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Une contribution à l'étude de certaine EDP non linéaire admettant des solutions solitons

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Some contribution of class for Non-Linear PDE Admitting Soliton Solutions

(in French)

Une contribution à l'étude de certaine EDP non linéaire admettant des solutions solitons.

October 9, 2016

À la mémoire de mon cher père défunt, que grâces et miséricordes soient sur lui.

À ma très chère mère .

À ma bien chère femme et mes bien chers enfants, Amine, Ayoub, Ansar, Raouf, Chorouk

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Chapter 1

Introduction

Mathematics consists initially of a language, which makes it possible to transcribe problems of quantitative nature: this is modeling. Once this transcription is made, tools are available to solve these problems, partially or completely. Then, one brings back the solution into its context of origin. Ordinary and partial differential equations (PDE's) are at the heart of mathematical modelization. They constitute the basic language in which most of the laws in physics or engineering can be written and one of the most important mathematical tools for modelling in the universe and socio-economical sciences. And they occur in many applications in chemistry to model reactions, to study economics market behavior, to study in finance and derivatives and in image processing to restore the damage…,etc. We make some simple basic assumptions, actually it is the physical properties of the universe, which are shared today by every fundamental theory in physics and which will have very deep consequences (see [9]).

(α) The universe is variational, that is we suppose that all the physical phenomena are governed by differential equations which admit a variational formulation. This variational principle is at once reasonable, if we think that all the fundamental equations of physics can be seen as the Euler-Lagrange equations of a suitable action functional.

Remark 1.0.1. *The variational principle has a very long history and can be traced back even to the ideas of Aristotle. However, its very discovery has been attributed to P.L.M. de Maupertuis (1698-1759), after he was engaged in polemics with the followers of G.W. Leibniz; in his work "Examen philosophique de la preuve de l'existence de Dieu" (1756), he stated the principle of minimal action as an evidence of rationality in the divine creation. It is well known that these metaphysical ideas were then formalized by L. Euler and G.L. Lagrange in the eighteenth century, but the ultimate reason for which the variational principle holds true in nature is still today a mystery. On the subject, we refer to the essay "Le Meilleur des mondes possibles. Mathématiques et destinée" by I. Ekeland [38].*

(β) The universe is invariant for the Poincare group, that is we suppose that all the equations of the universe are invariant with respect to the group generated by the following transformations:

- time translations, i.e. transformations depending on one parameter having the form
$$\begin{cases} x & \to & x \\ t & \to & t+t_0 \end{cases}$$

- space translations, i.e. transformations depending on three parameters having the form
$$\begin{cases} x & \to & x+x_0 \\ t & \to & t \end{cases}$$

- space rotations, i.e. transformations depending on three parameters having the form,
$$\begin{cases} x & \to & Rx, \ R\in O(3) \\ t & \to & t+t_0 \end{cases}$$

- Lorentz transformations, i.e. space-time rotations depending on one parameter ν having the form

$$\begin{cases} x_1 & \to & \gamma(x_1-\nu t) \\ x_2 & \to & x_2 \\ x_3 & \to x_3 & \\ t & \to & \gamma(t-\frac{\nu}{c^2}x_1) \end{cases} \begin{cases} x_1 & \to & x_1 \\ x_2 & \to & \gamma(x_2-\nu t) \\ x_3 & \to x_3 & \\ t & \to & \gamma(t-\frac{\nu}{c^2}x_2) \end{cases}$$

$$\begin{cases} x_1 & \to x_1 & \\ x_2 & \to & x_2 \\ x_3 & \to & \gamma(x_3-\nu t) \\ t & \to & \gamma(t-\frac{\nu}{c^2}x_3), \ where\ \gamma=\frac{c}{\sqrt{c^2-\nu^2}}, |\nu|<c \end{cases}$$

and c is a constant (dimensionally a velocity). Indeed, the Poincaré group P is the ten parameters Lie group generated by the above transformations together with the time and parity inversions $t\to -t$ and $x\to -x$.

The assumptions of the first three invariances cannot be omitted if we want to make a physical theory, for they express the possibility of repeating experiments. More precisely, translational invariances ask for time and space to be homogeneuos (i.e., whenever and wherever an experiment is performed, it gives the same results) and rotational invariance requires that the space is isotropic (i.e., there are no privileged directions in the universe). Finally, the Lorentz invariance is an empirical fact and we will see that it is the very cause of relativistic effects.

1.1 What is a soliton ?

Solitons are nonlinear waves. As a preliminary definition, a soliton is considered as a solitary, traveling wave pulse solution of a nonlinear partial differential equation (PDE). The soliton phenomenon was first described in 1834 by John Scott Russell (1808-1882) who observed a solitary wave on the canal from Edimburgh to Glasgow in 1834. Reporting to the British Association, he wrote [80]: "I believe I shall best introduce this phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that rare and beautiful phenomenon which I have called the Wave of Translation [...]."

A soliton is a solution of a field equation whose energy travels as a localized packet and which preserves its form under perturbations. The nonlinearity will play a significant role. For most dispersive evolution equations these solitary waves would scatter inelastically and lose 'energy' due to the radiation. Not so for the solitons: after a fully nonlinear interaction, the solitary waves remerge, retaining their identities with same speed and shape. It should have remarkable stability properties. In this respect solitons have a particle-like behavior. The soliton equations, in the mathematical sense, provide outstanding examples of systems completely integrable possdant an infinite number of degrees of freedom. That is why they so interest mathematicians.

1.2 History and Problematic

Solitons also concern physicists and they even become indispensable to explain and describe many phenomena, they occur in many areas of mathematical physics, such as classical and quantum field theory, non linear optics, fluid mechanics, plasma physics and in many models in chemistry and biology (see [38, 56, 57, 65, 70, 90]). Probably, the simplest equation which has soliton solutions is the sine-Gordon equation (see below in Chapter 1),

$$-\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial t^2}+\sin\psi=0,$$

where $\psi=\psi(x,t)$ is a scalar field, x,t are real numbers, representing, respectively, the space and the time variable. In 1964, Derrick, in a celebrated paper [40], considers the more realistic three-space-dimension model (see below in Chapter 2),

$$-\Delta\psi+\frac{\partial^2\psi}{\partial t^2}+V'(\psi)=0,$$

Δ being the 3-dimensional Laplace operator and V' is the gradient of a nonnegative C^1 real function V.

In [40] it is proved by a simple rescaling argument that the last equation in three-space-dimension does not possess any nontrivial finite-energy static solutions. Derrick proposed some possible ways out of this difficulty. The first proposal was to consider models which are the Euler-Lagrange equations of the action functional relative to the functional

$$S=\int\int\mathcal{L}dxdt.$$

The Lorentz invariant Lagrangian density proposed in [40] has the form

$$\mathcal{L}(\psi)=-\left(|\nabla\psi|^2-|\psi_t|^2\right)^{\frac{p}{2}},\quad p>3. \tag{1.2.1}$$

However, Derrick does not continue his analysis . He has been unable to demonstrate either the existence or nonexistence of stable solutions.
In this spirit and in recent years, a considerable amount of work has been done by V. Benci and collaborators (see [9–18, 27]), who which be the core of this work.

In this a thesis , we will be concerned about the generalization of some results of V. Benci in generalized Sobolev spaces about variable exponents.

This thesis work is structured into 5 chapters and each chapter contains additional sections. It is arranged as follows:
In **Chapter 1**, we introduce notations, definitions, lemmas and theorems which are used throughout this monograph.
In **Chapter 2**, we present the simplest equation admitting soliton solutions (Sine-Gordon equation). In the second part,we present the more realistic $3+1$ dimensional model admitting soliton solutions given by the nonlinear Klein-Gordon equation.
In **Chapter 3**, we introduce here an existence result for a $n+1$ dimensional model generalizing the one suggested by Derrick in his first proposal (Chapter 4).
In **Chapter 4**, the main purpose is to obtain soliton-like solutions with variable

exponent which generalize the results of Chapter 4.
In **Chapter 5**, the main purpose is to obtain soliton-like solutions with twice variable exponent which generalizing the Results of Chapter 5.

1.3 Motivation

In the mathematical models (soliton) studied in papers [10, 12] the space of the finite energy configurations (solution space) splits into infinitely many connected components according to the topological charge. They proved the existence of infinitely many solutions, which are constrained minima of the energy. More precisely, on every one connected component characterized by an topological charge equal to $n \in \mathbb{N}$ there exists a solution of charge n. Since p is arbitrary in static equation (see equation 4.1.10), so it is natural to considered $p = p(x)$ as a variable that depends on the connected component.
Our aim of this work is to carry out an existence analysis of the finite energy static solutions in more then one space dimension for a class of Lagrangian densities which include (1.2.1) and generalizing the results of Benci in his paper [10]. More precisely we are concerned with Generalized Sobolev Spaces with Variable Exponents.
The following examples give a more concrete notion of processes that can be described by a soliton.

Example 1.3.1. ***Equivalence between mass and energy(the celebrated Einstein equation):*** *([12])*

One of the main features of these soliton solutions is that they behave as relativistic particles. In fact, by using the Nöether theorem, we can introduce the energy $E(\psi)$ and the mass $m(\psi)$ and it can be proved that the celebrated Einstein relation $E(\psi) = m(\psi)c^2$ holds true.

We shall consider Lagrangian densities of the form

$$\mathcal{L}_1(\psi, \rho) = -\frac{1}{2}\alpha(\rho) - V(\psi), \tag{1.3.1}$$

where the function V is a real function defined in an open subset $\Omega \subset \mathbb{R}^m$ and α is a real function defined by

$$\alpha(\rho) = \rho + \frac{\varepsilon}{3}|\rho|^3, \quad \varepsilon > 0; \tag{1.3.2}$$

$$\rho = c^2|\nabla\psi|^2 - |\psi_t|^2.$$

The action functional related to (1.3.1)is

$$\begin{aligned} S_1(\psi) &= \int_{\mathbb{R}^{3+1}} \mathcal{L}_1(\psi,\rho)dxdt \\ &= \int_{\mathbb{R}^{3+1}} -\frac{1}{2}\alpha(\rho) - V(\psi)dxdt. \end{aligned}$$

The Euler-Lagrange equation is

$$\frac{\partial}{\partial t}((1+\varepsilon|\rho|^2)\psi_t) - c^2\nabla((1+\varepsilon|\rho|^2)\nabla\psi) + V'(\psi) = 0. \tag{1.3.3}$$

So the static solutions u solve the equation

$$-c^2\Delta u - c^6\varepsilon\Delta_6 u + V'(u) = 0. \tag{1.3.4}$$

Clearly (1.3.4) are the Euler-Lagrange equations with respect to the energy functional

$$E(u) = \int_{\mathbb{R}^3} \left(\frac{c^2}{2}|\nabla u|^2 + \varepsilon\frac{c^6}{6}|\nabla u|^6 + V(u)\right) dx. \tag{1.3.5}$$

Equation (1.3.3) probably is the simplest Lorentz invariant equation which has static solitons. Nevertheless, these solitons have some interesting properties since they behave as relativistic bodies, namely:

- *they experience a relativisic contraction in the direction of the motion;*
- *the rest mass is a scalar and not a tensor;*
- *the mass equals the energy;*
- *the mass increases with the velocity by the factor γ.*

Our Lagrangian (1.3.1) is Lorentz invariant, thus it is reasonable to expect at least some of these features. However, it is somewhat surprising that they can be deduced from a single equation without extra assumptions. Moreover, this equation might be interpreted as the equation of an "elastic medium" in a Newtonian space-time. Thus, this model shows, from a purely formal point of view, that the main features of the special relativity can be deduced from a partial differential equation in a Newtonian space-time.

Example 1.3.2. ***Solitons and the electromagnetic field:****([11])*
In the example 1.3.1, there has been introduced a Lorentz invariant equation in three space dimensions, having soliton like solutions, the equation introduced is the Euler Lagrange equation of an action functional

$$S_1(\psi) = \int_{t_0}^{t_1}\int_{\mathbb{R}^3} \mathcal{L}_1 dxdt.$$

The soliton solutions behave as relativistic particles. Moreover a topological invariant is associated to these solitons. If we interpret this invariant as the electric charge, it is natural to analyze the interaction between the soliton and the electromagnetic field and to try to construct a simple Lorentz invariant model for the electromagnetic theory namely a model describing particle-like matter interacting with the electromagnetic field through (deterministic) differential equations defined in a Newtonian space-time.

