find the solution $Q(t,s,\xi)$ to the system

$$D_t\big(E_d(t,s,\xi)Q(t,s,\xi)\big) = \big(\mathcal{D}(t,\xi)+\mathcal{R}_1(t,\xi)\big)E_d(t,s,\xi)Q(t,s,\xi), \quad Q(s,s,\xi) = I_2,$$

for all $t \geq s$ and $(s,\xi) \in Z_{hyp}(\varepsilon)$. By using the above system for E_d it is equivalent to the system

$$D_tQ(t,s,\xi) = P(t,s,\xi)Q(t,s,\xi), \quad Q(s,s,\xi) = I_2, \quad \text{for all} \quad t \geq s \ \text{ and } \ (s,\xi) \in Z_{hyp}(\varepsilon),$$

where

$$P(t,s,\xi) := E_d(s,t,\xi)\mathcal{R}_1(t,\xi)E_d(t,s,\xi).$$

Applying the Peano-Baker formula from Proposition 10.1.10 (see Appendix) we get

$$\|Q(t,s,\xi)\| \le \exp\Big(\int_s^t \|\mathcal{R}_1(\tau,\xi)\|d\tau\Big) \le C \quad \text{for all} \quad t \ge s \ \text{ and } \ (s,\xi) \in Z_{hyp}(\varepsilon).$$

Hence,

$$\|E_1(t,s,\xi)\| = \|E_d(t,s,\xi)\|\|Q(t,s,\xi)\| \le C \quad \text{for all} \quad t \ge s \ \text{ and } \ (s,\xi) \in Z_{hyp}(\varepsilon),$$

holds. Finally, we obtain the following estimate for the transformed micro-energy $V^{(1)}(t,\xi)$ in the hyperbolic zone for all $t \ge s$ and $(t,\xi) \in Z_{hyp}(\varepsilon)$:

$$|V^{(1)}(t,\xi)| \lesssim |V^{(1)}(s,\xi)|, \quad \Big|\begin{pmatrix} p(t,\xi)v(t,\xi) \\ D_t v(t,\xi)\end{pmatrix}\Big| \lesssim \Big|\begin{pmatrix} p(s,\xi)v(s,\xi) \\ D_t v(s,\xi)\end{pmatrix}\Big|$$

uniformly for all $t \ge s$ and $(t,\xi) \in Z_{hyp}(\varepsilon)$. From the backward transformation we use the same strategy as before in the Section 3.2.3 to get

$$\begin{aligned}
|\xi|^{|\beta|}|\hat{u}(t,\xi)| &\lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{2}\int_s^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(s,\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(s,\xi)|\Big) \\
&\quad \text{for } |\beta| \ge \sigma, \\
|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| &\lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{2}\int_s^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(s,\xi)| + |\xi|^{|\beta|}|\hat{u}_t(s,\xi)|\Big) \\
&\quad \text{for } |\beta| \ge 0.
\end{aligned}$$

In the following statement we set $s = t_2(|\xi|)$. Then all desired estimates are proved. □

Corollary 3.3.3. *We have the following representation of solution in hyperbolic zone for all* $t \ge s$ *and* $(s,\xi) \in Z_{hyp}(\varepsilon)$:

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ i\frac{b(t)|\xi|^{2\delta}}{2p(t,\xi)} & 1\end{pmatrix}\begin{pmatrix} p(t,\xi)\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix} = \\
&\qquad \exp\Big(-\tfrac{|\xi|^{2\delta}}{2}\textstyle\int_s^t b(\tau)d\tau\Big) M E_d(t,s,\xi)Q(t,s,\xi)M^{-1}\begin{pmatrix} p(s,\xi)\hat{u}(s,\xi) \\ D_t\hat{u}(s,\xi)\end{pmatrix}.
\end{aligned} \tag{3.3.2}$$

3.3.3 Treatment in the reduced zone

Proposition 3.3.4. *The following estimates hold for all* $t \in [t_1(|\xi|), t_2(|\xi|)]$:

$$\begin{aligned}
|\xi|^{|\beta|}|\hat{u}(t,\xi)| &\lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_1(|\xi|),\xi)|\Big) \\
&\quad \textit{for } |\beta| \ge \sigma, \\
|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| &\lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|}|\hat{u}_t(t_1(|\xi|),\xi)|\Big) \\
&\quad \textit{for } |\beta| \ge 0.
\end{aligned}$$

Proof. In the reduced zone we have $|\xi| \le C$ and $|\xi|^\sigma \sim \frac{b(t)}{2}|\xi|^{2\delta}$. We study directly

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + b(t)|\xi|^{2\delta}\hat{u}_t = 0.$$

Applying the transformation

$$\hat{u}(t,\xi) = \exp\Big(-\frac{|\xi|^{2\delta}}{2} \int_0^t b(\tau)d\tau\Big) v(t,\xi)$$

transfers the above equation into

$$v_{tt}(t,\xi) + \Big(|\xi|^{2\sigma} - \frac{b(t)^2}{4}|\xi|^{4\delta} - \frac{b'(t)}{2}|\xi|^{2\delta}\Big) v(t,\xi) = 0.$$

We have that the term with $b'(t)$ is a negligible term, this means, $|b'(t)| = \mathcal{O}(\varepsilon^2 b^2(t)|\xi|^{2\delta})$ as $t \to \infty$. Indeed, by using assumption (A5) we have

$$\frac{|b'(t)|}{\varepsilon^2 b^2(t)|\xi|^{2\delta}} = -\frac{b'(t)}{b(t)}\frac{1}{\varepsilon^2 b(t)|\xi|^{2\delta}} \lesssim \frac{b(t)}{\varepsilon^2 \Lambda(t) b(t)|\xi|^{2\delta}} \leq \frac{1}{\varepsilon^2 \Lambda(t)|\xi|^{2\delta}} \leq C(\varepsilon, N) \quad \text{as} \quad t \to \infty.$$

In this zone we can estimate

$$\Big| |\xi|^{2\sigma} - \frac{b^2(t)}{4}|\xi|^{4\delta} \Big| \leq \varepsilon^2 \frac{b^2(t)}{4}|\xi|^{4\delta}.$$

Thus we define the micro-energy

$$V(t,\xi) = \Big(\varepsilon \frac{b(t)}{2}|\xi|^{2\delta} v(t,\xi), D_t v(t,\xi)\Big)^T \quad \text{for all} \quad t \geq t_1(|\xi|) \;\; \text{and} \;\; (t,\xi) \in Z_{red}(\varepsilon),$$
$$V(t_1(|\xi|),\xi) = \Big(\varepsilon \frac{b(t_1(|\xi|))}{2}|\xi|^{2\delta} v(t_1(|\xi|),\xi), D_t v(t_1(|\xi|),\xi)\Big)^T,$$

where $V(t_1(|\xi|),\xi)$ is known from the elliptic zone $Z_{ell}(\varepsilon, N)$. We get the following system of first order:

$$D_t V(t,\xi) = \begin{pmatrix} \frac{D_t b(t)}{b(t)} & \varepsilon \frac{b(t)}{2}|\xi|^{2\delta} \\ \frac{|\xi|^{2\sigma} - \frac{1}{4}b^2(t)|\xi|^{4\delta} - \frac{1}{2}b'(t)|\xi|^{2\delta}}{\varepsilon \frac{b(t)}{2}|\xi|^{2\delta}} & \end{pmatrix} V(t,\xi). \tag{3.3.3}$$

To estimate the entries of this matrix we will use

1. $|b'(t)| = \mathcal{O}(\varepsilon^2 b^2(t)|\xi|^{2\delta})$,
2. $\Big| |\xi|^{2\sigma} - \frac{b^2(t)}{4}|\xi|^{4\delta} \Big| \leq \varepsilon^2 \frac{b^2(t)}{4}|\xi|^{4\delta}$,
3. consequently, $\frac{|\xi|^{2\sigma} - \frac{b^2(t)}{4}|\xi|^{4\delta} - \frac{b'(t)}{2}|\xi|^{2\delta}}{\varepsilon \frac{b(t)}{2}|\xi|^{2\delta}} \leq \varepsilon \frac{b(t)}{2}|\xi|^{2\delta} + \frac{|b'(t)|}{\varepsilon b(t)} \leq 2\varepsilon b(t)|\xi|^{2\delta}$.

Thus, we can estimate the norm of the coefficient matrix by $2\varepsilon b(t)|\xi|^{2\delta}$ for sufficiently large t. Summarizing the following statement holds:

Lemma 3.3.5. *The fundamental solution $E = E(t,s,\xi)$ to (3.3.3) is estimated by*

$$\|E(t,s,\xi)\| \leq \exp\Big(2\varepsilon|\xi|^{2\delta} \int_s^t b(\tau)d\tau\Big) \quad \textit{for all} \quad t \geq s \;\; \textit{and} \;\; (t,\xi),(s,\xi) \in Z_{red}(\varepsilon),$$

where $t_1(|\xi|)$ a sufficiently large.

From the backward transformation and the equivalence $\frac{b(t)}{2}|\xi|^{2\delta} \sim |\xi|^{\sigma}$ in $Z_{red}(\varepsilon)$ we conclude the next a-priori estimates for all $t \geq s$ and $(t,\xi),(s,\xi) \in Z_{red}(\varepsilon)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big((2\varepsilon-1)\frac{|\xi|^{2\delta}}{2}\int_s^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(s,\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(s,\xi)|\Big),$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big((2\varepsilon-1)\frac{|\xi|^{2\delta}}{2}\int_s^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(s,\xi)| + |\xi|^{|\beta|}|\hat{u}_t(s,\xi)|\Big).$$

In the following statement we set $s = t_1(|\xi|)$. Then these estimates imply the statements of the proposition. □

3.3.4 Treatment in the elliptic zone

Proposition 3.3.6. *The following estimates hold for all* $t \in [t_0(|\xi|), t_1(|\xi|)]$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_0(|\xi|),\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_0(|\xi|))}|\hat{u}_t(t_0(|\xi|),\xi)|\Big)$$
$$\text{for } |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\frac{1}{b(t)}\Big(|\xi|^{|\beta|+2\sigma-2\delta}|\hat{u}(t_0(|\xi|),\xi)|$$
$$+\frac{|\xi|^{|\beta|+2\sigma-4\delta}}{b(t_0(|\xi|))}|\hat{u}_t(t_0(|\xi|),\xi)|\Big) + \exp\Big(-|\xi|^{2\delta}\int_{t_0(|\xi|)}^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_t(t_0(|\xi|),\xi)|$$
$$\text{for } |\beta| \geq 0.$$

Proof. The proof is divided into several steps. *step 1*: A straight-forward estimate for the fundamental solution

Proposition 3.3.7. *The fundamental solution E satisfies for all $t \geq s$ and $(t,\xi),(s,\xi) \in Z_{ell}(\varepsilon,N)$ the following estimate* :

$$\begin{pmatrix} |E^{11}(t,s,\xi)| & |E^{12}(t,s,\xi)| \\ |E^{21}(t,s,\xi)| & |E^{22}(t,s,\xi)| \end{pmatrix} \lesssim \exp\Big(-C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\begin{pmatrix} 1 & \frac{1}{b(s)|\xi|^{2\delta-\sigma}} \\ b(t)|\xi|^{2\delta-\sigma} & \frac{b(t)}{b(s)} \end{pmatrix},$$

where the constant C is independent of $(s,\xi),(t,\xi) \in Z_{ell}(\varepsilon,N)$.

Proof. The characteristic roots of the matrix $A(t,\xi)$ are

$$\lambda_k(t,\xi) = \frac{ib(t)|\xi|^{2\delta} + (-1)^{k-1}i\sqrt{b^2(t)|\xi|^{4\delta} - 4|\xi|^{2\sigma}}}{2}, \quad k = 1,2.$$

In the further calculations we use the following properties of $\lambda_1(t,\xi)$ and $\lambda_2(t,\xi)$:

Lemma 3.3.8. *It holds*

1. $\Im\lambda_1(t,\xi) + \Im\lambda_2(t,\xi) = b(t)|\xi|^{2\delta}$, $\Im\lambda_1(t,\xi)\Im\lambda_2(t,\xi) = |\xi|^{2\sigma}$,
2. $\Im\lambda_1(t,\xi) \geq \Im\lambda_2(t,\xi) \geq 0, \quad |\lambda_1(t,\xi)| \geq |\lambda_2(t,\xi)|$,
3. $\frac{b(t)}{2}|\xi|^{2\delta} \leq \Im\lambda_1(t,\xi) \leq b(t)|\xi|^{2\delta}, \quad \frac{1}{b(t)}|\xi|^{2\sigma-2\delta} \leq \Im\lambda_2(t,\xi) \leq \frac{2}{b(t)}|\xi|^{2\sigma-2\delta}$.

Applying the diagonalization procedure (see in Appendix Section 10.1.1) we get after second step of diagonalization that the entries of the matrix $\mathcal{R}_2(t,\xi)$ are uniformly integrable over the elliptic zone. For this reason the matrix $\mathcal{R}_2(t,\xi)$ belongs to $L^1_{loc}\big(Z_{ell}(\varepsilon,N)\big)$. Hence, we can find the solution $U^{(1)}(t,\xi) =: \mathcal{N}_1(t,\xi)U^{(2)}(t,\xi)$, where $U^{(2)}(t,\xi)$ is the solution to the system

$$\big(D_t - \mathcal{D}(t,\xi) - \mathcal{F}^{(1)}(t,\xi) - \mathcal{R}_2(t,\xi)\big)U^{(2)}(t,\xi) = 0.$$

We can write $U^{(2)}(t,\xi) = E_2(t,s,\xi)U^{(2)}(s,\xi)$, where $E_2(t,s,\xi)$ is the fundamental solution, that is, the solution of the system

$$D_tE_2(t,s,\xi) = \big(\mathcal{D}(t,\xi) + \mathcal{F}^{(1)}(t,\xi) + \mathcal{R}_2(t,\xi)\big)E_2(t,s,\xi), \quad E_2(s,s,\xi) = I_2,$$

for all $t \geq s$ and $(t,\xi),(s,\xi) \in Z_{ell}(\varepsilon,N)$. First we estimate $E_d = E_d(t,s,\xi)$ as the fundamental solution of the diagonal part of this system, that is,

$$D_tE_d(t,s,\xi) = \big(\mathcal{D}(t,\xi) + \mathcal{F}^{(1)}(t,\xi)\big)E_d(t,s,\xi), \quad E_d(s,s,\xi) = I_2,$$

for all $t \geq s$ and $(t,\xi),(s,\xi) \in Z_{ell}(\varepsilon,N)$. Thus

$$\begin{aligned}
&E_d^{(11)}(t,s,\xi) = \exp\Big(\int_s^t -\frac{1}{2}\Big(1 + \frac{b'(\tau)|\xi|^{2\delta}}{b^2(\tau)|\xi|^{4\delta} - 4|\xi|^{2\sigma}}\Big)\Big(\sqrt{b^2(\tau)|\xi|^{4\delta} - 4|\xi|^{2\sigma}} + b(\tau)|\xi|^{2\delta}\Big)d\tau\Big),\\
&E_d^{(22)}(t,s,\xi) = \exp\Big(\int_s^t \frac{1}{2}\Big(1 + \frac{b'(\tau)|\xi|^{2\delta}}{b^2(\tau)|\xi|^{4\delta} - 4|\xi|^{2\sigma}}\Big)\Big(\sqrt{b^2(\tau)|\xi|^{4\delta} - 4|\xi|^{2\sigma}} - b(\tau)|\xi|^{2\delta}\Big)d\tau\Big),\\
&E_d^{(12)}(t,s,\xi) = E_d^{(21)}(t,s,\xi) = 0.
\end{aligned}$$

Lemma 3.3.9. *We have the following estimate in the elliptic zone for all $t \geq s$ and $(t,\xi),(s,\xi) \in Z_{ell}(\varepsilon,N)$:*

$$\|E_d(t,s,\xi)\| \lesssim \exp\Big(-C(\varepsilon,N)\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)$$

with a positive $C(\varepsilon,N)$ which is independent of (s,ξ) $(t,\xi) \in Z_{ell}(\varepsilon,N)$.

Proof. The estimate for E_d will be determined by the estimate of $E_d^{(22)}(t,s,\xi)$. By applying the definition of elliptic zone and Lemma 3.3.8 we get the following estimates

$$\begin{aligned}
&\frac{1}{2}\Big(\sqrt{b^2(t)|\xi|^{4\delta} - 4|\xi|^{2\sigma}} - b(t)|\xi|^{2\delta}\Big) \sim -\frac{|\xi|^{2\sigma-2\delta}}{b(t)},\\
&b^2(t)|\xi|^{4\delta} - 4|\xi|^{2\sigma} \sim b^2(t)|\xi|^{4\delta},\\
&\quad\text{and}\quad -\frac{b'(t)|\xi|^{2\delta}}{b^2(t)|\xi|^{4\delta} - 4|\xi|^{2\sigma}} \leq -\frac{b'(t)}{b^2(t)|\xi|^{2\delta}} \leq -\frac{b'(t)}{b(t)}\frac{1}{b(t)|\xi|^{2\delta}} \leq \frac{b(t)}{\Lambda(t)}\frac{1}{b(t)|\xi|^{2\delta}} \leq C_N.
\end{aligned}$$

The constant C_N is small if N is large. The desired statement follows immediately. □

The fundamental solution $E_2 = E_2(t,s,\xi)$ satisfies

$$\big(D_t - \mathcal{D}(t,\xi) - \mathcal{F}^{(1)}(t,\xi) - \mathcal{R}_2(t,\xi)\big)E_2(t,s,\xi) = 0, \;\; E_2(s,s,\xi) = I_2$$

for all $t \geq s$ and $(t,\xi),(s,\xi) \in Z_{ell}(\varepsilon,N)$. Thus we have

$$\begin{aligned}
&\partial_t\Big(\exp\Big(-i\int_s^t \big(\mathcal{D}(\tau,\xi) + \mathcal{F}^{(1)}(\tau,\xi)\big)d\tau\Big)E_2(t,s,\xi)\Big)\\
&\qquad = i\exp\Big(-i\int_s^t \big(\mathcal{D}(\tau,\xi) + \mathcal{F}^{(1)}(\tau,\xi)\big)d\tau\Big)\mathcal{R}_2(t,\xi)E_2(t,s,\xi),
\end{aligned}$$

and, consequently

$$E_2(t,s,\xi) = \exp\Big(i\int_s^t \big(\mathcal{D}(\tau,\xi)+\mathcal{F}^{(1)}(\tau,\xi)\big)d\tau\Big)E_2(s,s,\xi)$$
$$+i\int_s^t \exp\Big(i\int_\theta^t \big(\mathcal{D}(\tau,\xi)+\mathcal{F}^{(1)}(\tau,\xi)\big)d\tau\Big)\mathcal{R}_2(\theta,\xi)E_2(\theta,s,\xi)d\theta.$$

The statement of Lemma 3.3.9, in particular the structure of $E_d^{(22)}(t,s,\xi)$, motivates us to introduce the weight

$$w(t,\xi) = \frac{1}{2}\Big(1+\frac{b'(t)|\xi|^{2\delta}}{b^2(t)|\xi|^{4\delta}-4|\xi|^{2\sigma}}\Big)\Big(\sqrt{b^2(t)|\xi|^{4\delta}-4|\xi|^{2\sigma}}-b(t)|\xi|^{2\delta}\Big).$$

Let us define

$$Q(t,s,\xi) := \exp\Big(-\int_s^t w(\tau,\xi)d\tau\Big)E_2(t,s,\xi).$$

Then we get

$$Q(t,s,\xi) = \exp\Big(\int_s^t \big(i\mathcal{D}(\tau,\xi)+i\mathcal{F}^{(1)}(\tau,\xi)-w(\tau,\xi)I\big)d\tau\Big)$$
$$+\int_s^t \exp\Big(\int_\theta^t \big(i\mathcal{D}(\tau,\xi)+i\mathcal{F}^{(1)}(\tau,\xi)-w(\tau,\xi)I\big)d\tau\Big)\mathcal{R}_2(\theta,\xi)Q(\theta,s,\xi)d\theta.$$

Furthermore,

$$H(t,s,\xi) = \exp\Big(\int_s^t \big(i\mathcal{D}(\tau,\xi)+i\mathcal{F}^{(1)}(\tau,\xi)-w(\tau,\xi)I\big)d\tau\Big)$$
$$= \text{diag}\Big(\exp\Big(-\frac{1}{2}\int_s^t \Big(\sqrt{b^2(\tau)|\xi|^{4\delta}-4|\xi|^{2\sigma}}+\frac{b'(\tau)|\xi|^{2\delta}}{\sqrt{b^2(\tau)|\xi|^{4\delta}-4|\xi|^{2\sigma}}}\Big)d\tau\Big),1\Big).$$

Hence, the matrix H is uniformly bounded for (s,ξ), $(t,\xi)\in Z_{ell}(\varepsilon,N)$. Taking account of $\mathcal{R}_2(t,\xi)\in S_0\{-2\delta,-1,2\}$ the matrix $Q=Q(t,s,\xi)$ which is given by the matrizant representation, for $k=1$ we set $t_0=t$.

$$Q(t,s,\xi) = H(t,s,\xi)+\sum_{k=1}^{\infty}(-i)^k\int_s^t H(t,t_1,\xi)\mathcal{R}_2(t_1,\xi)\int_s^{t_1} H(t,t_2,\xi)\mathcal{R}_2(t_2,\xi)$$
$$\cdots\int_s^{t_{k-1}} H(t,t_{k-1},\xi)\mathcal{R}_2(t_{k-1},\xi)dt_k\cdots dt_2dt_1$$

is uniformly bounded in $Z_{ell}(\varepsilon,N)$. From the last consideration we may conclude

$$\begin{pmatrix} |E_2^{(11)}(t,s,\xi)| & |E_2^{(12)}(t,s,\xi)| \\ |E_2^{(21)}(t,s,\xi)| & |E_2^{(22)}(t,s,\xi)| \end{pmatrix} \lesssim \exp\Big(-C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

for all $t\geq s$ and $(t,\xi),(s,\xi)\in Z_{ell}(\varepsilon,N)$. From $U^{(2)}(t,\xi)=\mathcal{N}_1^{-1}(t,\xi)M^{-1}(t,\xi)U(t,\xi)$ the backward transformation gives the representation

$$E(t,s,\xi) = M(t,\xi)\mathcal{N}_1(t,\xi)E_2(t,s,\xi)\mathcal{N}_1^{-1}(s,\xi)M^{-1}(s,\xi).$$

Now we are only interested for $t\geq s$ and $(t,\xi),(s,\xi)\in Z_{ell}(\varepsilon,N)$. Due to Lemma 3.3.9 and the uniform bounded behavior of Q and $\mathcal{N}_1$ the statement of Proposition 3.3.7 follows if we take into consideration the estimate of the norm

$$\Big|M(t,\xi)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}M^{-1}(s,\xi)\Big| \text{ of the matrix } M(t,\xi)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}M^{-1}(s,\xi).$$

Using $|\lambda_1(t,\xi)| \sim b(t)|\xi|^{2\delta}$, $|\lambda_2(t,\xi)| \sim \frac{|\xi|^{2\sigma-2\delta}}{b(t)}$, and $|\det M(t,\xi)| \sim b(t)|\xi|^{2\delta-\sigma}$, then the definition of $Z_{ell}(\varepsilon, N)$, the fact that b is decreasing and $s \le t$ yield

$$\begin{aligned}
&\Big|M(t,\xi)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} M^{(11)}(s,\xi)^{-1}\Big| \lesssim 1,\\
&\Big|M(t,\xi)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} M^{(12)}(s,\xi)^{-1}\Big| \lesssim \frac{1}{b(s)|\xi|^{2\delta-\sigma}},\\
&\Big|M(t,\xi)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} M^{(21)}(s,\xi)^{-1}\Big| \lesssim b(t)|\xi|^{2\delta-\sigma},\\
&\Big|M(t,\xi)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} M^{(22)}(s,\xi)^{-1}\Big| \lesssim \frac{b(t)}{b(s)}.
\end{aligned}$$

These estimates give the desired statement. □

Remark 3.3.2. *Using Proposition 3.3.7 we obtain reasonable estimates for $|E^{11}|$ and $|E^{12}|$. The estimate for $|E^{22}|$ is only reasonable for decreasing coefficients $b(t)$. The estimate for $|E^{21}|$ is not optimal because the upper bound for $|E^{21}|$ is not bounded in the elliptic zone. This contradicts the damping character of our model. For this reason we need a refined estimate which we present in the next step.*

Step 2: A refined estimate for the fundamental solution

Lemma 3.3.10. *The fundamental solution E satisfies for all $t \ge s$ and $(t,\xi),(s,\xi) \in Z_{ell}(\varepsilon,N)$ the following estimate :*

$$\begin{aligned}
\begin{pmatrix}|E^{11}(t,s,\xi)| & |E^{12}(t,s,\xi)|\\ |E^{21}(t,s,\xi)| & |E^{22}(t,s,\xi)|\end{pmatrix} &\lesssim \exp\Big(-C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\begin{pmatrix}1 & \frac{1}{b(s)|\xi|^{2\delta-\sigma}}\\ \frac{1}{b(t)|\xi|^{2\delta-\sigma}} & \frac{1}{b(t)b(s)|\xi|^{4\delta-2\sigma}}\end{pmatrix}\\
&\quad + \exp\Big(-|\xi|^{2\delta}\int_s^t b(\tau)d\tau\Big)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},
\end{aligned}$$

where the constant C is independent of $(s,\xi),(t,\xi) \in Z_{ell}(\varepsilon,N)$.

Proof. If $\Phi_k(t,s,\xi), k = 1,2$, solves the equation $\Phi_{tt} + b(t)|\xi|^{2\delta}\Phi_t + |\xi|^{2\sigma}\Phi = 0$ with initial values $\Phi_k(s,s,\xi) = \delta_{1k}, \quad \partial_t\Phi_k(s,s,\xi) = \delta_{2k}$, we have

$$\begin{pmatrix}|\xi|^\sigma \hat{u}(t,\xi)\\ \\ D_t\hat{u}(t,\xi)\end{pmatrix} = \begin{pmatrix}\Phi_1(t,s,\xi) & i|\xi|^\sigma\Phi_2(t,s,\xi)\\ \\ \frac{D_t\Phi_1(t,s,\xi)}{|\xi|^\sigma} & iD_t\Phi_2(t,s,\xi)\end{pmatrix}\begin{pmatrix}|\xi|^\sigma\hat{u}(s,\xi)\\ \\ D_t\hat{u}(s,\xi)\end{pmatrix}.$$

Hence, it follows from Proposition 3.3.7

$$\begin{aligned}
&|\Phi_1(t,s,\xi)| \lesssim \exp\Big(-C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big),\\
&|\Phi_2(t,s,\xi)| \lesssim \frac{1}{b(s)|\xi|^{2\delta}}\exp\Big(-C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big),\\
&|\partial_t\Phi_1(t,s,\xi)| \lesssim b(t)|\xi|^{2\delta}\exp\Big(-C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big),\\
&|\partial_t\Phi_2(t,s,\xi)| \lesssim \frac{b(t)}{b(s)}\exp\Big(-C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big).
\end{aligned}$$

Let $\Psi_k(t,s,\xi)=\partial_t\Phi_k(t,s,\xi),\quad k=1,2$, then $\partial_t\Psi_k+b(t)|\xi|^{2\delta}\Psi_k=-|\xi|^{2\sigma}\Phi_k,\quad \Psi_k(s,s,\xi)=\delta_{2k}$. Standard calculations lead to

$$\begin{aligned}
\Psi_1(t,s,\xi)&=-|\xi|^{2\sigma}\int_s^t\Phi_1(\tau,s,\xi)\exp\Big(-|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)d\tau,\\
\Psi_2(t,s,\xi)&=\exp\Big(-|\xi|^{2\delta}\int_s^t b(\theta)d\theta\Big)-|\xi|^{2\sigma}\int_s^t\Phi_2(\tau,s,\xi)\exp\Big(-|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)d\tau.
\end{aligned}$$

Using the estimates for $|\Phi_1(t,s,\xi)|$ and $|\Phi_2(t,s,\xi)|$ we have

$$\begin{aligned}
|\Psi_1(t,s,\xi)|&\le C_1|\xi|^{2\sigma}\int_s^t\exp\Big(-C\int_s^\tau\frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta-|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)d\tau,\\
|\Psi_2(t,s,\xi)|&\le\exp\Big(-|\xi|^{2\delta}\int_s^t b(\theta)d\theta\Big)\\
&\qquad+C_1\frac{|\xi|^{2\sigma-2\delta}}{b(s)}\int_s^t\exp\Big(-C\int_s^\tau\frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta-|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)d\tau.
\end{aligned}$$

If we are able to derive the desired estimate for $|\Psi_1(t,s,\xi)|$, then we conclude immediately the desired estimate $|\Psi_2(t,s,\xi)|$. Applying partial integration we get

$$\begin{aligned}
|\Psi_1(t,s,\xi)|&\le C_1|\xi|^{2\sigma}\exp\Big(-C\int_s^t\frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta\Big)\Big(\int_s^t\exp\Big(-C\int_t^\tau\frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta\Big)\\
&\quad\times\exp\Big(-|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)d\tau\Big)\\
&\le C_1|\xi|^{2\sigma-2\delta}\exp\Big(-C\int_s^t\frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta\Big)\Big(\int_s^t\frac{1}{b(\tau)}\exp\Big(-C\int_t^\tau\frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta\Big)\\
&\quad\times\partial_\tau\exp\Big(-|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)d\tau\Big)\\
&\le C_1|\xi|^{2\sigma-2\delta}\exp\Big(-C\int_s^t\frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta\Big)\Big(\frac{1}{b(\tau)}\int_s^t\exp\Big(-C\int_t^\tau\frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta\Big)\\
&\quad\times\exp\Big(-|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)\Big|_s^t-\int_s^t\exp\Big(-|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)\\
&\quad\times\partial_\tau\Big(\frac{1}{b(\tau)}\int_s^t\exp\Big(-C\int_t^\tau\frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta\Big)\Big)d\tau\Big).
\end{aligned}$$

Using with a universal constant $C<1$ the estimate

$$\exp\Big(-C\int_\tau^t\frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta\Big)\le\exp\Big(-C|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big),$$

here we use the definition of the elliptic zone, gives

$$
\begin{aligned}
|\Psi_1(t,s,\xi)| &\leq C_1|\xi|^{2\sigma-2\delta}\exp\Big(-C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta\Big)\Big(\frac{1}{b(t)}+\frac{1}{b(s)}\exp\Big(-C\int_t^s \frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta\Big) \\
&\quad \times\exp\Big(-|\xi|^{2\delta}\int_s^t b(\theta)d\theta\Big)+\int_s^t \exp\Big(-|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big) \\
&\quad \times\exp\Big(-C\int_t^\tau \frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta\Big)\Big(-\frac{b'(\tau)}{b^2(\tau)}+\frac{|\xi|^{2\sigma-2\delta}}{b^2(\tau)}\Big)d\tau\Big) \\
&\leq C_1|\xi|^{2\sigma-2\delta}\exp\Big(-C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta\Big)\Big(\frac{1}{b(t)}+\frac{1}{b(s)}\exp\Big(-C|\xi|^{2\delta}\int_t^s b(\theta)d\theta\Big) \\
&\quad +\int_s^t \exp\Big(-C|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)\Big(-\frac{b'(\tau)}{b^2(\tau)}+\frac{|\xi|^{2\sigma-2\delta}}{b^2(\tau)}\Big)d\tau\Big) \\
&\leq C_1|\xi|^{2\sigma-2\delta}\exp\Big(-C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\theta)}d\theta\Big)\Big(\frac{1}{b(t)}+\int_s^t \exp\Big(-C|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big) \\
&\quad \times\Big(-\frac{b'(\tau)}{b^2(\tau)}+\frac{|\xi|^{2\sigma-2\delta}}{b^2(\tau)}\Big)d\tau\Big).
\end{aligned}
$$

Here we used that $b = b(t)$ is decreasing and $s \leq t$. Now we consider the part

$$
\Theta(t,s,\xi) := \int_s^t \exp\Big(-C|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)\Big(-\frac{b'(\tau)}{b^2(\tau)}+\frac{|\xi|^{2\sigma-2\delta}}{b^2(\tau)}\Big)d\tau.
$$

Using again partial integration we get

$$
\begin{aligned}
\Theta(t,s,\xi) &:= -C^{-1}\int_s^t \partial_\tau\exp\Big(-C|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)\Big(\frac{b'(\tau)}{b^3(\tau)|\xi|^{2\delta}}-\frac{|\xi|^{2\sigma-4\delta}}{b^3(\tau)}\Big)d\tau \\
&= -C^{-1}\exp\Big(-C|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)\Big(\frac{b'(\tau)}{b^3(\tau)|\xi|^{2\delta}}-\frac{|\xi|^{2\sigma-4\delta}}{b^3(\tau)}\Big)\Big|_s^t \\
&\quad -C^{-1}\int_s^t \exp\Big(-C|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)\partial_\tau\Big(-\frac{b'(\tau)}{b^3(\tau)|\xi|^{2\delta}}+\frac{|\xi|^{2\sigma-4\delta}}{b^3(\tau)}\Big)d\tau.
\end{aligned}
$$

We study the first and second term, respectively. Notice that

$$
-\frac{b'(\tau)}{b^3(\tau)|\xi|^{2\delta}} \leq \frac{C}{\Lambda(t)b(t)|\xi|^{2\delta}} \leq \frac{C}{Nb(t)}, \qquad \frac{1}{b^3(\tau)|\xi|^{4\delta-2\sigma}} \leq \frac{C}{b(t)},
$$

the first part can be estimated by $\frac{C}{b(t)}$. As for the second part, namely, for

$$
\int_s^t \exp\Big(-C|\xi|^{2\delta}\int_\tau^t b(\theta)d\theta\Big)\Big(-\frac{b''(\tau)}{b^3(\tau)|\xi|^{2\delta}}+\frac{3(b'(\tau))^2}{b^4(\tau)|\xi|^{2\delta}}-\frac{3b'(\tau)}{b^4(\tau)|\xi|^{4\delta-2\sigma}}\Big)d\tau,
$$

after taking into consideration assumption (A5) we may conclude

$$
\begin{aligned}
\frac{|b''(\tau)|}{b^3(\tau)|\xi|^{2\delta}} &\leq \frac{1}{b^2(\tau)|\xi|^{2\delta}}\frac{1}{(1+\tau)^2} \leq C\frac{1}{b^2(\tau)|\xi|^{2\delta}}\frac{b^2(\tau)}{\Lambda^2(\tau)} \leq C\frac{1}{b(\tau)\Lambda(\tau)|\xi|^{2\delta}}\frac{b(\tau)}{\Lambda(\tau)} \\
&\leq -C\frac{1}{b(\tau)\Lambda(\tau)|\xi|^{2\delta}}\frac{b'(\tau)}{b(\tau)} \leq -C\frac{b'(\tau)}{Nb^2(\tau)}
\end{aligned}
$$

and, together with the estimates

$$
\frac{(b'(\tau))^2}{b^4(\tau)|\xi|^{2\delta}} \leq -\frac{Cb'(\tau)}{Nb^2(\tau)}, \qquad -\frac{b'(\tau)|\xi|^{2\sigma-4\delta}}{b^4(\tau)} \leq -C_\varepsilon\frac{b'(\tau)}{b^2(\tau)},
$$

this term will be dominated by the left-hand side. Finally, we derived the following desired estimates for $|\Psi_1(t,s,\xi)|$ and $|E^{21}(t,s,\xi)|$, respectively:

$$|\Psi_1(t,s,\xi)| \lesssim \frac{|\xi|^{2\sigma-2\delta}}{b(t)} \exp\Big(- C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\theta)} d\theta\Big),$$
$$|E^{21}(t,s,\xi)| \lesssim \frac{|\xi|^{2\sigma-2\delta}}{b(t)} \exp\Big(- C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\theta)} d\theta\Big).$$

The estimates for $|\Psi_2(t,s,\xi)|$ and $|E^{22}(t,s,\xi)|$ follow immediately. They read as follows:

$$|\Psi_2(t,s,\xi)| = |E^{22}(t,s,\xi)| \lesssim \exp\Big(- C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\theta)} d\theta\Big)$$
$$+\frac{|\xi|^{2\sigma-4\delta}}{b(t)b(s)} \exp\Big(- C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\theta)} d\theta\Big).$$

Summarizing all desired estimates are proved. □

Remark 3.3.3. *We are able to derive a refined estimate for the fundamental solution because we use in the proof to Lemma 3.3.10 only the estimate for E^{11}, E^{12} from Proposition 3.3.7. Both estimates are optimal with our analytical tools.*

This finishes the proof to Proposition 3.3.6. □

Corollary 3.3.11. *We have the following representation of solution in the elliptic zone for all $t \geq s$ and (t,ξ), $(s,\xi) \in Z_{ell}(\varepsilon, N)$:*

$$\begin{pmatrix} |\xi|^\sigma \hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix} = \exp\big(\textstyle\int_s^t w(s,\xi)ds\big)$$
$$\times M(t,\xi)\mathcal{N}_1(t,\xi)Q(t,s,\xi)\mathcal{N}_1^{-1}(s,\xi)M^{-1}(s,\xi)\begin{pmatrix} |\xi|^\sigma \hat{u}(s,\xi) \\ D_t\hat{u}(s,\xi)\end{pmatrix}, \tag{3.3.4}$$

where $w(t,\xi) = \frac{1}{2}\Big(1 + \frac{b'(t)|\xi|^{2\delta}}{b^2(t)|\xi|^{4\delta}-4|\xi|^{2\sigma}}\Big)\Big(\sqrt{b^2(t)|\xi|^{4\delta}-4|\xi|^{2\sigma}} - b(t)|\xi|^{2\delta}\Big)$.

3.3.5 Treatment in the pseudo-differential zone

Proposition 3.3.12. *The following estimates hold for all $t \in [0, t_0(|\xi|)]$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}}(1+t)|\hat{u}_1(\xi)|$$
for $|\beta| \geq \sigma$,
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma}{2\delta}}(1+t)|\hat{u}_0(\xi)| + \exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_1(\xi)|$$
for $|\beta| \geq 0$.

Proof. Let us introduce the micro-energy $U = U(t,\xi)$

$$U(t,\xi) = \Big(\frac{N}{\Lambda^{\frac{\sigma}{2\delta}}(t)}\hat{u}(t,\xi), D_t\hat{u}(t,\xi)\Big)^T \quad \text{for all} \quad t \geq 0 \ \text{ and } \ (t,\xi) \in Z_{pd}(\varepsilon, N).$$

Then the transformed equation can be written for all $t \in (0, t_0(|\xi|)]$ in the form of the system of first order

$$D_t U(t,\xi) = \begin{pmatrix} \frac{\frac{1}{i}\partial_t\left(\frac{N}{\Lambda^{\frac{\sigma}{2\delta}}(t)}\right)}{\frac{N}{\Lambda^{\frac{\sigma}{2\delta}}(t)}} & \frac{N}{\Lambda^{\frac{\sigma}{2\delta}}(t)} \\ \frac{|\xi|^{2\sigma}\Lambda^{\frac{\sigma}{2\delta}}(t)}{N} & ib(t)|\xi|^{2\delta} \end{pmatrix} U(t,\xi), \quad U(0,\xi) = \big(N\hat{u}(0,\xi), D_t\hat{u}(0,\xi)\big)^T.$$

Thus the solution $U = U(t,\xi)$ can be represented as $U(t,\xi) = \mathcal{E}(t,s,\xi)U(s,\xi)$, where $\mathcal{E}(t,s,\xi)$ is the fundamental solution, that is, the solution to the system

$$\partial_t \mathcal{E}(t,s,\xi) = \begin{pmatrix} \frac{\partial_t\left(\frac{N}{\Lambda^{\frac{\sigma}{2\delta}}(t)}\right)}{\frac{N}{\Lambda^{\frac{\sigma}{2\delta}}(t)}} & i\frac{N}{\Lambda^{\frac{\sigma}{2\delta}}(t)} \\ i\frac{|\xi|^{2\sigma}\Lambda^{\frac{\sigma}{2\delta}}(t)}{N} & -b(t)|\xi|^{2\delta} \end{pmatrix} \mathcal{E}(t,s,\xi), \quad \mathcal{E}(s,s,\xi) = I_2 \tag{3.3.5}$$

for all $t \geq s$ and $(t,\xi) \in Z_{pd}(\varepsilon, N)$. In the following statement we set $s = 0$.

Lemma 3.3.13. *The entries $\mathcal{E}^{kl}(t,0,\xi)$, $k,l = 1,2$, of the fundamental solution $\mathcal{E}(t,0,\xi)$ satisfy the following system of Volterra integral equations :*

$$\begin{aligned}
\mathcal{E}^{11}(t,0,\xi) &= \frac{1}{\Lambda(t)^{\frac{\sigma}{2\delta}}} + i\frac{N}{\Lambda(t)^{\frac{\sigma}{2\delta}}}\int_0^t \mathcal{E}^{21}(\tau,0,\xi)d\tau, \\
\mathcal{E}^{21}(t,0,\xi) &= i\frac{|\xi|^{2\sigma}}{N\lambda^2(t,\xi)}\int_0^t \lambda^2(\tau,\xi)\Lambda(\tau)^{\frac{\sigma}{2\delta}}\mathcal{E}^{11}(\tau,0,\xi)d\tau, \\
\mathcal{E}^{12}(t,0,\xi) &= i\frac{N}{\Lambda(t)^{\frac{\sigma}{2\delta}}}\int_0^t \mathcal{E}^{22}(\tau,0,\xi)d\tau, \\
\mathcal{E}^{22}(t,0,\xi) &= \frac{1}{\lambda^2(t,\xi)} + i\frac{|\xi|^{2\sigma}}{N\lambda^2(t,\xi)}\int_0^t \lambda^2(\tau,\xi)\Lambda(\tau)^{\frac{\sigma}{2\delta}}\mathcal{E}^{12}(\tau,0,\xi)d\tau,
\end{aligned}$$

where $\lambda^2(t,\xi) = \exp\big(|\xi|^{2\delta}\int_0^t b(s)ds\big)$.

Proof. The statements follow by direct calculations. □

Now let us estimate the modulus $|\mathcal{E}^{kl}(t,0,\xi)|$, $k,l = 1,2$, of the entries of $\mathcal{E}(t,0,\xi)$.

Lemma 3.3.14. *We have the following estimates for all $t \in [0, t_0(|\xi|)]$:*

$$\begin{pmatrix} |\mathcal{E}^{11}(t,0,\xi)| & |\mathcal{E}^{12}(t,0,\xi)| \\ |\mathcal{E}^{21}(t,0,\xi)| & |\mathcal{E}^{22}(t,0,\xi)| \end{pmatrix} \lesssim \begin{pmatrix} \Lambda^{-\frac{\sigma}{2\delta}}(t) & (1+t)\Lambda^{-\frac{\sigma}{2\delta}}(t) \\ (1+t)\Lambda^{-\frac{\sigma}{\delta}}(t) & \exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big) \end{pmatrix}.$$

Proof. We have

$$\Lambda(t)^{\frac{\sigma}{2\delta}}\mathcal{E}^{11}(t,0,\xi) = 1 - |\xi|^{2\sigma}\int_0^t \frac{1}{\lambda^2(\tau,\xi)}\int_0^\tau \lambda^2(s,\xi)\Lambda(s)^{\frac{\sigma}{2\delta}}\mathcal{E}^{11}(s,0,\xi)dsd\tau,$$

hence,

$$|\Lambda(t)^{\frac{\sigma}{2\delta}}\mathcal{E}^{11}(t,0,\xi)| \leq 1 + |\xi|^{2\sigma}\int_0^t\int_0^\tau \frac{\lambda^2(s,\xi)}{\lambda^2(\tau,\xi)}|\Lambda(s)^{\frac{\sigma}{2\delta}}\mathcal{E}^{11}(s,0,\xi)|dsd\tau.$$

If we denote $\varphi(t,\xi)=\Lambda(t)^{\frac{\sigma}{2\delta}}\mathcal{E}^{11}(t,0,\xi)$, then we conclude from the about system

$$\varphi(t,\xi)=1-\int_0^t\int_0^\tau|\xi|^{2\sigma}\varphi(s,\xi)dsd\tau,$$

hence,

$$|\varphi(t,\xi)|\leq 1+\int_0^t\int_0^\tau|\xi|^{2\sigma}|\varphi(s,\xi)|dsd\tau.$$

Applying Lemma 10.1.6 from the Appendix to

$$|\varphi(t,\xi)|\leq 1+|\xi|^{2\sigma}\int_0^t\int_0^\tau|\varphi(s,\xi)|dsd\tau,$$

where

$$\varphi(t,\xi)=\Lambda^{\frac{\sigma}{2\delta}}(t)\mathcal{E}^{11}(t,0,\xi),\quad \text{it follows}\quad |\Lambda^{\frac{\sigma}{2\delta}}(t)\mathcal{E}^{11}(t,0,\xi)|\leq\exp\Big(\int_0^t\int_0^\tau|\xi|^{2\sigma}dsd\tau\Big).$$

Applying Lemma 3.3.1 gives

$$\int_0^t\int_0^\tau|\xi|^{2\sigma}dsd\tau\leq C|\xi|^{2\sigma}(1+t)^2\leq\frac{C_N(1+t)^2}{\Lambda(t)^{\frac{\sigma}{\delta}}}\leq C_N.$$

So we may conclude that

$$|\mathcal{E}^{11}(t,0,\xi)|\leq C\Lambda(t)^{-\frac{\sigma}{2\delta}}.$$

Now we consider

$$\mathcal{E}^{21}(t,0,\xi)=i\frac{|\xi|^{2\sigma}}{N}\int_0^t\frac{\lambda^2(\tau,\xi)}{\lambda^2(t,\xi)}\Lambda(\tau)^{\frac{\sigma}{2\delta}}\mathcal{E}^{11}(\tau,0,\xi)d\tau.$$

By using the estimate for $|\mathcal{E}^{11}(t,0,\xi)|$ we have

$$|\mathcal{E}^{21}(t,0,\xi)|\leq\frac{|\xi|^{2\sigma}}{N}\int_0^t|\Lambda(\tau)^{\frac{\sigma}{2\delta}}\mathcal{E}^{11}(\tau,0,\xi)|d\tau\leq C|\xi|^{2\sigma}(1+t)\leq\frac{C}{\Lambda(t)^{\frac{\sigma}{\delta}}(1+t)^{-1}}.$$

In this way we obtained

$$|\mathcal{E}^{21}(t,0,\xi)|\leq C(1+t)\Lambda(t)^{-\frac{\sigma}{\delta}}.$$

The representations for $\mathcal{E}^{12}$ and $\mathcal{E}^{22}$ imply

$$\mathcal{E}^{22}(t,0,\xi)=\frac{1}{\lambda^2(t,\xi)}-\frac{|\xi|^{2\sigma}}{\lambda^2(t,\xi)}\int_0^t\int_0^\tau\lambda^2(\tau,\xi)\mathcal{E}^{22}(s,0,\xi)dsd\tau.$$

Hence,

$$\begin{aligned}\lambda^2(t,\xi)\mathcal{E}^{22}(t,0,\xi)&=1-|\xi|^{2\sigma}\int_0^t\int_0^\tau\lambda^2(\tau,\xi)\mathcal{E}^{22}(s,0,\xi)dsd\tau\\&=1-|\xi|^{2\sigma}\int_0^t\int_0^\tau\frac{\lambda^2(\tau,\xi)}{\lambda^2(s,\xi)}\lambda^2(s,\xi)\mathcal{E}^{22}(s,0,\xi)dsd\tau.\end{aligned}$$

As in the previous step we estimate as follows :

$$|\lambda^2(t,\xi)\mathcal{E}^{22}(t,0,\xi)|\leq 1+|\xi|^{2\sigma}\int_0^t\int_0^\tau|\lambda^2(s,\xi)\mathcal{E}^{22}(s,0,\xi)|dsd\tau.$$

So we see that after setting $\varphi(t,\xi) = \lambda^2(t,\xi)\mathcal{E}^{22}(t,0,\xi)$ we are able to apply Lemma 10.1.6 from the Appendix. In the same way as we did it for $\mathcal{E}^{11}(t,0,\xi)$ it follows immediately $|\lambda^2(t,\xi)\mathcal{E}^{22}(t,0,\xi)| \le C$, thus

$$|\mathcal{E}^{22}(t,0,\xi)| \le \frac{C}{\lambda^2(t,\xi)} \le C\exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big).$$

In a similar way we also get

$$|\mathcal{E}^{12}(t,0,\xi)| \le C(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}}.$$

This completes the proof. □

Now let us come back to

$$U(t,\xi) = \mathcal{E}(t,0,\xi)U(0,\xi) \quad \text{for all} \quad t \in [0, t_0(|\xi|)]. \tag{3.3.6}$$

Because of $|\xi|^\sigma|\hat{u}(t,\xi)| \le \frac{N}{\Lambda(t)^{\frac{\sigma}{2\delta}}}|\hat{u}(t,\xi)|$ by using the definition of the pseudo-differential zone we conclude for the elastic energy and the kinetic energy of higher order the estimates from (3.3.6) and Lemma 3.3.14. The following statements can be concluded:

$$|\xi|^\sigma|\hat{u}(t,\xi)| \le N\Lambda(t)^{-\frac{\sigma}{2\delta}}|\hat{u}(t,\xi)| \le |\mathcal{E}^{11}(t,0,\xi)||N\hat{u}_0(\xi)| + |\mathcal{E}^{12}(t,0,\xi)||\hat{u}_1(\xi)| \tag{3.3.7}$$
$$\lesssim C\Lambda(t)^{-\frac{\sigma}{2\delta}}|\hat{u}_0(\xi)| + C\Lambda(t)^{-\frac{\sigma}{2\delta}}(1+t)|\hat{u}_1(\xi)|. \tag{3.3.8}$$

For the kinetic energy we may conclude

$$|\hat{u}_t(t,\xi)| \le |\mathcal{E}^{21}(t,0,\xi)||N\hat{u}_0(\xi)| + |\mathcal{E}^{22}(t,0,\xi)||\hat{u}_1(\xi)| \tag{3.3.9}$$
$$\lesssim C\Lambda(t)^{-\frac{\sigma}{\delta}}(1+t)|\hat{u}_0(\xi)| + \exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)|\hat{u}_1(\xi)|. \tag{3.3.10}$$

This completes the proof of the Proposition 3.3.12. □

3.3.6 Gluing procedure

Now we have to glue the estimates from the Propositions 3.3.2, 3.3.4, 3.3.6 and 3.3.12. If we consider the set $M_p := \{\xi : |\xi| \ge p\}$, then it is clear that an exponential type decay for the higher order energies follows from Proposition 3.3.2 under the usual regularity assumption for the data from the Cauchy problem for σ-evolution models. Thus the interesting case is to glue all estimates for small frequencies, let us say for $\{\xi : |\xi| \le p\}$, where p is sufficiently small. In this case all the zones restricted to small frequencies are unbounded.

Lemma 3.3.15. *The terms with phase functions have no meaning in the pseudo-differential zone, that is, it holds*

$$|\xi|^{2\delta}\int_0^{t_0(|\xi|)} b(\tau)d\tau \le N, \qquad |\xi|^{2\sigma-2\delta}\int_0^{t_0(|\xi|)}\frac{1}{b(\tau)}d\tau \le C.$$

Proof. By applying the definition of $t_0(|\xi|)$ we conclude

$$|\xi|^{2\delta}\int_0^{t_0(|\xi|)} b(\tau)d\tau \le |\xi|^{2\delta}\Big(1+\int_0^{t_0(|\xi|)} b(\tau)d\tau\Big) \le N.$$

After taking into consideration assumption (A5) we have for the second term

$$|\xi|^{2\sigma-2\delta}\int_0^{t_0(|\xi|)}\frac{1}{b(\tau)}d\tau \le C|\xi|^{2\sigma-2\delta}\int_0^{t_0(|\xi|)}\frac{1+\tau}{\Lambda(\tau)}d\tau.$$

Taking account of the condition $1+t\le C\Lambda(t)^\alpha$ from our assumption (A6) and the definition of $Z_{pd}(\varepsilon,N)$ we get

$$\begin{aligned}|\xi|^{2\sigma-2\delta}\int_0^{t_0(|\xi|)}\frac{1}{b(\tau)}d\tau &\lesssim |\xi|^{2\sigma-2\delta}\int_0^{t_0(|\xi|)}\Lambda(\tau)^{\alpha-1}d\tau \lesssim |\xi|^{2\sigma-2\delta}\int_0^{t_0(|\xi|)}|\xi|^{-2\delta(\alpha-1)}d\tau\\ &\lesssim |\xi|^{2\sigma-2\delta\alpha}(1+t_0(|\xi|)) \lesssim |\xi|^{2\sigma-2\delta\alpha}\Lambda(t_0(|\xi|))^\alpha \lesssim |\xi|^{2\sigma-4\delta\alpha}\lesssim 1.\end{aligned}$$

This we wanted to prove. □

Due to this statement it is allowed to extend the integrals in the phases from Proposition 3.3.6 Now we have to glue the estimates from Proposition 3.3.12 for $t=t_0(|\xi|)$ and from Proposition 3.3.6. Therefore we need the following statement.

