Contents

Introduction

Stabilization of evolution problems

Problems of global existence and stability in time of Partial Differential Equations made object, recently, of many work. In this thesis we were interested in study of the global existence and the stabilization of some evolution equations.

The purpose of stabilization is to attenuate the vibrations by feedback, thus it consists in guaranteeing the decrease of energy of the solutions to 0 in a more or less fast way by a mechanism of dissipation.

More precisely, the problem of stabilization consists in determining the asymptotic behaviour of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero.

This problem has been studied by many authors for various systems. In our study, we obtain severals type of stabilization

1) Strong stabilization: $E(t) \to 0$, as $t \to \infty$.

2) Logarithmic stabilization: $E(t) \leq c(\log(t))^{-\delta}, \forall t > 0, (c, \delta > 0)$.

3) polynomial stabilization: $E(t) \leq ct^{-\delta}, \forall t > 0, (c, \delta > 0)$

4) uniform stabilization: $E(t) \leq ce^{-\delta t}, \forall t > 0, (c, \delta > 0)$.

For wave equation with dissipation of the form $u'' - \Delta_x u + g(u') = 0$, stabilization problems have been investigated by many authors:
When $g : \mathbb{R} \to \mathbb{R}$ is continuous and increasing function such that $g(0) = 0$, global existence of solutions is known for all initial conditions (u_0, u_1) given in $H_0^1(\Omega) \times L^2(\Omega)$. This result is, for instance, a consequence of the general theory of nonlinear semi-groups of contractions generated by a maximal monotone operator (see Brézis **[10]**).

Moreover, if we impose on the control the condition $\forall \lambda \neq 0, g(\lambda) \neq 0$, then strong asymptotic stability of solutions occurs in $H_0^1(\Omega) \times L^2(\Omega)$, i.e.,

$$(u, u') \to (0, 0) \text{ strongly in } H_0^1(\Omega) \times L^2(\Omega),$$

without speed of convergence. These results follows, for instance, from the invariance principle of Lasalle (see for example C. M. Dafermos **[17]**, A. Haraux **[24]**, , F. Conrad, M. Pierre **[16]**). If the solution goes to 0 as time goes to ∞, how to get energy decay rates?

Dafermos has written in 1978 "Another advantage of this approach is that it is so simplistic that it requires only quite weak assumptions on the dissipative mechanism. The corresponding drawback is that the deduced information is also weak, never yielding, for example, decay rates of solutions."

Many authors have worked since then on energy decay rates. First results were obtained for linear stabilization, then for polynomial stabilization (see M. Nakao [**39**] A. Haraux [**24**], E. Zuazua [**50**] and V. Komornik [**26**]) and then extended to arbitrary growing feedbacks (close to 0). In the same time, geometrical aspects were considered.

By combining the multiplier method with the techniques of micro-local analysis, Lasiecka et al [**13**], [**18**], [**28**]-[**29**] have investigated different dissipative systems of partial differential equations (with Dirichlet and Neumann boundary conditions) under general geometrical conditions with nonlinear feedback without any growth restrictions near the origin or at infinity. The computation of decay rates is reduced to solving an appropriate explicitly given ordinary differential equation of monotone type. More precisely, the following explicit decay estimate of the energy is obtained:

$$E(t) \leq h(\frac{t}{t_0} - 1), \quad \forall t \geq t_0, \tag{1}$$

where $t_0 > 0$ and h is the solution of the following differential equation:

$$h'(t) + q(h(t)) = 0, \quad \forall t \geq 0 \quad \text{and} \quad h(0) = E(0) \tag{2}$$

and the function q is determined entirely from the behavior at the origin of the nonlinear feedback by proving that E satisfies

$$(Id - q)^{-1}\Big(E((m+1)t_0)\Big) \leq E(mt_0), \quad \forall m \in \mathbb{N}.$$

In this thesis, the main objective is to give a global existence and stabilization results.
This work consists in two chapter, the first, for wave equations with a constant weak delay term.
the second, for viscoelastic wave equations with a nonlinear delay term.

• In the chapter I, in the class $H_0^1 \cap H^2$, We prove the global existence to the solutions of wave equations with a weak linear dissipative term anda constant weak linear delay term in Sobolev spaces by means of the energy method combined with the Faedo-Galerkin procedure under a condition between the weight of the delay term in the feedback and the weight of the term without delay. We prove also the decay estimate f the energy using the multiplier method.

• In the chapter II, We prove the global existence to the viscoelastic wave equation in Sobolev spaces by means of the energy method combined with the Faedo-Galerkin procedure under a condition between the weight of the delay term in the feedback and the weight of the term without delay. We prove also the decay estimate of the energy using a perturbed energy method.

Chapter 1:Energy decay of solutions for a wave equation with a constant weak delay and a weak internal feedback

In this chapter we consider with a weak internal constant delay term

$$(P)\quad \begin{cases} u''(x,t) - \Delta_x u(x,t) + \mu_1(t)\ u'(x,t) + \mu_2(t)\ u'(x,t-\tau) = 0 & \text{in } \Omega\times]0,+\infty[,\\ u(x,t) = 0 & \text{on } \Gamma\times]0,+\infty[,\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{on } \Omega,\\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{on } \Omega\times]0,\tau[, \end{cases}$$

in a bounded domain. Under appropriate conditions on μ_1 and μ_2, we prove global existence of solutions by the Faedo-Galerkin method and establish a decay rate estimate for the energy using the multiplier method.

Chapter 2:Global existence and energy decay of solutions to a viscoelastic wave equation with a delay term in the nonlinear internal feedback

In this chapter we investigate the existence and decay properties of solutions for the initial boundary value problem of the nonlinear viscoelastic wave equation of the type

$$(P)\quad \begin{cases} u_{tt}(x,t) - \Delta_x u(x,t) + \displaystyle\int_0^t h(t-s)\Delta_x u(x,s)\,ds & \\ +\mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0 & \text{in } \Omega\times]0,+\infty[,\\ u(x,t) = 0 & \text{on } \Gamma\times]0,+\infty[,\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega,\\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau[, \end{cases}$$

and prove a global existence result using the energy method combined with the Faedo-Galerkin procedure under a condition between the weight of the delay term in the feedback and the weight of the term without delay. Furthermore, we study the asymptotic behavior of solutions using a perturbed energy method.

Chapter 3:Stability result of the wave equation with a time-varying delay term and a weak boundary feedback

In this chapter we consider a boundary stabilization problem for a nondissipative wave equation in a bounded domain with a time-varying delay term in the internal feedback. We use an approach introduced by Guesmia which leads to decay estimates (known in the dissipative case) when the integral inequalities method due to Haraux-Komornik **[24]**-**[26]** cannot be applied due to the lack of dissipativity.

Preliminaries

0.1 Sobolev spaces

We denote by Ω an open domain in $\mathbb{R}^n, n \geq 1$, with a smooth boundary $\Gamma = \partial\Omega$. In general, some regularity of Ω will be assumed. We will suppose that either

$$\Omega \text{ is Lipschitz,}$$

i.e., the boundary Γ is locally the graph of a Lipschitz function, or

$$\Omega \text{ is of class } \mathcal{C}^r, r \geq 1,$$

i.e., the boundary Γ is a manifold of dimension $n \geq 1$ of class $\mathcal{C}^r$. In both cases we assume that Ω is totally on one side of Γ. These definitions mean that locally the domain Ω is below the graph of some function ψ, the boundary Γ is represented by the graph of ψ and its regularity is determined by that of the function ψ. Moreover, it is necessary to note that a domain with a continuous boundary is never on both sides of its boundary at any point of this boundary and that a Lipschitz boundary has almost everywhere a unit normal vector ν.

We will also use the following multi-index notation for partial differential derivatives of a function:

$$\partial_i^k u = \frac{\partial^k u}{\partial x_i^k} \text{ for all } k \in \mathbb{N} \text{ and } i = 1, ..., n,$$
$$D^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} u = \frac{\partial^{\alpha_1 + \ldots + \alpha_n} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}},$$
$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + \ldots + \alpha_n.$$

We denote by $\mathcal{C}(D)$ (respectively $\mathcal{C}^k(D), k \in \mathbb{N}$ or $k = +\infty$) the space of real continuous functions on D (respectively the space of k times continuously differentiable functions on D), where D plays the role of Ω or its closure $\overline{\Omega}$. The space of real $\mathcal{C}^\infty$ functions on Ω with a compact support in Ω is denoted by $\mathcal{C}_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$ as in the distributions theory of Schwartz.The distributions space on Ω is denoted by $\mathcal{D}'(\Omega)$, i.e., the space of continuous linear form over $\mathcal{D}(\Omega)$.

For $1 \leq p \leq \infty$, we call $L^p(\Omega)$ the space of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

$$\|f\|_{L^\infty(\Omega)} = \sup_\Omega |f(x)| < +\infty \qquad \text{for} \quad p = +\infty$$

The space $L^p(\Omega)$ equipped with the norm $f \longrightarrow \|f\|_{L^p}$ is a Banach space: it is reflexive and separable for $1 < p < \infty$ (its dual is $L^{\frac{p}{p-1}}(\Omega)$), separable but not reflexive for $p = 1$ (its dual is $L^\infty(\Omega)$), and not separable, not reflexive for $p = \infty$ (its dual contains strictly $L^1(\Omega)$). In particular the space $L^2(\Omega)$ is a Hilbert space equipped with the scalar product defined by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

We denote by $L^p_{loc}(\Omega)$ the space of functions which are L^p on any bounded sub-domain of Ω.

Similar space can be defined on any open set other than Ω, in particular, on the cylinder set $\Omega \times]a, b[$ or on the set $\Gamma \times]a, b[$, where $a, b \in \mathbb{R}$ and $a < b$.

Let U be a Banach space, $1 < p < +\infty$ and $-\infty \leq a < b \leq +\infty$, then $L^p(a, b; U)$ is the space of L^p functions f from (a, b) into U which is a Banach space for the norm

$$\|f\|_{L^p(a,b;U)} = \left(\int_a^b \|f(x)\|_U^p \, dt\right)^{1/p} < +\infty \quad \text{for} \quad p < +\infty$$

and for the norm

$$\|f\|_{L^\infty(a,b;U)} = \sup_{t\in(a,b)} \|f(x)\|_U < +\infty \quad \text{for} \quad p = +\infty$$

Similarly, for a Banach space $U, k \in \mathbb{N}$ and $-\infty < a < b < +\infty$, we denote by $C([a, b]; U)$ (respectively $C^k([a, b]; U)$) the space of continuous functions (respectively the space of k times continuously differentiable functions) f from $[a, b]$ into U, which are Banach spaces, respectively, for the norms

$$\|f\|_{\mathcal{C}(a,b;U)} = \sup_{t\in(a,b)} \|f(x)\|_U, \quad \|f\|_{\mathcal{C}^k(a,b;U)} = \sum_{i=0}^k \left\|\frac{\partial^i f}{\partial t^i}\right\|_{\mathcal{C}(a,b;U)}$$

0.1.1 Definition of Sobolev Spaces

Now, we will introduce the Sobolev spaces: The Sobolev space $W^{k,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order k have a finite L^p norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega). \ \ \forall\alpha; |\alpha| \leq k\} \ ,$$

With this definition, the Sobolev spaces admit a natural norm,

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{1/p} , \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^\infty(\Omega)} , \text{ for } p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\| \,.\, \|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f,g)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)} \ .$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H_0^m(\Omega)$ (respectively $W_0^{k,p}(\Omega)$).

Now, we introduce a space of functions with values in a space X (a separable Hilbert space).

The space $L^2(a,b;X)$ is a Hilbert space for the inner product

$$(f,g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X \, dt$$

We note that $L^\infty(a,b;X) = (L^1(a,b;X))'$.

Now, we define the Sobolev spaces with values in a Hilbert space X

For $k \in \mathbb{N}$, $p \in [1,\infty]$, we set:

$$W^{k,p}(a,b;X) = \left\{ v \in L^p(a,b;X);\ \frac{\partial v}{\partial x_i} \in L^p(a,b;X).\ \ \forall i \leq k \right\} \ ,$$

The Sobolev space $W^{k,p}(a,b;X)$ is a Banach space with the norm

$$\begin{aligned} \|f\|_{W^{k,p}(a,b;X)} &= \left(\sum_{i=0}^{k} \left\| \frac{\partial f}{\partial x_i} \right\|^p_{L^p(a,b;X)} \right)^{1/p}, \ \text{for } p < +\infty \\ \|f\|_{W^{k,\infty}(a,b;X)} &= \sum_{i=0}^{k} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^\infty(a,b;X)}, \qquad \text{for } p = +\infty \end{aligned}$$

The spaces $W^{k,2}(a,b;X)$ form a Hilbert space and it is noted $H^k(0,T;X)$. The $H^k(0,T;X)$ inner product is defined by:

$$(u,v)_{H^k(a,b;X)} = \sum_{i=0}^{k} \int_a^b \left(\frac{\partial u}{\partial x^i}, \frac{\partial v}{\partial x^i} \right)_X dt \ .$$

Theorem 0.1.1 *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

where p^* is given by $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$ (where $p = n, p^* = \infty$). Moreover there exists a constant $C = C(p,n)$ such that

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)} \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corollary 0.1.1 *Let $1 \leq p < n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [p, p^*]$$

with continuous imbedding.

For the case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \forall q \in [n, +\infty[$$

Theorem 0.1.2 *Let $p > n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

with continuous imbedding.

Corollary 0.1.2 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{l} if \quad 1 \leq p < \infty, \text{ then } W^{1,p}(\Omega) \subset L^{p^*}(\Omega) \text{ where } \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if \quad p = n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if \quad p > n, \text{ then } W^{1,p}(\Omega) \subset L^\infty(\Omega) \end{array}$$

with continuous imbedding.
Moreover, if $p > n$, we have: $\forall u \in W^{1,p}(\Omega)$,

$$|u(x) - u(y)| \leq C|x - y|^\alpha \|u\|_{W^{1,p}(\Omega)} \ a.e\ x, y \in \Omega$$

with $\alpha = 1 - \dfrac{n}{p} > 0$ and C is a constant which depend on p, n and Ω. In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Corollary 0.1.3 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{array}{l} if \quad p < n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega) \forall q \in [1, p^*[\text{ where } \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \\ if \quad p = n, \text{ then } W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[. \\ if \quad p > n, \text{ then } W^{1,p}(\Omega) \subset C(\overline{\Omega}) \end{array}$$

with compact imbedding.

Remark 0.1.1 *We remark in particular that*

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

with compact imbedding for $1 \leq p \leq \infty$ and for $p \leq q < p^*$.

Corollary 0.1.4

$$\begin{array}{l} if \quad \dfrac{1}{p} - \dfrac{m}{n} > 0, \text{ then } W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \text{ where } \dfrac{1}{q} = \dfrac{1}{p} - \dfrac{m}{n}. \\ if \quad \dfrac{1}{p} - \dfrac{m}{n} = 0, \text{ then } W^{m,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [p, +\infty[. \\ if \quad \dfrac{1}{p} - \dfrac{m}{n} < 0, \text{ then } W^{m,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \end{array}$$

with continuous imbedding.

0.2 Weak convergence

Let $(E;\|.\|_E)$ a Banach space and E' its dual space, i.e., the Banach space of all continuous linear forms on E endowed with the norm $\|.\|'_E$ defined by

$$\|f\|_{E'} =: \sup_{x\neq 0} \frac{|\langle f,x\rangle|}{\|x\|}$$

; where $\langle f,x\rangle$; denotes the action of f on$x$, i.e.$\langle f,x\rangle := f(x)$. In the same way, we can define the dual space of E' that we denote by E''. (The Banach space E'' is also called the bi-dual space of E.) An element x of E can be seen as a continuous linear form on E' by setting $x(f) := \langle x,f\rangle$, which means that $E \subset E''$:

Definition 0.2.1 *The Banach space E is said to be reflexive if $E = E''$.*

Definition 0.2.2 *The Banach space E is said to be separable if there exists a countable subset D of E which is dense in E, i.e. $\overline{D} = E$.*

Theorem 0.2.1 *(Riesz). If $(H;\langle .,.\rangle)$ is a Hilbert space, $\langle .,.\rangle$ being a scalar product on H, then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $x \in H$ such that $f = \langle x,.\rangle$ and $\|f\|'_H = \|x\|_H$*

Remark : From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.

Proposition 0.2.1 *If E is reflexive and if F is a closed vector subspace of E, then F is reflexive.*

Corollary 0.2.1 *The following two assertions are equivalent: (i) E is reflexive; (ii) E' is reflexive.*

0.2.1 Weak, weak star and strong convergence

Definition 0.2.3 *(Weak convergence in E). Let $x \in E$ and let $\{x_n\} \subset E$. We say that $\{x_n\}$ weakly converges to x in E, and we write $x_n \rightharpoonup x$ in E, if*

$$\langle f,x_n\rangle \to \langle f,x\rangle$$

for all $f \in E'$.

Definition 0.2.4 *(weak convergence in E'). Let $f \in E'$ and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly converges to f in E', and we write $f_n \rightharpoonup f$ in E', if*

$$\langle f_n,x\rangle \to \langle f,x\rangle$$

for all $x \in E''$.

Definition 0.2.5 *(weak star convergence). Let $f \in E'$ and let $\{f_n\} \subset E'$. We say that $\{f_n\}$ weakly star converges to f in E', and we write $f_n \rightharpoonup *f$ in E' if;*

$$\langle f_n, x\rangle \to \langle f, x\rangle$$

for all $x \in E$.

Remark As $E \subset E''$ we have $f_n \rightharpoonup f$ in E' imply $f_n \rightharpoonup *f$ in E'. When E is reflexive, the last definitions are the same, i.e, weak convergence in E' and weak star convergence coincide.

Definition 0.2.6 *(strong convergence). Let $x \in E$(resp. $f \in E'$) and let $\{x_n\} \subset E$ (resp $\{f_n\} \subset E'$). We say that $\{x_n\}$ (resp. $\{f_n\}$) strongly converges to x (resp. f), and we write $x_n \to x$ in E (resp. $f_n \to f$ in E'), if*

$$\underset{n}{lim} \|x_n - x\|_E = 0; (resp.\ \underset{n}{lim} \|f_n - f\|'_E = 0)$$

Proposition 0.2.2 *Let $x \in E$, let $\{x_n\} \subset E$, let $f \in E'$ and let $\{f_n\} \subset E'$.*

i. If $x_n \to x$ in E then $x_n \rightharpoonup x$ in E.

ii. If $x_n \rightharpoonup x$ in E then $\{x_n\}$ is bounded.

iii. If $x_n \rightharpoonup x$ in E then $\liminf_{n\to\infty} \|x_n\|_E \geq \|x\|_E$

iv. If $f_n \to f$ in E' then $f_n \rightharpoonup f$ inE' (and so $f_n \overset{*}{\rightharpoonup} f$ in E').

v. If $f_n \rightharpoonup f$ in E' then $\{f_n\}$ is bounded.

vi. If $f_n \rightharpoonup f$ in E' then then $\liminf_{n\to\infty} \|f_n\|'_E \geq \|f\|'_E$

Proposition 0.2.3 *(finite dimension). If $\dim E < \infty$ then strong, weak and weak star convergence are equivalent.*

0.2.2 Weak and weak star compactness

In finite dimension, i.e, $\dim E < \infty$, we have Bolzano-Weierstrass's theorem (which is a strong compactness theorem).

Theorem 0.2.2 *(Bolzano-Weierstrass). If $\dim E < \infty$ and if $\{x_n\} \subset E$) is bounded, then there exist $\in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ strongly converges to x.*

The following two theorems are generalizations, in infinite dimension, of Bolzano- Weierstrass's theorem.

Theorem 0.2.3 *(weak star compactness, Banach-Alaoglu-Bourbaki). Assume that E is separable and consider $\{f_n\} \subset E'$) . If $\{x_n\}$ is bounded, then there exist $f \in E'$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ weakly star converges to f in E'.*

Theorem 0.2.4 *(weak compactness, Kakutani-Eberlein). Assume that E is reflexive and consider* $\{x_n\} \subset E$*). If* $\{x_n\}$ *is bounded, then there exist* $x \in E$ *and a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\{x_{n_k}\}$ *weakly converges to* x *in* E.

Weak, weak star convergence and compactness in $L^p(\Omega)$.

Definition 0.2.7 *(weak convergence in* $L^p(\Omega)$ *with* $1 \le p < \infty$ *). Let* Ω *an open subset of* $\mathbb{R}^n$ *. We say that the sequence* $\{f_n\}$ *of* $L^p(\Omega)$ *weakly converges to* $f \in L^p(\Omega)$*, if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^q;\ (\frac{1}{p} + \frac{1}{q} = 1)$$

Definition 0.2.8 *(weak star convergence in* $L^\infty(\Omega)$ *). We say that the sequence* $\{f_n\} \subset L^\infty(\Omega)$ *weakly star converges to* $f \in L^\infty(\Omega)$ *, if*

$$\lim_n \int_\Omega f_n(x)g(x)dx = \int_\Omega f(x)g(x)dx \text{ for all } g \in L^1(\Omega)$$

Theorem 0.2.5 *(weak compactness in* $L^p(\Omega)$*) with* $1 < p < \infty$*. Given* $\{f_n\} \subset L^p(\Omega)$ *, if* $\{f_n\}$ *is bounded, then there exist* $f \in L^p(\Omega)$ *and a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *such that* $f_n \rightharpoonup f$ *in* $L^p(\Omega)$*.*

Theorem 0.2.6 *(weak star compactness in* $L^\infty(\Omega)$*.*
Given $\{f_n\} \subset L^\infty(\Omega)$*, if* $\{f_n\}$ *is bounded, then there exist* $f \in L^\infty(\Omega)$ *and a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *such that* $f_n \overset{*}{\rightharpoonup} f$ *in* $L^\infty(\Omega)$*.*

Generalities. In what follows, Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and $1 \le p \le \infty$.

Weak and weak star convergence in Sobolev spaces

For $1 \le p \le \infty$, $W^{1;p}(\Omega)$ is a Banach space. Denote the space of all restrictions to Ω of C^1-differentiable functions from $\mathbb{R}^N$ to $\mathbb{R}$ with compact support in R^N by $C^1(\overline{\Omega})$.

Theorem 0.2.7 *for every* $1 \le p \le \infty$ $C^1(\overline{\Omega}) \subset W^{1;p}(\Omega) \subset L^p(\Omega)$ *, and, for* $1 < p < \infty$*,* $C^1(\overline{\Omega})$ *is dense in* $W^{1;p}(\Omega)$*.*

Definition 0.2.9 *(weak convergence in* $W^{1;p}(\Omega)$ *with* $1 \le p < \infty$*).)*
We say the $\{f_n\} \subset W^{1;p}(\Omega)$ *weakly converges to* $f \in W^{1;p}(\Omega)$*, and we write* $f_n \rightharpoonup f$ *in* $W^{1;p}(\Omega)$ *, if* $f_n \rightharpoonup f$ *in* $L^p(\Omega)$ *and* $\nabla f_n \rightharpoonup \nabla f$ *in* $L^p(\Omega;\ \mathbb{R}^N)$

Definition 0.2.10 *(weak convergence in* $W^{1;\infty}(\Omega)$
. We say the $\{f_n\} \subset W^{1;\infty}(\Omega)$ *weakly star converges to* $f \in W^{1;\infty}(\Omega)$*, and we write* $f_n \overset{*}{\rightharpoonup} f$ *in* $W^{1;\infty}(\Omega)$ *, if* $f_n \overset{*}{\rightharpoonup} f$ *in* $L^p(\Omega)$ *and* $\nabla f_n \overset{*}{\rightharpoonup} \nabla f$ *in* $L^\infty(\Omega;\ \mathbb{R}^N)$

Theorem 0.2.8 *(Rellich). Let* $1 \le p \le \infty$ *,* $\{f_n\} \subset W^{1;p}(\Omega)$ *and* $f \in W^{1;p}(\Omega)$*; if* $f_n \rightharpoonup f$ *in* $W^{1;p}(\Omega)$ *when* $1 \le p < \infty$ *(resp.*$f_n \overset{*}{\rightharpoonup} f$ *in* $W^{1;\infty}(\Omega)$*) when* $p = \infty$*) then* $f_n \to f$ *in* $L^p(\Omega)$*), which means that for every* $1 \le p \le \infty$*, the weak convergence in* $W^{1;p}(\Omega)$ *imply the strong convergence in* $L^p(\Omega)$*.*

Theorem 0.2.9 *Let $1 < p \leq \infty$ and let $\{f_n\} \subset W^{1;p}(\Omega)$. If $\{f_n\}$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \rightharpoonup f$ in $W^{1;p}(\Omega)$ when $1 < p < \infty$ (resp. $f_{n_k} \overset{*}{\rightharpoonup} f$ in $W^{1;\infty}(\Omega)$)*

As a consequence of this theorem we have

Corollary 0.2.2 *Let $1 < p \leq \infty$and let $\{f_n\} \subset W^{1;p}(\Omega)$. If $\{f_n\}$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ in $L^p(\Omega)$ and $\nabla f_{n_k} \rightharpoonup \nabla f$ in $L^p(\Omega)$ when $1 < p < \infty$ (resp.$\nabla f_{n_k} \overset{*}{\rightharpoonup} \nabla f$ in $L^\infty(\Omega)$)*

Theorem 0.2.10 *. If$N < p \leq \infty$ and if $\{f_n\} \subset W^{1;p}(\Omega)$ is bounded, then there exist $f \in W^{1;p}(\Omega)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ converges uniformly to f, and $\nabla f_{n_k} \rightharpoonup \nabla f$ in $W^{1;p}(\Omega)$ when $N < p < \infty$ (resp. $\nabla f_{n_k} \overset{*}{\rightharpoonup} \nabla f$ in $W^{1;\infty}$)*

0.3 Fadeo-Galerkin method

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$.

$$(P) \qquad \begin{cases} u''(t) + A(t)u(t) = f(t), \quad t \in [0,T] \\ (x,0) = u_0(x), \;\; u'(x,0) = u_1(x); \end{cases}$$

where u and f are unknown and given function, respectively, mapping the closed interval $[0,T] \subset$ IR into a real separable Hilbert space H ,$A(t)$ $(0 \leq t \leq T)$ are linear bounded operators in H acting in the energy space $V \subset H$.
Assume that $\langle A(t)u(t), v(t) \rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t; .,.)$ is a bilinear continuous in V.
The problem (P) can be formulated as: Found the solution $u(t)$ such that

$$(\widetilde{P}) \qquad \begin{cases} u \in C([0,T]; V), u' \in C([0,T]; H) \\ \langle u''(t), v \rangle + a(t; u(t), v) = \langle f, v \rangle \text{ in } D'(]0,T[) \\ u_0 \in V \;,\;\; u_1 \in H; \end{cases}$$

This problem can be resolved with the approximation process of Fadeo-Galerkin.