In the following, (A, ϕ) *will denote the gauge potentials associated to the electromagnetic field* $(E; H)$ *by the relations*

$$E = -(A_t + \nabla\phi) \tag{1.3.6}$$

$$H = \nabla \times A. \tag{1.3.7}$$

We need to define the Lagrangian density $\mathcal{L}_2$ *of the electromagnetic field and the Lagrangian density* $\mathcal{L}_3$ *describing the interaction between theme and the electromagnetic field.*

$$\begin{aligned}
\mathcal{L}_2 &= \frac{1}{8\pi}(|E|^2 - |H|^2) \\
&= \frac{1}{8\pi}\left(|A_t + \nabla\phi|^2 - |\nabla \times A|^2\right) \\
\mathcal{L}_3 &= (J(\psi, \nabla\psi, \psi_t) \mid A) - \varrho(\psi, \nabla\psi)\phi
\end{aligned}$$

For the definition of the dependence of electric current $J(\psi, \nabla\psi, \psi_t)$ *and the electric density* $\varrho(\psi, \nabla\psi)$ *on and its derivatives see ([11], Subsect. 1.2).*

The total action will be

$$\begin{aligned}
S &= S(\psi, A, \phi) \\
&= S_1(\psi) + S_2(A, \phi) + S_3(\psi, A, \phi)
\end{aligned}$$

with

$$S_i(\psi) = \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \mathcal{L}_i dxdt.$$

The model we introduce permits us to describe the interaction of a relativistic particle with an electromagnetic field by using only concepts of classical field theory. In this work we confine ourselves to analyze some mathematical questions related to the existence of solutions for this model. More precisely we prove the existence of static solutions (with non trivial charge) of the Euler-Lagrange equations

$$dS = 0 \tag{1.3.8}$$

namely solutions (u, A, ϕ) $(u = (u_1, \ldots, u_4), A = (A_1, A_2, A_3))$ *which traveling do not depend on* $t \in \mathbb{R}$*. Let us point out that these solutions give rise to travelling solutions* (ψ, A_v, ϕ_v) $(A_v = (A_{v,1}, A_{v,2}, A_{v,3}))$ *with velocity* $(v, 0, 0)$ *where*

$$\psi(x,t) = u\left(\frac{x_1 - vt}{\sqrt{1-(\frac{v}{c})^2}}, x_2, x_3\right)$$

$$\begin{aligned}
A_{v,1}(x,t) &= \frac{A_1\left(\frac{x_1-vt}{\sqrt{1-(\frac{v}{c})^2}}, x_2, x_3\right) - \frac{v}{c}\phi\left(\frac{x_1-vt}{\sqrt{1-(\frac{v}{c})^2}}, x_2, x_3\right)}{\sqrt{1-(\frac{v}{c})^2}} \\
A_{v,2}(x,t) &= A_2\left(\frac{x_1 - vt}{\sqrt{1-(\frac{v}{c})^2}}, x_2, x_3\right) \\
A_{v,3}(x,t) &= A_3\left(\frac{x_1 - vt}{\sqrt{1-(\frac{v}{c})^2}}, x_2, x_3\right)
\end{aligned}$$

$$\phi_v(x,t) = \frac{\phi\left(\frac{x_1-vt}{\sqrt{1-(\frac{v}{c})^2}}, x_2, x_3\right) - \frac{v}{c}A_1\left(\frac{x_1-vt}{\sqrt{1-(\frac{v}{c})^2}}, x_2, x_3\right)}{\sqrt{1-(\frac{v}{c})^2}}$$

ψ is a traveling soliton "surrounded" by the electromagnetic field (A_v, ϕ_v).

The aim of this work is to prove the existence of static solutions of the Euler-Lagrange equations relative to the action functional.

$$S = S(\psi, A, \phi).$$

First we take the variation with respect to A Therefore we get $dS[\delta A] = 0$ if and only if

$$\nabla \times (\nabla \times A) = 4\pi J(\psi, \nabla\psi, \psi_t) - \frac{\partial}{\partial t}(A_t + \nabla\phi). \tag{1.3.9}$$

Second we take the variation with respect to ϕ Therefore we get $dS[\delta\phi] = 0$ if and only if

$$-\nabla(A_t + \nabla\phi) = 4\pi\varrho(\psi, \nabla\psi). \tag{1.3.10}$$

By (1.3.6) and (1.3.7), we get

$$\nabla \times H = 4\pi J(\psi, \nabla\psi, \psi_t) + E_t \tag{1.3.11}$$

$$\nabla . E = 4\pi\varrho(\psi, \nabla\psi) \tag{1.3.12}$$

which complete the Maxwell equations (1.3.6) and (1.3.7).

Now, if we want to take the variation with respect to the $j-th$ component of ψ, we notice that it has a complicated form. Anyway we can write the equation

$$dS[\delta\psi] = 0$$

in the following form:

$$\Box\psi^j - \varepsilon\Box_6\psi^j + \frac{\partial V}{\partial \xi_j}(\psi) = F_j \tag{1.3.13}$$

where the left hand side derives from the variation of the action S_1 describing the matter field. The right hand side F_j of (1.3.13), which derives from the interaction term S_3, depends on (and its first and second derivatives) and on A and ϕ (and their first derivatives). such that

$$\Box_6\psi = \frac{\partial}{\partial t}[(c^2|\nabla\psi|^2 - |\psi_t|^2)^2\psi_t)] - c^2\nabla[(c^2|\nabla\psi|^2 - |\psi_t|^2)^2\nabla\psi]$$

$$\Box\psi = \psi_{tt} - c^2\Delta\psi.$$

We confine ourselves to search static solutions, namely fields ψ, A, ϕ which do not depend on t. We get immediately

$$J(\psi, \nabla\psi, \psi_t) = 0$$

then, (1.3.9), (1.3.10) and (1.3.13) give respectively

$$\Delta\phi = 4\pi\varrho(\psi, \nabla\psi) \tag{1.3.14}$$

$$\nabla \times (\nabla \times A) = 0 \tag{1.3.15}$$

$$\Delta\psi^j - \varepsilon\Delta_6\psi^j + \frac{\partial V}{\partial \xi_j}(\psi) = F_j \tag{1.3.16}$$

where G_j depends on ψ (and its first and second derivatives) and ϕ (and its first derivatives). Clearly $A = 0$ (as well as $A = \nabla h$) solves (1.3.15), so the unknowns of our problem are (ψ, ϕ). In particular, since our field ψ does not depend on t, from now on, we rename it u. Finally we can state our main result.

Theorem 1.3.1. *There exist two fields*

$$u : \mathbb{R}^3 \to \mathbb{R}^4$$

$$\phi : \mathbb{R}^3 \to \mathbb{R}$$

such that $ch(u) = 0$ and $(u, 0, \phi)$ is a (weak) static solution of the Euler-Lagrange equation (1.3.8).

Example 1.3.3. ***Soliton as a model for the dislocations in crystal***

In 1939,J. Frenkel and T. Kontrova [61] introduced the SG equation (see Chapter 3.1) as a model for the dislocations in a crystal. The displacement $\phi(x, t)$ of atoms connected by linear springs may propagate as a kink in the periodic *crystal field. Around 1960, J.K. Perring and Tony Skyrme [76] considered the SG equation, which is relativistic invariant, as a model for elementary particles (more rigorously, baryons). They examined collisions of kink-kink and kink-antikink and confirmed the particle-like stability of kinks (Historically, A. Seeger, H. Donth and A. Kochendörfer [78] found kink-kink solutions and kink-antikink solutions in the study of the SG equation as a dislocation model).*

Example 1.3.4. ***Soliton in the field of nonlinear optics.***
In 1967, S.L. Mc Call and E.L. Hahn [35] discovered an interesting phenomenon in the field of nonlinear optics. Coherent light propagating in the system of 2-level atoms obeys the SG equation when the spectral widths are neglected (perfect resonance). The observed soliton-like behavior is called self-induced transparency (SIT). The 2π-pulse is the soliton and 0π-pulse is the breather. In the other limit, that is, the interaction between the media and the light wave is not resonant, the envelope of the electric field is described by the NLS equation. Further extension of research along this line is the eletromagnetically induced transparency (EIT) where two coherent lights propagate in the system of 3-level atoms. EIT and soliton propagations have attracted much attention [86].

Now we illustrate some mathematical models with variable exponent. The equations to nonlinear partial differential equations involving the operator $p(x)$-Laplacian are modeling many physical phenomena such as elasticity nonlinearity, the electrorheological fluids (the interaction between fluids and EMF) and thermorheological, image restoration and propagation through porous medium.

Example 1.3.5. ***Image restoration*** *[31]: Image restoration is the adjustment of image, mesh or more generally of discrete data, by variational methods. These have been successfully applied to solve problems in different intervening computer vision, computer graphics or further data analysis. The aim of the regulation is to provide an approximation of the actual data from the observed data that suffered a deterioration from the environment (noise) or methods acquisition as quantification and discretization. We confine ourselves to the model [30] proposed by Blomgren, Chan, Mulet, Wong in 2000,*

$$min \int_{\Omega} |\nabla u|^{p(\nabla u)}$$

where $\lim_{s\to 0} p(s) = 2$, $\lim_{s\to\infty} p(s) = 1$, *and p is monotonically decreasing. An image, ($u : \Omega \to \mathbb{R}^n$), is recovered from an observed, noisy image, $\Omega \subset \mathbb{R}^2$ being the domain of the image.*

Example 1.3.6. ***Electrorheological fluids*** *[37]: is composed of fine particles dispersed in a dielectric liquid. Under the action of an electric field, the particles are attracted to form fibers connecting the electrodes parallel to the direction of* *the electric field giving the following equations given by Rajagopal an Ružička in 2001 (see [72]),*

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial E}{\partial x_i}, curl E = 0 \\ \\ \frac{\partial v}{\partial t} - \sum_{i=1}^{n} \frac{\partial S}{\partial x_i}(x, E, E(v)) + v|\nabla v| + \nabla \pi = g(x, E) \\ \\ \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} = 0 \end{cases}$$

where E is the electromagnetic fields. $v : \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity of fields. $\pi$the pressure, (v) is the symmetric part of the gradient, S is a tensor and its expression

$$S(x, E, z) = v(E)(1 + \|z\|^2)^{\frac{p-2}{2}} z, \quad \forall z \in \mathbb{R}^3,$$

$p = (\|E\|^2)$, *E depending on x.*

Chapter 2

Preliminaries

In this chapter, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this thesis.

$\Omega \subset \mathbb{R}^n$: open set in $\mathbb{R}^n, n \in \mathbb{N}^*$.

$x \in \mathbb{R}^n$: $x = (x_1, x_2, \ldots, x_n)$.

$\rightharpoonup$: weakly converges.

$\sigma(E, E^*)$: weak topology on E.

$\rightarrow$: strongly converges.

$$h' = \text{grad h} = \left(\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, \ldots, \frac{\partial h}{\partial x_n}\right).$$

$$L^p(\Omega) = \left\{h : \Omega \longrightarrow \mathbb{R}; u \text{ is measurable and } \int_\Omega |h(x)^p dx < +\infty\right\},$$
$1 \leq p < \infty$.

$L^\infty(\Omega) = \{h : \Omega \longrightarrow \mathbb{R}; h$ is measurable and $|h(x)| \leq c$ a.e. in Ω for some constant c$\}$.

$$\|h\|_{L^p} = \left(\int_\Omega |h(x)|^p dx\right)^{1/p}.$$

$\|h\|_{L^\infty} = \inf\{c\,; |u(x)| \leq c \quad a.e.\ on\ \Omega\}$.

$W^{1,p}(\Omega) = \{h \in L^p(\Omega); h' \in (L^p(\Omega))^n\}$.

$W_0^{1,p}(\Omega)$: the closure of $C_0^\infty(\Omega)\ in\ W^{1,p}(\Omega)$.

$\|h\|_{W^{1,p}} = \|h\|_{L^p} + \|h'\|_{L^p}$.

$C_0^k(\Omega)$: space of k times continuously differentiable functions with compact support in Ω.

$C^\infty(\Omega)$: functions which are continuously differentiable arbitrarily many times.