Lemma 3.3.16. *Let $b\in S_\alpha$ with $\alpha\in(1,\frac{\sigma}{2\delta}]$. Then the following estimate holds for the separating line $t_0(|\xi|)$ between the elliptic zone and the pseudo-differential zone :*

$$\frac{1}{b(t_0(|\xi|))}\lesssim \Lambda(t_0(|\xi|))^{\alpha-1}\lesssim |\xi|^{2\delta-2\delta\alpha}.$$

Proof. Applying assumption (A5) we may conclude

$$\frac{1}{b(t_0(|\xi|))}\lesssim\frac{1+t_0(|\xi|)}{\Lambda(t_0(|\xi|))},\quad \text{hence,}\quad \frac{1}{b(t_0(|\xi|))}\lesssim\big(1+t_0(|\xi|)\big)\Lambda(t_0(|\xi|))^{-1}.$$

Using the classification for $b\in S_\alpha$ from (A6) we get

$$\frac{1}{b(t_0(|\xi|))}\lesssim \Lambda(t_0(|\xi|))^{\alpha-1}\sim |\xi|^{2\delta-2\delta\alpha}.$$

This gives the desired statement. □

Using this lemma we conclude as follows:

Corollary 3.3.17. *The following estimates hold for all $t\in[t_0(|\xi|),t_1(|\xi|)]$:*

$$\begin{aligned}&|\xi|^{|\beta|}|\hat u(t,\xi)|\lesssim \exp\Big(-C\int_0^t\frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat u_0(\xi)|+|\xi|^{|\beta|-2\delta\alpha}|\hat u_1(\xi)|\Big)\quad \text{for }\ |\beta|\ge\sigma,\\ &|\xi|^{|\beta|}|\hat u_t(t,\xi)|\\ &\quad\lesssim \exp\Big(-C\int_0^t\frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\frac{1}{b(t)}\Big(|\xi|^{|\beta|+2\sigma-2\delta}|\hat u_0(\xi)|+|\xi|^{|\beta|+2\sigma-2\delta-2\delta\alpha}|\hat u_1(\xi)|\Big)\\ &\qquad+\exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\alpha}|\hat u_0(\xi)|+|\xi|^{|\beta|}|\hat u_1(\xi)|\Big)\quad\text{for }\ |\beta|\ge 0.\end{aligned}$$

Proof. Let us begin to estimate $|\xi|^{|\beta|}|\hat u(t,\xi)|$. The statement of Proposition 3.3.6 implies

$$|\xi|^{|\beta|}|\hat u(t,\xi)|\lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t\frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat u(t_0(|\xi|),\xi)|+\frac{|\xi|^{|\beta|-2\delta}}{b(t_0|\xi|)}|\hat u_t(t_0(|\xi|),\xi)|\Big).$$

Using the estimates for $|\xi|^{|\beta|}|\hat{u}(t_0(|\xi|),\xi)|$ and $|\hat{u}_t(t_0(|\xi|),\xi)|$ from Proposition 3.3.12 we have

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}}|\hat{u}_0(\xi)| + \Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}}$$
$$\times(1+t_0(|\xi|))|\hat{u}_1(\xi)| + \frac{\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}}(1+t_0(|\xi|))}{b(t_0(|\xi|))}|\hat{u}_0(\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_0(|\xi|))}|\hat{u}_1(\xi)|\Big).$$

Taking account of $1+t_0(|\xi|) \lesssim \Lambda(t_0(|\xi|))^\alpha$ from (A6) we get

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}}|\hat{u}_0(\xi)| + \Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}+\alpha}|\hat{u}_1(\xi)|$$
$$+\frac{\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}+\alpha}}{b(t_0(|\xi|))}|\hat{u}_0(\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_0(|\xi|))}|\hat{u}_1(\xi)|\Big).$$

The statement of Lemma 3.3.16 implies

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}}|\hat{u}_0(\xi)| + \Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}+\alpha}|\hat{u}_1(\xi)|$$
$$+\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}+2\alpha-1}|\hat{u}_0(\xi)| + \Lambda(t_0(|\xi|))^{\alpha-1}|\xi|^{|\beta|-2\delta}|\hat{u}_1(\xi)|\Big).$$

After using the definition of $t_0(|\xi|)$ and Lemma 3.3.15 we conclude

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta\alpha}|\hat{u}_1(\xi)|$$
$$+|\xi|^{|\beta|+2\sigma-4\delta\alpha}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta\alpha}|\hat{u}_1(\xi)|\Big).$$

Summarizing we have shown

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta\alpha}|\hat{u}_1(\xi)|\Big).$$

In the same way, the statement of Proposition 3.3.6 implies

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\frac{1}{b(t)}\Big(|\xi|^{|\beta|+2\sigma-2\delta}|\hat{u}(t_0(|\xi|),\xi)|$$
$$+\frac{|\xi|^{|\beta|+2\sigma-4\delta}}{b(t_0(|\xi|))}|\hat{u}_t(t_0(|\xi|),\xi)|\Big) + \exp\Big(-C|\xi|^{2\delta}\int_{t_0(|\xi|)}^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_t(t_0(|\xi|),\xi)|.$$

The estimates for $|\xi|^{|\beta|}|\hat{u}(t_0(|\xi|),\xi)|$ and $|\hat{u}_t(t_0(|\xi|),\xi)|$ from Proposition 3.3.12 yield

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\frac{1}{b(t)}\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}}|\hat{u}_0(\xi)|$$
$$+\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}}(1+t_0(|\xi|))|\hat{u}_1(\xi)| + \frac{\Lambda(t_0(|\xi|))^{-\frac{|\beta|+4\sigma-4\delta}{2\delta}}(1+t_0(|\xi|))}{b(t_0(|\xi|))}|\hat{u}_0(\xi)|$$
$$+\frac{|\xi|^{|\beta|+2\sigma-4\delta}}{b(t_0(|\xi|))}|\hat{u}_1(\xi)|\Big) + \exp\Big(-|\xi|^{2\delta}\int_{t_0(|\xi|)}^t b(\tau)d\tau\Big)\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma}{2\delta}}(1+t_0(|\xi|))|\hat{u}_0(\xi)|$$
$$+|\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big).$$

Taking account of $1+t_0(|\xi|)\lesssim \Lambda(t_0(|\xi|))^\alpha$ we get

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\frac{1}{b(t)}\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}}|\hat{u}_0(\xi)|$$
$$+\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}+\alpha}|\hat{u}_1(\xi)| + \frac{\Lambda(t_0(|\xi|))^{-\frac{|\beta|+4\sigma-4\delta}{2\delta}+\alpha}}{b(t_0(|\xi|))}|\hat{u}_0(\xi)| + \frac{|\xi|^{|\beta|+2\sigma-4\delta}}{b(t_0(|\xi|))}|\hat{u}_1(\xi)|\Big)$$
$$+\exp\Big(-|\xi|^{2\delta}\int_{t_0(|\xi|)}^t b(\tau)d\tau\Big)\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma}{2\delta}+\alpha}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big).$$

The statement of Lemma 3.3.16 implies

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\frac{1}{b(t)}\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}}|\hat{u}_0(\xi)|$$
$$+\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}+\alpha}|\hat{u}_1(\xi)| + \Lambda(t_0(|\xi|))^{-\frac{|\beta|+4\sigma-4\delta}{2\delta}+2\alpha-1}|\hat{u}_0(\xi)|$$
$$+\Lambda(t_0(|\xi|))^{\alpha-1}|\xi|^{|\beta|+2\sigma-4\delta}|\hat{u}_1(\xi)|\Big)$$
$$+\exp\Big(-|\xi|^{2\delta}\int_{t_0(|\xi|)}^t b(\tau)d\tau\Big)\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma}{2\delta}+\alpha}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big).$$

By using the definition of $t_0(|\xi|)$ and Lemma 3.3.15 we may conclude

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\frac{1}{b(t)}\Big(|\xi|^{|\beta|+2\sigma-2\delta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|+2\sigma-2\delta-2\delta\alpha}|\hat{u}_1(\xi)|$$
$$+|\xi|^{|\beta|+4\sigma-2\delta-4\delta\alpha}|\hat{u}_0(\xi)| + |\xi|^{|\beta|+2\sigma-2\delta-2\delta\alpha}|\hat{u}_1(\xi)|\Big)$$
$$+\exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\alpha}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big).$$

Summarizing we have shown the inequality

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\frac{1}{b(t)}\Big(|\xi|^{|\beta|+2\sigma-2\delta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|+2\sigma-2\delta-2\delta\alpha}|\hat{u}_1(\xi)|$$
$$+\exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\alpha}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big).$$

This completes the proof. □

Taking account of the Propositions 3.3.2 and 3.3.4 we obtain the following statement :

Corollary 3.3.18. *The following estimates hold for all* $t\in[t_1(|\xi|),\infty)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_1(|\xi|),\xi)|\Big)$$

for $|\beta|\geq\sigma$,

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|}|\hat{u}_t(t_1(|\xi|),\xi)|\Big)$$

for $|\beta|\geq 0$.

Finally, we have to glue the estimates from Corollary 3.3.17 for $t=t_1(|\xi|)$ and Corollary 3.3.18.

Corollary 3.3.19. *The following estimates hold for all $t \in [t_1(|\xi|), \infty)$:*

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}(t,\xi)| &\lesssim \exp\Big(-C\int_0^{t_1(|\xi|)} \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| \\
&\quad +|\xi|^{|\beta|-2\delta\alpha}|\hat{u}_1(\xi)|\Big) + \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma-2\delta\alpha}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big)
\end{aligned}
$$

for $|\beta| \geq \sigma$,

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| &\lesssim \exp\Big(-C\int_0^{t_1(|\xi|)} \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| \\
&\quad +|\xi|^{|\beta|+\sigma-2\delta\alpha}|\hat{u}_1(\xi)| + \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\alpha}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big)
\end{aligned}
$$

for $|\beta| \geq 0$.

Proof. Let us begin to estimate $|\xi|^{|\beta|}|\hat{u}(t,\xi)|$. The statement of Corollary 3.3.18 implies

$$
|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_1(|\xi|),\xi)|\Big).
$$

Using the estimates for $|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)|$ and $|\hat{u}_t(t_1(|\xi|),\xi)|$ from Corollary 3.3.17 we have

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}(t,\xi)| &\lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\exp\Big(-C\int_0^{t_1(|\xi|)} \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| \\
&\quad +|\xi|^{|\beta|-2\delta\alpha}|\hat{u}_1(\xi)| + \frac{1}{b(t_1(|\xi|))}\Big(|\xi|^{|\beta|+\sigma-2\delta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|+\sigma-2\delta-2\delta\alpha}|\hat{u}_1(\xi)|\Big)\Big) \\
&\quad + \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma-2\delta\alpha}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big).
\end{aligned}
$$

By using the definition of $t_1(|\xi|)$ we may conclude

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}(t,\xi)| &\lesssim \exp\Big(-C\int_0^{t_1(|\xi|)} \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| \\
&\quad +|\xi|^{|\beta|-2\delta\alpha}|\hat{u}_1(\xi)|\Big) + \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma-2\delta\alpha}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big).
\end{aligned}
$$

In the same way, the statement of Corollary 3.3.18 implies

$$
|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|}|\hat{u}_t(t_1(|\xi|),\xi)|\Big).
$$

The estimates for $|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)|$ and $|\hat{u}_t(t_1(|\xi|),\xi)|$ from Corollary 3.3.17 yield

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| &\lesssim \exp\Big(-C\int_0^{t_1(|\xi|)} \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| \\
&\quad +|\xi|^{|\beta|+\sigma-2\delta\alpha}|\hat{u}_1(\xi)| + \frac{1}{b(t_1(|\xi|))}\Big(|\xi|^{|\beta|+2\sigma-2\delta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|+2\sigma-2\delta-2\delta\alpha}|\hat{u}_1(\xi)|\Big)\Big) \\
&\quad + \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\alpha}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big).
\end{aligned}
$$

Applying the definition of $t_1(|\xi|)$ gives

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_0^{t_1(|\xi|)} \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big) \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|+\sigma-2\delta\alpha}|\hat{u}_1(\xi)|\Big) + \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\alpha}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big).$$

This completes the proof. □

3.3.7 Energy estimate

For large frequencies we may use the estimates from Proposition 3.3.2. They imply an exponential type decay. This we will show in the following statement.

Corollary 3.3.20. *For large frequencies $|\xi| \geq p$ the following estimates hold for all $t \in [0,\infty)$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C(1+t)^{\frac{1}{\alpha}}\big)\big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\big) \quad \text{for} \quad |\beta| \geq 0,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\big(-C(1+t)^{\frac{1}{\alpha}}\big)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big) \quad \text{for} \quad |\beta| \geq 0.$$

Proof. We use $\int_0^t b(\tau)d\tau \sim 1 + \int_0^t b(\tau)d\tau$ for large t. So, we get for large time t the estimate

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C\Lambda(t)\big)\big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\big).$$

The inequality $1+t \leq \Lambda(t)^\alpha$ from our condition (A6) gives

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C(1+t)^{\frac{1}{\alpha}}\big)\big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\big).$$

In the same way, we derive

$$\begin{aligned}|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| &\lesssim \exp\big(-C\Lambda(t)\big)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big) \\ &\lesssim \exp\big(-C(1+t)^{\frac{1}{\alpha}}\big)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big).\end{aligned}$$

Both estimates yield the exponential type decay. □

So we restrict ourselves to small frequencies. To derive the corresponding energy estimates from Corollary 3.3.19 we shall estimate the term

$$S_r(t,|\xi|) := |\xi|^r \exp\Big(-C_0|\xi|^{2\sigma-2\delta}\int_0^{t_1(|\xi|)} \frac{1}{b(\tau)}d\tau\Big) \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big).$$

This term explains the competition of influences from different phase functions.

Lemma 3.3.21. *If the constant C_0 is sufficiently small, then*

$$S_r(t,|\xi|) \lesssim \max_{\xi\in\mathbb{R}^n}\Big\{|\xi|^r \exp\Big(-C_0|\xi|^{2\sigma-2\delta}\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big\} \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{r}{2\sigma-2\delta}} \quad \text{for} \quad r \geq 0.$$

Proof. First we show that the first partial derivative $\partial_{|\xi|} S_r(t, |\xi|)$ is negative for $|\xi| \le \varepsilon_r$. For this reason we calculate

$$\begin{aligned}\partial_{|\xi|} S_r(t, |\xi|) &= S_r(t, |\xi|)\Big(\frac{r}{|\xi|} - C_0(2\sigma - 2\delta)|\xi|^{2\sigma-2\delta-1}\int_0^{t_1(|\xi|)}\frac{1}{b(\tau)}d\tau \\ &\quad -C_0|\xi|^{2\sigma-2\delta}\frac{1}{b(t_1(|\xi|))}d_{|\xi|}t_1(|\xi|) - \frac{2\delta}{4}|\xi|^{2\delta-1}\int_{t_1(|\xi|)}^t b(\tau)d\tau + \frac{1}{4}|\xi|^{2\delta}b(t_1(|\xi|))d_{|\xi|}t_1(|\xi|)\Big) \\ &\le S_r(t, |\xi|)\Big(\frac{r}{|\xi|} + \Big(\frac{1}{4}|\xi|^{2\delta}b(t_1(|\xi|)) - C_0|\xi|^{2\sigma-2\delta}\frac{1}{b(t_1(|\xi|))}\Big)d_{|\xi|}t_1(|\xi|)\Big).\end{aligned}$$

From

$$|\xi|^{2\delta}b(t_1(|\xi|)) = 2(1+\varepsilon)|\xi|^\sigma,\ \ |\xi|^{2\sigma-2\delta}\frac{1}{b(t_1(|\xi|))} = \frac{1}{2(1+\varepsilon)}|\xi|^\sigma,$$

and after choosing the constant C_0 sufficiently small we see, that the term $|\xi|^{2\delta}b(t_1(|\xi|))$ dominates the term $C_0|\xi|^{2\sigma-2\delta}\frac{1}{b(t_1(|\xi|))}$. After differentiation we get

$$\begin{aligned}d_{|\xi|}t_1(|\xi|)b'(t_1(|\xi|))|\xi|^{2\delta-\sigma} + (2\delta-\sigma)|\xi|^{2\delta-\sigma-1}b(t_1(|\xi|)) &= 0, \\ d_{|\xi|}t_1(|\xi|) &= \frac{(\sigma-2\delta)b(t_1(|\xi|))|\xi|^{2\delta-\sigma-1}}{b'(t_1(|\xi|))|\xi|^{2\delta-\sigma}},\end{aligned}$$

respectively. Using the assumption $-\frac{b'(t)}{b(t)} \le C\frac{b(t)}{\Lambda(t)}$ from (A5) implies

$$d_{|\xi|}t_1(|\xi|) \le -\frac{(\sigma-2\delta)\Lambda(t_1(|\xi|))|\xi|^{2\delta-\sigma-1}}{Cb(t_1(|\xi|))|\xi|^{2\delta-\sigma}}.$$

The increasing behavior of the function $\Lambda(t)$ and the definitions of $t_0(|\xi|)$ and $t_1(|\xi|)$ give

$$d_{|\xi|}t_1(|\xi|) \le -\frac{1}{C}(\sigma-2\delta)|\xi|^{2\delta-\sigma-1}\Lambda(t_0(|\xi|)) \le -\frac{1}{C}(\sigma-2\delta)N|\xi|^{-1-\sigma}.$$

Moreover, for a fixed r the term $\frac{r}{|\xi|}$ is dominated by the negative term

$$\frac{1}{4}|\xi|^{2\delta}b(t_1(|\xi|))d_{|\xi|}t_1(|\xi|) \le -\frac{1}{C}(\sigma-2\delta)N|\xi|^{-1}$$

if we choose N large enough. In order to complete the proof it is sufficient to study small frequencies with $|\xi| \le \varepsilon_r$. For $|\xi| \ge \varepsilon_r$ we have an exponential type decay from the hyperbolic zone. Let us now fix $t > 0$. Then the above term takes its maximum for the $|\tilde{\xi}|$ satisfying $t = t_1(|\tilde{\xi}|)$. For $t = t_1(|\tilde{\xi}|)$ the second integral vanishes in $S_r(t, |\xi|)$. Consequently,

$$\begin{aligned}S_r(t, |\xi|) &\le S_r(t_1(|\tilde{\xi}|), |\tilde{\xi}|) = |\tilde{\xi}|^r\exp\Big(-C|\tilde{\xi}|^{2\sigma-2\delta}\int_0^{t_1(|\tilde{\xi}|)}\frac{1}{b(\tau)}d\tau\Big) \\ &\lesssim \max_{\xi\in\mathbb{R}^n}\Big\{|\xi|^r\exp\Big(-C|\xi|^{2\sigma-2\delta}\int_0^t\frac{1}{b(\tau)}d\tau\Big)\Big\} \lesssim \Big(1 + \int_0^t\frac{1}{b(\tau)}d\tau\Big)^{-\frac{r}{2\sigma-2\delta}}.\end{aligned}$$

In this way the lemma is proved. □

By Corollary 3.3.19, Lemma 3.3.21 and Proposition 3.3.12 we obtain the following result:

Corollary 3.3.22. *The following estimates hold for small $|\xi|$ and for all $t \in [0,\infty)$:*

$$
\begin{aligned}
|\xi|^\beta |\hat{u}(t,\xi)| &\lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\alpha}{2\sigma-2\delta}}|\hat{u}_1(\xi)| \\
&\quad + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{2\delta}}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq \sigma, \\
|\xi|^\beta |\hat{u}_t(t,\xi)| &\lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+\sigma}{2\sigma-2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{2\sigma-2\delta}}|\hat{u}_1(\xi)| \\
&\quad + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\alpha}{2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq 0.
\end{aligned}
$$

3.3.8 Conclusion

Taking into consideration Corollary 3.3.20 with the exponential type decay estimate for large frequencies, Proposition 3.3.12, Corollary 3.3.17 and Corollary 3.3.22 for small frequencies we may conclude the following statement. Here on the one hand one has to pay attention that the estimates from Proposition 3.3.12 are dominated by the estimates of the Corollaries 3.3.17 and 3.3.22.

Theorem 3.3.23. *Let us consider the Cauchy problem (3.3.1), where the coefficient $b = b(t)$ satisfies the conditions (A1) to (A6). Additionally, $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = 0$. Then the solution $u = u(t,x)$ satisfies in the case $\delta \in (0,\sigma/2)$ the following estimates for the energies of higher order :*

$$
\begin{aligned}
\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 &\lesssim \max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{\sigma-\delta}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{\delta}}\Big\}\|u_0\|_{H^{|\beta|}}^2 \\
&\quad + \max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{\delta}}, \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\alpha}{\sigma-\delta}}\Big\}\|u_1\|_{H^{|\beta|-\sigma}}^2 \quad \text{for} \quad |\beta| \geq \sigma, \\
\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 &\lesssim \max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+\sigma}{\sigma-\delta}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\alpha}{\delta}}\Big\}\|u_0\|_{H^{|\beta|+\sigma}}^2 \\
&\quad + \max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\delta}}, \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{\sigma-\delta}}\Big\}\|u_1\|_{H^{|\beta|}}^2 \quad \text{for} \quad |\beta| \geq 0.
\end{aligned}
$$

Remark 3.3.4. *The estimates for the energies of higher order show the parabolic effect for the solutions to the Cauchy problem (3.3.1). The maximum of two decay functions in the estimates explains that we have in general no decay behavior of the classical elastic energy $\|\nabla^\beta u(t,\cdot)\|_{L^2}^2$ for $|\beta| = \sigma$ and of the classical kinetic energy $\|u_t(t,\cdot)\|_{L^2}^2$.*

Remark 3.3.5. *If we set formally $\sigma = 1$ and $b(t) = (1+t)^{-\gamma}$, $\gamma \in (0, 1-2\delta)$ in the estimates from Theorem 3.3.23, then $\alpha = \frac{1}{1-\gamma}$, and we get the estimates from Theorem 3.1 in [20].*

3.3.9 Some examples

Typical examples for coefficients $b = b(t)$ are

$$b_n(t) = \mu(1+t)^{-\gamma}\big(\log(e+t)\big)^{-\gamma_1}\cdots\big(\log^{[n]}(e^{[n]}+t)\big)^{-\gamma_n},\ \text{with nonnegative}$$

$$\mu,\ \gamma_i,\ i=1,\cdots,n,\ \text{and}\ \gamma \in \Big(0,\frac{\sigma-2\delta}{\sigma}\Big).\ \text{We use, } \log^{[0]}(x)=x,\ e^{[0]}=1\ \text{and}$$

$$\log^{[n+1]}(x) = \log\log^{[n]}(x),\ e^{[n+1]} = e^{e^{[n]}}.$$

Example 3.3.1. *If we choose $b(t) = (1+t)^{-\gamma}$, $\gamma \in \big(0,\frac{\sigma-2\delta}{\sigma}\big)$, then $b = b(t)$ satisfies the assumptions of Theorem 3.3.23. Consequently, the following estimates for the energies of higher order hold :*

$$\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 \lesssim \big(1+t\big)^{-\frac{|\beta|(1+\gamma)}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}} + \big(1+t\big)^{\max\big\{-(|\beta|-\sigma)\frac{1-\gamma}{\delta},-(|\beta|-\frac{2\delta}{1-\gamma})\frac{1+\gamma}{\sigma-\delta}\big\}}\|u_1\|^2_{H^{|\beta|-\sigma}}$$
$$\text{for}\quad |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 \lesssim \big(1+t\big)^{-(|\beta|+\sigma)\frac{1+\gamma}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}} + \big(1+t\big)^{\max\big\{-|\beta|\frac{1-\gamma}{\delta},-(|\beta|+\sigma-\frac{2\delta}{1-\gamma})\frac{1+\gamma}{\sigma-\delta}\big\}}\|u_1\|^2_{H^{|\beta|}}$$
$$\text{for}\quad |\beta| \geq 0.$$

Example 3.3.2. *We choose $b(t) = (1+t)^{-\gamma}\big(\log(e+t)\big)^{-\gamma_1}$, with $\gamma \in \big(0,\frac{\sigma-2\delta}{\sigma}\big)$ and nonnegative γ_1. Then $b = b(t)$ satisfies the assumptions of Theorem 3.3.23. Consequently, the following estimates for the energies of higher order hold :*

$$\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 \lesssim$$
$$\max\Big\{\Big((1+t)^{1+\gamma}\big(\log(e+t)\big)^{\gamma_1}\Big)^{-\frac{|\beta|}{\sigma-\delta}},\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|+\sigma-\frac{2\delta}{1-\gamma}}{\delta}}\Big\}\|u_0\|^2_{H^{|\beta|}}$$
$$+\max\Big\{\Big((1+t)^{1+\gamma}\big(\log(e+t)\big)^{\gamma_1}\Big)^{-\frac{|\beta|-\frac{2\delta}{1-\gamma}}{\sigma-\delta}},\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|-\sigma}{\delta}}\Big\}\|u_1\|^2_{H^{|\beta|-\sigma}}$$
$$\text{for}\quad |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 \lesssim$$
$$\max\Big\{\Big((1+t)^{1+\gamma}\big(\log(e+t)\big)^{\gamma_1}\Big)^{-\frac{|\beta|+\sigma}{\sigma-\delta}},\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|+2\sigma-\frac{2\delta}{1-\gamma}}{\delta}}\Big\}\|u_0\|^2_{H^{|\beta|}}$$
$$+\max\Big\{\Big((1+t)^{1+\gamma}\big(\log(e+t)\big)^{\gamma_1}\Big)^{-\frac{|\beta|+\sigma-\frac{2\delta}{1-\gamma}}{\sigma-\delta}},\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|}{\delta}}\Big\}\|u_1\|^2_{H^{|\beta|-\sigma}}$$
$$\text{for}\quad |\beta| \geq 0.$$

Now let us devote to the second case $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$.In this case we divide

the extended phase space only into the hyperbolic zone $Z_{hyp}(\varepsilon)$, the reduced zone $Z_{red}(\varepsilon)$ and the pseudo-differential zone $Z_{pd}(\varepsilon,N)$ (see Figure 10.3 in the Appendix).

3.3.10 Treatment in the hyperbolic zone

Proposition 3.3.24. *The following estimates hold for all* $t \in [t_2(|\xi|), \infty)$*, where* $t_2(|\xi|) = 0$ *for large frequencies* :

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{2}\int_{t_2(|\xi|)}^{t} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_2(|\xi|),\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_2(|\xi|),\xi)|\Big)$$
$$\text{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{2}\int_{t_2(|\xi|)}^{t} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(t_2(|\xi|),\xi)| + |\xi|^{|\beta|}|\hat{u}_t(t_2(|\xi|),\xi)|\Big)$$
$$\text{for} \quad |\beta| \geq 0.$$

Proof. The proof is the same as the proof to Proposition 3.3.2. □

3.3.11 Treatment in the reduced zone

Proposition 3.3.25. *The following estimates hold for all* $t \in [t_1(|\xi|), t_2(|\xi|)]$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^{t} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_1(|\xi|),\xi)|\Big)$$
$$\text{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^{t} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|}|\hat{u}_t(t_1(|\xi|),\xi)|\Big)$$
$$\text{for} \quad |\beta| \geq 0.$$

Proof. The proof is the same as the proof to Proposition 3.3.4. □

3.3.12 Treatment in the pseudo-differential zone

Now let us discuss the behavior of the solution in the pseudo-differential zone

$$Z_{pd}(\varepsilon, N) = \Big\{(t,\xi) : \frac{b(t)}{2}|\xi|^{2\delta-\sigma} \geq 1+\varepsilon \text{ and } \Lambda(t)|\xi|^{2\delta} \leq N\Big\}.$$

The separating line between the reduced zone and the elliptic zone from Section 3.3.1 is defined by $\frac{b(t)}{2}|\xi|^{2\delta-\sigma} = 1+\varepsilon$. The separating line between the pseudo-differential zone and the exterior of the pseudo-differential zone is defined by $\Lambda(t)|\xi|^{2\delta} = N$. If $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$, then for large time $t \geq C \gg 1$ the elliptic zone $Z_{ell}(\varepsilon, N)$ is contained in the pseudo-differential zone $Z_{pd}(\varepsilon, N)$. For this reason it is allowed to continue the integrals in the phases from Propositions 3.3.24 and 3.3.25 because due to Lemma 3.3.15 phases have no meaning in this zone. Moreover, the phase coming from the reduced zone dominates that one coming from the hyperbolic zone.

Proposition 3.3.26. *The following estimates hold for all $t \in [0, t_1(|\xi|)]$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq \sigma,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq 0.$$

Proof. We introduce the micro-energy

$$U(t,\xi) := \big(|\xi|^\sigma \hat{u}(t,\xi), D_t\hat{u}(t,\xi)\big)^T \quad \text{for all} \quad t \in [0, t_1(|\xi|)].$$

We obtain from (3.3.1) the system of first order

$$D_tU(t,\xi) = A(t,\xi)U(t,\xi),$$

where

$$A(t,\xi) := \begin{pmatrix} 0 & |\xi|^\sigma \\ |\xi|^\sigma & ib(t)|\xi|^{2\delta} \end{pmatrix}. \tag{3.3.11}$$

We have

$$\|A(t,\xi)\| \leq b(t)|\xi|^{2\delta} \left\| \begin{pmatrix} 0 & \frac{1}{b(t)|\xi|^{2\delta-\sigma}} \\ \frac{1}{b(t)|\xi|^{2\delta-\sigma}} & i \end{pmatrix} \right\| \lesssim b(t)|\xi|^{2\delta},$$

here we use the first condition in the definition of the pseudo-differential zone. Hence,

$$\int_0^{t_0(|\xi|)} \|A(\tau,\xi)\|d\tau \lesssim \int_0^{t_0(|\xi|)} b(\tau)|\xi|^{2\delta}d\tau \lesssim |\xi|^{2\delta}\Lambda(t_1(|\xi|)) = N.$$

We can write $U(t,\xi) = E(t,0,\xi)U(0,\xi)$, where $E(t,s,\xi)$ is the fundamental solution, that is, the solution of the system

$$D_tE(t,s,\xi) = A(t,\xi)E(t,s,\xi), \quad E(s,s,\xi) = I_2, \quad t \geq s \ \text{ and } \ (t,\xi) \in Z_{pd}(\varepsilon, N).$$

Then we get

$$\|E(t,s,\xi)\| \leq \exp\Big(\int_0^{t_0(|\xi|)} \|A(\tau,\xi)\|d\tau\Big) \leq C.$$

Finally, we obtain the following estimate for the transformed micro-energy $U(t,\xi)$ in the pseudo-differential zone for all $t \in [0, t_1(|\xi|)]$:

$$|U(t,\xi)| \lesssim C|U(0,\xi)|, \quad \Big|\begin{pmatrix} |\xi|^\sigma\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi) \end{pmatrix}\Big| \lesssim C\Big|\begin{pmatrix} |\xi|^\sigma\hat{u}(0,\xi) \\ D_t\hat{u}(0,\xi) \end{pmatrix}\Big| \quad \text{for} \ \ t \in [0, t_0(|\xi|)],$$

respectively, with a constant C which is independent of $t \in [0, t_0(|\xi|)]$. Consequently, by using the fact, that phases have due to Lemma 3.3.15 no meaning in the pseudo-differential zone we derived for $|\beta| \geq \sigma$ the a-priori estimates

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big).$$

In the same way we conclude for $|\beta| \geq 0$ the a-priori estimates

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big).$$

All desired estimates are proved. □

3.3.13 Energy estimates

Using the Propositions 3.3.24 to 3.3.26 we obtain the following result :

Corollary 3.3.27. *The following estimates hold for small frequencies $|\xi|$ and for all $t \in [0,\infty)$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{2\delta}}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq \sigma,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma}{2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq 0.$$

For large frequencies we may use the estimates from Proposition 3.3.24 because $t_2(|\xi|) = 0$. We will show in the following statement that in this part of the extended phase space we conclude an exponential type decay.

Corollary 3.3.28. *The following estimates hold for large frequencies and for all $t \in [0,\infty)$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C\Lambda(t)\big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq 0,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\big(-C\Lambda(t)\big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq 0.$$

3.3.14 Conclusion

Applying the Corollaries 3.3.27 and 3.3.28 we obtain the following result.

Theorem 3.3.29. *Let us consider the Cauchy problem (3.3.1) under the assumption $\lim_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$, where the coefficient $b = b(t)$ satisfies additionally the conditions (A1) to (A5). Then the solution $u = u(t,x)$ satisfies in the case $\delta \in (0,\sigma/2)$ the following estimates for the energies of higher order :*

$$\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\delta}}\|u_0\|^2_{H^{|\beta|}} + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}}$$
$$\text{for} \quad |\beta| \geq \sigma,$$

$$\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma}{\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}} + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\delta}}\|u_1\|^2_{H^{|\beta|}}$$
$$\text{for} \quad |\beta| \geq 0.$$

Remark 3.3.6. *If we set formally $\delta = \frac{\sigma}{2}$ in the estimates from Theorem 3.3.29, then we get the estimates from Theorem 3.2.5.*

Remark 3.3.7. *If we set formally $\sigma = 1$ and $b(t) = (1+t)^{-\gamma},\ \gamma \in [1-2\delta, 1]$ in the estimates from Theorem 3.3.29, then we get the estimates from Theorem 3.2 in [20].*

3.3.15 Some examples

Example 3.3.3. *We choose for example $b(t) = (1+t)^{-\gamma}$ with $\gamma \in \big[\frac{\sigma-2\delta}{\sigma}, 1\big]$. Then $b = b(t)$ satisfies the assumptions of Theorem 3.3.29. Consequently, the following estimates for the* *energies of higher order hold* :

$$
\begin{aligned}
&\gamma \in \big[\frac{\sigma-2\delta}{\sigma},1\big):\\
&\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \big(1+t\big)^{-\frac{|\beta|(1-\gamma)}{\delta}}\|u_0\|^2_{H^{|\beta|}} + \big(1+t\big)^{-\frac{(|\beta|-\sigma)(1-\gamma)}{\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}} \quad \text{for} \quad |\beta|\geq\sigma,\\
&\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \big(1+t\big)^{-\frac{(|\beta|+\sigma)(1-\gamma)}{\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}} + \big(1+t\big)^{-\frac{|\beta|(1-\gamma)}{\delta}}\|u_1\|^2_{H^{|\beta|}} \quad \text{for} \quad |\beta|\geq 0.\\
&\gamma=1:\\
&\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \big(\log(e+t)\big)^{-\frac{|\beta|}{\delta}}\|u_0\|^2_{H^{|\beta|}} + \big(\log(e+t)\big)^{-\frac{|\beta|-\sigma}{\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}} \quad \text{for} \quad |\beta|\geq\sigma,\\
&\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \big(\log(e+t)\big)^{-\frac{|\beta|+\sigma}{\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}} + \big(\log(e+t)\big)^{-\frac{|\beta|}{\delta}}\|u_1\|^2_{H^{|\beta|}} \quad \text{for} \quad |\beta|\geq 0.
\end{aligned}
$$

Example 3.3.4. *We choose* $b(t) = (1+t)^{-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-1}$ *with* $\gamma \in \big[\frac{\sigma-2\delta}{\sigma},1\big]$. *Then* $b=b(t)$ *satisfies the assumptions of Theorem 3.3.29. Consequently, the following estimates for the energies of higher order hold* :

$$
\begin{aligned}
&\gamma \in \big[\frac{\sigma-2\delta}{\sigma},1\big):\\
&\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-1}\Big)^{-\frac{|\beta|}{\delta}}\|u_0\|^2_{H^{|\beta|}}\\
&\qquad +\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-1}\Big)^{-\frac{|\beta|-\sigma}{\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}} \quad \text{for} \quad |\beta|\geq\sigma,\\
&\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-1}\Big)^{-\frac{|\beta|+\sigma}{\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}}\\
&\qquad +\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-1}\Big)^{-\frac{|\beta|}{\delta}}\|u_1\|^2_{H^{|\beta|}} \quad \text{for} \quad |\beta|\geq 0.\\
&\gamma=1:\\
&\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \Big(\log^{[3]}(e^{[3]}+t)\Big)^{-\frac{|\beta|}{\delta}}\|u_0\|^2_{H^{|\beta|}} + \Big(\log^{[3]}(e^{[3]}+t)\Big)^{-\frac{|\beta|-\sigma}{\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}}\\
&\qquad \text{for} \quad |\beta|\geq\sigma,\\
&\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \Big(\log^{[3]}(e^{[3]}+t)\Big)^{-\frac{|\beta|+\sigma}{\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}} + \Big(\log^{[3]}(e^{[3]}+t)\Big)^{-\frac{|\beta|}{\delta}}\|u_1\|^2_{H^{|\beta|}}\\
&\qquad \text{for} \quad |\beta|\geq 0.
\end{aligned}
$$

3.4 Time-dependent strictly decreasing dissipation – $\delta \in (\sigma/2, \sigma]$

In this section we consider for $\sigma > 1$ the model

$$
\begin{cases}
u_{tt} + (-\Delta)^\sigma u + b(t)(-\Delta)^\delta u_t = 0,\\
u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \sigma>1, \quad \delta\in(\sigma/2,\sigma].
\end{cases} \tag{3.4.1}
$$

After partial Fourier transformation we obtain the Cauchy problem

$$
\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + b(t)|\xi|^{2\delta}\hat{u}_t = 0, \quad \hat{u}(0,\xi)=\hat{u}_0(\xi), \quad \hat{u}_t(0,\xi)=\hat{u}_1(\xi), \quad \delta\in(\sigma/2,\sigma].
$$

3.4.1 Division of the extended phase space

We divide the extended phase space into the same zones as in Section 3.2.1. But there is a big difference in the geometry and localization of these zones. If we consider the most interesting part $\{(t,\xi) : |\xi| \leq p_0\}$, where p_0 is sufficiently small, then this part is completely contained in the hyperbolic zone, in other words, if a point (t,ξ) belongs to one of the other three zones we have $|\xi| \geq p_0$ with a suitable positive p_0. Moreover, the pseudo-differential zone degenerates to a compact set of the extended phase space. For this reason we have only to study the behavior of solutions in the hyperbolic zone, the reduced zone and the elliptic zone. The separating curves $t_1(|\xi|)$ and $t_2(|\xi|)$ are defined as in Section 3.2.1 (see Figure 10.4 in the Appendix).

3.4.2 Treatment in the hyperbolic zone

Proposition 3.4.1. *The following estimates hold for all* $t \in [t_2(|\xi|), \infty)$, *where* $t_2(|\xi|) = 0$ *for small frequencies* :

$$
\begin{aligned}
&|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{2}\int_{t_2(|\xi|)}^{t} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_2(|\xi|),\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_2(|\xi|),\xi)|\Big)\\
&\qquad \text{for} \quad |\beta| \geq \sigma,\\
&|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{2}\int_{t_2(|\xi|)}^{t} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(t_2(|\xi|),\xi)| + |\xi|^{|\beta|}|\hat{u}_t(t_2(|\xi|),\xi)|\Big)\\
&\qquad \text{for} \quad |\beta| \geq 0.
\end{aligned}
$$

Proof. The proof is the same as the proof to Proposition 3.3.2. □

3.4.3 Treatment in the reduced zone

Proposition 3.4.2. *The following estimates hold for all* $t \in [t_1(|\xi|), t_2(|\xi|)]$:

$$
\begin{aligned}
&|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^{t} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_1(|\xi|),\xi)|\Big)\\
&\qquad \text{for} \quad |\beta| \geq \sigma,\\
&|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^{t} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|}|\hat{u}_t(t_1(|\xi|),\xi)|\Big)\\
&\qquad \text{for} \quad |\beta| \geq 0.
\end{aligned}
$$

Proof. The proof is the same as the proof to Proposition 3.3.4. □

3.4.4 Treatment in the elliptic zone

Proposition 3.4.3. *The following estimates hold for all* $t \in [0, t_1(|\xi|)]$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \text{ for } \quad |\beta| \geq \sigma,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) + \exp\Big(-\frac{|\xi|^{2\delta}}{2}\int_0^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq 0.$$

Proof. The proof is the same as the proof to Proposition 3.3.6. But here we apply the definition of the elliptic zone to estimate some terms to above. This is reasonable because the difference of regularity of the data is already given in Proposition 3.4.1. □

3.4.5 Gluing procedure

For large frequencies we have to glue the statements from the Propositions 3.4.1, 3.4.2 and 3.4.3. We are able to extend the estimates from $Z_{hyp}(\varepsilon)$ in Proposition 3.4.1 to $Z_{red}(\varepsilon)$ in Proposition 3.4.2. For this reason we obtain for $t \geq t_1(|\xi|)$ the following statement :

Corollary 3.4.4. *The following estimates hold for all* $t \in [t_1(|\xi|), \infty)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_1(|\xi|),\xi)|\Big)$$
for $|\beta| \geq 0$,

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|}|\hat{u}_t(t_1(|\xi|),\xi)|\Big)$$
for $|\beta| \geq 0$.

Finally, we have to glue the estimates from Corollary 3.4.4 and the estimates from Proposition 3.4.3 for $t = t_1(|\xi|)$.

Corollary 3.4.5. *The following estimates hold for all* $t \in [t_1(|\xi|), \infty)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\exp\Big(-C|\xi|^{2\sigma-2\delta}\int_0^{t_1(|\xi|)}\frac{1}{b(\tau)}d\tau\Big) \times\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) + \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)|\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|$$
for $|\beta| \geq 0$,

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\exp\Big(-C\int_0^{t_1(|\xi|)}\frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big) \times\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) + \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_1(\xi)|$$
for $|\beta| \geq 0$.

Proof. Let us begin to estimate $|\xi|^{|\beta|}|\hat{u}(t,\xi)|$. The statement of Corollary 3.4.4 implies

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_1(|\xi|),\xi)|\Big).$$

Using the estimates for $|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)|$ and $|\hat{u}_t(t_1(|\xi|),\xi)|$ from Proposition 3.4.3 we have

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\exp\Big(-C\int_0^{t_1(|\xi|)}\frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) + \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)|\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|.$$

In the same way, we conclude

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)\exp\Big(-C\int_0^{t_1(|\xi|)}\frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big) \times\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) + \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_1(\xi)|.$$

This completes the proof. □

For small frequencies we may use the estimates from Proposition 3.4.1 because $t_2(|\xi|)=0$.

Corollary 3.4.6. *The following estimates hold for small frequencies and for all* $t\in[0,\infty)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{2}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta|\geq\sigma,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{2}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta|\geq 0.$$

3.4.6 Energy estimates

To derive the corresponding energy estimates from Corollary 3.4.5 for large frequencies we have to estimate the term

$$\exp\Big(-C\int_0^{t_1(|\xi|)}\frac{1}{b(\tau)}d\tau\Big)\exp\Big(-C_1\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big).$$

This term explains in this part of the extended phase space the competition of influences from different phase functions.

Lemma 3.4.7. *To a given positive constant C we can find a sufficiently small positive constant C_1 such that it holds for large time t*

$$\exp\Big(-C\int_0^{t_1(|\xi|)}\frac{1}{b(\tau)}d\tau\Big)\exp\Big(-C_1\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big) \lesssim \exp\big(-C_1\Lambda(t)\big).$$

Proof. Using the decreasing behavior of $b=b(t)$ implies for large time t

$$-C\frac{1}{b(t)} \leq -C_1 b(t), \quad \text{hence,} \quad -C\int_0^{t_1(|\xi|)}\frac{1}{b(\tau)}d\tau \leq -C_1\int_0^{t_1(|\xi|)} b(\tau)d\tau.$$

By applying the definition of $\Lambda = \Lambda(t)$ we get

$$-C\int_0^{t_1(|\xi|)} \frac{1}{b(\tau)} d\tau \le C_1(1-\Lambda(t_1(|\xi|))), \quad \text{and} \quad -C_1\int_{t_1(|\xi|)}^t b(\tau)d\tau = C_1(\Lambda(t_1(|\xi|))-\Lambda(t)).$$

In this way the lemma is proved. □

From Corollary 3.4.5 and Lemma 3.4.7 we obtain for large frequencies the following statement about "an exponential type decay" for large frequencies.

Corollary 3.4.8. *The following estimates hold for all $t \in [0,\infty)$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\big(-C\Lambda(t)\big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \ge 0,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\big(-C\Lambda(t)\big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \ge 0.$$

For small frequencies we may use the estimates from Corollary 3.4.6.

3.4.7 Conclusion

By Corollaries 3.4.8 and 3.4.6 we have the following result.

Theorem 3.4.9. *Let us consider the Cauchy problem (3.4.1), where the coefficient $b = b(t)$ satisfies the conditions (A1) to (A5). Then the solution $u = u(t,x)$ satisfies in the case $\delta \in (\sigma/2, \sigma]$ the following estimates for the energies of higher order :*

$$\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\delta}}\|u_0\|_{H^{|\beta|}}^2 + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{\delta}}\|u_1\|_{H^{|\beta|-\sigma}}^2$$
for $|\beta| \ge \sigma$,
$$\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma}{\delta}}\|u_0\|_{H^{|\beta|+\sigma}}^2 + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\delta}}\|u_1\|_{H^{|\beta|}}^2$$
for $|\beta| \ge 0$.

Remark 3.4.1. *If we set formally $\delta = \sigma/2$ in the estimates from Theorem 3.4.9, then we get the estimates from Theorem 3.2.5.*

Remark 3.4.2. *If we set formally $\sigma = 1$ and $b(t) = (1+t)^{-\gamma},\ \gamma \in (0,1]$, in the estimates from Theorem 3.4.9, then we get the estimates from Theorem 4.1 in [20].*

3.4.8 Some examples

Example 3.4.1. *Let us choose $b(t) = (1+t)^{-\gamma}$ with $\gamma \in (0,1]$. Then $b = b(t)$ satisfies the assumptions of Theorem 3.4.9. Consequently, the following estimates of the energy of higher* *order hold* :

$$\gamma \in (0,1):$$
$$\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \big(1+t\big)^{-\frac{|\beta|(1-\gamma)}{\delta}}\|u_0\|^2_{H^{|\beta|}} + \big(1+t\big)^{-\frac{(|\beta|-\sigma)(1-\gamma)}{\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}} \quad \text{for} \quad |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \big(1+t\big)^{-\frac{(|\beta|+\sigma)(1-\gamma)}{\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}} + \big(1+t\big)^{-\frac{|\beta|(1-\gamma)}{\delta}}\|u_1\|^2_{H^{|\beta|}} \quad \text{for} \quad |\beta| \geq 0.$$
$$\gamma = 1:$$
$$\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \big(\log(e+t)\big)^{-\frac{|\beta|}{\delta}}\|u_0\|^2_{H^{|\beta|}} + \big(\log(e+t)\big)^{-\frac{|\beta|-\sigma}{\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}} \quad \text{for} \quad |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \big(\log(e+t)\big)^{-\frac{|\beta|+\sigma}{\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}} + \big(\log(e+t)\big)^{-\frac{|\beta|}{\delta}}\|u_1\|^2_{H^{|\beta|}} \quad \text{for} \quad |\beta| \geq 0.$$

Example 3.4.2. *Let us choose* $b(t) = (1+t)^{-\gamma}\big(\log(e+t)\big)^{-\gamma_1}$ *with* $\gamma \in (0,1]$ *and* $\gamma_1 \in (0,1)$. *Then* $b = b(t)$ *satisfies the assumptions of Theorem 3.4.9. Consequently, the following estimates of the energy of higher order hold* :

$$\gamma \in (0,1):$$
$$\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|}{\delta}}\|u_0\|^2_{H^{|\beta|}}$$
$$+\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|-\sigma}{\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}} \quad \text{for} \quad |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|+\sigma}{\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}}$$
$$+\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|}{\delta}}\|u_1\|^2_{H^{|\beta|}} \quad \text{for} \quad |\beta| \geq 0.$$
$$\gamma = 1:$$
$$\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \Big(\log(e+t)\Big)^{-\frac{|\beta|(1-\gamma_1)}{\delta}}\|u_0\|^2_{H^{|\beta|}} + \Big(\log(e+t)\Big)^{-\frac{(|\beta|-\sigma)(1-\gamma_1)}{\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}}$$
$$\text{for} \quad |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \Big(\log(e+t)\Big)^{-\frac{(|\beta|+\sigma)(1-\gamma_1)}{\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}} + \Big(\log(e+t)\Big)^{-\frac{|\beta|(1-\gamma_1)}{\delta}}\|u_1\|^2_{H^{|\beta|}}$$
$$\text{for} \quad |\beta| \geq 0.$$

4 Time-dependent strictly increasing dissipation

In this section we devote to increasing coefficients in the structural damped σ evolution model. We study the model

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^{\sigma} u(t,x) + b(t)(-\Delta)^{\delta} u_t(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \sigma > 1, \quad \text{and} \quad \delta \in (0,\sigma). \end{cases} \tag{4.0.1}$$

For increasing coefficients $b = b(t)$. In Section 4.2 we shall consider the most technical case $\delta \in (\sigma/2, \sigma)$ which corresponds in some sense to the case $\delta \in (0, \sigma/2)$ for strictly decreasing $b(t)$. In Sections 4.3 and 4.4 we only sketch the considerations to study the cases $\delta \in (0, \sigma/2)$ and $\delta = \sigma/2$, respectively. The assumption that $b = b(t)$ is strictly increasing implies $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-1} = 0$. For the further approach we introduce a classification of admissible coefficients $b = b(t)$ (see condition (B6) below).

4.1 Objectives and strategies

We will study $L^2 - L^2$ decay estimates for the energy of the solution $u(t,\cdot)$ for structural damped $\sigma-$evolution models (4.0.1). As in the previous chapters we assume for the Cauchy data $u_0 \in H^{\sigma}(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. We are interested to understand the type of decay (decay function and decay rate) of the energy of solution $u(t,\cdot)$. Let us explain our strategy. It is divided into the following steps:

- In the first step we use the partial Fourier transformation to reduce the partial differential equation to an ordinary differential equation for $\hat{u}(t,\xi)$ parameterized by ξ.
- At first we want to consider the case $\delta \in (\sigma/2, \sigma)$ which corresponds in some sense to the case $\delta \in (0, \sigma/2)$ for strictly decreasing $b(t)$.
- We will divide the extended phase space into important zones, the hyperbolic, elliptic, reduced and pseudo-differential zone.
- If our considerations can be restricted to the hyperbolic or reduced zone, we use the "dissipative transformation", and then we will introduce an appropriate micro-energy to get for it a system of first order, in the hyperbolic zone after one step of diagonalization. The remainder becomes integrable. Here the remainder can be studied by the matrizant representation. Then we derive a representation of the fundamental solution.
- If our considerations can be restricted to the elliptic zone, then we use another micro-energy to get again a system of first order. This system should be diagonalized twice.

Then the remainder becomes integrable. We explain the matrix representation of the fundamental solution which entries can be estimated in a very effective (two steps) way.

- By using a gluing procedure we get the estimates for the elastic and the kinetic energy for small and large frequencies.

- By using the Plancherel theorem we obtain $L^2 - L^2$ estimates. Here we get two types of decay estimates, a "potential type decay" for small frequencies and an "exponential type decay" for large frequencies under additional regularity assumptions for the data.