0.3.1 General method

Let V_m a sub-space of V with the finite dimension d_m, and let $\{w_{jm}\}$ one basis of V_m.
we define the solution u_m of the approximate problem

$$(P_m) \qquad \begin{cases} u_m(t) = \sum_{j=1}^{d_m} g_j(t) w_{jm} \\ u_m \in C([0,T]; V_m), u'_m \in C([0,T]; V_m) \;\;, u_m \in L^2(0,T; V_m) \\ \langle u''_m(t), w_{jm} \rangle + a(t; u_m(t), w_{jm}) = \langle f, w_{jm} \rangle, \quad 1 \leq j \leq d_m \\ u_m(0) = \sum_{j=1}^{d_m} \xi_j(t) w_{jm} \;,\;\; u'_m(0) = \sum_{j=1}^{d_m} \eta_j(t) w_{jm} \end{cases}$$

where

$$\sum_{j=1}^{d_m} \xi_j(t) w_{jm} \longrightarrow u_0 \text{ in V as } m \longrightarrow \infty$$

$$\sum_{j=1}^{d_m} \eta_j(t) w_{jm} \longrightarrow u_1 \text{ in V as } m \longrightarrow \infty$$

By virtue of the theory of ordinary differential equations,the system (P_m) has unique local solution which is extend to a maximal interval $[0, t_m[$ by Zorn lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$, to obtain one solution defined for all $t > 0$.

0.3.2 A priori estimation and convergence

Using the following estimation

$$\|u_m\|^2 + \|u'_m\|^2 \leq C(\|u_m(0)\|^2 + \|u'_m(0)\|^2 + \int_0^T |f(s)|^2 ds) \; ; \quad 0 \leq t \leq T$$

and the Gronwall lemma we deduce that the solution u_m of the approximate problem (P_m) converges to the solution u of the initial problem (P).The uniqueness proves that u is the solution.

0.3.3 Gronwall lemma

Lemma 0.3.1 *Let $T > 0$, $g \in L^1(0, T)$, $g \geq 0$ a.e and c_1, c_2 are positives constants.Let $\varphi \in L^1(0, T)$ $\varphi \geq 0$ a.e such that $g\varphi \in L^1(0, T)$ and*

$$\varphi(t) \leq c_1 + c_2 \int_0^t g(s)\varphi(s) ds \quad a.e \ in \ (0, T).$$

then, we have

$$\varphi(t) \leq c_1 exp \ (c_2 \int_0^t g(s) ds) \quad a.e \ in \ (0, T).$$

0.4 Convex analysis

0.4.1 Fenchel conjugate functions

Let V be a topological vector space and let V' be its dual space with bilinear duality form $\langle ., . \rangle_{V,V'}$.

Definition 0.4.1 *(Conjugate function)*
Let $F : V \longrightarrow \overline{\rm I\!R}$ be an extend real valued function. The function $F^* : V' \longrightarrow \overline{\rm I\!R}$ defined by

$$F^*(f) = \sup_{u\in V}(\langle f, u\rangle_{V,V'} - F(u)), \quad \forall f \in V'$$

is said to by Fenchel (convex) conjugate or conjugate function of F.
The mapping $F \longrightarrow F^*$ is called the Legendre -Fenchel transformation.

Proposition 0.4.1 *Let $F : V \longrightarrow \overline{\rm I\!R}$ be a given extend real valued function, the following statements are true*

- *i.$F^*(f) + F(u) \geq \langle f, u\rangle_{V,V'}, \quad \forall f \in V', \ \forall u \in V$*
- *ii.Let f be in the dual $V'$of V and $\lambda \in$ I R ,the conjugate of affine function $u \longrightarrow (\langle f, u\rangle_{V,V'} - \lambda$ is less than F if and only if*

$$F^*(f) \leq \lambda$$

- *iii.If F is identically equal to $+\infty$ then F^* is identically equal to $-\infty$ .Moreover ,if F is proper ,then the relation:$F^*(f) = \sup_{u\in V}(\langle f, u\rangle_{V,V'} - F(u))$ may be restricted to the points u in the effective domain of F $(dom(F))$.*
- *iv.The function F^* is always in $\Gamma(V')$ (since F^* is the point-wise supremum of a family of affine continuous functions of v').Therefore, F^* is always o lower semi-continuous convex function on V'. Moreover, if F^*takes the value $-\infty$ then F^* is identically equal to $-\infty$.*

Proposition 0.4.2 *(i)Let F and G be tow given extend real valued functions of V into $\overline{\rm I\!R}$, the following properties hold:*

1. *$F^*(0) = -\inf_{u\in V} F(u)$.*
2. *If F is less than G then G^* is less than F^*.*
3. *If $G(u) = F(\alpha u), \ \forall u \in V$, with $\alpha \neq 0$ then $G^*(f) = F^*(f/\alpha)$,$\forall f \in V'$.*
4. *$(\alpha F)^*(f) = \alpha F^*(f/\alpha), \ \forall \in V', \ \ \forall \alpha > 0$.*
5. *$(F + \beta)^* = F^* - \beta, \ \ \forall \beta \in$ I R.*

(ii)Given a family $(F_i)_{i\in J}$ of functions from V into $\overline{\rm I\!R}$, we have

$$(\inf_{i\in J} F_i)^* = \sup_{i\in J} F_i^*$$

$$\sup_{i\in J} F_i^* \leq \inf_{i\in J}(F_i)^*$$

(iii)For every $a \in V$ we denote by F_a the translated function (i.e,$F_a(u) = F(u - a), \forall u \in V$).Then ${F_a}^*(f) = F^*(f) + \langle f, u\rangle_{V,V'}, \quad \forall f \in V'$.

Theorem 0.4.1 *(Fenchel duality)Let V be a locally convex Hausdorff topological vector space over $\mathbb{R}$ with its dual V'. Let F and G be two power convex functions of V into $\overline{\mathbb{R}}$ .Assume that there exists $u_0 \in dom(F) \cap dom(G)$ such that F is continuous in u_0 .Then*

$$\inf_{u\in V}(F(u)+G(u)) = \sup_{f\in V'}(-F^*(-f)-G^*(f)).$$

Proof:From Fenchel inequality, we have for any function H

$$H^*(f)+H(u) \geq (\langle f,u\rangle_{V,V'}, \quad \forall u \in V \ , \ \forall f \in V'$$

consequently, we have that

$$\inf_{u\in V}(F(u)+G(u)) \geq \sup_{f\in V'}(-F^*(-f)-G^*(f)).$$

(this fact is usually referred to as weak duality).
Denote $p := \inf_{u\in V}(F(u)+G(u))$, $q := \sup_{f\in V'}(-F^*(-f)-G^*(f))$ and $C := epiF$. To complete the proof, we show that $p \leq q$.
If $p=-\infty$ there is nothing to prove.Suppose now that $p \neq -\infty$.
It is clear that the interior of C : $intC$ is not empty (because F is continuous in u_0).
We introduce now the following sets:

$$A := intC,$$

$$B := \{(\lambda,u) \in V \times \mathbb{R} : \lambda \leq p - G(u)\}$$

The set A and B are convex (since F and G are convex) and disjoint (according to the definition of p), therefore, (because of Hahn-Banach's first geometric form) there exist a non zero continuous linear function $f \in V'$ and $(\alpha,\beta) \in \mathbb{R}^2$ such that

$$H = \{(\lambda,u) \in V \times \mathbb{R} : (\langle f,u\rangle_{V,V'} + \alpha\lambda = \beta\}$$

and

$$(3) \qquad \begin{array}{l} (\langle f,u\rangle_{V,V'} + \alpha\lambda \geq \beta, \forall (u,\lambda) \in C, \\ (\langle f,u\rangle_{V,V'} + \alpha\lambda \leq \beta, \forall (u,\lambda) \in B, \end{array}$$

By taking $u=u_0$ in the first part of the last inequality and by passing to the limit on $(\lambda \longrightarrow +\infty)$ we can deduce that $\alpha \geq 0$.
Prove now that $\alpha \neq 0$; for this we proceed by contradiction. Assume that $\alpha = 0$, then according to the last inequalities, we arrive at

$$\langle f,u\rangle_{V,V'} \geq \beta, \forall u \in dom(F), \text{ and }, \langle f,u\rangle_{V,V'} \leq \beta, \forall u \in dom(G).$$

In particular $\langle f,u_0\rangle_{V,V'} = \beta$ (since $u_0 \in dom(F) \cap dom(G)$) and then $\langle f, u-u_0\rangle_{V,V'} \geq 0$ for all u in $dom(F)$ .Consequently, $f=0$ since $dom(F)$ is neighborhood of u_0. We thus have $\alpha > 0$.
According to

$$(4) \qquad \begin{array}{l} \langle f,u\rangle_{V,V'} + \alpha\lambda \geq \beta, \forall (u,\lambda) \in C, \\ \langle f,u\rangle_{V,V'} + \alpha\lambda \leq \beta, \forall (u,\lambda) \in B, \end{array}$$

and dividing by $\alpha > 0$, we obtain easily that

$$F^*(-f_\alpha) \leq -\beta_\alpha,$$
$$G^*(f_\alpha) \leq \beta_\alpha - p$$

and then $f_\alpha = f/\alpha$ and $\beta_\alpha = \beta/\alpha$.
Therefore ,$p \leq q$.This complete the proof.

Examples

1.Let C be a non-empty subset of topological vector space V and χ_C be its indicator function. Then the conjugate function ${\chi_C}^*$ is defined by

$${\chi_C}^*(f) = \sup_{u \in C} \langle f, u \rangle_{V,V'}$$

and is called the support function of C. Moreover, if C is a closed and convex set, χ_C is closed and convex, and by the conjugacy theorem the conjugate of its support function is its indicator function.

2.Let $(V, \|.\|)$ be a Banach space, $(V', \|.\|_*)$ its dual, $\Psi_\alpha : t \in \mathbb{R} \longrightarrow |t|^\alpha/\alpha$ and $F_\alpha : V \longrightarrow \mathbb{R}$ such that $F_\alpha(u) = \Psi_\alpha(\|u\|)$, where $1 < \alpha < \infty$. Then

$$\begin{aligned} F^*_\alpha(f) &= \sup_{u \in V}(\langle f, u \rangle_{V,V'} - F_\alpha(u)) \\ &= \sup_{\lambda \geq 0}\left(\|f\|_*\lambda - \frac{\lambda^\alpha}{\alpha}\right) \end{aligned}$$

Hence (by analyzing the function $r(\lambda) := \theta\lambda - \lambda^\alpha/\alpha$ where $\theta := \|f\|_*$ and $\lambda \in [0, +\infty[$, $F^*_\alpha(f) = \|f\|_*^{\alpha^*}/\alpha^*$ where $1/\alpha + 1/\alpha^* = 1$.Consequently

$$F^*_\alpha(f) = \Psi_{\alpha^*}(\|f\|_*)$$

3.We finish with an interesting example for the boundary valued problems in a lemma form.

Lemma 0.4.1 *Let $(V, \|.\|)$ be a Banach space, $(V', \|.\|_*)$ its dual and C be a non-empty closed and convex subset or V. Consider the convex and lower semi-continuous real-valued function F on V given by*

$$F(v) := \langle f, v \rangle_{V,V'} + \chi_C(v - u) \quad \forall v \in V$$

where $u \in V$ and $f \in V'$ are given elements.
then the conjugate of F is

$$F^*(g) = \langle g - f, u \rangle_{V,V'} + \chi_{C^*}(g - f) \quad \forall g \in V'$$

where $C^* = \{g \in V' : \ \langle g, v \rangle_{V,V'} = 0 \ \ \forall v \in C\}$ (which is said to be the polar set of C)

Proof.Let $g \in V'$, we have

$$\begin{aligned} F^*(g) &= \sup_{v \in V}(\langle g, v \rangle_{V,V'} - \langle f, v \rangle_{V,V'} - \chi_C(v - u)) \\ &= \sup_{w \in C}\langle g - f, w + u \rangle_{V,V'} \\ &= \langle g - f, u \rangle_{V,V'} + \sup_{w \in C}\langle g - f, w \rangle_{V,V'} \end{aligned}$$

This completes the proof (since $\sup_{w \in C}\langle g - f, w \rangle_{V,V'} = \chi^*_C(g - f) = \chi_{C^*}(g - f)$).

0.4.2 Legendre transformation

In mathematics, the Legendre transformation or Legendre transform, named after Adrien-Marie Legendre, is an operation that transforms one real-valued function of a real variable into another. Specifically, the Legendre transform of a convex function F is the function F^* defined by

$$F^*(p) = \sup(px - F(x))$$

where "sup" represents the supremum. If F is differentiable, then $F^*(p)$ can be interpreted as the negative of the y-intercept of the tangent line to the graph of F that has slope p. In particular, the value of x that attains the maximum has the property :$F'(x) = p$

That is, the derivative of the function F becomes the argument to the function F^*. In particular, if F is convex (or concave up), then F^* satisfies the functional equation

$$F^*(F'(x)) = xF'(x) - F(x)$$

The Legendre transform is its own inverse. Like the familiar Fourier transform, the Legendre transform takes a function $F(x)$ and produces a function of a different variable p. However, while the Fourier transform consists of an integration with a kernel, the Legendre transform uses maximization as the transformation procedure. The transform is especially well behaved if $F(x)$ is a convex function. The Legendre transformation is an application of the duality relationship between points and lines. The functional relationship specified by $F(x)$ can be represented equally well as a set of (x, y) points, or as a set of tangent lines specified by their slope and intercept values. The Legendre transformation can be generalized to the Legendre-Fenchel transformation. It is commonly used in thermodynamics and in the Hamiltonian formulation of classical mechanics.

0.4.3 Jensen inequality

Let (Ω, A, μ) be a measure space, such that $\mu(\Omega) = 1$. If g is a real-valued function that is μ-integrable, and if is a convex function on the real line, then:

$$\varphi\left(\int_\Omega g\ d\mu\right) \le \int_\Omega \varphi \circ g\ d\mu$$

In real analysis, we may require an estimate on $\varphi\left(\int_a^b g(x)\ dx\right)$ where a, b are real numbers, and g is a non-negative real-valued function that is Lebesgue-integrable. In this case, the Lebesgue measure of $[a, b]$ don't need to be unity. However, by integration by substitution, the interval can be rescaled so that it has measure unity. Then Jensen's inequality can be applied to get

$$\varphi\left(\int_a^b g(x)\ dx\right) \le \frac{1}{b-a}\int_a^b \varphi((b-a)g(x))\ dx$$

0.5 Aubin -Lions lemma

The Aubin Lions lemma is a result in the theory of Sobolev spaces of Banach space-valued functions. More precisely, it is a compactness criterion that is very useful in the study of nonlinear evolutionary partial differential equations. The result is named after the French mathematicians Thierry Aubin and Jacques-Louis Lions. We complete the preliminaries by the useful inequalities of Gagliardo-Nirenberg and Sobolev-Poincaré.

Lemma 0.5.1 *LetX_0,X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Assume that$X_0$ is compactly embedded in X and that X is continuously embedded in X_1; assume also that X_0 and X_1 are reflexive spaces. For $1 < p, q < +\infty$, let*

$$W = \{u \in L^p([0,T];X_0)/ \quad \dot{u} \in L^q([0,T];X_1)\}$$

Then the embedding of W into $L^p([0,T];X)$ is also compact.

Lemma 0.5.2 (Gagliardo-Nirenberg) *Let $1 \leq r < q \leq +\infty$ and $p \leq q$. Then, the inequality*

$$\|u\|_{W^{m,q}} \leq C\|u\|_{W^{m,p}}^{\theta}\|u\|_r^{1-\theta} \quad \textit{for} \quad u \in W^{m,p} \bigcap L^r$$

holds with some $C > 0$ and

$$\theta = \left(\frac{k}{n} + \frac{1}{r} - \frac{1}{q}\right)\left(\frac{m}{n} + \frac{1}{r} - \frac{1}{p}\right)^{-1}$$

provided that $0 < \theta \leq 1$ (we assume $0 < \theta < 1$ if $q = +\infty$).

Lemma 0.5.3 (Sobolev-Poincaré inequality) *Let q be a number with $2 \leq q < +\infty$ ($n = 1, 2$) or $2 \leq q \leq 2n/(n-2)$ ($n \geq 3$), then there is a constant $c_* = c(\Omega, q)$ such that*

$$\|u\|_q \leq c_*\|\nabla u\|_2 \quad \textit{for} \quad u \in H_0^1(\Omega).$$

INTEGRAL INEQUALITIES

We will recall some fundamental integral inequalities introduced by A. Haraux ,V. Komornik and A.Guesmia to estimate the decay rate of the energy.

0.5.1 Case of exponential decay

The estimation of the energy decay for some dissipative problems is based on the following lemma:

Lemma 0.5.4 ([?]) *Let $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function and assume that there is a constant $A > 0$ such that*

$$\forall t \geq 0, \qquad \int_t^{+\infty} E(\tau)\, d\tau \leq \frac{1}{A} E(t). \tag{5}$$

Then we have

$$\forall t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}. \tag{6}$$

Proof of Lemma 0.5.4.
The inequality (6) is verified for $t \leq \frac{1}{A}$, this follows from the fact that E is a decreasing function. We prove that (6) is verified for $t \geq \frac{1}{A}$. Introduce the function

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_t^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (5) we find that

$$\forall t \geq 0, \quad h'(t) + Ah(t) \leq 0.$$

Let

$$T_0 = \sup\{t,\ h(t) > 0\}. \tag{7}$$

For every $t < T_0$, we have

$$\frac{h'(t)}{h(t)} \leq -A,$$

thus

$$h(0) \leq\ e^{-At} \leq \frac{1}{A} E(0)\, e^{-At}, \quad \text{for} \quad 0 \leq t < T_0. \tag{8}$$

Since $h(t) = 0$ if $t \geq T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Let $\varepsilon > 0$. As E is positive and decreasing, we deduce that

$$\forall t \geq \varepsilon, \quad E(t) \leq \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} E(\tau)\, d\tau \leq \frac{1}{\varepsilon} h(t-\varepsilon) \leq \frac{1}{A\varepsilon} E(0)\, e^{\varepsilon t}\, e^{-At}.$$

Choosing $\varepsilon = \frac{1}{A}$, we obtain

$$\forall t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}.$$

The proof of Lemma 0.5.4 is now completed.

0.5.2 Case of polynomial decay

Lemma 0.5.5 ([26]) *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ = [0, +\infty)$) be a non-increasing function and assume that there are two constants $q > 0$ and $A > 0$ such that*

$$\forall t \geq 0, \quad \int_{t}^{+\infty} E^{q+1}(\tau)\, d\tau \leq \frac{1}{A} E^q(0) E(t). \tag{9}$$

Then we have:

$$\forall t \geq 0, \quad E(t) \leq E(0) \left(\frac{1+q}{1+A\,q\,t} \right)^{1/q}. \tag{10}$$

Remark 0.5.1 It is clear that Lemma 0.5.4, is similar to Lemma 0.5.5 in the case of $q = 0$.

Proof of Lemma 0.5.5.

If $E(0) = 0$, then $E \equiv 0$ and there is nothing to prove. Otherwise, replacing the function E by the function $\dfrac{E}{E(0)}$ we may assume that $E(0) = 1$.

Introduce the function

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_{t}^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (9) we find that

$$\forall t \geq 0, \quad -h' \geq (Ah)^{1+q}.$$

where

$$T_0 = \sup\{t,\ h(t) > 0\}.$$

Integrating in $[0, t]$ we obtain that

$$\forall 0 \leq t < T_0, h(t)^{-q} - h(0)^{-\sigma} \geq \sigma\omega^{1+q} t,$$

hence

$$0 \leq t < T_0, \quad h(t) \leq \left(h^{-q}(0) + qA^{1+q}\, t \right)^{-1/q}. \tag{11}$$

Since $h(t) = 0$ if $t \geq T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Since

$$h(0) \leq \frac{1}{A} E(0)^{1+q} = \frac{1}{A},$$

by (9), the right-hand side of (11) is less than or equal to:

$$\left(h^{-q}(0) + qA^{1+q}\, t\right)^{-1/q} \leq \frac{1}{A}(1 + A\, q\, t)^{-1/q}, \tag{12}$$

From other hand, E being nonnegative and non-increasing, we deduce from the definition of h and the above estimate that:

$$\begin{aligned} \forall s \geq 0,\ E\left(\frac{1}{A} + (q+1)s\right)^{q+1} &\leq \frac{1}{\frac{1}{A} + q + 1} \int_s^{\frac{1}{A}+(q+1)s} E(\tau)^{q+1}\, d\tau \\ &\leq \frac{A}{1 + Aqs} h(s) \leq \frac{A}{1 + Aqs} \frac{1}{A} (1 + Aqs)^{-\frac{1}{q}}, \end{aligned}$$

hence

$$\forall S \geq 0, \quad E\left(\frac{1}{A} + (q+1)S\right) \leq \frac{1}{(1 + A\, q\, S)^{1/q}}.$$

Choosing $t = \dfrac{1}{A} + (1+q)s$ then the inequality (10) follows.Note that letting $q \to 0$ in this theorem we obtain (10).

0.6 New integral inequalities of P. Martinez

The above inequalities are verified only if the energy function is integrable, we will try to resolve this problem by introducing some weighted integral inequalities, so we can estimate the decay rate of the energy when it is slow. 0.5.4.

Lemma 0.6.1 ([33]) *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^1 function such that*

$$\phi(0) = 0 \quad \textit{and} \quad \phi(t) \to +\infty \quad \textit{when} \quad t \to +\infty. \tag{13}$$

Assume that there exist $q \geq 0$ and $A > 0$ such that

$$\int_S^{+\infty} E(t)^{q+1} \phi'(t)\, dt \leq \frac{1}{A} E(0)^q E(S), \quad 0 \leq S < +\infty. \tag{14}$$

then we have

$$\textit{if } q > 0, \quad \textit{then } E(t) \leq E(0) \left(\frac{1+q}{1 + q\, A\, \phi(t)}\right)^{\frac{1}{q}}, \ \forall t \geq 0,$$

$$\textit{if } q = 0, \quad \textit{then } E(t) \leq E(0)\, e^{1 - A\, \phi(t)}, \ \forall t \geq 0.$$

Proof of Lemma 0.6.1.
This Lemma is a generalization of Lemma 0.5.4, Let $f: \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$, (we notice that ϕ^{-1} has a meaning by the hypotheses assumed on ϕ). f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain

$$\begin{aligned}\int_{\phi(S)}^{\phi(T)} f(x)^{q+1}\, dx &= \int_{\phi(S)}^{\phi(T)} E\left(\phi^{-1}(x)\right)^{q+1}\, dx = \int_S^T E(t)^{q+1}\phi'(t)\, dt \\ &\leq \frac{1}{A}\, E(0)^q E(S) \\ &= \frac{1}{A}\, E(0)^q f(\phi(S)), \quad 0 \leq S < T < +\infty.\end{aligned}$$

Setting $s := \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\forall s \geq 0, \quad \int_s^{+\infty} f(x)^{q+1}\, dx \leq \frac{1}{A}\, E(0)^q f(s).$$

Thanks to Lemma 0.5.4, we deduce the desired results.

0.7 Generalized inequalities of A. Guesmia

Lemma 0.7.1 (Guesmia [21]) *Let $E: \mathbb{R}_+ \to \mathbb{R}_+$ differentiable function, $\lambda \in \mathbb{R}_+$ and $\Psi: \mathbb{R}_+ \to \mathbb{R}_+$ convex and increasing function such that $\Psi(0) = 0$. Assume that*

$$\textstyle\int_s^{+\infty} \Psi(E(t))\, dt \leq E(s), \quad \forall s \geq 0.$$

$$E'(t) \leq \lambda E(t), \quad \forall t \geq 0.$$

Then E satisfies the estimate

$$E(t) \leq e^{\tau_0 \lambda T_0} d^{-1}\Big(e^{\lambda(t-h(t))}\Psi\Big(\psi^{-1}\Big(h(t) + \psi(E(0))\Big)\Big)\Big), \quad \forall t \geq 0,$$

where

$$\psi(t) = \int_t^1 \frac{1}{\Psi(s)}\, ds, \quad \forall t > 0,$$

$$d(t) = \begin{cases} \Psi(t) & if \quad \lambda = 0, \\ \displaystyle\int_0^t \frac{\Psi(s)}{s}\, ds & if \quad \lambda > 0, \end{cases} \qquad \forall t \geq 0,$$

$$h(t) = \begin{cases} K^{-1}(D(t)), & if \quad t > T_0, \\ 0 & if \quad t \in [0, T_0], \end{cases}$$

$$K(t) = D(t) + \frac{\psi^{-1}(t+\psi(E(0)))}{\Psi(\psi^{-1}(t+\psi(E(0))))}\, e^{\lambda t}, \quad \forall t \geq 0,$$

$$D(t) = \int_0^t e^{\lambda s}\, ds, \quad \forall t \geq 0,$$

$$T_0 = D^{-1}\Big(\frac{E(0)}{\Psi(E(0))}\Big), \quad \tau_0 = \begin{cases} 0, & if \quad t > T_0, \\ 1, & if \quad t \in [0, T_0]. \end{cases}$$

Remark 0.7.1 If $\lambda = 0$ (that is E is non increasing), then we have

$$E(t) \leq \psi^{-1}\Big(h(t) + \psi(E(0))\Big), \quad \forall t \geq 0 \tag{15}$$

where $\psi(t) = \int_t^1 \frac{1}{\Psi(s)}, ds$ for $t > 0$, $h(t) = 0$ for $0 \leq t \leq \frac{E(0)}{\Psi(E(0))}$ and

$$h^{-1}(t) = t + \frac{\psi^{-1}\Big(t + \psi(E(0))\Big)}{\Psi\Big(\psi^{-1}\Big(t + \psi(E(0))\Big)\Big)}, \quad t > 0.$$

This particular result generalizes the one obtained by Martinez [33] in the particular case of $\Psi(t) = dt^{p+1}$ with $p \geq 0$ and $d > 0$, and improves the one obtained by Eller, Lagnese and Nicaise [20].