$C_0^\infty(\Omega)$: space of C^∞ continuous functions with compact support in Ω (some authors write $\mathcal{D}(\Omega)$ or $C_c^\infty(\Omega)$ instead of $C_0^\infty(\Omega)$.

$$
\begin{aligned}
\psi_t^j &= \frac{\partial \psi^j}{\partial t}. \\
\psi_i^j &= \frac{\partial \psi^j}{\partial x_i}. \\
u &= (u_0, \tilde{u}) \in \mathbb{R} \times \mathbb{R}^n. \\
\nabla u &: \text{denoting, the Jacobian.} \\
\|u\|_a &= a\|\nabla u\|_{L^2} + \|\nabla u\|_{L^p} + \|u\|_{L^2}, \ \ a > 0. \\
E_a = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_a} &: \text{the completion of } C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}) \text{ with the norm } \|\cdot\|_a\ . \\
\Delta_p u &= \nabla\big(|\nabla u|^{p-2}\nabla u\big). \\
\nabla(|\nabla u|^{p-2}\nabla u) &: \text{denotes the vector whose } j-th \text{ component is given by} \\
& \quad\ div(|\nabla u|^{p-2}\nabla u^j). \\
\triangle u &: \text{denotes the vector whose } j-th \text{ component is given by} \\
& \quad\ div(\nabla u^j). \\
\Lambda_a &= \left\{u \in E_a : u(x) \neq \eta, \text{ for all } x \in \mathbb{R}^n\right\}. \\
\partial\Lambda_a &= \left\{u \in E_a : \text{there exist } x \in \mathbb{R}^n \text{ such that } u(x) = \eta\right\}. \\
deg(h, \Omega, b) &= \sum_{x \in h^{-1}(b)\cap\Omega} sgn J_h(x), \\
J_h &: \text{denoting the determinate of the Jacobian matrix .} \\
ch(u) &:= \begin{cases} deg(\tilde{u}, K_u, 0) & \text{if } K_u \neq \emptyset, \\ & \\ 0 & \text{if } K_u = \emptyset. \end{cases} \\
{\Lambda^*}_q &= \{u \in \Lambda_a : ch(u) \neq 0\}. \\
\Lambda_a &= \bigcup_{q \in \mathbb{Z}} \Lambda_q. \\
O(n) &: \text{denotes the symmetry group of rotations and translations.} \\
\mathcal{P}(\Omega) := \mathcal{P}(\Omega, \mu) &: \text{the set of all } \ \mu\text{-measurable functions } \ p : \Omega \to [1, \infty]. \\
C_+(\mathbb{R}^n) &= \left\{p \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) : \ p(x) > 1 \ \text{ for all } x \in \mathbb{R}^n\right\}. \\
p^+ &= ess \sup_{x \in \mathbb{R}^n} p(x). \\
p^- &= ess \inf_{x \in \mathbb{R}^n} p(x). \\
L^{p(\cdot)}(\Omega) &= \left\{h : \Omega \longrightarrow \mathbb{R}; h \text{ is measurable and } \int_\Omega |h(x)|^{p(x)} dx < +\infty\right\}. \\
\rho_{p(\cdot)}(h) &= \int_\Omega |h(x)|^{p(x)} dx. \\
\|h\|_{p(\cdot)} &= \inf\left\{\sigma > 0 : \mathbb{R}^n \to \mathbb{R} : \int_{\mathbb{R}^n} \left|\frac{h(x)}{\sigma}\right|^{p(x)} dx \leq 1\right\}.
\end{aligned}
$$

$$
\begin{array}{rcl}
W^{1,p(\cdot)}(\Omega) & = & \{h \in L^{p(\cdot)}(\Omega); h' \in (L^{p(\cdot)}(\Omega))^N\}. \\
H_0^{1,p(\cdot)}(\Omega) & : & \text{the closure of } C_0^\infty(\Omega) \textit{ in } W^{1,p(\cdot)}(\Omega). \\
\|u\|_{a,p} & = & a\|\nabla u\|_{L^2} + \|\nabla u\|_{L^{p(\cdot)}} + \|u\|_{L^2}, \ \ a > 0. \\
E_{a,p} = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_{a,p}} & : & \text{The completion of } C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}), \\
 & & \text{with the norm} \| \cdot \|_{a,p}. \\
\Gamma_a & = & \{u \in E_{a,p} : u(x) \neq \eta, \text{ for all } x \in \mathbb{R}^n\}. \\
\partial\Gamma_a & = & \{u \in E_{a,p} : \text{there exist } x \in \mathbb{R}^n \text{ such that } u(x) = \eta\}. \\
\mathcal{L}(\mathbb{R}^n) & : & \text{The set of linear applications.}
\end{array}
$$

2.1 Classical Sobolev spaces

For all information of this section we see [28]. Let $\Omega \subset \mathbb{R}^N$ be an open set.

Definition 2.1.1. *Let $p \in \mathbb{R}$ with $1 \leq p < \infty$; we set*

$$L^p(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R}; u \textit{ is measurable and } \int_\Omega |u|^p dx < +\infty \right\},$$

with the norm

$$\|u\|_{L^p} = \|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{1/p}.$$

Definition 2.1.2. *We set*

$$L^\infty(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} : \begin{array}{l} u \textit{ is measurable and there is a constant } c \\ \textit{such that } |u(x)| \leq c \ \ \textit{a.e. on } \Omega \end{array} \right\}$$

with the norm

$$\|u\|_{L^\infty} = \inf\{c \, ; |u(x)| \leq c \quad \textit{a.e. on } \Omega\}.$$

- The space L^p is a separable Banach space; it is reflexive if $1 < p < \infty$ and the dual of L^p is isomorphic to the space $L^{p'}$ with

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

- The space L^1 is a separable Banach space; it is never reflexive and the dual of L^1 is isomorphic to the space L^∞.

- The space L^∞ is Banach space, it is not separable, it is not reflexive and the dual of L^∞ is strictly bigger than L^1.

Definition 2.1.3. *For $1 \leq p < \infty$ the Sobolev space $W^{1,p}(\Omega)$is defined by*

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \ \nabla u \in (L^p(\Omega))^n\}$$

equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|\nabla u\|_{L^p}.$$

we set

$$W^{1,2}(\Omega) = H^1(\Omega).$$

Remark 2.1.1.

$$\nabla u = gradu = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N}\right)$$

such that $\frac{\partial u}{\partial x_i}$ is derived in the sense of distribution, i.e.,

$$\exists \ \ g_i \in L^p(\Omega), \quad \int_\Omega u \frac{\partial \phi}{\partial x_i} = \int_\Omega g_i \phi \ \ \forall \phi \in C_0^\infty(\Omega).$$

Proposition 2.1.1. *([28], Proposition 8.1, page 203) The space $W^{1,p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$. It is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$. The space H^1 is a separable Hilbert space.*

Definition 2.1.4. *Let $1 \leq p < \infty$; $W_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.*
Set

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

The space $W_0^{1,p}$ equipped with the $W^{1,p}$ norm, is a separable Banach space; it is reflexive if $1 < p < \infty$. H_0^1 equipped with the H^1 scalar product, is a Hilbert space.

Corollary 2.1.2. *([28], Corollary 4.23,page 109) Let $\Omega \subset \mathbb{R}^N$ be an open set. Then $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for any*

$$1 \leq p < \infty.$$

Theorem 2.1.3. *([28], Theorem 9.2, page 265) Let $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$ then there exists a sequence (u_n) from $C_0^\infty(\mathbb{R}^N)$ such that*

$$u_{n/\Omega} \rightarrow u \ in \ L^p(\Omega)$$

and

$$\nabla u_{n/\omega} \rightarrow \nabla u_{/\omega} \ in \ (L^p(\Omega))^N \ \ for \ all \ \ \omega \subset\subset \Omega.$$

In case $\Omega = \mathbb{R}^N$and $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$. Then there exists a sequence (u_n) from $C_0^\infty(\mathbb{R}^N)$ such that

$$u_{n/\Omega} \rightarrow u \ in \ L^p(\mathbb{R}^N)$$

and

$$\nabla u_n \rightarrow \nabla u \ in \ (L^p(\mathbb{R}^N))^N.$$

$C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$.

Remark 2.1.2. *Since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, we have*

$$W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N).$$

Corollary 2.1.4. *([28],Corollary 9.8 (density), page 277) Assume that Ω is of class C^1, and let $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$. Then there exists a sequence (u_n) from $C_0^\infty(\mathbb{R}^N)$ such that $u_{n/\Omega} \to u$ in $W^{1,p}(\Omega)$. In other words, the restrictions to Ω of $C_0^\infty(\mathbb{R}^N)$ functions form a dense subspace of $W^{1,p}(\Omega)$.*

Theorem 2.1.5. *([28],Theorem 9.12, page 282) Let $p > N$. Then*

$$W^{1,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$$

with continuous injection. Furthermore, for all $u \in W^{1,p}(\mathbb{R}^N)$, we have

$$|u(x) - u(y)| \leq C^\alpha |x-y| \|\nabla u\|_{L^p} \quad a.e. \quad x, y \quad \in \mathbb{R}^N,$$

where $\alpha = 1 - (N/p)$ and C is a constant (depending only on p and N).

Furthermore $\lim\limits_{|x| \to +\infty} u(x) = 0$.

Corollary 2.1.6. *([28], Corollary 9.13, page 283) Let $m \geq 1$ be an integer and let $p \in [1, +\infty)$.*
We have

$$W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), where \ \ \frac{1}{q} = \frac{1}{p} - \frac{m}{N}, \ if \ \ \frac{1}{p} - \frac{m}{N} > 0$$

$$W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \forall q \in [p, +\infty), \ if \ \ \frac{1}{p} - \frac{m}{N} = 0$$

$$W^{m,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N), \qquad if \ \ \frac{1}{p} - \frac{m}{N} < 0$$

and all these injections are continuous.

Theorem 2.1.7. *([28], Theorem, page 285) Suppose that Ω is bounded and of class C^1. Then we have the following compact injections:*

$$W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [1, p^\star), where \ \ \frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N}, \ if \ \ p < N,$$

$$W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, \infty), \qquad if \ \ p = N,$$

$$W^{1,p}(\Omega) \subset C(\overline{\Omega}), \qquad if \ \ p = N.$$

In particular,$W^{1,p}(\Omega) \subset L^p(\Omega)$ with compact injection for all p (and all N).

Theorem 2.1.8. *([28], Theorem, page 278)*
Let $1 \leq p < N$. Then

$$W^{1,p}(\mathbb{R}^N) \subset L^{P^*}(\mathbb{R}^N), where \ \ p^* is\ given\ by \ \ \frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N}$$

and there exists a constant $c = c(p, N)$ such that

$$\|u\|_{L^{p^\star}} \leq c\|\nabla u\|_{L^p} \qquad \forall u \in W^{1,p}(\mathbb{R}^N).$$

2.2 Lebegue and Sobolev generalized spaces

Variable exponent Lebesgue spaces appeared in the literature for the first time in a 1931 article by Orlicz [75]. In that article the following question is considered: let (p_i) (with $p_i > 1$) and (x_i) be sequences of real numbers such that $\sum x^{p_i}$ converges. What are the necessary and sufficient conditions on (y_i) for $\sum_i x_i y_i$ to converge. It turns out that the answer is that $\sum(\lambda y_i)^{q_i}$ should converge for some $\lambda > 0$ and $p = \frac{q_i}{(p_i-1)}$. This is essentially Hölder's inequality in the space $\ell^{p()}$. Orlicz also was interested in the study of function spaces that contain all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega (\phi(\lambda|u(x)|)dx,$$

for some $\lambda > 0$ and ϕ. Satisfying some natural assumptions, where Ω is an open set in $\mathbb{R}^N$. This space is denoted by $L^{\phi(.)}$ and it is now called Orlicz space.

However, we point out that in [75] the case $|u(x)|^{p(x)}$ corresponding to variable exponents was not included. In the 1950's these problems were systematically studied by Nakano [68], who developed the theory of modular function spaces. Nakano explicitly mentioned variable exponent Lebesgue spaces as an example of more general spaces he considered, see Nakano ([68], p. 284). Later, Polish mathematicians investigated the modular function spaces, see Musielak [66]. Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. In that context, we refer to the work of Tsenov [89] and Sharapudinov [88].

In 1991, Kovacik and Rakosnik [60] established several basic properties of spaces $L^{p(x)}$ and $W^{1,p(x)}$ with variable exponents. Their results were extended by Fan and Zhao [54] in the framework of Sobolev spaces $W^{m,p(x)}$. Pioneering regularity results for functionals with nonstandard growth are due to Acerbi and Mingione [4]. Density of smooth functions in $W^{k,p(x)}(\Omega)$ and related Sobolev embedding properties are due to Edmunds and Rakosnik [51, 52].

The variable Lebesgue spaces, as their name implies, are a generalization of the classical Lebesgue spaces, replacing the constant exponent p with a variable exponent function $p(.)$. The resulting Banach function spaces $L^{p(.)}$ have many properties similar to the L^p spaces, but they also differ in surprising and subtle ways. For this reason the variable Lebesgue spaces have an intrinsic interest, but they are also very important for their applications to partial differential equations and variational integrals with non-standard growth conditions.

2.3 Lebesgue generalized space

In this section we shall introduce generalized Lebesgue space and state some of their basic properties.