4.2 Time-dependent strictly increasing dissipation– $\delta \in (\sigma/2, \sigma)$

Let us devote to the model

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + b(t)(-\Delta)^\delta u_t = 0, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \delta \in (\sigma/2, \sigma), \quad \sigma > 1. \end{cases} \tag{4.2.1}$$

After partial Fourier transformation we obtain the Cauchy problem

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + b(t)|\xi|^{2\delta}\hat{u}_t = 0, \;\; \hat{u}(0,\xi) = \hat{u}_0(\xi), \;\; \hat{u}_t(0,\xi) = \hat{u}_1(\xi) \;\; \sigma > 1, \;\; \delta \in (\sigma/2, \sigma). \tag{4.2.2}$$

In the further considerations we assume the following assumptions to the coefficient function $b = b(t)$:

(B1) *positivity* : $b(t) > 0$ for all $t \geq 0$,

(B2) *increasing behavior* : $b'(t) > 0$ for all $t \geq 0$,

(B3) *non-integrability* : $\int_0^\infty \frac{1}{b(\tau)} d\tau = \infty$,

(B4) *higher order derivatives* : it holds $|d_t^k b(t)| \leq C_k b(t)\big(\frac{1}{1+t}\big)^k$ for $k = 1, 2$,

(B5) *useful inequalities* : there exist positive constants C_0, C_1, C_2 which are independent of t such that

$$C_0 \frac{b(t)}{\Lambda(t)} \leq \frac{b'(t)}{b(t)} \leq C_1 \frac{1}{1+t} \leq C_2 \frac{b(t)}{\Lambda(t)} \;\text{ with }\; \Lambda(t) = 1 + \int_0^t b(\tau) d\tau,$$

(B6) *additional classification* : $b \in S_\eta$ for $\eta \in \big(0, \frac{\sigma}{2\delta}\big]$, where we introduce the family $\{S_\eta\}_\eta$ of classes

$$S_\eta := \Big\{ b = b(t) : \limsup_{t\to\infty} (1+t)\Lambda(t)^{-\eta} < \infty, \quad \lim_{t\to\infty} (1+t)\Lambda(t)^{-\beta} = \infty \text{ for all } \beta < \eta \Big\}.$$

4.2.1 Division of the extended phase space

We use zones which are already introduced in Section 3.3.1. But now the geometry and localization of these zones differs from those in Sections 3.3.1. On the one hand, all zones in general appear for small frequencies. On the other hand, large frequencies lie only in the elliptic zone. The separation lines differ to that ones from Section 3.3.1. We denote the separation line between the elliptic and the pseudo-differential zone by $t_0 = t_0(|\xi|)$, between the pseudo-differential and the reduced zone by $t_1 = t_1(|\xi|)$, and between the reduced and the hyperbolic zone by $t_2 = t_2(|\xi|)$.

Lemma 4.2.1. *If $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = 0$, then the pseudo-differential zone is no compact set. If $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$, then the pseudo-differential zone is a compact set.*

Proof. The proof coincides with the proof of Lemma 3.3.1. □

We divide the further considerations into the following two cases:
case 1: $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = 0$,
case 2: $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$.
Let us devote to the first case. In this case we divide the extended phase space into the hyperbolic zone $Z_{hyp}(\varepsilon)$, the reduced zone $Z_{red}(\varepsilon)$, the elliptic zone $Z_{ell}(\varepsilon, N)$, and the pseudo-differential zone $Z_{pd}(\varepsilon, N)$. Applying Lemma 4.2.1 the hyperbolic zone $Z_{hyp}(\varepsilon)$ and the reduced zone $Z_{red}(\varepsilon)$ are contained in the pseudo-differential region (see Figure 10.5 in the Appendix), where we introduce the *pseudo-differential region* by

$$R_{pd}(N) = \{(t,\xi) : \Lambda(t)|\xi|^{2\delta} \le N\}.$$

4.2.2 Treatment in the elliptic zone

Proposition 4.2.2. *The following estimates hold for all $t \in [t_0(|\xi|), \infty)$, where $t_0(|\xi|) = 0$ for large frequencies :*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_0(|\xi|),\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_0(|\xi|))}|\hat{u}_t(t_0(|\xi|),\xi)|\Big)$$
for $|\beta| \ge \sigma$,
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(\frac{|\xi|^{|\beta|+2\sigma-2\delta}}{b(t_0(|\xi|))}|\hat{u}(t_0(|\xi|),\xi)| + \frac{|\xi|^{|\beta|+2\sigma-4\delta}}{b^2(t_0(|\xi|))}|\hat{u}_t(t_0(|\xi|),\xi)|\Big) + \exp\Big(-|\xi|^{2\delta}\int_{t_0(|\xi|)}^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_t(t_0(|\xi|),\xi)| \text{ for } |\beta| \ge 0.$$

Proof. step 1: A straight-forward estimate for the fundamental solution

Proposition 4.2.3. *The fundamental solution E satisfies for all $t \ge s$ and $(t,\xi),(s,\xi) \in Z_{ell}(\varepsilon, N)$ the following estimate :*

$$\begin{pmatrix} |E^{11}(t,s,\xi)| & |E^{12}(t,s,\xi)| \\ |E^{21}(t,s,\xi)| & |E^{22}(t,s,\xi)| \end{pmatrix} \lesssim \exp\Big(-C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\begin{pmatrix} 1 & \frac{1}{b(s)|\xi|^{2\delta-\sigma}} \\ b(t)|\xi|^{2\delta-\sigma} & \frac{b(t)}{b(s)} \end{pmatrix},$$

where the constant C is independent of $(s,\xi),(t,\xi) \in Z_{ell}(\varepsilon, N)$.

Proof. The proof coincides with the proof to Proposition 3.3.7. □

Remark 4.2.1. *From Proposition 4.2.3 we obtain reasonable estimates for $|E^{11}|$ and $|E^{12}|$. We can not be satisfied with the estimates for $|E^{21}|$ and $|E^{22}|$ because the upper bounds are not uniformly bounded in the elliptic zone. For this reason we need a refined estimate which we present in the next step.*

step 2: A refined estimate for the fundamental solution

Lemma 4.2.4. *The fundamental solution E satisfies for all $t \geq s$ and $(t,\xi),(s,\xi) \in Z_{ell}(\varepsilon, N)$ the following estimates :*

$$\begin{pmatrix} |E^{11}(t,s,\xi)| & |E^{12}(t,s,\xi)| \\ |E^{21}(t,s,\xi)| & |E^{22}(t,s,\xi)| \end{pmatrix} \lesssim \exp\Big(-C\int_s^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big) \begin{pmatrix} 1 & \frac{1}{b(s)|\xi|^{2\delta-\sigma}} \\ \frac{1}{b(s)|\xi|^{2\delta-\sigma}} & \frac{1}{b^2(s)|\xi|^{4\delta-2\sigma}} \end{pmatrix}$$
$$+\exp\Big(-|\xi|^{2\delta}\int_s^t b(\tau)d\tau\Big)\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix},$$

where the constant C is independent of $(s,\xi),(t,\xi) \in Z_{ell}(\varepsilon, N)$.

Proof. The proof is the same as the proof to Lemma 3.3.10. There is only one difference. In the proof to Lemma 3.3.10 we used the decreasing behavior of $b(t)$ and $s \leq t$ to estimate

$$\frac{1}{b(t)} + \frac{1}{b(s)}\exp\Big(-C|\xi|^{2\delta}\int_s^t b(\theta)d\theta\Big) \lesssim \frac{1}{b(t)}.$$

But now we can only estimate

$$\frac{1}{b(t)} + \frac{1}{b(s)}\exp\Big(-C|\xi|^{2\delta}\int_s^t b(\theta)d\theta\Big) \lesssim \frac{1}{b(s)}.$$

For this reason the refined estimate for the entries of E differs to the estimate for the entries from Lemma 3.3.10. □

Remark 4.2.2. *We are able to derive a refined estimate for the fundamental solution because we use in the proof to Lemma 4.2.4 only the estimate for E^{11}, E^{12} from Proposition 4.2.3. Both estimates are optimal with our analytical tools.*

This completes the proof to Proposition 4.2.2. □

4.2.3 Treatment in the pseudo-differential region

Proposition 4.2.5. *The following estimates hold for all $t \in [0, t_0(|\xi|)]$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}}(1+t)|\hat{u}_1(\xi)| \text{ for } |\beta| \geq \sigma,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma}{2\delta}}(1+t)|\hat{u}_0(\xi)| + \exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_1(\xi)| \quad \text{for } |\beta| \geq 0.$$

Proof. The proof is the same as the proof to Proposition 3.3.12. □

4.2.4 Gluing procedure

We have to glue the estimates from the Propositions 4.2.2 and 4.2.5. We distinguish between *small frequencies* and *large frequencies*.
For large frequencies we have the estimate from Proposition 4.2.2 because $t_0(|\xi|) = 0$. For small frequencies we glue the statements from the Proposition 4.2.2 with the Proposition 4.2.5 for $t = t_0(|\xi|)$. Therefore we need the following statements.

Lemma 4.2.6. *Let $b \in S_\eta$. Then the following estimate holds for the separating line $t_0(|\xi|)$ between the elliptic zone and the pseudo-differential zone* :

$$\frac{1}{b(t_0(|\xi|))} \lesssim \Lambda(t_0(|\xi|))^{\eta-1} \sim |\xi|^{2\delta-2\delta\eta}.$$

Proof. Applying assumption (B5) we may conclude

$$\frac{1}{b(t_0(|\xi|))} \lesssim \frac{1+t_0(|\xi|)}{\Lambda(t_0(|\xi|))}, \quad \text{hence,} \quad \frac{1}{b(t_0(|\xi|))} \lesssim \big(1+t_0(|\xi|)\big)\Lambda(t_0(|\xi|))^{-1}.$$

Using the condition (B6) with $\eta \in (0, \frac{\sigma}{2\delta}]$ we get

$$\frac{1}{b(t_0(|\xi|))} \lesssim \Lambda(t_0(|\xi|))^{\eta-1} \sim |\xi|^{2\delta-2\delta\eta}.$$

This gives the desired statement. □

Lemma 4.2.7. *In the pseudo-differential zone the terms with phase functions have no meaning, that is, it holds*

$$|\xi|^{2\delta}\int_0^{t_0(|\xi|)} b(\tau)d\tau \le N, \qquad \text{and} \qquad \int_0^{t_0(|\xi|)} \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau \le C.$$

Proof. By applying the definition of pseudo-differential zone we conclude

$$|\xi|^{2\delta}\int_0^{t_0(|\xi|)} b(\tau)d\tau \le \Lambda(t_0(|\xi|))|\xi|^{2\delta} = N.$$

Applying the definition of pseudo-differential zone we have

$$\int_0^{t_0(|\xi|)} \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau \le |\xi|^\sigma \int_0^{t_0(|\xi|)} \frac{1}{b(\tau)|\xi|^{2\delta-\sigma}}d\tau \le |\xi|^\sigma \int_0^{t_0(|\xi|)} C(\varepsilon)d\tau.$$

Using the condition (B6) with $\eta \in (0, \frac{\sigma}{2\delta}]$ and the definition of $t_0(|\xi|)$ we get

$$\int_0^{t_0(|\xi|)} \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau \le C(\varepsilon)|\xi|^\sigma(1+t_0(|\xi|)) \le C(\varepsilon)|\xi|^{\sigma-2\delta\eta} \le C.$$

In this way the lemma is proved. □

Using this lemma we conclude as follows:

Corollary 4.2.8. *The following estimates hold for all* $t \in [t_0(|\xi|), \infty)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta\eta}|\hat{u}_1(\xi)|\Big) \quad \textit{for} \quad |\beta| \geq \sigma,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\eta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|+2\sigma-4\delta\eta}|\hat{u}_1(\xi)|\Big)$$
$$+\exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\eta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) \quad \textit{for} \quad |\beta| \geq 0.$$

Proof. Let us begin to estimate $|\xi|^{|\beta|}|\hat{u}(t,\xi)|$. The statement of Proposition 4.2.2 implies

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_0(|\xi|),\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_0(|\xi|))}|\hat{u}_t(t_0(|\xi|),\xi)|\Big).$$

Using the estimates for $|\xi|^{|\beta|}|\hat{u}(t_0(|\xi|),\xi)|$ and $|\xi|^{|\beta|}|\hat{u}_t(t_0(|\xi|),\xi)|$ from Proposition 4.2.5 we have

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}}|\hat{u}_0(\xi)| + \Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}}$$
$$\times(1+t_0(|\xi|))|\hat{u}_1(\xi)| + \frac{\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}}(1+t_0(|\xi|))}{b(t_0(|\xi|))}|\hat{u}_0(\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_0(|\xi|))}|\hat{u}_1(\xi)|\Big).$$

Here we recall Lemma 4.2.7 that phase functions bring no essential contribution to estimates in the pseudo-differential zone. Taking account of $1+t_0(|\xi|) \lesssim \Lambda(t_0(|\xi|))^{\eta}$ we get

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}}|\hat{u}_0(\xi)| + \Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}+\eta}$$
$$\times|\hat{u}_1(\xi)| + \frac{\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}+\eta}}{b(t_0(|\xi|))}|\hat{u}_0(\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_0(|\xi|))}|\hat{u}_1(\xi)|\Big).$$

Finally, the definition of $t_0(|\xi|)$ and Lemma 4.2.6 imply

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta\eta}|\hat{u}_1(\xi)|$$
$$+|\xi|^{|\beta|+2\sigma-4\delta\eta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta\eta}|\hat{u}_1(\xi)|\Big).$$

We conclude immediately

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta\eta}|\hat{u}_1(\xi)|\Big).$$

In the same way we can show the inequality

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\eta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|+2\sigma-4\delta\eta}|\hat{u}_1(\xi)|\Big)$$
$$+\exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\eta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big).$$

This completes the proof. □

4.2.5 Energy estimates

Corollary 4.2.8 and Proposition 4.2.5 imply the desired estimates for the energies in the extended phase space. For small frequencies we have the following statement :

Corollary 4.2.9. *The following estimates hold for all* $t \in [0, \infty)$:

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}(t,\xi)| &\lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\eta}{2\sigma-2\delta}}|\hat{u}_1(\xi)| \\
&\quad \text{for} \quad |\beta| \geq \sigma, \\
|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| &\lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{2\sigma-2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-4\delta\eta}{2\sigma-2\delta}}|\hat{u}_1(\xi)| \\
&\quad + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq 0.
\end{aligned}
$$

For large frequencies we may use the estimates from Proposition 4.2.2, because $t_0(|\xi|) = 0$. These statements imply an "exponential type decay". This we will show in the following statement.

Corollary 4.2.10. *The following estimates hold for all* $t \in [0, \infty)$:

$$
\begin{aligned}
|\xi|^{|\beta|}|\hat{u}(t,\xi)| &\lesssim \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq 0, \\
|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| &\lesssim \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|+2\sigma-4\delta}|\hat{u}_1(\xi)|\Big) \\
&\quad + \exp\Big(-C\int_0^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq 0.
\end{aligned}
$$

4.2.6 Conclusion

Taking into consideration all these estimates and the fact, that the statements from Corollaries 4.2.9 and 4.2.10 determine the decay estimates and regularity of the data we may conclude the following result.

Theorem 4.2.11. *The solution* $u = u(t,x)$ *to (4.2.1) with* $b = b(t)$ *satisfies the following (B1) to (B6) and* $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = 0$. *Then the solution satisfies the following estimates for the energies of higher order in the case* $\delta \in (\sigma/2, \sigma)$:

$$
\begin{aligned}
\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 &\lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{\sigma-\delta}}\|u_0\|_{H^{|\beta|}}^2 + \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\eta}{\sigma-\delta}}\|u_1\|_{H^{|\beta|-2\delta}}^2 \\
&\quad \text{for} \quad |\beta| \geq 2\delta, \\
\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 &\lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{\sigma-\delta}}\|u_0\|_{H^{|\beta|}}^2 + \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\eta}{\sigma-\delta}}\|u_1\|_{L^2}^2 \\
&\quad \text{for} \quad |\beta| \in [\sigma, 2\delta], \\
\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 &\lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{\sigma-\delta}}\|u_0\|_{H^{|\beta|+2\sigma-2\delta}}^2 \\
&\quad + \max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-4\delta\eta}{\sigma-\delta}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\delta}}\Big\}\|u_1\|_{H^{|\beta|}}^2 \quad \text{for} \quad |\beta| \geq 0.
\end{aligned}
$$

Remark 4.2.3. *If we set formally $\sigma = 1$ and $b(t) = (1+t)^\gamma,\ \gamma \in [2\delta-1,1]$ in the estimates from Theorem 4.2.11, then we get the estimates from Theorem 4.2 in [36].*

4.2.7 Some examples

Typical examples for coefficients $b = b(t)$ are

$$b_n(t) = \mu(1+t)^\gamma\big(\log(e+t)\big)^{-\gamma_1}\cdot\cdot\cdot\big(\log^{[n]}(e^{[n]}+t)\big)^{-\gamma_n},\ \text{with nonnegative}$$

$$\mu,\ \gamma_i,\ \ i=1,\cdot\cdot\cdot,n,\ \text{ and } \gamma \in \Big[\frac{2\delta-\sigma}{\sigma},1\Big]. \text{ We use, } \log^{[0]}(x)=x,\ e^{[0]}=1 \text{ and}$$

$$\log^{[n+1]}(x) = \log\log^{[n]}(x),\ e^{[n+1]} = e^{e^{[n]}}.$$

Example 4.2.1. *Let us choose $b(t) = (1+t)^\gamma$ with $\gamma \in \big[\frac{2\delta-\sigma}{\sigma},1\big]$. Then $b=b(t)$ satisfies the assumptions of Theorem 4.2.11. Consequently, the following estimates for the energies of higher order hold :*

$$\begin{aligned}
&\gamma \in \Big[\frac{2\delta-\sigma}{\sigma},1\Big):\\
&\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \big(1+t\big)^{-|\beta|\frac{1-\gamma}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}} + \big(1+t\big)^{-(|\beta|-\frac{2\delta}{1+\gamma})\frac{1-\gamma}{\sigma-\delta}}\|u_1\|^2_{H^{|\beta|-2\delta}} \quad \text{for} \quad |\beta|\geq 2\delta,\\
&\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \big(1+t\big)^{-|\beta|\frac{1-\gamma}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}} + \big(1+t\big)^{-(|\beta|-\frac{2\delta}{1+\gamma})\frac{1-\gamma}{\sigma-\delta}}\|u_1\|^2_{L^2} \quad \text{for} \quad |\beta|\in[\sigma,2\delta],\\
&\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \big(1+t\big)^{-(|\beta|+2\sigma-\frac{2\delta}{1+\gamma})\frac{1-\gamma}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|+2\sigma-2\delta}}\\
&\qquad\qquad + \big(1+t\big)^{\max\big\{-|\beta|\frac{1+\gamma}{\delta},-(|\beta|+2\sigma-\frac{4\delta}{1+\gamma})\frac{1-\gamma}{\sigma-\delta}\big\}}\|u_1\|^2_{H^{|\beta|}} \quad \text{for} \quad |\beta|\geq 0.\\
&\gamma = 1:\\
&\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \big(\log(e+t)\big)^{-\frac{|\beta|}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}} + \big(\log(e+t)\big)^{-\frac{|\beta|-\delta}{\sigma-\delta}}\|u_1\|^2_{H^{|\beta|-2\delta}} \quad \text{for} \quad |\beta|\geq 2\delta,\\
&\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \big(\log(e+t)\big)^{-\frac{|\beta|}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}} + \big(\log(e+t)\big)^{-\frac{|\beta|-\delta}{\sigma-\delta}}\|u_1\|^2_{L^2} \quad \text{for} \quad |\beta|\in[\sigma,2\delta],\\
&\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \big(\log(e+t)\big)^{-\frac{|\beta|+2\sigma-\delta}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|+2\sigma-2\delta}}\\
&\qquad\qquad + \max\Big\{\big(\log(e+t)\big)^{-\frac{|\beta|+2\sigma-2\delta}{\sigma-\delta}},(1+t)^{-2\frac{|\beta|}{\delta}}\Big\}\|u_1\|^2_{H^{|\beta|}} \quad \text{for} \quad |\beta|\geq 0.
\end{aligned}$$

Example 4.2.2. *Let us choose $b(t) = (1+t)^\gamma\log(e+t)$ with $\gamma \in \big[\frac{2\delta-\sigma}{\sigma},1\big]$. Then $b=b(t)$ satisfies the assumptions of Theorem 4.2.11. Consequently, the following estimates for the* *energies of higher order hold* :

$$
\begin{aligned}
&\gamma \in \big[\frac{2\delta-\sigma}{\sigma}, 1\big) :\\
&\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\Big)^{-\frac{|\beta|}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}}\\
&\qquad\qquad + \Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\Big)^{-\frac{|\beta|-\frac{2\delta}{1+\gamma}}{\sigma-\delta}}\|u_1\|^2_{H^{|\beta|-2\delta}} \quad \text{for} \quad |\beta| \geq 2\delta,\\
&\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\Big)^{-\frac{|\beta|}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}}\\
&\qquad\qquad + \Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\Big)^{-\frac{|\beta|-\frac{2\delta}{1+\gamma}}{\sigma-\delta}}\|u_1\|^2_{L^2} \quad \text{for} \quad |\beta| \in [\sigma, 2\delta],\\
&\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\Big)^{-\frac{|\beta|+2\sigma-\frac{2\delta}{1+\gamma}}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|+2\sigma-2\delta}}\\
&+\max\Big\{\Big((1+t)^{1+\gamma}\big(\log(e+t)\big)\Big)^{-\frac{|\beta|}{\delta}}, \Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\Big)^{-\frac{|\beta|+2\sigma-\frac{4\delta}{1+\gamma}}{\sigma-\delta}}\Big\}\|u_1\|^2_{H^{|\beta|}}\\
&\qquad\qquad \text{for} \quad |\beta| \geq 0.\\
&\gamma = 1 :\\
&\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \big(\log^{[2]}(e^{[2]}+t)\big)^{-\frac{|\beta|}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}} + \big(\log^{[2]}(e^{[2]}+t)\big)^{-\frac{|\beta|-\delta}{\sigma-\delta}}\|u_1\|^2_{H^{|\beta|-2\delta}}\\
&\qquad\qquad \text{for} \quad |\beta| \geq 2\delta,\\
&\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \big(\log^{[2]}(e^{[2]}+t)\big)^{-\frac{|\beta|}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}} + \big(\log^{[2]}(e^{[2]}+t)\big)^{-\frac{|\beta|-\delta}{\sigma-\delta}}\|u_1\|^2_{L^2}\\
&\qquad\qquad \text{for} \quad |\beta| \in [\sigma, 2\delta],\\
&\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \big(\log^{[2]}(e^{[2]}+t)\big)^{-\frac{|\beta|+2\sigma-\delta}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|+2\sigma-2\delta}}\\
&\qquad + \max\Big\{\big(\log^{[2]}(e^{[2]}+t)\big)^{-\frac{|\beta|+2\sigma-2\delta}{\sigma-\delta}}, \big((1+t)^2\log(e+t)\big)^{-\frac{|\beta|}{\delta}}\Big\}\|u_1\|^2_{H^{|\beta|}} \quad \text{for} \quad |\beta| \geq 0.
\end{aligned}
$$

Let us devote to the second case $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$.

In this case by using Lemma 4.2.1 the hyperbolic zone $Z_{hyp}(\varepsilon)$ and the reduced zone $Z_{red}(\varepsilon)$ are not contained in the pseudo-differential region. The pseudo-differential zone is only a compact set. Consequently we divide the extended phase space into the hyperbolic zone $Z_{hyp}(\varepsilon)$, the reduced zone $Z_{red}(\varepsilon)$, and the elliptic zone $Z_{ell}(\varepsilon, N)$ (see Figure 10.6 in the Appendix).

4.2.8 Treatment in the hyperbolic and reduced zone

The treatment in the hyperbolic and reduced zone is the same as in the Sections 3.3.2 and 3.3.3 respectively. We are able to extend the estimates from $Z_{hyp}(\varepsilon)$ to $Z_{red}(\varepsilon)$. For this reason, we obtain for $t \leq t_1(|\xi|)$, where $t_1 = t_1(|\xi|)$ is the separating line between the reduced and the elliptic zone the following a-priori estimates :

Proposition 4.2.12. *The following estimates hold for all* $t \in [0, t_1(|\xi|)]$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq 0.$$

4.2.9 Treatment in the elliptic zone

Proposition 4.2.13. *The following estimates hold for all* $t \in [t_1(|\xi|), \infty)$*, where* $t_1(|\xi|) = 0$ *for large frequencies* :

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_1(|\xi|))}|\hat{u}_t(t_1(|\xi|),\xi)|\Big)$$
$$\text{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(\frac{|\xi|^{|\beta|+2\sigma-2\delta}}{b(t_1(|\xi|))}|\hat{u}(t_1(|\xi|),\xi)|$$
$$+\frac{|\xi|^{|\beta|+2\sigma-4\delta}}{b^2(t_1(|\xi|))}|\hat{u}_t(t_1(|\xi|),\xi)|\Big) + \exp\Big(-|\xi|^{2\delta}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_t(t_1(|\xi|),\xi)| \ \text{for} \ |\beta| \geq 0.$$

Proof. The proof is the same as the proof to Proposition 4.2.2. □

4.2.10 Gluing procedure

For large frequencies we may use the estimate from Proposition 4.2.13 because of $t_1(|\xi|) = 0$. We glue for small frequencies the estimates from Propositions 4.2.12 for $t = t_1(|\xi|)$ with the estimates from Proposition 4.2.13.

Corollary 4.2.14. *The following estimates hold for all* $t \in [t_1(|\xi|), \infty)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^{t_1(|\xi|)} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)|$$
$$+|\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^{t_1(|\xi|)} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)|$$
$$+|\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) + \exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big) \quad \text{for} \quad |\beta| \geq 0.$$

Proof. Let us begin to estimate $|\xi|^{|\beta|}|\hat{u}(t,\xi)|$. The statement of Proposition 4.2.13 implies

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_1(|\xi|))}|\hat{u}_t(t_1(|\xi|),\xi)|\Big).$$

By using the definition of $t_1(|\xi|)$ we may conclude

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_1(|\xi|),\xi)|\Big).$$

Using the estimates for $|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)|$ and $|\xi|^{|\beta|}|\hat{u}_t(t_1(|\xi|),\xi)|$ from Propositions 4.2.12 we have

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^{t_1(|\xi|)} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)|$$
$$+|\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)| + |\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big).$$

We conclude immediately

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^{t_1(|\xi|)} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)|$$
$$+|\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big).$$

In the same way, we have the following statement

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^{t_1(|\xi|)} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)|$$
$$+|\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) + \exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big).$$

This completes the proof. □

4.2.11 Energy estimates

For small frequencies we have to understand the interplay between the two phase functions in the function

$$S_r(t,|\xi|) := |\xi|^r \exp\Big(-C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(-C_1|\xi|^{2\delta}\int_0^{t_1(|\xi|)} b(\tau)d\tau\Big).$$

Lemma 4.2.15. *To a given positive constant C there exists a positive and, in general, small constant C_1 such that for large t it holds*

$$S_r(t,|\xi|) \lesssim \max_{\xi\in\mathbb{R}^n}\Big\{|\xi|^r \exp\Big(-C_1|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)\Big\} \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{r}{2\delta}} \quad \text{for} \quad r \geq 0.$$

Proof. Here we follow ideas from the proof to Lemma 3.3.21. To estimate the term $S_r(t,|\xi|)$ it is important that the first partial derivative $\partial_{|\xi|}S_r(t,|\xi|)$ is negative for $|\xi| \leq \varepsilon_r$. This follows from

$$\partial_{|\xi|}S_r(t,|\xi|) = S_r(t,|\xi|)\Big(\frac{r}{|\xi|} - C(2\sigma-2\delta)\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta-1}}{b(\tau)}d\tau + C\frac{|\xi|^{2\sigma-2\delta}}{b(t_1(|\xi|))}d_{|\xi|}t_1(|\xi|)$$
$$-2\delta C_1|\xi|^{2\delta-1}\int_0^{t_1(|\xi|)} b(\tau)d\tau - C_1|\xi|^{2\delta}b(t_1(|\xi|))d_{|\xi|}t_1(|\xi|)$$
$$\leq S_r(t,|\xi|)\Big(\frac{r}{|\xi|} + \Big(-C_1|\xi|^{2\delta}b(t_1(|\xi|)) + C\frac{|\xi|^{2\sigma-2\delta}}{b(t_1(|\xi|))}\Big)d_{|\xi|}t_1(|\xi|)\Big),$$

and from

$$|\xi|^{2\delta}b(t_1(|\xi|)) = 2(1+\varepsilon)|\xi|^{\sigma},\ \frac{|\xi|^{2\sigma-2\delta}}{b(t_1(|\xi|))} = \frac{1}{2(1+\varepsilon)}|\xi|^{\sigma}.$$

If we choose the constant C_1 sufficiently small, then the term $C|\xi|^{2\sigma-2\delta}\frac{1}{b(t_1(|\xi|))}$ dominates the term $C_1|\xi|^{2\delta}b(t_1(|\xi|))$. Moreover, after differentiation we get

$$d_{|\xi|}t_1(|\xi|)b'(t_1(|\xi|))|\xi|^{2\delta-\sigma} + (2\delta-\sigma)|\xi|^{2\delta-\sigma-1}b(t_1(|\xi|)) = 0,$$

$$d_{|\xi|}t_1(|\xi|) = \frac{-(2\delta-\sigma)b(t_1(|\xi|))|\xi|^{2\delta-\sigma-1}}{b'(t_1(|\xi|))|\xi|^{2\delta-\sigma}},$$

respectively. Using the assumption $\frac{b'(t)}{b(t)} \leq C_2\frac{b(t)}{\Lambda(t)}$ from (B5) yields

$$d_{|\xi|}t_1(|\xi|) \leq -C\frac{(2\delta-\sigma)\Lambda(t_1(|\xi|))|\xi|^{2\delta-\sigma-1}}{b(t_1(|\xi|))|\xi|^{2\delta-\sigma}}.$$

The increasing behavior of the function $\Lambda(t)$ and the definitions of $t_0(|\xi|)$ and $t_1(|\xi|)$ give

$$d_{|\xi|}t_1(|\xi|) \leq -C_2(2\delta-\sigma)|\xi|^{2\delta-\sigma-1}\Lambda(t_0(|\xi|)) \leq -C_2(2\delta-\sigma)N|\xi|^{-1-\sigma}.$$

Moreover, for a fixed r the term $\frac{r}{|\xi|}$ is dominated by the negative term

$$\frac{1}{4}|\xi|^{2\delta}b(t_1(|\xi|))d_{|\xi|}t_1(|\xi|) \leq -C(2\delta-\sigma)N|\xi|^{-1}$$

if we choose N large enough. In order to complete the proof it is sufficient to study small frequencies with $|\xi| \leq \varepsilon_r$. For $|\xi| \geq \varepsilon_r$ we have an "exponential type decay" from the elliptic zone. Let us now fix $t > 0$. Then the above term takes its maximum for the $|\tilde{\xi}|$ satisfying $t = t_1(|\tilde{\xi}|)$. For $t = t_1(|\tilde{\xi}|)$ the first integral vanishes in $S_r(t,|\xi|)$. Consequently,

$$\begin{aligned} S_r(t,|\xi|) &\leq S_r(t_1(|\tilde{\xi}|),|\tilde{\xi}|) = |\tilde{\xi}|^r \exp\Big(-C|\tilde{\xi}|^{2\delta}\int_0^{t_1(|\tilde{\xi}|)} b(\tau)d\tau\Big) \\ &\lesssim \max_{\xi\in\mathbb{R}^n}\Big\{|\xi|^r\exp\Big(-C|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)\Big\} \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{r}{2\delta}}. \end{aligned}$$

In this way the lemma is proved. □

Using Lemma 4.2.15 we get the following statement.

Corollary 4.2.16. *The following estimates hold for all $t \in [0,\infty)$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{2\delta}}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq \sigma,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma}{2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq 0.$$

For large frequencies we may use the estimates from Proposition 4.2.13 because of $t_1(|\xi|) = 0$. These estimates imply an "exponential type decay".

Corollary 4.2.17. *The following estimates hold for all $t \in [0,\infty)$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq 0,$$

$$\begin{aligned} |\xi|^{|\beta|}|\hat{u}_t(t,\xi)| &\lesssim \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|+2\sigma-4\delta}|\hat{u}_1(\xi)|\Big) \\ &\quad + \exp\Big(-C\int_0^t b(\tau)d\tau\Big)|\xi|^{|\beta}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq 0. \end{aligned}$$

4.2.12 Conclusion

Taking into consideration all these estimates and the fact, that the statements from Proposition 4.2.12, Corollaries 4.2.16 and 4.2.17 determine the decay estimates and regularity of the data, respectively, we may conclude the following result:

Theorem 4.2.18. *Let us consider the Cauchy problem for the structural damped σ evolution model (4.2.1), where the coefficient $b = b(t)$ satisfies the conditions (B1) to (B6) and $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$ and where $\delta \in (\sigma/2, \sigma)$. Then the solution to (4.0.1) satisfies the following estimates for the energies of higher order :*

$$
\begin{aligned}
\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 &\lesssim \max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\delta}}, \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big\}\|u_0\|_{H^{|\beta|}}^2 \\
&\quad + \max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{\delta}}, \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big\}\|u_1\|_{H^{|\beta|-2\delta}}^2 \quad \text{for} \quad |\beta| \geq 2\delta, \\
\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 &\lesssim \max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\delta}}, \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big\}\|u_0\|_{H^{|\beta|}}^2 \\
&\quad + \max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{\delta}}, \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big\}\|u_1\|_{L^2}^2 \quad \text{for} \quad |\beta| \in [\sigma, 2\delta], \\
\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 &\lesssim \max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma}{\delta}}, \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big\}\|u_0\|_{H^{|\beta|+2\sigma-2\delta}}^2 \\
&\quad + \max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\delta}}, \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big\}\|u_1\|_{H^{|\beta|}}^2 \quad \text{for} \quad |\beta| \geq 0.
\end{aligned}
$$

Remark 4.2.4. *If we set formally $\sigma = 1$ and $b(t) = (1+t)^\gamma$, $\gamma \in (0, 2\delta - 1)$, in the estimates from Theorem 4.2.18, then we get the estimates from Theorem 4.1 in [36].*

4.2.13 Some examples

Example 4.2.3. *Let us choose $b(t) = (1+t)^\gamma$ with $\gamma \in \big(0, \frac{2\delta-\sigma}{\sigma}\big)$. Then $b = b(t)$ satisfies the assumptions of Theorem 4.2.18. Consequently, the following estimates for the energies of higher order hold :*

$$
\begin{aligned}
\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 &\lesssim \big(1+t\big)^{-\frac{|\beta|(1+\gamma)}{\delta}}\|u_0\|_{H^{|\beta|}}^2 + \big(1+t\big)^{-\frac{(|\beta|-\sigma)(1+\gamma)}{\delta}}\|u_1\|_{H^{|\beta|-2\delta}}^2 \quad \text{for} \quad |\beta| \geq 2\delta, \\
\|\nabla^\beta u(t,\cdot)\|_{L^2}^2 &\lesssim \big(1+t\big)^{-\frac{|\beta|(1+\gamma)}{\delta}}\|u_0\|_{H^{|\beta|}}^2 + \big(1+t\big)^{-\frac{(|\beta|-\sigma)(1+\gamma)}{\delta}}\|u_1\|_{L^2}^2 \quad \text{for} \quad |\beta| \in [\sigma, 2\delta], \\
\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2 &\lesssim \big(1+t\big)^{-\frac{(|\beta|+\sigma)(1+\gamma)}{\delta}}\|u_0\|_{H^{|\beta|+2\sigma-2\delta}}^2 + \big(1+t\big)^{-\frac{|\beta|(1+\gamma)}{\delta}}\|u_1\|_{H^{|\beta|}}^2 \quad \text{for} \quad |\beta| \geq 0.
\end{aligned}
$$

Example 4.2.4. *Let us choose $b(t) = (1+t)^\gamma\big(\log(e+t)\big)^{-\gamma_1}$ with $\gamma \in \big(0, \frac{2\delta-\sigma}{\sigma}\big)$ and non-negative γ_1. Then $b = b(t)$ satisfies the assumptions of Theorem 4.2.18. Consequently, the* *following estimates for the energies of higher order hold* :

$$\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \Big((1+t)^{1+\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|}{\delta}}\|u_0\|^2_{H^{|\beta|}}$$
$$+\Big((1+t)^{1+\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|-\sigma}{\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}} \quad \textit{for} \quad |\beta|\geq 2\delta,$$
$$\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \Big((1+t)^{1+\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|}{\delta}}\|u_0\|^2_{H^{|\beta|}}$$
$$+\Big((1+t)^{1+\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|-\sigma}{\delta}}\|u_1\|^2_{L^2} \quad \textit{for} \quad |\beta|\in[\sigma,2\delta],$$
$$\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \Big((1+t)^{1+\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|+\sigma}{\delta}}\|u_0\|^2_{H^{|\beta|+2\sigma-2\delta}}$$
$$+\Big((1+t)^{1+\gamma}\big(\log(e+t)\big)^{-\gamma_1}\Big)^{-\frac{|\beta|}{\delta}}\|u_1\|^2_{H^{|\beta|}} \quad \textit{for} \quad |\beta|\geq 0.$$

4.3 Time-dependent strictly increasing dissipation– $\delta \in (0,\sigma/2)$

We consider with $\sigma > 1$ the Cauchy problem for the σ- evolution model

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + b(t)(-\Delta)^\delta u_t = 0, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x) \quad \sigma>1, \quad \delta\in(0,\sigma/2). \end{cases} \tag{4.3.1}$$

After partial Fourier transformation we obtain the Cauchy problem

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + b(t)|\xi|^{2\delta}\hat{u}_t = 0, \quad \hat{u}(0,\xi)=\hat{u}_0(\xi), \quad \hat{u}_t(0,\xi)=\hat{u}_1(\xi), \quad \delta\in(0,\sigma/2).$$

We choose the assumptions (B1) to (B6) for the time-dependent coefficient function $b = b(t)$ as in the Section 4.2.

Division of the extended phase space into zones

We divide the extended phase space into the same zone as in Section 4.2.1. But there is big difference in the localization of these zones. Now the separating lines satisfy $t_2(|\xi|) < t_1(|\xi|)$ for large frequencies. Small frequencies belong only to the elliptic region (see Figure 10.7 in the Appendix).

Treatment in the hyperbolic and reduced zone

The treatment in the hyperbolic and reduced zone is the same as in the Sections 3.3.2 and 3.3.3, respectively. We are able to extend the estimate from $Z_{hyp}(\varepsilon)$ to $Z_{red}(\varepsilon)$. For this reason we obtain for $t \leq t_1(|\xi|)$, where $t_1 = t_1(|\xi|)$ is the separating line between the reduced and the elliptic zone the following a-priori estimates:

Proposition 4.3.1. *The following estimates hold for all* $t \in [0, t_1(|\xi|)]$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq 0,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq 0.$$

Treatment in the elliptic zone

If we choose the constant N in the definition of the pseudo-differential zone sufficiently large, then we have to care for two cases. The first one is $t \geq t_1(|\xi|)$ for large frequencies, the second one is $t \geq t_0(|\xi|)$ for small frequencies.

Proposition 4.3.2. *The following estimates hold for all* $t \in [t_0(|\xi|), \infty)$ *where* $t_0(|\xi|) = t_1(|\xi|)$ *for large frequencies* :

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_0(|\xi|),\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_0(|\xi|))}|\hat{u}_t(t_0(|\xi|),\xi)|\Big)$$
$$\text{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(\frac{|\xi|^{|\beta|+2\sigma-2\delta}}{b(t_0(|\xi|))}|\hat{u}(t_0(|\xi|),\xi)| + \frac{|\xi|^{|\beta|+2\sigma-4\delta}}{b^2(t_0(|\xi|))}$$
$$\times|\hat{u}_t(t_0(|\xi|),\xi)|\Big) + \exp\Big(-|\xi|^{2\delta}\int_{t_0(|\xi|)}^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_t(t_0(|\xi|),\xi)| \quad \text{for} \quad |\beta| \geq 0.$$

Proof. The proofs are the same as the proof to Proposition 4.2.2. □

Treatment in the pseudo-differential zone

Proposition 4.3.3. *The following estimate hold for all* $t \in [0, t_0(|\xi|)]$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}}(1+t)|\hat{u}_1(\xi)|$$
$$\text{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma}{2\delta}}(1+t)|\hat{u}_0(\xi)| + \exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_1(\xi)|$$
$$\text{for} \quad |\beta| \geq 0.$$

Proof. The proof is analogous to the proof to Proposition 3.3.12. □

Gluing procedure

For small frequencies we glue the estimates from the Proposition 4.3.2 with the estimates from Proposition 4.3.3 for $t = t_0(|\xi|)$. Therefore we need the following statements.

Lemma 4.3.4. *Let us assume $b \in S_\eta$. Then the following estimate holds for the separating line $t_0(|\xi|)$ between the elliptic zone and the pseudo-differential zone* :

$$\frac{1}{b(t_0(|\xi|))} \lesssim \Lambda(t_0(|\xi|))^{\eta-1} \sim |\xi|^{2\delta-2\delta\eta}.$$

Proof. The proof coincides with the proof of Lemma 4.2.6. □

Lemma 4.3.5. *In the pseudo-differential zone the terms with phase functions have no meaning, that is, it holds*

$$|\xi|^{2\delta}\int_0^{t_0(|\xi|)} b(\tau)d\tau \le N, \qquad \text{and} \qquad \int_0^{t_0(|\xi|)} \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau \le C.$$

Proof. The proof coincides with the proof of Lemma 4.2.7. □

Using these lemmas we conclude as follows:

Corollary 4.3.6. *The following estimates hold for all $t \in [t_0(|\xi|), \infty)$:*

$$|\xi|^{|\beta|}|\hat u(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat u_0(\xi)| + |\xi|^{|\beta|-2\delta\eta}|\hat u_1(\xi)|\Big) \quad \text{for} \quad |\beta| \ge \sigma,$$

$$|\xi|^{|\beta|}|\hat u_t(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\eta}|\hat u_0(\xi)| + |\xi|^{|\beta|+2\sigma-4\delta\eta}|\hat u_1(\xi)|\Big)$$
$$+\exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\eta}|\hat u_0(\xi)| + |\xi|^{|\beta|}|\hat u_1(\xi)|\Big) \quad \text{for} \quad |\beta| \ge 0.$$

Proof. Let us begin to estimate $|\xi|^{|\beta|}|\hat u(t,\xi)|$. The statement of Proposition 4.3.2 implies

$$|\xi|^{|\beta|}|\hat u(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat u(t_0(|\xi|),\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_0(|\xi|))}|\hat u_t(t_0(|\xi|),\xi)|\Big).$$

Using the estimates for $|\xi|^{|\beta|}|\hat u(t_0(|\xi|),\xi)|$ and $|\xi|^{|\beta|-2\delta}|\hat u_t(t_0(|\xi|),\xi)|$ from Proposition 4.3.3 we have

$$|\xi|^{|\beta|}|\hat u(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}}|\hat u_0(\xi)| + \Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}}$$
$$(1+t_0(|\xi|)) \times |\hat u_1(\xi)| + \frac{\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}}(1+t_0(|\xi|))}{b(t_0(|\xi|))}|\hat u_0(\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_0(|\xi|))}|\hat u_1(\xi)|\Big).$$

Taking account of $1+t_0(|\xi|) \lesssim \Lambda(t_0(|\xi|))^\eta$ we get

$$|\xi|^{|\beta|}|\hat u(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}}|\hat u_0(\xi)| + \Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}+\eta}$$
$$\times|\hat u_1(\xi)| + \frac{\Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}+\eta}}{b(t_0(|\xi|))}|\hat u_0(\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_0(|\xi|))}|\hat u_1(\xi)|\Big).$$

The statement of Lemma 4.3.4 implies

$$|\xi|^{|\beta|}|\hat u(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(\Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}}|\hat u_0(\xi)| + \Lambda(t_0(|\xi|))^{-\frac{|\beta|}{2\delta}+\eta}$$
$$\times|\hat u_1(\xi)| + \Lambda(t_0(|\xi|))^{-\frac{|\beta|+2\sigma-2\delta}{2\delta}+2\eta-1}|\hat u_0(\xi)| + \Lambda(t_0(|\xi|))^{\eta-1}|\xi|^{|\beta|-2\delta}|\hat u_1(\xi)|\Big).$$

The definition of $t_0(|\xi|)$ and Lemma 4.3.5 yield

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(- C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta\eta}|\hat{u}_1(\xi)| \\ +|\xi|^{|\beta|+2\sigma-4\delta\eta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta\eta}|\hat{u}_1(\xi)|\Big).$$

Hence, we may conclude immediately

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(- C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-2\delta\eta}|\hat{u}_1(\xi)|\Big).$$

In the same way one can show the inequality

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(- C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\eta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|+2\sigma-4\delta\eta}|\hat{u}_1(\xi)|\Big) \\ + \exp\Big(- |\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-2\delta\eta}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big).$$

This completes the proof. □

For large frequencies we have to glue the estimates from Proposition 4.3.1 for $t = t_1(|\xi|)$ and Proposition 4.3.2 for $t_0(|\xi|) = t_1(|\xi|)$.

Corollary 4.3.7. *The following estimates hold for all* $t \in [t_1(|\xi|), \infty)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(- C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(- \frac{|\xi|^{2\delta}}{4}\int_0^{t_1(|\xi|)} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| \\ +|\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \quad \text{for} \quad |\beta| \geq 0,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(- C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(- \frac{|\xi|^{2\delta}}{4}\int_0^{t_1(|\xi|)} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| \\ +|\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) + \exp\Big(- |\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\big) \quad \text{for} \quad |\beta| \geq 0.$$

Proof. Let us begin to estimate $|\xi|^{|\beta|}|\hat{u}(t,\xi)|$. The statement of Proposition 4.3.2 for $t_0(|\xi|) = t_1(|\xi|)$ implies

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(- C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)| + \frac{|\xi|^{|\beta|-2\delta}}{b(t_1(|\xi|))}|\hat{u}_t(t_1(|\xi|),\xi)|\Big).$$

Applying the definition of $b(t_1(|\xi|))$ we may conclude

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(- C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_1(|\xi|),\xi)|\Big).$$

Using the estimates for $|\xi|^{|\beta|}|\hat{u}(t_1(|\xi|),\xi)|$ and for $|\xi|^{|\beta|-\sigma}|\hat{u}_t(t_1(|\xi|),\xi)|$ from Proposition 4.3.1 we have

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(- C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(- \frac{|\xi|^{2\delta}}{4}\int_0^{t_1(|\xi|)} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| \\ +|\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)| + |\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big).$$

We conclude immediately

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^{t_1(|\xi|)} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| +|\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big).$$

In the same way we are able to derive the estimate

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_{t_1(|\xi|)}^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^{t_1(|\xi|)} b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| +|\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) + \exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)\big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big).$$

This completes the proof. □

Energy estimate

For large frequencies we have to estimate the term

$$\exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^{t_1(|\xi|)} b(\tau)d\tau\Big)\exp\Big(-C|\xi|^{2\sigma-2\delta}\int_{t_1(|\xi|)}^t \frac{1}{b(\tau)}d\tau\Big).$$

This term explains in this part of the extended phase space the competition of influences from different phase functions.

Lemma 4.3.8. *To a given positive constant C we can find a sufficiently small positive constant C_1 such that for large t it holds*

$$\exp\Big(-\int_0^{t_1(|\xi|)} b(\tau)d\tau\Big)\exp\Big(-C\int_{t_1(|\xi|)}^t \frac{1}{b(\tau)}d\tau\Big) \leq \exp\Big(-C_1\int_0^t \frac{1}{b(\tau)}d\tau\Big).$$

Proof. Using the increasing behavior of $b = b(t)$ implies for large time t the following relations:

$$-b(t) \leq -C_1\frac{1}{b(t)}, \quad \text{hence,} \quad \int_0^{t_1(|\xi|)} -b(\tau)d\tau \leq -C_1\int_0^{t_1(|\xi|)} \frac{1}{b(\tau)}d\tau.$$

In this way the lemma is proved. □

From Corollary 4.3.7 and Lemma 4.3.8 we obtain for large frequencies the following statements explaining an "exponential type decay".

Corollary 4.3.9. *The following estimates hold for all $t \in [0,\infty)$:*

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \quad \textit{for} \quad |\beta| \geq 0,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) \quad \textit{for} \quad |\beta| \geq 0.$$

For small frequencies we conclude immediately from Corollary 4.3.6 and Proposition 4.3.3 the following statement.

Corollary 4.3.10. *The following estimates hold for all* $t \in [0,\infty)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\eta}{2\sigma-2\delta}}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq \sigma,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{2\sigma-2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-4\delta\eta}{2\sigma-2\delta}}|\hat{u}_1(\xi)|$$
$$+\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{2\delta}}|\hat{u}_0(\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}}|\hat{u}_1(\xi)| \quad \text{for} \quad |\beta| \geq 0.$$

Conclusion

Taking into consideration all these estimates and the fact, that the statements from Corollaries 4.3.9 and 4.3.10 determine the regularity of the data and decay estimates for the solutions respectively, we have proved the main result about the behavior of $\|\nabla^\beta u\|^2_{L^2}$ and $\|\nabla^\beta u_t\|^2_{L^2}$.