Proof of Lemma 0.7.1 Because $E'(t) \leq \lambda E(t)$ imply $E(t) \leq e^{\lambda(t-t_0)}E(t_0)$ for all $t \geq t_0 \geq 0$, then, if $E(t_0) = 0$ for some $t_0 \geq 0$, then $E(t) = 0$ for all $t \geq t_0$, and then there is nothing to prove in this case. So we assume that $E(t) > 0$ for all $t \geq 0$ without loss of generality. Let:

$$L(s) = \int_s^{+\infty} \Psi(E(t))\, dt, \quad \forall s \geq 0.$$

We have, $L(s) \leq E(s)$, for all $s \geq 0$. The function L is positive, decreasing and of class $C^1(\mathbb{R}_+)$ satisfying

$$-L'(s) = \Psi(E(s)) \geq \Psi(L(s)), \quad \forall s \geq 0.$$

The function ψ is decreasing, then

$$\Big(\psi(L(s))\Big)' = \frac{-L'(s)}{\Psi(L(s))} \geq 1, \quad \forall s \geq 0.$$

Integration on $[0, t]$, we obtain

$$\psi(L(t)) \geq t + \psi(E(0)), \quad \forall t \geq 0. \tag{16}$$

Since Ψ is convex and $\Psi(0) = 0$, we have

$$\Psi(s) \leq \Psi(1)s, \ \forall s \in [0,1] \quad and \quad \Psi(s) \geq \Psi(1)s, \ \forall s \geq 1,$$

then $\underset{t\to 0}{lim}\,\psi(t) = +\infty$ and $[\psi(E(0)), +\infty[\subset Image\,(\psi)$. Then (16) imply that

$$L(t) \leq \psi^{-1}\Big(t + \psi(E(0))\Big), \quad \forall t \geq 0. \tag{17}$$

Now, for $s \geq 0$, let

$$f_s(t) = e^{-\lambda t}\int_s^t e^{\lambda\tau}\,d\tau, \quad \forall t \geq s.$$

The function f_s is increasing on $[s, +\infty[$ and strictly positive on $]s, +\infty[$ such that

$$f_s(s) = 0 \quad and \quad f_s'(t) + \lambda f_s(t) = 1, \quad \forall t \geq s \geq 0,$$

and the function d is well defined, positive and increasing such that:

$$d(t) \leq \Psi(t) \quad and \quad \lambda t d'(t) = \lambda\Psi(t), \quad \forall t \geq 0,$$

then

$$\begin{aligned}\partial_\tau\Big(f_s(\tau)d(E(\tau))\Big) &= f_s'(\tau)d(E(\tau)) + f_s(\tau)E'(\tau)d'(E(\tau))\\ &\leq \Big(1 - \lambda f_s(\tau)\Big)\Psi(E(\tau)) + \lambda f_s(\tau)\Psi(E(\tau))\\ &= \Psi(E(\tau)), \quad \forall \tau \geq s \geq 0.\end{aligned}$$

Integrating on $[s, t]$, we obtain

$$L(s) \geq \int_s^t \Psi(E(\tau))\,d\tau \geq f_s(t)d(E(t)), \quad \forall t \geq s \geq 0. \tag{18}$$

Since $\underset{t\to+\infty}{lim}\, d(s) = +\infty$, $d(0) = 0$ and d is increasing, then (17) and (18) imply

$$E(t) \leq d^{-1}\left(\inf_{s\in[0,t[}\frac{\psi^{-1}\Big(s + \psi(E(0))\Big)}{f_s(t)}\right), \quad \forall t > 0. \tag{19}$$

Now, let $t > T_0$ and

$$J(s) = \frac{\psi^{-1}\Big(s + \psi(E(0))\Big)}{f_s(t)}, \quad \forall s \in [0, t[.$$

The function J is differentiable and we have

$$J'(s) = f_s^{-2}(t)\Big[e^{-\lambda(t-s)}\psi^{-1}\Big(s + \psi(E(0))\Big) - f_s(t)\Psi\Big(\psi^{-1}\Big(s + \psi(E(0))\Big)\Big)\Big].$$

Then

$$J'(s) = 0 \;\Leftrightarrow\; K(s) = D(t) \quad and \quad J'(s) < 0 \;\Leftrightarrow\; K(s) < D(T).$$

Since $K(0) = \dfrac{E(0)}{\Psi(E(0))}$, $D(0) = 0$ and K and D are increasing (because ψ^{-1} is decreasing and $s \mapsto \dfrac{s}{\Psi(s)}$, $s > 0$, is non increasing thanks to the fact that Ψ is convex). Then, for $t > T_0$,

$$\inf_{s\in[0,t[} J(s) = J\Big(K^{-1}(D(t))\Big) = J(h(t)).$$

Since h satisfies $J'(h(t)) = 0$, we conclude from (19) our desired estimate for $t > T_0$.

For $t \in [0, T_0]$, we have just to note that $E'(t) \leq \lambda E(t)$ and the fact that $d \leq \Psi$ implies

$$E(t) \leq e^{\lambda t}E(0) \leq e^{\lambda T_0}E(0) \leq e^{\lambda T_0}\Psi^{-1}\Big(e^{\lambda t}\Psi(E(0))\Big) \leq e^{\lambda T_0}d^{-1}\Big(e^{\lambda t}\Psi(E(0))\Big).$$

Remark 0.7.2 Under the hypotheses of Lemma 0.7.1, we have $\underset{t\to+\infty}{lim} E(t) = 0$. Indeed, we have just to choose $s = \frac{1}{2}t$ in (19) instead of $h(t)$ and note that $d^{-1}(0) = 0$, $\underset{t\to+\infty}{lim} \psi^{-1}(t) = 0$ and $\underset{t\to+\infty}{lim} f_{\frac{1}{2}t}(t) > 0$.

Lemma 0.7.2 (Guesmia [21]) *Let $E : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a differentiable function , $a : \mathbb{R}^+ \longrightarrow \mathbb{R}^{+*}$ and $\lambda : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ two continuous functions .Assume that there exist $r \geq 0$ such that*

$$\int_s^{+\infty} E^{r+1}(t)dt \leq a(s)E(s), \forall s \geq 0 \tag{20}$$

$$E'(t) \leq \lambda(t)E(t), \qquad \forall t \geq 0 \tag{21}$$

Then E verifies ,for all $t \geq 0$,

$$E(t) \leq \frac{E(0)}{\omega(0)}\omega(h(t)exp(\widetilde{\lambda}(t) - \widetilde{\lambda}(h(t)))exp(-\int_0^{h(t)} \omega(\tau)d\tau), \ \ if\ r = 0$$

and

$$E(t) \leq \omega(h(t))exp(\widetilde{\lambda}(t) - \widetilde{\lambda}(h(t)))\left[\left(\frac{\omega(0)}{E(0)}\right)^r + r\int_0^{h(t)} \omega(\tau)^{r+1}d\tau\right]^{-1/r} \quad if\ r > 0$$

where $\widetilde{\lambda}(t) = \int_0^t \lambda(\tau)d\tau$

Proof If $E(s) = 0$ or $a(s) = 0$ for one $s \geq 0$,the first inequality implies $E(t) = 0$ for $t \geq s$,we suppose then that $E(t) > 0$ and $a(t) > 0$ for $t \geq 0$

Put $\omega = \frac{1}{a}$ and $\Psi(s) = \int_s^{+\infty} E^{r+1}(t)dt$; we have

$$\Psi(s) \leq \frac{1}{\omega(s)}E(s), \qquad \forall s \geq 0. \tag{22}$$

the function Ψ is decreasing ,positive and of class C^1 on $\mathbb{R}^+$ and verifies:

$$\Psi'(s) = -E^{r+1}(s) \leq -(\omega(s)\Psi(s))^{r+1}, \quad \forall s \geq 0$$

then

$$\Psi(s) \leq \Psi(0)exp\left(\int_0^s \omega(\tau)d\tau\right) \leq \frac{E(0)}{\omega(0)}exp\left(\int_0^s \omega(\tau)d\tau\right) \qquad \text{if } r = 0 \tag{23}$$

$$\Psi(s) \leq \left((\frac{\omega(0)}{E(0)})^r + \int_0^s (\omega(\tau))^{r+1}d\tau\right)^{-1/r} \qquad \text{if } r > 0 \tag{24}$$

Now we put for all $s \geq 0$,

$$f_s(t) = exp(-(r+1)\widetilde{\lambda}(t)) \int_s^t exp((r+1)\widetilde{\lambda}(\tau))d\tau, \quad \forall t \geq s \tag{25}$$

where $f_s(s) = 0$ and $f_s'(t) + (r+1)\lambda(t) f_s(t) = 1$, $\forall t \geq s \geq 0$.
Under the second hypothesis in the lemma, we deduce

$$E^{r+1}(t) \geq \partial_t(f_s(t)E^{r+1}(t)); \forall t \geq s \geq 0 \tag{26}$$

hence

$$\Psi(s) \geq \int_s^{g(s)} E^{r+1}(t) \geq f_s(g(s))E^{r+1}(g(s))); \forall s \geq 0 \tag{27}$$

where $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^{+*}$ with $I_s(g(s)) = 0$, I_s is defined by

$$I_s(t) = (\omega(s))^{r+1} \int_s^t exp((r+1)\widetilde{\lambda}(\tau))d\tau$$

Let $t > g(0)$ and $s = h(t)$ with

$$h(t) = \begin{cases} 0, \text{ if } & t \in [0, g(0)] \\ \max g^{-1(t)} \text{ if } & t \in]g(0), +\infty[\end{cases}$$

Hence we have $g(s) = t$ and we deduce from (27 that, for all $t \geq g(0)$,

$$\Psi(h(t)) \geq f_{h(t)}(t)E^{r+1}(t) = \left(exp(-(r+1)\widetilde{\lambda}(t)) \int_{h(t)}^t exp((r+1)\widetilde{\lambda}(\tau))d\tau \right) E^{r+1}(t)$$

We conclude from (23 and (24 that ,for all $t > g(0)$,

$$E(t) \leq \frac{E(0)}{\omega(0)} exp(\widetilde{\lambda}(t)) \left(\int_{h(t)}^t exp(\widetilde{\lambda}(\tau))d\tau \right)^{-1} exp\left(-\int_0^{h(t)} \omega(\tau)d\tau \right) \text{ if } r = 0$$

and

$$E(t) \leq exp(\widetilde{\lambda}(t)) \left(\int_{h(t)}^t exp((r+1)\widetilde{\lambda}(\tau))d\tau \right)^{\frac{-1}{r+1}} \times$$
$$\left((\tfrac{\omega(0)}{E(0)})^r + r\int_0^{h(t)} (\omega(\tau))^{r+1} d\tau \right)^{\frac{-1}{r(r+1)}} \text{ if } r > 0$$

The fact that $I_{h(t)}^t = I_s(g(s)) = 0$, we obtain the result of the lemma for $t > g(0)$.
If $t \in [0, g(0)]$ the second inequality of the lemma implies that

$$E(t) \leq E(0)exp(\widetilde{\lambda}(t))$$

Since $h(t) = 0$ on $[0, g(0)]$, $E(0)exp(\widetilde{\lambda}(t))$ is identically equal to the left hand side of the results of the lemma.That conclude the proof.

Lemma 0.7.3 (Guesmia [21]) *Let* $E: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *be a differentiable function ,* $a_1, a_2 \in \mathbb{R}^{+*}$ *and* $a_3, \lambda, r, p \in \mathbb{R}^+$ *such that*

$$a_3\lambda(r+1) < 1$$

and for all $0 \leq s \leq T < +\infty$ *,*

$$\int_s^T E^{r+1}(t)dt \leq a_1(s)E(s) + a_2E^{p+1}(s) + a_3E^{r+1}(T),$$

$$E'(t) \leq \lambda E(t), \quad \forall t \geq 0$$

Then there exist two positive constants ω *and* c *such that ,for all* $t \geq 0$*,*

$$E(t) \leq ce^{-\omega t}, \text{ if } r = 0$$

$$E(t) \leq c(1+t)^{-1/r}, \text{ if } r > 0 \quad \text{and } \lambda = 0$$

$$E(t) \leq c(1+t)^{\frac{-1}{r(r+1)}}, \text{ if } r > 0 \quad \text{and } \lambda > 0$$

Proof:
We show that E verifies the inequality (20).Applying the lemma (0.7.2),we have

$$a_3E^{r+1}(T) = a_3\int_s^T E'^{r+1}(t)dt + a_3E^{r+1}(s)$$
$$\leq a_3(r+1)\int_s^T \lambda E^{r+1}(t)dt + a_3E^{r+1}(s)$$

Under (20),we obtain:

$$\int_s^{+\infty} E^{r+1}(t)dt \leq b(s)E(s), \quad \forall s \geq 0 \tag{28}$$

where

$$b(s) = \frac{a_1 + a_2E^p(s) + a_3E^r(s)}{1 - a_3\lambda(r+1)}, \quad \forall s \geq 0$$

We consider the function f_0 defined in (25)and integrating on $[0, s]$ the inequality

$$E^{r+1}(t) \geq \partial_t(f_0(t)E^{r+1}(t)), \quad \forall t \geq 0$$

we obtain under (28)

$$b(0)E(0) \geq \int_0^s E^{r+1}(t)dt \geq f_0(s)E^{r+1}(s), \quad \forall s \geq 0$$

then

$$E(s) \leq \left(\frac{b(0)E(0)}{f_0(s)}\right)^{\frac{1}{r+1}}, \quad \forall s \geq 0$$

on the other hand, the conditions of the lemma implies that

$$E(s) \leq E(0)exp(\widetilde{\lambda}(s) \quad \forall s \geq 0$$

Hence

$$E(s) \leq \min\left\{E(0)exp(\widetilde{\lambda}(s), \left(\frac{b(0)E(0)}{f_0(s)}\right)^{\frac{1}{r+1}}\right\} = d(s) \quad \forall s \geq 0$$

d is continuous and positive and

$$b(s) \leq \frac{a_1 + a_2(d(s))^p + a_3(d(s))^r}{1 - a_3\lambda(r+1)}, \quad \forall s \geq 0$$

Hence we can conclude from (28) the first inequality (20) of the lemma (0.7.2) with

$$a(s) = \frac{a_1 + a_2(d(s))^p + a_3(d(s))^r}{1 - a_3\lambda(r+1)}, \quad \forall s \geq 0$$

This completes the proof.

Chapter 1

ENERGY DECAY OF SOLUTIONS FOR A WAVE EQUATION WITH A CONSTANT WEAK DELAY AND A WEAK INTERNAL FEEDBACK

1.1 Introduction

In this chapter we investigate the decay properties of solutions for the initial boundary value problem for the linear wave equation of the form

$$(P) \quad \begin{cases} u''(x,t) - \Delta_x u(x,t) + \mu_1(t)\ u'(x,t) + \mu_2(t)\ u'(x,t-\tau) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{on } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{on } \Omega\times]0,\tau[, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$, $\tau > 0$ is a time delay and the initial data (u_0, u_1, f_0) belong to a suitable function space.

In absence of delay ($\mu_2 = 0$), the energy of problem (P) is exponentially decaying to zero provided that μ_1 is constant, see, for instance, **[14]**, **[15]**, **[26]**, **[27]** and **[38]**. On the contrary, if $\mu_1 = 0$ and $\mu_2 > 0$ (a constant weight), that is, there exits only the internal delay, the system (P) becomes unstable (see, for instance **[19]**). In recent years, the PDEs with time delay effects have become an active area of research since they arise in many pratical problems (see, for example, **[1]**, **[47]**). In **[19]**, it has been shown that a small delay in a boundary control could turn a well-behave hyperbolic system into a wild one and, therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see **[40]**, **[41]**, **[49]**). For instance, the authors of **[40]** studied the wave equation with a linear internal damping term with constant delay ($\tau = \mathit{const}$ in the problem (P)) and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (P) will be instable if $\mu_2 \geq \mu_1$. The main approach used in [**40**] is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay are acting on the boundary. We also recall the result by Xu, Yung and Li [**49**], where the authors proved a result similar to the one in [**40**] for the one-space dimension by adopting the spectral analysis approach.

In [**44**], Nicaise, Pignotti and Valein extended the above result to higher space dimensions and established an exponential decay.

Our purpose in this chapter is to give an energy decay estimate of the solution of problem (P) in the presence of a delay term with a weight depending on time. We use the Galerkin approximation scheme and the multiplier technique to prove our results.

1.2 Preliminaries and main results

First assume the following hypotheses:
(H1) $\mu_1 : \mathbb{R}_+ \to]0, +\infty[$ is a non-increasing function of class $C^1(\mathbb{R}_+)$ satisfying

$$\left|\frac{\mu_1'(t)}{\mu_1(t)}\right| \leq M, \tag{1.1}$$

(H2) $\mu_2 : \mathbb{R}_+ \to \mathbb{R}$ is a function of class $C^1(\mathbb{R}_+)$, which is not necessarily positive or monotone, such that

$$|\mu_2(t)| \leq \beta\mu_1(t), \tag{1.2}$$

$$|\mu_2'(t)| \leq \tilde{M}\mu_1(t), \tag{1.3}$$

for some $0 < \beta < 1$ and $\tilde{M} > 0$.

We now state a Lemma needed later.

Lemma 1.2.1 (Martinez[33]) *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^1 function such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Assume that there exist $\sigma > -1$ and $\omega > 0$ such that

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)\,dt \leq \frac{1}{\omega}E^\sigma(0)E(S). \quad 0 \leq S < +\infty, \tag{1.4}$$

Then

$$E(t) = 0 \quad \forall t \geq \frac{E(0)^\sigma}{\omega|\sigma|}, \quad \text{if} \quad -1 < \sigma < 0 \tag{1.5}$$

$$E(t) \leq E(0)\left(\frac{1+\sigma}{1+\omega\sigma\phi(t)}\right)^{\frac{1}{\sigma}} \forall t \geq 0, \quad \text{if} \quad \sigma > 0, \tag{1.6}$$

$$E(t) \leq E(0)e^{1-\omega\phi(t)} \forall t \geq 0, \quad \text{if} \quad \sigma = 0. \tag{1.7}$$

We introduce, as in **[40]**, the new variable

$$z(x,\rho,t) = u_t(x, t-\tau\rho), \quad x\in\Omega,\ \rho\in(0,1), \quad t>0. \tag{1.8}$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega\times(0,1)\times(0,+\infty). \tag{1.9}$$

Therefore, problem (P) takes the form:

$$\begin{cases} u''(x,t) - \Delta_x u(x,t) + \mu_1(t)u'(x,t) + \mu_2(t)z(x,1,t) = 0, & x\in\Omega, t>0,\\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x\in\Omega, \rho\in(0,1), t>0,\\ u(x,t) = 0, & x\in\partial\Omega, t>0,\\ z(x,0,t) = u'(x,t) & x\in\Omega, t>0,\\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & x\in\Omega,\\ z(x,\rho,0) = f_0(x,-\tau\rho) & x\in\Omega, \rho\in(0,1). \end{cases} \tag{1.10}$$

Let $\overline{\xi}$ be a positive constant such that

$$\tau\beta < \overline{\xi} < \tau(2-\beta). \tag{1.11}$$

We define the energy of the solution by:

$$E(t) = \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{2}\|\nabla_x u(t)\|_2^2 + \frac{\xi(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx, \tag{1.12}$$

where

$$\xi(t) = \overline{\xi}\mu_1(t).$$

We have the following theorem.

Theorem 1.2.1 *Let $(u_0,u_1,f_0)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\times H_0^1(\Omega;H^1(0,1))$ satisfy the compatibility condition*

$$f_0(.,0) = u_1.$$

Assume that $(H1)$ and $(H2)$ hold. Then problem (P) admits a unique global weak solution

$$u\in L^\infty_{loc}((-\tau,\infty);H^2(\Omega)\cap H_0^1(\Omega)), u'\in L^\infty_{loc}((-\tau,\infty);H_0^1(\Omega)), u''\in L^\infty_{loc}((-\tau,\infty);L^2(\Omega)).$$

Moreover, for some positive constants c,ω, we obtain the following decay property:

$$E(t)\le cE(0)e^{-\omega\int_0^t \mu_1(s)\,ds}, \qquad \forall\, t\ge 0. \tag{1.13}$$

Lemma 1.2.2 *Let (u,z) be a solution to the problem (1.10). Then, the energy functional defined by (1.12) satisfies*

$$E'(t) \le -\left(\mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{|\mu_2(t)|}{2}\right)\|u_t(x,t)\|_2^2 - \left(\frac{\xi(t)}{2\tau} - \frac{|\mu_2(t)|}{2}\right)\|z(x,1,t)\|_2^2$$

$$\le 0. \tag{1.14}$$

Proof. Multiplying the first equation in (1.10) by $u_t(x,t)$, integrating over Ω and using Green's identity, we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\|u_t(x,t)\|_2^2 \;+\; \frac{1}{2}\frac{\partial}{\partial t}\|\nabla u(x,t)\|_2^2\\&+\mu_1(t)\;\|u_t(x,t)\|_2^2+\;\mu_2(t)\int_\Omega u_t(x,t-\tau)u_t(x,t)\;dx=0.\end{aligned} \tag{1.15}$$

We multiply the second equation in (1.10) by $\xi(t)z$ and integrate over $\Omega\times(0,1)$ to obtain:

$$\xi(t)\tau\int_\Omega\int_0^1 z_t(x,\rho,t)z(x,\rho,t)\,d\rho\,dx+\xi(t)\int_\Omega\int_0^1 z_\rho(x,\rho,t)z(x,\rho,t))\,d\rho\,dx=0. \tag{1.16}$$

This yields

$$\frac{\xi(t)\tau}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx+\frac{\xi(t)}{2}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}z^2(x,\rho,t))\,d\rho\,dx=0,$$

which gives

$$\begin{aligned}&\frac{\tau}{2}\left[\frac{d}{dt}\left(\xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\right)-\xi'(t)\int_0^1\int_\Omega z^2(x,\rho,t)\,d\rho dx\right]\\&+\frac{\xi(t)}{2}\int_\Omega z^2(x,1,t)\,dx-\frac{\xi(t)}{2}\int_\Omega u_t^2(x,t)\,dx=0.\end{aligned}$$

Consequently,

$$\begin{aligned}&\frac{\tau}{2}\frac{d}{dt}\left(\xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\right)=\\&=\frac{\tau}{2}\xi'(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho dx-\frac{\xi(t)}{2}\int_\Omega z^2(x,1,t)\,dx+\frac{\xi(t)}{2}\int_\Omega u_t^2(x,t)\,dx.\end{aligned} \tag{1.17}$$

Combination of (1.15) and (1.17) leads to

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\left[\|u_t(x,t)\|_2^2+\|\nabla u(x,t)\|_2^2+\xi(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\right]\\&=-\mu_1(t)\|u_t(x,t)\|_2^2-\mu_2(t)\int_\Omega z(x,1,t)u_t(x,t)\,dx\\&+\frac{1}{2}\xi'(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho dx-\frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)\,dx+\frac{\xi(t)}{2\tau}\|u_t(x,t)\|_2^2.\end{aligned}$$

Recalling the definition of $E(t)$ in (1.12), we arrive at

$$\begin{aligned}E'(t)\;&=\;-\left(\mu_1(t)\;-\;\frac{\xi(t)}{2\tau}\right)\|u_t(x,t)\|_2^2-\mu_2(t)\int_\Omega z(x,1,t)u_t(x,t)\,dx\\&\quad+\frac{1}{2}\xi'(t)\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho dx-\frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)\,dx.\\&\le\;-\left(\mu_1(t)-\frac{\xi(t)}{2\tau}\right)\|u_t(x,t)\|_2^2-\mu_2(t)\int_\Omega z(x,1,t)u_t(x,t)\,dx\\&\quad-\frac{\xi(t)}{2\tau}\int_\Omega z^2(x,1,t)\,dx.\end{aligned} \tag{1.18}$$

Due to Young's inequality, we have

$$\int_\Omega z(x,1,t)u_t(x,t)\,dx \le \frac{1}{2}\|u_t(x,t)\|_2^2 + \frac{1}{2}\|z(x,1,t)\|_2^2. \tag{1.19}$$

Inserting (1.19) into (1.18), we obtain

$$\begin{aligned} E'(t) &\le -\left(\mu_1(t) - \frac{\xi(t)}{2\tau} - \frac{|\mu_2(t)|}{2}\right)\|u_t(x,t)\|_2^2 - \left(\frac{\xi(t)}{2\tau} - \frac{|\mu_2(t)|}{2}\right)\|z(x,1,t)\|_2^2 \\ &\le -\mu_1(t)\left(1 - \frac{\overline{\xi}}{2\tau} - \frac{\beta}{2}\right)\|u_t(x,t)\|_2^2 - \mu_1(t)\left(\frac{\overline{\xi}}{2\tau} - \frac{\beta}{2}\right)\|z(x,1,t)\|_2^2 \le 0. \end{aligned} \tag{1.20}$$

This completes the proof of the lemma.

1.3 Global Existence

Throughout this section we assume $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$, $f_0 \in L^2(\Omega; H^1(0,1))$.

We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \dots, w_k\}$ where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2(\Omega) \cap H_0^1(\Omega)$.

Now, we define for $1 \le j \le k$ the sequence $\phi_j(x,\rho)$ as follows:

$$\phi_j(x,0) = w_j.$$

Then, we may extend $\phi_j(x,0)$ by $\phi_j(x,\rho)$ over $L^2(\Omega\times(0,1))$ such that $(\phi_j)_j$ form a basis of $L^2(\Omega; H^1(0,1))$ and denote by Z_k the space generated by $\{\phi_1, \phi_2, \dots, \phi_k\}$.

We construct approximate solutions $(u_k, z_k), k = 1,2,3,\dots,$ in the form

$$u_k(t) = \sum_{j=1}^k g_{jk}(t)w_j, \quad z_k(t) = \sum_{j=1}^k h_{jk}(t)\phi_j,$$

where g_{jk} and h_{jk} $(j = 1,2,\dots,k)$ are determined by the following system of ordinary differential equations:

$$\begin{cases} (u_k''(t), w_j) + (\nabla_x u_k(t), \nabla_x w_j) + \mu_1(t)(u_k', w_j) + \mu_2(t)(z_k(.,1), w_j) = 0, \\ 1 \le j \le k, \\ z_k(x,0,t) = u_k'(x,t), \end{cases} \tag{1.21}$$

associated with the initial conditions

$$u_k(0) = u_{0k} = \sum_{j=1}^k (u_0, w_j)w_j \to u_0 \text{ in } H^2(\Omega)\cap H_0^1(\Omega) \text{ as } k \to +\infty, \tag{1.22}$$

$$u_k'(0) = u_{1k} = \sum_{j=1}^k (u_1, w_j)w_j \to u_1 \text{ in } H_0^1(\Omega) \text{ as } k \to +\infty, \tag{1.23}$$

and

$$\begin{cases} (\tau z_{kt} + z_{k\rho}, \phi_j) = 0, \\ 1 \le j \le k, \end{cases} \tag{1.24}$$

$$z_k(\rho, 0) = z_{0k} = \sum_{j=1}^{k} (f_0, \phi_j)\phi_j \to f_0 \text{ in } L^2(\Omega; H^1(0,1)) \text{ as } k \to +\infty. \tag{1.25}$$

By virtue of the theory of ordinary differential equations, the system (1.21)-(1.25) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \le +\infty$) by Zorn lemma. Note that $u_k(t)$ is of class C^2.

In the next step, we obtain a priori estimates for the solution of the system (1.21)-(1.25), so that it can be extended beyond $[0, T_k[$ to obtain a solution defined for all $t > 0$. Then, we utilize a standard compactness argument for the limiting procedure.

The first estimate. Since the sequences u_{0k}, u_{1k} and z_{0k} converge, then from (1.14) we can find a positive constant C independent of k such that

$$E_k(t) + \int_0^t a_1(s)\|u_k'(s)\|_2^2\, ds + \int_0^t a_2(s)\|z_k(x,1,s)\|_2^2\, ds \le E_k(0) \le C, \tag{1.26}$$

where

$$E_k(t) = \frac{1}{2}\|u_k'(t)\|_2^2 + \frac{1}{2}\|\nabla_x u_k(t)\|_2^2 + \frac{\xi(t)}{2}\int_\Omega \int_0^1 z_k^2(x,\rho,t)\, d\rho\, dx,$$

$$a_1(t) = \mu_1(t)\left(1 - \frac{\overline{\xi}}{2\tau} - \frac{\beta}{2}\right) \text{ and } a_2(t) = \mu_1(t)\left(\frac{\underline{\xi}}{2\tau} - \frac{\beta}{2}\right).$$

These estimates imply that the solution (u_k, z_k) exists globally in $[0, +\infty[$.