Definition 2.3.1. *Let $(\mathcal{A}, \sum, \mu)$ be a σ-finite, complete measure space. We define $\mathcal{P}(A, \mu)$ to be the set of all μ-measurable functions $p : A \to [1, \infty]$. Functions $p \in \mathcal{P}(A, \mu)$ are called variable exponents on A.*

In the special case that μ is the n-dimensional Lebesgue measure and Ω is an open subset of $\mathbb{R}^n$, we abbreviate $\mathcal{P}(\Omega) := \mathcal{P}(\Omega, \mu)$. For $p \in L^\infty(\mathbb{R}^n)$, with $1 < p^-$,

$$L^{p(x)}(\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \to \mathbb{R} : \int_{\mathbb{R}^n} |u(x)|^{p(x)} dx < \infty \right\},$$

which is a Banach space when furnished with the Luxemburg norm

$$|u|_{p(x)} = \inf \left\{ \sigma > 0 : \mathbb{R}^n \to \mathbb{R} : \int_{\mathbb{R}^n} \left| \frac{u(x)}{\sigma} \right|^{p(x)} dx \leq 1 \right\}.$$

For all $u \in L^{p(x)}(\mathbb{R}^n)$, the relation between modular and Luxemburg norm is clarified by following proposition.

Proposition 2.3.1. *Let $u \in L^{p(x)}(\mathbb{R}^n)$ and (u_m) be a sequence in $L^{p(x)}(\mathbb{R}^n)$, then*

(1) $|u|_{p(x)} < 1 (= 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1 (= 1, > 1)$.

(2) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} < \rho_{p(x)}(u) < |u|_{p(x)}^{p^+}$.

(3) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} < \rho_{p(x)}(u) < |u|_{p(x)}^{p^-}$.

(4) $|u - u_m|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u - u_m) \to 0$.

(5) $|u|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u) \to 0$, $|u|_{p(x)} \to \infty \Leftrightarrow \rho_{p(x)}(u) \to \infty$.

Remark 2.3.1. *While this more technical definition is necessary when $p(.)$ is unbounded, we can simplify it when $p_+ < \infty$.*

Denoting by $q(x)$ the conjugate exponent of $p(x)$, namely the function satisfying

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1 \text{ pointwise in } \mathbb{R},$$

let $\Omega \subset \mathbb{R}^n$ be a measurable subset and $meas\ \Omega > 0$. The following Hölder-type inequality holds for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$

$$\int_\Omega uvdx \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)}|v|_{q(x)} \leq 2|u|_{p(x)}|v|_{q(x)}.$$

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_{\mathbb{R}^n} |u(x)|^{p(x)}dx,$$

where the $\rho_{p(x)}$ are convex and continuous modular, (see [41, Theorem 3.4.1 (p. 87) and Theorem 3.4.9 (p. 89)]) i.e. $\rho_{p(.)}$ verifies the following properties

- $\rho_{p(.)}(u) = 0 \Leftrightarrow u = 0;$
- $\rho_{p(.)}(u) = \rho_{p(.)}(-u),$
- $\rho_{p(.)}(\alpha u + \beta v) \leq \alpha\rho_{p(.)}(u) + \beta\rho_{p(.)}(v),\ \forall u, v \in E,\ \forall \alpha, \beta \geq 0,\ \alpha + \beta = 1,$ where
$$E = \{u : \Omega \to \mathbb{R} :\ u \text{ is a measurable function in } \Omega\}.$$

Theorem 2.3.2. *Let $p \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$. Then the set $C(\Omega) \cap L^{p(x)}(\Omega)$ is dense in $L^{p(x)}(\Omega)$. If, moreover, Ω is open, then the set $C(\Omega)$ is dense in $L^{p(x)}(\Omega)$.*

Proposition 2.3.3. *Given Ω and $p(.) \in \mathcal{P}(\Omega)$, if $p_+ < \infty$, then $f \in L^{p(\cdot)}(\Omega)$ if and only if*

$$\rho_{p(.)}(u) < \infty.$$

Lemma 2.3.4. *([41, Lemma 3.2.12 (page 78)]) Let $s \in \mathcal{P}(A, \mu)$ then*

$$\frac{1}{2}\min\left\{\mu(A)^{\frac{1}{s^+}}, \mu(A)^{\frac{1}{s^-}}\right\} \leq \|1\|_{L^{s(\cdot)}(A,\mu)} \leq 2\max\left\{\mu(A)^{\frac{1}{s^+}}, \mu(A)^{\frac{1}{s^-}}\right\}.$$

2.4 Sobolev space with variable exponent

In this section we define the variable exponent Sobolev space by

$$W^{1,p(x)}(\mathbb{R}^n) = \left\{u \in L^{p(x)}(\mathbb{R}^n) : \nabla u \in L^{p(x)}(\mathbb{R}^n)\right\},$$

which is a Banach space equipped with the norm

$$|u|_{1,p} = |u|_{p(x)} + |\nabla u|_{p(x)} \text{ and it is reflexive, } 1 < p^- \leq p^+ < \infty.$$

Definition 2.4.1. *([41, Definition 11.2.1 (page 346)]) Let $p \in \mathcal{P}(\Omega)$ and $k \in \mathbb{N}$. The space $H_0^{k,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$.*

Theorem 2.4.1. *Let $p \in \mathcal{P}(\Omega)$. Then $H_0^{k,p(\cdot)}(\Omega)$ is Banach space. If $p(\cdot)$ is bounded, $H_0^{k,p(.)}(\Omega)$ is separable and if $1 < p^- \leq p^+ < \infty$, then it is reflexive and uniformly convex.*

Definition 2.4.2. *([41, Definition 4.1.1 (page 100)]) We say that a function $\alpha : \Omega \to \mathbb{R}$ is locally log-Hölder Continuous on Ω if there exists $c_1 > 0$ such that*

$$|\alpha(x) - \alpha(y)| < \frac{c_1}{log\,|e + 1/|y - x||}$$

for all $x, y \in \Omega$. We say that α satisfies the log if there exist an $\alpha_\infty \in \mathbb{R}$ and a constant $c_2 > 0$ such that

$$|\alpha(x) - \alpha_\infty| < \frac{c_2}{log\,|e + |x||}$$

for all $x \in \Omega$. We say that α is globally log-Hölder Continuous in Ω if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition.

Definition 2.4.3. *([41, Definition 4.1.4]) We define the following class of variable exponents*

$$\mathcal{P}^{log}(\Omega) := \{p \in \mathcal{P}(\Omega) : \frac{1}{p} \ \ is\ globally\ log - Hölder\ Continuous\}$$

Remark 2.4.1. *([41, Remark 4.1.5(page 101)] If $p \in \mathcal{P}(\Omega)$ with $p^+ < \infty$, then $p \in \mathcal{P}^{log}(\Omega)$ if and only if p is globally Hölder continuous.*

Lemma 2.4.2. *([41, Lemma 4.1.6]) Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ be continuous and bounded, i.e., $-\infty < \alpha^- \leq \alpha+ < +\infty$. The flowing conditions are equivalent:*

(a) α is locally log-Hölder Continuous.

(b) for all balls we have $|B|^{\alpha_B^- - \alpha_B^+} \leq c$.

(c) for all $x \in B$ we have $|B|^{\alpha_B^- - \alpha(x)} \leq c$.

(d) for all $x \in B$ we have $|B|^{\alpha(x) - \alpha_B^+} \leq c$.
Instead of balls it is also possible to use cubes.

Density results for generalized Lebesgue and Sobolev spaces.

Corollary 2.4.3. *Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ be a bounded exponent. Then the set $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$.*

Theorem 2.4.4. *Letp $\in \mathcal{P}(\Omega)$ be a bounded exponent. If $p \in E$ or $p \in \mathcal{P}^{log}(\Omega)$, $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$.*

2.5 Convergence theorems

The next Theorems collect analogues of the classical Lebesgue integral convergence results.

Theorem 2.5.1. *Given Ω and $p(.) \in \mathcal{P}(\Omega)$, let $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ be a sequence of non-negative functions such that f_k increases to a function f pointwise a.e. Then either $f \in L^{p(\cdot)}(\Omega)$ and $\|f_k\|_{1,p(.)} \to \|f\|_{1,p}$, or $f \notin L^{p(\cdot)}(\Omega)$ and $\|f_k\|_{1,p(.)} \to \infty$.*

The next result is the analog of Fatou's Lemma.. It is proved in [64].

Lemma 2.5.2. *(Fatou's lemma). Given Ω and $p(.) \in \mathcal{P}(\Omega)$, suppose the sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ be a sequence $f_k f$ such that pointwise a.e. If*

$$f(x) = \liminf_{k\to\infty} f_k(x) < +\infty.$$

Then $f \in L^{p(\cdot)}(\Omega)$ and

$$\int_\Omega f(x)dx \leq \liminf_{k\to\infty} \int_\Omega f_k(x)dx.$$

Theorem 2.5.3. *(dominated convergence theorem, Lebesgue). Given Ω and $p(.) \in \mathcal{P}(\Omega)$, suppose the sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ be a sequence $f_k \to f$ such that pointwise a.e. and there is a function $g \in L^{p(\cdot)}(\Omega, \mathbb{R}_+)$ such that for all $k, |f_k(x)| \leq g(x)$ a.e. on Ω. Then $f \in L^{p(\cdot)}$ and*

$$\|f_k - f\|_{L^{p(.)}} \to 0 \quad \text{as } k \to \infty.$$

The final convergence result shows that norm convergence yields pointwise convergence on subsequences. The proof depends on showing that norm convergence implies convergence in measure; see [64] for details.

Theorem 2.5.4. *Given Ω and $p(.) \in \mathcal{P}(\Omega)$, suppose the sequence $f_k \to f$ in norm in $L^{p(\cdot)}(\Omega)$, then there exists a subsequence $\{f_{k_j}\}$ that converges pointwise a.e. to f.*

2.6 Lower and upper semicontinuous

Definition 2.6.1. *Let $f : X \to [-\infty, \infty]$ be a function. We define the following set*

$$dom(f) = \{x \in X | f(x) < \infty\}.$$

We say that the function f is proper if $dom(f) \neq \emptyset$.

Definition 2.6.2. *Let X be a topological space. A function $f : X \to [-\infty, \infty]$ is called lower(upper) semicontinuous at a point $x_0 \in X$, abbreviated l.s.c.(u.s.c.) if for each $a \in \mathbb{R}$ with $f(x_0) > a(f(x_0) < a)$ there exists a neighbourhood U of x_0 such that the following implication holds true*

$$x \in U \Rightarrow f(x) > a(x \in U \Rightarrow f(x) < a).$$

We say that f is l.s.c.(u.s.c.) on a set $M \subseteq X$ if it is such at each point of the set M.

The next proposition gives the characterization of local semicontinuity of functions.

Proposition 2.6.1. *[58] Let X be a Hausdorff space, let $f : X \to [-\infty, \infty]$, and $x_0 \in X$. Then*

(i) f is l.s.c at x_0, if for each net $(x_\alpha)_{\alpha \in I} \to x_0$ in X then

$$\liminf_{i \in I} f(x_0) \geq f(x_0);$$

(ii) f is u.s.c at x_0 , whenever $x_\alpha \to x_0$ in X then

$$\limsup_{i \in} f(x_0) \leq f(x_0),$$

(iii) f is l.s.c. if and only if $-f$ is u.s.c.

The following result is a closely related characterization of the local semicontinuity

Proposition 2.6.2. *For any function $f : X \to [-\infty, \infty]$ and $x_0 \in X$ the following assertions are true:*

(i) f is l.s.c at x_0 if and only if $\liminf_{x \to x_0} f(x) = f(x_0)$,

(ii) f is u.s.c at x_0 if and only if $\limsup_{x \to x_0} f(x) = f(x_0)$.

The classical Weierstrass theorem states that a continuous function defined on a compact set achieves its minimum and its maximum on that set. The following refinement is a fundamental tool in proving the existence of solutions to minimization problems.

Theorem 2.6.3. *(Weierstrass) [58] Let X be a Hausdorff space, let $f : X \to [-\infty, \infty]$ be l.s.c., and let C be a compact subset of X. Suppose that $C \cap dom f \neq \emptyset$. Then f achieves its infimum over C.*

2.7 Ekeland's variational principle

The variational principle which provides an approximate minimizer of a bounded below and lower semicontinuous function in a given neighborhood of a point, was discovered introduced by Ekeland [48] in 1972 (see also, [49, 50]. It is known as Ekeland's variational principle (in short,*E.V.P.*). In 1981, Sullivan [82] established the validity of the E.V.P. statement on a metric space (X, d) is equivalent to the completeness of the metric space (X, d). In 1982, McLinden [67] showed how E.V.P., or more precisely the augmented form of it provided by Rockafellor [74], can be adapted to extremum problems of minimax type. In this section, we present several forms of Ekeland's variational principle. From Weirstrass theorem 2.6.3, for each $C \subset X$ compact, and if f is lower semicontinuous, then the following constrained optimization problem.

$$\inf_{x \in C} f(x), \tag{2.7.1}$$

has a solution. Note only that, the solution set of (2.7.1) is compact. Now the question is "Can we achieve the infimum of the following optimization problem

$$\inf_{x \in X} f(x), \tag{2.7.2}$$

or of (2.7.2) without the compactness assumption? The answer is "yes". But we need some kind of coercivity assumption as well as convexity structure on C. But we can always obtain an approximately ϵ-solution, that is, a point x_ϵ for $\epsilon > 0$ satisfying

$$\inf_{x \in X} f(x) \leq \inf f(x_\epsilon) \leq \inf_{x \in X} f(x) + \epsilon.$$

The Ekeland's variational principle guarantees the existence of such an ϵ-solution where neither compactness nor convexity on the underlying space is needed.