Theorem 4.3.11. *Let us consider the Cauchy problem (4.3.1) with* $\delta \in (0,\frac{\sigma}{2})$*, where the time-dependent coefficient* $b = b(t) \in S_\eta$*, satisfies the conditions (B1) to (B6). Then the solution* $u = u(t,x)$ *satisfies the following estimates for the energies of higher order* :

$$\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}} + \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\eta}{\sigma-\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}}$$
$$\text{for} \quad |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \Big(1+\int_0^t \frac{d\tau}{b(\tau)}\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}}$$
$$+\max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-4\delta\eta}{\sigma-\delta}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\delta}}\Big\}\|u_1\|^2_{H^{|\beta|}} \quad \text{for} \quad |\beta| \geq 0.$$

Remark 4.3.1. *If we set formally* $\sigma = 1$ *and* $b(t) = (1+t)^\gamma,\ \gamma \in (0,1]$*, in the estimates from Theorem 4.3.11, then we get the estimates from Theorem 2.1 in [36].*

Some examples

Example 4.3.1. *Let us choose* $b(t) = (1+t)^\gamma$ *with* $\gamma \in (0,1]$*. Then* $b = b(t)$ *satisfies the assumptions of Theorem 4.3.11. Consequently, the following estimates for the energies of higher order hold* :

$$\gamma \in (0,1):$$
$$\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim (1+t)^{-|\beta|\frac{1-\gamma}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}} + (1+t)^{-(|\beta|-\frac{2\delta}{1+\gamma})\frac{1-\gamma}{\sigma-\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}} \quad \text{for} \quad |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim (1+t)^{-(|\beta|+2\sigma-\frac{2\delta}{1+\gamma})\frac{1-\gamma}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}}$$
$$+(1+t)^{\max\{-|\beta|\frac{1+\gamma}{\delta},-(|\beta|+2\sigma-\frac{4\delta}{1+\gamma})\frac{1-\gamma}{\sigma-\delta}\}}\|u_1\|^2_{H^{|\beta|}} \quad \text{for} \quad |\beta| \geq 0.$$
$$\gamma = 1:$$
$$\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \big(\log(e+t)\big)^{-\frac{|\beta|}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}} + \big(\log(e+t)\big)^{-\frac{|\beta|-\sigma}{\sigma-\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}} \quad \text{for} \quad |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \big(\log(e+t)\big)^{-\frac{|\beta|+2\sigma-\delta}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}}$$
$$+\max\Big\{\big(\log(e+t)\big)^{-\frac{|\beta|+2\sigma-2\delta}{\sigma-\delta}}, (1+t)^{-2\frac{|\beta|}{\delta}}\Big\}\|u_1\|^2_{H^{|\beta|}} \quad \text{for} \quad |\beta| \geq 0.$$

Example 4.3.2. *Let us choose* $b(t)=(1+t)^\gamma\log(e+t)\log^{[2]}(e^{[2]}+t)$ *with* $\gamma\in\big(0,1\big]$*. Then* $b=b(t)$ *satisfies the assumptions of Theorem 4.3.11. Consequently, the following estimates for the energies of higher order hold :*

$$\gamma\in\big(0,1\big):$$

$$\|\nabla^\beta u(t,\cdot)\|^2_{L^2}\lesssim\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-1}\Big)^{-\frac{|\beta|}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}}$$

$$+\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-1}\Big)^{-\frac{|\beta|-\frac{2\delta}{1+\gamma}}{\sigma-\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}}\quad\text{for}\quad|\beta|\geq\sigma,$$

$$\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2}\lesssim\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-1}\Big)^{-\frac{|\beta|+2\sigma-\frac{2\delta}{1+\gamma}}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}}$$

$$+\max\Big\{\Big((1+t)^{1+\gamma}\log(e+t)\log^{[2]}(e^{[2]}+t)\Big)^{-\frac{|\beta|}{\delta}},$$

$$\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-1}\Big)^{-\frac{|\beta|+2\sigma-\frac{4\delta}{1+\gamma}}{\sigma-\delta}}\Big\}\|u_1\|^2_{H^{|\beta|}}\quad\text{for}\quad|\beta|\geq 0.$$

$$\gamma=1:$$

$$\|\nabla^\beta u(t,\cdot)\|^2_{L^2}\lesssim\big(\log^{[3]}(e^{[3]}+t)\big)^{-\frac{|\beta|}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|}}+\big(\log^{[3]}(e^{[3]}+t)\big)^{-\frac{|\beta|-\sigma}{\sigma-\delta}}\|u_1\|^2_{H^{|\beta|-\sigma}}$$

$$\text{for}\quad|\beta|\geq\sigma,$$

$$\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2}\lesssim\big(\log^{[3]}(e^{[3]}+t)\big)^{-\frac{|\beta|+2\sigma-\delta}{\sigma-\delta}}\|u_0\|^2_{H^{|\beta|+\sigma}}$$

$$+\max\Big\{\big(\log^{[3]}(e^{[3]}+t)\big)^{-\frac{|\beta|+2\sigma-2\delta}{\sigma-\delta}},\big((1+t)^2\log(e+t)\log^{[2]}(e^{[2]}+t)\big)^{-\frac{|\beta|}{\delta}}\Big\}\|u_1\|^2_{H^{|\beta|}}$$

$$\text{for}\quad|\beta|\geq 0.$$

4.4 Time-dependent strictly increasing dissipation– $\delta=\sigma/2$

In this section we study the Cauchy problem (4.0.1) for $\delta=\sigma/2$ and $b(t)$ strictly increasing, that is, the model

$$\begin{cases} u_{tt}+(-\Delta)^\sigma u+b(t)(-\Delta)^{\sigma/2}u_t=0, \\ u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x),\quad\sigma>1. \end{cases}\tag{4.4.1}$$

This model corresponds to the non-constant distributed resistance of typical semiconductors [?]. In this section we assume the following conditions for $b=b(t)$:

(B1) *positivity :* $b(t)>0$ and there exists a positive constant T_c such that $b(t)>2$ for all $t\geq T_c$,

(B2) *increasing behavior :* $b'(t)>0$ for all $t\geq 0$,

(B3) *non-integrability :* $\int_0^\infty\frac{1}{b(\tau)}d\tau=\infty$,

(B4) *higher order derivatives* : it holds $|d_t^k b(t)|\leq C_k b(t)\big(\frac{1}{1+t}\big)^k$ for $k=1,2,$

(B5) *useful inequalities* : there exist positive constants C_0, C_1, C_2 which are independent of t such that

$$C_2 \frac{b(t)}{\Lambda(t)} \le \frac{b'(t)}{b(t)} \le C_0 \frac{1}{1+t} \le C_1 \frac{b(t)}{\Lambda(t)} \quad \text{with} \quad \Lambda(t) = 1 + \int_0^t b(\tau) d\tau,$$

(B6) *additional classification* : $b \in S_\eta$ for $\eta \in (0,1]$, where we introduce the family $\{S_\eta\}_\eta$ of classes

$$S_\eta := \Big\{ b = b(t) : \limsup_{t\to\infty} (1+t)\Lambda(t)^{-\eta} < \infty, \quad \lim_{t\to\infty} (1+t)\Lambda(t)^{-\beta} = \infty \text{ for all } \beta < \eta \Big\}.$$

After partial Fourier transformation $\hat{u}(t,\xi) = F_{x\to\xi}(u)(t,\xi)$ in (4.4.1) we obtain the Cauchy problem

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + b(t)|\xi|^\sigma \hat{u}_t = 0, \quad \hat{u}(0,\xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi), \quad \sigma > 1.$$

Division of the extended phase space into zones

In this case we have a very simple division of the extended phase space $\{(t,\xi) \in [0,\infty) \times \mathbb{R}^n\}$. We divide it into the following zones with $\Lambda(t) = 1 + \int_0^t b(\tau)d\tau$, where ε is small:

$$\text{remaining zone} \quad Z_{rem}(\varepsilon) = \Big\{ (t,\xi) : \frac{b(t)}{2} \le 1 + \varepsilon \Big\},$$

$$\text{elliptic zone} \quad Z_{ell}(\varepsilon, N) = \Big\{ (t,\xi) : \frac{b(t)}{2} \ge 1 + \varepsilon \text{ and } \Lambda(t)|\xi|^\sigma \ge N \Big\},$$

$$\text{pseudo-differential zone} \quad Z_{pd}(\varepsilon, N) = \Big\{ (t,\xi) : \frac{b(t)}{2} \ge 1 + \varepsilon \text{ and } \Lambda(t)|\xi|^\sigma \le N \Big\}.$$

We introduce separating lines. We denote the separating lines t_1 between the elliptic zone and the remaining zone and $t_0(|\xi|)$ between the pseudo-differential zone and the elliptic zone. We mention, that $Z_{rem}(\varepsilon) = [0, t_1] \times \mathbb{R}^n$ if there exists a positive constant t_1 satisfying $1 + \varepsilon = \frac{b(t_1)}{2}$. It turns out that it is enough to study the behavior of solutions for large times. For this reason we may restrict ourselves to the pseudo-differential and to the elliptic zone (see Figure 10.8 in the Appendix).

Treatment in the remaining zone

Proposition 4.4.1. *The Cauchy problem (4.4.1) is well-posed, that is, to every data* $u_0 \in H^{|\beta|}$ *and* $u_1 \in H^{|\beta|-\sigma}$, $|\beta| \ge \sigma$, *there are exists a unique solution* $u \in C([0,\infty), H^{|\beta|})$ *with* $u_t \in C([0,\infty), H^{|\beta|-\sigma})$. *The solution depends continuously on the data. For the energy of higher order*

$$E^{|\beta|}[u](t) := \|u(t,\cdot)\|^2_{H^{|\beta|}} + \|u_t(t,\cdot)\|^2_{H^{|\beta|-\sigma}}$$

with $|\beta| \ge \sigma$ *we have the estimate*

$$E^{|\beta|}[u](t) \lesssim C(t)\big(\|u_0\|^2_{H^{|\beta|}} + \|u_1\|^2_{H^{|\beta|-\sigma}}\big). \tag{4.4.2}$$

Proof. The proof is the same as the proof to Proposition 3.2.1. □

Treatment in the pseudo-differential zone

Proposition 4.4.2. *The following estimates hold for all* $t \in [t_1, t_0(|\xi|)]$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \Big(1+\int_{t_1}^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\sigma}}|\hat{u}(t_1,\xi)| + \Big(1+\int_{t_1}^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\sigma}}(1+t)|\hat{u}_t(t_1,\xi)|$$

for $|\beta| \geq \sigma$,

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \Big(1+\int_{t_1}^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma}{\sigma}}(1+t)|\hat{u}(t_1,\xi)| + \exp\Big(-|\xi|^\sigma\int_{t_1}^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_t(t_1,\xi)| \quad \text{for} \quad |\beta| \geq 0.$$

Proof. The proof is the same as the proof to Proposition 3.3.12. □

Treatment in the elliptic zone

Proposition 4.4.3. *The following estimates hold for all* $t \in [t_0(|\xi|), \infty)$*, where* $t_0(|\xi|) = t_1$ *for large frequencies* :

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^\sigma}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_0(|\xi|),\xi)| + \frac{|\xi|^{|\beta|-\sigma}}{b(t_0(|\xi|)}|\hat{u}_t(t_0(|\xi|),\xi)|\Big)$$

for $|\beta| \geq \sigma$,

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_{t_0(|\xi|)}^t \frac{|\xi|^\sigma}{b(\tau)}d\tau\Big)\Big(\frac{|\xi|^{|\beta|+\sigma}}{b(t_0(|\xi|))}|\hat{u}(t_0(|\xi|),\xi)| + \frac{|\xi|^{|\beta|}}{b^2(t_0(|\xi|))} \times|\hat{u}_t(t_0(|\xi|),\xi)|\Big) + \exp\Big(-|\xi|^\sigma\int_{t_0(|\xi|)}^t b(\tau)d\tau\Big)|\xi|^{|\beta|}|\hat{u}_t(t_0(|\xi|),\xi)| \quad \text{for} \quad |\beta| \geq 0.$$

Proof. The proof is the same as the proof to Proposition 4.2.2. □

Gluing procedure

For small frequencies we glue the estimates from the Propositions 4.4.2 and 4.4.3 by taking into consideration that phase functions have no meaning in the pseudo-differential region.

Corollary 4.4.4. *The following estimates hold for all* $t \in [t_1, \infty)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^\sigma}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1,\xi)| + |\xi|^{|\beta|-\sigma\eta}|\hat{u}_t(t_1,\xi)|\Big) \quad \text{for} \quad |\beta| \geq \sigma,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{|\xi|^\sigma}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-\sigma\eta}|\hat{u}(t_1,\xi)| + |\xi|^{|\beta|+2\sigma-2\sigma\eta}|\hat{u}_t(t_1,\xi)|\Big) + \exp\Big(-|\xi|^\sigma\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+2\sigma-\sigma\eta}|\hat{u}(t_1,\xi)| + |\xi|^{|\beta|}|\hat{u}_t(t_1,\xi)|\Big) \quad \text{for} \quad |\beta| \geq 0.$$

Proof. The proof coincides with the proof to Corollary 4.3.6. □

Energy estimate

For large frequencies we have Proposition 4.4.3. The statements imply an "exponential type decay". Taking account of the integrals in the phases can be continued up to $t = 0$ we conclude the following result :

Corollary 4.4.5. *The following estimates hold for all* $t \in [0, \infty)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}(t_1,\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_t(t_1,\xi)|\Big) \quad \text{for} \quad |\beta| \geq 0,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}(t_1,\xi)| + |\xi|^{|\beta|}|\hat{u}_t(t_1,\xi)|\Big) \quad \text{for} \quad |\beta| \geq 0.$$

For small frequencies we have Corollary 4.4.4. We obtain the following result :

Corollary 4.4.6. *The following estimates hold for all* $t \in [0, \infty)$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{\sigma}}|\hat{u}(t_1,\xi)| + \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-\sigma\eta}{\sigma}}|\hat{u}_t(t_1,\xi)|$$
$$\text{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-\sigma\eta}{\sigma}}|\hat{u}(t_1,\xi)| + \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-2\sigma\eta}{\sigma}}|\hat{u}_t(t_1,\xi)|$$
$$+\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-\sigma\eta}{\sigma}}|\hat{u}(t_1,\xi)| + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\sigma}}|\hat{u}_t(t_1,\xi)| \quad \text{for} \quad |\beta| \geq 0.$$

Conclusion

Taking into consideration all these estimates and the fact, that the statements from Corollaries 4.4.5 and 4.4.6 determine the decay estimates and regularity of the data, respectively, after applying Proposition 4.4.1 we conclude estimates for the energies of higher order $\|\nabla^\beta u\|^2_{L^2}$ and $\|\nabla^\beta u_t\|^2_{L^2}$.

Theorem 4.4.7. *Let us consider the Cauchy problem (4.4.1) with* $\delta = \frac{\sigma}{2}$*, where the time-dependent coefficient* $b = b(t) \in S_\eta$ *satisfies the conditions (B1) to (B6). Then the solution* $u = u(t,x)$ *satisfies the following estimates for the energies of higher order* :

$$\|\nabla^\beta u(t,\cdot)\|^2_{L^2} \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-2\frac{|\beta|}{\sigma}}\|u_0\|^2_{H^{|\beta|}} + \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-2\frac{|\beta|-\sigma\eta}{\sigma}}\|u_1\|^2_{H^{|\beta|-\sigma}}$$
$$\text{for} \quad |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2} \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-2\frac{|\beta|+2\sigma-\sigma\eta}{\sigma}}\|u_0\|^2_{H^{|\beta|+\sigma}}$$
$$+\max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-2\frac{|\beta|+2\sigma-2\sigma\eta}{\sigma}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-2\frac{|\beta|}{\sigma}}\Big\}\|u_1\|^2_{H^{|\beta|}} \quad \text{for} \quad |\beta| \geq 0.$$

Remark 4.4.1. *If we set formally* $\delta = \frac{\sigma}{2}$ *in the estimates from Theorems 4.2.11 and 4.3.11, then we get the estimates from Theorem 4.4.7.*

Remark 4.4.2. *If we set formally* $\sigma = 1$ *and* $b(t) = (1+t)^\gamma,\ \gamma \in (0,1]$*, in the estimates from Theorem 4.4.7, then we get the estimates from Theorem 3.1 in [36].*

Some examples

Example 4.4.1. *Let us choose* $b(t)=(1+t)^\gamma$ *with* $\gamma\in(0,1]$. *Then* $b=b(t)$ *satisfies the assumptions of Theorem 4.4.7. Consequently, the following estimates for the energies of higher order hold* :

$$\begin{aligned}
&\gamma\in(0,1):\\
&\|\nabla^\beta u(t,\cdot)\|_{L^2}^2\lesssim\big(1+t\big)^{-2|\beta|\frac{1-\gamma}{\sigma}}\|u_0\|^2_{H^{|\beta|}}+\big(1+t\big)^{-2(|\beta|-\frac{\sigma}{1+\gamma})\frac{1-\gamma}{\sigma}}\|u_1\|^2_{H^{|\beta|-\sigma}}\quad\text{for}\quad|\beta|\geq\sigma,\\
&\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2\lesssim\big(1+t\big)^{-2(|\beta|+2\sigma-\frac{\sigma}{1+\gamma})\frac{1-\gamma}{\sigma}}\|u_0\|^2_{H^{|\beta|+\sigma}}\\
&\qquad\quad+\big(1+t\big)^{\max\left\{-2|\beta|\frac{1+\gamma}{\sigma},-(|\beta|+2\sigma-\frac{2\sigma}{1+\gamma})\frac{1-\gamma}{\sigma}\right\}}\|u_1\|^2_{H^{|\beta|}}\quad\text{for}\quad|\beta|\geq 0.\\
&\gamma=1:\\
&\|\nabla^\beta u(t,\cdot)\|_{L^2}^2\lesssim\big(\log(e+t)\big)^{-\frac{|\beta|}{\sigma}}\|u_0\|^2_{H^{|\beta|}}+\big(\log(e+t)\big)^{-\frac{2|\beta|-\sigma}{\sigma}}\|u_1\|^2_{H^{|\beta|-\sigma}}\quad\text{for}\quad|\beta|\geq\sigma,\\
&\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2\lesssim\big(\log(e+t)\big)^{-\frac{2|\beta|+3\sigma}{\sigma}}\|u_0\|^2_{H^{|\beta|+\sigma}}\\
&\qquad+\max\Big\{\big(\log(e+t)\big)^{-2\frac{|\beta|+\sigma}{\sigma}},(1+t)^{-4\frac{|\beta|}{\sigma}}\Big\}\|u_1\|^2_{H^{|\beta|}}\quad\text{for}\quad|\beta|\geq 0.
\end{aligned}$$

Example 4.4.2. *Let us choose* $b(t)=(1+t)^\gamma\log(e+t)\log^{[2]}(e^{[2]}+t)$ *with* $\gamma\in\big(0,1\big]$. *Then* $b=b(t)$ *satisfies the assumptions of Theorem 4.4.7. Consequently, the following estimates for the energies of higher order hold* :

$$\begin{aligned}
&\gamma\in\big(0,1\big):\\
&\|\nabla^\beta u(t,\cdot)\|_{L^2}^2\lesssim\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-1}\Big)^{-2\frac{|\beta|}{\sigma}}\|u_0\|^2_{H^{|\beta|}}\\
&\quad+\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-1}\Big)^{-2\frac{|\beta|-\frac{\sigma}{1+\gamma}}{\sigma}}\|u_1\|^2_{H^{|\beta|-\sigma}}\quad\text{for}\quad|\beta|\geq\sigma,\\
&\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2\lesssim\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-1}\Big)^{-2\frac{|\beta|+2\sigma-\frac{\sigma}{1+\gamma}}{\sigma}}\|u_0\|^2_{H^{|\beta|+\sigma}}\\
&+\max\Big\{\Big((1+t)^{1+\gamma}\log(e+t)\log^{[2]}(e^{[2]}+t)\Big)^{-2\frac{|\beta|}{\sigma}},\\
&\quad\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-1}\Big)^{-2\frac{|\beta|+2\sigma-\frac{2\sigma}{1+\gamma}}{\sigma}}\Big\}\|u_1\|^2_{H^{|\beta|}}\quad\text{for}\quad|\beta|\geq 0.\\
&\gamma=1:\\
&\|\nabla^\beta u(t,\cdot)\|_{L^2}^2\lesssim\big(\log^{[3]}(e^{[3]}+t)\big)^{-2\frac{|\beta|}{\sigma}}\|u_0\|^2_{H^{|\beta|}}+\big(\log^{[3]}(e^{[3]}+t)\big)^{-\frac{2|\beta|-\sigma}{\sigma}}\|u_1\|^2_{H^{|\beta|-\sigma}}\\
&\qquad\text{for}\quad|\beta|\geq\sigma,\\
&\|\nabla^\beta u_t(t,\cdot)\|_{L^2}^2\lesssim\big(\log^{[3]}(e^{[3]}+t)\big)^{-\frac{2|\beta|+3\sigma}{\sigma}}\|u_0\|^2_{H^{|\beta|+\sigma}}\\
&+\max\Big\{\big(\log^{[3]}(e^{[3]}+t)\big)^{-2\frac{|\beta|+\sigma}{\sigma}},\big((1+t)^2\log(e+t)\log^{[2]}(e^{[2]}+t)\big)^{-4\frac{|\beta|}{\sigma}}\Big\}\|u_1\|^2_{H^{|\beta|}}\\
&\qquad\text{for}\quad|\beta|\geq 0.
\end{aligned}$$

5 Smoothing properties

In Chapters 2, 3 and 4 we used representation formulas for the solutions to structural damped σ-evolution models to prove estimates for the energies of higher order. In this way we explained the parabolic effect. After the study of decay estimates for energies of higher order for solutions to structural damped $\sigma-$evolution models with dissipation terms $b(t)(-\Delta)^{\delta}u_t$, where $b = b(t)$ is a monotonous positive function satisfying $\int_0^\infty b(\tau)d\tau = \infty$ or $\int_0^\infty \frac{1}{b(\tau)}d\tau = \infty$ we will study in this section another property of solutions to model (1.3.1), the so-called smoothing effect. The model of interest is

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^{\sigma}u(t,x) + b(t)(-\Delta)^{\delta}u_t(t,x) = 0, & (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \delta \in (0,\sigma), & \text{and} \quad \sigma > 1. \end{cases} \tag{5.0.1}$$

Thus, applying the partial Fourier transformation and the "dissipative" transformation leads to the following Cauchy problem:

$$v_{tt}(t,\xi) + m(t,\xi)v(t,\xi) = 0, \quad v(0,\xi) =: v_0(\xi), \quad v_t(0,\xi) =: v_1(\xi), \quad \delta \in (0,\sigma), \tag{5.0.2}$$

with mass term

$$m(t,\xi) = |\xi|^{2\sigma} - \frac{b^2(t)}{4}|\xi|^{4\delta} - \frac{b'(t)}{2}|\xi|^{2\delta}. \tag{5.0.3}$$

For the behaviour of the solutions to this ordinary differential equation the sign of the coefficient $m = m(t,\xi)$ is important. We divide our considerations into the cases

1. $b(t)$ is strictly decreasing, that is, $b'(t) < 0$ for $t > 0$,

2. $b(t)$ is strictly increasing, that is, $b'(t) > 0$ for $t > 0$.

5.1 Objectives and strategies

We will study the regularity of the solution $u(t,\cdot)$ for structural damped $\sigma-$evolution models (5.0.1). As in the previous chapters we assume for the Cauchy data $u_0 \in H^{\sigma}(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. We are interested to understand to which Gevrey space $\Gamma^{a,s}(\mathbb{R}^n)$ our solution $u(t,\cdot)$ belongs to. The study of regularity properties for $u(t,\cdot)$ for $t > 0$ allows to restrict our considerations to large frequencies in the extended phase space. Let us explain our strategy. It is divided into the following steps:

- We use the partial Fourier transformation and the "dissipative" transformation to reduce the partial differential equation into an ordinary differential equation for $\hat{u}(t,\xi)$ parameterised by the frequency parameter ξ.

- The natural starting point is to rewrite the second order equation as a system for the micro-energy $(|\xi|^\sigma \hat{u}, D_t\hat{u})^T$ or a modified one and to use a diagonalization technique to simplify the structure and to estimate its fundamental solution.

- To study the Cauchy problem for the ordinary differential equation we will divide the extended phase space into two important zones, the hyperbolic and the elliptic zone, the reduced zone brings no new contribution.

- If our considerations can be restricted to the hyperbolic zone, then we will introduce an appropriate micro-energy to get for it a system of first order after one step of diagonalization. The remainder becomes integrable. Here the remainder can be studied by the matrizant representation. Then we derive a representation of the fundamental solution.

- If our considerations can be restricted to the elliptic zone, then we use another micro-energy to get again a system of first order. This system should be diagonalized twice. Then the remainder becomes integrable. We explain the matrix representation of the fundamental solution which entries can be estimated in a very effective (two steps) way.

- We restrict our considerations to $\{\xi : |\xi| \geq M\}$, where $M = M(t)$ is large and is allowed to depend on the time variable. This allows us to work completely either in $Z_{hyp}(\varepsilon)$ or in $Z_{ell}(\varepsilon)$.

- We make statements about the smoothing effect for $u(t,\cdot)$ by using the representation formula in the zone of interest for every fixed time t.

5.2 Gevrey space regularity

We devote to definitions of the Gevrey space regularity and qualitative properties such as the regularity of a function which we will define in the following.

Definition 5.2.1. *A given function $u : \mathbb{R}^n \to \mathbb{R}$ belongs to the Gevrey space $\Gamma^{a,s}(\mathbb{R}^n)$ if and only if there exist positive real constants a and s such that $F_{x\to\xi}(u)(\xi)\exp\left(a\langle\xi\rangle^{\frac{1}{s}}\right) \in L^2(\mathbb{R}^n)$. Here $F_{x\to\xi}(u)$ denotes the Fourier transform of u. We write $u \in \Gamma^{a,s}(\mathbb{R}^n)$. By $\Gamma^s(\mathbb{R}^n)$ we denote the inductive limit of all spaces $\Gamma^{a,s}(\mathbb{R}^n)$, that is, $\Gamma^s(\mathbb{R}^n) := \bigcup_{a>0}\Gamma^{a,s}(\mathbb{R}^n)$.*

More precisely, we can define the regularity of Gevrey-Sobolev spaces.

Definition 5.2.2. *A given function $u : \mathbb{R}^n \to \mathbb{R}$ belongs to the Gevrey-Sobolev space $\Gamma^{a,s,\rho}(\mathbb{R}^n)$ if and only if there exist positive constants a, s and a real constant $\rho \in \mathbb{R}$ such that $F_{x\to\xi}(u)(\xi)\langle\xi\rangle^\rho \exp\left(a\langle\xi\rangle^{\frac{1}{s}}\right) \in L^2(\mathbb{R}^n)$. We write $u \in \Gamma^{a,s,\rho}(\mathbb{R}^n)$. By $\Gamma^{s,\rho}(\mathbb{R}^n)$ we denote the inductive limit of all spaces $\Gamma^{a,s,\rho}(\mathbb{R}^n)$, that is, $\Gamma^{s,\rho}(\mathbb{R}^n) := \bigcup_{a>0}\Gamma^{a,s,\rho}(\mathbb{R}^n)$.*

Remark 5.2.1. *If $u \in \Gamma^{a,s,\rho}(\mathbb{R}^n)$ we can modify the constant a such that $u \in \Gamma^{a,s}(\mathbb{R}^n)$.*

5.3 Smoothing effect for decreasing coefficients in the dissipation

In this section we study the special case of (5.0.1) if $b = b(t)$ is a decreasing positive function in the model

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + b(t)(-\Delta)^\delta u_t = 0, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \delta \in (0,\sigma], \quad \sigma > 1. \end{cases} \tag{5.3.1}$$

5.3.1 Treatment in the case $\delta \in (0, \sigma/2]$

In this section we study the special case of (5.3.1) for $\delta \in (0, \sigma/2]$. Applying the partial Fourier transformation and the "dissipative" transformation to (5.3.1) leads to a Cauchy problem with micro-local mass term $m(t,\xi) = |\xi|^{2\sigma} - \frac{b^2(t)}{4}|\xi|^{4\delta} - \frac{b'(t)}{2}|\xi|^{2\delta}$, where the sign of the coefficient $m = m(t,\xi)$ is positive for large frequencies. For this reason we can completely work in $Z_{hyp}(\varepsilon)$ (see also Figure 10.1, 10.2 or 10.3 in the Appendix). To derive the representation of solution for large frequencies we only need the following properties for the coefficient $b = b(t)$:

(A1) *positivity* : $b(t) > 0$ for all $t \geq 0$,

(A2) *decreasing* : $b'(t) < 0$ for all $t \geq 0$.

By the treatment in the hyperbolic zone from Section 3.3.2 we get the following representation of the solution

$$\begin{pmatrix} p(t,\xi)v(t,\xi) \\ D_t v(t,\xi) \end{pmatrix} = T(t,0,\xi) \begin{pmatrix} p(0,\xi)v(0,\xi) \\ D_t v(0,\xi) \end{pmatrix}, \tag{5.3.2}$$

where we introduce $T(t,0,\xi) := ME_d(t,0,\xi)Q(t,0,\xi)M^{-1}$. The matrices $E_d = E_d(t,0,\xi)$, $Q = Q(t,0,\xi)$ and M are defined in Section 3.3.2. So, we can write

$$p(t,\xi)v(t,\xi) = T^{11}(t,0,\xi)p(0,\xi)v_0(\xi) + T^{12}(t,0,\xi)D_t v(0,\xi). \tag{5.3.3}$$

By the backward "dissipative" transformation we have

$$\begin{aligned} &\exp\Big(\frac{|\xi|^{2\delta}}{2}\int_0^t b(\tau)d\tau\Big)p(t,\xi)\hat{u}(t,\xi) \\ &\quad = \Big(T^{11}(t,0,\xi)p(0,\xi) - iT^{12}(t,0,\xi)\frac{b(0)}{2}|\xi|^{2\delta}\Big)\hat{u}(0,\xi) - iT^{12}(t,0,\xi)\hat{u}_t(0,\xi). \end{aligned}$$

Using this representation and the equivalence $p(t,\xi) \sim |\xi|^\sigma$ in $Z_{hyp}(\varepsilon)$ the solution $\hat{u} = \hat{u}(t,\xi)$ satisfies the following estimate for large frequencies:

$$\int_{|\xi|\geq M} \exp\Big(|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)(1+|\xi|^2)^\sigma|\hat{u}(t,\xi)|^2 d\xi \lesssim \int_{|\xi|\geq M}(1+|\xi|^2)^\sigma|\hat{u}_0(\xi)|^2 d\xi + \int_{|\xi|\geq M}|\hat{u}_1(\xi)|^2 d\xi.$$

By Definition 5.2.2 we can conclude immediately the following statement:

Theorem 5.3.1. *Let us consider the Cauchy problem (5.3.1) in the case $\delta \in (0, \sigma/2]$, where $b = b(t)$ satisfies the assumptions (A1) and (A2). For the data we assume $u_0 \in H^\sigma$ and $u_1 \in L^2$. Then the solution $u(t,\cdot)$ belongs for all $t > 0$ to the Gevrey-Sobolev space $\Gamma^{a(t),\frac{1}{2\delta},\sigma}(\mathbb{R}^n)$, where $a(t) := \frac{1}{2}\int_0^t b(\tau)d\tau$.*

5.3.2 Treatment in the case $\delta \in (\sigma/2, \sigma]$

In this section we study the special case of (5.3.1) for $\delta \in (\sigma/2, \sigma)$. Using the same procedure we get that the mass term $m = m(t,\xi)$ is negative for large frequencies. For this reason for every fixed time t we restrict our considerations to $\{\xi : |\xi| \geq M(t)\}$ with $M(t)$ large, such that we work completely in $Z_{ell}(\varepsilon)$ (see also the Figure 10.4 in the Appendix). To derive the representation of solution we need the following properties for the coefficient $b = b(t)$:

(A1) *positivity* : $b(t) > 0$ for all $t \geq 0$,

(A2) *decreasing behavior* : $b'(t) < 0$ for all $t \geq 0$,

(A4) *higher order derivatives* : it holds $|d_t^k b(t)| \leq C_k b(t)\big(\frac{1}{1+t}\big)^k$ for $k = 1, 2$,

(A5) *useful inequalities* : there exist positive constants C_0, C_1, C_2 which are independent of t such that

$$C_0 \frac{b(t)}{\Lambda(t)} \leq -\frac{b'(t)}{b(t)} \leq C_1 \frac{1}{1+t} \leq C_2 \frac{b(t)}{\Lambda(t)} \quad \text{with} \quad \Lambda(t) = 1 + \int_0^t b(\tau)d\tau.$$

By the treatment in the elliptic zone from Section 3.3.4 we get with a suitable weight function $w = v(t,\xi)$ the representation

$$\begin{pmatrix} |\xi|^\sigma \hat{u}(t,\xi) \\ D_t \hat{u}(t,\xi) \end{pmatrix} = \exp\Big(\int_0^t w(\tau,\xi)d\tau\Big) G(t,0,\xi) \begin{pmatrix} |\xi|^\sigma \hat{u}(0,\xi) \\ D_t \hat{u}(0,\xi) \end{pmatrix}, \tag{5.3.4}$$

where

$$G(t,0,\xi) := M(t,\xi)\mathcal{N}_1(t,\xi)Q(t,0,\xi)\mathcal{N}_1^{-1}(0,\xi)M^{-1}(0,\xi).$$

The matrices $M(t,\xi)$, $Q = Q(t,0,\xi), \mathcal{N}_1(t,\xi)$ and the weight function $w = w(t,\xi)$ are defined in Section 3.3.4. Here, we can write

$$\exp\Big(-\int_0^t w(\tau,\xi)d\tau\Big)|\xi|^\sigma \hat{u}(t,\xi) = G^{11}(t,0,\xi)|\xi|^\sigma \hat{u}_0(\xi) + G^{12}(t,0,\xi)D_t\hat{u}(0,\xi). \tag{5.3.5}$$

Using the proof to Lemma 3.3.9 we have the equivalence $w(t,\xi) \sim -\frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}$ in $Z_{ell}(\varepsilon)$, and, the solution $\hat{u} = \hat{u}(t,\xi)$ satisfies the following estimate for large frequencies:

$$\int_{|\xi|\geq M(t)} \exp\Big(2|\xi|^{2\sigma-2\delta}\int_0^t \frac{1}{b(\tau)}d\tau\Big)(1+|\xi|^2)^\sigma |\hat{u}(t,\xi)|^2 d\xi \lesssim \int_{|\xi|\geq M(t)} (1+|\xi|^2)^\sigma |\hat{u}_0(\xi)|^2 d\xi + \int_{|\xi|\geq M(t)} |\hat{u}_1(\xi)|^2 d\xi.$$

By Definition 5.2.2 we can conclude immediately the following statement:

Theorem 5.3.2. *Let us consider the Cauchy problem (5.3.1) in the case $\delta \in (\sigma/2, \sigma)$, where $b = b(t)$ satisfies the assumptions (A1), (A2), (A4) and (A5). For the data we assume $u_0 \in H^\sigma$ and $u_1 \in L^2$. Then the solution $u(t,\cdot)$ belongs for all $t > 0$ to the Gevrey space $\Gamma^{\frac{1}{2\sigma-2\delta}}(\mathbb{R}^n)$.*

5.4 Smoothing effect for increasing coefficients in the dissipation

In this section we study the special case of (5.0.1) if $b = b(t)$ is a increasing positive function in the model

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + b(t)(-\Delta)^\delta u_t = 0, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \delta \in (0,\sigma), \quad \sigma > 1. \end{cases} \tag{5.4.1}$$

5.4.1 Treatment in the case $\delta \in (0, \sigma/2]$

In this section we study the special case of (5.4.1) for $\delta \in (0,\sigma/2]$. Analogous to the previous subsections we get that the coefficient $m = m(t,\xi)$ is positive for large frequencies. For every fixed time t we restrict our considerations to $\{\xi : |\xi| \geq M\}$, where $M = M(t)$ is large, such that we work completely in $Z_{hyp}(\varepsilon)$ (see also the Figure 10.7 in the Appendix). To derive the representation of solution we need the following properties for the coefficient $b = b(t)$:

(B1) *positivity* : $b(t) > 0$ for all $t \geq 0$,

(B2) *increasing behavior* : $b'(t) > 0$ for all $t \geq 0$.

In the same way as in Section 5.3.1 we get the following estimate for large frequencies:

$$\int_{|\xi|\geq M(t)} \exp\Big(|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)(1+|\xi|^2)^\sigma|\hat{u}(t,\xi)|^2 d\xi \lesssim \int_{|\xi|\geq M(t)}(1+|\xi|^2)^\sigma|\hat{u}_0(\xi)|^2 d\xi$$
$$+ \int_{|\xi|\geq M(t)} |\hat{u}_t(0,\xi)|^2 d\xi \lesssim \int_{\mathbb{R}^n}\big((1+|\xi|^2)^\sigma|\hat{u}_0(\xi)|^2 + |\hat{u}_t(0,\xi)|^2\big)d\xi.$$

By Definition 5.2.2 we can conclude immediately the following statement:

Theorem 5.4.1. *Let us consider the Cauchy problem (5.4.1) in the case $\delta \in (0,\sigma/2)$, where $b = b(t)$ satisfies the assumptions (B1) and (B2). For the data we assume $u_0 \in H^\sigma$ and $u_1 \in L^2$. Then the solution $u(t,\cdot)$ belongs for all $t > 0$ to the Gevrey space $\Gamma^{\frac{1}{2\delta}}(\mathbb{R}^n)$.*

5.4.2 Treatment in the case $\delta \in (\sigma/2, \sigma)$

In this section we study the special case of (5.4.1) for $\delta \in (\sigma/2,\sigma)$. By the same technique we get that the coefficient $m = m(t,\xi)$ is negative for large frequencies. For this reason we can completely work in $Z_{ell}(\varepsilon)$ (see also the Figure 10.5, 10.6 or 10.8 in the Appendix). To derive the representation of solution we need the following properties for the coefficient $b = b(t)$:

(B1) *positivity* : $b(t) > 0$ for all $t \geq 0$,

(B2) *increasing behavior*: $b'(t) > 0$ for all $t \geq 0$,

(B4) *higher order derivatives* : it holds $|d_t^k b(t)| \leq C_k b(t)\big(\frac{1}{1+t}\big)^k$ for $k = 1, 2$, (B5) *useful inequalities* : there exist positive constant C_0, C_1, C_2 which are independent of t such that

$$C_0 \frac{b(t)}{\Lambda(t)} \leq \frac{b'(t)}{b(t)} \leq C_1 \frac{1}{1+t} \leq C_2 \frac{b(t)}{\Lambda(t)} \quad \text{with} \quad \Lambda(t) = 1 + \int_0^t b(\tau) d\tau.$$

In the same way as in Section 5.3.2 we get the following estimate for large frequencies:

$$\int_{|\xi|\geq M} \exp\Big(2|\xi|^{2\sigma-2\delta} \int_0^t \frac{1}{b(\tau)} d\tau\Big)(1+|\xi|^2)^\sigma |\hat{u}(t,\xi)|^2 d\xi \lesssim \int_{|\xi|\geq M} (1+|\xi|^2)^\sigma |\hat{u}_0(\xi)|^2 d\xi + \int_{|\xi|\geq M} |\hat{u}_1(\xi)|^2 d\xi.$$

By Definition 5.2.2 we can conclude immediately the following statement:

Theorem 5.4.2. *Let us consider the Cauchy problem (5.4.1) in the case $\delta \in (\sigma/2, \sigma)$, where $b = b(t)$ satisfies the assumptions (B1), (B2), (B4) and (B5). For the data we assume $u_0 \in H^\sigma$ and $u_1 \in L^2$. Then the solution $u(t,\cdot)$ belongs for all $t > 0$ to the Gevrey-Sobolev space $\Gamma^{a(t), \frac{1}{2\sigma-2\delta}, \sigma}(\mathbb{R}^n)$, where $a(t) := \int_0^t \frac{1}{b(\tau)} d\tau$.*

Remark 5.4.1. *To derive estimates for the energies of higher order we assumed that b or $1/b$ are non-integrable. But to explain smoothing property we do not exclude integrable b or $1/b$. For such integrable functions we have no parabolic effect (see the main results of Chapters 3 and 4), but we have Gevrey-Sobolev smoothing property. In the last cases the time-dependent parameter $a(t)$ tends to a finite limit in the suitable Gevrey-Sobolev spaces.*

6 Optimality – Scale invariant models

The question for the optimality of the results from Chapters 2 to 5 remained open up to now. In this chapter we devote to give an answer. For this reason we consider scale invariant models.

6.1 Scale invariant models

In this chapter we focus our considerations to scale-invariant models. It turns out that this class of models provides us with a lot of ideas and gives some feeling for optimality of the general results given in Chapters 3 and 4. We are concerned with the following structural damped $\sigma-$evolution models :

$$\begin{cases} u_{tt}(t,x)+(-\Delta)^\sigma u(t,x)+\mu(1+t)^\gamma(-\Delta)^\delta u_t(t,x)=0, \qquad (t,x)\in(0,\infty)\times\mathbb{R}^n, \\ u(0,x)=:u_0(x), \quad u_t(0,x)=:u_1(x), \quad \delta\in(0,\sigma),\ \sigma>1, \quad \text{and} \quad \gamma\in\mathbb{R}. \end{cases} \tag{6.1.1}$$

The coefficient μ is a positive real number. This Cauchy problem is of particular interest for several reasons. At first this equation has more symmetries than other problems with time-dependent dissipation. Let $u=u(t,x)$ be a solution to the equation from (6.1.1). If we introduce the change of variables according to

$$u(1+t^*,x^*):=u(\lambda(1+t),\lambda^p x),$$

then we try to find a parameter p such that the function $u(1+t^*,x^*):=u(\lambda(1+t),\lambda^p x)$ is a solution as well. For those parameters the equation from (6.1.1) becomes scale-invariant. As we will see later this implies that we can compute explicit representations of solutions in terms of known special functions.

Definition 6.1.1. *The energy of higher order for solutions to (6.1.1) is defined for $m\geq\sigma$ by*

$$E^m[u](t):=\sum_{|\beta|=m}\|\nabla^\beta u_t(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}+\sum_{|\beta|=m}\|\nabla^\beta|D|^\sigma u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}. \tag{6.1.2}$$

6.2 Objectives and strategies

The main objective in this chapter is the proof of the optimality of the energies estimates. These estimates rely on more structural properties of representations of solutions than estimates in the L^2-scale and can not be deduced by the same methods as the above mentioned results. Let us explain our strategy. It is divided into the following steps:

- First we find scale invariant models.
- Then we transform these models to ordinary differential equations. So, the asymptotic properties of solutions to these models are of interest.
- If our considerations can be restricted to the hyperbolic zone, then we prove the optimality of these estimates by applying scattering theory.
- If our considerations can be restricted to the elliptic zone, then we prove the optimality of these estimates by using asymptotic theory for solutions to ordinary differential equations.
- The obtained asymptotic estimates for solutions to ordinary differential equations are transferred to solutions to the scale-invariant cases for (6.1.1).

6.3 Reduction to ordinary differential equations

Definition 6.3.1. *The differential equation from (6.1.1) becomes scale-invariant if there exist real parameters δ, γ and p such that $u(1+t,x)$ and $u(1+t^*,x^*) := u(\lambda(1+t),\lambda^p x)$ are solutions to the equation from (6.1.1) as well for $\lambda \neq 0$.*

Suppose that $u = u(t,x)$ is a solution of the equation from (6.1.1), that is, it solves

$$u_{tt}(t,x) + (-\Delta)^\sigma u(t,x) + \mu(1+t)^\gamma(-\Delta)^\delta u_t(t,x) = 0.$$

Then we ask for which δ and γ we can find a p such that $u(1+t^*,x^*) := u(\lambda(1+t),\lambda^p x)$ is a solution as well. Some straight-forward calculations imply

$$\begin{aligned}&\big(\partial_t^2 + (-\Delta_x)^\sigma + \mu(1+t)^\gamma(-\Delta_x)^\delta\partial_t\big)u(1+t^*,x^*)\\ &\quad= \big(\lambda^2\partial_{t^*}^2 + \lambda^{2p\sigma}(-\Delta_{x^*})^\sigma + \lambda^{2p\delta+1-\gamma}\mu(1+t^*)^\gamma(-\Delta_{x^*})^\delta\partial_{t^*}\big)u(1+t^*,x^*).\end{aligned}$$

Hence, $u(1+t^*,x^*)$ is a solution of the equation from (6.1.1) if

$$\begin{cases} 2p\sigma = 2,\\ 2p\delta+1-\gamma = 2.\end{cases} \qquad \text{Hence, we get the coherence} \quad \gamma = \frac{2\delta-\sigma}{\sigma}.$$

Corollary 6.3.1. *The following models (6.1.1) become scale-invariant :*

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\sigma u(t,x) + \mu(1+t)^{\frac{2\delta-\sigma}{\sigma}}(-\Delta)^\delta u_t(t,x) = 0, \qquad (t,x)\in(0,\infty)\times\mathbb{R}^n,\\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x),\ \delta\in(0,\sigma),\ \sigma>1.\end{cases} \tag{6.3.1}$$

Applying the partial Fourier transformation $\hat{u}(t,\xi) = F_{x\to\xi}(u)(t,\xi)$ to Cauchy problem (6.3.1) we obtain

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + \mu|\xi|^{2\delta}(1+t)^{\frac{2\delta-\sigma}{\sigma}}\hat{u}_t = 0, \quad \hat{u}(0,\xi) := \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) := \hat{u}_1(\xi). \tag{6.3.2}$$

In the following we shall restrict ourselves to solutions $\hat{u}$ which are constant on manifolds described through the equation $\tau = (1+t)|\xi|^\sigma$. For this reason we propose the transformations

$$\tau := (1+t)|\xi|^\sigma, \qquad \hat{u}(t,\xi) := \tilde{v}((1+t)|\xi|^\sigma). \tag{6.3.3}$$

Applying these transformations to (6.3.2) we get the Cauchy problem

$$\tilde{v}_{\tau\tau}(\tau)+\tilde{v}(\tau)+\mu\tau^{\frac{2\delta-\sigma}{\sigma}}\tilde{v}_\tau(\tau)=0,\quad \tilde{v}(r):=\tilde{v}_0(r),\quad \tilde{v}_\tau(r):=\tilde{v}_1(r),\quad r=|\xi|^\sigma\geq 0.$$

Corollary 6.3.2. *Applying the transformations (6.3.3) the Cauchy problem*

$$\hat{u}_{tt}+|\xi|^{2\sigma}\hat{u}+\mu|\xi|^{2\delta}(1+t)^{\frac{2\delta-\sigma}{\sigma}}\hat{u}_t=0,\quad \hat{u}(0,\xi):=\hat{u}_0(|\xi|^\sigma),\quad \hat{u}_t(0,\xi):=\hat{u}_1(|\xi|^\sigma)$$

is equivalent to the following Cauchy problem with data at $\tau=r=|\xi|^\sigma$:

$$\begin{cases} \tilde{v}_{\tau\tau}(\tau)+\mu\tau^{\frac{2\delta-\sigma}{\sigma}}\tilde{v}_\tau(\tau)+\tilde{v}(\tau)=0,\quad \tau\in[r,\infty),\\ \tilde{v}(r):=\tilde{v}_0(r),\quad \tilde{v}_\tau(r):=\tilde{v}_1(r),\quad \delta\in(0,\sigma),\ \sigma>1. \end{cases} \tag{6.3.4}$$

6.4 Estimates for energies of higher order - Optimality

6.4.1 Treatment in the case $\delta\in(0,\sigma/2]$

In this section we study the special case of (6.3.1) for $\delta\in(0,\sigma/2],\ \sigma>1$. The corresponding Cauchy problem is

$$\tilde{v}_{\tau\tau}(\tau)+\mu\tau^{\frac{2\delta-\sigma}{\sigma}}\tilde{v}_\tau(\tau)+\tilde{v}(\tau)=0,\quad \tilde{v}(r):=\tilde{v}_0(r),\quad \tilde{v}_\tau(r):=\tilde{v}_1(r),\quad \delta\in(0,\sigma/2]. \tag{6.4.1}$$

To get appropriate estimates we can choose N large such that $N>\big(2/\mu\big)^{\frac{\sigma}{2\delta-\sigma}}$. We divide our considerations into two cases : $N\geq r$ or $N\leq r$.

1. In the case $N\geq r$ we divide the domain of τ into two zones $Z_1:=\{\tau:\ \tau\geq N\}$ and $Z_2:=\{\tau:\ r\leq\tau\leq N\}$.
2. In the case $N\leq r$ we divide the domain of admissible τ in one zone $Z_1:=\{\tau:\ \tau\geq r\}$.