Estimate (1.26) yields

$$(u_k) \text{ is bounded in } L^\infty_{loc}(0,\infty; H_0^1(\Omega)), \tag{1.27}$$

$$(u_k') \text{ is bounded in } L^\infty_{loc}(0,\infty; L^2(\Omega)), \tag{1.28}$$

$$\mu_1(t)(u_k'^2(t)) \text{ is bounded in } L^1(\Omega \times (0,T)), \tag{1.29}$$

$$\mu_1(t)(z_k^2(x,\rho,t)) \text{ is bounded in } L^\infty_{loc}(0,\infty; L^1(\Omega \times (0,1))), \tag{1.30}$$

$$\mu_1(t)(z_k^2(x,1,t)) \text{ is bounded in } L^1(\Omega \times (0,T)). \tag{1.31}$$

The second estimate. We first estimate $u_k''(0)$. Replacing w_j by $u_k''(t)$ in (1.21) and taking $t = 0$, we obtain:

$$\begin{aligned} \|u_k''(0)\|_2 &\le \|\Delta_x u_{0k}\|_2 + \mu_1(0)\|u_{1k}\|_2 + |\mu_2(0)|\|z_{0k}\|_2 \\ &\le \|\Delta_x u_0\|_2 + \mu_1(0)\|u_1\|_2 + |\mu_2(0)|\|z_0\|_2 \\ &\le C. \end{aligned}$$

Differentiating (1.21) with respect to t, we get

$$(u_k'''(t) + \Delta_x u_k'(t) + \mu_1(t)u_k''(t) + \mu_1'(t)u_k'(t) + \mu_2(t)z_k'(1,t) + \mu_2'(t)z_k(1,t), w_j) = 0.$$

Multiplying by $g''_{jk}(t)$, summing over j from 1 to k, it follows that

$$\begin{aligned}(1.32)\quad &\frac{1}{2}\frac{d}{dt}\left(\|u''_k(t)\|_2^2+\|\nabla_x u'_k(t)\|_2^2\right)+\mu_1(t)\int_\Omega u''^2_k(t)\,dx+\mu'_1(t)\int_\Omega u''_k(t)u'_k(t)\,dx\\ &+\mu_2(t)\int_\Omega u''_k(t)z'_k(x,1,t)\,dx+\mu'_2(t)\int_\Omega u''_k(t)z_k(x,1,t)\,dx=0.\end{aligned}$$

Differentiating (1.24) with respect to t, we get

$$\left(\tau z''_k(t)+\frac{\partial}{\partial\rho}z'_k,\phi_j\right)=0.$$

Multiplying by $h'_{jk}(t)$, summing over j from 1 to k, it follows that

$$(1.33)\qquad \frac{\tau}{2}\frac{d}{dt}\|z'_k(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z'_k(t)\|_2^2=0.$$

Taking the sum of (1.32) and (1.33), we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u''_k(t)\|_2^2+\|\nabla_x u'_k(t)\|_2^2+\tau\int_0^1\|z'_k(x,\rho,t)\|^2_{L^2(\Omega)}\,d\rho\right)+\mu_1(t)\int_\Omega u''^2_k(t)\,dx\\ &\qquad+\frac{1}{2}\int_\Omega|z'_k(x,1,t)|^2\,dx\\ &=-\mu_2(t)\int_\Omega u''_k(t)z'_k(x,1,t)\,dx-\mu'_1(t)\int_\Omega u''_k(t)u'_k(t)\,dx-\mu'_2(t)\int_\Omega u''_k(t)z_k(x,1,t)\,dx\\ &\qquad+\frac{1}{2}\|u''_k(t)\|_2^2\end{aligned}$$

Using **(H1)**, **(H2)**, Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u''_k(t)\|_2^2+\|\nabla_x u'_k(t)\|_2^2+\int_0^1\tau\|z'_k(x,\rho,t)\|^2_{L^2(\Omega)}\,d\rho\right)+\mu_1(t)\int_\Omega u''^2_k(t)\,dx\\ &\qquad+\frac{1}{2}\int_\Omega|z'_k(x,1,t)|^2\,dx\\ &\le|\mu_2(t)|\|u''_k(t)\|_2\|z'_k(x,1,t)\|_2+|\mu'_1(t)|\|u''_k(t)\|_2\|u'_k(t)\|_2+|\mu'_2(t)|\|u''_k(t)\|_2\|z_k(x,1,t)\|_2\\ &\qquad+\frac{1}{2}\|u''_k(t)\|_2^2\\ &\le\frac{|\mu_2(t)|^2}{2}\|u''_k(t)\|_2^2+\frac{1}{2}\|z'_k(x,1,t)\|_2^2+\frac{|\mu'_1(t)|}{4}\|u''_k(t)\|_2^2+|\mu'_1(t)|\|u'_k(t)\|_2^2\\ &\qquad+\frac{|\mu'_2(t)|}{4}\|u''_k(t)\|_2^2+|\mu'_2(t)|\|z_k(x,1,t)\|_2^2+\frac{1}{2}\|u''_k(t)\|_2^2\\ &\le c'\|u''_k(t)\|_2^2+|\mu'_1(t)|\|u'_k\|_2^2+|\mu'_2(t)|\|z_k(x,1,t)\|_2^2+\frac{1}{2}\|z'_k(x,1,t)\|_2^2.\\ &\le c'\|u''_k(t)\|_2^2+M\mu_1(t)\|u'_k\|_2^2+\tilde{M}\mu_1(t)\|z_k(x,1,t)\|_2^2+\frac{1}{2}\|z'_k(x,1,t)\|_2^2.\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using (1.26), we get

$$\begin{aligned}
&\left(\|u''_k(t)\|_2^2 + \|\nabla_x u'_k(t)\|_2^2 + \tau\int_0^1 \|z'_k(x,\rho,t)\|^2_{L^2(\Omega)}\,d\rho\right)\\
&\le \left(\|u''_k(0)\|_2^2 + \|\nabla_x u'_k(0)\|_2^2 + \tau\int_0^1 \|z'_k(x,\rho,0)\|^2_{L^2(\Omega)}\,d\rho\right) + 2M\int_0^t \mu_1(s)\|u'_k(s)\|_2^2\,ds\\
&+2\tilde{M}\int_0^t \mu_1(s)\|z_k(x,1,s)\|_2^2\,ds + 2c'\int_0^t \|u''_k(s)\|_2^2\,ds\\
&\le C + C'\int_0^t \left(\|u''_k(s)\|_2^2 + \|\nabla_x u'_k(s)\|_2^2 + \tau\int_0^1 \|z'_k(x,\rho,s)\|^2_{L^2(\Omega)}\,d\rho\right)\,ds.
\end{aligned}$$

Using Gronwall's lemma, we deduce that

$$\|u''_k(t)\|_2^2 + \|\nabla_x u'_k(t)\|_2^2 + \tau\int_0^1 \|z'_k(x,\rho,t)\|^2_{L^2(\Omega)}\,d\rho \le Ce^{C'T}$$

for all $t \in \mathbb{R}^+$, therefore, we conclude that

$$(u''_k) \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2(\Omega)), \tag{1.34}$$

$$(u'_k) \text{ is bounded in } L^\infty_{loc}(0,\infty;H^1_0(\Omega)), \tag{1.35}$$

$$(\tau z'_k) \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2(\Omega\times(0,1))). \tag{1.36}$$

Applying Dunford-Pettis' theorem, we deduce from (1.27),(1.28), (1.29), (1.30),(1.31), (1.34), (1.35) and (1.36), replacing the sequence u_k with a subsequence if necessary, that

$$u_k \to u \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty;H^2(\Omega)\cap H^1_0(\Omega)), \tag{1.37}$$

$$\begin{aligned}
u'_k &\to u' \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty;H^1_0(\Omega)),\\
u''_k &\to u'' \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty;L^2(\Omega)),\\
u'_k &\to \chi \ \text{ weak in } \ L^2(\Omega\times(0,T);\mu_1(t)),
\end{aligned} \tag{1.38}$$

$$\begin{aligned}
z_k &\to z \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty;H^1_0(\Omega;L^2(0,1)),\\
z'_k &\to z' \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty;L^2(\Omega\times(0,1))),\\
z_k(x,1,t) &\to \psi \ \text{ weak in } \ L^2(\Omega\times(0,T),\mu_1(t))
\end{aligned} \tag{1.39}$$

for suitable functions $u \in L^\infty(0,T;H^2(\Omega)\cap H^1_0(\Omega)), z \in L^\infty(0,T;L^2(\Omega\times(0,1)))$, $\chi \in L^2(\Omega\times(0,T);\mu_1(t)), \psi \in L^2(\Omega\times(0,T);\mu_1(t))$, for all $T \ge 0$. We have to show that u is a solution of (P).

From (1.35) we have that (u'_k) is bounded in $L^\infty(0,T;H^1_0(\Omega))$. Then (u'_k) is bounded in $L^2(0,T;H^1_0(\Omega))$. Since (u''_k) is bounded in $L^\infty(0,T;L^2(\Omega))$, then (u''_k) is bounded in $L^2(0,T;L^2(\Omega))$. Consequently, (u'_k) is bounded in $H^1(Q)$.
Since the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact, using Aubin-Lions theorem **[31]**, we can extract a subsequence (u_ς) of (u_k) such that

$$u'_\varsigma \to u' \text{ strongly in } \ L^2(Q). \tag{1.40}$$

Therefore

$$u'_\varsigma \to u' \text{ strongly and a.e in } Q. \tag{1.41}$$

Similarly we obtain

$$z_\varsigma \to z \text{ strongly in } \; L^2(\Omega \times (0,1) \times (0,T)) \tag{1.42}$$

and

$$z_\varsigma \to z \text{ strongly and a.e in } \Omega \times (0,1) \times (0,T). \tag{1.43}$$

It follows at once, from (1.37), (1.38), (1.39), (1.40) and (1.27), that for each fixed $v \in L^2(0,T;L^2(\Omega))$ and $w \in L^2(0,T;L^2(\Omega) \times (0,1))$

$$\begin{aligned} &\int_0^T \int_\Omega (u''_\varsigma - \Delta_x u_\varsigma + \mu_1(t)u'_\varsigma + \mu_2(t)z_\varsigma)v\,dx\,dt \\ &\to \int_0^T \int_\Omega (u'' - \Delta_x u + \mu_1(t)u' + \mu_2(t)z)v\,dx\,dt \end{aligned} \tag{1.44}$$

$$\int_0^T \int_0^1 \int_\Omega (\tau z'_\varsigma + \frac{\partial}{\partial\rho} z_\varsigma) w\,dx\,d\rho\,dt \to \int_0^T \int_0^1 \int_\Omega (\tau z' + \frac{\partial}{\partial\rho} z) w\,dx\,d\rho\,dt \tag{1.45}$$

as $\varsigma \to +\infty$. Thus the problem (P) admits a global weak solution u.

Uniqueness. Let (u_1, z_1) and (u_2, z_2) be two solutions of problem (1.10). Then $(w, \tilde{w}) = (u_1, z_1) - (u_2, z_2)$ satisfies

$$\begin{cases} w''(x,t) - \Delta_x w(x,t) + \mu_1(t)w'(x,t) & \\ \qquad +\mu_2(t)\tilde{w}(x,1,t) = 0, & \text{in } \Omega \times]0,+\infty[, \\ \tau\tilde{w}'(x,\rho,t) + \tilde{w}_\rho(x,\rho,t) = 0, & \text{in } \Omega \times]0,1[\times]0,+\infty[\\ w(x,t) = 0, & \text{on } \partial\Omega \times]0,+\infty[\\ \tilde{w}(x,0,t) = w'(x,t) & \text{on } \Omega \times]0,+\infty[\\ w(x,0) = 0, \;\; w'(x,0) = 0 & \text{in } \Omega \\ \tilde{w}(x,\rho,0) = 0 & \text{in } \Omega \times]0,1[\end{cases} \tag{1.46}$$

Multiplying the first equation in (1.46) by w', integrating over Ω and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}(\|w'\|_2^2 + \|\nabla_x w\|_2^2) + \mu_1(t)\|w'\|_2^2 + \mu_2(t)(\tilde{w}(x,1,t), w') = 0. \tag{1.47}$$

Multiplying the second equation in (1.46) by $\tilde{w}$, integrating over $\Omega \times (0,1)$ and using integration by parts, we get

$$\frac{\tau}{2}\frac{d}{dt}\|\tilde{w}\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|\tilde{w}\|_2^2 = 0. \tag{1.48}$$

Then

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1 \|\tilde{w}\|_2^2\,d\rho + \frac{1}{2}(\|\tilde{w}(x,1,t)\|_2^2 - \|w'\|_2^2) = 0. \tag{1.49}$$

From (1.47), (1.49), using Cauchy-Schwarz inequality we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|w'\|_2^2 + \|\nabla_x w\|_2^2 + \tau\int_0^1 \|\tilde{w}\|_2^2\,d\rho\right) + \mu_1(t)\|w'\|_2^2 + \frac{1}{2}\|\tilde{w}(x,1,t)\|_2^2 \\ &= -\mu_2(t)(\tilde{w}(x,1,t), w') + \frac{1}{2}\|w'\|_2^2 \\ &\le |\mu_2(t)|\|\tilde{w}(x,1,t)\|_2\|w'\|_2 + \frac{1}{2}\|w'\|_2^2. \end{aligned}$$

Using Young inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|w'\|_2^2+\|\nabla_x w\|_2^2+\tau\int_0^1\|\tilde{w}\|_2^2\,d\rho\right)\leq c\|w'\|_2^2,$$

where c is a positive constant. Then integrating over $(0,t)$, using Gronwalls lemma, we conclude that

$$\|w'\|_2^2+\|\nabla_x w\|_2^2+\tau\int_0^1\|\tilde{w}\|_2^2\,d\rho=0.$$

Hence, uniqueness follows.

1.4 Asymptotic Behavior

From now on, we denote by c various positive constants which may be different at different occurrences. We multiply the first equation of (1.10) by $\phi' E^q u$, where ϕ is a bounded function satisfying all the hypotheses of Lemma 1.2.1. We obtain

$$\begin{aligned}
0 &= \int_S^T E^q\phi'\int_\Omega u\Big(u''-\Delta u+\mu_1(t)u'+\mu_2(t)z(x,1,t)\Big)\,dx\,dt\\
&= \left[E^q\phi'\int_\Omega uu'\,dx\right]_S^T-\int_S^T(qE'E^{q-1}\phi'+E^q\phi'')\int_\Omega uu'\,dxdt\\
&\quad -2\int_S^T E^q\phi'\int_\Omega u'^2\,dx\,dt+\int_S^T E^q\phi'\int_\Omega\left(u'^2+|\nabla u|^2\right)\,dxdt\\
&\quad +\int_S^T E^q\phi'\mu_1(t)\int_\Omega uu'\,dx\,dt+\int_S^T E^q\phi'\mu_2(t)\int_\Omega uz(x,1,t)\,dx\,dt.
\end{aligned}$$

Similarly, we multiply the second equation of (1.10) by $E^q\phi'\xi(t)e^{-2\tau\rho}z(x,\rho,t)$ and get

$$\begin{aligned}
0 &= \int_S^T E^q\phi'\int_\Omega\int_0^1 e^{-2\tau\rho}\xi(t)z\Big(\tau z_t+z_\rho\Big)\,dxd\rho dt\\
&= \left[\frac{1}{2}E^q\phi'\xi(t)\tau\int_\Omega\int_0^1 e^{-2\tau\rho}z^2\,dxd\rho\right]_S^T\\
&\quad -\frac{1}{2}\int_S^T\int_\Omega\int_0^1\left(E^q\phi'\xi(t)\tau e^{-2\tau\rho}\right)'z^2\,dxd\rho dt\\
&\quad +\int_S^T E^q\phi'\int_\Omega\int_0^1\xi(t)\left(\frac{1}{2}\frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}z^2\right)+\tau e^{-2\tau\rho}z^2\right)dxd\rho dt\\
&= \left[\frac{1}{2}E^q\phi'\xi(t)\tau\int_\Omega\int_0^1 e^{-2\tau(t)\rho}z^2\,dxd\rho\right]_S^T\\
&\quad -\frac{\tau}{2}\int_S^T(E^q\phi'\xi(t))'\int_\Omega\int_0^1 e^{-2\tau\rho}z^2\,dxd\rho dt\\
&\quad +\frac{1}{2}\int_S^T E^q\phi'\xi(t)\int_\Omega\left(e^{-2\tau}z^2(x,1,t)-z^2(x,0,t)\right)dxdt\\
&\quad +\int_S^T E^q\phi'\xi(t)\tau\int_0^1\int_\Omega e^{-2\tau\rho}z^2\,dxd\rho dt.
\end{aligned}$$

Taking their sum, we obtain

$$\begin{aligned}
&A\int_S^T E^{q+1}\phi'\,dt \le -\left[E^q\phi'\int_\Omega uu'\,dx\right]_S^T + \int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_\Omega uu'\,dxdt \\
&+2\int_S^T E^q\phi'\int_\Omega u'^2\,dxdt - \int_S^T \mu_1(t)E^q\phi'\int_\Omega uu'\,dxdt - \int_S^T \mu_2(t)E^q\phi'\int_\Omega uz(x,1,t)\,dxdt \\
&-\left[\frac{1}{2}E^q\phi'\xi(t)\tau\int_\Omega\int_0^1 e^{-2\tau\rho}z^2\,dxd\rho\right]_S^T + \frac{1}{2}\int_S^T (E^q\phi'\xi(t))'\,\tau\int_\Omega\int_0^1 e^{-2\tau\rho}z^2\,dxd\rho dt \\
&-\frac{1}{2}\int_S^T E^q\phi'\xi(t)\int_\Omega \left(e^{-2\tau(t)}z^2(x,1,t) - z^2(x,0,t)\right)dxdt,
\end{aligned} \tag{1.50}$$

where $A = 2\min\{1, e^{-2\tau_1}\}$. Using the Cauchy-Schwarz and Poincaré's inequalities and the definition of E and assuming that ϕ' is a bounded non-negative function on $\mathbb{R}^+$, we get

$$\left|E^q(t)\phi'\int_\Omega uu'\,dx\right| \le cE(t)^{q+1}.$$

By recalling (1.14), we have

$$\int_S^T \left|qE'E^{q-1}\phi'\int_\Omega uu'\,dx\right|dt \le c\int_S^T E^q(t)|E'(t)|\,dt \le c\int_S^T E^q(t)(-E'(t))\,dt \le cE^{q+1}(S),$$

$$\int_S^T E^q\phi''\int_\Omega uu'\,dx\,dt \le c\int_S^T E^{q+1}(t)(-\phi'') \le cE^{q+1}(S)\int_S^T (-\phi'')\,dt \le cE^{q+1}(S),$$

and

$$\begin{aligned}
\int_S^T E^q\phi'\int_\Omega u'^2\,dxdt &\le c\int_S^T E^q\phi'\frac{1}{\mu_1(t)}\int_\Omega \mu_1(t)u'^2\,dxdt \\
&\le \int_S^T E^q\frac{\phi'}{\mu_1(t)}\left(-E'\right)\,dt.
\end{aligned} \tag{1.51}$$

Define

$$\phi(t) = \int_0^t \mu_1(s)\,ds. \tag{1.52}$$

It is clear that ϕ is a non-decreasing function of class C^1 on $\mathbb{R}^+, \phi$ is bounded and

$$\phi(t) \to +\infty \text{ as } t \to +\infty. \tag{1.53}$$

So, we deduce, from (1.51), that

$$\int_S^T E^q\phi'\int_\Omega u'^2\,dxdt \;\le\; c\int_S^T E^q(-E')\,dt \;\le\; cE^{q+1}(S), \tag{1.54}$$

By the hypothesis (**H1**), Young's and Poincaré's inequality and (1.14), we have

$$\begin{aligned}
\left|\int_S^T E^q\phi'\int_\Omega uu'\,dx\,dt\right| &\le c\int_S^T E^q\phi'\|u\|_2\|u'\|_2\,dt \\
&\le c\varepsilon'\int_S^T E^q\phi'\|u\|_2^2\,dt + c(\varepsilon')\int_S^T E^q\phi'\|u'\|_2^2\,dt \\
&\le \varepsilon' c_*\int_S^T E^q\phi'\|\nabla_x u\|_2^2\,dt + c(\varepsilon')\int_S^T E^q\phi'\|u'\|_2^2\,dt \\
&\le \varepsilon' c_*\int_S^T E^{q+1}\phi'\,dt + cE^{q+1}(S).
\end{aligned}$$

Recalling that $\xi' \leq 0$ and the definition of E, we have

$$\begin{aligned}\int_S^T (E^q\xi(t))'\,\tau \int_\Omega \int_0^1 e^{-2\tau\rho} z^2\, dx d\rho dt &\leq \int_S^T (E^q)'\,\xi(t)\tau \int_\Omega \int_0^1 e^{-2\tau\rho} z^2\, dx d\rho dt \\ &\leq c\int_S^T E^q|E'|\, dt \\ &\leq c\int_S^T E^q(-E'(t))\, dt \\ &\leq cE^{q+1}(S)\end{aligned}$$

$$\begin{aligned}\int_S^T E^q\xi(t) \int_\Omega e^{-2\tau} z^2(x,1,t)\, dx dt &\leq c\int_S^T E^q\xi(t) \int_\Omega z^2(x,1,t)\, dx dt \\ &\leq c\int_S^T E^q\left(-E'\right)\, dt \\ &\leq cE^{q+1}(S)\end{aligned}$$

$$\begin{aligned}\int_S^T E^q\xi(t) \int_\Omega z^2(x,0,t)\, dx dt &= \int_S^T E^q\xi(t) \int_\Omega u'^2(x,t)\, dx dt \\ &\leq cE^{q+1}(S),\end{aligned}$$

where we have also used the Cauchy-Schwarz inequality. Combining these estimates and choosing ε' sufficiently small, we conclude from (1.50) that

$$\int_S^T E^{q+1}\phi'\, dt \leq CE^{q+1}(S).$$

This ends the proof of Theorem 1.2.1.

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Chapter 2

GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS TO A VISCOELASTIC WAVE EQUATION WITH A DELAY TERM IN THE NONLINEAR INTERNAL FEEDBACK

2.1 Introduction

In this chapter we investigate the existence and decay properties of solutions for the initial boundary value problem of the nonlinear viscoelastic wave equation of the type

$$(P) \qquad \begin{cases} u_{tt}(x,t) - \Delta_x u(x,t) + \int_0^t h(t-s)\Delta_x u(x,s)\,ds \\ +\mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau[, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$, h is a positive nonincreasing function defined on $\mathbb{R}^+$, g_1 and g_2 are two functions, $\tau > 0$ is a time delay, μ_1 and μ_2 are positive real numbers, and the initial data (u_0, u_1, f_0) belong to a suitable function space.

In the absence of the viscoelastic term (that is, if $h = 0$), problem (P) has been studied by many mathematicians. It is well known that in the further absence of a damping mechanism, the delay term causes instability of system (see, for instance **[19]**). In the contrast, in the absence of the delay term, the damping term assures global existence for arbitrary initial data and energy decay estimates depending on the rate of growth of g_1 (see **[2]**,**[6]**, **[24]**, [**26**] and [**30**]). In recent years, the PDEs with time delay effects have become an active area of research and arise in many pratical real world problems (see for example [**1**], [**47**]). To stabilize a hyperbolic system involving delay terms, additional control terms are necessary (see [**40**], [**41**] and [**49**]). In [**40**] the authors examined problem (P) in the linear situation (that is, if $g_1(s) = g_2(s) = s\ \forall s \in \mathbb{R}$) and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they found a sequence of delays for which the corresponding solution of (P) will be instable if $\mu_2 \geq \mu_1$. The main approach used in [**40**] is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay acting in the boundary domain. We also recall the result by Xu, Yung and Li [**49**], where the authors proved the same result as in [**40**] for the one space dimension by adopting the spectral analysis approach. Very recently, Benaissa and Louhibi [**7**] extended the result of [**40**] to the nonlinear case.

In the presence of the viscoelastic term ($h \neq 0$), Cavalcanti et al. [**12**] studied (P) for $g_2 \equiv 0$ and in the presence of a linear localized frictional damping $(a(x)u_t)$. They obtained an exponential rate of decay by assuming that the kernel h is of exponential decay. This work was later improved by Berrimi and Messaoudi [**9**] by introducing a different functional, which allowed them to weaken the conditions on h. In [**36**], Messaoudi investigated the decay rate to (P) under a more general condition on h and improved earlier results in which only the exponential and polynomial rates were considered. Kirane and Said Houari [**7**] extended the result in [**36**] to the case when g_1, g_2 are linear and $\mu_1 \geq \mu_2$.

Our purpose in this chapter is to give a global solvability and energy decay estimates of the solutions of problem (P) when h is of general decay rate and g_1, g_2 are nonlinear. To obtain global solutions of problem (P), we use the Galerkin approximation scheme (see [**31**]) together with the energy estimate method. The technic based on the theory of nonlinear semigroups used in [**40**] does not seem to be applicable in the nonlinear case.

To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by Lasiecka et al. [**13**], [**18**], [**28**], [**29**] and [**30**], and used by Liu and Zuazua [**32**], Eller et al [**20**] and Alabau-Boussouira [**2**].

2.2 Preliminaries and main results

For the relaxation function, the damping and the delay functions, we make the following hypotheses:

(H1) $(*)$ $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^2 function satisfying

$$h(0) = h_0 > 0, \qquad l = \int_0^{+\infty} h(s)ds < 1.$$

$(**)$ There exists a non-increasing differentiable function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$h'(s) \leq -\zeta(s)h(s), \qquad \forall s \geq 0.$$

and

$$\int_0^{+\infty} \zeta(s)\,ds = +\infty.$$

(H2) $g_1 : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function of class $C(\mathbb{R})$ such that there exist $\epsilon', c_1, c_2, \alpha_1, \alpha_2 > 0$ and a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of the class $C^1(\mathbb{R}_+) \cap C^2(]0, \infty[)$ satisfying $H(0) = 0$, and H linear on $[0, \epsilon']$ or ($H'(0) = 0$ and $H'' > 0$ on $]0, \epsilon']$), such that

$$|g_1(s)| \leq c_2|s|, \quad \text{if } |s| \geq \epsilon'. \tag{2.1}$$

$$g_1^2(s) \leq H^{-1}(sg_1(s)), \quad \text{if } |s| \leq \epsilon'. \tag{2.2}$$

$g_2 : \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of the class $C^1(\mathbb{R})$ such that there exist $c_3, \alpha_1, \alpha_2 > 0$

$$|g_2'(s)| \leq c_3 \tag{2.3}$$

$$\alpha_1\; sg_2(s) \leq G_2(s) \leq \alpha_2\; sg_1(s), \tag{2.4}$$

where

$$G_2(s) = \int_0^s g_2(r)\,dr.$$

$$\alpha_2\mu_2 < \alpha_1\mu_1. \tag{2.5}$$

Remark 2.2.1 *1.By the mean value Theorem for integrals and the monotonicity of g_2, we find that*

$$G_2(s) = \int_0^s g_2(r)\,dr \leq sg_2(s).$$

Then, $\alpha_1 \leq 1$.

2. We need condition (2.3) only to prove global existence. For the energy decay, we can replace the linear growth of the function $g_2(s)$, for large $|s|$, by nonlinear polynomial growth.

We also state a Lemma which will be needed later.