Theorem 2.7.1. *(Strong Form of Ekeland's Variational Principle) [49] Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper, bounded below and l.s.c. functional. Let $\epsilon > 0$ and $x_* \in X$ be given such that*

$$f(x_*) \leq \inf_{x \in x} f(x) + \epsilon.$$

Then for a given $\lambda > 0$, there exists $\bar{x} \in X$ such that

(a) $f(\bar{x}) \leq f(x_*)$

(b) $d(x_*, \bar{x}) \leq \lambda$

(c) $f(\bar{x}) < f(x) + \frac{\epsilon}{\lambda} d(x, \bar{x})$ *for all* $x \in X \backslash \{\bar{x}\}$.

Aubin and Frankowska [7] established the following form of Ekeland's variational principle which is equivalent to Theorem 2.7.1.

Theorem 2.7.2. *Let (X, d) be a complete metric space and f a lower semi-continuous map from X to $\mathbb{R}$. We assume that f is lower bounded and we set $c := \inf_{x\in X} f(x)$. Then for all $\epsilon > 0$; there exist u_ϵ such that*

$$\begin{cases} \quad c \leq f(u_\epsilon) \leq c + \epsilon, \\ \forall x \in X, x \neq u_\epsilon \quad f(x) - f(u_\epsilon) + \epsilon d(x, u_\epsilon) > 0. \end{cases}$$

We now present, the so called, weak formulation of Ekeland's variational principle.

Corollary 2.7.3. *(Weak form of Ekeland's variational principle) [5] Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ be proper, bounded below and l.s.c. Then for any given $\epsilon > 0$, there exists $\bar{x} \in X$ such that*

$$f(\bar{x}) \leq \inf_{x\in x} f(x) + \epsilon$$

and

$$f(\bar{x}) < f(x) + \epsilon d(x, x_*) \text{ for all } x \in X \backslash \{\bar{x}\}.$$

The property of Ekeland's variational principle for proper but, extended real-valued lower semicontinuous and bounded below functions, on a metric space characterizes compactness of the metric space.

Theorem 2.7.4. *(Converse of E.V.P.) [5] A metric space (X, d) is complete if for every functional $f : X \to \mathbb{R} \cup \{\infty\}$ which is a proper, bounded below and l.s.c. then for any given $\epsilon > 0$, there exists $\bar{x} \in X$ such that*

$$f(\bar{x}) \leq \inf_{x\in x} f(x) + \epsilon$$

and

$$f(\bar{x}) < f(x) + \epsilon d(x, x_*) \text{ for all } x \in X \backslash \{\bar{x}\}.$$

2.8 Topological Degree

For more information on this topic see [43] and [69] .

Now, we give the construction of Brouwer degree in this section as follows:

Definition 2.8.1. *([43],Definition 1.2.1,page 4).*

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and $f \in C^1(\overline{\Omega})$. If $b \notin f(\partial\Omega)$ and $J_f(b) \neq 0$, then we define

$$deg(f, \Omega, b) = \sum_{x\in f^{-1}(b)} sgn J_f(x),$$

where $\quad deg(f, \Omega, b) = 0 \quad$ *if* $\quad f^{-1}(b) = \emptyset.$

The definition of the degree can be extended to functions that are only continuous and also to non-regular values;

- Let $f \in C(\overline{\Omega})$, then there exists a sequence $f_k \in C^1(\overline{\Omega})$ such that
$$\|f_k - f\|_\infty := \sup_{x\in\overline{\Omega}} \|f_k(x) - f(x)\| \to 0 \ \ on \ \ \overline{\Omega},$$
and we can show that
$$deg(f, \Omega, b) = \lim_{k\to\infty} deg(f_k, \Omega, b).$$

- Let $f \in C(\overline{\Omega})$ and $b \notin f(\partial\Omega)$ not necessarily a regular value.
Then there is sequence b_k, (regular values of f)such that $b_k \to b$
and we can show that
$$deg(f, \Omega, b) = \lim_{k\to\infty} deg(f, \Omega, b_k),$$
and the limit is independent of the sequence b_k.

Theorem 2.8.1. *([43],Theorem 1.2.6,page 7) Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset and $f : \overline{\Omega} \longrightarrow \mathbb{R}^N$ be a continuous mapping. If $b \notin f(\partial\Omega)$, then there exists an integer $deg(f, \Omega, b)$ satisfying the following properties:*

(1) *(Normality) $deg(I, \Omega, b) = 1$ if and only if $b \in \Omega$, where I denotes the identity mapping;*

(2) *(Solvability) $deg(f, \Omega, b) \neq 0$ then $f(x) = b$ has a solution in Ω;*

(3) *(Additivity) . Suppose that $\Omega_1, \Omega_2$are two disjoint open subsets of Ω and $b \notin f(\overline{\Omega} - \Omega_1 \cup \Omega_2)$. Then $deg(f, \Omega, b) = deg(f, \Omega_1, b) + deg(f, \Omega_2, b)$;*

(4) *(Homotopy) $If \ \ f_t(x) : [0\ 1]\times\overline{\Omega} \longrightarrow \mathbb{R}^N$ is continuous and $b \notin \cup_{t\in[0\ \ 1]} f_t(\partial\Omega)$ then $deg(f_t, \Omega, b)$ does not depend on $t \in [0, 1]$;*

(5) *$deg(f, \Omega, b)$is a constant on any connected component of $\mathbb{R}^N \setminus f(\partial\Omega)$.*

Another properties include (see [69]):

(6) (excision) let $A \subset \overline{\Omega}$ an compact set and $b \notin f(A)$ then
$$deg(f, \Omega, b) = deg(f, \Omega/A, b).$$

(7) (stability):topological degree with respect to the uniform convergence
Let $f, g \in C(\overline{\Omega})$, $b \notin f(\partial\Omega) \cup g(\partial\Omega)$.
$$If \ \ \|g - f\|_\infty \leq \frac{1}{4} d\left(b, f(\partial\Omega) \cup g(\partial\Omega)\right) \ then \ deg(f, \Omega, b) = deg(g, \Omega, b).$$

(8) If $f = g$ *on* $\partial\Omega$ *then* $deg(f, \Omega, b) = deg(g, \Omega, b)$.

Chapter 3

Solitons in one and three dimensions of spaces

3.1 Solitons in one dimension of space: Sine-Gordon equation (SG)

In this section, we consider the $1+1$ dimensional sine-Gordon equation

$$\psi_{tt} - \psi_{xx} + \sin\psi = 0, \tag{3.1.1}$$

where $\psi = \psi(x,t)$ is a scalar field, x, t are real numbers.

This is probably the simplest equation admitting soliton solutions and can be seen as a pattern for our study, to see more clearly soliton's properties and characteristics. Its name was coined by J. Rubinstein [73] as a pun on "Klein-Gordon" and it arises in the study of surfaces with constant negative Gaussian curvature in differential geometry and also in many physical applications, such as two-dimensional models of elementary particles, stability of fluid motions, propagation of crystal dislocations (see [1, 2, 8, 29, 33, 38, 42, 59, 63, 70, 76] and [81] for exhaustive discussions and references).

(3.1.1) is the Euler-Lagrange equation of the action functional

$$S(\psi) = \int_{\mathbb{R}\times\mathbb{R}} \left[\tfrac{1}{2}(\psi_t^2 - \psi_x^2) - V(\psi)\right] dx\, dt, \tag{3.1.2}$$

where we can choose $V(\psi) = 1 - \cos\psi$ so to obtain $V \geq 0$, and then the related energy functional is

$$E(\psi) = \frac{1}{2}\int_{\mathbb{R}} \psi_t^2\, dx + \frac{1}{2}\int_{\mathbb{R}} \psi_x^2\, dx + \frac{1}{2}\int_{\mathbb{R}} V(\psi)\, dx. \tag{3.1.3}$$

Note that the potential V has a discrete infinite set of degenerate minima $2\pi\mathbb{Z}$, where it vanishes. Obviously $\psi(x) = k\pi$ is a trivial solution of (3.1.2) for every

$k \in \mathbb{Z}$, but of course we are interested in nontrivial solutions. In particular, we will concern ourselves with nonsingular finite-energy solutions (of which solitary waves are special cases).

So, let ψ be a classical solution with $E(\psi) < \infty$. By this we mean of course that all the integrals in (3.1.3) are finite , and this implies $1 - \cos\psi(.,t) \in H^1(\mathbb{R})$ for all fixed t, being $[1 - \cos\psi(.,t)]^2 \leq 2[1 - \cos\psi(.,t)] \in L^1$ and $|\frac{d}{dx}\cos\psi(.,t)| \leq |\psi_x(.,t)| \in L^2$.

Hence $\lim_{x\to\pm\infty}[1 - \cos\psi(x,t)] = 0$ and from this we deduce that every configuration of u satisfies the following asymptotic conditions

$$\psi(\pm\infty, t) = \lim_{x\to\pm\infty} \psi(x,t) \in 2\pi\mathbb{Z}. \tag{3.1.4}$$

Moreover, if we assume that $\psi_t \in L^\infty(\mathbb{R}^2)$ (which is necessarily the case of solitary waves) then the functions of the variable t defined by the left-hand side of (3.1.4), being continuous and discrete-valued, must be constant

$$\psi(\pm\infty, t) \equiv \psi(\pm\infty) \in 2\pi\mathbb{Z}. \tag{3.1.5}$$

i.e., ψ preserves its asymptotic values as t varies. These facts suggest to consider the sets

$$H_{(k_1,k_2)} = \{f \in C^2(\mathbb{R};\mathbb{R}) \mid \lim_{x\to+\infty} f(x,t) = 2k_1\pi, \lim_{x\to-\infty} f(x,t) = 2k_2\pi\}$$

with $k_1, k_2 \in \mathbb{Z}$, and the topological space

$$H = \bigcup_{(k_1,k_2)\in\mathbb{Z}^2} H_{(k_1,k_2)} \subset L^\infty(\mathbb{R}).$$

It is easy to see that $(k_1,k_2) \neq (h_1,h_2) \implies H_{(k_1,k_2)} \cap H_{(k_1,k_2)} = \emptyset$ is an open path-connected subset of H, called a sector. The property (3.1.5) implies that, for a solution ψ, the function $x \mapsto \psi(x,t)$ (which we call a configuration of ψ) stays always in the same connected component $H_{(k_1,k_2)}$ as time evolves. This fact allows a topological classification of the finite-energy nonsingular solutions to the equation (3.1.1) satisfying (3.1.5), each bearing thereby the pair of indices (k_1,k_2). Moreover, consistent with the invariance of the equation (3.1.1) (and also of the action (3.1.2) under the change $\psi \mapsto \psi + 2\pi k$, we can fix k_1 and relate to such solutions a single integer index given by the difference $k_2 - k_1$, namely

$$Q(\psi) := \frac{1}{2\pi}\int_{\mathbb{R}} \psi_x(x,t)\, dx.$$

$Q(\psi)$ defines a topological index, called topological charge. Note that Q is essentially a boundary condition which is constant in time because of the finiteness of energy, in contrast with the other more familiar conserved quantities (see Subsection 1.2.1 in [9]) coming from the symmetries of the action functional. Now, we turn to the particular case of static (i.e. t -independent) nonsingular finite-energy solutions. They solve the equation

$$-u^{''} + \sin u = 0, \quad u : \mathbb{R} \to \mathbb{R} \tag{3.1.6}$$

which can be interpreted as a conservative system (or, by a mechanical analogy, as the equation of motion for a unit-mass point particle). The mechanical energy

$$E_M = \frac{1}{2}(u^{'})^2 - V(u)$$

(in the analogy, kinetic energy plus potential energy) is constant with respect to x and must equal zero. Indeed $u(\pm\infty) \in 2\pi\mathbb{Z}$ implies $E_M = \lim_{x\to\pm\infty}(u^{'})^2/2$, so $E_M < \infty$ implies $E_M = 0$. Hence in the phase plane we get the zero-energy orbits, that is, the solutions u for which

- $\forall x \in \mathbb{R}\ u^{'}(x) = \pm sin\frac{u(x)}{2}$
- $\exists k \in \mathbb{Z}\ \forall x \in \mathbb{R},\ 2\pi k < u(x) < 2\pi(k+1)$
- u is monotone and either $\lim_{x\to-\infty} u(x) = 2k\pi$ and $\lim_{x\to+\infty} u(x) = 2(k+1)\pi$ or $\lim_{x\to-\infty} u(x) = 2(k+1)\pi$ and $\lim_{x\to+\infty} u(x) = 2k\pi$.