Let us devote to the second case.
Treatment in Z_1. We investigate the "dissipative" transformed problem

$$\begin{cases} v_{\tau\tau}(\tau)+\underbrace{\big(1-\frac{\mu^2}{4}\tau^{\frac{4\delta-2\sigma}{\sigma}}\big)}_{=:a^2(\tau)}v(\tau)-\underbrace{\frac{\mu(2\delta-\sigma)}{2\sigma}\tau^{\frac{2\delta-2\sigma}{\sigma}}}_{:=b(\tau)}v(\tau)=0,\qquad \tau\in[r,\infty),\\ v(r)=\exp\big(\frac{\mu\sigma}{4\delta}r^{\frac{2\delta}{\sigma}}\big)\tilde{v}_0(r),\quad v_\tau(r)=\frac{\mu}{2}r^{\frac{2\delta-\sigma}{\sigma}}\exp\big(\frac{\mu\sigma}{4\delta}r^{\frac{2\delta}{\sigma}}\big)\tilde{v}_0(r)+\exp\big(\frac{\mu\sigma}{4\delta}r^{\frac{2\delta}{\sigma}}\big)\tilde{v}_1(r), \end{cases} \tag{6.4.2}$$

where $v(\tau):=\exp\big(\frac{\mu}{2}\int_0^\tau s^{\frac{2\delta-\sigma}{\sigma}}ds\big)\tilde{v}(\tau)$ is the "dissipative" transformation and $\tilde{v}(\tau)$ solves the Cauchy problem (6.3.4) for $\delta\in(0,\sigma/2]$. By applying the diagonalization procedure (see in Appendix Section 10.1.1) we get after first step of diagonalization that the entries of the matrix $\mathcal{R}_1(\tau)$ are uniformly integrable. For this reason the matrix $\mathcal{R}_1(\tau)$ belongs to $L^1_{loc}\big(Z_1\big)$. We can write $V^{(1)}(\tau)=E_1(\tau,s)V^{(1)}(s)$, where $E_1(\tau,s)=E_d(\tau,s)Q(\tau,s)$ is the fundamental solution, that is the solution to the system

$$d_\tau E_1(\tau,s)=\big(\mathcal{D}(\tau)+\mathcal{R}(\tau)\big)E_1(\tau,s),\quad E_1(s,s)=I_2\quad \text{for all}\ \ s,\tau\geq r. \tag{6.4.3}$$

First we take the fundamental solution $E_d = E_d(\tau, s)$ of the "diagonal part", that is,

$$d_\tau E_d(\tau, s) = \mathcal{D}(\tau) E_d(\tau, s), \quad E_d(s, s) = I_2 \quad \text{for all} \;\; s, \tau \geq r. \tag{6.4.4}$$

Then the following estimate holds:

$$\|E_d(\tau, s)\| \leq C \quad \text{for all} \;\; s, \tau \geq r.$$

The matrix $Q(\tau, s)$ satisfies

$$d_\tau Q(\tau, s) = P_1(\tau, s) Q(\tau, s), \quad Q(s, s) = I_2 \quad \text{for all} \;\; s, \tau \geq r,$$

where $P_1(\tau, s) := E_d^{-1}(\tau, s)\mathcal{R}(\tau) E_d(\tau, s)$. Applying the Peano-Baker formula from Proposition 10.1.10 (see Appendix) we get

$$\|Q(\tau, s)\| \leq \exp\Big(\int_s^\tau \|\mathcal{R}(\tau)\| d\tau \Big) \leq C \quad \text{for all} \;\; s, \tau \geq r.$$

Hence,

$$\|E_1(\tau, s)\| = \|E_d(\tau, s) Q(\tau, s)\| \leq C \quad \text{for all} \;\; s, \tau \geq r.$$

In Z_1 we get

$$V(\tau) = E(\tau, r) V(r) \quad \text{for all} \;\; \tau \geq r, \tag{6.4.5}$$

where

$$E(\tau, r) := \mathcal{N} E_d(\tau, r) Q(\tau, r) \mathcal{N}^{-1} \quad \text{for all} \;\; \tau \geq r. \tag{6.4.6}$$

From the backward transformation $\tilde{v}(\tau) := \exp\big(-\frac{\mu}{2} \int_0^\tau s^{\frac{2\delta-\sigma}{\sigma}} ds \big) v(\tau)$ we have

$$|\tilde{v}(\tau)| \lesssim \exp\big(-\frac{\mu}{2}(\tau^{\frac{2\delta}{\sigma}} - r^{\frac{2\delta}{\sigma}})\big)\big(|\tilde{v}_0(r)| + |\tilde{v}_1(r)|\big),$$
$$|\tilde{v}_\tau(\tau)| \lesssim \exp\big(-\frac{\mu}{2}(\tau^{\frac{2\delta}{\sigma}} - r^{\frac{2\delta}{\sigma}})\big)\big(|\tilde{v}_0(r)| + |\tilde{v}_1(r)|\big).$$

Let us consider the first case.
Treatment in Z_1. The same approach leads to the estimates

$$|\tilde{v}(\tau)| \lesssim \exp\big(-\frac{\mu}{2}(\tau^{\frac{2\delta}{\sigma}} - N^{\frac{2\delta}{\sigma}})\big)\big(|\tilde{v}_0(N)| + |\tilde{v}_1(N)|\big),$$
$$|\tilde{v}_\tau(\tau)| \lesssim \exp\big(-\frac{\mu}{2}(\tau^{\frac{2\delta}{\sigma}} - N^{\frac{2\delta}{\sigma}})\big)\big(|\tilde{v}_0(N)| + |\tilde{v}_1(N)|\big).$$

Treatment in Z_2. Here we introduce the micro-energy $\tilde{V}(\tau) := \Big(\tilde{v}(\tau), \tilde{v}_\tau(\tau)\Big)^T$. It satisfies the system of first order

$$\partial_\tau \tilde{V}(\tau) = \begin{pmatrix} 0 & 1 \\ -1 & -\mu\tau^{\frac{2\delta-\sigma}{\sigma}} \end{pmatrix} \tilde{V}(\tau), \qquad \tilde{V}_0(r) := \Big(\tilde{v}_0, \tilde{v}_1\Big)^T.$$

For bounded τ the matrix $A(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & -\mu\tau^{\frac{2\delta-\sigma}{\sigma}} \end{pmatrix}$ is integrable, $A(\tau) \in L^1(Z_2)$. Due to this fact we obtain

$$\Big| \begin{pmatrix} \tilde{v}(\tau) \\ \tilde{v}_\tau(\tau) \end{pmatrix} \Big| \leq C_N \Big| \begin{pmatrix} \tilde{v}_0(r) \\ \tilde{v}_1(r) \end{pmatrix} \Big| \quad \text{for} \quad r \leq \tau \leq N. \tag{6.4.7}$$

Summarizing we arrived at the following statement:

Proposition 6.4.1. *The following estimate holds of the Cauchy problem (6.4.1) for* $\tau \geq r \geq 0$ *:*

$$|\tilde{v}(\tau)| \lesssim \exp\big(-\frac{\mu}{2}(\tau^{\frac{2\delta}{\sigma}} - r^{\frac{2\delta}{\sigma}})\big)\big(|\tilde{v}_0(r)| + |\tilde{v}_1(r)|\big),$$
$$|\tilde{v}_\tau(\tau)| \lesssim \exp\big(-\frac{\mu}{2}(\tau^{\frac{2\delta}{\sigma}} - r^{\frac{2\delta}{\sigma}})\big)\big(|\tilde{v}_0(r)| + |\tilde{v}_1(r)|\big).$$

Scattering results

Our goal is to show that the behavior of the solution $v(\tau)$ of (6.4.2) with Cauchy data (v_0, v_1) coincides asymptotically with the behavior of the solution for some reference system with related data (v_0^1, v_1^1) in a special sense. We will use that the energy of the solution for those reference system is constant. The operator relating the data (v_0, v_1) to (v_0^1, v_1^1) will be denoted as Mfiller wave operator following the conventions from the scattering theory for wave equations. Now we define another reference system

$$\bar{V}_\tau(\tau) = \begin{pmatrix} 0 & a(\tau) \\ -a(\tau) & 0 \end{pmatrix} \bar{V}(\tau), \quad \bar{V}(r) = \bar{V}_0 \quad \text{for} \quad \tau \geq r. \tag{6.4.8}$$

Using the same procedure as before after a single step of diagonalization we get

$$\bar{V}(\tau) = \bar{E}(\tau, s)\bar{V}(s) \quad \text{for all} \;\; s, \tau \geq r, \tag{6.4.9}$$

where

$$\bar{E}(\tau, s) := \mathcal{N} E_d(\tau, s)\mathcal{N}^{-1} \quad \text{for all} \;\; s, \tau \geq r. \tag{6.4.10}$$

The matrix-valued functions $\mathcal{N}$ and $E_d = E_d(\tau, s)$ coincide with those from above. We construct an operator mapping the Cauchy data $V(r) = (a(r)v(r), v_\tau(r))^T$ from (??) to the Cauchy data $\bar{V}(r) = (a(r)\bar{v}(r), \bar{v}_\tau(r))^T$ to the reference system (6.4.8). For this reason we introduce

$$W_+(r) := \lim_{\tau\to\infty} \bar{E}^{-1}(\tau, r)E(\tau, r) = \lim_{\tau\to\infty} \mathcal{N} E_d(r, \tau)E_1(\tau, r)\mathcal{N}^{-1}.$$

Taking account of

$$\mathcal{N} E_d(r, \tau)E_1(\tau, r)\mathcal{N}^{-1} = \mathcal{N} Q(\tau, r)\mathcal{N}^{-1}$$

brings

$$W_+(r) := \mathcal{N} \lim_{\tau\to\infty} Q(\tau, r)\mathcal{N}^{-1}.$$

To prove the existence of $W_+(r)$ as an operator from $L(\mathbb{R}^2, \mathbb{R}^2)$ we show that $\{Q(\tau_k, r)\}_k$ is a Cauchy sequence for all sequences $\{\tau_k\}_k$ with $\tau_k \to \infty$. Let us fix a constant R large enough and $\tau, \tau' \geq R$. We obtain

$$\|Q(\tau, r) - Q(\tau', r)\| = \Big\| \sum_{k=1}^{\infty} \int_r^{\tau} P(\tau_1, r)J_k(\tau_1, r)d\tau_1 - \int_r^{\tau'} P(\tau_1, r)J_k(\tau_1, r)d\tau_1 \Big\|$$
$$= \Big\| \sum_{k=1}^{\infty} \int_{\tau'}^{\tau} P(\tau_1, r)J_k(\tau_1, r)d\tau_1 \Big\| \leq \exp\Big(\int_{\tau'}^{\tau} \|\mathcal{R}(\tau)\| d\tau \Big) \leq \varepsilon(R)$$

uniformly for all τ, $\tau' \geq R$. Consequently, the Mfiller wave operator $W_+(r) = \mathcal{N} \lim_{\tau\to\infty} Q(\tau, r)\mathcal{N}^{-1}$ exists in $L(\mathbb{R}^2, \mathbb{R}^2)$. To complete the scattering we examine the behavior of V and $\bar{V}$ by taking the limit

$$\begin{aligned}
\lim_{\tau\to\infty} \|V(\tau) - \bar{V}(\tau)\| &= \lim_{\tau\to\infty} \|E(\tau, r)V(r) - \bar{E}(\tau, r)\bar{V}(r)\| \\
&= \lim_{\tau\to\infty} \|\mathcal{N} E_1(\tau, r)\mathcal{N}^{-1}V(r) - \mathcal{N} E_d(\tau, r)\mathcal{N}^{-1}W_+(r)V(r)\| \\
&= \lim_{\tau\to\infty} \|\mathcal{N} E_d(\tau, r)Q(\tau, r)\mathcal{N}^{-1}V(r) - \mathcal{N} E_d(\tau, r) \lim_{\tau'\to\infty} E_d(r, \tau')E_1(\tau', r)\mathcal{N}^{-1}V(r)\| \\
&= \lim_{\tau\to\infty} \|\mathcal{N} E_d(\tau, r)\mathcal{L}(\tau, r)\mathcal{N}^{-1}V(r)\|
\end{aligned}$$

with

$$\mathcal{L}(\tau,r) := Q(\tau,r) - \lim_{\tau'\to\infty} E_d(r,\tau')E_1(\tau',r).$$

Due to the existence of $W_+(r)$ we arrive at $\lim_{\tau\to\infty}\mathcal{L}(\tau,r)=0$ uniformly. By Definition 6.1.1 we can conclude immediately the following statement :

Theorem 6.4.2. *Let us consider the Cauchy problem (6.3.1) in the case $\delta\in(0,\sigma/2]$. Then the solution $u=u(t,x)$ satisfies the following estimates for the energies of higher order :*

$$E^m[u](t) \lesssim (1+t)^{-2\frac{m+\sigma}{\sigma}}\|u_0\|^2_{H^{m+\sigma}(\mathbb{R}^n)} + (1+t)^{-2(\frac{m}{\sigma})}\|u_1\|^2_{H^m(\mathbb{R}^n)}.$$

Remark 6.4.1. *This result coincides with the estimates for $E^m[u](t)$ in the case of decreasing dissipation from Theorem 3.3.29. Furthermore, we can see that for the scale-invariant models the decay function for $E^m[u](t)$ is independent of the parameters μ, δ.*

6.4.2 Treatment in the case $\delta\in(\sigma/2,\sigma)$

In this section we study the special case (6.3.1) for $\delta\in(\sigma/2,\sigma)$. The corresponding Cauchy problem is

$$\tilde v_{\tau\tau}(\tau) + \mu\tau^{\frac{2\delta-\sigma}{\sigma}}\tilde v_\tau(\tau) + \tilde v(\tau) = 0, \quad \tilde v(r) := \tilde v_0(r), \quad \tilde v_\tau(r) := \tilde v_1(r), \quad \delta\in(\sigma/2,\sigma), \ \sigma>1. \tag{6.4.11}$$

We proceed as in the other case. So we investigate the "dissipative" transformed problem

$$\begin{cases} v_{\tau\tau}(\tau) - \underbrace{\Big(\frac{\mu^2}{4}\tau^{\frac{4\delta-2\sigma}{\sigma}} - 1\Big)}_{=:a^2(\tau)} v(\tau) - \underbrace{\frac{\mu(2\delta-\sigma)}{2\sigma}\tau^{\frac{2\delta-2\sigma}{\sigma}}}_{:=b(\tau)} v(\tau) = 0, \qquad \tau\in[r,\infty), \\ v(r) = \exp\big(\frac{\mu\sigma}{4\delta}r^{\frac{2\delta}{\sigma}}\big)\tilde v_0(r), \quad v_\tau(r) = \frac{\mu}{2}r^{\frac{2\delta-\sigma}{\sigma}}\exp\big(\frac{\mu\sigma}{4\delta}r^{\frac{2\delta}{\sigma}}\big)\tilde v_0(r) + \exp\big(\frac{\mu\sigma}{4\delta}r^{\frac{2\delta}{\sigma}}\big)\tilde v_1(r) \end{cases} \tag{6.4.12}$$

where $v(\tau) := \exp\big(\frac{\mu}{2}\int_0^\tau s^{\frac{2\delta-\sigma}{\sigma}}ds\big)\tilde v(\tau)$ is the "dissipative" transformation and $\tilde v(\tau)$ solves the Cauchy problem (6.3.4) for $\delta\in(\sigma/2,\sigma)$. By applying the diagonalization procedure (see in Appendix Section 10.1.1) we get after second step of diagonalization that the entries of the matrix $\mathcal{R}_2(\tau)$ are uniformly integrable. For this reason the matrix $\mathcal{R}_2(\tau)$ belongs to $L^1_{loc}(Z_1)$. For large τ gives that $\mathcal{R}_2(\tau)$ is integrable. So, the remainder $\mathcal{R}_2(\tau)$ is a "better remainder" than the non-integrable $\mathcal{R}_1(\tau)$. Here, we can write $V^{(2)}(\tau) = E_2(\tau,r)V_0^{(2)}(r)$, where $E_2(\tau,s) = E_d(\tau,s)Q(\tau,s)$ is the fundamental solution, that is, the solution to

$$d_\tau E_2(\tau,s) = \big(\mathcal{D}(\tau) + \mathcal{F}^{(1)}(\tau) + \mathcal{R}_2(\tau)\big)E_2(\tau,s), \quad E_2(s,s) = I_2 \quad \text{for all } \ s,\tau\geq r. \tag{6.4.13}$$

First we take the fundamental solution $E_d = E_d(\tau,s)$ of the "diagonal part", that is, to

$$d_\tau E_d(\tau,s) = \big(\mathcal{D}(\tau) + \mathcal{F}^{(1)}(\tau)\big)E_d(\tau,s), \quad E_d(s,s) = I_2 \quad \text{for all } \ s,\tau\geq r. \tag{6.4.14}$$

Thus

$$\begin{aligned} E_d^{(11)}(\tau,r) &= \exp\Big(\int_r^\tau \Big(-a(s) + \frac{a_\tau(s)-b(s)}{2a(s)}\Big)ds, \\ E_d^{(22)}(\tau,r) &= \exp\Big(\int_r^\tau \Big(a(s) + \frac{a_\tau(s)+b(s)}{2a(s)}\Big)ds, \\ E_d^{(12)}(\tau,r) &= E_d^{(21)}(\tau,r) = 0. \end{aligned}$$

We have to find the solution $Q(\tau, r)$ to

$$d_\tau\big(E_d(\tau,r)Q(\tau,r)\big) = \big(\mathcal{D}(\tau) + \mathcal{F}^{(1)}(\tau) + \mathcal{R}_2(\tau)\big)E_d(\tau,r)Q(\tau,r), \quad Q(r,r) = I,$$

or, equivalently, to

$$d_\tau Q(\tau,r) = E_d(r,\tau)\mathcal{R}_2(\tau)E_d(\tau,r)Q(\tau,r), \quad Q(r,r) = I.$$

To rewrite this system for $Q = Q(\tau, r)$ as an integral equation for $E_2(\tau, r)$ we integrate over $[r, \tau]$ and apply the ansatz to substitute $Q(\tau, r)$. Hence, we have

$$E_2(\tau,r) = E_d(\tau,r) + \int_r^\tau E_d(r,s)\mathcal{R}_2(s)E_2(s,r)ds.$$

We see that $E_d(\tau, r)$ is not bounded. For this reason we suggest to multiply $E_d(\tau, r)$ by a weight-function $\exp\Big(-\int_r^\tau w(s)ds\Big)$ in order to recalculate an integral equation for

$$E_2^{(1)}(\tau,r) = E_2(\tau,r)\exp\Big(-\int_r^\tau w(s)ds\Big), \quad \text{where} \quad w(\tau) = a(\tau) + \frac{a_\tau(\tau)+b(\tau)}{2a(\tau)}.$$

As a result we have to consider

$$E_2^{(1)}(\tau,r) = E_d^{(1)}(\tau,r) + \int_r^\tau E_d^{(1)}(r,s)\mathcal{R}_2(s)E_2^{(1)}(s,r)ds, \quad E_2^{(1)}(r,r) = I_2$$

with

$$E_d^{(1)}(\tau,r) := \operatorname{diag}\Big(\exp\Big(\int_r^\tau\Big(-a(s) + \frac{a_\tau(s)-b(s)}{2a(s)} - w(s)\Big)ds\Big), \\ \exp\Big(\int_r^\tau\Big(a(s) + \frac{a_\tau(s)+b(s)}{2a(s)} - w(s)\Big)ds\Big)\Big).$$

Applying Gronwall's inequality we obtain $\|E_2^{(1)}(\tau,r)\| \le C$. This estimate implies

$$|V(\tau)| \le C_N \exp\big(\int_r^\tau w(s)ds\big)|V(r)|, \tag{6.4.15}$$

hence,

$$\Big|\begin{pmatrix} a(\tau)v(\tau) \\ v_\tau(\tau)\end{pmatrix}\Big| \le C_N \exp\Big(\int_r^\tau w(s)ds\Big)\Big|\begin{pmatrix} a(r)v_0(r) \\ v_1(r)\end{pmatrix}\Big| \quad \text{for all} \quad \tau \ge r.$$

From the backward transformation $v(\tau) = \exp\big(\frac{\mu\sigma}{4\delta}\tau^{2\frac{\delta}{\sigma}}\big)\tilde{v}(\tau)$ we have

$$\begin{pmatrix} a(\tau)v(\tau) \\ v_\tau(\tau)\end{pmatrix} = \exp\big(\frac{\mu\sigma}{4\delta}\tau^{2\frac{\delta}{\sigma}}\big)\begin{pmatrix} 1 & 0 \\ \frac{\mu\tau^{\frac{2\delta-\sigma}{\sigma}}}{2a(\tau)} & 1\end{pmatrix}\begin{pmatrix} a(\tau)\tilde{v}(\tau) \\ \tilde{v}_\tau(\tau)\end{pmatrix}. \tag{6.4.16}$$

Using the estimate (6.4.15) gives

$$\Big|\begin{pmatrix} 1 & 0 \\ \frac{\mu\tau^{\frac{2\delta-\sigma}{\sigma}}}{2a(\tau)} & 1\end{pmatrix}\begin{pmatrix} a(\tau)\tilde{v}(\tau) \\ \tilde{v}_\tau(\tau)\end{pmatrix}\Big| \lesssim \exp\Big(\int_r^\tau\Big(a(s) + \frac{a_\tau(s)+b(s)}{2a(s)} - \frac{\mu}{2}s^{\frac{2\delta-\sigma}{\sigma}}\Big)ds - \frac{\mu\sigma}{4\delta}r^{2\frac{\delta}{\sigma}}\Big) \\ \times\Big|\begin{pmatrix} a(r)v(r) \\ v_\tau(r)\end{pmatrix}\Big|.$$

At this point we use the asymptotic equivalence

$$a(s) \sim \frac{\mu}{2}s^{\frac{2\delta-\sigma}{\sigma}} - \frac{1}{\mu}s^{\frac{-2\delta+\sigma}{\sigma}}, \quad \text{and} \quad \frac{a_\tau(s)-b(s)}{2a(s)} \sim s^{-1},$$

and we get

$$\Big|\begin{pmatrix} 1 & 0 \\ \frac{\mu\tau^{\frac{2\delta-\sigma}{\sigma}}}{2a(\tau)} & 1 \end{pmatrix}\begin{pmatrix} a(\tau)\tilde{v}(\tau) \\ \tilde{v}_\tau(\tau) \end{pmatrix}\Big| \lesssim \Big(\frac{\tau}{r}\Big)^{\frac{2\delta-\sigma}{\sigma}} \exp\Big(-\int_r^\tau \frac{1}{\mu}s^{\frac{-2\delta+\sigma}{\sigma}}ds\Big)\exp\big(-\frac{\mu\sigma}{4\delta}r^{2\frac{\delta}{\sigma}}\big)\Big|\begin{pmatrix} a(r)v(r) \\ v_\tau(r) \end{pmatrix}\Big|.$$

After integration and using (6.4.16) we may conclude

$$\Big|\begin{pmatrix} 1 & 0 \\ \frac{\mu\tau^{\frac{2\delta-\sigma}{\sigma}}}{2a(\tau)} & 1 \end{pmatrix}\begin{pmatrix} a(\tau)\tilde{v}(\tau) \\ \tilde{v}_\tau(\tau) \end{pmatrix}\Big| \lesssim \exp\Big(-\frac{\sigma}{\mu(2\sigma-2\delta)}\Big(\tau^{\frac{-2\delta+2\sigma}{\sigma}} - r^{\frac{-2\delta+2\sigma}{\sigma}}\Big)\Big)\Big|\begin{pmatrix} a(r)\tilde{v}_0(r) \\ \tilde{v}_1(r) \end{pmatrix}\Big|.$$

This yields

$$a(\tau)|\tilde{v}(\tau)| \lesssim \exp\Big(-\frac{\sigma}{\mu(2\sigma-2\delta)}\Big(\tau^{\frac{-2\delta+2\sigma}{\sigma}} - r^{\frac{-2\delta+2\sigma}{\sigma}}\Big)\Big)\big(a(r)|\tilde{v}_0(r)| + |\tilde{v}_1(r)|\big).$$

In the same way we derive

$$|v_\tau(\tau,\xi)| \lesssim \exp\Big(-\frac{\sigma}{\mu(2\sigma-2\delta)}\Big(\tau^{\frac{-2\delta+2\sigma}{\sigma}} - r^{\frac{-2\delta+2\sigma}{\sigma}}\Big)\Big)\big(a(r)|\tilde{v}_0(r)| + |\tilde{v}_1(r)|\big).$$

In the case $r \leq N$ we can use the latter estimate in order to obtain together with

$$\Big|\begin{pmatrix} \tilde{v}(N) \\ \tilde{v}_\tau(N) \end{pmatrix}\Big| \leq C_N\Big|\begin{pmatrix} \tilde{v}_0(r) \\ \tilde{v}_1(r) \end{pmatrix}\Big|.$$

Summarizing we arrived at the following statement:

Proposition 6.4.3. *The following estimates hold* :

$$\begin{aligned} a(\tau)|\tilde{v}(\tau)| &\lesssim \exp\Big(-\frac{\sigma}{\mu(2\sigma-2\delta)}\Big(\tau^{\frac{-2\delta+2\sigma}{\sigma}} - r^{\frac{-2\delta+2\sigma}{\sigma}}\Big)\Big)\big(a(r)|\tilde{v}_0(r)| + |\tilde{v}_1(r)|\big), \\ |v_\tau(\tau,\xi)| &\lesssim \exp\Big(-\frac{\sigma}{\mu(2\sigma-2\delta)}\Big(\tau^{\frac{-2\delta+2\sigma}{\sigma}} - r^{\frac{-2\delta+2\sigma}{\sigma}}\Big)\Big)\big(a(r)|\tilde{v}_0(r)| + |\tilde{v}_1(r)|\big). \end{aligned}$$

Proof of the optimality

In this section we want to prove the optimality of the estimate (6.4.15). To do so we shall show that at least one component of $V^{(2)}(\tau)$ can not be estimated to below better than $\exp\Big(\int_r^\tau w(s)ds\Big)$. Let us devote to the system

$$V_\tau^{(2)}(\tau) = \Big(\mathcal{D}(\tau) + \mathcal{F}^{(1)}(\tau) + \mathcal{R}_2(\tau)\Big)V^{(2)}(\tau).$$

Here we denote by $r_{kl}(\tau)$ with $k,l = 1,2,$ the entries of $\mathcal{R}_2(\tau)$ from the last section and we denote by $\varphi_1(\tau),\ \varphi_2(\tau)$ appearing in the matrix

$$\mathcal{D}(\tau) + \mathcal{F}^{(1)}(\tau) + \mathcal{R}_2(\tau) := \begin{pmatrix} \varphi_1(\tau) & r_{12}(\tau) \\ r_{21}(\tau) & \varphi_2(\tau) \end{pmatrix}.$$

We shall examine the behavior of the components of the fundamental solution $E(\tau,r)$ as solution to

$$d_\tau E(\tau,r)=\begin{pmatrix}\varphi_1(\tau) & r_{12}(\tau)\\ r_{21}(\tau) & \varphi_2(\tau)\end{pmatrix}E(\tau,r),\quad E(r,r)=I_2\quad\text{for all}\quad \tau\geq r.$$

We have the following representations for its entries $E^{kl}(\tau,r),\ k,l=1,2$:

$$E^{11}(\tau,r)=\exp\Big(\int_r^\tau\varphi_1(s)ds\Big)+\int_r^\tau r_{12}(s)E^{21}(s,r)\exp\Big(\int_s^\tau\varphi_1(\theta)d\theta\Big)ds, \tag{6.4.17}$$

$$E^{12}(\tau,r)=\int_r^\tau r_{12}(s)E^{22}(s,r)\exp\Big(\int_s^\tau\varphi_1(\theta)d\theta\Big)ds, \tag{6.4.18}$$

$$E^{21}(\tau,r)=\int_r^\tau r_{21}(s)E^{11}(s,r)\exp\Big(\int_s^\tau\varphi_2(\theta)d\theta\Big)ds, \tag{6.4.19}$$

$$E^{22}(\tau,r)=\exp\Big(\int_r^\tau\varphi_2(s)ds\Big)+\int_r^\tau r_{21}(s)E^{12}(s,r)\exp\Big(\int_s^\tau\varphi_2(\theta)d\theta\Big)ds. \tag{6.4.20}$$

Substituting the representations into each other we can establish integral equations for each component. Let us begin with the component E^{21}.
$E^{21}(\tau,r)$: To understand the behavior of this component in correspondence with

$$\exp\Big(\int_r^\tau\varphi_2(s)ds\Big)=\exp\Big(\int_r^\tau w(s)+r_{22}(s)ds\Big)\leq C_N\exp\Big(\int_r^\tau w(s)ds\Big)$$

we investigate

$$\exp\Big(-\int_r^\tau\varphi_2(\theta)d\theta\Big)E^{21}(\tau,r)=\int_r^\tau r_{21}(s)E^{11}(s,r)\exp\Big(-\int_r^s\varphi_2(\theta)d\theta\Big)ds.$$

Using the representation for $E^{11}(\tau,r)$ from formula (6.4.17) we get

$$\begin{aligned}\underbrace{\exp\Big(-\int_r^\tau\varphi_2(\theta)d\theta\Big)E^{21}(\tau,r)}_{:=q_{21}(\tau,r)}&=\underbrace{\int_r^\tau r_{21}(s)\exp\Big(\int_r^s\big(\varphi_1(\theta)-\varphi_2(\theta)\big)d\theta\Big)ds}_{:=\alpha(\tau,r)}\\ &+\int_r^\tau\int_r^s r_{21}(s)r_{12}(\theta)\exp\Big(\int_\theta^s\big(\varphi_1(\rho)-\varphi_2(\rho)\big)d\rho\Big)\underbrace{\exp\Big(-\int_r^\theta\varphi_2(\rho)d\rho\Big)E^{21}(\theta,r)}_{:=q_{21}(\theta,r)}\,d\theta ds.\end{aligned}$$

Here we introduced the notations

$$q_{21}(\tau,r):=\exp\Big(-\int_r^\tau\varphi_2(\theta)d\theta\Big)E^{21}(\tau,r),$$
$$\text{and}\quad\alpha(\tau,r):=\int_r^\tau r_{21}(s)\exp\Big(\int_r^s\big(\varphi_1(\theta)-\varphi_2(\theta)\big)d\theta\Big)ds.$$

With this we get an integral equation for $q_{21}(\tau,r)$:

$$q_{21}(\tau,r)=\alpha(\tau,r)+\int_r^\tau\int_r^s r_{21}(s)r_{12}(\theta)\exp\Big(\int_\theta^s\big(\varphi_1(\rho)-\varphi_2(\rho)\big)d\rho\Big)q_{21}(\theta,r)d\theta ds.$$

Introducing the notation

$$\beta(s,r):=r_{21}(s)\int_r^s r_{12}(\theta)\exp\Big(\int_\theta^s\big(\varphi_1(\rho)-\varphi_2(\rho)\big)d\rho\Big)d\theta$$

leads to

$$q_{21}(\tau,r)=\alpha(\tau,r)+\int_r^\tau \beta(s,r)q_{21}(s,r)ds.$$

Hence, we can write

$$|q_{21}(\tau,r)|\le|\alpha(\tau,r)|+\int_r^\tau |\beta(s,r)||q_{21}(s,r)|ds.$$

Taking into consideration $|\alpha(\tau,r)|\lesssim N^{-\frac{2\delta}{\sigma}}$ and $|\beta(s,r)|\lesssim N^{-\frac{2\delta}{\sigma}}s^{-\frac{2\delta+\sigma}{\sigma}}$ uniformly for all large constants N the application of Gronwall's lemma brings

$$|q_{21}(\tau,r)|\le C_N,\quad \text{hence,}\quad |E^{21}(\tau,r)|\le C_N\exp\Big(\int_r^\tau w(s)ds\Big),$$

where the constant C_N tends to 0 for N to ∞. So we are not able to prove that this component behaves better than $\exp\big(\int_r^\tau w(s)ds\big)$. We may derive same estimates for $|E^{11}(\tau,r)|$ and $|E^{12}(\tau,r)|$, where C_N has the above behavior.
$E^{22}(\tau,r)$: Only for this component we can show that it does not behave better than $C_N\exp\big(\int_r^\tau w(s)ds\big)$ with a constant C_N which tends to 1 for N to ∞. Our starting point is the integral equation

$$q_{22}(\tau,r)=\alpha(\tau,r)+\int_r^\tau \beta(s,r)q_{22}(s,r)ds,$$

where

$$q_{22}(\tau,r):=\exp\Big(-\int_r^\tau \varphi_2(\theta)d\theta\Big)E^{22}(\tau,r),\qquad \alpha(\tau,r)=1,$$

and

$$\beta(s,r):=r_{21}(s)\int_r^s r_{12}(\theta)\exp\Big(\int_\theta^s\big(\varphi_1(\rho)-\varphi_2(\rho)\big)d\rho\Big)d\theta.$$

We use the following representation of $q_{22}(\tau,r)$. For $n=1$ we set $\tau_0=\tau$:

$$q_{22}(\tau,r)=1+\sum_{n=1}^\infty\int_r^\tau\beta(\tau,\tau_1)\int_r^{\tau_1}\beta(\tau_1,\tau_2)\cdots\int_r^{\tau_{n-1}}\beta(\tau_{n-1},\tau_n)d\tau_n\cdots d\tau_1.$$

This representation allows to prove $|q_{22}(\tau,r)|\ge\frac12$ for a suitable choice of N. We begin with

$$q_{22}(\tau,r)\ge 1-\sum_{n=1}^\infty\Big|\int_r^\tau\beta(\tau,\tau_1)\int_r^{\tau_1}\beta(\tau_1,\tau_2)\cdots\int_r^{\tau_{n-1}}\beta(\tau_{n-1},\tau_n)d\tau_n\cdots d\tau_1\Big|$$

and show that

$$\Big|\int_r^\tau\beta(\tau,\tau_1)\int_r^{\tau_1}\beta(\tau_1,\tau_2)\cdots\int_r^{\tau_{n-1}}\beta(\tau_{n-1},\tau_n)d\tau_n\cdots d\tau_1\Big|\le\frac{1}{3^n},\quad n\ge1.$$

Summarizing we have shown

$$|E^{22}(\tau,r)|\ge\frac12\exp\Big(\int_r^\tau w(s)ds\Big).$$

Hence, this component does note behave better than the weight term $\exp\big(\int_r^\tau w(s)ds\big)$. Finally, if N is large enough and if we choose data

$$V^{(2)}(r)=\big(V_1^{(2)}(r),V_2^{(2)}(r)\big)\ \text{ with }\ |V_2^{(2)}(r)|\ge\frac{1}{\varepsilon}|V_1^{(2)}(r)|$$

for a suitable small ε we can estimate

$$|V^{(2)}(\tau)|\ge\frac14\exp\Big(\int_r^\tau w(s)ds\Big)|V^{(2)}(r)|,\quad |V(\tau)|\ge C_0\exp\Big(\int_r^\tau w(s)ds\Big)|V(r)|,$$

respectively, where C_0 is a positive constant. By Definition 6.1.1 we can conclude immediately the following statement:

Theorem 6.4.4. *Let us consider the Cauchy problem (6.3.1) in the case $\delta \in (\sigma/2, \sigma)$. Then the solution $u = u(t,x)$ satisfies the following estimates for the energies of higher order :*

$$E^m[u](t) \lesssim (1+t)^{-2\frac{m+\sigma}{\sigma}} \|u_0\|^2_{H^{m+\sigma}(\mathbb{R}^n)} + (1+t)^{-2(\frac{m}{\sigma})} \|u_1\|^2_{H^m(\mathbb{R}^n)}.$$

Remark 6.4.2. *This result coincides with the estimates for $E^m[u](t)$ in the case of increasing dissipation from Theorem 4.2.18. Furthermore, we can see that for the scale-invariant models the decay function for $E^m[u](t)$ is independent of the parameters μ, δ.*

7 $L^p - L^q$ estimates on the conjugate line

In this chapter we are interested in $L^p - L^q$ decay estimates on the conjugate line. It turns out that this class of models provides us with a lot of ideas and gives some feeling for expected results from the results of Chapters 3 and 4. We are concerned with the following structural damped $\sigma-$evolution models :

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\sigma u(t,x) + b(t)(-\Delta)^\delta u_t(t,x) = 0, \qquad (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \sigma > 1, \quad \delta \in (0,\sigma), \end{cases} \tag{7.0.1}$$

where $b = b(t)$ is a strictly monotonous and positive function.

7.1 Objectives and strategies

The main objective in this chapter is the derivation of $L^p - L^q$ decay estimates for the higher order energies on the conjugate line. These estimates rely on more structural properties of representations of solutions and can not be deduced by the same methods as the above mentioned results from Chapters 3 and 4. Let us explain our strategy. It is divided into the following steps:

- In the first step we use the partial Fourier transformation to reduce the partial differential equation to an ordinary differential equation for $\hat{u}(t,\xi)$ parameterized by ξ.
- We divide the extended phase space $[0,\infty)\times\mathbb{R}^n$ into zones to find a WKB- representation for the solution of (7.0.1) in all zones.
- By using techniques for Fourier multipliers we obtain $L^1 - L^\infty$ estimates. In the Fourier multipliers appear characteristic functions χ_A of sets A from the extended phase space. We get two type of decay estimates, a "potential type decay" for small frequencies and an "exponential type decay" for large frequencies under additional regularity assumptions for the data.
- By using the Plancherel theorem we obtain $L^2 - L^2$ estimates. Here we can use the results from Chapters 3 and 4. We get also two types of decay estimates, a "potential type decay" for small frequencies and an "exponential type decay" for large frequencies under additional regularity assumptions for the data.
- The general statement follows by an interpolation argument. We apply the Riesz-Thorin interpolation theorem to interpolate the $L^1 - L^\infty$ and $L^2 - L^2$ estimates.

7.2 $L^p - L^q$ estimates for decreasing dissipation

Let us consider the special case of (7.0.1), that is, the model

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\sigma u(t,x) + b(t)(-\Delta)^\delta u_t(t,x) = 0, \qquad (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \sigma > 1, \quad \delta \in (0,\sigma), \end{cases} \tag{7.2.1}$$

where $b = b(t)$ is a strictly decreasing and positive function.

Lemma 7.2.1. *The following estimate holds for $|\beta| \geq 0$ and for large time t :*

$$\Big\| |\xi|^{|\beta|} \exp\Big(-|\xi|^{2\delta} \int_0^t b(\tau)d\tau \Big) \Big\|_{L^1} \lesssim \Big(1 + \int_0^t b(\tau)d\tau \Big)^{-\frac{|\beta|}{2\delta} - \frac{n}{2\delta}}.$$

Proof. The proof follows from

$$\begin{aligned} \Big\| |\xi|^{|\beta|} \exp\Big(-|\xi|^{2\delta} \int_0^t b(\tau)d\tau \Big) \Big\|_{L^1} &\lesssim \int_0^\infty |\xi|^{|\beta|+n-1} \exp\Big(-|\xi|^{2\delta} \int_0^t b(\tau)d\tau \Big) d|\xi| \\ &\lesssim \Big(\int_0^t b(\tau)d\tau \Big)^{-\frac{|\beta|+n}{2\delta}} \int_0^\infty \eta^{\frac{|\beta|+n}{2\delta}-1} \exp\big(-\eta\big) d\eta \end{aligned}$$

after setting $\eta = |\xi|^{2\delta} \int_0^t b(\tau)d\tau$ with $d\eta = 2\delta|\xi|^{2\delta-1}\big(\int_0^t b(\tau)d\tau\big)d|\xi|$. □

Remark 7.2.1. *If we estimate $\| \exp\big(-\frac{|\xi|^{2\delta}}{2} \int_0^t b(\tau)d\tau \big)|\xi|^{|\beta|} \|_{L^1}$, then this brings a coefficient like*

$$\Big(\int_0^t b(\tau)d\tau \Big)^{-\frac{|\beta|}{2\delta} - \frac{n}{2\delta}}.$$

This coefficient becomes singular for $t \to +0$. For this reason we will often distinguish in the further considerations between the two cases $t \in (0,1]$ and $t \in [1,\infty)$.

7.2.1 Treatment in the case $\delta \in (0, \sigma/2)$

In this section we study the special case of (7.2.1) for $\delta \in (0,\sigma/2)$, that is, the model

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + b(t)(-\Delta)^\delta u_t = 0, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \sigma > 1, \quad \delta \in (0,\sigma/2), \end{cases} \tag{7.2.2}$$

To derive a suitable WKB representation for the solution of (7.2.2) we divide the extended phase space $[0,\infty)\times\mathbb{R}^n_\xi$ into the same zones as before in Section 3.3.1. The part of the extended phase space containing only large frequencies belongs completely to the hyperbolic zone. By applying Lemma 3.3.1 the part of the extended phase space containing only small frequencies is divided into the following zones:

case 1: If $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = 0$, then this part of the extended phase space is divided into the pseudo-differential zone , the elliptic zone , the reduced zone and the hyperbolic zone.

case 2: If $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$, then this part of the extended phase space is divided into the pseudo-differential zone, the reduced zone and the hyperbolic zone.

Let us devote to the first case:

$L^1 - L^\infty$ estimates

Large frequencies. For $\{(t,\xi) : |\xi| \geq C_1\}$ with C_1 sufficiently large this part of the extended phase space belongs to the hyperbolic zone.

Proposition 7.2.2. *The following estimates hold for large frequencies :*

$$\int_{\mathbb{R}^n} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{|\xi|\geq C_1\}}d\xi \lesssim \exp\Big(-C\int_0^t b(\tau)d\tau\Big)\Big(\|u_0\|_{H^{|\beta|+m,1}} + \|u_1\|_{H^{|\beta|-\sigma+m,1}}\Big)$$
$$\text{for} \quad m>n, \quad |\beta|\geq\sigma,$$
$$\int_{\mathbb{R}^n} |\xi|^{|\beta|}|\hat{u}_t(t,\xi)|\chi_{\{|\xi|\geq C_1\}}d\xi \lesssim \exp\Big(-C\int_0^t b(\tau)d\tau\Big)\Big(\|u_0\|_{H^{|\beta|+\sigma+m,1}} + \|u_1\|_{H^{|\beta|+m,1}}\Big)$$
$$\text{for} \quad m>n, \quad |\beta|\geq 0.$$

Proof. From Proposition 3.3.2 we have

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{|\xi|\geq C_1\}} \lesssim \exp\Big(-\tfrac{|\xi|^{2\delta}}{4}\textstyle\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)|\Big) \text{ for } |\beta|\geq\sigma,$$

$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)|\chi_{\{|\xi|\geq C_1\}} \lesssim \exp\Big(-\tfrac{|\xi|^{2\delta}}{4}\textstyle\int_0^t b(\tau)d\tau\Big)\Big(|\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)|\Big) \text{ for } |\beta|\geq 0.$$

Elastic energy. Taking account of Remark 7.2.1 we distinguish between the two cases small and large times.
Small times. If we would estimate the L^1 norm of $|\xi|^{|\beta|}\exp\Big(-\frac{|\xi|^\sigma}{2}\int_0^t b(\tau)d\tau\Big)$, then this becomes singular for $t\to+0$. For this reason we assume additional regularity for the given Cauchy data to conclude

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{|\xi|\geq C_1\}}d\xi$$
$$\lesssim \Big(\sup_{\xi\in\mathbb{R}^n}\big|(1+|\xi|^2)^{\frac{|\beta|+m}{2}}\hat{u}_0(\xi)\big| + \sup_{\xi\in\mathbb{R}^n}\big|(1+|\xi|^2)^{\frac{|\beta|-\sigma+m}{2}}\hat{u}_1(\xi)\big|\Big)\int_{\mathbb{R}^n_\xi}\frac{1}{(1+|\xi|^2)^{\frac{m}{2}}}d\xi.$$

Applying the $L^1 - L^\infty$ property of Fourier multipliers and $m>n$ we arrive for $t\in(0,1]$ at the inequality

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{|\xi|\geq C_1\}}d\xi \lesssim \|u_0\|_{H^{|\beta|+m,1}} + \|u_1\|_{H^{|\beta|-\sigma+m,1}} \text{ for } m>n. \tag{7.2.3}$$

Large times. We can apply two different strategies. First we estimate

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{|\xi|\geq C_1\}}d\xi \lesssim \|\hat{u}_0\|_{L^\infty}\int_{\mathbb{R}^n_\xi}|\xi|^{|\beta|}\exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)d\xi$$
$$+\|\hat{u}_1\|_{L^\infty}\int_{\mathbb{R}^n_\xi}|\xi|^{|\beta|-\sigma}\exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)d\xi.$$

Applying Lemma 7.2.1 gives

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{|\xi|\geq C_1\}}d\xi \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}}\|u_0\|_{L^1} + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{2\delta}-\frac{n}{2\delta}}\|u_1\|_{L^1}.$$

This implies a "potential type decay" for large times without assuming additional regularity for the data. The second strategy is to use the additional regularity from small times. In this way we may derive From Proposition 3.3.2 we derive

$$\int_{\mathbf{R}^n} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \exp\Big(- C\int_0^t b(\tau)d\tau\Big)\Big(\sup_{\xi\in\mathbf{R}^n} \big|(1+|\xi|^2)^{\frac{|\beta|+m}{2}} \hat{u}_0(\xi)\big| + \sup_{\xi\in\mathbf{R}^n} \big|(1+|\xi|^2)^{\frac{|\beta|-\sigma+m}{2}} \hat{u}_1(\xi)\big|\Big) \int_{\mathbf{R}^n_\xi} \frac{1}{(1+|\xi|^2)^{\frac{m}{2}}} d\xi.$$

Applying a $L^1 - L^\infty$ property of Fourier multipliers and $m > n$ we have

$$\int_{\mathbf{R}^n} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \exp\Big(- C\int_0^t b(\tau)d\tau\Big)\Big(|u_0\|_{H^{|\beta|+m,1}} + \|u_1\|_{H^{|\beta|-\sigma+m,1}}\Big) \quad \text{for } \ m > n.$$

This implies a "exponential type decay". This implies an "exponential type decay".

Kinetic energy. As for the elastic energy we distinguish between the two cases small and large times.

Small times. Under additional regularity assumptions for the data we conclude as follows:

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \Big(\sup_{\xi\in\mathbb{R}^n} \big|(1+|\xi|^2)^{\frac{|\beta|+\sigma+m}{2}} \hat{u}_0(\xi)\big| + \sup_{\xi\in\mathbb{R}^n} \big|(1+|\xi|^2)^{\frac{|\beta|+m}{2}} \hat{u}_1(\xi)\big|\Big) \int_{\mathbb{R}^n_\xi} \frac{1}{(1+|\xi|^2)^{\frac{m}{2}}} d\xi.$$

Applying the $L^1 - L^\infty$ property of Fourier multipliers and $m > n$ we have

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \|u_0\|_{H^{|\beta|+\sigma+m,1}} + \|u_1\|_{H^{|\beta|+m,1}} \quad \text{for } \ m > n.$$

Large times. Here we use

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \|\hat{u}_0\|_{L^\infty} \int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|+\sigma} \exp\Big(- |\xi|^{2\delta} \int_0^t b(\tau)d\tau\Big) d\xi + \|\hat{u}_1\|_{L^\infty} \int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} \exp\Big(- |\xi|^{2\delta} \int_0^t b(\tau)d\tau\Big) d\xi.$$

Applying Lemma 7.2.1 and the $L^1 - L^\infty$ property of Fourier multipliers we get

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma}{2\delta}-\frac{n}{2\delta}} \|u_0\|_{L^1} + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}} \|u_1\|_{L^1}.$$

This gives a "potential type decay". The second strategy

$$\int_{\mathbf{R}^n} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \preceq \exp\Big(- C\int_0^t b(\tau)d\tau\Big)\Big(\sup_{\xi\in\mathbf{R}^n} \big|(1+|\xi|^2)^{\frac{|\beta|+\sigma+m}{2}} \hat{u}_0(\xi)\big| + \sup_{\xi\in\mathbf{R}^n} \big|(1+|\xi|^2)^{\frac{|\beta|+m}{2}} \hat{u}_1(\xi)\big|\Big) \int_{\mathbf{R}^n_\xi} \frac{1}{(1+|\xi|^2)^{\frac{m}{2}}} d\xi.$$

Applying a $L^1 - L^\infty$ property of Fourier multipliers and $m > n$ we arrive at

$$\int_{\mathbf{R}^n} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \preceq \exp\Big(- C\int_0^t b(\tau)d\tau\Big)\Big(|u_0\|_{H^{|\beta|+\sigma+m,1}} + \|u_1\|_{H^{|\beta|+m,1}}\Big) \quad \text{for } \ m > n.$$

This gives a "exponential type decay" and completes the proof. □

Small frequencies. We feel small frequencies in several zones separated by the lines $t_0(|\xi|)$, $t_1(|\xi|)$ and $t_2(|\xi|)$. For this reason we distinguish three case.

Treatment in the case $t \in (0, t_0(|\xi|)]$ In the following statement there appears the parameter α which appears in condition (A6) from Section 3.3.1.

Proposition 7.2.3. *The following estimates hold for small frequencies in the case* $t \in (0, t_0(|\xi|)]$:

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{pd\}\cap\{|\xi|\le C_2\}}(t,\xi) d\xi \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}} \|u_0\|_{L^1}$$
$$+\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-2\delta\alpha}{2\delta}-\frac{n}{2\delta}} \|u_1\|_{L^1} \text{ for } |\beta| \ge \sigma,$$
$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{pd\}\cap\{|\xi|\le C_2\}}(t,\xi) d\xi \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\alpha}{2\delta}-\frac{n}{2\delta}} \|u_0\|_{L^1}$$
$$+\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}} \|u_1\|_{L^1} \quad \text{for} \quad |\beta| \ge 0.$$

Proof. Here we work completely in the pseudo-differential zone, see Section 3.3.5. By taking into consideration the phase function $-|\xi|^{2\delta}\int_0^t b(\tau)d\tau$ have non meaning in the pseudo-differential zone and using the formulas (3.3.7) and (3.3.9) we have From
Elastic energy. Using the estimates (3.3.7) and taking into consideration Remark 7.2.1 we distinguish between the two cases for small and large times.
Small times. In this case we have compact set, then we conclude immediately the following estimate:

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{pd\}\cap\{|\xi|\le C_2\}}(t,\xi) d\xi \lesssim \|u_0\|_{L^1} + \|u_1\|_{L^1}.$$

Large times. For large times it follows from (3.3.7) :

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{pd\}\cap\{|\xi|\le C_2\}}(t,\xi) d\xi$$
$$\lesssim \int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|-\sigma} \exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big) d\xi \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{\sigma}{2\delta}} \|\hat{u}_0\|_{L^\infty}$$
$$+\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|-\sigma-2\delta\alpha} \exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big) d\xi \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{\sigma}{2\delta}} \|\hat{u}_1\|_{L^\infty}.$$

Applying Lemma 7.2.1 and a $L^1 - L^\infty$ property of Fourier multipliers gives

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{pd\}\cap\{|\xi|\le C_2\}}(t,\xi) d\xi \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}} \|u_0\|_{L^1}$$
$$+\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-2\delta\alpha}{2\delta}-\frac{n}{2\delta}} \|u_1\|_{L^1}.$$

This brings a "potential type decay".

Kinetic energy.
Small times. As for the elastic energy we get

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{pd\}\cap\{|\xi|\le C_2\}}(t,\xi) d\xi \lesssim \|u_0\|_{L^1} + \|u_1\|_{L^1}.$$

Large times. For large times it follows from (3.3.9) :

$$\begin{aligned} &\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{pd\}\cap\{|\xi|\le C_2\}}(t,\xi) d\xi \\ &\quad\lesssim \int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|-2\delta\alpha} \exp\Big(-|\xi|^{2\delta} \int_0^t b(\tau)d\tau \Big) d\xi \Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{\sigma}{\delta}} \|\hat{u}_0\|_{L^\infty} \\ &\quad + \int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} \exp\Big(-|\xi|^{2\delta} \int_0^t b(\tau)d\tau \Big) d\xi \, \|\hat{u}_1\|_{L^\infty}. \end{aligned}$$

Applying Lemma 7.2.1 and the $L^1 - L^\infty$ property of Fourier multipliers gives

$$\begin{aligned} \int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{pd\}\cap\{|\xi|\le C_2\}}(t,\xi) d\xi &\lesssim \Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\alpha}{2\delta} - \frac{n}{2\delta}} \|u_0\|_{L^1} \\ &+ \Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta} - \frac{n}{2\delta}} \|u_1\|_{L^1}. \end{aligned}$$

This yields a "potential type decay". In this way the proposition is proved. □

Treatment in the case $t \in [t_0(|\xi|), t_1(|\xi|)]$

Proposition 7.2.4. *The following estimates hold for small frequencies in the case $t \in [t_0(|\xi|), t_1(|\xi|)]$* :

$$\begin{aligned} &\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{ell\cup pd\}\cap\{|\xi|\le C_2\}}(t,\xi) d\xi \lesssim \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}} \|u_0\|_{L^1} \\ &\qquad + \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|-2\delta\alpha}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}} \|u_1\|_{L^1} \qquad \text{for} \quad |\beta| \ge \sigma, \\ &\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{ell\cup pd\}\cap\{|\xi|\le C_2\}}(t,\xi) d\xi \\ &\lesssim \max\Big\{ \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|+\sigma}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}}, \Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\alpha}{2\delta} - \frac{n}{2\delta}} \Big\} \|u_0\|_{L^1} \\ &+ \max\Big\{ \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}}, \Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta} - \frac{n}{2\delta}} \Big\} \|u_1\|_{L^1} \;\; \text{for} \;\; |\beta| \ge 0. \end{aligned}$$

Proof. Elastic energy.
Small times. In this case we work in a compact set, then we conclude

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{ell\cup pd\}\cap\{|\xi|\le C_2\}}(t,\xi) d\xi \lesssim \|u_0\|_{L^1} + \|u_1\|_{L^1}.$$

Large times. By taking into consideration the gluing procedure between the pseudo-differential zone and the elliptic zone from Corollary 3.3.17, the decreasing behavior of $b = b(t)$ and the definition of the elliptic zone we have

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{ell\cup pd\}\cap\{|\xi|\le C_2\}}(t,\xi)d\xi \lesssim \int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}\exp\Big(-|\xi|^{2\sigma-2\delta}\int_0^t \frac{1}{b(\tau)}d\tau\Big)d\xi\,\|\hat{u}_0\|_{L^\infty}$$
$$+\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|-2\delta\alpha}\exp\Big(-|\xi|^{2\sigma-2\delta}\int_0^t \frac{1}{b(\tau)}d\tau\Big)d\xi\,\|\hat{u}_1\|_{L^\infty}.$$

Applying Lemma 7.2.1 and the $L^1 - L^\infty$ property of Fourier multipliers we arrive at

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{ell\cup pd\}\cap\{|\xi|\le C_2\}}(t,\xi)d\xi \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}\|u_0\|_{L^1}$$
$$+\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\alpha}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}\|u_1\|_{L^1}.$$

This brings a "potential type decay".*Kinetic energy.* Here we use again the estimates from Corollary 3.3.17.
Small times. As before we have

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}_t(t,\xi)|\chi_{\{ell\cup pd\}\cap\{|\xi|\le C_2\}}(t,\xi)d\xi \lesssim \|u_0\|_{L^1}+\|u_1\|_{L^1}.$$

Large times. From Corollary 3.3.17, the decreasing behavior of $b=b(t)$ and the definition of the elliptic zone we have it follows

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}_t(t,\xi)|\chi_{\{ell\cup pd\}\cap\{|\xi|\le C_2\}}(t,\xi)d\xi \lesssim \max\Big\{\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|+2\sigma-2\delta\alpha}\exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)d\xi,$$
$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|+\sigma}\exp\Big(-|\xi|^{2\sigma-2\delta}\int_0^t \frac{1}{b(\tau)}d\tau\Big)d\xi\Big\}\|\hat{u}_0\|_{L^\infty}$$
$$+\max\Big\{\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}\exp\Big(-|\xi|^{2\delta}\int_0^t b(\tau)d\tau\Big)d\xi,$$
$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|+\sigma-2\delta\alpha}\exp\Big(-|\xi|^{2\sigma-2\delta}\int_0^t \frac{1}{b(\tau)}d\tau\Big)d\xi\Big\}\|\hat{u}_1\|_{L^\infty}.$$

Applying Lemma 7.2.1 and the $L^1 - L^\infty$ property of Fourier multipliers we get

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}_t(t,\xi)|\chi_{\{ell\cup pd\}\cap\{|\xi|\le C_2\}}(t,\xi)d\xi$$
$$\lesssim +\max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+\sigma}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}},\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\alpha}{2\delta}-\frac{n}{2\delta}}\Big\}\|u_0\|_{L^1}$$
$$\max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}},\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}}\Big\}\|u_1\|_{L^1}.$$

This implies a "potential type decay". In this way the proposition is proved. □

Treatment in the case $t\in[t_1(|\xi|),\infty)$

Proposition 7.2.5. *The following estimates hold for small frequencies in the case* $t \in [t_1(|\xi|), \infty)$:

$$\int_{\mathbb{R}^n} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{red \cup hyp\} \cap \{|\xi| \le C_2\}}(t,\xi) d\xi$$
$$\lesssim \max\Big\{ \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}}, \Big(1 + \int_0^t b(\tau) d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{2\delta} - \frac{n}{2\delta}} \Big\} \|u_0\|_{L^1}$$
$$+ \max\Big\{ \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|-2\delta\alpha}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}}, \Big(1 + \int_0^t b(\tau) d\tau\Big)^{-\frac{|\beta|-\sigma}{2\delta} - \frac{n}{2\delta}} \Big\} \|u_1\|_{L^1} \text{ for } |\beta| \ge \sigma,$$
$$\int_{\mathbb{R}^n} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{red \cup hyp\} \cap \{|\xi| \le C_2\}}(t,\xi) d\xi$$
$$\lesssim \max\Big\{ \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|+\sigma}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}}, \Big(1 + \int_0^t b(\tau) d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\alpha}{2\delta} - \frac{n}{2\delta}} \Big\} \|u_0\|_{L^1}$$
$$+ \max\Big\{ \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}}, \Big(1 + \int_0^t b(\tau) d\tau\Big)^{-\frac{|\beta|}{2\delta} - \frac{n}{2\delta}} \Big\} \|u_1\|_{L^1} \text{ for } |\beta| \ge 0.$$

Proof. We use the gluing procedure between the pseudo-differential zone, the elliptic zone, the reduced zone, and the hyperbolic zone, as proposed in Section 3.3.6 (see Corollary 3.3.19 for $t \in [t_1(|\xi|), \infty)$) :
Elastic energy.
Small times. As in the other cases we derive

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{red \cup hyp\} \cap \{|\xi| \le C_2\}}(t,\xi) d\xi \lesssim \|u_0\|_{L^1} + \|u_1\|_{L^1}.$$

Large times. From Corollary 3.3.19 it follows

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{red\} \cup \{hyp\}}(t,\xi) d\xi \lesssim$$
$$\sup_{\xi \in \mathbb{R}^n} \Big| (1+|\xi|^2)^{\frac{|\beta|+m}{2}} \exp\Big(-\frac{|\xi|^{2\delta}}{4} \int_{t_1(|\xi|)}^t b(\tau) d\tau \Big) \exp\Big(-|\xi|^{2\sigma-2\delta} \int_0^{t_1(|\xi|)} \frac{1}{b(\tau)} d\tau \Big) \Big| \|\hat{u}_0\|_{L^\infty}$$
$$\times \int_{\mathbb{R}^n_\xi} \frac{1}{(1+|\xi|^2)^{\frac{m}{2}}} d\xi + \Big\| (1+|\xi|^2)^{\frac{|\beta|+\sigma-2\delta\alpha}{2}} \exp\Big(-\frac{|\xi|^{2\delta}}{4} \int_0^t b(\tau) d\tau \Big) \Big\|_{L^1} \|\hat{u}_0\|_{L^\infty}$$
$$+ \sup_{\xi \in \mathbb{R}^n} \Big| (1+|\xi|^2)^{\frac{|\beta|-2\delta\alpha+m}{2}} \exp\Big(-\frac{|\xi|^{2\delta}}{4} \int_{t_1(|\xi|)}^t b(\tau) d\tau \Big) \exp\Big(-|\xi|^{2\sigma-2\delta} \int_0^{t_1(|\xi|)} \frac{1}{b(\tau)} d\tau \Big) \Big| \|\hat{u}_1\|_{L^\infty}$$
$$\times \int_{\mathbb{R}^n_\xi} \frac{1}{(1+|\xi|^2)^{\frac{m}{2}}} d\xi + \Big\| (1+|\xi|^2)^{\frac{|\beta|-\sigma}{2}} \exp\Big(-\frac{|\xi|^{2\delta}}{4} \int_0^t b(\tau) d\tau \Big) \Big\|_{L^1} \|\hat{u}_1\|_{L^\infty} \quad \text{for} \quad m > n.$$

Applying Lemma 7.2.1, Lemma 3.3.21, $m > n$ and a $L^1 - L^\infty$ property of Fourier multipliers the last inequality implies

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{red\} \cup \{hyp\}}(t,\xi) d\xi$$
$$\lesssim \max\Big\{ \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}}, \Big(1 + \int_0^t b(\tau) d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{2\delta} - \frac{n}{2\delta}} \Big\} \|u_0\|_{L^1}$$
$$+ \max\Big\{ \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|-2\delta\alpha}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}}, \Big(1 + \int_0^t b(\tau) d\tau\Big)^{-\frac{|\beta|-\sigma}{2\delta} - \frac{n}{2\delta}} \Big\} \|u_1\|_{L^1}.$$

This gives a "potential type decay".
Kinetic energy.
Small times. By the same arguments as before we get

$$\int_{\mathbf{R}^n} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{red\}\cup\{hyp\}}(t,\xi)|d\xi \preceq \|u_0\|_{L^1} + \|u_1\|_{L^1}.$$

Large times. We use again Corollary 3.3.19 we have

$$\begin{aligned}
&\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{red\}\cup\{hyp\}}(t,\xi) d\xi \lesssim \\
&\sup_{\xi\in\mathbb{R}^n} \Big| (1+|\xi|^2)^{\frac{|\beta|+\sigma+m}{2}} \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big) \exp\Big(-|\xi|^{2\sigma-2\delta}\int_0^{t_1(|\xi|)} \frac{1}{b(\tau)}d\tau\Big)\Big| \\
&\times \|\hat{u}_0\|_{L^\infty} \int_{\mathbb{R}^n_\xi} \frac{1}{(1+|\xi|^2)^{\frac{m}{2}}} d\xi + \Big\|(1+|\xi|^2)^{\frac{|\beta|+2\sigma-2\delta\alpha}{2}} \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big\|_{L^1} \|\hat{u}_0\|_{L^\infty} \\
&+ \sup_{\xi\in\mathbb{R}^n} \Big| (1+|\xi|^2)^{\frac{|\beta|+\sigma-2\delta\alpha+m}{2}} \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_{t_1(|\xi|)}^t b(\tau)d\tau\Big) \exp\Big(-|\xi|^{2\sigma-2\delta}\int_0^{t_1(|\xi|)} \frac{1}{b(\tau)}d\tau\Big)\Big| \\
&\times \|\hat{u}_1\|_{L^\infty} \int_{\mathbb{R}^n_\xi} \frac{1}{(1+|\xi|^2)^{\frac{m}{2}}} d\xi + \Big\|(1+|\xi|^2)^{\frac{|\beta|}{2}} \exp\Big(-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau\Big)\Big\|_{L^1} \|\hat{u}_1\|_{L^\infty} \\
&\qquad \text{for} \quad m > n.
\end{aligned}$$

Applying Lemma 7.2.1, Lemma 3.3.21, $m > n$ and the $L^1 - L^\infty$ property of Fourier multipliers the last inequality implies

$$\begin{aligned}
&\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{red\}\cup\{hyp\}}(t,\xi) d\xi \\
&\lesssim \max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+\sigma}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\alpha}{2\delta}-\frac{n}{2\delta}}\Big\} \|u_0\|_{L^1} \\
&\quad + \max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}}\Big\} \|u_1\|_{L^1}.
\end{aligned}$$

This brings a "potential type decay" and the proposition is proved. □

Conclusion By the statements from Propositions 7.2.2 to 7.2.5 we obtain the desired $L^1 - L^\infty$ estimates for the energies of higher order of solutions to the Cauchy problem (7.2.2).