Lemma 2.2.1 (Sobolev-Poincaré's inequality) *Let q be a number with $2 \leq q < +\infty$ ($n = 1, 2$) or $2 \leq q \leq 2n/(n-2)$ ($n \geq 3$). Then there is a constant $c_* = c_*(\Omega, q)$ such that*

$$\|u\|_q \leq c_*\|\nabla u\|_2 \quad \textit{for} \quad u \in H_0^1(\Omega).$$

We introduce as in **[40]** the new variable

$$z(x,\rho,t) = u_t(x, t - \tau\rho),\; x \in \Omega,\; \rho \in]0,1[, \quad t > 0. \tag{2.6}$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega\times]0,1[\times]0,+\infty[. \tag{2.7}$$

Therefore, problem (P) takes the form:

$$
(2.8)\quad \begin{cases}
u_{tt}(x,t) - \Delta_x u(x,t) + \int_0^t h(t-s)\Delta_x u(x,s)ds + \mu_1 g_1(u_t(x,t)) & \\
\qquad\qquad +\mu_2 g_2(z(x,1,t)) = 0, & \text{in } \Omega\times]0,+\infty[,\\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & \text{in } \Omega\times]0,1[\times]0,+\infty[,\\
u(x,t) = 0, & \text{on } \partial\Omega\times[0,+\infty[,\\
z(x,0,t) = u_t(x,t), & \text{on } \Omega\times]0,+\infty[,\\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega,\\
z(x,\rho,0) = f_0(x,-\rho\tau), & \text{in } \Omega\times]0,1[.
\end{cases}
$$

Let ξ be a positive constant such that

$$
(2.9)\qquad \tau\frac{\mu_2(1-\alpha_1)}{\alpha_1} < \xi < \tau\frac{\mu_1-\alpha_2\mu_2}{\alpha_2}.
$$

We define the energy associated to the solution of problem (2.8) by the following formula:

$$
(2.10)\qquad \begin{aligned}
E(t) = \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{2}\left(1-\int_0^t h(s)\,ds\right)\|\nabla_x u(t)\|_2^2 + \frac{1}{2}(h\circ\nabla u)(t)\\
+\xi\int_\Omega\int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx.
\end{aligned}
$$

We have the following theorem.

Theorem 2.2.1 *Let* $(u_0,u_1,f_0)\in H^2(\Omega)\cap H_0^1(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega;H^1(0,1))$ *satisfy the compatibility condition*

$$f_0(.,0) = u_1.$$

Assume that the hypotheses $(H1)-(H2)$ *hold. Then problem* (P) *admits a unique weak solution*

$$u\in L^\infty_{loc}((-\tau,+\infty);H^2(\Omega)\cap H_0^1(\Omega)), u'\in L^\infty_{loc}((-\tau,+\infty);H_0^1(\Omega)), u''\in L^\infty_{loc}((-\tau,+\infty);L^2(\Omega))$$

and, for some constants ω,ϵ_0, *we obtain the following decay property:*

$$
(2.11)\qquad E(t)\le H_1^{-1}\left(\omega\int_0^t\zeta(s)\,ds\right),\quad \forall t>0,
$$

where

$$
(2.12)\qquad H_1(t) = \int_t^1\frac{1}{H_2(s)}\,ds
$$

and

$$
H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0,\epsilon'],\\ tH'(\epsilon_0 t) & \text{if } H'(0)=0 \text{ and } H''>0 \text{ on }]0,\epsilon'].\end{cases}
$$

Example. Let g be given by $g(s) = s^p(-\ln s)^q$, where $0 \le p \le 1$ and $q \in \mathbb{R}$ on $(0, \epsilon_1]$. Then $g'(s) = s^{p-1}(-\ln s)^{q-1}(p(-\ln s) - q)$ which is an increasing function in a right neighborhood of 0 (if $q = 0$ we can take $\epsilon_1 = 1$). The function H is defined in the neighborhood of 0 by

$$H(s) = \begin{cases} cs^{\frac{p+1}{2p}}(-\ln s)^{-\frac{q}{p}} & \text{if } 0 < p < 1, \quad q \in \mathbb{R} \\ cs(-\ln s)^{-q} & \text{if } p = 1, \quad q > 0 \\ c\sqrt{s}\ e^{-s^{\frac{1}{2q}}} & \text{if } p = 0, \quad q < 0 \end{cases}$$

and we have

$$H'(s) = \begin{cases} cs^{\frac{1-p}{2p}}(-\ln s)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}(-\ln s) + \frac{q}{p}\right) & \text{if } 0 < p \le 1, \quad q \in \mathbb{R} \\ c\frac{1}{\sqrt{s}}\left(1 - \frac{1}{q}\ s^{\frac{1}{2q}}\right) e^{-s^{\frac{1}{2q}}} & \text{if } p = 0, \quad q < 0 \end{cases} \quad \text{when } s \text{ is near } 0.$$

Thus

$$\varphi(s) = \begin{cases} cs^{\frac{p+1}{2p}}(-\ln s)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}(-\ln s) + \frac{q}{p}\right) & \text{if } 0 < p \le 1, \quad q \in \mathbb{R} \\ c\sqrt{s}\left(1 - \frac{1}{q}\ s^{\frac{1}{2q}}\right) e^{-s^{\frac{1}{2q}}} & \text{if } p = 0, \quad q < 0 \end{cases} \quad \text{when } s \text{ is near } 0.$$

and

$$\begin{aligned} \psi(t) &= c\int_t^1 \frac{1}{s^{\frac{p+1}{2p}}(-\ln s)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}(-\ln s) + \frac{q}{p}\right)}\,ds \\ &= c\int_1^{\frac{1}{t}} \frac{z^{\frac{1-3p}{2p}}}{(\ln z)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}\ln z + \frac{q}{p}\right)}\,dz \text{ when } t \text{ is near } 0. \end{aligned}$$

and

$$\begin{aligned} \psi(t) &= c\int_t^1 \frac{1}{\sqrt{s}\left(1 - \frac{1}{q}\ s^{\frac{1}{2q}}\right) e^{-s^{\frac{1}{2q}}}}\,ds \\ &= c\int_1^{\frac{1}{t}} \frac{e^{(\frac{1}{z})^{\frac{1}{2q}}}}{z^{\frac{3}{2}}(1 - \frac{1}{q}\ (\frac{1}{z})^{\frac{1}{2q}})}\,dz, \quad p = 0, q < 0, \text{ when } t \text{ is near } 0 \end{aligned}$$

We obtain in a neighborhood of 0

$$\psi(t) \equiv \begin{cases} ct^{\frac{p-1}{2p}}(-\ln t)^{\frac{q}{p}} & \text{if } 0 < p < 1, \quad q \in \mathbb{R} \\ c(-\ln t)^{1+q} & \text{if } p = 1,\ q > 0, \\ ct^{\frac{q-2}{2q}}e^{t^{\frac{1}{2q}}} & \text{if } p = 0,\ q < 0 \end{cases}$$

and then in a neighborhood of $+\infty$ (see Appendix)

$$\psi^{-1}(t) \equiv \begin{cases} ct^{\frac{2p}{p-1}}(\ln t)^{-\frac{2q}{p-1}} & \text{if } 0 < p \le 1, \quad q \in \mathbb{R}, \\ ce^{-t^{\frac{1}{1+q}}} & \text{if } p = 1,\ q > 0, \\ c(\ln t)^{2q} & \text{if } p = 0,\ q < 0. \end{cases}$$

Using the fact that $h(t) = t$ as t goes to infinity, then

$$E(t) \leq \begin{cases} c\tilde{\xi}(t)^{-\frac{2}{p-1}}(\ln\tilde{\xi}(t))^{-\frac{2q}{p-1}} & \text{if } 0 < p \leq 1, \quad q \in \mathbb{R}, \\ ce^{-\tilde{\xi}(t)^{\frac{1}{1+q}}} & \text{if } p = 1,\ q < 1, \\ c(\ln\tilde{\xi}(t))^{2q} & \text{if } p = 0,\ q < 0, \\ ce^{-\tilde{\xi}(t)} & \text{if } p > 1 \text{ or } p = 1 \text{ and } q \leq 0. \end{cases}$$

We finish this section by giving an explicit upper bound for the derivative of the energy.

Lemma 2.2.2 *Let (u, z) be a solution to the problem (2.6). Then, the energy functional defined by (2.10) satisfies*

$$\begin{aligned} E'(t) &\leq -\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_\Omega u' g_1(u')\,dx \\ &- \left(\frac{\xi}{\tau}\alpha_1 - \mu_2(1-\alpha_1)\right)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx - \frac{1}{2}h(t)\|\nabla u\|_2^2 + \frac{1}{2}(h'\circ\nabla u)(t) \\ &\leq 0. \end{aligned} \tag{2.13}$$

Proof. Multiplying the first equation in (2.8) by $u_t(x,t)$, and integrating the result over Ω, to obtain:

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}(\|u_t(x,t)\|_2^2 + \|\nabla u(x,t)\|_2^2) + \mu_1\int_\Omega g_1(u_t(x,t))u_t(x,t)\,dx \\ &+\mu_2\int_\Omega g_2(z(x,1,t))u_t(x,t)\,dx = \int_\Omega\int_0^t h(t-s)\nabla u(x,s)\nabla u_t(x,t)\,ds\,dx. \end{aligned} \tag{2.14}$$

The term in the right-hand side of (2.14) can be rewritten as follows

$$\begin{aligned} &\int_\Omega\int_0^t h(t-s)\nabla u(x,s)\nabla u_t(x,t)\ ds\ dx + \frac{1}{2}h(t)\ \|\nabla u(x,t)\|_2^2 \\ &= \tfrac{1}{2}\tfrac{d}{dt}\left[\int_0^t h(s)\,ds\|\nabla u(x,t)\|_2^2 - (h\circ\nabla u)(t)\right] + \tfrac{1}{2}(h'\circ\nabla u)(t). \end{aligned}$$

Consequently, equality (2.14) becomes

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\left[\|u_t(x,t)\|_2^2 + \left(1 - \int_0^t h(s)\,ds\right)\|\nabla u(x,t)\|_2^2 + (h\circ\nabla u)(t)\right] = -\mu_1\int_\Omega g_1(u_t(x,t))u_t(x,t)\,dx \\ &-\mu_2\int_\Omega g_2(z(x,1,t))u_t(x,t)\,dx - \frac{1}{2}h(t)\|\nabla u(x,t)\|_2^2 + \frac{1}{2}(h'\circ\nabla u)(t). \end{aligned} \tag{2.15}$$

We multiply the second equation in (2.8) by $\xi g_2(z(x,\rho,t))$ and integrate over $\Omega\times]0,1[$ to obtain:

$$\begin{aligned} \xi\int_\Omega\int_0^1 z_t(x,\rho,t))g_2(z(x,\rho,t))\,d\rho\,dx &= -\frac{\xi}{\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}G_2(z(x,\rho,t))\,d\rho\,dx \\ &= -\frac{\xi}{\tau}\int_\Omega(G_2(z(x,1,t)) - G_2(z(x,0,t)))\,dx. \end{aligned} \tag{2.16}$$

Hence

$$\xi \frac{d}{dt}\int_\Omega \int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx + \frac{\xi}{\tau}\int_\Omega G_2(z(x,1,t))\,dx - \frac{\xi}{\tau}\int_\Omega G_2(u_t(x,t))\,dx = 0. \tag{2.17}$$

From (2.15), (2.17) and use of Young's inequality, we get

$$E'(t) = \frac{1}{2}(h' \circ \nabla u)(t) - \frac{1}{2}h(t)\|\nabla u(x,t)\|_2^2 - \mu_1 \int_\Omega g_1(u_t(x,t))u_t(x,t)\,dx$$
$$-\mu_2 \int_\Omega g_2(z(x,1,t))u_t(x,t)\,dx - \frac{\xi}{\tau}\int_\Omega G_2(z(x,1,t))\,dx + \frac{\xi}{\tau}\int_\Omega G_2(u_t(x,t))\,dx.$$

By recalling (2.4), we arrive at

$$\begin{aligned} E'(t) \le & -\left(\mu_1 - \frac{\xi\alpha_2}{\tau}\right)\int_\Omega g_1(u_t(x,t))u_t(x,t)\;dx - \frac{1}{2}(h' \circ \nabla u)(t) - \frac{1}{2}h(t)\;\|\nabla u(x,t)\|_2^2 \\ & -\mu_2 \int_\Omega g_2(z(x,1,t))u_t(x,t)\,dx - \frac{\xi}{\tau}\int_\Omega G_2(z(x,1,t))\,dx. \end{aligned} \tag{2.18}$$

Let us denote by G_2^* the conjugate function of the convex function G_2, i.e.,
$G_2^*(s) = \sup_{t\in\mathbb{R}^+}(st - G_2(t))$. Then G_2^* is the Legendre transform of G_2, which is given by

$$G_2^*(s) = s(G_2')^{-1}(s) - G_2[(G_2')^{-1}(s)], \quad \forall s \ge 0 \tag{2.19}$$

and satisfies the following inequality

$$st \le G_2^*(s) + G_2(t), \quad \forall s,t \ge 0. \tag{2.20}$$

(see Arnold [**4**], p. 61-62, and Lasiecka [**13**], [**18**], [**28**]-[**29**] for more information).

Then, from the definition of G_2, we get

$$G_2^*(s) = s g_2^{-1}(s) - G_2(g_2^{-1}(s)).$$

Hence

$$G_2^*(g_2(z(x,1,t))) = z(x,1,t)g_2(z(x,1,t)) - G_2(z(x,1,t)). \tag{2.21}$$

Making use of (2.18), (2.20) and (2.21), we arrive at

$$\begin{aligned} E'(t) \;\; \le \;\; & -\left(\mu_1 - \frac{\xi\alpha_2}{\tau}\right)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx - \frac{\xi\alpha_1}{\tau}\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \\ & +\mu_2 \int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \; + \mu_2 \int_\Omega G_2(u_t(x,t))dx - \mu_2 \int_\Omega G_2(z(x,1,t))dx \\ & +\frac{1}{2}(h' \circ \nabla u)(t) - \frac{1}{2}h(t)\|\nabla u(x,t)\|_2^2. \end{aligned} \tag{2.22}$$

Again, use of (2.4) yields

$$\begin{aligned} E'(t) \;\; \le \;\; & -\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx \\ & -\left(\frac{\xi\alpha_1}{\tau} - \mu_2 + \mu_2\alpha_1\right)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \\ & +\tfrac{1}{2}(h' \circ \nabla u)(t) - \tfrac{1}{2}h(t)\;\|\nabla u(x,t)\|_2^2 \le 0. \end{aligned} \tag{2.23}$$

2.3 Global Existence

We are now ready to prove Theorem 2.2.1 in the next two sections.

Throughout this section we assume $u_0 \in H^2(\Omega) \cap H_0^1(\Omega), u_1 \in H_0^1(\Omega)$ and $f_0 \in H_0^1(\Omega; H^1(0,1))$.

We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \dots, w_k\}$ where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2(\Omega) \cap H_0^1(\Omega)$.

Now, we define, for $1 \leq j \leq k$, the sequence $\phi_j(x, \rho)$ as follows:

$$\phi_j(x, 0) = w_j.$$

Then, we may extend $\phi_j(x, 0)$ by $\phi_j(x, \rho)$ over $L^2(\Omega \times (0,1))$ such that $(\phi_j)_j$ form a basis of $L^2(\Omega; H^1(0,1))$ and denote Z_k the space generated by $\{\phi_1, \phi_2, \dots, \phi_k\}$.

We construct approximate solutions $(u_k, z_k), k = 1, 2, 3, \dots$, in the form

$$u_k(t) = \sum_{j=1}^{k} g_{jk}(t) w_j,$$

$$z_k(t) = \sum_{j=1}^{k} h_{jk}(t) \phi_j,$$

where g_{ik} and $h_{ik}, j = 1, 2, \dots, k$, are determined by the following ordinary differential equations:

$$(2.24)\quad \begin{cases} (u_k''(t), w_j) + (\nabla_x u_k(t), \nabla_x w_j) - \displaystyle\int_0^t h(t-s)(\nabla_x u_k(s), \nabla_x w_j)\, ds + \mu_1(g_1(u_k'), w_j) \\ \qquad\qquad +\mu_1(g_2(z_k(.,1)), w_j) = 0, \quad 1 \leq j \leq k, \\ z_k(x, 0, t) = u_k'(x, t) \end{cases}$$

$$(2.25)\quad u_k(0) = u_{0k} = \sum_{j=1}^{k} (u_0, w_j) w_j \to u_0 \text{ in } H^2(\Omega) \cap H_0^1(\Omega) \text{ as } k \to +\infty,$$

$$(2.26)\quad u_k'(0) = u_{1k} = \sum_{j=1}^{k} (u_1, w_j) w_j \to u_1 \text{ in } H_0^1(\Omega) \text{ as } k \to +\infty$$

and

$$(2.27)\quad (\tau z_{kt} + z_{k\rho}, \phi_j) = 0, \quad 1 \leq j \leq k,$$

$$(2.28)\quad z_k(\rho, 0) = z_{0k} = \sum_{j=1}^{k} (f_0, \phi_j) \phi_j \to f_0 \text{ in } H_0^1(\Omega; H^1(0,1)) \text{ as } k \to +\infty.$$

By virtue of the theory of ordinary differential equations, the system (2.24)-(2.28) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \leq +\infty$) by Zorn lemma since the nonlinear terms in (2.24) are locally Lipschitz continuous. Note that $u_k(t)$ is C^2-class.

In the next step, we obtain a priori estimates for the solution, so that it can be extended outside $[0, T_k[$ to obtain one solution defined for all $t > 0$.

In order to use a standard compactness argument with the limiting procedure, it suffices to derive some a priori estimates for (u_k, z_k).

The first estimate. Since the sequences u_{0k}, u_{1k} and z_{0k} converge, then standard calculations, using (2.24)-(2.28), similar to those used to derive (2.13), yield

$$E_k(t) + a_1 \int_0^t u_k'(x,s) g_1(u_k'(x,s))\, ds + a_2 \int_0^t \sigma(s) z_k(x,1,s) g_2(z_k(x,1,s))\, ds \le E_k(0) \le C, \tag{2.29}$$

where

$$E_k(t) = \frac{1}{2}\|u_k'(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t h(s)\, ds\right) \|\nabla_x u_k(t)\|_2^2 + \xi \int_\Omega \int_0^1 G_2(z_k(x,\rho,t))\, d\rho\, dx,$$

$$a_1 = \mu_1 - \frac{\xi \alpha_2}{\tau} - \mu_2 \alpha_2 \text{ and } a_2 = \frac{\xi \alpha_1}{\tau} - \mu_2(1 - \alpha_1).$$

for some C independent of k. These estimates imply that the solution (u_k, z_k) exists globally in $[0, +\infty[$.

Estimate (2.29) yields

$$u_k \text{ is bounded in } L^\infty_{loc}(0,\infty; H_0^1(\Omega)), \tag{2.30}$$

$$u_k' \text{ is bounded in } L^\infty_{loc}(0,\infty; L^2(\Omega)), \tag{2.31}$$

$$u_k'(t) g_1(u_k'(t)) \text{ is bounded in } L^1(\Omega \times (0,T)), \tag{2.32}$$

$$G_2(z_k(x,\rho,t)) \text{ is bounded in } L^\infty_{loc}(0,\infty; L^1(\Omega \times (0,1))), \tag{2.33}$$

$$z_k(x,1,t) g_2(z_k(x,1,t)) \text{ is bounded in } L^1(\Omega \times (0,T)). \tag{2.34}$$

The second estimate. first, we estimate $u_k''(0)$. Testing (2.24) by $g_{jk}''(t)$ and choosing $t = 0$, we obtain:

$$\|u_k''(0)\|_2 \le \|\Delta_x u_{0k}\|_2 + \mu_1 \|g_1(u_{1k})\|_2 + \mu_2 \|g_2(z_{0k})\|_2.$$

Since $g_1(u_{1k}), g_2(z_{0k})$ are bounded in $L^2(\Omega)$ hence, from (2.25), (2.26) and (2.28),

$$\|u_k''(0)\|_2 \le C.$$

Differentiating (2.24) with respect to t, we get

$$\left(u_k'''(t) + \Delta_x u_k'(t) + \frac{d}{dt}\left(\int_0^t h(t-s) \Delta_x u_k(s)\, ds\right) + \mu_1 u_k''(t) g_1'(u_k') + \mu_2 z_k' g_2'(z_k), w_j\right) = 0.$$

Multiplying by $g''_{jk}(t)$ and summing over j from 1 to k, it follows that

$$\begin{aligned}(2.35)\quad &\frac{1}{2}\frac{d}{dt}\left(\|u''_k(t)\|_2^2+\|\nabla_x u'_k(t)\|_2^2\right)-h(0)\frac{d}{dt}(\nabla_x u_k(t),\nabla_x u'_k(t))+h(0)\|\nabla_x u'_k(t)\|_2^2\\ &-\frac{d}{dt}\int_0^t h'(t-s)(\nabla_x u_k(s),\nabla_x u'_k(t))\,ds+h'(0)(\nabla_x u_k(t),\nabla_x u'_k(t))\\ &+\int_0^t h''(t-s)(\nabla_x u_k(s),\nabla_x u'_k(t))\,ds+\mu_1\int_\Omega {u''}_k^2(t)g'_1(u'_k(t))\,dx\\ &\qquad+\mu_2\int_\Omega {u''}_k(t)z'_k(x,1,t)g'_2(z_k(x,1,t))\,dx=0.\end{aligned}$$

Differentiating (2.27) with respect to t, we get

$$(\tau z''_k(t)+\frac{\partial}{\partial\rho}z'_k,\phi_j)=0.$$

Multiplying by $h'_{jk}(t)$ and summing over j from 1 to k, it follows that

$$(2.36)\quad \frac{1}{2}\tau\frac{d}{dt}\|z'_k(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z'_k(t)\|_2^2=0.$$

Taking the sum of (2.35) and (2.36), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u''_k(t)\|_2^2+\|\nabla_x u'_k(t)\|_2^2+\tau\|z'_k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\right)+h(0)\|\nabla_x u'_k(t)\|_2^2\\ &+\mu_1\int_\Omega {u''}_k^2(t)g'_1(u'_k(t))\,dx+\frac{1}{2}\int_\Omega|z'_k(x,1,t)|^2\,dx=h(0)\frac{d}{dt}(\nabla_x u_k(t),\nabla_x u'_k(t))\\ &+\frac{d}{dt}\int_0^t h'(t-s)(\nabla_x u_k(s),\nabla_x u'_k(t))\,ds-h'(0)(\nabla_x u_k(t),\nabla_x u'_k(t))\\ &-\int_0^t h''(t-s)(\nabla_x u_k(s),\nabla_x u'_k(t))\,ds-\mu_2\int_\Omega {u''}_k(t)z'_k(x,1,t)g'_2(z_k(x,1,t))\,dx+\frac{1}{2}\|u''_k(t)\|_2^2.\end{aligned}$$

Using (2.3), Cauchy-Schwarz and Young's inequalities, we obtain

$$|h'(0)(\nabla_x u_k(t),\nabla_x u'_k(t))|\le\varepsilon\|\nabla_x u_k(t)\|_2^2+\frac{h'(0)^2}{4\varepsilon}\|\nabla_x u'_k(t)\|_2^2,$$

$$\begin{aligned}\left|\int_0^t h''(t-s)(\nabla_x u_k(s),\nabla_x u'_k(t))\,ds\right| &\le \|\nabla_x u'_k(t)\|_2\int_0^t|h''(t-s)|\|\nabla_x u_k(s)\|_2\,ds\\ &\le \frac{1}{4\varepsilon}\|\nabla_x u'_k(t)\|_2^2+\varepsilon\|h''\|_{L^1}\int_0^t|h''(t-s)|\|\nabla_x u_k(s)\|_2^2\,ds,\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u''_k(t)\|_2^2+\|\nabla_x u'_k(t)\|_2^2+\tau\|z'_k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\right)+\mu_1\int_\Omega {u''}_k^2(t)g'_1(u'_k(t))\,dx\\ &+c\int_\Omega|z'_k(x,1,t)|^2\,dx+h(0)\|\nabla_x u'_k(t)\|_2^2\\ &\le c'\|u''_k(t)\|_2^2+\varepsilon\|\nabla_x u_k(t)\|_2^2+\frac{h'(0)^2}{4\varepsilon}\|\nabla_x u'_k(t)\|_2^2+\frac{1}{4\varepsilon}\|\nabla_x u'_k(t)\|_2^2\\ &+\varepsilon\|h''\|_{L^1}\int_0^t|h''(t-s)|\|\nabla_x u_k(s)\|_2^2\,ds+h(0)\frac{d}{dt}(\nabla_x u_k(t),\nabla_x u'_k(t))\\ &\qquad+\frac{d}{dt}\int_0^t h'(t-s)(\nabla_x u_k(s),\nabla_x u'_k(t))\,ds.\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's lemma, we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\|u_k''(t)\|_2^2+\|\nabla_x u_k'(t)\|_2^2+\tau\|z_k'(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\right)+\mu_1\int_0^t\int_\Omega {u''}_k^2(t)g_1'(u_k'(s))\,dx\,ds\\
&+c\int_0^t\int_\Omega |z_k'(x,1,s)|^2\,dx\,ds\le\\
&\frac{1}{2}\left(\|u_k''(0)\|_2^2+\|\nabla_x u_k'(0)\|_2^2+\tau\|z_k'(x,\rho,0)\|^2_{L^2(\Omega\times(0,1))}\right)+h(0)(\nabla_x u_k(t),\nabla_x u_k'(t))\\
&-h(0)(\nabla_x u_k(0),\nabla_x u_k'(0))+\int_0^t h'(t-s)(\nabla_x u_k(s),\nabla_x u_k'(t))\,ds\\
&+\left(\frac{1}{4\varepsilon}+\frac{h'(0)^2}{4\varepsilon}-h(0)\right)\int_0^t\|\nabla_x u_k'(s)\|_2^2\,ds+\left(\varepsilon+\varepsilon\|h''\|^2_{L^1}\right)\int_0^t\|\nabla_x u_k(s)\|_2^2\,ds\\
&+c'\int_0^t\|u_k''(s)\|_2^2\,ds,
\end{aligned}\tag{2.37}$$

$$h(0)(\nabla_x u_k(t),\nabla_x u_k'(t))\le\varepsilon\|\nabla_x u_k'(t)\|_2^2+\frac{h(0)^2}{4\varepsilon}\|\nabla_x u_k(t)\|_2^2,$$

$$\int_0^t h'(t-s)(\nabla_x u_k(s),\nabla_x u_k'(t))\,ds\le\varepsilon\|\nabla_x u_k'(t)\|_2^2+\frac{\xi(0)\|h\|_{L^1}\|h\|_{L^\infty}}{4\varepsilon}\int_0^t\|\nabla_x u_k(s)\|_2^2\,ds.$$

Then from (2.37), choosing ε small enough and using Gronwall's lemma, we obtain

$$\begin{aligned}
\|u_k''(t)\|_2^2+\|\nabla_x u_k'(t)\|_2^2+\tau\|z_k'(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}+\mu_1\int_0^t\int_\Omega {u''}_k^2(t)g_1'(u_k'(s))\,dx\,ds\\
+c\int_0^t\int_\Omega|z_k'(x,1,s)|^2\,dx\,ds\le M,
\end{aligned}\tag{2.38}$$

for all $t\in[0,T]$ and M is a positive constant independent of $k\in\mathbb{N}$. Therefore, we conclude that

$$u_k'' \text{ is bounded in } L^\infty_{loc}(0,+\infty;L^2(\Omega)),\tag{2.39}$$

$$u_k' \text{ is bounded in } L^\infty_{loc}(0,+\infty;H_0^1(\Omega)),\tag{2.40}$$

$$z_k' \text{ is bounded in } L^\infty_{loc}(0,+\infty;L^2(\Omega\times(0,1))).\tag{2.41}$$