This implies that $Q(u) = \pm 1$ for these solutions.
Finally, upon integration,

$$x - x_0 = \pm \int_{x_0}^{x} \frac{du(x)}{u^{'}(x)}$$

and

$$(u^{-1}(u(x)))^{'} = u^{-1^{'}}(u).u^{'}(x) = 1,$$

then

$$x - x_0 = \pm \int_{u(x_0)}^{u(x)} u^{-1^{'}}(u)du \ = \pm u^{-1}(u)|_{u(x_0)}^{u(x)}.$$

So we have $x - x_0 = \pm \int_{u(x_0)}^{u(x)} \frac{du}{2\sin(u/2)} \ = \pm \int_{u(x_0)}^{u(x)} \frac{d(\tan(u/4))}{\tan(u/4)} \ = \pm \ln \frac{\tan[u(x)/4]}{\tan[u(x_0)/4]},$

$$\forall x \in \mathbb{R}.$$

Using again the invariance $u \mapsto u + 2\pi k$, we impose $u(x_0) = \pi$ and we get the explicit solutions

$$u_K(x) = 4 \arctan e^{x-x_0} \;\; and \;\; u_A(x) = -4 \arctan e^{x-x_0} \qquad (3.1.7)$$

which are the so-called kink and antikink, respectively, and carry $Q(u_K) = 1$ and $Q(u_A) = -1$. Note that the translational invariance of (3.1.6) is reflected by the fact that a different choice of the arbitrary constant x_0 only brings the solution to shift in space.

Given on the space-time $\mathbb{R} \times \mathbb{R}$ a nonlinear equation with associated energy functional (see Section 1.2 in [9])

$$E(\psi) = \int_{\mathbb{R}} \varepsilon_\psi(x,t)$$

we call solitary wave any nonsingular solution whose energy density has a space-time dependence of the form

$$\varepsilon_\psi(x,t) = \tilde{\varepsilon}_\psi(x - vt) \qquad (3.1.8)$$

where $\tilde{\varepsilon}_\psi$ is a localized function and v is velocity in the direction of the motion. The energy density of both kink and antikink is given by the localized function

$$\tilde{\varepsilon}(x) = \frac{16e^{2(x-x_0)}}{[1 + e^{2(x-x_0)}]^2}$$

and hence they are static solitary waves (i.e. corresponding to $v = 0$ in (3.1.8)).

By the Lorentz invariance of (3.1.2), traveling solitary waves can be trivially obtained on Lorentz-transforming (3.1.7) and their energy density turns out to be

$$\varepsilon(x,t) = \gamma^2 \tilde{\varepsilon}(\gamma[x - vt])$$

which represents a single bump traveling undistorted with uniform velocity.

3.2 Solitons in three dimensions of space : Derricś Problem

In 1963, attempting to find a model for extended elementary particles in contrast with point particles, U. Enz [44] was led to study an equation like (3.1.1). He proved the existence of nonsingular time-independent solutions with energy density localized about a point on the x axis, and under a further request of stability, he found that the energy is bound to assume only certain discrete values, which can be seen as corresponding to the rest energies of elementary particles. Moving beyond this work, G.H. Derrick proposed, in a celebrated paper [40], the more realistic 3+1 dimensional model given by the nonlinear Klein-Gordon equation

$$\Box\psi + W^{'}(\psi) = 0, \tag{3.2.1}$$

where

$$\Box\psi = -\Delta\psi + \frac{\partial^2\psi}{\partial t^2}$$

(Δ being the 3-dimensional Laplace operator) and $W^{'}$ is the gradient of a non-negative C^1 real function W.

Owing to the relativistic invariance of (3.2.1), moving waves can be trivially obtained from static solutions by boosting, i.e., turning to a moving coordinate frame by applying a Lorentz transformation. Thus, we are led to concern ourselves with finite-energy static solutions (of which solitary waves are a particular case) $u = u(x), x \in \mathbb{R}^3$ with

$$E(u) = \int_{\mathbb{R}^N} [\frac{1}{2}|\nabla u|^2 + W(u)]\, dx < \infty \tag{3.2.2}$$

where $N = 3$, solve the equation

$$\Delta u + W^{'}(u) = 0$$

which is the also the Euler-Lagrange equation of the energy functional (3.2.2).

In [40] Derrick showed that, if the potential W is nonnegative, any finite-energy static solution of (3.2.1) is necessarily trivial, namely it takes a constant value which is a minimum point of W. On the other hand, if the nonnegativity of W is not required, no stable finite-energy static solution is permitted to the equation (3.2.1). In fact, the following theorem holds.

Theorem 3.2.1. *Let $N \geq 3$. The energy functional (3.2.2) has no nontrivial local minima, i.e.,*

$$\delta^2 E(u) \geq 0 \textit{ with } u \textit{ non constant } \Rightarrow \delta E(u) \neq 0.$$

Moreover, if $W > 0$ then E does not have any nontrivial critical point at all, namely

$$\delta E(u) = 0 \Rightarrow u \equiv u_0 \, with \, W(u_0) = 0.$$

Proof. Using Derrick's simple rescaling argument, we set $u_\lambda(x) := u(\lambda x)$ and

$$E(u) = \frac{1}{2\lambda^{N-2}} \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{1}{\lambda^N} \int_{\mathbb{R}^N} W(u)\, dx := \frac{1}{\lambda^{N-2}} I_1 + \frac{1}{\lambda^N} I_2.$$

If $\delta E(u)h = 0$ for any variation h, we have in particular

$$\frac{d}{d\lambda} E(u_\lambda)\bigg|_{\lambda=1} = \frac{2-N}{2} I_1 + N I_2 = 0, \tag{3.2.3}$$

and therefore

$$\frac{d^2}{d\lambda^2}E(u_\lambda)\Big|_{\lambda=1} = \frac{(2-N)(1-N)}{2}I_1 + N(N+1)I_2 = (2-N)I_1.$$

Hence the second variation of E at any nonconstant critical point u is negative for a variation corresponding to a uniform stretching of u. Finally, if $W > 0$ then both I_1 and I_2 are nonnegative and from (3.2.3) we deduce $I_1 = I_2 = 0$.

□

Remark 3.2.1. *According to Enźs results as well as to our previous discussion on the equation (3.1.1), the above argument is not applicable to the 1+1 dimensional case: if* $N = 1$ *we obtain* $E(u_\lambda)/ = \lambda I_1/2 + I_2/\lambda$ *yielding on differentiation* $I_1 = 2I_2$*, which gives no contraddiction.*

On the other hand, if we consider non-positive potential, we are forced to seek saddle points, instead of minima, and for these static solutions we have lack of stability. As an example we recall that, if we take

$$W(\xi) = \frac{1}{2}\xi^2 - \frac{1}{4}\xi^4$$

critical points of the energy functional

$$E(u) = \int_{\mathbb{R}^3} [\frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - \frac{1}{4}u^4]\, dx$$

have been found in [26] and [83] and for more general potentials in [25, 79]; but in [3] and [24] it has been proved that these static solutions are not stable.

In [40], these facts led Derrick to say :"We are thus faced with the disconcerting fact that no equation of type "(3.2.1) " has any time-independent solutions which could reasonably be interpreted as elementary particles."

Derrick proposed some possible ways out of this difficulty. The first proposal was to consider models which are the Euler-Lagrange equations of the action functional relative to the functional

$$S = \int\int \mathcal{L} dx dt.$$

The Lorentz invariant Lagrangian density proposed in [40] has the form

$$\mathcal{L}(\psi) = -\big(|\nabla\psi|^2 - |\psi_t|^2\big)^{\frac{p}{2}}. \tag{3.2.4}$$

For $p = 2$, the Euler Lagrange equations reduce to (3.2.1). For every integer $p > 3$, the nonexistence proof in [40] for the finite energy static solutions fails. However, Derrick does not continue his analysis and he concludes that a Lagrangian density of type (3.2.4) leads to a very complicated differential equation.

He has been unable to demonstrate either the existence or nonexistence of stable solutions.

In this spirit, a considerable amount of work has been done by V. Benci and collaborators and a model equation proposed in [12] will be the topic of the next Chapter.

Chapter 4

Solitons in several space dimensions

We introduce here an existence result for a $3+1$ dimensional model generalizing the one suggested by Derrick in his first proposal. A first existence result is stated in [12], which also gives a topological classification of static solutions by means of a topological invariant: the topological charge. In order to prove the existence of static solutions with nontrivial charge, a study of the behaviour of sequences of bounded energy is needed, in the spirit of the concentration-compactness principle. A further generalization is carried out in [10], which develops an existence analysis of the finite-energy static solutions in higher spatial dimension and for a larger class of Lorentz invariant Lagrangian densities.

4.1 Statement of the problem

The class of Lagrangian densities we consider generalizes the problem studied in [12], in such a way as to include the Derrick proposal. First we introduce some notation. If n, m are positive integers, $\mathbb{R}^{n+1}$ and $\mathbb{R}^m$ will denote respectively the physical space-time (typically $n = 3$) and the internal parameters space .We are interested in the multidimensional case, so we assume that

$$n \geq 2.$$

A point in $\mathbb{R}^{n+1}$ will be denoted by (x, t), where $x \in \mathbb{R}^n$ denotes the space variable and $t \in \mathbb{R}$ denotes the time variable. The fields we are interested in are maps $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^m$, $\psi = (\psi_1, \ldots, \psi_m)$. We set

$$\rho = |\nabla \psi|^2 - |\psi_t|^2,$$

$\nabla\psi$ and ψ_t denoting, respectively, the Jacobian with respect to x and the derivative with respect to t, such that

$$|\nabla\psi|^2 = \sum_{1\le j\le m, 1\le i\le n} |\psi_i^j|^2,$$

$$|\psi_t|^2 = \sum_{1\le j\le m} |\psi_t^j|^2.$$

We shall consider Lagrangian densities of the form

$$\mathcal{L}(\psi,\rho) = -\frac{1}{2}\alpha(\rho) - V(\psi), \tag{4.1.1}$$

where the function V is a real function defined in an open subset $\Omega \subset \mathbb{R}^m$ and α is a real function defined by

$$\alpha(\rho) = a\rho + b|\rho|^{\frac{p}{2}},\ p > n, \tag{4.1.2}$$

where $a \ge 0$ and $b > 0$.

The results of [12] were concerned with the case: $a = 1, n = 3$ and $p = 6$. If $a = 0$ and $n = 3$, (4.1.1) is equivalent to the Lagrangian density (3.2.4) proposed by Derrick in [40], when we look for static solutions.

The action functional related to (4.1.1) is

$$\begin{aligned} S(\psi) &= \int_{\mathbb{R}^{n+1}} \mathcal{L}(\psi,\rho)\,dxdt \\ &= \int_{\mathbb{R}^{n+1}} -\frac{1}{2}\alpha(\rho) - V(\psi)\,dxdt. \end{aligned}$$

So the Euler-Lagrange equations are (system of m scalar equations in $n+1$ dimension) see [32]

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\psi_t^j}\right) + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{L}}{\partial\psi_i^j}\right) - \frac{\partial\mathcal{L}}{\partial\psi^j} = 0 \tag{4.1.3}$$

where ($\psi_i^j = \frac{\partial\psi^j}{\partial x_i}$), $1 \le i \le n$ and $1 \le j \le m$

$$\frac{\partial\mathcal{L}}{\partial\psi^j} = -\frac{1}{2}\alpha'(\rho)\underbrace{\frac{\partial\rho}{\partial\psi^j}}_{=0} - \frac{\partial V}{\partial\xi_j}(\psi) \tag{4.1.4}$$

$$\frac{\partial\mathcal{L}}{\partial\psi_i^j} = -\frac{1}{2}\alpha'(\rho)\frac{\partial\rho}{\partial\psi_i^j} = -\alpha'(\rho)\psi_i^j \tag{4.1.5}$$

$$\frac{\partial\mathcal{L}}{\partial\psi_t^j} = -\frac{1}{2}\alpha'(\rho)\frac{\partial\rho}{\partial\psi_t^j} = \alpha'(\rho)\psi_t^j. \tag{4.1.6}$$

Substituting (4.1.4),(4.1.5) and (4.1.6) into (4.1.3), we get

$$\frac{\partial}{\partial t}(\alpha'(\rho)\psi_t^j) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\alpha'(\rho)\psi_i^j) + \frac{\partial V}{\partial \xi_j}(\psi) = 0,\ 1 \leq i \leq n, \quad 1 \leq j \leq m. \quad (4.1.7)$$

So we have

$$\frac{\partial}{\partial t}(\alpha'(\rho)\psi_t^j) - div(\alpha'(\rho)\nabla\psi^j) + \frac{\partial V}{\partial \xi_j}(\psi) = 0,$$

$$1 \leq j \leq m.$$

Then we have

$$\frac{\partial}{\partial t}(\alpha'(\rho)\psi_t) - \nabla(\alpha'(\rho)\nabla\psi) + V'(\psi) = 0 \ \ in\, \mathbb{R}^m \quad (4.1.8)$$

where $\nabla(\alpha'(\rho)\nabla\psi)$ denotes the vector whose $j-th$ component is given by

$$div(\alpha'(\rho)\nabla\psi^j)$$

and V' denotes the gradient of V

Remark 4.1.1. *Lorentz transformations, i.e. space-time rotations depending on one parameter v have the form*

$$\begin{cases} x_1 \mapsto \gamma(x_1 - vt) \\ x_2 \mapsto x_2 \\ x_3 \mapsto x_3 \\ t \mapsto \gamma(t - \frac{v}{c^2}x_1) \end{cases}$$

where $\gamma = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}$, $|v| < c$ and c is a constant (dimensionally a velocity), for sake of simplicity we assume $c = 1$.