Theorem 7.2.6. *The following $L^1 - L^\infty$ estimates hold for the energies of higher order of* *solutions to the Cauchy problem (7.2.2) :*

$$\|\nabla^\beta u(t,\cdot)\|_{L^\infty}$$
$$\lesssim \max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{2\delta}-\frac{n}{2\delta}}\Big\}\|u_0\|_{H^{|\beta|+m,1}}$$
$$+\max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\alpha}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{2\delta}-\frac{n}{2\delta}}\Big\}\|u_1\|_{H^{|\beta|-\sigma+m,1}}$$
for $\quad m>n, \quad |\beta|\geq\sigma,$

$$\|\nabla^\beta u_t(t,\cdot)\|_{L^\infty}$$
$$\lesssim \max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+\sigma}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\alpha}{2\delta}-\frac{n}{2\delta}}\Big\}\|u_0\|_{H^{|\beta|+\sigma+m,1}}$$
$$+\max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}}\Big\}\|u_1\|_{H^{|\beta|+m,1}}$$
for $\quad m>n, \quad |\beta|\geq 0.$

Remark 7.2.2. *We want to pint out that in the above estimates the decay comes from the WKB representations of solutions for small frequencies and large times, the regularity comes from the large frequencies and from the small times.*

Interpolation

To get the desired $L^p - L^q$ decay estimates on the conjugate line we apply the Riesz-Thorin interpolation theorem between the $L^2 - L^2$ estimates from Theorem 3.3.23 in Section 3.3.5 and the $L^1 - L^\infty$ estimates from Theorem 7.2.6.

Theorem 7.2.7. *Let us consider the Cauchy problem (7.2.2), where $b=b(t)$ satisfies the following assumptions : (A1) to (A6) and $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}}=0$. Then the energies of higher order of the solutions $u=u(t,x)$ satisfy the following $L^p - L^q$ decay estimates on the conjugate line* :

$$\|\nabla^\beta u(t,\cdot)\|_{L^q}$$
$$\lesssim \max\Big\{\Big(1+\textstyle\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}(\frac{1}{p}-\frac{1}{q})}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\Big\}\|u_0\|_{H^{|\beta|+N_p,p}}$$
$$+\max\Big\{\Big(1+\textstyle\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\alpha}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}(\frac{1}{p}-\frac{1}{q})}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\Big\}\|u_1\|_{H^{|\beta|-\sigma+N_p,p}}$$
for $\quad |\beta|\geq\sigma,$

$$\|\nabla^\beta u_t(t,\cdot)\|_{L^q}$$
$$\lesssim \max\Big\{\Big(1+\textstyle\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+\sigma}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}(\frac{1}{p}-\frac{1}{q})}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\alpha}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\Big\}\|u_0\|_{H^{|\beta|+\sigma+N_p,p}}$$
$$+\max\Big\{\Big(1+\textstyle\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+\sigma-2\delta\alpha}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}(\frac{1}{p}-\frac{1}{q})}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\Big\}\|u_1\|_{H^{|\beta|+N_p,p}}$$
for $\quad |\beta|\geq 0,$

where $p\in[1,2],\ pq=p+q$ and $N_p>n(\frac{1}{p}-\frac{1}{q})$.

Remark 7.2.3. *We have seen in Chapter 3 that our model of interest (7.2.2) possesses the parabolic effect. The last result shows, moreover, that this model is parabolic like from the point of view of $L^p - L^q$ decay estimates on the conjugate line.*

Following the same approach we are able to prove the following corresponding result to Theorem 7.2.7 for the second case.

Theorem 7.2.8. *Let us consider the Cauchy problem (7.2.2), where $b = b(t)$ satisfies the following assumptions : (A1) to (A6) and $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$. Then the energies of higher order of the solution $u = u(t,x)$ satisfy the following $L^p - L^q$ decay estimates on the conjugate line :*

$$\begin{aligned}
\|\nabla^\beta u(t,\cdot)\|_{L^q} &\lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{H^{|\beta|+N_p,p}} \\
&\quad+\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\|u_1\|_{H^{|\beta|-\sigma+N_p,p}} \quad \text{for} \quad |\beta|\geq\sigma, \\
\|\nabla^\beta u_t(t,\cdot)\|_{L^q} &\lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{H^{|\beta|+\sigma+N_p,p}} \\
&\quad+\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\|u_1\|_{H^{|\beta|+N_p,p}} \quad \text{for} \quad |\beta|\geq 0.
\end{aligned}$$

where $p\in[1,2]$, $pq = p+q$, and $N_p > n\big(\frac{1}{p}-\frac{1}{q}\big)$.

7.2.2 Treatment in the case $\delta \in (\sigma/2, \sigma)$

Following the same approach we are able to prove the following corresponding result to Theorem 7.2.7.

Theorem 7.2.9. *Let us consider the Cauchy problem (7.2.1), where $b = b(t)$ satisfies the following assumptions : (A1) to (A5). Then the energies of higher order of the solution $u = u(t,x)$ satisfy in the case $\delta \in (\sigma/2,\sigma)$ the following $L^p - L^q$ decay estimates on the conjugate line :*

$$\begin{aligned}
\|\nabla^\beta u(t,\cdot)\|_{L^q} &\lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{H^{|\beta|+N_p,p}} \\
&\quad+\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\|u_1\|_{H^{|\beta|-\sigma+N_p,p}} \quad \text{for} \quad |\beta|\geq\sigma, \\
\|\nabla^\beta u_t(t,\cdot)\|_{L^q} &\lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{H^{|\beta|+\sigma+N_p,p}} \\
&\quad+\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\|u_1\|_{H^{|\beta|+N_p,p}} \quad \text{for} \quad |\beta|\geq 0.
\end{aligned}$$

where $p\in[1,2]$, $pq = p+q$, and $N_p > n\big(\frac{1}{p}-\frac{1}{q}\big)$.

7.3 $L^p - L^q$ estimates for increasing dissipation

Let us consider the special case of (7.0.1), that is, the model

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\sigma u(t,x) + b(t)(-\Delta)^\delta u_t(t,x) = 0, & (t,x)\in(0,\infty)\times\mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x),, \quad \sigma>1, \quad \delta\in(0,\sigma), \end{cases} \tag{7.3.1}$$

where $b = b(t)$ is a strictly increasing and positive function.

7.3.1 Treatment in the case $\delta \in (\sigma/2, \sigma)$

In this section we study the special case of (7.0.1) for $\delta \in (\sigma/2, \sigma)$, that is, the model

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + b(t)(-\Delta)^\delta u_t = 0, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \delta \in (\sigma/2, \sigma). \end{cases} \tag{7.3.2}$$

We divide again the extended phase space as proposed in Section 3.3.1. The geometry of the zones is the same as in Section 4.3. The part of the extended phase space containing large frequencies is completely contained in the elliptic zone. By taking account of Lemma 3.3.1 we distinguish for small frequencies our considerations into the following cases:
case 1: If $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = 0$, then this part of the extended phase space is divided into the pseudo-differential region and the elliptic zone.
case 2: If $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$, then this part of the extended phase space is divided into the hyperbolic zone, the reduced zone, and the elliptic zone. The pseudo-differential zone is a compact set.
Let us devote to the first case :

$L^1 - L^\infty$ estimates

Large frequencies. The part of the extended phase space $\{(t,\xi) : |\xi| \geq C_1\}$ with C_1 sufficiently large belongs to the elliptic zone.

Proposition 7.3.1. *The following estimates hold for large frequencies*

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{|\xi|\geq C_1\}}d\xi \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}\|u_0\|_{H^{|\beta|+m,1}}$$
$$+\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}\|u_1\|_{H^{|\beta|-2\delta+m,1}} \quad \text{for} \quad m>n, \quad |\beta|\geq 2\delta,$$

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}_t(t,\xi)|\chi_{\{|\xi|\geq C_1\}}d\xi \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}\|u_0\|_{H^{|\beta|+2\sigma-2\delta+m,1}}$$
$$+\max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-4\delta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}}\Big\}\|u_1\|_{H^{|\beta|+m,1}}$$
$$\text{for} \quad m>n, \quad |\beta|\geq 0.$$

Proof. In this case we already derived the representation of solution in Section 4.2.2. Following Proposition 4.2.2
Elastic energy.
Small times. Here we need again more regularity for the Cauchy data. This additional regularity allows to conclude

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{|\xi|\geq C_1\}}d\xi \lesssim \int_{\mathbb{R}^n_\xi}\exp\Big(-C\int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)}d\tau\Big)\Big(|\xi|^{|\beta|}|\hat{u}_0(\xi)|$$
$$+|\xi|^{|\beta|-2\delta}|\hat{u}_1(\xi)|\Big)d\xi$$
$$\lesssim \Big(\sup_{\xi\in\mathbb{R}^n}\big|(1+|\xi|^2)^{\frac{|\beta|+m}{2}}\hat{u}_0(\xi)\big| + \sup_{\xi\in\mathbb{R}^n}\big|(1+|\xi|^2)^{\frac{|\beta|-2\delta+m}{2}}\hat{u}_1(\xi)\big|\Big)\int_{\mathbb{R}^n_\xi}\frac{1}{(1+|\xi|^2)^{\frac{m}{2}}}d\xi.$$

Applying the $L^1 - L^\infty$ property of Fourier multipliers gives

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \|u_0\|_{H^{|\beta|+m,1}} + \|u_1\|_{H^{|\beta|-2\delta+m,1}} \quad \text{for } \; m > n.$$

Large times. From the Proposition 4.2.2 and for $\{|\xi| \geq C_1\}$ it follows

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \sup_{\xi\in\mathbb{R}^n} |\hat{u}_0(\xi)| \int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} \exp\Big(- C \int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)} d\tau \Big) d\xi$$
$$+ \sup_{\xi\in\mathbb{R}^n} |\hat{u}_1(\xi)| \int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|-2\delta} \exp\Big(- C \int_0^t \frac{|\xi|^{2\sigma-2\delta}}{b(\tau)} d\tau \Big) d\xi.$$

Applying Lemma 7.2.1 and the $L^1 - L^\infty$ property of Fourier multipliers we get

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}} \|u_0\|_{L^1}$$
$$+ \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|-2\delta}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}} \|u_1\|_{L^1}.$$

So we have a "potential type decay". If we would use the additional regularity of data from considerations for small times, then we could even derive an "exponential type decay".

Kinetic energy.

Small times. With the additional regularity for the Cauchy data we have

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi$$
$$\lesssim \Big(\sup_{\xi\in\mathbb{R}^n} \big|(1+|\xi|^2)^{\frac{|\beta|+2\sigma-2\delta+m}{2}} \hat{u}_0(\xi)\big| + \sup_{\xi\in\mathbb{R}^n} \big|(1+|\xi|^2)^{\frac{|\beta|+2\sigma-4\delta+m}{2}} \hat{u}_1(\xi)\big| \Big) \int_{\mathbb{R}^n_\xi} \frac{1}{(1+|\xi|^2)^{\frac{m}{2}}} d\xi$$
$$+ \sup_{\xi\in\mathbb{R}^n} \big|(1+|\xi|^2)^{\frac{|\beta|+m}{2}} \hat{u}_1(\xi)\big| \int_{\mathbb{R}^n_\xi} \frac{1}{(1+|\xi|^2)^{\frac{m}{2}}} d\xi,$$

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \|u_0\|_{H^{|\beta|+2\sigma-2\delta+m,1}} + \|u_1\|_{H^{|\beta|+m,1}} \quad \text{for } \; m > n.$$

Large times. From the Proposition 4.2.2 and for $\{|\xi| \geq C_1\}$ we have

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \sup_{\xi\in\mathbb{R}^n} |u_0(\xi)| \int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|+2\sigma-2\delta} \exp\Big(- C|\xi|^{2\sigma-2\delta} \int_0^t \frac{1}{b(\tau)} d\tau \Big) d\xi$$
$$+ \sup_{\xi\in\mathbb{R}^n} |u_1(\xi)| \Big(\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|+2\sigma-4\delta} \exp\Big(- C|\xi|^{2\sigma-2\delta} \int_0^t \frac{1}{b(\tau)} d\tau \Big) d\xi$$
$$+ \int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} \exp\Big(- |\xi|^{2\delta} \int_0^t b(\tau) d\tau \Big) d\xi \Big).$$

Applying Lemma 7.2.1 and the $L^1 - L^\infty$ property of Fourier multipliers brings

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}} \|u_0\|_{L^1}$$
$$+ \max\Big\{ \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|+2\sigma-4\delta}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}}, \Big(1 + \int_0^t b(\tau) d\tau\Big)^{-\frac{|\beta|}{2\delta} - \frac{n}{2\delta}} \Big\} \|u_1\|_{L^1}.$$

This yields a "potential type decay". If we would use the additional regularity of data from considerations for small times, then we could even derive an "exponential type decay". The proposition is proved. □

Small frequencies. For small frequencies we have only the separating line $t_0(|\xi|)$. For this reason we distinguish two cases.

Treatment in the case $t \in (0, t_0(|\xi|)]$

Proposition 7.3.2. *The following estimate holds for small frequencies in the case* $t \in (0, t_0(|\xi|)]$:

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{pd\}\cap\{|\xi|\leq C_2\}}(t,\xi) d\xi \lesssim \Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta} - \frac{n}{2\delta}} \|u_0\|_{L^1}$$
$$+ \Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-2\delta\eta}{2\delta} - \frac{n}{2\delta}} \|u_1\|_{L^1} \quad \text{for} \quad |\beta| \geq \sigma,$$
$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{pd\}\cap\{|\xi|\leq C_2\}}(t,\xi) d\xi \lesssim \Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{2\delta} - \frac{n}{2\delta}} \|u_0\|_{L^1}$$
$$+ \Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta} - \frac{n}{2\delta}} \|u_1\|_{L^1} \quad \text{for} \quad |\beta| \geq 0.$$

Proof. Here we work completely in the pseudo-differential region, see Section 4.2.3. Applying the same strategy as before from the proof of Proposition 7.2.3. By taking into consideration the phase function $-\frac{|\xi|^{2\delta}}{4}\int_0^t b(\tau)d\tau$ have non meaning in the pseudo-differential zone and the formulas (3.3.7), (3.3.9). In this way the proposition is proved. □

Treatment in the case $t \in [t_0(|\xi|), \infty)$

Proposition 7.3.3. *The following estimates hold for the solution to the Cauchy problem (7.3.2) in the case* $t \in [t_0(|\xi|), \infty)$ *for small frequencies* :

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{ell\cup pd\}\cap\{|\xi|\leq C_2\}}(t,\xi) d\xi \lesssim \Big(1 + \int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}} \|u_0\|_{L^1}$$
$$+ \Big(1 + \int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\eta}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}} \|u_1\|_{L^1} \text{ for } |\beta| \geq 2\delta,$$
$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{ell\cup pd\}\cap\{|\xi|\leq C_2\}}(t,\xi) d\xi \lesssim \Big(1 + \int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}} \|u_0\|_{L^1}$$
$$+ \max\Big\{\Big(1 + \int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-4\delta\eta}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}}, \Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta} - \frac{n}{2\delta}}\Big\} \|u_1\|_{L^1} \text{ for } \|\beta| \geq 0.$$

Proof. Here we use the gluing procedure between the pseudo-differential zone and the elliptic zone as proposed in Section 4.2.4. Applying the same strategy as before from the proof of Proposition 7.2.4. By using the Corollary 4.2.8 : In this way the proposition is proved. □

Conclusion The main result of this section follows from Propositions 7.3.1, 7.3.2 and 7.3.3.

Theorem 7.3.4. *The following $L^1 - L^\infty$ estimates hold for the energies of higher order of the solutions to the Cauchy problem (7.3.2) :*

$$\|\nabla^\beta u(t,\cdot)\|_{L^\infty} \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}} \|u_0\|_{H^{|\beta|+m,1}}$$
$$+\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\eta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}} \|u_1\|_{H^{|\beta|-2\delta+m,1}} \quad \text{for} \quad m>n, \quad |\beta|\geq 2\delta,$$
$$\|\nabla^\beta u_t(t,\cdot)\|_{L^\infty} \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}} \|u_0\|_{H^{|\beta|+2\sigma-2\delta+m,1}}$$
$$+\max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-4\delta\eta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}}\Big\} \|u_1\|_{H^{|\beta|+m,1}}$$
$$\text{for} \quad m>n, \quad |\beta|\geq 0.$$

Proof. Taking into consideration all these estimates in the extended phase space we arrive at the decay of the energies of higher order of solutions to the Cauchy problem (7.3.2). We have "potential type decay" because the decay for large frequencies ("exponential type decay") is faster than the decay for small frequencies ("potential type decay"). By taking into consideration the increasing behavior of $b = b(t)$ we have that the decay in the pseudo-differential is faster than the decay in the elliptic zone. In this way we may conclude that the decay is coming from the elliptic zone from Proposition 7.3.3 and the regularity of the data is coming for large frequencies from Proposition 7.3.1. In this way we have the desired statements. □

Interpolation

An interpolation argument between $L^1 - L^\infty$ estimates from Theorem 7.3.4 and the $L^2 - L^2$ estimates from Theorem 4.2.11 implies the following result.

Theorem 7.3.5. *Let us consider the Cauchy problem (7.3.1), where $b = b(t)$ satisfies the following assumptions : (B1) to (B6) and $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = 0$. The parameter η is defined in assumption (B6). Then the energies of higher order of the solution $u = u(t,x)$ satisfy the following $L^p - L^q$ decay estimates on the conjugate line :*

$$\|\nabla^\beta u(t,\cdot)\|_{L^q} \lesssim \Big(1+\int_0^t \tfrac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{H^{|\beta|+N_p,p}}$$
$$+\Big(1+\int_0^t \tfrac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\eta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}(\frac{1}{p}-\frac{1}{q})} \|u_1\|_{H^{|\beta|-2\delta+N_p,p}} \quad \text{for} \quad |\beta|\geq 2\delta,$$
$$\|\nabla^\beta u_t(t,\cdot)\|_{L^q} \lesssim \Big(1+\int_0^t \tfrac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{H^{|\beta|+2\sigma-2\delta+N_p,p}}$$
$$+\max\Big\{\Big(1+\int_0^t \tfrac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-4\delta\eta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}(\frac{1}{p}-\frac{1}{q})}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\Big\} \|u_1\|_{H^{|\beta|+N_p,p}}$$
$$\text{for} \quad |\beta|\geq 0,$$

where $p\in[1,2]$, $pq = p+q$ and $N_p > n\big(\frac{1}{p}-\frac{1}{q}\big)$.

Following the same approach we are able to prove the following corresponding result to Theorem 7.3.5 for the second case.

Theorem 7.3.6. *Let us consider the Cauchy problem (7.3.1), where $b = b(t)$ satisfies the following assumptions : (B1) to (B6) and $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$. Then the energies of higher order of the solution $u = u(t,x)$ satisfy the following $L^p - L^q$ decay estimates on the conjugate line, where C is a suitable positive constant.*

$$\|\nabla^\beta u(t,\cdot)\|_{L^q} \lesssim \max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}, \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big\}\|u_0\|_{H^{|\beta|+N_p,p}}$$
$$+\max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}, \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big\}\|u_1\|_{H^{|\beta|-2\delta+N_p,p}} \text{ for } |\beta| \geq 2\delta,$$
$$\|\nabla^\beta u(t,\cdot)\|_{L^q} \lesssim \max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}, \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big\}\|u_0\|_{H^{|\beta|+N_p,p}}$$
$$+\max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}, \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big\}\|u_1\|_{H^{|\beta|(\frac{1}{p}-\frac{1}{q})+N_p,p}} \text{ for } |\beta| \in [\sigma, 2\delta],$$
$$\|\nabla^\beta u_t(t,\cdot)\|_{L^q} \lesssim \max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}, \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big\}\|u_0\|_{H^{|\beta|+2\sigma-2\delta+N_p,p}}$$
$$+\max\Big\{\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}, \exp\Big(-C\int_0^t \frac{1}{b(\tau)}d\tau\Big)\Big\}\|u_1\|_{H^{|\beta|+N_p,p}} \quad \text{for} \quad |\beta| \geq 0.$$

7.3.2 Treatment of the case $\delta \in (0, \sigma/2)$

In this section we study the special case of (7.0.1) for $\delta \in (0,\sigma/2)$ and $b = b(t)$ is strictly increasing, that is, the model

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + b(t)(-\Delta)^\delta u_t = 0, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \sigma > 1, \quad \delta \in (0,\sigma/2). \end{cases} \tag{7.3.3}$$

We follow the definition of zones from Section 3.3.1. The geometry of these zones is the same as in Section 4.3. The extended phase space is divided for small frequencies into the pseudo-differential zone and the elliptic zone. For large frequencies the extended phase space is divided into the hyperbolic zone, the reduced zone and the elliptic zone.

$L^1 - L^\infty$ estimates

Large frequencies. Only the separating line $t_1(|\xi|)$ is of importance. For this reason we distinguish between the two cases $t \in (0, t_1(|\xi|)]$ and $t \in (0,\infty)$.

Treatment in the case $t \in (0, t_1(|\xi|)]$

Proposition 7.3.7. *The following estimates hold for large frequencies in the case $t \in (0, t_1(|\xi|)]$:*

$$\int_{\mathbb{R}^n} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{hyp\cup red\}}(t,\xi)d\xi$$
$$\lesssim \exp\Big(-C\int_0^t b(\tau)d\tau\Big)\Big(\|u_0\|_{H^{|\beta|+m,1}} + \|u_1\|_{H^{|\beta|-\sigma+m,1}}\Big) \text{ for } m > n, \ |\beta| \geq \sigma,$$
$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}_t(t,\xi)|\chi_{\{hyp\cup red\}}(t,\xi)d\xi$$
$$\lesssim \exp\Big(-C\int_0^t b(\tau)d\tau\Big)\Big(\|u_0\|_{H^{|\beta|+\sigma+m,1}} + \|u_1\|_{H^{|\beta|+m,1}}\Big) \text{ for } \quad m > n, \quad |\beta| \geq 0.$$

Proof. The proof is the same as the proof to Proposition 7.2.2. □

Treatment in the case $t \in (0,\infty)$

Proposition 7.3.8. *The following estimates hold for large frequencies in the case* $t \in (0,\infty)$:

$$\int_{\mathbb{R}^n} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \exp\Big(-C \int_0^t \frac{1}{b(\tau)} d\tau \Big) \Big(\|u_0\|_{H^{|\beta|+m,1}} + \|u_1\|_{H^{|\beta|-\sigma+m,1}} \Big)$$
$$\text{for} \quad m > n, \quad |\beta| \geq \sigma,$$
$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \exp\Big(-C \int_0^t \frac{1}{b(\tau)} d\tau \Big) \Big(\|u_0\|_{H^{|\beta|+\sigma+m,1}} + \|u_1\|_{H^{|\beta|+m,1}} \Big)$$
$$\text{for} \quad m > n, \quad |\beta| \geq 0.$$

Proof. Elastic energy. From Corollary 4.3.9 under additional assumptions for the regularity of the Cauchy data we get

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \int_{\mathbb{R}^n_\xi} \exp\Big(-C \int_0^t \frac{1}{b(\tau)} d\tau \Big) \Big(|\xi|^{|\beta|} |\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma} |\hat{u}_1(\xi)| \Big) d\xi$$
$$\lesssim \exp\big(-C \int_0^t \frac{1}{b(\tau)} d\tau \big) \Big(\sup_{\xi\in\mathbb{R}^n} \big| (1+|\xi|^2)^{\frac{|\beta|+m}{2}} \hat{u}_0(\xi) \big| + \sup_{\xi\in\mathbb{R}^n} \big| (1+|\xi|^2)^{\frac{|\beta|-\sigma+m}{2}} \hat{u}_1(\xi) \big| \Big)$$
$$\times \int_{\mathbb{R}^n_\xi} \frac{1}{(1+|\xi|^2)^{\frac{m}{2}}} d\xi.$$

Applying the $L^1 - L^\infty$ property of Fourier multipliers and the increasing behavior of $b(t)$ we obtain for $t \in (0,\infty)$

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \exp\Big(-C \int_0^t \frac{1}{b(\tau)} d\tau \Big) \Big(\|u_0\|_{H^{|\beta|+m,1}} + \|u_1\|_{H^{|\beta|-\sigma+m,1}} \Big) \text{ for } m > n.$$

These estimates imply an "exponential type decay".

Kinetic energy. From Corollary 4.3.9 again we may conclude

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \exp\big(-C \int_0^t \frac{1}{b(\tau)} d\tau \big) \int_{\mathbb{R}^n_\xi} \frac{1}{(1+|\xi|^2)^{\frac{m}{2}}} d\xi$$
$$\times \Big(\sup_{\xi\in\mathbb{R}^n} \big| (1+|\xi|^2)^{\frac{|\beta|+\sigma+m}{2}} \hat{u}_0(\xi) \big| + \sup_{\xi\in\mathbb{R}^n} \big| (1+|\xi|^2)^{\frac{|\beta|+m}{2}} \hat{u}_1(\xi) \big| \Big).$$

Applying the $L^1 - L^\infty$ property of Fourier multipliers and the increasing behavior of $b(t)$ we get for $t \in (0,\infty)$

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \exp\Big(-C \int_0^t \frac{1}{b(\tau)} d\tau \Big) \Big(\|u_0\|_{H^{|\beta|+\sigma+m,1}} + \|u_1\|_{H^{|\beta|+m,1}} \Big) \text{ for } m > n.$$

These estimates imply an "exponential type decay". This completes the proof. □

Small frequencies. Only the separating line $t_0(|\xi|)$ is of importance. For this reason we distinguish between the two cases $t \in (0, t_0(|\xi|)]$ and $t \in (0,\infty)$.

Treatment in the case $t \in (0, t_0(|\xi|)]$

Proposition 7.3.9. *The following estimates hold for small frequencies in the case* $t \in (0, t_0(|\xi|)]$:

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{pd\}\cap\{|\xi|\le C_2\}}(t,\xi) d\xi \lesssim \Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta} - \frac{n}{2\delta}} \|u_0\|_{L^1}$$
$$+\Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-2\delta\eta}{2\delta} - \frac{n}{2\delta}} \|u_1\|_{L^1} \quad \text{for} \quad |\beta| \ge \sigma,$$
$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{pd\}\cap\{|\xi|\le C_2\}}(t,\xi) d\xi \lesssim \Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{2\delta} - \frac{n}{2\delta}} \|u_0\|_{L^1}$$
$$+\Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta} - \frac{n}{2\delta}} \|u_1\|_{L^1} \quad \text{for} \quad |\beta| \ge 0.$$

Proof. The proof coincides with the proof of Proposition 7.3.2. □

Treatment in the case $t \in (0, \infty)$

Proposition 7.3.10. *The following estimates hold for small frequencies in the case* $t \in (0, \infty)$:

$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{|\xi|\le C_2\}} d\xi \lesssim \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}} \|u_0\|_{L^1}$$
$$+\Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|-2\delta\eta}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}} \|u_1\|_{L^1} \text{ for } \quad |\beta| \ge \sigma,$$
$$\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\le C_2\}} d\xi \lesssim \Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}} \|u_0\|_{L^1}$$
$$+\max\Big\{\Big(1 + \int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|+2\sigma-4\delta\eta}{2\sigma-2\delta} - \frac{n}{2\sigma-2\delta}}, \Big(1 + \int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta} - \frac{n}{2\delta}}\Big\} \|u_1\|_{L^1} \text{ for } |\beta| \ge 0.$$

Proof. The proof is the same as the proof to Proposition 7.2.4. By taking into consideration the increasing behavior of $b = b(t)$ and $\delta \in (0, \sigma/2)$ we conclude that the decay in the pseudo-differential is faster than the decay coming from the elliptic zone. □

Conclusion By using Propositions 7.3.8 and 7.3.10 we conclude the following result.

Theorem 7.3.11. *The following* $L^1 - L^\infty$ *estimates hold for the energies of higher order of* *the solutions to the Cauchy problem (7.3.3) :*

$$\|\nabla^\beta u(t,\cdot)\|_{L^\infty} \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}} \|u_0\|_{H^{|\beta|+m,1}}$$
$$+\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\eta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}} \|u_1\|_{H^{|\beta|-\sigma+m,1}} \text{ for } \quad m>n, \quad |\beta|\geq\sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|_{L^\infty} \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}} \|u_0\|_{H^{|\beta|+\sigma+m,1}}$$
$$+\max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-4\delta\eta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}}\Big\} \|u_1\|_{H^{|\beta|+m,1}}$$
$$\text{for } \ m>n, \ |\beta|\geq 0.$$

Interpolation

The Riesz-Thorin interpolation argument between the L^1-L^∞ estimates from Theorem 7.3.11 and the L^2-L^2 estimates from Theorem 4.3.11 yields the main result of this section.

Theorem 7.3.12. *Let us consider the Cauchy problem (7.3.3), where $b=b(t)$ satisfies the following assumptions : (B1) to (B6), where η is defined in (B6). Then the energies of higher order of the solutions $u=u(t,x)$ satisfy the following estimates :*

$$\|\nabla^\beta u(t,\cdot)\|_{L^q} \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{H^{|\beta|+N_p,p}}$$
$$+\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-2\delta\eta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}(\frac{1}{p}-\frac{1}{q})} \|u_1\|_{H^{|\beta|-\sigma+N_p,p}} \quad \text{for} \quad |\beta|\geq\sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|_{L^q} \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-2\delta\eta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{H^{|\beta|+\sigma+N_p,p}}$$
$$+\max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-4\delta\eta}{2\sigma-2\delta}-\frac{n}{2\sigma-2\delta}(\frac{1}{p}-\frac{1}{q})}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{2\delta}-\frac{n}{2\delta}(\frac{1}{p}-\frac{1}{q})}\Big\} \|u_1\|_{H^{|\beta|+N_p,p}}$$
$$\text{for} \quad |\beta|\geq 0,$$

where $p\in[1,2]$, $pq=p+q$ and $N_p > n\big(\frac{1}{p}-\frac{1}{q}\big)$.

7.4 Treatment of the case $\delta = \sigma/2$

7.4.1 Strictly increasing coefficient

In this section we study the special case of (7.0.1) for $\delta=\sigma/2$ and $b=b(t)$ is strictly increasing, that is, the model

$$\begin{cases} u_{tt}+(-\Delta)^\sigma u + b(t)(-\Delta)^{\sigma/2}u_t = 0, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \sigma>1. \end{cases} \tag{7.4.1}$$

We define the zones as before in Section 4.4. The extended phase space is divided for small frequencies into the remaining zone, the pseudo-differential zone and the elliptic zone. For large frequencies the extended phase space is divided into the remaining zone and the elliptic zone.

Treatment in the remaining zone

Proposition 7.4.1. *The following estimates hold for all* $t \in [0, t_1]$:

$$|\xi|^{|\beta|}|\hat{u}(t,\xi)| \lesssim |\xi|^{|\beta|}|\hat{u}_0(\xi)| + |\xi|^{|\beta|-\sigma}|\hat{u}_1(\xi)| \quad \textit{for} \quad |\beta| \geq \sigma,$$
$$|\xi|^{|\beta|}|\hat{u}_t(t,\xi)| \lesssim |\xi|^{|\beta|+\sigma}|\hat{u}_0(\xi)| + |\xi|^{|\beta|}|\hat{u}_1(\xi)| \quad \textit{for} \quad |\beta| \geq 0.$$

Proof. We introduce the energy $2E[u](t) := |\xi|^{2\sigma}\hat{u}(t,\xi)^2 + \hat{u}_t(t,\xi)^2$. Differentiation of the energy $E(u)(t)$ with respect to t gives

$$2d_tE[u](t) = 2|\xi|^{2\sigma}\hat{u}_t(t,\xi)\hat{u}(t,\xi) + 2\hat{u}_{tt}(t,\xi)\hat{u}_t(t,\xi) = 2\hat{u}_t(t,\xi)\big(|\xi|^{2\sigma}\hat{u}(t,\xi) + \hat{u}_{tt}(t,\xi)\big).$$

Together with (7.4.1) it follows

$$2d_tE[u](t) = -2b(t)|\xi|^{\sigma}|\hat{u}_t(t,\xi)|^2 \leq 0.$$

Thus the energy is decreasing for increasing t. We conclude

$$|\xi|^{2\sigma}|\hat{u}(t,\xi)|^2 + |\hat{u}_t(t,\xi)|^2 \lesssim |\xi|^{2\sigma}|\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2.$$

In this way we obtain the desired statement and we completed the proof of the proposition. □

$L^1 - L^\infty$ estimates

Large frequencies. For large frequencies we have the separating line $t = t_1$. Hence we study two cases $t \in (0, t_1]$ and $t \in [t_1, \infty)$.

Treatment in the case $t \in (0, t_1]$

Proposition 7.4.2. *The following estimates hold for large frequencies in the case* $t \in (0, t_1]$:

$$\int_{\mathbb{R}^n} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{|\xi|\geq C_1\}}d\xi \lesssim \|u_0\|_{H^{|\beta|+m,1}} + \|u_1\|_{H^{|\beta|-\sigma+m,1}} \quad \textit{for} \quad m > n, \quad |\beta| \geq \sigma,$$
$$\int_{\mathbb{R}^n} |\xi|^{|\beta|}|\hat{u}_t(t,\xi)|\chi_{\{|\xi|\geq C_1\}}d\xi \lesssim \|u_0\|_{H^{|\beta|+\sigma+m,1}} + \|u_1\|_{H^{|\beta|+m,1}} \quad \textit{for} \quad m > n, \quad |\beta| \geq 0.$$

Proof. We use the estimates from Proposition 7.4.1.
Elastic energy. We have

$$\int_{\mathbb{R}^n} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{|\xi|\geq C_1\}}d\xi$$
$$\lesssim \Big(\sup_{\xi\in\mathbb{R}^n}\big|(1+|\xi|^2)^{\frac{|\beta|+m}{2}}\hat{u}_0(\xi)\big| + \sup_{\xi\in\mathbb{R}^n}\big|(1+|\xi|^2)^{\frac{|\beta|-\sigma+m}{2}}\hat{u}_1(\xi)\big|\Big)\int_{\mathbb{R}^n}\frac{1}{(1+|\xi|^2)^{\frac{m}{2}}}d\xi.$$

Applying the $L^1 - L^\infty$ property of Fourier multipliers we get the inequality

$$\int_{\mathbb{R}^n} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \|u_0\|_{H^{|\beta|+m,1}} + \|u_1\|_{H^{|\beta|-\sigma+m,1}} \quad \text{for} \quad m>n, \quad |\beta|\geq\sigma.$$

Kinetic energy. We have

$$\begin{aligned}&\int_{\mathbb{R}^n} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \\ &\lesssim \Big(\sup_{\xi\in\mathbb{R}^n} \big|(1+|\xi|^2)^{\frac{|\beta|+\sigma+m}{2}} \hat{u}_0(\xi)\big| + \sup_{\xi\in\mathbb{R}^n} \big|(1+|\xi|^2)^{\frac{|\beta|+m}{2}} \hat{u}_1(\xi)\big| \Big) \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^{\frac{m}{2}}} d\xi.\end{aligned}$$

Applying the $L^1 - L^\infty$ property of Fourier multipliers we get the inequality

$$\int_{\mathbb{R}^n} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi \lesssim \|u_0\|_{H^{|\beta|+\sigma+m,1}} + \|u_1\|_{H^{|\beta|+m,1}} \quad \text{for} \quad m>n, \quad |\beta|\geq 0.$$

In this way the proposition is proved. □

Treatment in the case $t \in [t_1, \infty)$

Proposition 7.4.3. *The following estimates hold for large frequencies in the case $t \in [t_1,\infty)$:*

$$\begin{aligned}\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi &\lesssim \Big(1+\int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|}{\sigma}-\frac{n}{\sigma}} \|u_0\|_{L^1} \\ &\quad + \Big(1+\int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|-\sigma}{\sigma}-\frac{n}{\sigma}} \|u_1\|_{L^1} \quad \text{for} \quad m>n, \quad |\beta|\geq\sigma, \\ \int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{|\xi|\geq C_1\}} d\xi &\lesssim \Big(1+\int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|+\sigma}{\sigma}-\frac{n}{\sigma}} \|u_0\|_{L^1} \\ &\quad + \Big(1+\int_0^t \frac{1}{b(\tau)} d\tau\Big)^{-\frac{|\beta|}{\sigma}-\frac{n}{\sigma}} \|u_1\|_{L^1} \quad \text{for} \quad m>n, \quad |\beta|\geq 0.\end{aligned}$$

Small frequencies. For small frequencies we have the separating lines $t = t_1$ and $t_0(|\xi|)$. For this reason we study the three cases $t \in (0, t_1]$, $t \in [t_1, t_0(|\xi|)]$ and $t \in [t_0(|\xi|), \infty)$.

Treatment in the case $t \in (0, t_1]$

Proposition 7.4.4. *The following estimates hold for small frequencies in the case $t \in (0,t_1]$:*

$$\begin{aligned}&\int_{\mathbb{R}^n} |\xi|^{|\beta|} |\hat{u}(t,\xi)| \chi_{\{rem\}\cap\{|\xi|\leq C_2\}} d\xi \lesssim \|u_0\|_{L^1} + \|u_1\|_{L^1} \quad \text{for} \quad |\beta|\geq\sigma, \\ &\int_{\mathbb{R}^n} |\xi|^{|\beta|} |\hat{u}_t(t,\xi)| \chi_{\{rem\}\cap\{|\xi|\leq C_2\}} d\xi \lesssim \|u_0\|_{L^1} + \|u_1\|_{L^1} \quad \text{for} \quad |\beta|\geq 0.\end{aligned}$$

Treatment in the case $t \in [t_1, t_0(|\xi|)]$

Proposition 7.4.5. *The following estimates hold for small frequencies in the case* $t \in [t_1, t_0(|\xi|)]$:

$$\begin{aligned}
&\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{pd\cup rem\}\cap\{|\xi|\le C_2\}}(t,\xi)d\xi \\
&\quad\lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\sigma}-\frac{n}{\sigma}}\|u_0\|_{L^1} + \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma\eta}{\sigma}-\frac{n}{\sigma}}\|u_1\|_{L^1} \text{ for } |\beta|\ge\sigma,\\
&\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}_t(t,\xi)|\chi_{\{pd\cup rem\}\cap\{|\xi|\le C_2\}}(t,\xi)d\xi \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+2\sigma-\sigma\eta}{\sigma}-\frac{n}{\sigma}}\|u_0\|_{L^1}\\
&\quad+\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\sigma}-\frac{n}{\sigma}}\|u_1\|_{L^1} \text{ for } |\beta|\ge 0.
\end{aligned}$$

Proof. The proof coincides with the proof of Proposition 7.3.2. □

Treatment in the case $t \in [t_0(|\xi|), \infty)$

Proposition 7.4.6. *The following estimates hold for small frequencies in the case* $t \in [t_0(|\xi|), \infty)$:

$$\begin{aligned}
&\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}(t,\xi)|\chi_{\{|\xi|\le C_2\}}(t,\xi)d\xi \\
&\quad\lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{\sigma}-\frac{n}{\sigma}}\|u_0\|_{L^1} + \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-\sigma\eta}{\sigma}-\frac{n}{\sigma}}\|u_1\|_{L^1} \text{ for } |\beta|\ge\sigma,\\
&\int_{\mathbb{R}^n_\xi} |\xi|^{|\beta|}|\hat{u}_t(t,\xi)|\chi_{\{|\xi|\le C_1\}}(t,\xi)d\xi \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-\sigma\eta}{\sigma}-\frac{n}{\sigma}}\|u_0\|_{L^1}\\
&\quad+\max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-2\sigma\eta}{\sigma}-\frac{n}{\sigma}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\sigma}-\frac{n}{\sigma}}\Big\}\|u_1\|_{L^1} \text{ for } |\beta|\ge 0.
\end{aligned}$$

Proof. The proof is the same as the proof to Proposition 7.2.4. □

Conclusion To formulate the main result of this section we summarize the statements from Propositions 7.4.2 to 7.4.6.

Theorem 7.4.7. *The following* $L^1 - L^\infty$ *estimates hold for the energies of higher order for the solutions to the Cauchy problem (7.4.1)* :

$$\begin{aligned}
\|\nabla^\beta u(t,\cdot)\|_{L^\infty} &\lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{\sigma}-\frac{n}{\sigma}}\|u_0\|_{H^{|\beta|+m,1}}\\
&\quad+\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-\sigma\eta}{\sigma}-\frac{n}{\sigma}}\|u_1\|_{H^{|\beta|-\sigma+m,1}} \quad\text{for}\quad m>n,\quad |\beta|\ge\sigma,\\
\|\nabla^\beta u_t(t,\cdot)\|_{L^\infty} &\lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-\sigma\eta}{\sigma}-\frac{n}{\sigma}}\|u_0\|_{H^{|\beta|+\sigma+m,1}}\\
&+\max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-2\sigma\eta}{\sigma}-\frac{n}{\sigma}}, \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\sigma}-\frac{n}{\sigma}}\Big\}\|u_1\|_{H^{|\beta|+m,1}}\\
&\text{for}\quad m>n,\quad |\beta|\ge 0.
\end{aligned}$$

Interpolation

An interpolation argument between the $L^1 - L^\infty$ estimates from Theorem 7.4.7 and the $L^2 - L^2$ estimates from Theorem 4.4.7 implies the desired $L^p - L^q$ estimates on the conjugate line.

Theorem 7.4.8. *Let us consider the Cauchy problem (7.4.1), where $b = b(t)$ satisfies the following assumptions : (B1) to (B6). Then the energies of higher order for the solutions $u = u(t,x)$ satisfy the following $L^p - L^q$ decay estimates on the conjugate line :*

$$\|\nabla^\beta u(t,\cdot)\|_{L^q} \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|}{\sigma}-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{H^{|\beta|+N_p,p}}$$
$$+\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|-\sigma\eta}{\sigma}-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})}\|u_1\|_{H^{|\beta|-\sigma+N_p,p}} \quad \text{for} \quad |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|_{L^q} \lesssim \Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-\sigma\eta}{\sigma}-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{H^{|\beta|+\sigma+N_p,p}}$$
$$+\max\Big\{\Big(1+\int_0^t \frac{1}{b(\tau)}d\tau\Big)^{-\frac{|\beta|+2\sigma-2\sigma\eta}{\sigma}-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})},\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\sigma}-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})}\Big\}\|u_1\|_{H^{|\beta|+N_p,p}}$$
$$\text{for} \quad |\beta| \geq 0,$$

where $p \in [1,2]$, $pq = p+q$ and $N_p > n\big(\frac{1}{p}-\frac{1}{q}\big)$.

7.4.2 Strictly decreasing coefficient

Theorem 7.4.9. *Let us consider the Cauchy problem (7.2.1), where $b = b(t)$ satisfies the following assumptions (A1) to (A3) : Then the energies of higher order of the solution $u = u(t,x)$ satisfy in the case $\delta = \sigma/2$ the following $L^p - L^q$ decay estimates on the conjugate line :*

$$\|\nabla^\beta u(t,\cdot)\|_{L^q} \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\sigma}-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{H^{|\beta|+N_p,p}}$$
$$+\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|-\sigma}{\sigma}-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})}\|u_1\|_{H^{|\beta|-\sigma+N_p,p}} \quad \text{for} \quad |\beta| \geq \sigma,$$
$$\|\nabla^\beta u_t(t,\cdot)\|_{L^q} \lesssim \Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|+\sigma}{\sigma}-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{H^{|\beta|+\sigma+N_p,p}}$$
$$+\Big(1+\int_0^t b(\tau)d\tau\Big)^{-\frac{|\beta|}{\sigma}-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})}\|u_1\|_{H^{|\beta|+N_p,p}} \quad \text{for} \quad |\beta| \geq 0,$$

where $p \in [1,2]$, $pq = p+q$ and $N_p > n\big(\frac{1}{p}-\frac{1}{q}\big)$.

8 $L^1 - L^1$ estimates

In Chapter 5 we explained smoothing properties for structurally damped σ-evolution models. The main goal in this chapter is to derive $L^1 - L^1$ estimates for the elastic energy of the solution to (8.0.1) in the following sense:

$$\||D|^\sigma u(t,\cdot)\|_{L^1} \lesssim C_0(t)\|u_0\|_{L^1} + C_1(t)\|u_1\|_{L^1}.$$

We are interested to explain the behavior of the functions $C_0(t)$ and $C_1(t)$ for $t \to 0^+$ and $t \to \infty$. We are concerned with the following structural damped σ-evolution model :

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\sigma u(t,x) + b(t)(-\Delta)^{\sigma/2} u_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \sigma > 1, \end{cases} \tag{8.0.1}$$

where $b = b(t)$ is a positive decreasing function with $b(t) \in (0,2)$.

8.1 Objectives and strategies

Let us explain our strategy. It is divided into the following steps:

- We restrict ourselves to a class of models which allows to restrict our considerations to the hyperbolic zone.
- We use the main ideas from the previous chapters to derive estimates for the fundamental solution.
- The radial symmetry of the fundamental solution allows to apply the theory of modified Bessel functions. Our considerations will be divided into the cases that the radial parameter r is small or r is large.
- To derive the desired estimates we estimate oscillating integrals and apply Young's inequality.

8.2 Special models with strictly decreasing coefficient in the damping term

In this section we study (8.0.1) under the following conditions to the coefficient function $b = b(t)$:

(A1) *positivity* : $b(t) \in (0,2)$ for all $t \geq 0$,

(A2) *strictly decreasing behavior* : $b'(t) < 0$ for all $t \geq 0$.

Under these assumptions the extended phase space $\{(t,\xi) \in [0,\infty) \times \mathbb{R}^n_\xi\}$ consists only of the hyperbolic zone.