The third estimate. Replacing w_j by $-\Delta_x w_j$ in (2.24), multiplying by $g_{jm}'(t)$, summing over j from 1 to k, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla_x u_k'(t)\|_2^2+\|\Delta_x u_k(t)\|_2^2\right)-\int_0^t h(t-s)(\Delta_x u(s),\Delta_x u'(t))\,ds\\
&+\mu_1\int_\Omega|\nabla_x u_k'(t)|^2 g_1'(u_k'(t))\,dx+\mu_2\int_\Omega\nabla_x {u'}_k(t)\nabla_x z_k'(x,1,t)g_2'(z_k(x,1,t))\,dx=0,
\end{aligned}\tag{2.42}$$

$$\begin{aligned}
&\int_0^t h(t-s)(\Delta_x u(s),\Delta_x u'(t))\,ds+\frac{1}{2}h(t)\|\Delta_x u(t)\|_2^2\\
&=\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(s)\,ds\|\Delta_x u(t)\|_2^2-(h\circ\Delta_x u)(t)\right]+\frac{1}{2}(h'\circ\Delta_x u)(t).
\end{aligned}$$

Consequently, equality (2.42) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla_x u_k'(t)\|_2^2+\left(1-\int_0^t h(s)\,ds\right)\|\Delta_x u_k(t)\|_2^2+(h\circ\Delta_x u)(t)\right)+h(t)\|\nabla_x u_k(t)\|_2^2\\
&-\frac{1}{2}(h'\circ\Delta_x u)(t)+\mu_1\int_\Omega|\nabla_x u_k'(t)|^2 g_1'(u_k'(t))\,dx\\
&+\mu_2\int_\Omega\nabla_x {u'}_k(t)\nabla_x z_k'(x,1,t)g_2'(z_k(x,1,t))\,dx=0.
\end{aligned}\tag{2.43}$$

Replacing ϕ_j by $-\Delta_x\phi_j$ in (2.27), multiplying by $h_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\tau\frac{d}{dt}\|\nabla_x z_k(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|\nabla_x z_k(t)\|_2^2=0. \tag{2.44}$$

From (2.42) and (2.44), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla_x u_k'(t)\|_2^2+\left(1-\int_0^t h(s)\,ds\right)\|\Delta_x u_k(t)\|_2^2+\tau\|\nabla_x z_k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\right)\\
&+h(t)\|\nabla_x u_k(t)\|_2^2-\frac{1}{2}(h'\circ\Delta_x u)(t)+\mu_1\int_\Omega|\nabla_x u_k'(t)|^2 g_1'(u_k'(t))\,dx+\frac{1}{2}\int_\Omega|\nabla_x z_k(x,1,t)|^2\,dx\\
&=-\mu_2\int_\Omega\nabla_x u'_k(t)\nabla_x z_k'(x,1,t)g_2'(z_k(x,1,t))\,dx+\frac{1}{2}\|\nabla_x u_k'(t)\|_2^2.
\end{aligned}$$

Using (2.3), Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla_x u_k'(t)\|_2^2+\left(1-\int_0^t h(s)\,ds\right)\|\Delta_x u_k(t)\|_2^2+\tau\|\nabla_x z_k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\right)\\
&+\mu_1\int_\Omega|\nabla_x u_k'(t)|^2 g_1'(u_k'(t))\,dx+c\int_\Omega|\nabla_x z_k(x,1,t)|^2\,dx\le c'\|\nabla_x u_k'(t)\|_2^2.
\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's lemma, we obtain

$$\begin{aligned}
&\|\nabla_x u_k'(t)\|_2^2+\left(1-\int_0^t h(s)\,ds\right)\|\Delta_x u_k(t)\|_2^2+\tau\|\nabla_x z_k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\le\\
&\qquad e^{cT}\left(\|\nabla_x u_k'(0)\|_2^2+\|\Delta_x u_k'(0)\|_2^2+\tau\|\nabla_x z_k(x,\rho,0)\|^2_{L^2(\Omega\times(0,1))}\right)
\end{aligned}$$

for all $t\in\mathbb{R}_+$, therefore, we conclude that

$$u_k \text{ is bounded in } L^\infty_{loc}(0,+\infty;H^2(\Omega)\cap H_0^1(\Omega)), \tag{2.45}$$

$$z_k \text{ is bounded in } L^\infty_{loc}(0,+\infty;H_0^1(\Omega;L^2(0,1))). \tag{2.46}$$

Applying Dunford-Petti's theorem, we conclude from (2.30),(2.31), (2.32), (2.33), (2.39), (2.40), (2.41) (2.45) and (2.46), replacing the sequences u_k and z_k with subsequence if necessary, that

$$u_k\to u \quad\text{weak-star in}\quad L^\infty_{loc}(0,+\infty;H^2(\Omega)\cap H_0^1(\Omega)), \tag{2.47}$$

$$u_k'\to u' \quad\text{weak-star in}\quad L^\infty_{loc}(0,+\infty;H_0^1(\Omega)),$$

$$u''_k\to u'' \quad\text{weak-star in}\quad L^\infty_{loc}(0,+\infty;L^2(\Omega)), \tag{2.48}$$

$$g_1(u_k')\to\chi \quad\text{weak in}\quad L^2(\Omega\times(0,T)),$$

$$z_k\to z \quad\text{weak-star in}\quad L^\infty_{loc}(0,+\infty;H_0^1(\Omega;L^2(0,1)),$$

$$z_k'\to z' \quad\text{weak-star in}\quad L^\infty_{loc}(0,+\infty;L^2(\Omega\times(0,1))), \tag{2.49}$$

$$g_2(z_k(x,1,t))\to\psi \quad\text{weak in}\quad L^2(\Omega\times(0,T))$$

for suitable functions $u\in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), z\in L^\infty(0,T;L^2(\Omega\times(0,1)))$, $\chi\in L^2(\Omega\times(0,T)),\psi\in L^2(\Omega\times(0,T))$ for all $T\ge 0$. We have to show that (u,z) is a solution of (2.8).

From (2.30) and (2.31) we have (u'_k) is bounded in $L^\infty(0,T;H^1_0(\Omega))$. Then (u'_k) is bounded in $L^2(0,T;H^1_0(\Omega))$. Since (u''_k) is bounded in $L^\infty(0,T;L^2(\Omega))$, then (u''_k) is bounded in $L^2(0,T;L^2(\Omega))$. Consequently, (u'_k) is bounded in $H^1(Q)$.
Since the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact, using Aubin-Lions' theorem [**31**], we can extract a subsequence (u_ν) of (u_k) such that

$$u'_\nu \to u' \text{ strongly in } \; L^2(Q).$$

Therefore

$$u'_\nu \to u' \text{ a.e in } Q. \tag{2.50}$$

Similarly we obtain

$$z_\nu \to z \text{ a.e in } Q. \tag{2.51}$$

Lemma 2.3.1 *For each $T > 0$, $g(u'), g(z(x,1,t)) \in L^1(Q)$ and $\|g(u')\|_{L^1(Q)}, \|g(z(x,1,t))\|_{L^1(Q)} \leq K_1$, where K_1 is a constant independent of t.*

Proof: By (**H2**) and (2.50) we have

$$g_1(u'_k(x,t)) \to g_1(u'(x,t)) \text{ a.e. in } Q,$$

$$0 \leq g_1(u'_k(x,t))u'_k(x,t) \to g_1(u'(x,t))u'(x,t) \text{ a.e. in } Q$$

Hence, by (2.32) and Fatou's lemma we have

$$\int_0^T \int_\Omega u'(x,t) g_1(u'(x,t))\, dx\, dt \leq K \text{ for } T > 0. \tag{2.52}$$

By Cauchy-Schwarz inequality, using (2.52), we have

$$\begin{aligned} \int_0^T \int_\Omega |g_1(u'(x,t))|\, dx\, dt &\leq c|Q|^{\frac{1}{2}} \left(\int_0^T \int_\Omega u' g_1(u')\, dx\, dt \right)^{\frac{1}{2}} \\ &\leq c|Q|^{\frac{1}{2}} K^{\frac{1}{2}} \equiv K_1 \end{aligned}$$

Lemma 2.3.2 *$g(u'_k) \to g(u')$ in $L^1(\Omega \times (0,T))$ and $g(z_k) \to g(z)$ in $L^1(\Omega \times (0,T))$.*

Proof: Let $E \subset \Omega \times [0,T]$ and set

$$E_1 = \left\{ (x,t) \in E;\ |g_1(u'_k(x,t))| \leq \frac{1}{\sqrt{|E|}} \right\}, \quad E_2 = E \setminus E_1,$$

where $|E|$ is the measure of E. If $M(r) := \inf\{|s|;\ s \in \mathbb{R} \text{ and } |g_1(s)| \geq r\}$,

$$\int_E |g_1(u'_k)|\, dxdt \leq c\sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}} \right) \right)^{-1} \int_{E_2} |u'_k g_1(u'_k)|\, dxdt.$$

Applying (2.32) we deduce that $\sup_k \int_E |g_1(u_k')|\,dxdt \to 0$ as $|E| \to 0$. From Vitali's convergence theorem we deduce that $g_1(u_k') \to g_1(u')$ in $L^1(\Omega \times (0,T))$, hence

$$g_1(u_k') \to g_1(u') \text{ weak in } L^2(Q).$$

Similarly, we have

$$g_2(z_k') \to g_2(z') \text{ weak in } L^2(Q),$$

and this implies that

$$\int_0^T \int_\Omega g_1(u_k')v\,dx\,dt \to \int_0^T \int_\Omega g_1(u')v\,dx\,dt, \text{ for all } v \in L^2(0,T;H_0^1) \tag{2.53}$$

$$\int_0^T \int_\Omega g_2(z_k)v\,dx\,dt \to \int_0^T \int_\Omega g_2(z)v\,dx\,dt, \text{ for all } v \in L^2(0,T;H_0^1) \tag{2.54}$$

as $k \to +\infty$. It follows at once from (2.47), (2.48), (2.53), (2.54) and (2.49) that for each fixed $v \in L^2(0,T;H_0^1)$ and $w \in L^2(0,T;H_0^1(\Omega \times (0,1)))$

$$\begin{aligned}
&\int_0^T \int_\Omega (u_k'' - \Delta_x u_k + + \int_0^t h(t-s)\Delta_x u_k(x,s)ds + \mu_1 g_1(u_k') + \mu_2 g_2(z_k))v\,dx\,dt \\
&\to \int_0^T \int_\Omega (u'' - \Delta_x u + \int_0^t h(t-s)\Delta_x u(x,s)ds + \mu_1 g_1(u') + \mu_2 g_2(z))v\,dx\,dt
\end{aligned}$$

$$\int_0^T \int_0^1 \int_\Omega (\tau z_k' + \frac{\partial}{\partial \rho} z_k)w\,dx\,d\rho\,dt \to \int_0^T \int_0^1 \int_\Omega (\tau z' + \frac{\partial}{\partial \rho} z)w\,dx\,d\rho\,dt$$

as $k \to +\infty$. Hence

$$\int_0^T \int_\Omega (u'' + \Delta_x u + \int_0^t h(t-s)\Delta_x u_k(x,s)ds + \mu_1 g_1(u') + \mu_2 g_2(z))v\,dx\,dt = 0, \qquad \forall v \in L^2(0,T;H_0^1).$$

$$\int_0^T \int_0^1 \int_\Omega (\tau u' + \frac{\partial}{\partial \rho} z)w\,dx\,d\rho\,dt = 0, \qquad \forall w \in L^2(0,T;H_0^1(\Omega \times (0,1))).$$

Thus the problem (P) admits a global weak solution u.

2.4 Asymptotic behavior

For $M > 0$ and $\varepsilon_1, \varepsilon_2 > 0$, we define the perturbed modified energy by

$$L(t) = ME(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 I(t) + \chi(t), \tag{2.55}$$

where

$$\Psi(t) = \int_\Omega u_t(x,t)\,u(x,t)\;dx, \tag{2.56}$$

$$I(t) = \int_\Omega \int_0^1 e^{-2\tau\rho} G_2(z(x,\rho,t))\,d\rho\;dx, \tag{2.57}$$

$$\chi(t) = -\int_\Omega u_t(x,t) \int_0^t h(t-s)(u(t) - u(s))\,ds\,dx. \tag{2.58}$$

Lemma 2.4.1 *There exist two positive constants B_1 and B_2 depending on $\varepsilon_1, \varepsilon_2$ and M such that for all $t > 0$*

$$B_1 E(t) \leq L(t) \leq B_2 E(t).$$

Proof: We consider the functional

$$K(t) = \varepsilon_1 \Psi(t) + \varepsilon_2 I(t) + \chi(t)$$

and show that

$$|K(t)| \leq CE(t), \quad C > 0. \tag{2.59}$$

Using Young's inequality and Poincaré's inequality, we obtain

$$\begin{aligned} |\chi(t)| &= \left| \int_\Omega u_t(x,t) \int_0^t h(t-s)(u(t)-u(s))\, ds\, dx \right| \\ &\leq \frac{1}{2}\int_\Omega u_t^2\, dx + \frac{1}{2}\int_\Omega \left(\int_0^t h(t-s)(u(t)-u(s))\, ds \right)^2 dx \\ &\leq \frac{1}{2}\int_\Omega u_t^2\, dx + \frac{1}{2} l\ c_*^2 (h \circ \nabla u)(t). \end{aligned} \tag{2.60}$$

Similarly, we have

$$\begin{aligned} |\varepsilon_1 \Psi(t) + \varepsilon_2 I(t)| &\leq \varepsilon_1 \int_\Omega |u_t||u|\, dx + \varepsilon_2 \int_\Omega \int_0^1 e^{-2\tau\rho} G_2(z(x,\rho,t))\, d\rho\ dx \\ &\leq \frac{\varepsilon_1}{2}\int_\Omega u_t^2\, dx + \frac{\varepsilon_1}{2} c_*^2 \int_\Omega |\nabla u|^2\, dx + \varepsilon_2 \int_\Omega \int_0^1 G_2(z(x,\rho,t))\, d\rho\ dx. \end{aligned} \tag{2.61}$$

Using $1 - \int_0^t h(s)\, ds \geq 1 - l$, (2.10), (2.60) and (2.61), we get (2.59) for some positive constant C. By choosing M large enough, our result follows from (2.55), (2.59).

Proposition 2.4.1 *For each $t_0 > 0$ and sufficiently large $M > 0$ and appropriately small $\varepsilon_1, \varepsilon_2 > 0$, there exist positive constants $C_3, C_4,$ and C_5 such that*

$$\frac{d}{dt} L(t) \leq -C_3 E(t) + C_4 (h \circ \nabla u)(t) + C_5 \|g_1(u_t)\|_2^2 \quad \forall t \geq t_0.$$

The proof of this proposition will be carried out through three lemmas.

Lemma 2.4.2 *Let (u, z) be the solution of (2.8), then for any $\gamma > 0$, we have*

$$\begin{aligned} \Psi'(t) \leq\ & \|u_t\|_2^2 - (1 - l - \gamma - \gamma c_*^2((\mu_1 + \mu_2)) \|\nabla u\|_2^2 \\ & + \tfrac{\mu_1}{4\gamma} \int_\Omega |g_1(u_t(x,t))|^2\, dx + \frac{\mu_2}{4\gamma} \int_\Omega |g_2(z(x,1,t))|^2\, dx + \frac{l}{4\gamma} (h \circ \nabla u)(t), \end{aligned} \tag{2.62}$$

Proof. Using the first equation in (2.8), a direct computation leads to

$$
\begin{aligned}
\Psi'(t) &= \int_\Omega u_t^2(x,t)\,dx + \int_\Omega u_{tt}(x,t)u(x,t)\,dx \\
&= \|u_t\|_2^2 + \int_\Omega \Big[\Delta u(x,t) - \int_0^t h(t-s)\Delta u(x,s)ds \\
&\quad -\mu_1 g_1(u_t(x,t)) - \mu_2 g_2(u_t(x,t-\tau))\Big]\, u(x,t)\,dx \\
&= \|u_t\|_2^2 - \|\nabla u\|_2^2 + \int_\Omega \nabla u(x,t)\int_0^t h(t-s)\nabla u(x,s)\,ds\,dx \\
&\quad -\mu_1\int_\Omega g_1(u_t(x,t))u(x,t)\,dx - \mu_2\int_\Omega g_2(z(x,1,t))u(x,t)\,dx.
\end{aligned}
\tag{2.63}
$$

Now, the third term in the right-hand side of (2.63) can be estimated as follows:

$$
\begin{aligned}
&\int_\Omega \nabla u(x,t)\int_0^t h(t-s)\nabla u(x,s)\,ds\,dx \\
= {}& \int_\Omega\int_0^t h(t-s)\Big[\nabla u(x,s)-\nabla u(x,t)\Big]\nabla u(x,t)\,ds\,dx + \int_\Omega\int_0^t h(t-s)|\nabla u(x,t)|^2\,ds\,dx \\
\le {}& l\|\nabla u(x,t)\|_2^2 + \int_\Omega |\nabla u(x,t)|\int_0^t h(t-s)|\nabla u(x,s)-\nabla u(x,t)|\,ds\,dx \\
\le {}& l\|\nabla u(x,t)\|_2^2 + \|\nabla u(x,t)\|_2\Big(\int_\Omega\Big(\int_0^t h(t-s)|\nabla u(x,s)-\nabla u(x,t)|ds\Big)^2dx\Big)^{\frac12} \\
\le {}& l\|\nabla u(x,t)\|_2^2 \\
& +\|\nabla u(x,t)\|_2\Big[\int_\Omega\Big(\int_0^t h(t-s)\,ds\int_0^t h(t-s)|\nabla u(x,s)-\nabla u(x,t)|^2ds\Big)\,dx\Big]^{\frac12} \\
\le {}& l\|\nabla u(x,t)\|_2^2 + l^{\frac12}\|\nabla u(x,t)\|_2\Big[\int_\Omega\Big(\int_0^t h(t-s)|\nabla u(x,s)-\nabla u(x,t)|^2ds\Big)\,dx\Big]^{\frac12} \\
\le {}& l\|\nabla u(x,t)\|_2^2 + l^{\frac12}\|\nabla u(x,t)\|_2(h\circ\nabla u)^{\frac12}(t) \\
\le {}& l\|\nabla u(x,t)\|_2^2 + \gamma\|\nabla u(x,t)\|_2^2 + \tfrac{l}{4\gamma}(h\circ\nabla u)(t) \\
\le {}& (l+\gamma)\|\nabla u(x,t)\|_2^2 + \tfrac{l}{4\gamma}(h\circ\nabla u)(t),
\end{aligned}
$$

then we conclude

$$
\begin{aligned}
\Psi'(t) \le {}& \|u_t\|_2^2 - \|\nabla u\|_2^2 + (l+\gamma)\|\nabla u\|_2^2 + \tfrac{l}{4\gamma}(h\circ\nabla u)(t) \\
& +\mu_1\int_\Omega |g_1(u_t(x,t))||u(x,t)|\,dx + \mu_2\int_\Omega |g_2(z(x,1,t))||u(x,t)|\,dx.
\end{aligned}
$$

Since

$$
\int_\Omega |g_1(u_t(x,t))||u(x,t)|\,dx \le \gamma c_*^2\|\nabla u\|_2^2 + \frac{1}{4\gamma}\int_\Omega |g_1(u_t(x,t))|^2\,dx
$$

$$
\int_\Omega |g_2(z(x,1,t))||u(x,t)|\,dx \le \gamma c_*^2\|\nabla u\|_2^2 + \frac{1}{4\gamma}\int_\Omega |g_2(z(x,1,t))|^2\,dx
$$

we obtain

$$
\begin{aligned}
\Psi'(t) \le {}& \|u_t\|_2^2 - (1-l-\gamma-\gamma c_*^2(\mu_1+\mu_2))\|\nabla u\|_2^2 \\
& +\frac{\mu_1}{4\gamma}\int_\Omega |g_1(u_t(x,t))|^2\,dx + \frac{\mu_2}{4\gamma}\int_\Omega |g_2(z(x,1,t))|^2\,dx + + \frac{l}{4\gamma}(h\circ\nabla u)(t).
\end{aligned}
$$

Lemma 2.4.3 *Let (u,z) be the solution of (2.8), then we have*

$$\frac{d}{dt}I(t) \leq -2I(t) - \frac{e^{-2\tau}}{\tau}\int_\Omega G_2(z(x,1,t))dx + \frac{1}{\tau}\int_\Omega G_2(u_t(x,t))\,dx. \tag{2.64}$$

Proof. Differentiating (2.57) and using the second equation in (2.8), we have

$$\begin{aligned}
\tfrac{d}{dt}I(t) &= \int_\Omega\int_0^1 e^{-2\tau\rho}z_t(x,\rho,t)g_2(z(x,\rho,t))\,d\rho\,dx\\
&= -\tfrac{1}{\tau}\int_\Omega\int_0^1 e^{-2\tau\rho}z_\rho(x,\rho,t)g_2(z(x,\rho,t))\,d\rho\,dx\\
&= -\tfrac{1}{\tau}\int_\Omega\int_0^1 e^{-2\tau\rho}\frac{d}{d\rho}G_2(z(x,\rho,t))\,d\rho\,dx\\
&= -\tfrac{1}{\tau}\int_\Omega\int_0^1 \left[\frac{d}{d\rho}\Big(e^{-2\tau\rho}G_2(z(x,\rho,t))\Big) + 2\tau e^{-2\tau\rho}G_2(z(x,\rho,t))\right]d\rho\,dx\\
&= -\tfrac{1}{\tau}\int_\Omega\left[e^{-2\tau}G_2(z(x,1,t)) - G_2(u_t(x,t))\right]dx\\
&\quad -2\int_\Omega\int_0^1 e^{-2\tau\rho}G_2(z(x,\rho,t))\,d\rho\,dx\\
&\leq -2\int_\Omega\int_0^1 e^{-2\tau\rho}G_2(z(x,\rho,t))\,d\rho\,dx - \frac{1}{\tau}\int_\Omega e^{-2\tau}G_2(z(x,1,t))dx\\
&\quad +\tfrac{1}{\tau}\int_\Omega G_2(u_t(x,t))dx\,dx\\
&\leq -2I(t) - \tfrac{e^{-2\tau}}{\tau}\int_\Omega G_2(z(x,1,t))dx + \frac{1}{\tau}\int_\Omega G_2(u_t(x,t))\,dx.
\end{aligned}$$

Lemma 2.4.4 *Let (u,z) be the solution of (2.8), then we have the estimate*

$$\begin{aligned}
&\frac{d}{dt}\chi(t) \leq \eta(1+l)\|\nabla u\|_2^2 - \left(\left(\int_0^t h(s)\,ds\right) - \eta\right)\|u_t\|_2^2\\
&+\left(l + \frac{l}{4\eta} + \frac{lc^*}{4\eta}(\mu_1+\mu_2) + \frac{l^2}{4\eta}\right)(h\circ\nabla u)(t) - \frac{h_0c_*}{4\eta}(h'\circ\nabla u)(t)\\
&+\eta\mu_1\|g_1(u_t)\|_2^2 + \eta\mu_2\|g_2(z(x,1,t))\|_2^2,
\end{aligned} \tag{2.65}$$

for any η a positive constant.

Proof. A differentiation of (2.58) leads to

$$\chi(t) = -\int_\Omega u_t(x,t)\int_0^t h(t-s)\Big(u(t)-u(s)\Big)ds\,dx,$$

we have

$$
\begin{aligned}
\chi'(t) &= -\int_\Omega u_{tt}(x,t)\int_0^t h(t-s)(u(t)-u(s))ds\ dx \\
&\quad -\int_\Omega u_t(x,t)\Big[u_t(x,t)\int_0^t h(t-s)\,ds\ +\ \int_0^t h'(t-s)(u(t)-u(s))ds\Big]dx, \\
&= -\int_\Omega \Big[\Big(\Delta_x u(x,t)-\int_0^t h(t-s)\Delta_x u(x,s)ds-\mu_1\, g_1(u_t(x,t)) \\
&\quad -\mu_2\, g_2(z(x,1,t))\Big)\int_0^t h(t-s)(u(t)-u(s))ds\Big]dx \\
&\quad -\int_0^t h(s)\,ds\,\|u_t\|_2^2 - \int_\Omega u_t(x,t)\int_0^t h'(t-s)(u(t)-u(s))ds\ dx, \\
&= \int_\Omega \nabla u(x,t)\int_0^t h(t-s)(\nabla u(x,t)-\nabla u(x,s))ds\,dx \\
&\quad -\int_\Omega \Big[\int_0^t h(t-s)\nabla u(x,s)\,ds\Big]\Big[\int_0^t h(t-s)\,(\nabla u(x,t)-\nabla u(x,s))\,ds\Big]dx \\
&\quad +\mu_1\int_\Omega g_1(u_t(x,t))\int_0^t h(t-s)(u(x,t)-u(x,s))ds\,dx \\
&\quad +\mu_2\int_\Omega g_1(z(x,1,t))\int_0^t h(t-s)(u(x,t)-u(x,s))ds\,dx \\
&\quad -\Big(\int_0^t h(s)\,ds\Big)\|u_t\|_2^2 - \int_\Omega u_t(x,t)\int_0^t h'(t-s)(u(t)-u(s))ds\ dx.
\end{aligned}
\tag{2.66}
$$

Using Young's inequality and the embedding $H_0^1(\Omega)\to L^2(\Omega)$, we infer

$$
\begin{aligned}
&\int_\Omega u_t(x,t)\int_0^t h'(t-s)\Big(u(t)-u(s)\Big)ds\ dx \\
\leq\ & \eta\|u_t\|_2^2+\frac{1}{4\eta}\Big(\int_0^t -h'(t-\tau)\|u(t)-u(\tau)\|_2\,d\tau\Big)^2 \\
\leq\ & \eta\|u_t\|_2^2+\frac{1}{4\eta}\Big(\int_0^t h'(\tau)\,d\tau\Big)\int_0^t h'(t-\tau)\|u(t)-u(\tau)\|_2^2\,d\tau \\
\leq\ & \eta\|u_t\|_2^2-\frac{h_0c_*^2}{4\eta}(h'\circ\nabla u)(t),
\end{aligned}
$$

$$
\Big|\int_\Omega \nabla u(x,t)\int_0^t h(t-s)\Big(\nabla u(x,t)-\nabla u(x,s)\Big)ds\,dx\Big| \leq \eta\|\nabla u\|_2^2\ +\ \frac{l}{4\eta}(h\circ\nabla u)(t),
$$

$$
\begin{aligned}
&\mu_1\int_\Omega g_1(u_t(x,t))\int_0^t h(t-s)(u(x,t)-u(x,s))ds\,dx \\
&\qquad \leq \eta\mu_1\|g_1(u_t(x,t))\|_2^2\ +\ \tfrac{l\mu_1c_*^2}{4\eta}(h\circ\nabla u)(t),
\end{aligned}
$$

$$
\begin{aligned}
&\mu_2\int_\Omega g_2(z(x,1,t))\int_0^t h(t-s)(u(x,t)-u(x,s))ds\,dx \\
&\qquad \leq \eta\mu_2\|g_2(z(x,1,t))\|_2^2\ +\ \tfrac{l\mu_2c_*^2}{4\eta}(h\circ\nabla u)(t)
\end{aligned}
$$

and

$$
\begin{aligned}
& \left|\int_\Omega \left(\int_0^t h(t-s)\nabla u(x,s)\,ds\right)\left(\int_0^t h(t-s)(\nabla u(x,t)-\nabla u(x,s))ds\right)dx\right| \\
= & \left|\int_\Omega \left[\int_0^t h(t-s)\Big(\nabla u(x,t)-\nabla u(x,s)\Big)\,ds - \int_0^t h(t-s)\nabla u(x,t)\,ds\right]\right. \\
& \left.\times\left[\int_0^t h(t-s)\Big(\nabla u(x,t)-\nabla u(x,s)\Big)ds\right]dx\right| \\
\le & \int_\Omega \left(\int_0^t h(t-s)\Big|\nabla u(x,t)-\nabla u(x,s)\Big|\,ds\right)^2 dx \\
& + \int_\Omega \Big|\nabla u(x,t)\Big|\left(\int_0^t h(s)\,ds\right)\left(\int_0^t h(t-s)\Big|\nabla u(x,t)-\nabla u(x,s)\Big|ds\right)dx, \\
\le & \int_\Omega \left(\int_0^t h(t-s)\,ds\right)\left(\int_0^t h(t-s)\Big|\nabla u(x,t)-\nabla u(x,s)\Big|^2 ds\right)dx \\
& + \left(\int_0^t h(s)\,ds\right)\int_\Omega \Big|\nabla u(x,t)\Big|\left(\int_0^t h(t-s)\Big|\nabla u(x,t)-\nabla u(x,s)\Big|ds\right)dx \\
\le & \left(\int_0^t h(s)\,ds\right)(h\circ\nabla u)(t) + \left(\int_0^t h(s)\,ds\right) \\
& \times \int_\Omega \Big|\nabla u(x,t)\Big|\left(\int_0^t h(t-s)ds\right)^{\frac12}\left(\int_0^t h(t-s)\Big|\nabla u(x,t)-\nabla u(x,s)\Big|^2 ds\right)^{\frac12} dx \\
\le & \left(\int_0^t h(s)\,ds\right)(h\circ\nabla u)(t) + \left(\int_0^t h(s)\,ds\right)\|\nabla u\|_2\left[\left(\int_0^t h(s)ds\right)(h\circ\nabla u)(t)\right]^{\frac12} \\
\le & \left(\int_0^t h(s)\,ds\right)(h\circ\nabla u)(t) \\
& + \left(\int_0^t h(s)\,ds\right)\left[\eta\|\nabla u\|_2^2 + \frac{1}{4\eta}\left(\int_0^t h(s)ds\right)((h\circ\nabla u)(t)\right] \\
\le & \ l(h\circ\nabla u)(t) + \eta l\|\nabla u\|_2^2 + \tfrac{l^2}{4\eta}\Big((h\circ\nabla u)(t) \\
\le & \ \eta l\|\nabla u(x,t)\|_2^2 + \left(l + \frac{l^2}{4\eta}\right)(h\circ\nabla u)(t).
\end{aligned}
$$