The equation(4.1.8) is Lorentz invariant (see below in subsection 4.1.1). The static solutions $\psi(x,t) = u(x)$ of (4.1.8) solve the equation

$$-\nabla(\alpha'(\rho)\nabla u) + V'(u) = 0. \quad (4.1.9)$$

Using (4.1.2), (4.1.9) becomes

$$-a\Delta u - b\frac{p}{2}\Delta_p + V'(u) = 0, \quad (4.1.10)$$

where

$$\Delta_p u = \nabla\big(|\nabla u|^{p-2}\nabla u\big).$$

It is easy to verify that, if $u = u(x)$ is a solution of the (4.1.10) and $\mathrm{v} = (v, 0, 0, \ldots, 0)$ with $|\mathrm{v}| < 1$, the field

$$\psi_{\mathrm{v}}(x,t) = u(\gamma(x_1 - vt), x_2, \ldots, x_n) \tag{4.1.11}$$

is a solution of (4.1.8) (see below in subsection 4.1.2). Notice that the function ψ_{v} experiences a contraction by a factor

$$\gamma = \frac{1}{\sqrt{1 - v^2}}$$

in the direction of the motion; this is a consequence of the fact that (4.1.8) is Lorentz invariant. Clearly (4.1.10) are the Euler-Lagrange equations with respect to the energy functional

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^2 + \frac{b}{2}|\nabla u|^p + V(u) \right) dx, \tag{4.1.12}$$

where $m = n + 1$, so the time independent fields u are maps

$$u : \mathbb{R}^n \to \mathbb{R}^m.$$

For every $\xi \in \mathbb{R}^{n+1}$, we write

$$\xi = (\xi_0, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^n.$$

As to the function V, we assume that it is defined on

$$V : \Omega \to \mathbb{R}$$

where $\Omega = \mathbb{R}^{n+1} \setminus \{\eta\}$, $\eta = (1, 0)$, and V is positive and singular in η. More precisely we assume:

$(V1)$ $V \in C^1(\Omega, \mathbb{R})$.

$(V2)$ $V(\xi) \geq V(0) = 0$.

$(V3)$ V is twice differentiable in 0 and the Hessian matrix $V''(0)$ is nondegenerate.

$(V4)$ There exist $c, \rho > 0$ such that if $|\xi| < \rho$ then

$$V(\eta + \xi) \geq c|\xi|^{-q}, \tag{4.1.13}$$

where

$$\frac{1}{q} = \frac{1}{n} - \frac{1}{p}.$$

$(V5)$ For every $\xi \in \Omega \backslash \{0\}$ we have

$$V(\xi) > 0, \text{ and } \quad \lim_{|\xi| \to \infty} \inf V(\xi) = v > 0.$$

Taking $a = 1$ we observe that, for $j = 1, \ldots, n+1$,

$\alpha(0) = a = 1$ and, since 0 is a minimum for V , we can choose a base in $\mathbb{R}^{n+1}$ which diagonalizes $V''(0)$ so that

$$\begin{pmatrix} m_1^2 & & 0 \\ & \ddots & \\ 0 & & m_{n+1}^2 \end{pmatrix}$$

such that

$$V'(\xi) = V''(0)\xi + \circ(\xi) \simeq V''(0)\xi$$

in a neighborhood of 0.

Then, linearizing (4.1.8) at 0 and taking $a = 1$, we get a system of Klein-Gordon equations

$$\Box \psi^j + m_j^2 \psi^j = 0,$$
$$1 \leq j \leq n+1,$$

where m_j^2 denote the eigenvalues of $V''(0)$ and $\Box \psi = \frac{\partial \psi}{\partial t^2} - \triangle \psi$.

Example 4.1.1. *A potential satisfying the assumptions* $(V_1) - (V_5)$ *is*

$$V(\xi) = \omega^2 \Big(|\xi|^2 + \frac{|\xi|^4}{|\xi - \eta|^q} \Big),$$

where $q = np/(p-n)$.

Definition 4.1.1. *We call* soliton *a solution of equation (4.1.8) having the form of equation (4.1.11), where u is a local minimum of the energy functional (4.1.12).*

4.1.1 Lorentz invariant field equations

For simplicity we take $n = 1$ and $m = 1$ so equation (4.1.8) in $\mathbb{R}^2$ becomes

$$\frac{\partial}{\partial t}(\alpha'(\rho)\psi_t) - \frac{\partial}{\partial x}(\alpha'(\rho)\psi_x) + V'(\psi) = 0 \tag{4.1.14}$$

where

$$\frac{\partial}{\partial t}(\alpha'(\rho)\psi_t) = \alpha''(\rho)\rho_t\psi_t + \alpha'(\rho)\psi_{tt},$$

$$\frac{\partial}{\partial x}(\alpha'(\rho)\psi_x) = \alpha''(\rho)\rho_x\psi_x + \alpha'(\rho)\psi_{xx}.$$

Then the equation (4.1.14) becomes

$$\alpha''(\rho)\underbrace{(\rho_t\psi_t - \rho_x\psi_x)}_{A} + \alpha'(\rho)\underbrace{(\psi_{tt} - \psi_{xx})}_{B} + V'(\psi) = 0.$$

To prove the equation (4.1.14) is Lorentz invariant, it is sufficient to prove that the parts A and B are invariant

$$\text{Lorentz transformations:} \begin{cases} X = \gamma(x - vt) \\ T = \gamma(t - vx) \end{cases}$$

$$\psi(X, T) = \psi(\gamma(x - vt), \gamma(t - vx)).$$

First we show that the part B is Lorentz invariant we have

$$\psi_x = \gamma\psi_X - v\gamma\psi_T,$$

$$\psi_t = -v\gamma\psi_X + \gamma\psi_T.$$

Then we have

$$\psi_{xx} = (\gamma\psi_X - v\gamma\psi_T)_x = \gamma(\psi_X)_x - v\gamma(\psi_T)_x,$$

$$\psi_{tt} = (-v\gamma\psi_X + \gamma\psi_T)_t = -v\gamma(\psi_X)_t + \gamma(\psi_T)_t,$$

which implies

$$\psi_{xx} = \gamma^2\psi_{XX} + (v\gamma)^2\psi_{TT} - 2v\gamma^2\psi_{TX},$$
$$\psi_{tt} = (v\gamma)^2\psi_{XX} + \gamma^2\psi_{TT} - 2v\gamma^2\psi_{TX}.$$

So,

$$\psi_{tt} - \psi_{xx} = \psi_{TT} - \psi_{XX}.$$

The proof of the part A invariant follows from the same arguments used in the proof of B invariant.

4.1.2 Static solution and stability

For simplicity and with no loss of generality we take $n = 1$ and $m = 1$ so equation (4.1.10) in $\mathbb{R}^2$ becomes

$$-au_{xx} - b\frac{p}{2}\frac{\partial}{\partial x}(|u_x|^{p-2}u_x) + V'(u) = 0. \qquad (4.1.15)$$

Let u be a solution of equation (4.1.15).
We show that $\psi(x, t) = u(\gamma(x_1 - vt))$ is a solution of (4.1.14).

By easy calculation we have

$$\begin{cases} \rho = |\psi_x|^2 - |\psi_t|^2 = {u_x}^2 & \\ \psi_x = \gamma u_x, & \psi_t = (\gamma v) u_x \\ \psi_{xx} = \gamma^2 u_{xx}, & \psi_{tt} = (\gamma v)^2 u_{xx} \\ \rho_x = \gamma ({u_x}^2)_x, & \rho_t = (\gamma v)({u_x}^2)_x. \end{cases} \tag{4.1.16}$$

Recall that $\gamma = \frac{1}{\sqrt{1-v^2}}$.

We have

$$\frac{\partial}{\partial t}(\alpha'(\rho)\psi_t) - \frac{\partial}{\partial x}(\alpha'(\rho)\psi_x) = \alpha''(\rho)(\rho_t\psi_t - \rho_x\psi_x) + \alpha'(\rho)(\psi_{tt} - \psi_{xx}),$$

and using (4.1.16) we get

$$\begin{aligned} \frac{\partial}{\partial t}(\alpha'(\rho)\psi_t) - \frac{\partial}{\partial x}(\alpha'(\rho)\psi_x) &= -\alpha''({u_x}^2)(({u_x}^2)_x u_x) - \alpha'({u_x}^2)(u_{xx}) \\ &= -\frac{\partial}{\partial x}(\alpha'({u_x}^2)u_x) \\ &= -\frac{\partial}{\partial x}(au_x + b\frac{p}{2}|u_x|^{P-2}u_x). \end{aligned}$$

So the conclusion follows and $\psi(x,t) = u(\gamma(x_1 - vt))$ is a solution of (4.1.14).

4.2 Functional setting

Let $p > n \geq 2$, and with no loss of generality, we can consider the functional (4.1.12) with $b = 1$. It will be convenient to introduce the following notation:

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^2 + \frac{1}{2}|\nabla u|^p + V(u) \right) dx,$$

and we define the space E_a to be the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|_a = a\|\nabla u\|_{L^2} + \|\nabla u\|_{L^p} + \|u\|_{L^2}, \quad p > n \geq 2, \ a \geq 0,$$

i.e.,

$$E_a = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_a};$$

$$\|u\|_{L^2} = \left(\sum_{j=1}^{n+1} \|u_j\|_{L^2}^2 \right)^{\frac{1}{2}},$$

$$\|\nabla u\|_{L^2} = \left(\sum_{j=1}^{n+1} \|\nabla u_j\|_{L^2}^2 \right)^{\frac{1}{2}},$$

and

$$\|\nabla u\|_{L^p} = \left(\sum_{j=1}^{n+1} \|\nabla u_j\|_{L^p}^p \right)^{\frac{1}{p}}.$$

For every $a > 0$, the norms $\|\cdot\|_a$ are equivalent, so we have to study only two cases: $a = 0,\ a > 0$.