8.2.1 Representation of solution in the extended phase space

Following the treatment in the hyperbolic zone from Section 3.2.1 we may conclude the following representation of the solution:

$$\begin{pmatrix} p(t,\xi)v(t,\xi) \\ D_t v(t,\xi) \end{pmatrix} = M E_d(t,0,\xi) Q(t,0,\xi) M^{-1} \begin{pmatrix} p(0,\xi)v(0,\xi) \\ D_t v(0,\xi) \end{pmatrix}, \tag{8.2.1}$$

where the matrices

$$E_d(t,0,\xi) = \begin{pmatrix} \exp\Big(i\int_0^t p(s,\xi)ds\Big) & 0 \\ 0 & \exp\Big(-i\int_0^t p(s,\xi)ds\Big) \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and

$$Q(t,0,\xi) = I_2 + \sum_{k=1}^{\infty} \int_0^t iP(t_1,0,\xi) \int_0^{t_1} iP(t_2,0,\xi) \cdots \int_0^{t_{k-1}} iP(t_k,0,\xi) dt_k \cdots dt_2 dt_1. \tag{8.2.2}$$

Here $p(s,\xi) = |\xi|^\sigma \sqrt{1 - \frac{b^2(s)}{4}}$ and $P(t_k,0,\xi) = E_d(0,t_k,\xi)\mathcal{R}(t_k)E_d(t_k,0,\xi)$. Consequently, all the terms and matrices (besides data and solution) depend on t and $|\xi|$. For this reason we can write $p =: p(s,|\xi|)$, $E_d =: E_d(t,0,|\xi|)$, $P =: P(t,0,|\xi|)$ and $Q =: Q(t,0,|\xi|)$. Straightforward calculations imply

$$\begin{aligned} v(t,\xi) = &\Big(\tilde{h}_{11}(t,|\xi|)\exp\big(ic_2|\xi|^\sigma\tau(t)\big) + \tilde{h}_{12}(t,|\xi|)\exp\big(-ic_2|\xi|^\sigma\tau(t)\big)\Big)\tfrac{p(0,|\xi|)}{p(t,|\xi|)}v_0(\xi) \\ &+\Big(\tilde{h}_{21}(t,|\xi|)\exp\big(ic_2|\xi|^\sigma\tau(t)\big) + \tilde{h}_{22}(t,|\xi|)\exp\big(-ic_2|\xi|^\sigma\tau(t)\big)\Big)\tfrac{1}{ip(t,|\xi|)}v_1(\xi), \end{aligned} \tag{8.2.3}$$

where all $\tilde{h}_{kl}(t,|\xi|)$ depend on the entries of the matrix $Q(t,0,|\xi|)$ for $k,l = 1,2$ and $\tau(t) := \int_0^t \sqrt{1 - \frac{b^2(s)}{4}}ds$. After backward "dissipative" transformation the solution $\hat{u} = \hat{u}(t,\xi)$ satisfies the following representation:

$$\begin{aligned} |\xi|^\sigma\hat{u}(t,\xi) = &\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\Big(\Big(\tilde{h}_{11}(t,|\xi|)\exp\big(ic_2|\xi|^\sigma\tau(t)\big) + \tilde{h}_{12}(t,|\xi|)\exp\big(-ic_2|\xi|^\sigma\tau(t)\big)\Big) \\ &\times\tfrac{p(0,|\xi|)|\xi|^\sigma}{p(t,|\xi|)}\hat{u}_0(\xi) + \Big(\tilde{h}_{21}(t,|\xi|)\exp\big(ic_2|\xi|^\sigma\tau(t)\big) + \tilde{h}_{22}(t,|\xi|)\exp\big(-ic_2|\xi|^\sigma\tau(t)\big)\Big)\tfrac{b(0)|\xi|^{2\sigma}}{2ip(t,|\xi|)}\hat{u}_0(\xi) \\ &+\Big(\tilde{h}_{21}(t,|\xi|)\exp\big(ic_2|\xi|^\sigma\tau(t)\big) + \tilde{h}_{22}(t,|\xi|)\exp\big(-ic_2|\xi|^\sigma\tau(t)\big)\Big)\tfrac{|\xi|^\sigma}{ip(t,|\xi|)}\hat{u}_1(\xi)\Big), \end{aligned}$$

where $\Lambda(t) = \int_0^t b(s)ds$. By using the explicit representation of $p = p(t,|\xi|)$ for all $t \geq 0$ we get

$$\begin{aligned} &|\xi|^\sigma\hat{u}(t,\xi) = \\ &\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\Big(h_{11}(t,|\xi|)\exp\big(ic_2|\xi|^\sigma\tau(t)\big) + h_{12}(t,|\xi|)\exp\big(-ic_2|\xi|^\sigma\tau(t)\big)\Big)|\xi|^\sigma\hat{u}_0(\xi) \\ &+\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\Big(h_{21}(t,|\xi|)\exp\big(ic_2|\xi|^\sigma\tau(t)\big) + h_{22}(t,|\xi|)\exp\big(-ic_2|\xi|^\sigma\tau(t)\big)\Big)\hat{u}_1(\xi), \end{aligned} \tag{8.2.4}$$

where $h_{kl}(t,|\xi|)$ depend on the all entries of the matrix $Q(t,0,|\xi|)$ and $b = b(t)$ for $k,l = 1,2$.

8.2.2 L^p estimates for model oscillating integrals

In this section we derive L^p estimates for the oscillating integrals

$$F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big), \tag{8.2.5}$$

$$F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\cos\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big), \tag{8.2.6}$$

where all $h(t,|\xi|)$ depend on the entries of the matrix $Q(t,0,|\xi|)$ and $b(t)$. To derive L^p estimates we need the following statements.

Lemma 8.2.1. *The following estimates hold for all t and small frequencies $|\xi|$:*

$$\Big|\frac{\partial^k}{\partial|\xi|^k}P(t,0,|\xi|)\Big| \lesssim \tau(t)^k\|\mathcal{R}(t)\||\xi|^{\sigma-k} \quad \textit{for all} \quad k\geq 1. \tag{8.2.7}$$

Proof. The matrix $P:=P(t,0,|\xi|)$ is defined by $P(t,0,|\xi|)=E_d(0,t,|\xi|)\mathcal{R}(t)E_d(t,0,|\xi|)$. By Leibniz formula we get

$$\frac{\partial^k}{\partial|\xi|^k}P(t,0,|\xi|)=\sum_{j=0}^{k}C_{j,k}\frac{\partial^{k-j}}{\partial|\xi|^{k-j}}E_d(0,t,|\xi|)\mathcal{R}(t)\frac{\partial^j}{\partial|\xi|^j}E_d(t,0,|\xi|).$$

Taking into consideration the estimates $\big|\frac{\partial^j}{\partial|\xi|^j}E_d(t,0,|\xi|)\big|\lesssim\tau(t)^j|\xi|^{\sigma-j}$ for small frequencies we conclude the desired estimate. □

Lemma 8.2.2. *The following estimates hold for all t and for large frequencies $|\xi|$:*

$$\Big|\frac{\partial^k}{\partial|\xi|^k}P(t,0,|\xi|)\Big| \lesssim \tau(t)^k\|\mathcal{R}(t)\||\xi|^{k\sigma-k} \quad \textit{for all} \quad k\geq 1. \tag{8.2.8}$$

Proof. The proof follows the same strategy as in the proof to Lemma 8.2.1 by taking into consideration that $|\xi|$ is large. □

Corollary 8.2.3. *The solution $Q=Q(t,0,|\xi|)$ satisfies the following a-priori estimates :*

- *for small frequencies :*

$$\Big|\frac{\partial^k}{\partial|\xi|^k}Q(t,0,|\xi|)\Big| \lesssim \tau(t)^k|\xi|^{\sigma-k} \quad \textit{for all} \quad k\geq 1, \tag{8.2.9}$$

- *for large frequencies :*

$$\Big|\frac{\partial^k}{\partial|\xi|^k}Q(t,0,|\xi|)\Big| \lesssim \tau(t)^k|\xi|^{k\sigma-k} \quad \textit{for all} \quad k\geq 1. \tag{8.2.10}$$

Proof. We use the explicit representation for $Q=Q(t,0,|\xi|)$ from (8.2.2). Moreover, we use the estimates for $\Big|\frac{\partial^k}{\partial|\xi|^k}P(t,0,|\xi|)\Big|$ and take into consideration the cases $|\xi|$ large and small. □

To estimate $F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)$ in the L^p norm we use the theory of modified Bessel functions. Firstly, we will study the 3d case and then the 2d case. We will explain how the higher-dimensional case can be reduced to one of these two basic cases.

Lemma 8.2.4. *The following estimates hold in* $\mathbb{R}^3$:

$$\Big\|F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)\Big\|_{L^p(\mathbb{R}^3)} \lesssim \Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3\Lambda(t)^{-\frac{3}{\sigma}(1-\frac{1}{p})},$$
$$\Big\|F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\cos\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)\Big\|_{L^p(\mathbb{R}^3)} \lesssim \Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3\Lambda(t)^{-\frac{3}{\sigma}(1-\frac{1}{p})},$$

for $p\in[1,\infty]$ *and* $t>0$*. Here* c_1 *and* c_2 *are supposed to be positive constants.*

Proof. Using the radial symmetry of $\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)$ we have

$$\begin{aligned}F^{-1}_{\xi\to x}&\Big(\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)\\ &=\int_0^\infty \exp\big(-c_1 r^\sigma\Lambda(t)\big)\sin\big(c_2 r^\sigma\tau(t)\big)h(t,r)r^2 J_{1/2}(r|x|)dr,\end{aligned}$$

where $J_{1/2}(r|x|)$ is a modified Bessel function.
Our strategy is the following : In a first step we prove

$$\Big|F^{-1}_{\eta\to x}\Big(\exp\big(-c_1|\eta|^\sigma\big)\sin\Big(\frac{c_2|\eta|^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{|\eta|}{\Lambda(t)^{1/\sigma}}\Big)\Big)\Big| \lesssim \Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3\frac{1}{\langle x\rangle^4} \quad \text{for all} \quad x\in\mathbb{R}^3.$$

In a second step we shall apply the change of variables $\eta=\xi\Lambda(t)^{\frac{1}{\sigma}}$ and derive the representation

$$F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)=\frac{1}{\Lambda(t)^{\frac{3}{\sigma}}}G\Big(\frac{x}{\Lambda(t)^{\frac{1}{\sigma}}}\Big),$$

where

$$G(y)=\int_{\mathbb{R}^3}\exp(iy\eta)\exp\big(-c_1|\eta|^\sigma\big)\sin\Big(\frac{c_2|\eta|^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{|\eta|}{\Lambda(t)^{1/\sigma}}\Big)d\eta.$$

So, from the first step we have

$$\|G(\cdot)\|_{L^p(\mathbb{R}^3_y)}\lesssim\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3 \quad \text{for all} \quad p\in[1,\infty]$$

and after backward transformation we may conclude

$$\Big\|F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)\Big\|_{L^1(\mathbb{R}^3_x)} \lesssim \Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3,$$
$$\Big\|F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)\Big\|_{L^p(\mathbb{R}^3_x)} \lesssim \Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3\Lambda(t)^{-\frac{3}{\sigma}(1-\frac{1}{p})},$$

for all $p\in[1,\infty]$, respectively. Let us devote how to show the basic estimate

$$\Big|F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\big)\sin\Big(\frac{c_2|\xi|^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{|\xi|}{\Lambda(t)^{1/\sigma}}\Big)\Big)\Big| \lesssim \Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3\frac{1}{\langle x\rangle^4}.$$

For $|x|\le 1$ we have

$$\Big|F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\big)\sin\Big(\frac{c_2|\xi|^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{|\xi|}{\Lambda(t)^{1/\sigma}}\Big)\Big| \lesssim \int_{\mathbb{R}^n}\exp\big(-c_1|\xi|^\sigma\big)d\xi\lesssim 1.$$

Now let us restrict ourselves to $\{|x|>1\}$. In this case we take account of the radial symmetry and study the integral

$$\int_0^\infty \exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)r^2J_{1/2}(r|x|)dr.$$

Using the relation

$$J_{\frac{1}{2}}(r|x|)=-\frac{1}{r|x|^2}\partial_rJ_{-\frac{1}{2}}(r|x|)$$

and the explicit representation $J_{-\frac{1}{2}}(r|x|):=\cos(r|x|)$ we arrive at

$$\begin{aligned}
&\int_0^\infty \exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)r^2J_{1/2}(r|x|)dr\\
&=\frac{1}{|x|^2}\int_0^\infty \partial_r\Big(r\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\Big)\cos(r|x|)dr\\
&\quad=\frac{1}{|x|^2}\int_0^\infty(1-c_1\sigma r^\sigma)\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\
&\qquad+\frac{1}{|x|^2}\int_0^\infty c_2\sigma\frac{\tau(t)r^\sigma}{\Lambda(t)}\exp\big(-c_1r^\sigma\big)\cos\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\
&\qquad+\frac{1}{|x|^2}\int_0^\infty r\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial}{\partial r}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr.
\end{aligned}$$

To generate the decay rate $\langle x\rangle^{-4}$ we will apply two more steps of partial integration. After the second step of partial integration we obtain the sum of integrals

$$\begin{aligned}
&-\frac{1}{|x|^3}\int_0^\infty \partial_r\Big((1-c_1\sigma r^\sigma)\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\Big)\sin(r|x|)dr\\
&\quad-\frac{1}{|x|^3}\int_0^\infty \partial_r\Big(c_2\sigma\frac{\tau(t)r^\sigma}{\Lambda(t)}\exp\big(-c_1r^\sigma\big)\cos\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\Big)\sin(r|x|)dr\\
&\qquad-\frac{1}{|x|^3}\int_0^\infty \partial_r\Big(r\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial}{\partial r}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\Big)\sin(r|x|)dr.
\end{aligned}$$

A formal calculation gives

$$\begin{aligned}
I_1=&-\frac{1}{|x|^3}\int_0^\infty c_1\sigma\big(-(\sigma+1)+c_1\sigma r^\sigma\big)r^{\sigma-1}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\sin(r|x|)dr\\
&-\frac{1}{|x|^3}\int_0^\infty c_2\sigma(1-c_1\sigma r^\sigma)\frac{\tau(t)}{\Lambda(t)}r^{\sigma-1}\exp\big(-c_1r^\sigma\big)\cos\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\sin(r|x|)dr\\
&-\frac{1}{|x|^3}\int_0^\infty(1-c_1\sigma r^\sigma)\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial}{\partial r}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\sin(r|x|)dr,
\end{aligned}$$

$$\begin{aligned}
I_2=&-\frac{1}{|x|^3}\int_0^\infty c_2\sigma^2\big(1-c_1r^\sigma\big)\frac{\tau(t)}{\Lambda(t)}r^{\sigma-1}\exp\big(-c_1r^\sigma\big)\cos\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\sin(r|x|)dr\\
&+\frac{1}{|x|^3}\int_0^\infty c_2^2\sigma^2\frac{\tau(t)^2}{\Lambda(t)^2}r^{2\sigma-1}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\sin(r|x|)dr\\
&-\frac{1}{|x|^3}\int_0^\infty c_2\sigma\frac{\tau(t)}{\Lambda(t)}r^\sigma\exp\big(-c_1r^\sigma\big)\cos\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial}{\partial r}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\sin(r|x|)dr,
\end{aligned}$$

$$\begin{aligned}I_3 = &-\frac{1}{|x|^3}\int_0^\infty (1-c_1\sigma r^{\sigma-1})\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial}{\partial r}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\sin(r|x|)dr\\ &-\frac{1}{|x|^3}\int_0^\infty c_2\sigma\frac{\tau(t)}{\Lambda(t)}r^\sigma\exp\big(-c_1r^\sigma\big)\cos\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial}{\partial r}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\sin(r|x|)dr\\ &-\frac{1}{|x|^3}\int_0^\infty r\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial^2}{\partial r^2}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\sin(r|x|)dr.\end{aligned}$$

Finally, after the third step of partial integration we have the integrals

$$\begin{aligned}I_{1,1} = &-\tfrac{1}{|x|^4}\int_0^\infty c_1\sigma\big(-(\sigma-1)(\sigma+1)+(2c_1\sigma^2-c_1\sigma+\sigma+1)r^\sigma-c_1\sigma r^{2\sigma}\big)r^{\sigma-2}\\ &\quad\times\exp\big(-c_1r^\sigma\big)\sin\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &-\tfrac{1}{|x|^4}\int_0^\infty c_1c_2\sigma^2\big(-\sigma-1+c_1\sigma r^\sigma\big)\tfrac{\tau(t)}{\Lambda(t)}r^{2\sigma-2}\exp\big(-c_1r^\sigma\big)\cos\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &-\tfrac{1}{|x|^4}\int_0^\infty c_1\sigma\big(-\sigma-1+c_1\sigma r^\sigma\big)r^{\sigma-1}\exp\big(-c_1r^\sigma\big)\sin\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\tfrac{\partial}{\partial r}h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr,\end{aligned}$$

$$\begin{aligned}I_{1,2} = &-\tfrac{1}{|x|^4}\int_0^\infty c_2\sigma\big(\sigma-1-2c_1\sigma^2r^\sigma+c_1^2\sigma^2r^{2\sigma}\big)\tfrac{\tau(t)}{\Lambda(t)}r^{\sigma-2}\\ &\quad\times\exp\big(-c_1r^\sigma\big)\cos\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &\quad+\tfrac{1}{|x|^4}\int_0^\infty c_2^2\sigma^2(1-c_1\sigma r^\sigma)\tfrac{\tau(t)^2}{\Lambda(t)^2}r^{2\sigma-2}\exp\big(-c_1r^\sigma\big)\sin\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &\quad-\tfrac{1}{|x|^4}\int_0^\infty c_2\sigma(1-c_1\sigma r^\sigma)\tfrac{\tau(t)}{\Lambda(t)}r^{\sigma-1}\exp\big(-c_1r^\sigma\big)\cos\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\tfrac{\partial}{\partial r}h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr,\end{aligned}$$

$$\begin{aligned}I_{1,3} = &-\tfrac{1}{|x|^4}\int_0^\infty c_1\sigma\big(-\sigma-1+c_1\sigma r^\sigma\big)r^{\sigma-1}\exp\big(-c_1r^\sigma\big)\sin\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\tfrac{\partial}{\partial r}h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &\quad-\tfrac{1}{|x|^4}\int_0^\infty c_2\sigma(1-c_1\sigma r^\sigma)\tfrac{\tau(t)}{\Lambda(t)}r^{\sigma-1}\exp\big(-c_1r^\sigma\big)\cos\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\tfrac{\partial}{\partial r}h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &\quad-\tfrac{1}{|x|^4}\int_0^\infty(1-c_1\sigma r^\sigma)\exp\big(-c_1r^\sigma\big)\sin\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\tfrac{\partial^2}{\partial r^2}h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr,\end{aligned}$$

$$\begin{aligned}I_{2,1} = &-\tfrac{1}{|x|^4}\int_0^\infty c_2\sigma^2\big(\sigma-1-c_1(2\sigma-1)r^\sigma-c_1\sigma r^\sigma+c_1^2\sigma r^{2\sigma}\big)\tfrac{\tau(t)}{\Lambda(t)}r^{\sigma-2}\\ &\quad\times\exp\big(-c_1r^\sigma\big)\cos\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &\quad+\tfrac{1}{|x|^4}\int_0^\infty c_2^2\sigma^3\Big(1-c_1r^\sigma\Big)\tfrac{\tau(t)^2}{\Lambda(t)^2}r^{2\sigma-2}\exp\big(-c_1r^\sigma\big)\sin\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &\quad-\tfrac{1}{|x|^4}\int_0^\infty c_2\sigma^2\Big(1-c_1r^\sigma\Big)\tfrac{\tau(t)}{\Lambda(t)}r^{\sigma-1}\exp\big(-c_1r^\sigma\big)\cos\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\tfrac{\partial}{\partial r}h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr,\end{aligned}$$

$$\begin{aligned}I_{2,2} = &+\tfrac{1}{|x|^4}\int_0^\infty c_2^2\sigma^2\big(2\sigma-1-c_1\sigma r^\sigma\big)\tfrac{\tau(t)^2}{\Lambda(t)^2}r^{2\sigma-2}\exp\big(-c_1r^\sigma\big)\sin\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &\quad+\tfrac{1}{|x|^4}\int_0^\infty c_2^3\sigma^3\tfrac{\tau(t)^3}{\Lambda(t)^3}r^{3\sigma-2}\exp\big(-c_1r^\sigma\big)\cos\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &\quad+\tfrac{1}{|x|^4}\int_0^\infty c_2^2\sigma^2\tfrac{\tau(t)^2}{\Lambda(t)^2}r^{2\sigma-1}\exp\big(-c_1r^\sigma\big)\sin\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\tfrac{\partial}{\partial r}h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr,\end{aligned}$$

$$\begin{aligned}I_{2,3} = &-\tfrac{1}{|x|^4}\int_0^\infty c_2\sigma^2(1-c_1r^\sigma)\tfrac{\tau(t)}{\Lambda(t)}r^{\sigma-1}\exp\big(-c_1r^\sigma\big)\cos\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\tfrac{\partial}{\partial r}h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &\quad+\tfrac{1}{|x|^4}\int_0^\infty c_2^2\sigma^2\tfrac{\tau(t)^2}{\Lambda(t)^2}r^{2\sigma-1}\exp\big(-c_1r^\sigma\big)\sin\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\tfrac{\partial}{\partial r}h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &\quad-\tfrac{1}{|x|^4}\int_0^\infty c_2\sigma\tfrac{\tau(t)}{\Lambda(t)}r^\sigma\exp\big(-c_1r^\sigma\big)\cos\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\tfrac{\partial^2}{\partial r^2}h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr,\end{aligned}$$

$$\begin{aligned}I_{3,1} = &-\tfrac{1}{|x|^4}\int_0^\infty c_1\sigma\big(-(\sigma-1)-r+c_1\sigma r^\sigma\big)r^{\sigma-2}\exp\big(-c_1r^\sigma\big)\sin\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\tfrac{\partial}{\partial r}h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &\quad-\tfrac{1}{|x|^4}\int_0^\infty c_2\sigma(1-c_1\sigma r^{\sigma-1})\tfrac{\tau(t)}{\Lambda(t)}r^{\sigma-1}\exp\big(-c_1r^\sigma\big)\cos\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\tfrac{\partial}{\partial r}h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr\\ &\quad-\tfrac{1}{|x|^4}\int_0^\infty(1-c_1\sigma r^{\sigma-1})\exp\big(-c_1r^\sigma\big)\sin\Big(\tfrac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\tfrac{\partial^2}{\partial r^2}h\Big(t,\tfrac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr,\end{aligned}$$

$$I_{3,2} = -\frac{1}{|x|^4}\int_0^\infty c_2\sigma\big(\sigma-1-c_1\sigma r^\sigma\big)\frac{\tau(t)}{\Lambda(t)}r^{\sigma-1}\exp\big(-c_1r^\sigma\big)\cos\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial}{\partial r}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr$$
$$+\frac{1}{|x|^4}\int_0^\infty c_2^2\sigma^2\frac{\tau(t)^2}{\Lambda(t)^2}r^{2\sigma-1}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial}{\partial r}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr$$
$$-\frac{1}{|x|^4}\int_0^\infty c_2\sigma\frac{\tau(t)}{\Lambda(t)}r^\sigma\exp\big(-c_1r^\sigma\big)\cos\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial^2}{\partial r^2}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr,$$

$$I_{3,3} = -\frac{1}{|x|^4}\int_0^\infty(1-c_1\sigma r^\sigma)\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial^2}{\partial r^2}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr$$
$$-\frac{1}{|x|^4}\int_0^\infty c_2\sigma\frac{\tau(t)}{\Lambda(t)}r^\sigma\exp\big(-c_1r^\sigma\big)\cos\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial^2}{\partial r^2}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr$$
$$-\frac{1}{|x|^4}\int_0^\infty r\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial^3}{\partial r^3}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr.$$

Now we will estimate all these integrals. We divide them in $\int_0^1\cdots dr$ and $\int_1^\infty\cdots dr$. By using Corollary 8.2.3, $p(t,\xi)\sim|\xi|^\sigma$, all these integrals can be estimated by the following sum

$$\frac{1}{|x|^4}\int_0^1\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3r^{\sigma-2}\exp\big(-c_1r^\sigma\big)dr+\frac{1}{|x|^4}\int_1^\infty\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3r^{3\sigma-2}\exp\big(-c_1r^\sigma\big)dr.$$

Therefore we have

$$\frac{1}{|x|^4}\int_0^1\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3r^{\sigma-2}\exp\big(-c_1r^\sigma\big)dr\lesssim\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3\frac{1}{|x|^4},$$
$$\frac{1}{|x|^4}\int_1^\infty\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3r^{3\sigma-2}\exp\big(-c_1r^\sigma\big)dr\lesssim\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3\frac{1}{|x|^4}.$$

Summarizing we have shown $\|G(\cdot)\|_{L^p(\mathbb{R}^3)}\lesssim\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3$. This completes the proof. □

Moreover, we can conclude :

Lemma 8.2.5. *The following estimates hold in $\mathbb{R}^3$:*

$$\big\|F^{-1}_{\xi\to x}\Big(|\xi|^a\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)\big\|_{L^p(\mathbb{R}^3)}\lesssim\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^3\Lambda(t)^{-\frac{a}{\sigma}-\frac{3}{\sigma}(1-\frac{1}{p})},$$

where $\sigma>1,\ \ p\in[1,\infty)$ and $t>0$. Here c_1, c_2 and a are supposed to be positive constants.

Proof. The proof is same as before in Lemma 8.2.4. □

Lemma 8.2.6. *The following estimates hold in $\mathbb{R}^2$:*

$$\Big\|F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)\Big\|_{L^p(\mathbb{R}^2)}\lesssim\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^2\Lambda(t)^{-\frac{2}{\sigma}(1-\frac{1}{p})},$$

where $\sigma>1,\ \ p\in[1,\infty]$ and $t>0$. Here c_1 and c_2 are supposed to be positive constants.

Proof. Let us use a similar strategy to that in the proof to Lemma 8.2.4. We study the integral

$$\int_0^\infty\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)rJ_0(r|x|)dr.$$

From the relation $J_0(s)=2J_1(s)+sd_sJ_1(s)$ it follows immediately

$$\int_0^\infty r\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\big(2J_1(r|x|)+r\partial_rJ_1(r|x|)\big)dr$$
$$=-\int_0^\infty r^2\partial_r\Big(\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\Big)J_1(r|x|)dr.$$

Straight-forward calculations lead to the sum of integrals

$$\int_0^\infty c_1\sigma r^{\sigma+1}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)J_1(r|x|)dr$$
$$-\int_0^\infty c_2\sigma\frac{\tau(t)}{\Lambda(t)}r^{\sigma+1}\exp\big(-c_1r^\sigma\big)\cos\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)J_1(r|x|)dr$$
$$-\int_0^\infty r^2\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial}{\partial r}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)J_1(r|x|)dr.$$

We only consider the case $|x|\geq 1$. All integrals $\int_0^\infty$ we divide into $\int_0^{\frac{1}{|x|}}+\int_{\frac{1}{|x|}}^\infty$. In this way we have

$$\int_0^{\frac{1}{|x|}} c_1\sigma r^{\sigma+1}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)J_1(r|x|)dr$$
$$-\int_0^{\frac{1}{|x|}} c_2\sigma\frac{\tau(t)}{\Lambda(t)}r^{\sigma+1}\exp\big(-c_1r^\sigma\big)\cos\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)J_1(r|x|)dr$$
$$-\int_0^{\frac{1}{|x|}} r^2\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial}{\partial r}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)J_1(r|x|)dr$$
$$+\int_{\frac{1}{|x|}}^\infty c_1\sigma r^{\sigma+1}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)J_1(r|x|)dr$$
$$-\int_{\frac{1}{|x|}}^\infty c_2\sigma\frac{\tau(t)}{\Lambda(t)}r^{\sigma+1}\exp\big(-c_1r^\sigma\big)\cos\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)J_1(r|x|)dr$$
$$-\int_{\frac{1}{|x|}}^\infty r^2\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial}{\partial r}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)J_1(r|x|)dr.$$

Using the boundedness of $J_1(s)$ for $s\in[0,1]$ the first three integrals can be estimated by $\langle x\rangle^{-(2+\sigma)}$. To estimate the second three integrals we apply the following asymptotic formula for $J_1(s)$ for $s\in[1,\infty)$:

$$J_1(s)=C_1\frac{1}{s^{\frac{3}{2}}}\cos\Big(s-\frac{3}{4}\pi\Big)+\mathcal{O}\Big(\frac{1}{|s|^{\frac{5}{2}}}\Big).$$

Using the remainder gives

$$\int_{\frac{1}{|x|}}^\infty r^{\sigma+1}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\mathcal{O}\Big(\frac{1}{(r|x|)^{\frac{5}{2}}}\Big)dr\lesssim\frac{\tau(t)}{\Lambda(t)^{1/\sigma}\langle x\rangle^{5/2}}.$$

It remains to estimate the model integrals

$$\frac{1}{|x|^{3/2}}\int_{\frac{1}{|x|}}^\infty\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}r^{\sigma-1/2}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr$$

and

$$\frac{1}{|x|^{3/2}}\int_{\frac{1}{|x|}}^\infty\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}r^{\sigma-1/2}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\sin(r|x|)dr.$$

Here we proceed as in the proof to Lemma 8.2.4. We explain only the first integral (after dividing again into two integrals)

$$\frac{1}{|x|^{3/2}}\int_{\frac{1}{|x|}}^1\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}r^{\sigma-1/2}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr$$
$$+\frac{1}{|x|^{3/2}}\int_1^\infty\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}r^{\sigma-1/2}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr.$$

The first integral is equal to

$$\frac{1}{|x|^{5/2}}\int_{\frac{1}{|x|}}^{1}\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}r^{\sigma-1/2}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\partial_r\sin(r|x|)dr.$$

After partial integration the limit terms behave as $\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\langle x\rangle^{-(2+\sigma)}$. The new model integral can be estimated as follows:

$$\frac{1}{|x|^{5/2}}\int_{\frac{1}{|x|}}^{1}\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^2r^{\sigma-3/2}\exp\big(-c_1r^\sigma\big)dr\lesssim\frac{\tau(t)^2}{\Lambda(t)^{2/\sigma}\langle x\rangle^{5/2}}.$$

The second integral is equal to

$$\frac{1}{|x|^{5/2}}\int_{1}^{\infty}\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}r^{\sigma-1/2}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\partial_r\sin(r|x|)dr.$$

It can be estimated by $\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^2\langle x\rangle^{-5/2}$. In the same way we treat the integral

$$\frac{1}{|x|^{3/2}}\int_{\frac{1}{|x|}}^{\infty}\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}r^{\sigma-1/2}\exp\big(-c_1r^\sigma\big)\sin\Big(\frac{c_2r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\sin(r|x|)dr.$$

This completes the proof. □

Lemma 8.2.7. *The following estimates hold in* $\mathbb{R}^2$:

$$\Big\|F^{-1}_{\xi\to x}\Big(|\xi|^a\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)\Big\|_{L^p(\mathbb{R}^2)}\lesssim\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^2\Lambda(t)^{-\frac{a}{\sigma}-\frac{2}{\sigma}(1-\frac{1}{p})},$$

where $\sigma>1,\ \ p\in[1,\infty)$ *and* $t>0$*. Here* c_1*,* c_2 *and* a *are supposed to be positive constants.*

Proof. The proof is similar to the proofs in Lemmas 8.2.6 and 8.2.5. □

Finally, let us generalize the statements of Lemmas 8.2.4 and 8.2.6 to higher dimensions.

Lemma 8.2.8. *The following estimates hold in* $\mathbb{R}^n,\quad n\geq 4$:

$$n=2k+1:$$
$$\Big\|F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)\Big\|_{L^p(\mathbb{R}^n)}\lesssim\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^{\frac{n+3}{2}}\Lambda(t)^{-\frac{n}{\sigma}(1-\frac{1}{p})},$$
$$n=2k:$$
$$\Big\|F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)\Big\|_{L^p(\mathbb{R}^n)}\lesssim\Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^{\frac{n+2}{2}}\Lambda(t)^{-\frac{n}{\sigma}(1-\frac{1}{p})},$$

where $\sigma>1,\ \ p\in[1,\infty]$ *and* $t>0$*. Here* c_1 *and* c_2 *are supposed to be positive constants.*

Proof. If n is odd, then we carry out $\frac{n+1}{2}$ steps of partial integration. In $\frac{n-1}{2}$ steps we apply the rule

$$J_{\mu+1}(r|x|)=-\frac{1}{r|x|^2}\partial_rJ_\mu(r|x|).$$

Introducing the vector field $Xf(r) := \frac{d}{dr}\big(\frac{1}{r}f(r)\big)$ we obtain

$$X^k\big(r^l f(r)\big) := \sum_{j=0}^{k} c_{k,j} d^j f(r) r^{l-2k+j} \qquad \text{for all } r > 0. \tag{8.2.11}$$

By using the same strategy as in the proof to Lemma 8.2.4 for real non-negative μ we have

$$\begin{aligned}
&F^{-1}_{\xi\to x}\Big(\exp\big(-c_1|\xi|^\sigma\big)\sin\Big(\frac{c_2\tau(t)|\xi|^\sigma}{\Lambda(t)}\Big)h\Big(t,\frac{|\xi|}{\Lambda(t)}\Big)\Big)\\
&\quad= \int_0^\infty \exp\big(-c_1 r^\sigma\big)\sin\Big(\frac{c_2 r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)r^{n-1}J_{\frac{n}{2}-1}(r|x|)dr\\
&\quad= \frac{(-1)^{\frac{n-1}{2}}}{|x|^{n-1}}\int_0^\infty X^{\frac{n-1}{2}}\Big(r^{n-1}\exp\big(-c_1 r^\sigma\big)\sin\Big(\frac{c_2 r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\Big)J_{1/2}(r|x|)dr.
\end{aligned}$$

Applying formula (8.2.11) we get

$$\frac{(-1)^{\frac{n-1}{2}}}{|x|^{n-1}}\int_0^\infty \sum_{k=0}^{\frac{n-1}{2}} c_{n,k} r^k \frac{\partial^k}{\partial r^k}\Big(\exp\big(-c_1 r^\sigma\big)\sin\Big(\frac{c_2 r^\sigma\tau(t)}{\Lambda(t)}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\Big)J_{1/2}(r|x|)dr.$$

The application of Leibniz formula implies the integral

$$\frac{(-1)^{\frac{n-1}{2}}}{|x|^{n-1}}\int_0^\infty \sum_{k=0}^{\frac{n-1}{2}}\sum_{j=0}^{k}\sum_{i=0}^{j} c_{n,k,i,j} r^k \frac{\partial^{k-j}}{\partial r^{k-j}}\exp\big(-c_1 r^\sigma\big)\frac{\partial^{j-i}}{\partial r^{j-i}}\sin\Big(\frac{c_2 r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial^i}{\partial r^i}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)J_{1/2}(r|x|)dr.$$

Taking account of Faa di Bruno formula the last integral is equal to

$$\begin{aligned}
&\frac{(-1)^{\frac{n-1}{2}}}{|x|^{n-1}}\int_0^\infty \sum_{k=0}^{\frac{n-1}{2}}\sum_{j=0}^{k}\sum_{i=0}^{j} c_{n,k,i,j} r^k \sum_{p=1}^{k-j}\exp\big(-c_1 r^\sigma\big) r^{p\sigma-(k-j)}\\
&\qquad\times\sum_{l=1}^{j-i}\sin\Big(\frac{c_2 r^\sigma\tau(t)}{\Lambda(t)}+l\frac{\pi}{2}\Big)r^{l\sigma-(j-i)}\frac{\partial^i}{\partial r^i}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)J_{1/2}(r|x|)dr
\end{aligned}$$

$$\begin{aligned}
&+\frac{(-1)^{\frac{n-1}{2}}}{|x|^{n-1}}\int_0^\infty \sum_{k=0}^{\frac{n-1}{2}}\sum_{i=0}^{k} c_{n,k,i} r^k \exp\big(-c_1 r^\sigma\big)\frac{\partial^{k-i}}{\partial r^{k-i}}\sin\Big(\frac{c_2 r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial^i}{\partial r^i}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)J_{1/2}(r|x|)dr\\
&+\frac{(-1)^{\frac{n-1}{2}}}{|x|^{n-1}}\int_0^\infty \sum_{k=0}^{\frac{n-1}{2}}\sum_{j=0}^{k} c_{n,k,j} r^k \frac{\partial^{k-j}}{\partial r^{k-j}}\exp\big(-c_1 r^\sigma\big)\sin\Big(\frac{c_2 r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial^j}{\partial r^j}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)J_{1/2}(r|x|)dr,
\end{aligned}$$

where $\sum_{i=1}^{k-j} m_i = p, \quad \sum_{i=1}^{k-j} i m_i = k-j$ and $\sum_{i=1}^{j-i} m_i = l, \quad \sum_{i=1}^{j-i} i m_i = j-i$. Among all integrals the integrals

$$\frac{\tilde c}{|x|^{n-1}}\int_0^1 \big(1+cr^\sigma\big)\exp\big(-c_1 r^\sigma\big)\sin\Big(\frac{c_2 r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial^{\frac{n-1}{2}}}{\partial r^{\frac{n-1}{2}}}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr$$

and

$$\frac{\tilde c}{|x|^{n-1}}\int_1^\infty r^{\frac{n-1}{2}\sigma}\exp\big(-c_1 r^\sigma\big)\sin\Big(\frac{c_2 r^\sigma\tau(t)}{\Lambda(t)}\Big)\frac{\partial^{\frac{n-1}{2}}}{\partial r^{\frac{n-1}{2}}}h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)h\Big(t,\frac{r}{\Lambda(t)^{1/\sigma}}\Big)\cos(r|x|)dr,$$

where $\tilde{c}$ is a universal constant, have a dominant influence on the final estimates. Following the approach as in the proof to Lemma 8.2.4 (two steps more of partial integration and so on) gives immediately $\|G(\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim \Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^{\frac{n+3}{2}}$. This completes the proof for odd $n \geq 4$. Analogous to the odd case we can carry out $\frac{n}{2}-1$ steps of partial integration. Following the ideas for the odd case gives together with the approach from Lemma 8.2.6 immediately $\|G(\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim \Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^{\frac{n+2}{2}}$. This completes the proof. □

Lemma 8.2.9. *The following estimates hold in* $\mathbb{R}^n, \quad n \geq 4:$

$$n = 2k+1:$$
$$\Big\|F^{-1}_{\xi\to x}\Big(|\xi|^a \exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)\Big\|_{L^p(\mathbb{R}^n)} \lesssim \Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^{\frac{n+3}{2}}\Lambda(t)^{-\frac{a}{\sigma}-\frac{n}{\sigma}(1-\frac{1}{p})},$$
$$n = 2k:$$
$$\Big\|F^{-1}_{\xi\to x}\Big(|\xi|^a \exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)\sin\big(c_2|\xi|^\sigma\tau(t)\big)h(t,|\xi|)\Big)\Big\|_{L^p(\mathbb{R}^n)} \lesssim \Big(\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}\Big)^{\frac{n+2}{2}}\Lambda(t)^{-\frac{a}{\sigma}-\frac{n}{\sigma}(1-\frac{1}{p})},$$

where $\sigma > 1, \;\; p \in [1,\infty)$ *and* $t > 0$. *Here* c_1, c_2 *and* a *are supposed to be positive constants.*

Proof. The proof is similar to the proofs to Lemmas 8.2.8 and 8.2.5. □

8.2.3 $L^p - L^q$ estimates for the energy of solutions to Cauchy problems

Theorem 8.2.10. *Let us consider the Cauchy problem (8.0.1), where the coefficient* $b = b(t)$ *satisfies the conditions (A1), (A2). Then the solution* $u = u(t,x)$ *satisfies the following* $L^p - L^q$ *estimates for* $k \in \mathbb{N}:$

$$n = 2k+1:$$
$$\||D|^\sigma u(t,\cdot)\|_{L^p} \lesssim \tau(t)^{\frac{n+3}{2}}\Lambda(t)^{-1-\frac{n(3r-2)+3r}{2r\sigma}}\|u_0\|_{L^q} + \tau(t)^{\frac{n+3}{2}}\Lambda(t)^{-\frac{n(3r-2)+3r}{2r\sigma}}\|u_1\|_{L^q},$$
$$n = 2k:$$
$$\||D|^\sigma u(t,\cdot)\|_{L^p} \lesssim \tau(t)^{\frac{n+2}{2}}\Lambda(t)^{-1-\frac{n(3r-2)+2r}{2r\sigma}}\|u_0\|_{L^q} + \tau(t)^{\frac{n+2}{2}}\Lambda(t)^{-\frac{n(3r-2)+2r}{2r\sigma}}\|u_1\|_{L^q},$$

where
$$\tau(t) =: \int_0^t \sqrt{1-\frac{b^2(s)}{4}}ds, \quad \Lambda(t) = \int_0^t b(s)ds,$$
$1 \leq q \leq p \leq \infty$ *and* $1+\frac{1}{p} = \frac{1}{r}+\frac{1}{q}$.

Proof. By using the explicit representation of solution (8.2.4) and applying the properties of Fourier multipliers we get

$$F^{-1}_{\xi\to x}\Big(|\xi|^\sigma\hat{u}(t,\xi)\Big) =$$
$$\sum_{k=1}^{2}\sum_{l=1}^{2}F^{-1}_{\xi\to x}\Big(|\xi|^{(2-k)\sigma}\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)h_{kl}(t,|\xi|)\exp\big((-1)^{l+1}ic_2|\xi|^\sigma\tau(t)\big)\Big) * F^{-1}_{\xi\to x}\big(\hat{u}_{k-1}(\xi)\big).$$

Taking account the L^p norm and applying Young's inequality we get

$$\||D|^\sigma u(t,\cdot)\|_{L^p} \lesssim$$
$$\sum_{k=1}^{2}\sum_{l=1}^{2}\Big\|F^{-1}_{\xi\to x}\Big(|\xi|^{(2-k)\sigma}\exp\big(-c_1|\xi|^\sigma\Lambda(t)\big)h_{kl}(t,|\xi|)\exp\big((-1)^{l+1}ic_2|\xi|^\sigma\tau(t)\big)\Big)(t,\cdot)\Big\|_{L^r}\|u_{k-1}\|_{L^q},$$

where $1 \le q \le p \le \infty$ and $1+\frac{1}{p}=\frac{1}{r}+\frac{1}{q}$. Taking account the estimates from Lemmas 8.2.8 and 8.2.9 we get the desired results. In this way the Theorem is proved. □

In the same way we can prove for the kinetic energy the following result :

Theorem 8.2.11. *Let us consider the Cauchy problem (8.0.1), where the coefficient $b=b(t)$ satisfies the conditions (A1) and (A2). Then the kinetic energy of solution $u=u(t,x)$ satisfies the following $L^p - L^q$ estimates for $k \in \mathbb{N}$:*

$$\begin{aligned}
&n = 2k+1 :\\
&\|u_t(t,\cdot)\|_{L^p} \lesssim \tau(t)^{\frac{n+3}{2}}\Lambda(t)^{-1-\frac{n(3r-2)+3r}{2r\sigma}}\|u_0\|_{L^q} + \tau(t)^{\frac{n+3}{2}}\Lambda(t)^{-\frac{n(3r-2)+3r}{2r\sigma}}\|u_1\|_{L^q},\\
&n = 2k :\\
&\|u_t(t,\cdot)\|_{L^p} \lesssim \tau(t)^{\frac{n+2}{2}}\Lambda(t)^{-1-\frac{n(3r-2)+2r}{2r\sigma}}\|u_0\|_{L^q} + \tau(t)^{\frac{n+2}{2}}\Lambda(t)^{-\frac{n(3r-2)+2r}{2r\sigma}}\|u_1\|_{L^q},
\end{aligned}$$

where

$$\tau(t) =: \int_0^t \sqrt{1-\frac{b^2(s)}{4}}ds, \quad \Lambda(t) = \int_0^t b(s)ds,$$

$1 \le q \le p \le \infty$ *and* $1+\frac{1}{p}=\frac{1}{r}+\frac{1}{q}$.

Remark 8.2.1. *From the assumptions (A1) and (A2) we conclude that $\frac{\tau(t)}{\Lambda(t)^{1/\sigma}}$ behaves like the term $t\Lambda(t)^{-1/\sigma}$.*

Remark 8.2.2. *If we set formally $\sigma = 1$ and $b(t) = \mu$, $\mu \in (0,2)$, in the estimates from Theorem 8.2.10, then the approach from [NaRei12] brings the same result.*

8.2.4 Some examples

Typical examples for $b=b(t)$ are

$$\begin{aligned}
&b_n(t) = \mu(1+t)^{-\gamma}\big(\log(e+t)\big)^{-\gamma_1}\cdots\big(\log^{[n]}(e^{[n]}+t)\big)^{-\gamma_n}, \;\; \mu\in(0,2), \;\; \gamma\in(0,1),\\
&\text{and} \qquad \gamma_i > 0, \;\; \text{for} \;\; i = 1,\cdots,n,
\end{aligned}$$

where the functions $log^{[n+1]}$, $e^{[n+1]}$ are defined in Section 3.2.7.

Example 8.2.1. *If we choose $b(t) = (1+t)^{-\gamma}$, $\gamma \in (0,1)$, then $b=b(t)$ satisfies the assumptions of Theorem 8.2.10. The solution $u=u(t,x)$ to (8.0.1) satisfies the $L^p - L^q$ estimates for $k \in \mathbb{N}$:*

$$\begin{aligned}
&n = 2k+1 :\\
&\||D|^\sigma u(t,\cdot)\|_{L^p} \lesssim t^{\frac{n+3}{2}}\big((1+t)^{1-\gamma}-1\big)^{-1-\frac{n(3r-2)+3r}{2r\sigma}}\|u_0\|_{L^q} + t^{\frac{n+3}{2}}\big((1+t)^{1-\gamma}-1\big)^{-\frac{n(3r-2)+3r}{2r\sigma}}\|u_1\|_{L^q},\\
&n = 2k :\\
&\||D|^\sigma u(t,\cdot)\|_{L^p} \lesssim t^{\frac{n+2}{2}}\big((1+t)^{1-\gamma}-1\big)^{-1-\frac{n(3r-2)+2r}{2r\sigma}}\|u_0\|_{L^q} + t^{\frac{n+2}{2}}\big((1+t)^{-\gamma}-1\big)^{1-\frac{n(3r-2)+2r}{2r\sigma}}\|u_1\|_{L^q},
\end{aligned}$$

where $\gamma \in (0,1)$, $\quad 1 \le q \le p \le \infty$ *and* $1+\frac{1}{p}=\frac{1}{r}+\frac{1}{q}$.

Example 8.2.2. *If we choose $b(t) = (1+t)^{-\gamma}(\log(e+t))^{-1}$ with $\gamma \in (0,1)$, then $b=b(t)$ satisfies the assumptions of Theorem 8.2.10. The solution $u=u(t,x)$ to (8.0.1) satisfies the*

$L^p - L^q$ estimates for $k \in \mathbb{N}$:

$$n = 2k+1:$$
$$\||D|^\sigma u(t,\cdot)\|_{L^p} \lesssim t^{\frac{n+3}{2}}\big((1+t)^{1-\gamma}(\log(e+t))^{-1}-1\big)^{-1-\frac{n(3r-2)+3r}{2r\sigma}}\|u_0\|_{L^q}$$
$$+t^{\frac{n+3}{2}}\big((1+t)^{1-\gamma}(\log(e+t))^{-1}-1\big)^{-\frac{n(3r-2)+3r}{2r\sigma}}\|u_1\|_{L^q},$$
$$n = 2k:$$
$$\||D|^\sigma u(t,\cdot)\|_{L^p} \lesssim t^{\frac{n+2}{2}}\big((1+t)^{1-\gamma}(\log(e+t))^{-1}-1\big)^{-1-\frac{n(3r-2)+2r}{2r\sigma}}\|u_0\|_{L^q}$$
$$+t^{\frac{n+2}{2}}\big((1+t)^{1-\gamma}(\log(e+t))^{-1}-1\big)^{-\frac{n(3r-2)+2r}{2r\sigma}}\|u_1\|_{L^q}.$$

where $\gamma \in (0,1), \quad 1 \leq q \leq p \leq \infty$ and $1+\frac{1}{p}=\frac{1}{r}+\frac{1}{q}$.

Example 8.2.3. *If we choose $b(t)=(1+t)^{-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-\gamma_2}$ with $\gamma \in (0,1)$ and $\gamma_2>0$, then $b=b(t)$ satisfies the assumptions of Theorem 8.2.10. The solution $u=u(t,x)$ to (8.0.1) satisfies the $L^p - L^q$ estimates for $k \in \mathbb{N}$:*

$$n = 2k+1:$$
$$\||D|^\sigma u(t,\cdot)\|_{L^p} \lesssim t^{\frac{n+3}{2}}\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-\gamma_2}-1\Big)^{-1-\frac{n(3r-2)+3r}{2r\sigma}}\|u_0\|_{L^q}$$
$$+t^{\frac{n+3}{2}}\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-\gamma_2}-1\Big)^{-\frac{n(3r-2)+3r}{2r\sigma}}\|u_1\|_{L^q},$$
$$n = 2k:$$
$$\||D|^\sigma u(t,\cdot)\|_{L^p} \lesssim t^{\frac{n+2}{2}}\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-\gamma_2}-1\Big)^{-1-\frac{n(3r-2)+2r}{2r\sigma}}\|u_0\|_{L^q}$$
$$+t^{\frac{n+2}{2}}\Big((1+t)^{1-\gamma}\big(\log(e+t)\big)^{-1}\big(\log^{[2]}(e^{[2]}+t)\big)^{-\gamma_2}-1\Big)^{-\frac{n(3r-2)+2r}{2r\sigma}}\|u_1\|_{L^q},$$

where $\gamma \in (0,1), \quad 1 \leq q \leq p \leq \infty$ and $1+\frac{1}{p}=\frac{1}{r}+\frac{1}{q}$.

9 Concluding remarks and open problems

In this chapter we give an overview about open problems in connection with the considerations of this thesis. The list is not complete in any sense, it should only give some hints of possible generalizations, applications and also parallel developments.

9.1 $L^p - L^q$ estimates on the conjugate line

In the future we are interested in linear Cauchy problems of the form

$$\begin{cases} u_{tt}(t,x) + a^2(t)(-\Delta)^\sigma u(t,x) + b(t)(-\Delta)^\delta u_t(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \delta \in (0,\sigma], \quad \sigma > 1. \end{cases} \tag{9.1.1}$$

In Chapters 2 to 7 we developed an approach for deriving estimates for higher order energies and $L^p - L^q$ estimates on the conjugate line for $a(t) \equiv 1$. In the future we are interested to find an approach which allows to study above models with $a(t)$ having a monotonic behavior. Here the question for the influence of $a(t)$ on the proposed approach from Chapters 2 to 7 is of importance. The case $\sigma = 1$ and $\delta = 0$ is studied in the thesis of Bui Tang Bao Ngoc [?] or in the paper [4].

9.2 $L^p - L^q$ estimates away from the conjugate line

In Chapter 8 we prove $L^1 - L^1$ estimates and $L^p - L^q$ estimates away from the conjugate line for solutions to (9.1.1) only in the case $a(t) \equiv 1$, $b(t)$ is a strictly decreasing function with $b(t) \in (0,2)$ and $\delta = \sigma/2$. In a forthcoming project one should study at first the general strictly decreasing case for $b(t)$, then the case for strictly increasing $b(t)$ and, finally, to generalize the case $\delta = \sigma/2$ to $\delta \in (0,\sigma)$. Moreover, one can try to attack this issue also for special classes of coefficients $a(t)$.

9.3 Semi-linear models

Here we are interested in the semi-linear Cauchy problem

$$\begin{cases} u_{tt}(t,x) + a^2(t)(-\Delta)^\sigma u(t,x) + b(t)(-\Delta)^\delta u_t(t,x) = f(u), & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) =: u_0(x), \quad u_t(0,x) =: u_1(x), \quad \delta \in (0,\sigma], \quad \sigma > 1. \end{cases} \tag{9.3.1}$$

In a forthcoming project one should study the existence of a critical exponent for global existence of small data solutions. An effective tool are $L^p - L^q$ estimates away from the conjugate line. Precise estimates for the corresponding linear model allow to prove by fixed point methods local (in time) solutions and global (in time) small data solutions as well. Here $f(u) \approx |u|^p$. Finally, the optimality of a critical exponent should be explained.