Combining all estimates above, we get

$$
\begin{aligned}
& \chi'(t) \le \eta(1+l)\|\nabla u\|_2^2 - \left(\left(\int_0^t h(s)\,ds\right) - \eta\right)\|u_t\|_2^2 \\
& + \left(l + \frac{l}{4\eta} + \frac{lc_*^2}{4\eta}(\mu_1+\mu_2) + \frac{l^2}{4\eta}\right)(h\circ\nabla u)(t) - \frac{h_0c_*^2}{4\eta}(h'\circ\nabla u)(t) \\
& + \eta\mu_1\|g_1(u_t)\|_2^2 + \eta\mu_2\|g_2(z(x,1,t))\|_2^2.
\end{aligned} \tag{2.67}
$$

Proof of proposition 2.4.1. Since h is positive, then for any $t_0 > 0$ we have $\int_0^t h(s)\,ds \ge \int_0^{t_0} h(s)\,ds = \tilde{h}_0$ for all $t \ge t_0$. Thus, making use of this and combining (2.23), (2.62), (2.64) and (2.67) we have

$$
\begin{aligned}
L'(t) \leq & -\left(\tilde{h}_0 - \eta - \varepsilon_1\right)\|u_t\|_2^2 + \left(\frac{M}{2} - \frac{h_0 c_*^2}{4\eta}\right)(h' \circ \nabla u)(t) \\
& -\left[\varepsilon_1(1 - l - \gamma - \gamma c_*^2(\mu_1 + \mu_2)) - \eta(1+l)\right]\|\nabla_x u\|_2^2 - 2\varepsilon_2 I(t) \\
& -\left(MC + \varepsilon_2 \frac{e^{-2\tau}}{\tau}\alpha_1 - \left(\eta\mu_2 + \frac{\varepsilon_1\mu_2}{4\gamma}\right)c_3\right)\int_\Omega g_2(z(x,1,t))z(x,1,t)\,dx \\
& -\left(MC - \frac{\varepsilon_2}{\tau}\alpha_2\right)\int_\Omega g_1(u_t)u_t\,dx \\
& +\left(l + \frac{l\varepsilon_1}{4\gamma} + \frac{l}{4\eta} + \frac{lc_*^2}{4\eta}(\mu_1+\mu_2) + \frac{l^2}{4\eta}\right)(h \circ \nabla u)(t) + \left(\eta\mu_1 + \frac{\varepsilon_1\mu_1}{4\gamma}\right)\|g_1(u_t)\|_2^2.
\end{aligned} \tag{2.68}
$$

At this point, we choose, first, $\varepsilon_1 > 0$ so small that

$$\tilde{h}_0 - \varepsilon_1 > 0.$$

Next, we choose $\gamma > 0$ so small such that

$$1 - l - \gamma - \gamma c_*^2(\mu_1 + \mu_2) > 0$$

and $\eta > 0$ sufficiently small such that

$$\varepsilon_1(1 - l - \gamma - \gamma c_*^2(\mu_1 + \mu_2)) - \eta(1+l) > 0$$

and

$$\tilde{h}_0 - \eta - \varepsilon_1 > 0.$$

Then, we pick $M > 0$ sufficiently large so that

$$
\begin{aligned}
& \frac{M}{2} - \frac{h_0 c_*^2}{4\eta} > 0 \\
& MC + \varepsilon_2 \frac{e^{-2\tau}}{\tau}\alpha_1 - \left(\eta\mu_2 + \frac{\varepsilon_1\mu_2}{4\gamma}\right)c_3 > 0 \\
& MC - \frac{\varepsilon_2}{\tau}\alpha_2 > 0
\end{aligned}
$$

Therefore, (2.68) takes the form

$$\frac{d}{dt}L(t) \leq -C_3 E(t) + C_4 (h \circ \nabla u)(t) + C_5\|g_1(u_t)\|_2^2 \tag{2.69}$$

where C_3, C_4 and C_5 are three positive constants. This completes the proof of Proposition 2.4.1.

Now, we estimate the last term in the right hand side of (2.69). We denote by

$$\Omega^+ = \{x \in \Omega : |u'| \geq \varepsilon'\}, \quad \Omega^- = \{x \in \Omega : |u'| \leq \varepsilon'\}.$$

From (2.1) and (2.2), it follows that

$$\int_{\Omega^+} |g_1(u')|^2\,dx \le \mu_1 \int_{\Omega^+} u' g_1(u')\,dx \le -\mu_1 E'(t). \tag{2.70}$$

Case 1: H is linear on $[0,\varepsilon']$. In this case one can easily check that there exists $\mu_1' > 0$, such that $|g_1(s)| \le \mu_1'|s|$ for all $|s| \le \varepsilon'$, and thus

$$\int_{\Omega^-} |g_1(u')|^2\,dx \le \mu_1' \int_{\Omega^-} u' g_1(u')\,dx \le -\mu_1' E'(t). \tag{2.71}$$

Substitution of (2.70) and (2.71) into (2.69) gives

$$(L(t) + \mu E(t))' \le -c_1 H_2(E(t)) + C_4 h \circ \nabla u \tag{2.72}$$

where $\mu = C_5(\mu_1 + \mu_1')$ and here and in the sequel we take C_i to be a generic positive constant.
Case 2: $H'(0) = 0$ and $H'' > 0$ on $]0,\varepsilon']$.

Since H is convex and increasing, H^{-1} is concave and increasing. By (2.1), the reversed Jensen's inequality for concave function, and (2.23), it follows that

$$\begin{aligned} \int_{\Omega^-} |g_1(u')|^2\,dx &\le \int_{\Omega^-} H^{-1}(u' g_1(u'))\,dx \\ &\le |\Omega| H^{-1}\left(\frac{1}{|\Omega|}\int_{\Omega^-} u' g_1(u')\,dx\right) \le C H^{-1}(-C'E'(t)). \end{aligned} \tag{2.73}$$

A combination of (2.69), (2.70) and (2.73) yields

$$(L(t) + C_5\mu_1 E(t))' \le -C_3 E(t) + C_4(h\circ\nabla u)(t) + \tilde{C}_5 H^{-1}(-C'E'(t)), \quad t \ge t_0. \tag{2.74}$$

Let us denote by H^* the conjugate function of the convex function H, i.e.,

$$H^*(s) = \sup_{t\in\mathbb{R}_+} (st - H(t)).$$

Then H^* is the Legendre transform of H, which is given by

$$H^*(s) = s(H')^{-1}(s) - H[(H')^{-1}(s)], \quad \forall s \ge 0 \tag{2.75}$$

and satisfies the following inequality

$$st \le H^*(s) + H(t), \quad \forall s,t \ge 0. \tag{2.76}$$

The relation (2.75), the fact that $H'(0) = 0$ and $(H')^{-1}, H$ are increasing functions yield

$$H^*(s) \le s(H')^{-1}(s), \quad \forall s \ge 0. \tag{2.77}$$

Making use of $E'(t) \le 0, H''(t) \ge 0$, (2.74) and (2.77) we derive for $\varepsilon_0 > 0$ small enough

$$\begin{aligned} & [H'(\varepsilon_0 E(t))\{L(t) + C_5\mu_1 E(t)\} + \tilde{C}_5 C' E(t)]' \\ = \;& \varepsilon_0 E'(t) H''(\varepsilon_0 E(t))(L(t) + C_5\mu_1 E(t)) + H'(\varepsilon_0 E(t))(L'(t) + C_5\mu_1 E'(t)) + \tilde{C}_5 C' E'(t) \\ \le \;& -C_3 H'(\varepsilon_0 E(t))E(t) + C_4 H'(\varepsilon_0 E(t))(h\circ\nabla u)(t) \\ & \qquad + \tilde{C}_5 H'(\varepsilon_0 E(t)) H^{-1}(-C'E'(t)) + \tilde{C}_5 C' E'(t) \\ \le \;& -C_3 H'(\varepsilon_0 E(t))E(t) + \tilde{C}_5 H^*(H'(\varepsilon_0 E(t))) + C_4 H'(\varepsilon_0 E(0))(h\circ\nabla u)(t) \\ \le \;& -C_3 H'(\varepsilon_0 E(t))E(t) + \tilde{C}_5 H'(\varepsilon_0 E(t))\varepsilon_0 E(t) + C_4 H'(\varepsilon_0 E(0))(h\circ\nabla u)(t) \\ \le \;& -\tilde{C}_3 H'(\varepsilon_0 E(t))E(t) + C_4 H'(\varepsilon_0 E(0))(h\circ\nabla u)(t) \\ = \;& -\tilde{C}_3 H_2(E(t)) + C_4 H'(\varepsilon_0 E(0))(h\circ\nabla u)(t). \end{aligned} \tag{2.78}$$

We note that, in the second inequality, we have used (2.76) and $0 \le H'(\varepsilon_0 E(t)) \le H'(\varepsilon_0 E(0))$.

Let

$$\tilde{L}(t) = \begin{cases} L(t) + \mu E(t) & \text{if } H \text{ is linear on } [0, \varepsilon'] \\ H'(\varepsilon_0 E(t))\{L(t) + C_5\mu_1 E(t)\} + \tilde{C}_5 C' E(t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \varepsilon'], \end{cases} \tag{2.79}$$

then from (2.72) and (2.78), it holds that

$$\tilde{L}'(t) \le -c_4 H_2(E(t)) + c_5 (h \circ \nabla u)(t), \quad \forall t \ge t_0. \tag{2.80}$$

On the other hand, by choosing $M > 0$ larger if needed, we can observe from Lemma 2.4.1 that $L(t)$ is equivalent to $E(t)$. So, $\tilde{L}(t)$ is also equivalent to $E(t)$. Moreover, because $\zeta(t) \le \zeta(0)$, there exists $\overline{\varepsilon} > 0$, such that

$$\zeta(t)\tilde{L}(t) + 2c_5 E(t) \le \overline{\varepsilon} E(t), \quad \forall t \ge t_0. \tag{2.81}$$

Finally, let

$$\mathcal{L}(t) = \varepsilon(\zeta(t)\tilde{L}(t) + 2c_5 E(t)), \qquad \text{for } 0 < \varepsilon < \frac{1}{\overline{\varepsilon}},$$

then we observe, from (2.80), (H1), (2.23) and (2.81), that

$$\begin{aligned} \mathcal{L}'(t) &= \varepsilon(\zeta'(t)\tilde{L}(t) + \zeta(t)\tilde{L}'(t) + 2c_5 E'(t)) \\ &\le -c_4\varepsilon\zeta(t)H_2(E(t)) + c_5\varepsilon\zeta(t)(h \circ \nabla u)(t) + 2c_5\varepsilon E'(t) \\ &\le -c_4\varepsilon\zeta(t)H_2(E(t)) - c_5\varepsilon(h' \circ \nabla u)(t) + 2c_5\varepsilon E'(t) \\ &\le -c_4\varepsilon\zeta(t)H_2(E(t)) \le -c_4\varepsilon\zeta(t)H_2\left(\frac{1}{\overline{\varepsilon}}\left(\zeta(t)\tilde{L}(t) + 2c_5 E(t)\right)\right) \\ &\le -c_4\varepsilon\zeta(t)H_2(\varepsilon(\zeta(t)\tilde{L}(t) + 2c_5 E(t))) = -c_4\varepsilon\zeta(t)H_2(\mathcal{L}(t)). \end{aligned} \tag{2.82}$$

We have used the fact H_2 is increasing in the last two inequalities. Noting that $H_1' = -1/H_2$ (see (2.12)), we infer from (2.82)

$$\mathcal{L}'(t)H_1'(\mathcal{L}(t)) \ge c_4\varepsilon\zeta(t), \quad \forall t \ge t_0.$$

A simple Integration over (t_0, t) then yields

$$H_1(\mathcal{L}(t)) \ge H_1(\mathcal{L}(t_0)) + c_4\varepsilon\int_0^t \zeta(s)\,ds - c_4\varepsilon\int_0^{t_0} \zeta(s)\,ds.$$

Choose $\varepsilon > 0$ sufficiently small so that $H_1(\mathcal{L}(t_0)) - c_4\varepsilon\int_0^{t_0}\zeta(s)\,ds > 0$, then, thanks to the fact H_1^{-1} is decreasing, we infer

$$\begin{aligned} \mathcal{L}(t) &\le H_1^{-1}\left(H_1(\mathcal{L}(t_0)) - c_4\varepsilon\int_0^{t_0}\zeta(s)\,ds + c_4\varepsilon\int_0^t \zeta(s)\,ds\right) \\ &\le H_1^{-1}\left(c_4\varepsilon\int_0^t \zeta(s)\,ds\right). \end{aligned}$$

Consequently, the equivalence of $\mathcal{L}, \tilde{L}, L$ and E, yield

$$E(t) \le C_0 H_1^{-1}\left(\omega\int_0^t \zeta(s)\,ds\right).$$

Chapter 3

STABILITY RESULT OF THE WAVE EQUATION WITH A TIME-VARYING DELAY TERM AND A WEAK BOUNDARY FEEDBACK

3.1 Introduction

In this chapter we investigate the boundary stabilization of the weakly nonlinear wave equation in bounded domain Ω of $\mathbb{R}^n, n \geq 2$, with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0, Γ_1 are closed subsets of $\partial\Omega$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$. Moreover we assume $meas\Gamma_0 > 0$.
The system in given by:

$$(P) \begin{cases} u_{tt}(x,t) - \Delta_x u(x,t) + \mu_2 \sigma(t) u_t(x, t-\tau(t)) + \theta(t) h(\nabla_x u(x,t)) = 0 & \text{in } \Omega \times]0, +\infty[, \\ u(x,t) = 0 & \text{on } \Gamma_0 \times]0, +\infty[, \\ \frac{\partial u}{\partial \nu}(x,t) = -\mu_1 \sigma(t) u_t(x,t) & \text{on } \Gamma_1 \times]0, +\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{on } \Omega, \\ u_t(x, t-\tau(0)) = f_0(x, t-\tau(0)) & \text{on } \Omega \times]0, \tau(0)[, \end{cases}$$

where ν stands for the unit normal vector of $\partial\Omega$ pointing towards the exterior of Ω and $\frac{\partial u}{\partial \nu}$ is the normal derivative. Moreover, $\tau > 0$ is a time delay, μ_1 and μ_2 are positive real numbers, and the initial data (u_0, u_1, f_0) belong to a suitable function space.

Denoting by m the standard multiplier, that is $m(x) = x - x_0$, we assume

$$m(x) \cdot \nu(x) \leq 0, \quad x \in \Gamma_0, \qquad \text{and} \qquad m(x) \cdot \nu(x) \geq \delta > 0, \quad x \in \Gamma_1.$$

In recent years, the PDEs with time delay effects have become an active area of research since they arise in many pratical problems (see, for example, [**1**], [**47**]).

When $h \equiv 0$, it is well known that, in absence of delay ($\mu_2 = 0$), the energy of problem (P) is exponentially decaying to zero. See, for instance, [**14**], [**15**], [**26**], [**27**] and [**38**]. On the contrary, if $\mu_1 = 0$ and $\mu_2 > 0$, that is, there exits only the delay part in the internal, the system (P) becomes unstable (see, for instance [**19**]). To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [**40**], [**41**], [**49**]). For instance, in [**40**] the authors studied the wave equation with a linear internal damping term with constant delay ($\sigma(t) \equiv 1, \tau(t) = const$ in the problem (P)) and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (P) will be instable if $\mu_2 \geq \mu_1$. The main approach used in [**40**] is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay are acting on the boundary.

The case of time-varying delay in the wave equation has been studied recently by Nicaise, Valein and Fridman [**44**] ($\sigma(t) \equiv 1$ in the problem (P)). They proved an exponential stability result under the condition

$$\mu_2 < \sqrt{1-d}\mu_1$$

where the fuction τ satisfies

$$\tau'(t) \leq d, \quad \forall t > 0$$

for a constant $d < 1$.

When $h \not\equiv 0$, in the case $\mu_2 = 0$, very little is known in the literature (see [**5**], [**11**], [**23**], [**22**], [**21**]). In [**23**], Guesmia established well posedness and energy decay estimates in the case of constant coefficients ($\sigma \equiv 1$ and $\theta \equiv 1$). He used a new approach based on a combination of some ideas given in his paper [**22**] and the multiplier method. In [**5**], the authors proved the same result in the case of an unbounded domain and variable coefficients. in [**8**], the authors extend the result of Guesmia to the case of presence of time delay. They established well posedness and energy decay estimates.

Our purpose in this chapter is to give an energy decay estimate of the solution of the problem (P) in the case when h is nonlinear and linear in the presence of a time-varying delay term in the internal feedback. We use the ideas given in Benaissa and Messaoudi in [**8**] and Ammari et al. in in [**3**].

3.2 Preliminaries and main results

First assume the following hypotheses:
(H1) $\sigma, \theta : \mathbb{R}_+ \to]0, +\infty[$ are non increasing functions of class $C^1(\mathbb{R}_+)$ satisfying

$$\int_0^{+\infty} \sigma(s)\, ds = +\infty, \tag{3.1}$$

$$|\sigma'(t)| \leq c\sigma(t). \tag{3.2}$$

$$|\theta'(t)| \leq c\theta(t). \tag{3.3}$$

$$\theta(t) \leq c\sigma(t). \tag{3.4}$$

(H2) τ is a function such that

$$\tau \in W^{2,\infty}([0,T]), \forall T > 0 \tag{3.5}$$

$$0 < \tau_0 \le \tau(t) \le \tau_1, \quad \forall t > 0 \tag{3.6}$$

$$\tau'(t) \le d < 1, \quad \forall t > 0, \tag{3.7}$$

where τ_0 and τ_1 are two positive constants.
(H3)

$$\mu_2 < \sqrt{1-d}\mu_1. \tag{3.8}$$

(H4) $h: \mathbb{R}^n \to \mathbb{R}$ is a C^1 function such that ∇h is bounded and there exists $\beta > 0$ such that

$$|h(\zeta)| \le \beta|\zeta|, \forall \zeta \in \mathbb{R}^n. \tag{3.9}$$

We introduce, as in **[40]**, the new variable

$$z(x,\rho,t) = u_t(x, t-\tau(t)\rho), \quad x \in \Omega,\ \rho \in (0,1), \quad t > 0. \tag{3.10}$$

Then, we have

$$\tau(t) z_t(x,\rho,t) + (1-\tau'(t)\rho)\, z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,+\infty). \tag{3.11}$$

Therefore, problem (P) is equivalent to:

$$\begin{cases} u_{tt}(x,t) - \Delta_x u(x,t) + \mu_2 \sigma(t) z(x,1,t) + \theta(t) h(\nabla_x u(x,t)) = 0, & x \in \Omega, t > 0, \\ \tau(t) z_t(x,\rho,t) + (1-\tau'(t)\rho) z_\rho(x,\rho,t) = 0, & x \in \Omega, \rho \in (0,1), t > 0, \\ u(x,t) = 0, & x \in \Gamma_0, t > 0, \\ \frac{\partial u}{\partial \nu}(x,t) = -\mu_1 \sigma(t) u_t(x,t) & x \in \Gamma_1, t > 0, \\ z(x,0,t) = u_t(x,t) & x \in \Omega, t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & x \in \Omega, \\ z(x,\rho,0) = f_0(x, -\tau(0)\rho) & x \in \Omega, \rho \in (0,1). \end{cases} \tag{3.12}$$

Let $\bar{\xi}$ be a positive constant. We define the energy of the solution by:

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla_x u(t)\|_2^2 + \frac{\xi(t)\tau(t)}{2} \int_\Omega \int_0^1 z^2(x,\rho,t)\, d\rho\, dx, \tag{3.13}$$

where

$$\xi(t) = \bar{\xi}\sigma(t).$$

We have the following theorem.

Theorem 3.2.1 *Let* $(u_0, u_1, f_0) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times H_0^1(\Omega; H^1(0,1))$ *satisfy the compatibility condition*

$$f_0(.,0) = u_1.$$

Assume that the hypotheses $(H1)-(H4)$ *hold with* β *small enough. Then problem* (P) *admits a unique weak solution*

$$u \in L^\infty_{loc}((-\tau(0),\infty); H^2(\Omega)\cap H^1_0(\Omega)), u_t \in L^\infty_{loc}((-\tau(0),\infty); H^1_0(\Omega)), u_{tt} \in L^\infty_{loc}((-\tau(0),\infty); L^2(\Omega)).$$

Moreover, the energy satisfies for $t \geq 0$

$$E(t) \leq cE(0)e^{-\omega\tilde{\sigma}(t)}, \quad \forall t \geq 0, \tag{3.14}$$

Lemma 3.2.1 *Let* (u,z) *be a solution to the problem (3.12). Then, the energy functional defined by (3.13) satisfies*

$$\begin{aligned} E'(t) \leq & -\mu_1\sigma(t)\int_{\Gamma_1} u_t^2(x,t)\,d\Gamma + \frac{\xi(t)+\mu_2\sigma(t)}{2}\int_\Omega u_t^2(x,t)\,dx \\ & + \frac{\mu_2\sigma(t)-\xi(t)(1-\tau'(t))}{2}\int_\Omega z^2(x,1,t)\,dx \\ & -\theta(t)\int_\Omega h(\nabla_x u(x,t))u_t(x,t)\,dx. \end{aligned} \tag{3.15}$$

Proof. Multiplying the first equation in (3.12) by $u_t(x,t)$, integrating over Ω and using integration by parts, we get

$$(u_{tt}(x,t),u_t(x,t))-(\Delta_x u(x,t),u_t(x,t))+(\mu_2\sigma(t)z(x,1,t),u_t(x,t))+(\theta(t)h(\nabla_x u(x,t)),u_t(x,t)) = 0,$$

$$\Rightarrow \int_\Omega u_{tt}(x,t)u_t(x,t)dx - \int_\Omega \Delta_x u(x,t)u_t(x,t)dx + \mu_2\sigma(t)\int_\Omega z(x,1,t)u_t(x,t)dx$$
$$+\theta(t)\int_\Omega h(\nabla_x u(x,t))u_t(x,t)dx = 0,$$

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial t}\|u_t(x,t)\|_2^2 + \int_\Omega \nabla_x u(x,t)\nabla_x u_t(x,t)dx - \int_{\partial\Omega}\nabla_x u(x,t)u_t(x,t)\nu d\Gamma$$
$$+\mu_2\sigma(t)\int_\Omega z(x,1,t)u_t(x,t)dx + \theta(t)\int_\Omega h(\nabla_x u(x,t))u_t(x,t)dx = 0,$$

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial t}\|u_t(x,t)\|_2^2 + \frac{1}{2}\frac{\partial}{\partial t}\|\nabla_x u(x,t)\|_2^2 - \int_{\Gamma_0}\nabla_x u(x,t)u_t(x,t)\nu d\Gamma - \int_{\Gamma_1}\nabla_x u(x,t)u_t(x,t)\nu d\Gamma$$
$$+\mu_2\sigma(t)\int_\Omega z(x,1,t)u_t(x,t)dx + \theta(t)\int_\Omega h(\nabla_x u(x,t))u_t(x,t)dx = 0,$$

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial t}\Big(\|u_t(x,t)\|_2^2 + \|\nabla_x u(x,t)\|_2^2\Big) - \int_{\Gamma_0} u_t(x,t)\frac{\partial u}{\partial\nu}(x,t)d\Gamma - \int_{\Gamma_1} u_t(x,t)\frac{\partial u}{\partial\nu}(x,t)d\Gamma$$
$$+\mu_2\sigma(t)\int_\Omega z(x,1,t)u_t(x,t)dx + \theta(t)\int_\Omega h(\nabla_x u(x,t))u_t(x,t)dx = 0,$$

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial t}\Big(\|u_t(x,t)\|_2^2 + \|\nabla_x u(x,t)\|_2^2\Big) - \int_{\Gamma_1} u_t(x,t)(-\mu_1\sigma(t)u_t(x,t))d\Gamma$$
$$+\mu_2\sigma(t)\int_\Omega z(x,1,t)u_t(x,t)dx + \theta(t)\int_\Omega h(\nabla_x u(x,t))u_t(x,t)dx = 0,$$

then, we have

$$
\begin{aligned}
(3.16)\qquad &\frac{1}{2}\frac{\partial}{\partial t}\Big(\|u_t(x,t)\|_2^2+\|\nabla_x u(x,t)\|_2^2\Big)+\mu_1\sigma(t)\int_{\Gamma_1}u_t^2(x,t)d\Gamma\\
&+\mu_2\sigma(t)\int_\Omega z(x,1,t)u_t(x,t)dx+\theta(t)\int_\Omega h(\nabla_x u(x,t))u_t(x,t)dx=0.
\end{aligned}
$$

We multiply the second equation in (3.12) by $\xi(t)z$ and integrate over $\Omega\times(0,1)$ to obtain:

$$\xi(t)\tau(t)\int_\Omega\int_0^1 z_t(x,\rho,t)z(x,\rho,t)\,d\rho\,dx+\xi(t)\int_\Omega\int_0^1(1-\tau'(t)\rho)z_\rho(x,\rho,t)z(x,\rho,t)\,d\rho\,dx=0.$$

$$\Rightarrow\quad \frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1\frac{\partial}{\partial t}z^2(x,\rho,t)\,d\rho\,dx+\frac{\xi(t)}{2}\int_\Omega\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho,t)\,d\rho\,dx=0.$$