10 Notation-Guide to the reader

10.0.1 Preliminaries

10.0.2 Frequently used function spaces

We collect some of the function spaces occurring in this thesis together with a short definition:

$L^p(\mathbb{R}^n)$	$L^p(\mathbb{R}^n) = \Big\{u : \ \mathbb{R}^n \to \mathbb{R} \ : \ \int_{\mathbb{R}^n} \lvert u(x)\rvert^p dx < \infty\Big\}$ where the function u are Lebesgue measurable and $1 \leq p < \infty$,
$\lVert\cdot\rVert_{L^p}$	to denotes the L^p norms $\lVert u\rVert_{L^p} = \Big(\int_{\mathbb{R}^n} \lvert u(x)\rvert^p dx\Big)^{\frac{1}{p}}$,
$L^\infty(\mathbb{R}^n)$	$L^\infty(\mathbb{R}^n) = \Big\{u : \ \mathbb{R}^n \to \mathbb{R} \ : \ \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \lvert u(x)\rvert < \infty\Big\}$ where the function u are Lebesgue measurable,
$\lVert\cdot\rVert_{L^\infty}$	$\lVert u\rVert_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \lvert u(x)\rvert = \inf\Big\{C > 0 : \sup_{x\in\Omega\subset\mathbb{R}^n} \lvert u(x)\rvert \geq C \text{such that} \mu(\Omega) = 0\Big\}$,
$H^{m,p}(\mathbb{R}^n)$	$H^{m,p}(\mathbb{R}^n) = \Big\{u \in L^p(\mathbb{R}^n) : \ \partial_x^\alpha u \in\in L^p(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{N}^n \text{ with } \lvert\alpha\rvert \leq m\Big\}$ where the function u are Lebesgue measurable, $m \in \mathbb{N}$ and $1 \leq p \leq \infty$,
$H^m(\mathbb{R}^n)$	to denote $H^m(\mathbb{R}^n) := H^{m,2}(\mathbb{R}^n)$
$\lVert\cdot\rVert_{H^{m,p}}$	$\lVert u\rVert_{H^{m,p}} = \sum_{\lvert\alpha\rvert\leq m}\Big(\int_{\mathbb{R}^n} \lvert\partial_x^\alpha u(x)\rvert^p dx\Big)^{\frac{1}{p}}$ for all $1 \leq p < \infty$,
$\lVert\cdot\rVert_{H^{m,\infty}}$	$\lVert u\rVert_{H^{m,\infty}} = \max_{\lvert\alpha\rvert\leq m} \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \big\lvert\partial_x^\alpha u(x)\big\rvert$,
$C^k(\mathbb{R}^n)$	space of k-times continuously differentiable functions,
$C^\infty(\mathbb{R}^n)$	inductive limit $C^\infty(\mathbb{R}^n) = \bigcap_{k=0}^\infty C^k(\mathbb{R}^n)$,
$S(\mathbb{R}^n)$	Schwartz space of rapidly decaying functions,
	$S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \ : \ x^\alpha D_x^\beta f(x) \in L^\infty \quad \forall\alpha,\beta\}$,
$F_{x\to\xi}$	Fourier transformation $F_{x\to\xi}[u](t,\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(t,x)dx$,
$F^{-1}_{\xi\to x}$	inverse Fourier transformation $F^{-1}_{x\to\xi}[u](t,x) = \int_{\mathbb{R}^n} e^{i\xi\cdot x} u(t,\xi)d\xi$,
$L^{2,\lvert\beta\rvert}$	to denote the image of $H^{\lvert\beta\rvert}$ by Fourier Transforms,

$\langle\cdot\rangle$	which stands for $\langle x\rangle = \sqrt{1+\|x\|^2}$,
$\|\cdot\|$	to denotes the absolute value,
$\\|\cdot\\|$	to denotes the norms for a vector or a matrix,
D_t	to denote $-i\partial_t \quad t \in \mathbb{R}$,
$\Delta := \Delta_x$	to denotes the Laplace operator with respect to $x \in \mathbb{R}^n \,:\, \Delta_x := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_{ii}^2}$,
∂_x^β	to denote if $\beta = (\beta_1, \beta_2, \cdot, \cdot, \cdot, \beta_n)^T \in \mathbb{R}^n$, then $\partial_x^\beta := \partial_x^\beta \partial_x^\beta \cdots \partial_x^\beta$,
	where $\|\beta\| := \beta_1 + \beta_2 + \cdots + \beta_n$.

The matrix norm has to be distinguished from norms in certain function spaces or operator norms. The corresponding space is used as index of this norm. Exceptions are the frequently used Lebesgue and Bessel potential spaces, where we set

10.0.3 Symbols used throughout the thesis

Some of the symbols are used in all chapters of the thesis and for convenience of the reader we will collect them here. The following list can also be seen as a list of definitions for these auxiliary functions. If the symbols are related to a particular chapter, we give also the corresponding reference. Our aim is the investigation of the Cauchy problem with time-dependent dissipation term $b(t)(\Delta)^{\delta}u_t$. Related to it we use

$V(t,\xi)$	micro-energy in the hyperbolic zone $V(t,\xi) = \big(p(t,\xi)v, D_t v\big)^T$, satisfies $D_t V = A_1(t,\xi)V$,
$E_1(t,s,\xi)$	fundamental solution to $D_t - \mathcal{D}(t,\xi) - \mathcal{R}_1(t,\xi)$,
$E_d(t,s,\xi)$	fundamental solution to diagonal part $D_t - \mathcal{D}(t,\xi)$,
$U(t,\xi)$	micro-energy in the elliptic zone, $U = \big(\lvert\xi\rvert^{\sigma}\hat{u}, D_t\hat{u}\big)^T$, satisfies $D_t U = A_2(t,\xi)U$,
$E_2(t,s,\xi)$	fundamental solution to $D_t - \mathcal{D}(t,\xi) - \mathcal{F}^{(1)}(t,\xi) - \mathcal{R}_2(t,\xi)$,
$E_d(t,s,\xi)$	fundamental solution to diagonal part $D_t - \mathcal{D}(t,\xi) - \mathcal{F}^{(1)}(t,\xi)$,
$U(t,\xi)$	micro-energy in the pseudo-differential zone $U(t,\xi) = \big(\frac{N}{\Lambda(t)^{\frac{\sigma}{2\delta}}}\hat{u}, D_t\hat{u}\big)^T$, satisfies $D_t U = A_3(t,\xi)U$,
$\mathcal{E}(t,s,\xi)$	fundamental solution to $D_t - A(t,\xi)$,
$\mathcal{R}_k(t,\xi)$	the remainder matrix after $k+1$ step of diagonalization procedure,
$\Lambda(t)$	$\Lambda(t) = \int_0^t b(\tau)d\tau$,
$\lambda(t,\xi)$	$\lambda(t,\xi) = \exp\Big(\frac{\lvert\xi\rvert^{2\delta}}{2}\int_0^t b(\tau)d\tau\Big)$,
$Z_{hyp}(\varepsilon)$	hyperbolic zone,
$R_{hyp}(\varepsilon)$	hyperbolic region,
$Z_{red}(\varepsilon)$	reduced zone,
$t_2(\lvert\xi\rvert)$	separating line between reduced and hyperbolic zone is solution to $\frac{b(t)}{2}\lvert\xi\rvert^{2\delta-\sigma} = 1-\varepsilon$,
$Z_{ell}(\varepsilon,N)$	elliptic zone,

$t_1(\|\xi\|)$	separating line between elliptic and reduced zone is solution to $\frac{b(t)}{2}\|\xi\|^{2\delta-\sigma} = 1+\varepsilon$,
$R_{ell}(\varepsilon, N)$	elliptic region,
$Z_{pd}(\varepsilon, N)$	pseudo-differential zone,
$t_0(\|\xi\|)$	separating line between pseudo-differential and elliptic zone is solution to $\Lambda(t)\|\xi\|^{2\delta} = N$,
$R_{pd}(N)$	pseudo-differential region,
$Z_{rem}(\varepsilon)$	remaining zone,
t_0	separating line between remaining and hyperbolic zone is solution to $b(t) = 2(1-\varepsilon)$,
t_1	separating line between remaining and elliptic zone is solution to $b(t) = 2(1+\varepsilon)$,
$\Im\lambda_k(t,\xi)$	Imaginary part of $\lambda_k(t,\xi)$,
$\Re\lambda_k(t,\xi)$	Real part of $\lambda_k(t,\xi)$,
$E[v](t,\xi)$	energy of v,
$E^{\|\beta\|}[v](t,\xi)$	energy of higher order of v,
S_α	classification of $b = (t)$ in the case of decreasing of $b = b(t)$,
S_η	classification of $b = b(t)$ in the case of increasing of $b = b(t)$,
α	$\alpha \in \big(1, \frac{\sigma}{2\delta}\big]$,
η	$\eta \in \big(0, \frac{\sigma}{2\delta}\big]$,
$E_k(t,s,\xi)$	fundamental solution to $D_t - \mathcal{D}(t,\xi) - \sum_0^{k-1} \mathcal{F}^{(p)}(t,\xi) - \mathcal{R}_{k-1}(t,\xi)$,
$\lesssim$	instead of $\leq C$, where C is a positive constant,
$E^{kl}(t,s,\xi)$	the entries of matrix $E(t,s,\xi)$ in line k and cologne l,
$\operatorname{diag}\mathcal{R}$	to denote the diagonal matrix with entries $r_{jj}, j = 1, \cdots, n$,
δ_{ij}	to denote $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$ for $i, j \in \mathbb{N}$,
$B_R(x_0)$	to denote the ball in $\mathbb{R}^n$ of radius R with center $x_0 \in \mathbb{R}^n$,

10.1 Basic tools

10.1.1 Diagonalization procedure

Representation of solution in the hyperbolic zone

In order to give a representation for solution to (3.0.1) in $Z_{hyp}(\varepsilon)$ we carry out one step of diagonalisation procedure. We denote by $v(t,\xi)$ the "dissipative transformation" $\hat{u}(t,\xi) = \exp\Big(-\frac{|\xi|^{2\delta}}{2}\int_0^t b(\tau)d\tau\Big)v(t,\xi)$. In the extended phase space the Cauchy problem for the damped $\sigma-$evolution problem (3.0.1) is written as the following Cauchy problem :

$$\begin{cases} v_{tt}(t,\xi) + \underbrace{\Big(|\xi|^{2\sigma} - \frac{b^2(t)}{4}|\xi|^{4\delta}\Big)}_{:=p^2(t,\xi)} v(t,\xi) - \underbrace{\frac{b'(t)}{2}|\xi|^{2\delta}}_{:=m(t,\xi)} v(t,\xi) = 0, \\ \\ v(s,\xi) =: v_0(\xi), \quad v_t(s,\xi) =: v_1(\xi), \end{cases} \tag{10.1.1}$$

where $p(t,\xi) \sim |\xi|^\sigma$ uniformly in the hyperbolic zone. Moreover, the Cauchy data

$$v_0(\xi) := \exp\Big(\frac{|\xi|^{2\delta}}{2}\int_0^s b(\tau)d\tau\Big)\hat{u}(s,\xi) \text{ and } v_1(\xi) := \exp\Big(\frac{|\xi|^{2\delta}}{2}\int_0^s b(\tau)d\tau\Big)\Big(\frac{b(s)}{2}|\xi|^{2\delta}\hat{u}(s,\xi)+\hat{u}_t(s,\xi)\Big),$$

at time level $t = s$.
We introduce the micro-energy $V = V(t,\xi)$ in $Z_{hyp}(\varepsilon)$ by

$$V(t,\xi) = \big(p(t,\xi)v(t,\xi), D_t v(t,\xi)\big)^T \quad \text{for all} \quad t \geq s \ \text{ and } \ (s,\xi) \in Z_{hyp}(\varepsilon).$$

The corresponding first-order system of problem (10.1.1), with respect to the micro-energy V, is stated as

$$D_t V(t,\xi) = \begin{pmatrix} 0 & p(t,\xi) \\ p(t,\xi) & 0 \end{pmatrix} V(t,\xi) + \frac{1}{p(t,\xi)}\begin{pmatrix} D_t p(t,\xi) & 0 \\ m(t,\xi) & 0 \end{pmatrix} V(t,\xi). \tag{10.1.2}$$

We shall use the notations

$$A = A(t,\xi) := \begin{pmatrix} 0 & p(t,\xi) \\ p(t,\xi) & 0 \end{pmatrix}, \qquad \mathcal{R} = \mathcal{R}(t,\xi) := \frac{1}{p(t,\xi)}\begin{pmatrix} D_t p(t,\xi) & 0 \\ m(t,\xi) & 0 \end{pmatrix}.$$

Let us carry out the first step of diagonalization. We want to diagonalize the matrix A. Since the eigenvalues of the first matrix are $\tau_1 = \tau_1(t,\xi) = -p(t,\xi)$ and $\tau_2 = \tau_2(t,\xi) = p(t,\xi)$ we introduce the corresponding matrix of eigenvectors M and its inverse matrix M^{-1} as

$$M := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \text{and} \quad M^{-1} := \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Setting $V^{(1)} = V^{(1)}(t,\xi) := M^{-1}V(t,\xi)$ for all $t \geq s$ and $(t,\xi) \in Z_{hyp}(\varepsilon)$ we obtain the following system for $V^{(1)}$:

$$D_t V^{(1)}(t,\xi) = M^{-1}A(t,\xi)M\ V^{(1)}(t,\xi) + M^{-1}\mathcal{R}(t,\xi)M\ V^{(1)}(t,\xi). \tag{10.1.3}$$

Straight-forward calculations imply

$$\mathcal{D}(t,\xi) := M^{-1}A(t,\xi)M = \begin{pmatrix} -p(t,\xi) & 0 \\ 0 & p(t,\xi) \end{pmatrix},$$
$$\mathcal{R}_1(t,\xi) := M^{-1}\mathcal{R}(t,\xi)M = \tfrac{1}{2p(t,\xi)} \begin{pmatrix} D_t p(t,\xi) - m(t,\xi) & D_t p(t,\xi) - m(t,\xi) \\ D_t p(t,\xi) + m(t,\xi) & D_t p(t,\xi) + m(t,\xi) \end{pmatrix}. \tag{10.1.4}$$

The entries of the second matrix $\mathcal{R}_1(t,\xi)$ are uniformly integrable over the hyperbolic zone. Taking into consideration the monotonic behavior of the function p the function $\partial_t p(t,\xi)$ does not change its sign and, therefore, for $t \geq s$ and $(s,\xi) \in Z_{hyp}(\varepsilon)$

$$\int_s^t \Big|\frac{D_\tau p(\tau,\xi)}{p(\tau,\xi)}\Big| d\tau \leq \Big|\int_s^t \frac{\partial_\tau p(\tau,\xi)}{p(\tau,\xi)} d\tau\Big| = \Big|\log \frac{p(t,\xi)}{p(s,\xi)}\Big|,$$

and $p(t,\xi) \sim |\xi|^\sigma$ uniformly in the hyperbolic zone. Furthermore,

$$\frac{m(t,\xi)}{p(t,\xi)} = \frac{-\partial_t q(t,\xi)}{\sqrt{|\xi|^{2\sigma} - q^2(t,\xi)}}, \quad q(t,\xi) := \frac{b(t)}{2}|\xi|^{2\delta}.$$

In the following we use that

$$\int_s^t \Big|\frac{\partial_\tau q(\tau,\xi)}{\sqrt{|\xi|^{2\sigma} - q^2(\tau,\xi)}}\Big| d\tau \lesssim \frac{1}{|\xi|^\sigma}\int_s^t -\partial_\tau q(\tau,\xi) d\tau \lesssim C_\varepsilon$$

is uniformly bounded with respect to $(s,\xi), (t,\xi) \in Z_{hyp}(\varepsilon)$. This and $p(t,\xi) \sim |\xi|^\sigma$ in the hyperbolic zone allow to stop the diagonalization procedure after the first step.

Representation of solution in the elliptic zone

In order to give a representation for the solution to (3.0.1) in $Z_{ell}(\varepsilon,N)$, we carry out a diagonalisation procedure with suitable remainder at each step. The following definition of symbol classes characterizes the necessary properties of the remainder.

Symbol classes in the elliptic zone

Definition 10.1.1. *Let us define the following classes of symbols related to the properties of $b=b(t)$ and $Z_{ell}(\varepsilon,N)$:*

$$S_{ell}^{l,m}\{m_1,m_2,m_3\} := \Big\{a(t,\xi) \in C^l\big((0,T];C^m(Z_{ell}(\varepsilon,N))\big) :$$
$$|\partial_t^k \partial_{|\xi|}^p a(t,\xi)| \leq C_{k,p}|\xi|^{m_1-p} b(t)^{m_2}\Big(\frac{1}{1+t}\Big)^{m_3+k}$$
$$\text{for all non-negative integers } k,\ p \text{ such that } k \leq l,\ p \leq m\Big\}.$$

The further considerations are basing on the following properties of the symbol classes:

Lemma 10.1.1. *The family of symbol classes $S_{ell}^{l,m}\{m_1,m_2,m_3\}$ generates a hierarchy having the following properties:*

- $S^{l,m}_{ell}\{m_1,m_2,m_3\} \subset S^{l,m}_{ell}\{m_1+2k\delta, m_2+k, m_3-k\}$ for $k \geq 0$,

- $S^{l,m}_{ell}\{m_1,m_2+k,m_3\} \subset S^{l,m}_{ell}\{m_1,m_2,m_3\}$ for $k \geq 0$,

- $S^{k_1,p_1}_{ell}\{m_1,m_2,m_3\} \subset S^{k_2,p_2}_{ell}\{m_1,m_2,m_3\}$ for $k_2 \leq k_1$, and $p_2 \leq p_1$,

- *if* $a \in S^{l,m}_{ell}\{m_1,m_2,m_3\}$ *and* $b \in S^{l,m}_{ell}\{k_1,k_2,k_3\}$, *then* $ab \in S^{l,m}_{ell}\{m_1+k_1, m_2+k_2, m_3+k_3\}$,

- *if* $a \in S^{l,m}_{ell}\{m_1,m_2,m_3\}$, *then* $\partial_t^k a \in S^{l-k,m}_{ell}\{m_1,m_2,m_3+k\}$ *and* $\partial^p_{|\xi|} a \in S^{l,m-p}_{ell}\{m_1-p,m_2,m_3\}$ *for* $k \leq l$ *and* $p \leq m$.

Diagonalization modulo $S^{l,m}_{ell}\{0,0,1\}$

After partial Fourier transformation the damped $\sigma-$evolution problem (3.0.1) is written as the following Cauchy problem at time level $t = s$:

$$\begin{cases} \hat{u}_{tt}(t,\xi) + |\xi|^{2\sigma}\hat{u}(t,\xi) + b(t)|\xi|^{2\delta}\hat{u}_t(t,\xi) = 0, \\ \hat{u}(s,\xi), \quad \hat{u}_t(s,\xi), \quad \delta \in (0,\sigma), \quad \sigma > 1. \end{cases} \tag{10.1.5}$$

We introduce the micro-energy $U = U(t,\xi)$ in the $Z_{ell}(\varepsilon, N)$ by

$$U(t,\xi) = \big(|\xi|^\sigma \hat{u}(t,\xi), D_t\hat{u}(t,\xi)\big)^T \quad \text{for all} \quad t \geq s \quad \text{and} \quad (t,\xi),(s,\xi) \in Z_{ell}(\varepsilon,N).$$

The corresponding first-order system of problem (10.1.5), with respect to the micro-energy U, is stated as

$$D_tU(t,\xi) = \begin{pmatrix} 0 & |\xi|^\sigma \\ |\xi|^\sigma & ib(t)|\xi|^{2\delta} \end{pmatrix} U(t,\xi), \qquad U(s,\xi) = \big(|\xi|^\sigma \hat{u}(s,\xi), D_t\hat{u}(s,\xi)\big)^T. \tag{10.1.6}$$

We shall use the notation

$$A = A(t,\xi) := \begin{pmatrix} 0 & |\xi|^\sigma \\ |\xi|^\sigma & ib(t)|\xi|^{2\delta} \end{pmatrix}.$$

Let us carry out the first step of diagonalization, we want to diagonalize the matrix A. Since the eigenvalues of the first matrix $A = A(t,\xi)$ are

$$\lambda_k(t,\xi) = \frac{ib(t)|\xi|^{2\delta} + (-1)^{k-1} i\sqrt{b^2(t)|\xi|^{4\delta} - 4|\xi|^{2\sigma}}}{2}, \quad k = 1,2,$$

we introduce the corresponding matrix of eigenvectors M and its inverse matrix M^{-1} as

$$M(t,\xi) := \begin{pmatrix} 1 & 1 \\ \lambda_1(t,\xi)|\xi|^{-\sigma} & \lambda_2(t,\xi)|\xi|^{-\sigma} \end{pmatrix},$$
$$\text{and} \quad M^{-1}(t,\xi) := \frac{1}{\sqrt{b^2(t)|\xi|^{4\delta-2\sigma} - 4}} \begin{pmatrix} \lambda_2(t,\xi)|\xi|^{-\sigma} & -1 \\ -\lambda_1(t,\xi)|\xi|^{-\sigma} & 1 \end{pmatrix}.$$

Setting $U^{(1)} = U^{(1)}(t,\xi) := M^{-1}U(t,\xi)$ for all $t \geq s$ and $(t,\xi) \in Z_{ell}(\varepsilon, N)$ we obtain the following system for $U^{(1)}$:

$$D_t U^{(1)}(t,\xi) = M^{-1}(t,\xi)A(t,\xi)M(t,\xi)\; U^{(1)}(t,\xi) + M^{-1}(t,\xi)D_t M(t,\xi)\; U^{(1)}(t,\xi). \quad (10.1.7)$$

Straight-forward calculations imply

$$\mathcal{D}(t,\xi) := M^{-1}(t,\xi)A(t,\xi)M(t,\xi) = \begin{pmatrix} \lambda_1(t,\xi) & 0 \\ 0 & \lambda_2(t,\xi) \end{pmatrix}, \quad (10.1.8)$$

$$\mathcal{R}_1(t) := M^{-1}(t,\xi)D_t M(t,\xi) = -\frac{1}{2}\begin{pmatrix} a+b & a-b \\ \\ -a-b & -a+b \end{pmatrix}, \quad (10.1.9)$$

$$\text{where} \qquad a := \frac{b'(t)|\xi|^{2\delta}}{i\sqrt{b^2(t)|\xi|^{4\delta} - 4|\xi|^{2\sigma}}}, \qquad b := \frac{b(t)b'(t)|\xi|^{4\delta}}{i(b^2(t)|\xi|^{4\delta} - 4|\xi|^{2\sigma})}. \quad (10.1.10)$$

The system (10.1.7) has diagonal "principal" part $\mathcal{D} \in S_{ell}^{l,m}\{2\delta, 1, 0\}$ with remainder $\mathcal{R}_1 \in S_{ell}^{l,m}\{0,0,1\}$. Thus, we have obtained in $Z_{ell}(\varepsilon, N)$ the diagonalization of the system (10.1.6) modulo remainder $\mathcal{R}_1 \in S_{ell}^{l,m}\{0,0,1\}$.

Diagonalization modulo $S_{ell}^{l-1,m}\{-2\delta, -1, 2\}$

We carry out one more step of diagonalization procedure. Let

$$\mathcal{N}^{(1)} = \mathcal{N}^{(1)}(t,\xi) := \begin{pmatrix} 0 & -\frac{\mathcal{R}_1^{12}}{\tau_1 - \tau_2} \\ -\frac{\mathcal{R}_1^{21}}{\tau_2 - \tau_1} & 0 \end{pmatrix} \in S_{ell}^{l,m}\{-2\delta, -1, 1\}.$$

Now we set $\mathcal{N}_1 = \mathcal{N}_1(t,\xi) := I_2 + \mathcal{N}^{(1)}(t,\xi)$. This matrix is invertible since

$$\|\mathcal{N}^{(1)}(t,\xi)\| \leq \frac{1}{|\xi|^\sigma(1+t)} \leq \frac{C}{N} \qquad \text{by the definition of elliptic zone } Z_{ell}(\varepsilon, N).$$

We can choose a sufficiently large N such that $\|\mathcal{N}_1(t,\xi) - I\| < \frac{1}{2}$ in $Z_{ell}(\varepsilon, N)$. Hence, the matrix $\mathcal{N}_1(t,\xi)$ is invertible.

Proposition 10.1.2. *Let* $\mathcal{R}_1 := -\mathcal{N}_1^{-1}\underbrace{\Big(\big(D_t - \mathcal{R}_1\big)\mathcal{N}^{(1)} + \mathcal{N}^{(1)}\mathcal{F}^{(1)}\Big)}_{:=\mathcal{B}^{(1)}}$. *Then in* $Z_{ell}(\varepsilon, N)$ *we have* $\mathcal{R}_2 \in S_{ell}^{l,m}\{-2\delta, 1, 2\}$ *and the following identity holds* :

$$\big(D_t - \mathcal{D} - \mathcal{R}_1\big)\mathcal{N}_1 = \mathcal{N}_1\big(D_t - \mathcal{D} - \mathcal{F}^{(1)} - \mathcal{R}_2\big),$$

where $\mathcal{F}^{(1)} = \operatorname{diag} \mathcal{R}_1$ *and* $\mathcal{D}$, $\mathcal{R}_1$ *are defined in (10.1.8).*

Proof. The representation of $\mathcal{R}_1$ follows immediately from the observation that $[\mathcal{N}^{(1)}, \mathcal{D}] = \mathcal{R}_1 - \mathcal{F}^{(1)}$. By taking into consideration the matrices $\mathcal{F}^{(1)} \in S_{ell}^{l,m}\{0,0,1\}$, $\mathcal{N}^{(1)} \in S_{ell}^{l,m}\{-2\delta,-1,1\}$ and $\mathcal{R}_1 \in S_{ell}^{l,m}\{0,0,1\}$ and using the propriety of symbol classes from Lemma 10.1.1 we may conclude $\mathcal{R}_2 \in S_{ell}^{l-1,m}\{-2\delta, -1, 2\}$. □

Thus, we have obtained in $Z_{ell}(\varepsilon,N)$ the diagonalization of the system (10.1.6) modulo remainder $\mathcal{R}_2 \in S_{ell}^{l-1,m}\{-2\delta,-1,2\}$. Let $U^{(2)} = U^{(2)}(t,\xi) := \mathcal{N}_1^{-1}(t,\xi)M^{-1}(t,\xi)U(t,\xi)$, then we obtain the following equivalent problem to (10.1.6) for $U^{(2)} = U^{(2)}(t,\xi)$ for all $t \geq s$ and $(t,\xi),(s,\xi) \in Z_{ell}(\varepsilon,N)$:

$$\big(D_t - \mathcal{D} - \mathcal{F}^{(1)} - \mathcal{R}_2\big)U^{(2)} = 0, \qquad U^{(2)}(s,\xi) := \mathcal{N}_1^{-1}(s,\xi)M^{-1}(s,\xi)U(s,\xi), \tag{10.1.11}$$

where $\mathcal{R}_2 \in S_{ell}^{l-1,m}\{-2\delta,-1,2\}$. Thus, we have obtained in $Z_{ell}(\varepsilon,N)$ the diagonalization of the system (10.1.6) modulo remainder $\mathcal{R}_2 \in S_{ell}^{l-1,m}\{-2\delta,-1,2\}$.

Diagonalization modulo $S_{ell}^{l-k,m}\{-2k\delta,-k,k+1\}$

We carry out $k+1$ steps of the diagonalization procedure.

Proposition 10.1.3. *For each $k \in \mathbb{N}$ with $k \leq l$ let $\mathcal{B}^{(k)} = \big(D_t - \mathcal{R}_1\big)\mathcal{N}^{(k)} + \mathcal{N}^{(k)}\sum_{p=1}^{k}\mathcal{F}^{(p)} + \sum_{p=1}^{k-1}\mathcal{N}^{(p)}\mathcal{F}^{(k)}$. Then the following identity holds in $Z_{ell}(\varepsilon,N)$:*

$$\big(D_t - \mathcal{D} - \mathcal{R}_1\big)\mathcal{N}_k = \mathcal{N}_k\Big(D_t - \mathcal{D} - \sum_{p=1}^{k}\mathcal{F}^{(p)} - \mathcal{R}_{k+1}\Big), \tag{10.1.12}$$

where $\mathcal{R}_{k+1} := -\mathcal{N}_k^{-1}\mathcal{B}^{(k)}$ and the matrix-valued functions for all $p = 1,2,\cdots,k$ satisfy :

- $\mathcal{N}^{(p)} = \mathcal{N}^{(p)}(t,\xi) \in S_{ell}^{l-p+1,m}\{-2p\delta,-p,p\}$,
- $\mathcal{F}^{(p)} = \mathcal{F}^{(p)}(t,\xi) \in S_{ell}^{l-p,m}\{-2(p-1)\delta,-(p-1),p\}$,
- $\mathcal{R}_{p+1} = \mathcal{R}_{p+1}(t,\xi) \in S_{ell}^{l-p-1,m}\{-2p\delta,-p,p+1\}$,

where the matrix $\mathcal{N}_k := \sum_{p=0}^{k}\mathcal{N}^{(p)}$, $\mathcal{N}^{(0)} := I_2$ is invertible and $\mathcal{D}$, $\mathcal{R}_1$ are defined in (10.1.8).

Proof. We apply the principle of induction. We see that the representation of the matrices $\mathcal{B}^{(1)}$ and $\mathcal{R}_1$ given in Proposition 10.1.2 also satisfy the matrix structure displayed in the claim of Proposition 10.1.3. Let us assume that the statements of Proposition 10.1.3 hold true for an integer $j \leq k-1$. Thus, the matrices $\mathcal{N}^{(j)} \in S_{ell}^{l-j+1,m}\{-2j\delta,-j,j\}$, $\mathcal{F}^{(j)} \in S_{ell}^{l-j+1,m}\{-2(j-1)\delta,-(j-1),j\}$, and the matrices $\mathcal{B}^{(k-1)}$ and $\mathcal{R}_{k-1}$ satisfy the given structures. Then by (10.1.12) we immediately get

$$\mathcal{B}^{(k)} := \big(D_t - \mathcal{D} - \mathcal{R}_1\big)\mathcal{N}_k - \mathcal{N}_k\Big(D_t - \mathcal{D} - \sum_{p=1}^{k}\mathcal{F}^{(p)}\Big).$$

We seek representation for $\mathcal{N}_k$, $\mathcal{F}_k$ in the form

$$\mathcal{N}_k = \sum_{p=0}^{k}\mathcal{N}^{(p)}, \qquad \mathcal{F}_k = \sum_{p=1}^{k}\mathcal{F}^{(p)}.$$

To do this, define inductively the matrix-valued function in the manner below :

$$\mathcal{N}^{(0)} := I_2, \quad \mathcal{B}^{(0)} := -\mathcal{R}_1, \quad \mathcal{F}^{(p+1)} := -\operatorname{diag}\mathcal{B}^{(p)}, \quad \mathcal{N}^{(p+1)} := \begin{pmatrix} 0 & \frac{\mathcal{B}_{12}^{(p)}}{\tau_1-\tau_2} \\ \frac{\mathcal{B}_{21}^{(p)}}{\tau_2-\tau_1} & 0 \end{pmatrix}.$$

By using $\mathcal{N}_k = \mathcal{N}_{k-1} + \mathcal{N}^{(k)}$ and the definition of $\mathcal{B}^{(k-1)}$ we get

$$\begin{aligned}\mathcal{B}^{(k)} &= \mathcal{B}^{(k-1)} + \big(D_t - \mathcal{D} - \mathcal{R}_1\big)\mathcal{N}^{(k)} + \sum_{p=0}^{k-1} \mathcal{N}^{(p)}\mathcal{F}^{(k)} - \mathcal{N}^{(k)}\Big(D_t - \mathcal{D} - \sum_{p=1}^{k} \mathcal{F}^{(p)}\Big)\\ &= \mathcal{B}^{(k-1)} + \mathcal{F}^{(k)} - [\mathcal{D}, \mathcal{N}^{(k)}] + \mathcal{S},\end{aligned}$$

where $\mathcal{S} = D_t\mathcal{N}^{(k)} - \mathcal{R}_1\mathcal{N}^{(k)} + \sum_{p=1}^{k-1}\mathcal{N}^{(p)}\mathcal{F}^{(k)} + \mathcal{N}^{(k)}\sum_{p=1}^{k}\mathcal{F}^{(p)}$, by the definition of $\mathcal{F}^{(k)}$ and $\mathcal{N}^{(k)}$ we have

$$\mathcal{B}^{(k-1)} + \mathcal{F}^{(k)} - [\mathcal{D}, \mathcal{N}^{(k)}] = 0.$$

Therefor $\mathcal{B}^{(k)} = \mathcal{S}$ which lies in $S_{ell}^{l-k,m}\{-k\sigma, 1, k+1\}$ by the induction hypothesis and the rules of the symbolic calculus of Lemma 10.1.1. Setting $\mathcal{R}_{k+1} := -\mathcal{N}_k^{-1}\mathcal{B}^{(k)}$ this matrix belongs to $S_{ell}^{l-k,m}\{-2k\delta, -k, k+1\}$. Finally, $\mathcal{F}^{(k)} \in S_{ell}^{l-k+1,m}\{-2(k-1)\delta, -(k-1), k\}$ and $\mathcal{N}^{(k)} \in S_{ell}^{l-k+1,m}\{-2k\delta, -k, k\}$. So the claim is proved. Now we claim that $\mathcal{N}_k(t,\xi) = \sum_{p=0}^{m}\mathcal{N}^{(p)}(t,\xi)$ is invertible; this is true because

$$\|\mathcal{N}^{(p)}(t,\xi)\| \le \frac{1}{|\xi|^{\sigma p}(1+t)^p} \le \frac{C_p}{\big(\Lambda(t)|\xi|^{\sigma}\big)^p} \le \frac{C_p}{N^p},$$

by the definition of $Z_{ell}(\varepsilon, N)$. Choose N in the definition of $Z_{ell}(\varepsilon, N)$ so that

$$\frac{C_p}{N^p} \le \frac{1}{2^{p+1}} \qquad \text{for} \quad p = 1, 2, \cdots, m.$$

The value of k shall be chosen later, but since it is fixed, this fixes $\mathcal{N}(t,\xi)$. Hence,

$$\|\mathcal{N}_k(t,\xi) - I\| \le \sum_{p=1}^{m} \|\mathcal{N}^{(p)}(t,\xi)\| \le \sum_{p=1}^{m} \frac{1}{2^{p+1}} \le \frac{1}{2},$$

thus proving the invertibility oh $\mathcal{N}_k$. In this way we completed the proof of the proposition. □

Now we set $U^{(k+1)} = U^{(k+1)}(t,\xi) := \mathcal{N}_k^{-1}(t,\xi)M^{-1}(t,\xi)U(t,\xi)$ for all $t \ge s,\ (t,\xi) \in Z_{ell}(\varepsilon, N)$, and see that the system (10.1.6) for U is equivalent in $Z_{ell}(\varepsilon, N)$ to

$$\Big(D_t - \mathcal{D} - \sum_{p=1}^{k}\mathcal{F}^{(p)} - \mathcal{R}_{k+1}\Big)U^{(k+1)} = 0, \quad U^{(k+1)}(s,\xi) = \mathcal{N}_k^{-1}(t,s)M^{-1}(t,s)U(s,\xi) \tag{10.1.13}$$

Thus, we have obtained in $Z_{ell}(\varepsilon, N)$ the diagonalization of the system (10.1.6) modulo remainder $\mathcal{R}_{k+1} \in S_{ell}^{l-k,m}\{-2k\delta, -k, k+1\}$. Consequently, we have obtained in $Z_{ell}(\varepsilon, N)$ the diagonalization of the system (10.1.6) modulo remainder $\mathcal{R}_{k+1} \in S_{ell}^{l-k,m}\{-2k\delta, -k, k+1\}$.

10.1.2 Modified Bessel functions

Here we summarize some rules for modified Bessel functions. Let $\tilde{J}_\mu = \tilde{J}_\mu(s)$ be the Bessel function of order $\mu \in (-\infty, \infty)$, and let $J_\mu(s) := \tilde{J}_\mu(s)/s^\mu$ when μ is not a negative integer.

Lemma 10.1.4. *Let $f \in L^p(\mathbb{R}^n),\ p \in [1,2]$, be a radial function. Then the Fourier transform $F(f)$ is also a radial function and it satisfies*

$$F(f)(\xi) = c\int_0^\infty g(r)r^{n-1}J_{\frac{n}{2}-1}(r|\xi|)dr, \qquad g(|x|) := f(x).$$

Lemma 10.1.5. *Assume that μ is not a negative integer. Then the following rules hold :*

- $J_{\mu-1}(r|x|) = rdrJ_\mu(r|x|) + 2\mu J_\mu(r|x|)$,
- $J_{\mu+1}(r|x|) = -\frac{1}{r|x|^2}\partial_r J_\mu(r|x|)$, for $r \neq 0,\ x \neq 0$,
- $J_{-\frac{1}{2}}(r|x|) = \sqrt{\frac{\pi}{2}}\cos(r|x|)$,
- *we have for any μ the relations*
  - $\big|J_\mu(r|x|)\big| \leq C\exp\big(\pi|\Im\mu|\big)$ for $r|x| \leq 1$,
  - $J_\mu(r|x|) = C(r|x|)^{-1/2-\mu}\cos\big(r|x| - \frac{2\mu+1}{4}\pi\big) + \mathcal{O}\big((r|x|)^{-3/2-\mu}\big)$ for $r|x| \geq 1$.

10.1.3 Further auxiliary lemmas

A useful tool for energy estimates is Gronwall's inequality.

Lemma 10.1.6. ***(Gronwall's inequality)**. Suppose that $a < b$ and let $g,\ h$ and f be non-negative continuous functions defined on the interval $[a,b]$. Moreover, suppose that g is differentiable on (a,b) with non-negative continuous derivative g'. If for all $t \in [a,b]$*

$$f(t) \leq g(t) + \int_0^t h(\tau)f(\tau)d\tau,$$

then

$$f(t) \leq g(t)\exp\Big(-\int_a^t h(\tau)d\tau\Big) \qquad \text{for all} \quad t \in [a,b].$$

Let us consider the homogeneous linear system of ordinary differential equations

$$D_tU = A(t)U \qquad \text{for} \quad t \in \mathbb{R}_+. \tag{10.1.14}$$

Lemma 10.1.7. ***(Liouville's formula)**. Suppose that $E = E(t,s)$ is a matrix-valued solution of the system (10.1.14) on $\mathbb{R}_+$. Then*

$$\det E(t,s) = \det E(s,s)\exp\Big(i\int_s^t \operatorname{tr} A(\tau)d\tau\Big)$$

for $0 \leq s,t$.

We define a notion of higher-order directional derivative of a smooth function and use it to establish three simple formulae for the n^{th} derivative of the composition of two functions

Lemma 10.1.8. ***(Francesco Faa di Bruno, who published it in 1857)[12]** Let $f,g \in C^n(\mathbb{R})$ then we have the following*

$$\frac{d^n}{dt^n}f(g(t)) = \sum \frac{1}{k_1!k_2!\cdots k_n!}n!f^{(k)}(g(t))\prod_{i=1}^n\Big(\frac{g^{(i)}(t)}{i!}\Big)^{k_i},$$

where the sum is over all nonnegative integer solutions of the Diophantine equation $k := k_1 + k_2 + \cdots + k_n$ and $k_1 + 2k_2 + \cdots + nk_n = n$.

10.1.4 The Peano-Baker formula

First order systems of ordinary differential equations

$$\frac{d}{dt}u = A(t)u, \qquad u(0) = u_0 \in \mathbb{C}^n$$

are solved in terms of the fundamental solution $E(t,s)$ as $u(t) = E(t,0)u_0$. The matrix function $E(t,s)$ is the solution to

$$\frac{d}{dt}E(t,s) = A(t)E(t,s), \qquad E(s,s) = I_2 \in \mathbb{C}^{n\times n}.$$

It is well-known, that for a constant matrix this fundamental solution can be expressed in terms of the matrix exponential,

$$E(t,s) = \exp\big((t-s)A\big), \qquad \exp(A) = I + \sum_{k=1}^{\infty} \frac{1}{k!}A^k.$$

For variable coefficients this representation is not valid any more. For the sake of completeness, we give the representation used several times throughout our calculations.

Theorem 10.1.9. *Let $A \in L^1_{loc}(\mathbb{R}, \mathbb{C}^{n\times n})$. Then the fundamental solution $E(t,s)$ to $\partial_t - A(t)$ is given by the Peano-Baker formula*

$$E(t,s) = I + \sum_{k=1}^{\infty} \int_s^t A(t_1) \int_s^{t_1} A(t_2) \cdots \int_s^{t_{k-1}} A(t_k) dt_k \cdots dt_2 dt_1.$$

The proof follows by differentiating the series term by term. To prove the convergence of the series and its formal derivative one uses the domination by the exponential series following from Proposition 10.1.10.

Proposition 10.1.10. *Assume $A \in L^1_{loc}(\mathbb{R})$. Then*

$$\int_s^t |A(t_1)| \int_s^{t_1} |A(t_2)| \cdots \int_s^{t_{k-1}} |A(t_k)| dt_k \cdots dt_1 \le \frac{1}{k!}\Big(\int_s^t |A(\tau)| d\tau\Big)^k \qquad \textit{for all} \qquad k \in \mathbb{N}.$$

The proof follows by induction over k.

Corollary 10.1.11. *Let $A \in L^1_{loc}(\mathbb{R}, \mathbb{C}^{n\times n})$. Then the fundamental matrix $E(t,s)$ satisfies*

$$\|E(t,s)\| \le \exp\Big(\int_s^t \|A(\tau)\| d\tau\Big).$$

In several applications we need not only estimates for the fundamental solution, but also statements about its asymptotic behaviour and invertibility. It is convenient to use the Theorem of Liouville in the following form.

Theorem 10.1.12. *Let $A \in L^1_{loc}(\mathbb{R}, \mathbb{C}^{n\times n})$. Then the fundamental solution $E(t,s)$ satisfies*

$$\det E(t,s) = \exp\Big(\int_s^t \operatorname{tr} A(\tau) d\tau\Big).$$

10.1.5 Remarks on Volterra integral equations

The estimate of Corollary 10.1.11 is in general not sharp, to obtain better estimates, we are interested in solutions to the Volterra equation

$$f(t,\xi) + \int_0^t k(t,\tau,\xi) f(\tau,\xi) d\tau = \psi(t,\xi), \tag{10.1.15}$$

with kernel $k = k(t,\tau,\xi)$ and right-hand side $\psi(t,\xi)$, both depending on some parameter $\xi \in P \subseteq \mathbb{R}^n$.

Theorem 10.1.13. *Assume* $\psi \in L^\infty(\mathbb{R}_+ \times P),\ \ k \in L^\infty(\mathbb{R}_+^2 \times P)$ *and*

$$\int_0^t \|k(\cdot,\tau,\xi)\|_{L^\infty} d\tau \in L^\infty(\mathbb{R}_+ \times P).$$

Then, there exists a (unique) solution $f(t,\xi)$ *of (10.1.15) in* $L^\infty(\mathbb{R}_+ \times P)$*,*

$$\operatorname{ess\,sup}_{t\in\mathbb{R}_+,\xi\in P} |f(t,\xi)| < \infty.$$

Proof. Sketch of the proof. Uniqueness of the solution follows for small t by the contraction mapping principle. It remains to show the global bound on the solution. We may represent the solutions to this integral equation by the Neumann series

$$f(t,\xi) = \psi(t,\xi) + \sum_{k=1}^\infty (-1)^k \int_0^t k(t,t_1,\xi) \int_0^{t_1} k(t_1,t_2,\xi)$$
$$\cdots \int_0^{t_{k-1}} k(t_{k-1},t_k,\xi)\psi(t_k,\xi) dt_k \cdots dt_2 dt_1$$

and use Proposition 10.1.10 to conclude

$$\begin{aligned}\|f(t,\xi)\|_{L^\infty} = \|\psi\|_{L^\infty}\Big(1 + \sum_{k=1}^\infty \int_0^t \|k(\cdot,t_1,\xi)\|_{L^\infty} \int_0^{t_1} \|k(\cdot,t_2,\xi)\|_{L^\infty} \\ \cdots \int_0^{t_{k-1}} \|k(\cdot,t_k,\xi)\|_{L^\infty} \psi(t_k,\xi) dt_k \cdots dt_2 dt_1, \\ \lesssim \|\psi\|_{L^\infty} \exp\Big(\int_0^t \|k(\cdot,\tau,\xi)\|_{L^\infty} d\tau\Big).\end{aligned}$$

For results under weaker assumptions on the integral kernel we refer to the treatment of G. Gripenberg, S.-O. Londen and O. Staffans, [GLS90]. For the applications we may take also domains for the parameter ξ depending on both variables t and τ . In this case one can trivially extent the kernel function $k(t,\tau,\xi)$ by zero to a larger common parameter domain without changing the solution. This will be the case in most of the applications. Due to *its importance for the understanding of the results in pseudo-differential zone* we give one auxiliary application of this theorem. □

Theorem 10.1.14. *Assume*

$$A(t,\xi) \in L^\infty(P, L^1_{loc}(\mathbb{R}_+, \mathbb{C}^{n\times n})), diagonal, \Re A(t,\xi) \le a(t,\xi) I,$$
$$B(t,\xi) \in L^\infty(P, L^1(\mathbb{R}_+, \mathbb{C}^{n\times n})).$$

Then the fundamental solution $E(t,s,\xi)$ *to* $\partial_t - A(t,\xi) - B(t,\xi)$ *satisfies*

$$\|E(t,s,\xi)\| \lesssim \exp\Big(\int_s^t a(\tau,\xi) d\tau\Big).$$

Proof. Sketch of proof. In order to prove this, we consider the fundamental solution $E_d(t,s,\xi)$ to the system $\partial_t - A(t,\xi)$ and conclude from

$$E_d(t,s,\xi) = \exp\Big(\int_s^t A(\tau,\xi)d\tau\Big), \qquad \partial_t E_d^{-1}(t,s,\xi) = -E_d^{-1}(t,s,\xi)A(t,\xi),$$

that

$$\partial_t(E_d^{-1}(t,s,\xi)E(t,s,\xi)) = E_d^{-1}(t,s,\xi)B(t,\xi)E(t,s,\xi).$$

Thus, we obtain the integral equation

$$\begin{aligned} E(t,s,\xi) &= I + E_d(t,s,\xi)\int_s^t E_d^{-1}(\tau,s,\xi)B(\tau,\xi)E(\tau,s,\xi)d\tau \\ &= I + \int_s^t E_d^{-1}(t,\tau,\xi)B(\tau,\xi)E(\tau,s,\xi)d\tau, \end{aligned}$$

which can be transformed to

$$\begin{aligned} \exp\Big(-\int_s^t a(\tau,\xi)d\tau\Big)E(t,s,\xi) &= \exp\Big(-\int_s^t a(\tau,\xi)d\tau\Big) \\ &\quad + \int_s^t \exp\Big(\int_s^t \big(A(\tau,\xi) - a(\tau,\xi)I\big)d\tau\Big)B(\tau,\xi)E(\tau,s,\xi)d\tau. \end{aligned}$$

Now the exponential term is bounded by a constant. Moreover, the assumptions on $B(t,\xi)$, in particular, the L^1 property can be used to conclude the boundedness of $\exp\Big(-\int_s^t a(\tau,\xi)d\tau\Big)E(t,s,\xi)$ by Theorem 10.1.13. □

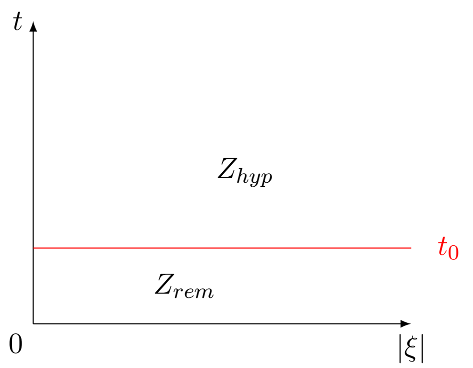

Figure10.1: Decreasing behavior of $b = b(t)$ and $\delta = \sigma/2$.

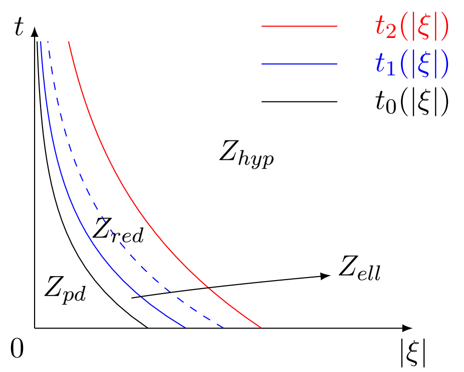

Figure10.2: Decreasing behavior of $b = b(t)$, $\delta \in (0, \sigma/2)$ and $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = 0$.

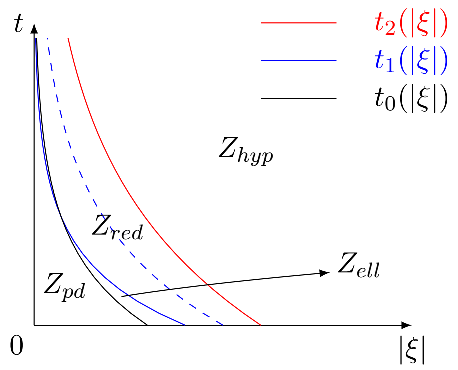

Figure10.3: Decreasing behavior of $b = b(t)$, $\delta \in (0, \sigma/2)$ and $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$.

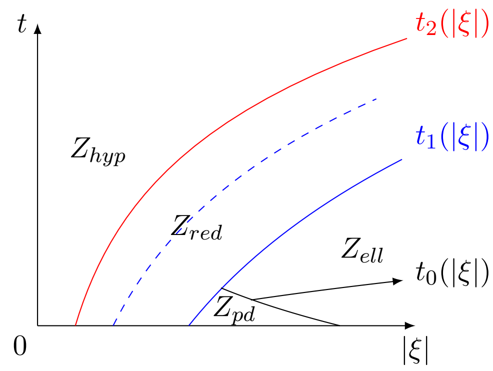

Figure10.4: Decreasing behavior of $b = b(t)$ and $\delta \in (\sigma/2, \sigma)$.

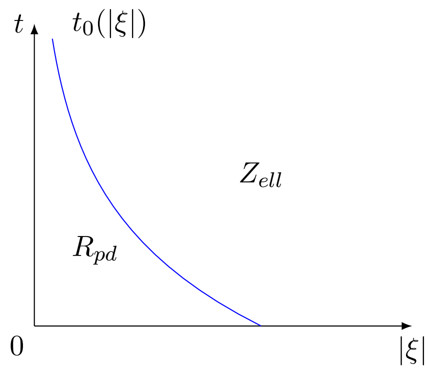

Figure10.5: Increasing behavior of $b = b(t)$, $\delta \in (\sigma/2, \sigma)$ and $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = 0$.

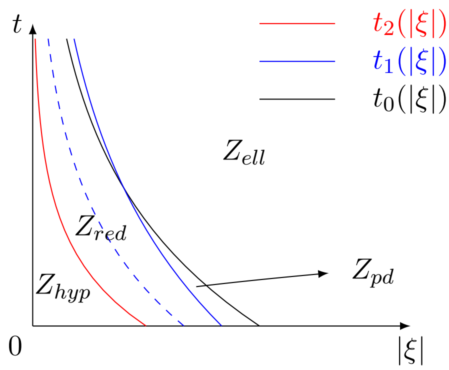

Figure10.6: Increasing behavior of $b = b(t)$, $\delta \in (\sigma/2, \sigma)$ and $\limsup_{t\to\infty}(1+t)\Lambda(t)^{-\frac{\sigma}{2\delta}} = \infty$.

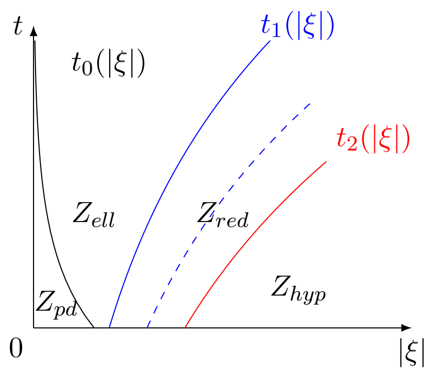

Figure10.7: Increasing behavior of $b = b(t)$ and $\delta \in (0, \sigma/2)$.

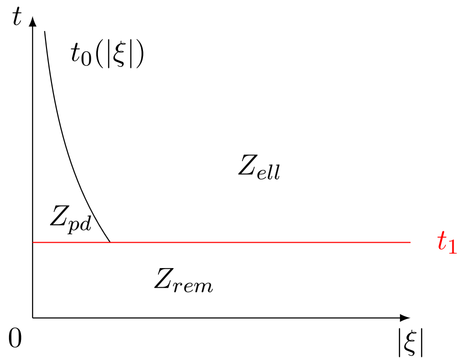

Figure10.8: Increasing behavior of $b = b(t)$ and $\delta = \sigma/2$.

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