Then

$$(3.17)\quad \frac{\xi(t)\tau(t)}{2}\frac{\partial}{\partial t}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx+\frac{\xi(t)}{2}\int_\Omega\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho,t)\,d\rho\,dx=0.$$

We have

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big[\frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\Big]&=\frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx+\\
&\frac{\xi(t)\tau'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx+\frac{\xi(t)\tau(t)}{2}\frac{\partial}{\partial t}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad \frac{\xi(t)\tau(t)}{2}\frac{\partial}{\partial t}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx&=\frac{\partial}{\partial t}\Big[\frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\Big]\\
&-\frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx-\frac{\xi(t)\tau'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx.
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\xi(t)}{2}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}\Big[(1-\tau'(t)\rho)z^2(x,\rho,t)\Big]d\rho\,dx&=\frac{\xi(t)}{2}\int_\Omega\int_0^1(-\tau'(t))z^2(x,\rho,t)\,d\rho\,dx\\
&+\frac{\xi(t)}{2}\int_\Omega\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho,t)\,d\rho\,dx
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad \frac{\xi(t)}{2}\int_\Omega\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho,t)\,d\rho\,dx&=\frac{\xi(t)}{2}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}\Big[(1-\tau'(t)\rho)z^2(x,\rho,t)\Big]d\rho\,dx\\
&+\frac{\xi(t)\tau'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx
\end{aligned}
$$

$$\Rightarrow \frac{\xi(t)}{2}\int_\Omega\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho,t)\,d\rho\,dx = \frac{\xi(t)\tau'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx$$
$$+\frac{\xi(t)}{2}\int_\Omega(1-\tau'(t))z^2(x,1,t)\,dx - \frac{\xi(t)}{2}\int_\Omega z^2(x,0,t)\,dx$$
$$\Rightarrow \frac{\xi(t)}{2}\int_\Omega\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho,t)\,d\rho\,dx = \frac{\xi(t)\tau'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx$$
$$+\frac{\xi(t)}{2}\int_\Omega(1-\tau'(t))z^2(x,1,t)\,dx - \frac{\xi(t)}{2}\int_\Omega u_t^2(x,t)\,dx$$

Consequently,

$$\begin{aligned}0 \;=\; & \frac{\xi(t)\tau(t)}{2}\frac{\partial}{\partial t}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx + \frac{\xi(t)}{2}\int_\Omega\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho,t)\,d\rho\,dx\\
=\; & \frac{\partial}{\partial t}\Big[\frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\Big] - \frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\\
& -\frac{\xi(t)\tau'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx.\\
& +\frac{\xi(t)\tau'(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx + \frac{\xi(t)}{2}\int_\Omega(1-\tau'(t))z^2(x,1,t)\,dx - \frac{\xi(t)}{2}\int_\Omega u_t^2(x,t)\,dx\\
=\; & \frac{\partial}{\partial t}\Big[\frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\Big] - \frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\\
& +\frac{\xi(t)}{2}\int_\Omega(1-\tau'(t))z^2(x,1,t)\,dx - \frac{\xi(t)}{2}\int_\Omega u_t^2(x,t)\,dx\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(\frac{1}{2}\|u_t(x,t)\|_2^2+\frac{1}{2}\|\nabla_x u(x,t)\|_2^2+\frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\Big)+\mu_1\sigma(t)\int_{\Gamma_1}u_t^2(x,t)d\Gamma\\
&+\mu_2\sigma(t)\int_\Omega z(x,1,t)u_t(x,t)dx+\theta(t)\int_\Omega h(\nabla_x u(x,t))u_t(x,t)dx-\frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\\
&+\frac{\xi(t)}{2}\int_\Omega(1-\tau'(t))z^2(x,1,t)\,dx - \frac{\xi(t)}{2}\int_\Omega u_t^2(x,t)\,dx = 0.\end{aligned}$$

(3.18)

We obtain

$$\begin{aligned}E'(t) \;=\; & -\mu_1\sigma(t)\int_{\Gamma_1}u_t^2(x,t)d\Gamma-\mu_2\sigma(t)\int_\Omega z(x,1,t)u_t(x,t)dx-\theta(t)\int_\Omega h(\nabla_x u(x,t))u_t(x,t)dx\\
&+\frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx - \frac{\xi(t)}{2}\int_\Omega(1-\tau'(t))z^2(x,1,t)\,dx + \frac{\xi(t)}{2}\int_\Omega u_t^2(x,t)\,dx.\end{aligned}$$

(3.19)

Consequently,

$$\begin{aligned}E'(t) \;=\; & -\mu_1\sigma(t)\int_{\Gamma_1}u_t^2(x,t)d\Gamma-\mu_2\sigma(t)\int_\Omega z(x,1,t)u_t(x,t)dx-\theta(t)\int_\Omega h(\nabla_x u(x,t))u_t(x,t)dx\\
&+\frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx - \frac{\xi(t)}{2}\int_\Omega z^2(x,1,t)\,dx\\
&+\frac{\xi(t)\tau'(t)}{2}\int_\Omega z^2(x,1,t)\,dx + \frac{\xi(t)}{2}\int_\Omega u_t^2(x,t)\,dx.\end{aligned}$$

(3.20)

3.3 Asymptotic behavior

For any regular solution of problem (P), we have differntiating and integrating by parts we obtain

$$
\begin{aligned}
&\frac{1}{dt}\Big(\int_\Omega \Big[2m\cdot\nabla u+(n-1)u\Big]\ u_t\,dx\Big)\\
&= \int_\Omega \Big[2m\cdot\nabla u+(n-1)u\Big]u_{tt}\,dx+\int_\Omega \Big[2m\cdot\nabla u_t+(n-1)u_t\Big]u_t\,dx,\\
&= \int_\Omega \Big[2m\cdot\nabla u+(n-1)u\Big]\Big[\Delta u-\mu_2\sigma(t)z(x,1,t)-\theta(t)h(\nabla_x u)\Big]\,dx\\
&\quad +\int_\Omega \Big[2m\cdot\nabla u_t+(n-1)u_t\Big]u_t\,dx,\\
&= \int_\Omega \Big[2m\cdot\nabla u+(n-1)u\Big]\Delta u\,dx-\mu_2\sigma(t)\int_\Omega \Big[2m\cdot\nabla u+(n-1)u\Big]z(x,1,t)\,dx\\
&\quad -\theta(t)\int_\Omega \Big[2m\cdot\nabla u+(n-1)u\Big]h(\nabla u)\,dx+\int_\Omega \Big[2m\cdot\nabla u_t+(n-1)u_t\Big]u_t\,dx,\\
&= \int_{\partial\Omega} \Big[2m\cdot\nabla u+(n-1)u\Big]\nabla u\,\nu\,d\Gamma-\int_\Omega \nabla\Big[2m\cdot\nabla u+(n-1)u\Big]\nabla u\,dx\\
&\quad -\mu_2\sigma(t)\int_\Omega \Big[2m\cdot\nabla u+(n-1)u\Big]z(x,1,t)\,dx+\int_\Omega \Big[2m\cdot\nabla u_t+(n-1)u_t\Big]u_t\,dx\\
&\quad -\theta(t)\int_\Omega \Big[2m\cdot\nabla u+(n-1)u\Big]h(\nabla u)\,dx,\\
&= \int_{\Gamma_0} \Big[2m\cdot\nabla u+(n-1)u\Big]\frac{\partial u}{\partial\nu}\,d\Gamma+\int_{\Gamma_1} \Big[2m\cdot\nabla u+(n-1)u\Big]\frac{\partial u}{\partial\nu}\,d\Gamma\\
&\quad -\int_\Omega \nabla\Big[2m\cdot\nabla u+(n-1)u\Big]\nabla u\,dx+\int_\Omega \Big[2m\cdot\nabla u_t+(n-1)u_t\Big]u_t\,dx\\
&\quad -\mu_2\sigma(t)\int_\Omega \Big[2m\cdot\nabla u+(n-1)u\Big]z(x,1,t)\,dx-\theta(t)\int_\Omega \Big[2m\cdot\nabla u+(n-1)u\Big]h(\nabla u)\,dx,\\
&= \int_{\Gamma_1} \Big[2m\cdot\nabla u+(n-1)u\Big]\frac{\partial u}{\partial\nu}\,d\Gamma\\
&\quad -\int_\Omega \nabla\Big[2m\cdot\nabla u+(n-1)u\Big]\nabla u\,dx+\int_\Omega \Big[2m\cdot\nabla u_t+(n-1)u_t\Big]u_t\,dx\\
&\quad -\mu_2\sigma(t)\int_\Omega \Big[2m\cdot\nabla u+(n-1)u\Big]z(x,1,t)\,dx-\theta(t)\int_\Omega \Big[2m\cdot\nabla u+(n-1)u\Big]h(\nabla u)\,dx,\\
&= \int_{\Gamma_1} \Big(2m\cdot\nabla u\Big)\frac{\partial u}{\partial\nu}\,d\Gamma+(n-1)\int_{\Gamma_1} u\frac{\partial u}{\partial\nu}\,d\Gamma\\
&\quad -\int_\Omega \nabla\Big(2m\cdot\nabla u\Big)\nabla u\,dx-(n-1)\int_\Omega |\nabla u|^2dx+\int_\Omega \Big(2m\cdot\nabla u_t\Big)u_t\,dx+(n-1)\int_\Omega u_t^2dx\\
&\quad -\mu_2\sigma(t)\int_\Omega \Big(2m\cdot\nabla u\Big)z(x,1,t)\,dx-(n-1)\mu_2\sigma(t)\int_\Omega uz(x,1,t)\,dx\\
&\quad -\theta(t)\int_\Omega \Big(2m\cdot\nabla u\Big)h(\nabla u)\,dx-(n-1)\theta(t)\int_\Omega u\,h(\nabla u)\,dx,
\end{aligned}
$$

Now since $u=0$ on Γ_0 and $m\cdot\nu\leq 0$ on Γ_0, from $(*)$ we deduce

$$\begin{aligned}
&\frac{1}{dt}\Big(\int_\Omega\Big[2m\cdot\nabla u+(n-1)u\Big]\;u_t\,dx\Big)\\
&\leq\;-\int_\Omega\Big\{u_t^2+|\nabla u|^2\Big\}dx-2\mu_2\sigma(t)\int_\Omega\Big(m\cdot\nabla u\Big)z(x,1,t)\,dx-(n-1)\mu_2\sigma(t)\int_\Omega uz(x,1,t)\,dx\\
&\quad+\|m\|_\infty\int_{\Gamma_1}u_t^2\,d\Gamma-\delta\int_{\Gamma_1}|\nabla u|^2d\Gamma+2\int_{\Gamma_1}\Big(m\cdot\nabla u\Big)\frac{\partial u}{\partial\nu}\,d\Gamma+(n-1)\int_{\Gamma_1}u\frac{\partial u}{\partial\nu}\,d\Gamma\\
&\quad-\theta(t)\int_\Omega\Big(2m\cdot\nabla u\Big)h(\nabla u)\,dx-(n-1)\theta(t)\int_\Omega u\,h(\nabla u)\,dx,
\end{aligned}$$

We have

$$\begin{aligned}
2\int_{\Gamma_1}\Big(m\cdot\nabla u\Big)\frac{\partial u}{\partial\nu}\,d\Gamma &\leq \frac{\delta}{2}\int_{\Gamma_1}|\nabla u|^2d\Gamma+2\frac{\|m\|_\infty^2}{\delta}\int_{\Gamma_1}\Big|\frac{\partial u}{\partial\nu}\Big|^2\,d\Gamma\\
&\leq \frac{\delta}{2}\int_{\Gamma_1}|\nabla u|^2d\Gamma+\frac{2}{\delta}\|m\|_\infty^2\mu_1^2\sigma^2(t)\int_{\Gamma_1}u_t^2\,d\Gamma.
\end{aligned}$$

Moreover

$$\begin{aligned}
(n-1)\int_{\Gamma_1}u\frac{\partial u}{\partial\nu}\,d\Gamma &\leq \frac{\varepsilon}{2}\int_{\Gamma_1}u^2d\Gamma+\frac{(n-1)^2}{2\varepsilon}\int_{\Gamma_1}\Big|\frac{\partial u}{\partial\nu}\Big|^2\,d\Gamma\\
&\leq \frac{\varepsilon}{2}\,C(P)\int_\Omega|\nabla u|^2dx+\frac{(n-1)^2}{2\varepsilon}\mu_1^2\sigma^2(t)\int_{\Gamma_1}u_t^2\,d\Gamma.
\end{aligned}$$

Remark. In the above proposition $C(P)$ is the smallest positive constant such that

$$\int_{\Gamma_1}\varphi^2(x)\,d\Gamma\leq C(P)\int_\Omega|\nabla\varphi(x)|^2dx,\quad\forall\varphi\in H^1_{\Gamma_0}(\Omega).$$

Corollary. For any regular solution of $(*)$

$$\begin{aligned}
&\frac{1}{dt}\Big(\int_\Omega\Big[2m\cdot\nabla u+(n-1)u\Big]\;u_t\,dx\Big)\\
&\leq\;-\int_\Omega u_t^2dx-\Big(1-\frac{\varepsilon}{2}\,C(P)-\mu_2\|m\|_\infty^2-\frac{\mu_2}{2}(n-1)^2C_0(P)\Big)\int_\Omega|\nabla u|^2dx\\
&\quad+\Big(\frac{2}{\delta}\|m\|_\infty^2\mu_1^2\sigma^2(t)+\frac{(n-1)^2}{2\varepsilon}\mu_1^2\sigma^2(t)+\|m\|_\infty\Big)\int_{\Gamma_1}u_t^2\,d\Gamma+\frac{\varepsilon}{2}\mu_2\sigma^2(t)\int_\Omega|z(x,1,t)|^2\,dx\\
&\quad-\theta(t)\int_\Omega\Big(2m\cdot\nabla u\Big)h(\nabla u)\,dx-(n-1)\theta(t)\int_\Omega u\,h(\nabla u)\,dx,
\end{aligned}$$

Now, let us introduce the functional $S(t)=\displaystyle\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)\,d\rho\;dx.$

We use $s=t-\rho\tau(t)$, then, we have $S(t)=\displaystyle\frac{1}{\tau(t)}\int_\Omega\int_{t-\tau(t)}^t e^{2(s-t)}u_t^2(x,s)\,ds\,dx.$

We can easily estimate

$$\begin{aligned}
S'(t) &= -\frac{\tau'(t)}{\tau^2(t)}\int_\Omega\int_{t-\tau(t)}^t e^{2(s-t)}u_t^2(x,s)\,ds\,dx+\frac{1}{\tau(t)}\int_\Omega u_t^2(x,t)\,dx-\frac{e^{-2\tau(t)}}{\tau(t)}\int_\Omega u_t^2(x,t-\tau(t))\,dx,\\
&= \frac{1}{\tau(t)}\int_\Omega u_t^2(x,t)\,dx-\frac{e^{-2\tau(t)}}{\tau(t)}\int_\Omega u_t^2(x,t-\tau(t))\,dx-\frac{\tau'(t)}{\tau(t)}S(t),
\end{aligned}$$

Let us introduce the Lyapunov functional

$$\mathcal{E}(t) = ME(t) + \varepsilon_1 \int_\Omega \Big[2m \cdot \nabla u + (n-1)u\Big]\ u_t\, dx + \varepsilon_2 S(t), \tag{3.21}$$

where M, ε_1 and ε_2 are suitable positive small constants that will be precised later on. Note that $\mathcal{E}(t)$ is equivalent to the energy $E(t)$ if ε_1 is small enough.

Lemma 3.3.1 *There exist two positive constants α_1 and α_2 depending on $\varepsilon_1, \varepsilon_2$ and M such that for all $t > 0$*

$$\alpha_1 E(t) \leq \mathcal{E}(t) \leq \alpha_2 E(t). \tag{3.22}$$

Proof: We consider the functional

$$K(t) = \varepsilon_1 \int_\Omega \Big[2m \cdot \nabla u + (n-1)u\Big]\ u_t\, dx + \varepsilon_2 S(t),$$

and show that

$$|K(t)| \leq CE(t), \quad C > 0. \tag{3.23}$$

We have

$$\begin{aligned} K(t) &= \varepsilon_1 \int_\Omega \Big[2m \cdot \nabla u + (n-1)u\Big]\ u_t\, dx + \varepsilon_2 \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2(x,\rho,t)\, d\rho\, dx, \\ &= \varepsilon_1 \int_\Omega \Big(2m \cdot \nabla u\Big) u_t\, dx + \varepsilon_1 (n-1) \int_\Omega u u_t\, dx + \varepsilon_2 \int_\Omega \int_0^1 e^{-2\rho\tau(t)} z^2(x,\rho,t)\, d\rho\, dx. \end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned} |K(t)| &\leq \frac{\varepsilon_1}{2} \int_\Omega u_t^2\, dx + 2\varepsilon_1 \|m\|_\infty^2 \int_\Omega |\nabla u|^2\, dx + \frac{\varepsilon_1(n-1)}{2} \int_\Omega u_t^2\, dx + \frac{\varepsilon_1(n-1)}{2} \int_\Omega u^2\, dx \\ &\quad + \varepsilon_2 \int_\Omega \int_0^1 z^2(x,\rho,t)\, d\rho\, dx, \\ &\leq \frac{\varepsilon_1 n}{2} \int_\Omega u_t^2\, dx + 2\varepsilon_1 \|m\|_\infty^2 \int_\Omega |\nabla u|^2\, dx + \frac{\varepsilon_1(n-1)}{2} \int_\Omega u^2\, dx + \varepsilon_2 \int_\Omega \int_0^1 z^2(x,\rho,t)\, d\rho\, dx, \\ &\leq CE(t) + \frac{\varepsilon_1(n-1)}{2} \int_\Omega u^2\, dx, \end{aligned}$$

such that $C = \max\left(\varepsilon_1 n,\ 4\varepsilon_1 \|m\|_\infty^2,\ \sup_{t \in [0,+\infty[} \dfrac{2\varepsilon_2}{\xi(t)\tau(t)}\right)$.

If ε_1 is small enough, we get (3.23) for some positive constant C. By choosing M large enough, our result follows from (3.21), (3.23).

Proposition 3.3.1 *For each $t_0 > 0$ and sufficiently large $M > 0$ and appropriately small $\varepsilon_1, \varepsilon_2 > 0$, there exist positive constants C_1 such that*

$$\frac{d}{dt}\mathcal{E}(t) \leq -C_1 \sigma(t) E(t), \quad \forall t \geq t_0.$$

Proof. We have,

$$\frac{1}{dt}\Big(\varepsilon_1\int_\Omega \Big[2m\cdot\nabla u+(n-1)u\Big]\; u_t\,dx+\varepsilon_2 S(t)\Big)$$

$$\begin{aligned}
\leq\;& \varepsilon_1\frac{2n-1}{2}\int_\Omega u_t^2\,dx+\frac{\varepsilon_1}{2}\Big[3-2n+4\mu_2\sigma(t)\|m\|_\infty^2+4\beta\theta(t)\|m\|_\infty+(n-1)\beta^2\theta(t)\Big]\int_\Omega|\nabla u|^2\,dx\\
&+\frac{\varepsilon_1(n-1)(\theta(t)+\mu_2\sigma(t))}{2}\int_\Omega u^2\,dx+2\varepsilon_1\|m\|_\infty^2\int_\Omega|\nabla u_t|^2\,dx+\frac{\varepsilon_1 n\mu_2\sigma(t)}{2}\int_\Omega z^2(x,1,t)\,dx\\
&+\frac{\varepsilon_1}{2}\int_\Omega|\nabla(2m\cdot\nabla u)|^2\,dx+\frac{\varepsilon_1(n-1)}{2}\int_{\Gamma_1}u^2\,d\Gamma+\frac{\varepsilon_1}{2}\int_{\Gamma_1}|\nabla u|^2\,d\Gamma\\
&+\frac{\varepsilon_1\mu_1^2\sigma^2(t)\Big(n-1+4\|m\|_\infty^2\Big)}{2}\int_{\Gamma_1}u_t^2\,d\Gamma-\frac{\varepsilon_2\tau'(t)}{\tau^2(t)}\int_\Omega\int_{t-\tau(t)}^t e^{2(s-t)}u_t^2(x,s)\,ds\,dx\\
&+\frac{\varepsilon_2}{\tau(t)}\int_\Omega u_t^2(x,t)\,dx-\frac{\varepsilon_2 e^{-2\tau(t)}}{\tau(t)}\int_\Omega u_t^2(x,t-\tau(t))\,dx\\
\leq\;& \varepsilon_1\frac{2n-1}{2}\int_\Omega u_t^2\,dx+\frac{\varepsilon_1}{2}\Big[3-2n+4\mu_2\sigma(t)\|m\|_\infty^2+4\beta\theta(t)\|m\|_\infty+(n-1)\beta^2\theta(t)\Big]\int_\Omega|\nabla u|^2\,dx\\
&+\frac{\varepsilon_1(n-1)(\theta(t)+\mu_2\sigma(t))}{2}\int_\Omega u^2\,dx+2\varepsilon_1\|m\|_\infty^2\int_\Omega|\nabla u_t|^2\,dx+\frac{\varepsilon_1 n\mu_2\sigma(t)}{2}\int_\Omega z^2(x,1,t)\,dx\\
&+\frac{\varepsilon_1}{2}\int_\Omega|\nabla(2m\cdot\nabla u)|^2\,dx+\frac{\varepsilon_1(n-1)}{2}\int_{\Gamma_1}u^2\,d\Gamma+\frac{\varepsilon_1}{2}\int_{\Gamma_1}|\nabla u|^2\,d\Gamma\\
&+\frac{\varepsilon_1\mu_1^2\sigma^2(t)\Big(n-1+4\|m\|_\infty^2\Big)}{2}\int_{\Gamma_1}u_t^2\,d\Gamma+\frac{\varepsilon_2}{\tau(t)}\int_\Omega u_t^2(x,t)\,dx\\
&-\frac{\varepsilon_2 e^{-2\tau(t)}}{\tau(t)}\int_\Omega u_t^2(x,t-\tau(t))\,dx-\frac{\varepsilon_2\tau'(t)}{\tau(t)}\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)\,d\rho\;dx,\\
\leq\;& \varepsilon_1\frac{2n-1}{2}\int_\Omega u_t^2\,dx+\frac{\varepsilon_1}{2}\Big[3-2n+4\mu_2\sigma(t)\|m\|_\infty^2+4\beta\theta(t)\|m\|_\infty+(n-1)\beta^2\theta(t)\Big]\int_\Omega|\nabla u|^2\,dx\\
&+\frac{\varepsilon_1(n-1)(\theta(t)+\mu_2\sigma(t))}{2}\int_\Omega u^2\,dx+2\varepsilon_1\|m\|_\infty^2\int_\Omega|\nabla u_t|^2\,dx+\frac{\varepsilon_1 n\mu_2\sigma(t)}{2}\int_\Omega z^2(x,1,t)\,dx\\
&+\frac{\varepsilon_1}{2}\int_\Omega|\nabla(2m\cdot\nabla u)|^2\,dx+\frac{\varepsilon_1(n-1)}{2}\int_{\Gamma_1}u^2\,d\Gamma+\frac{\varepsilon_1}{2}\int_{\Gamma_1}|\nabla u|^2\,d\Gamma\\
&+\frac{\varepsilon_1\mu_1^2\sigma^2(t)\Big(n-1+4\|m\|_\infty^2\Big)}{2}\int_{\Gamma_1}u_t^2\,d\Gamma+\frac{\varepsilon_2}{\tau(t)}\int_\Omega u_t^2(x,t)\,dx\\
&-\frac{\varepsilon_2\tau'(t)}{\tau(t)}\int_\Omega\int_0^1 e^{-2\rho\tau(t)}z^2(x,\rho,t)\,d\rho\;dx,\\
\leq\;& \varepsilon_1\frac{2n-1}{2}\int_\Omega u_t^2\,dx+\frac{\varepsilon_1}{2}\Big[3-2n+4\mu_2\sigma(t)\|m\|_\infty^2+4\beta\theta(t)\|m\|_\infty+(n-1)\beta^2\theta(t)\Big]\int_\Omega|\nabla u|^2\,dx\\
&+\frac{\varepsilon_1(n-1)(\theta(t)+\mu_2\sigma(t))}{2}\int_\Omega u^2\,dx+2\varepsilon_1\|m\|_\infty^2\int_\Omega|\nabla u_t|^2\,dx+\frac{\varepsilon_1 n\mu_2\sigma(t)}{2}\int_\Omega z^2(x,1,t)\,dx\\
&+\frac{\varepsilon_1}{2}\int_\Omega|\nabla(2m\cdot\nabla u)|^2\,dx+\frac{\varepsilon_1(n-1)}{2}\int_{\Gamma_1}u^2\,d\Gamma+\frac{\varepsilon_1}{2}\int_{\Gamma_1}|\nabla u|^2\,d\Gamma
\end{aligned}$$

$$
\begin{aligned}
& +\frac{\varepsilon_1 \mu_1^2 \sigma^2(t)\left(n-1+4\|m\|_\infty^2\right)}{2} \int_{\Gamma_1} u_t^2 \, d\Gamma + \frac{\varepsilon_2}{\tau(t)} \int_\Omega u_t^2(x,t)\, dx + \frac{\varepsilon_2 |\tau'(t)|}{\tau(t)} \int_\Omega \int_0^1 z^2(x,\rho,t)\, d\rho \, dx, \\
\leq \; & C_1 E(t) + \frac{\varepsilon_1 (n-1)(\theta(t)+\mu_2\sigma(t))}{2} \int_\Omega u^2 \, dx + 2\varepsilon_1 \|m\|_\infty^2 \int_\Omega |\nabla u_t|^2 \, dx \\
& + \frac{\varepsilon_1 n \mu_2 \sigma(t)}{2} \int_\Omega z^2(x,1,t)\, dx + \frac{\varepsilon_1}{2} \int_\Omega |\nabla(2m\cdot\nabla u)|^2 \, dx + \frac{\varepsilon_1(n-1)}{2} \int_{\Gamma_1} u^2 \, d\Gamma \\
& + \frac{\varepsilon_1}{2} \int_{\Gamma_1} |\nabla u|^2 \, d\Gamma + \frac{\varepsilon_1 \mu_1^2 \sigma^2(t)\left(n-1+4\|m\|_\infty^2\right)}{2} \int_{\Gamma_1} u_t^2 \, d\Gamma,
\end{aligned}
$$

$$
\begin{aligned}
C_1 \; = \; & \max\Bigg(\varepsilon_1(2n-1) + \frac{2\varepsilon_2}{\tau_0}, \; \varepsilon_1\Big[3 - 2n + 4\mu_2\sigma(t)\|m\|_\infty^2 + 4\beta\theta(t)\|m\|_\infty + (n-1)\beta^2\theta(t)\Big], \\
& \sup_{t\in[0,+\infty[} \frac{2\varepsilon_2|\tau'(t)|}{\xi(t)\tau^2(t)}\Bigg),
\end{aligned}
$$

for each $t_0 > 0$ and sufficiently large $M > 0$ and appropriately small $\varepsilon_1, \varepsilon_2 > 0$, consequently,

$$
\frac{d}{dt}\mathcal{E}(t) \leq -C_1\sigma(t)E(t), \quad \forall t \geq t_0.
$$

This ends the proof of Theorem 1.

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