Proposition 4.2.1. *The Banach space E_0 is continuously embedded in $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$, for every $s \in [2, \infty]$.*

Proof. The space E_0 is continuously embedded in $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, therefore it is sufficient to show that E_0 is embedded also in $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Since $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in E_0, and also in L^s (see Corollary 2.1.2). So it is sufficient to prove that there exists $c > 0$ such that, for every $u \in C_0(\mathbb{R}^n, \mathbb{R}^{n+1})$, we have

$$\|u\|_{L^\infty} \leq c\|u\|_0.$$

We fix $u \in C_0(\mathbb{R}^n, \mathbb{R}^{n+1})$ and consider a family of cubes $Q_k \subset \mathbb{R}^n$ such that

$$mes(Q_k) = 1,\ \cup_{k\in\mathbb{N}} Q_k = \mathbb{R}^n.$$

Then, by a well-known inequality (see [28] page 283), for every $k \in \mathbb{N}$ and $Q_k \subset \mathbb{R}^n$,

$$|u(x)| \leq \left| \int_{Q_k} u dy \right| + M\|\nabla u\|_{L^p(Q_k)}, \tag{4.2.1}$$

where $M \geq 0$ being independent of u. Thus

$$\begin{aligned} |u(x)| &\leq mes(Q_k)\|u\|_{L^2} + M\|\nabla u\|_{L^p(Q_k)} \\ &\leq \|u\|_{L^2(\mathbb{R}^n)} + M\|\nabla u\|_{L^p(\mathbb{R}^n)} \\ &\leq (1 + M)\|u\|_0. \end{aligned}$$

Hence

$$\|u\|_{L^\infty} \leq c\|u\|_0, \quad c = 1 + M.$$

□

Corollary 4.2.2. *The Banach space E_0 is continuously embedded in $W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1})$.*

Proof. By definition of the space E_0, we have for every $u \in E_0$

$$\|u\|_0 > \|\nabla u\|_{L^p}.$$

From Proposition 4.2.1 there exists $c_1 > 0$ such that

$$c_1\|u\|_0 > \|u\|_{L^p},$$

and so

$$\|u\|_0 > c\|u\|_{W^{1,p}}.$$

□

Corollary 4.2.3. *For every $a > 0$, the space E_a can be identified with the Banach space*

$$W = W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1}) \cap W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1}),$$

equipped with the usual norm

$$\|u\|_W = \|u\|_{W^{1,2}} + \|u\|_{W^{1,p}}.$$

Proof. $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$ and also in $W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1})$; see Theorem 2.1.8. For any $u \in E_a$ we have

$$\|u\|_a \leq \sup(1, a)\|u\|_W.$$

From Corollary 4.2.2, there exists $c > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$, we have

$$\|u\|_a \geq c(\|u\|_{W^{1,2}} + \|u\|_{W^{1,p(\cdot)}}).$$

□

By Proposition 3.1 and well-known Sobolev embeddings we have the following:

Remark 4.2.1. *Since $p > n$, by the preceding Corollaries and well-known Sobolev embeddings (see Theorem 2.1.5), we get easily some useful properties of the Banach space E_a:*

(1) *We have*

$$E_a \subset W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1}) \subset L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}), \tag{4.2.2}$$

if $\{u_k\}$ converges weakly in E_a to u , then it converges uniformly on every compact set contained in $\mathbb{R}^n$.

(2) *Furthermore they are Hölder continuous of order $(p - n)/p$*

$$|u(x) - u(y)| = C^{(p-n)/p}|x - y|\|\nabla u\|_{L^p}, \tag{4.2.3}$$

i.e.

$$E_a \subset C^{0,(p-n)/p}(\mathbb{R}^n, \mathbb{R}^{n+1})$$

is a locally compact injection.

(3) *For every value $a \geq 0$, the functions in E_a are bounded and decay to zero at infinity,*

$$\lim_{|x|\to\infty} u(x) = 0. \tag{4.2.4}$$

The presence of $\triangle_p$ in (4.1.8) implies that the functions u on which the energy f_a is finite are continuous and decay to 0 at infinity; the presence of the singular term $V'(u)$ implies that such maps u take values in Ω. So the nontrivial topological properties of Ω (namely $\pi_n(\Omega) = \mathbb{Z}$) permit, as in the sine-Gordon equation (Chapter 2), to give a topological classification of the static configurations. This classification is carried out by means of a topological invariant, the topological charge (see Definition 4.3.1), which depends only on the region where the function is concentrated, namely the support. We point out that in other models (see [19, 51, 53, 84]), the topological classification follows from the fact that the field u takes values in suitable manifolds.

Recall that η is a singular point of the potential V, so it is reasonable to consider in space E_a, the open subset

$$\Lambda_a = \{u \in E_a : u(x) \neq \eta, \text{ for all } x \in \mathbb{R}^n\}.$$

In fact, if $u \in \Lambda_a$, by Remark 4.2.1, we have

$$\inf_{x\in\mathbb{R}^n} |u(x) - \eta| = d > 0.$$

Then, by using Proposition 4.2.1 (E_0 is continuously embedded in L^∞), we deduce that there exists a small neighborhood of u contained in Λ_a.
The boundary of Λ_a is given by

$$\partial\Lambda_a = \{u \in E_a : \text{there exist } x \in \mathbb{R}^n \text{ such that } u(x) = \eta\}.$$

We can show that Λ_a has a rich topological structure, more precisely it consists of infinitely many connected components. These components are identified by the topological charge we are going to introduce.

4.3 Topological charge and connected components of Λ_a

For the sake of simplicity, we consider the function space

$$C = \left\{u : \mathbb{R}^n \to \mathbb{R}^{n+1}\backslash\{\eta\} \text{ is continuous and } \lim_{|x|\to\infty} u(x) = 0\right\}$$

where $\eta = (1, 0)$. Every function $u \in C$ we write in the form $u(x) = (u_0(x), \tilde{u}(x)) \in \mathbb{R}^{n+1}$ where $u_0 : \mathbb{R}^n \to \mathbb{R}$ and $\tilde{u} : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 4.3.1. *For every function $u \in C$ we define the support of u*

$$K_u = \{x \in \mathbb{R}^n : u_0(x) > 1\}.$$

Then we define the topological charge of u

$$ch(u) := \begin{cases} deg(\tilde{u}, K_u, 0) & \text{if } K_u \neq \emptyset, \\ 0 & \text{if } K_u = \emptyset, \end{cases}$$

such that

$$deg(\tilde{u}, K_u, 0) = \sum_{x \in \tilde{u}^{-1}(0)} sgn J_{\tilde{u}}(x).$$

(Brouwer degree) For more information about this subject, see Section 2.8; where $J_{\tilde{u}}$ *denotes the determinante of the Jacobian matrix.*

We notice that the above definition is well posed. Indeed, since

$$\lim_{|x| \to \infty} u(x) = 0,$$

we have that K_u is an open, bounded set; moreover, for every $x \in \partial K_u$, we have, together with $u(x) \neq \eta$, that $\tilde{u}(x) \neq 0$.

Proposition 4.3.1. *For every* $u = (u_0, \tilde{u}) \in \Lambda_a$ *there exists* $l \in \mathbb{N}$ *and there exist* $x^1, \ldots, x^l \in \mathbb{R}^n$, $R_1, \ldots, R_l > 0$.
We set $B^i = B(x^i, R_i)$, *such that*

$$B^i \cap B^j = \emptyset, \quad i \neq j;$$

$$\forall x \in \mathbb{R}^n \setminus \bigcup_{i=1}^{l} B^i, \qquad u_0(x) < 1;$$

$$K_u \subset \bigcup_{i=1}^{l} B^i; \tag{4.3.1}$$

$$ch(u) = deg(\tilde{u}, \bigcup_{i=1}^{l} B^i, 0) = \sum_{i=1}^{l} deg(\tilde{u}, B^i, 0). \tag{4.3.2}$$

Proof. Let $u \in \Lambda_a$

$$ch(u) := \begin{cases} deg(\tilde{u}, K_u, 0) & \text{if } K_u \neq \emptyset, \\ 0 & \text{if } K_u = \emptyset. \end{cases}$$

If $K_u = \emptyset$, we have

$$deg(\tilde{u}, K_u, 0) = 0.$$

We consider $K_u \neq \emptyset$.
Recall that E_a is a reflexive Banach space and continuously embedded in $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$.

Let $x^1 \in \mathbb{R}^n$ be a maximum point for u_0. By Remark 4.2.1, x^1 always exists. Since $K_u \neq \emptyset$ then $u_0(x^1) > 1$.
We take R_1 such that $\forall x \in B(x^1, R_1)$, $u_0(x) > 1$.
For simplicity we set$B^1 = B(x^1, R_1)$. Now we distinguish two cases

$(A1)$ $\forall x \in \mathbb{R}^N/B^1$, $u_0(x) \leq 1$

or

$(B1)$ $\exists x \in \mathbb{R}^N/B^1$, $u_0(x) > 1$.

In the case $(A1)$ the proposition is proved with $l = 1$, indeed by additivity properties we have

$$\deg(\tilde{u}, B^1, 0) = deg(\tilde{u}, K_u, 0) + deg(\tilde{u}, B^1/\overline{K}_u, 0).$$

Since $B^1/\overline{K}_u = \emptyset$, thus

$$deg(\tilde{u}, B^1/\overline{K}_u, 0) = 0.$$

Then

$$ch(u) = \deg(\tilde{u}, B^1, 0). \tag{4.3.3}$$

Let us consider the subcase $(B1)$. Let x^2 be a maximum point for u_0 in $\mathbb{R}^N/B^1$; we have that $u_0(x^2) > 1$. We set $B^2 = B(x^2, R_2)$ and we take R_1 such that

$$\forall x \in B^2 \Rightarrow u_0(x) > 1.$$

Also in this second step we have an alternative: either

$(A2)$ $\forall x \in \mathbb{R}^N \setminus (B^1 \cup B^2)$, $u_0(x) \leq 1$

or

$(B2)$ $\exists x \in \mathbb{R}^N/(B^1 \cup B^2)$, $u_0(x) > 1$

If case $(A2)$ holds true, the proposition is proved with $l = 2$; endded the spheres B^1 and B^2 are disjoint for R_1, R_2 sufficiently small, with the same arguments in (4.3.3)

$$ch(u) = \deg(\tilde{u}, B^1, 0) + \deg(\tilde{u}, B^2, 0),$$

such that

$$K_u \subset B^1 \cup B^2.$$

In case $(B2)$ we consider a maximum point of u_0 in $\mathbb{R}^N \setminus (B^1 \cup B^2)$ and we repeat the same argument used in the case $(B1)$.
By Remark 4.2.1 this alternative process terminates in a finite number of steps.

$\square$

The open set $K_u := \{x \in \mathbb{R}^N : u_0(x) > 1\}$ is the support of u; by Proposition 4.3.1 there exists $R(u)$, $B(0, R) \subset \mathbb{R}^N$ such that $K_u := B(0, R)$.

Theorem 4.3.2. *For every $u \in \Lambda_a$ there exists $r = r(u) > 0$ such that; for every $v \in \Lambda_a$*

$$\|u - v\|_\infty \leq r \Longrightarrow ch(u) = ch(v). \tag{4.3.4}$$

Proof. Let $u^n = (u_0^n, \tilde{u^n}) \in \Lambda_a$ be uniformly convergent to $u = (u_0, \tilde{u})$.

By Proposition 4.3.1 there exists $l \in \mathbb{N}$. such that $ch(u) = deg(\tilde{u}, \bigcup_{i=1}^l B^i, 0) = \sum_{i=1}^l deg(\tilde{u}, B^i, 0)$. We shall show that, for n sufficiently large,

$$ch(u^n) = ch(u).$$

First we show that

$$K_{(u^n)} \subset \bigcup_{i=1}^l B^i.$$

Let $x \notin \bigcup_{i=1}^l B^i$ then $u_0(x) < 1$ since u_0^n be uniformly convergent to u_0; for n sufficiently large, $u_0^n(x) < 1$ so,

$$x \notin K_{(u^n)}.$$

Using the excision property of the degree, we have

$$deg(\tilde{u^n}, \bigcup_{i=1}^l B^i, 0) = \underbrace{deg(\tilde{u^n}, K_{(u^n)}, 0)}_{=ch(u^n)} + \underbrace{deg(\tilde{u^n}, \bigcup_{i=1}^l B^i \setminus \overline{K}_{(u^n)}, 0)}_{=0} \tag{4.3.5}$$

where the second term on the right-hand side of (4.3.5) is 0. Indeed, let $x \in \bigcup_{i=1}^l B^i \setminus \overline{K}_{(u^n)}$. then $u_0^n(x) < 1$ and $u_0(x) > 1$ for n sufficiently large. We get a contradiction, $u_0(x) < 1$ and $u_0(x) > 1$. So,

$$\bigcup_{i=1}^l B^i \setminus \overline{K}_{(u^n)} = \emptyset$$

then

$$deg(\tilde{u^n}, \bigcup_{i=1}^l B^i \setminus \overline{K}_{(u^n)}, 0) = 0. \tag{4.3.6}$$

From (4.3.5), (4.3.6) and by additivity properties we have

$$ch(u^n) = deg(\tilde{u^n}, \bigcup_{i=1}^l B^i, 0) = \sum_{i=1}^l deg(\tilde{u^n}, B^i, 0).$$

Now, using the previous proposition and the continuity of topological degree with respect to the uniform convergence, we get, for n sufficiently large,

$$ch(u^n) = \sum_{i=1}^l deg(\tilde{u}, B^i, 0) = ch(u),$$

$$ch(u) = deg(\tilde{u}, B(0,R), 0). \tag{4.3.7}$$

□

We recall that the topological charge is continuous with respect to the uniform convergence. Now, for every $q \in \mathbb{Z}$ we set

$$\Lambda_q = \{u \in \Lambda_a : ch(u) = q\}.$$

Since the topological charge is continuous with respect to the uniform convergence (see Theorem 4.3.2) and the continuity of the embeddings E_a in L^∞ (see Proposition 4.2.1) assure that the topological charge is continuous on Λ_a, it follows that Λ_q is open in E_a, since we have also

- $\Lambda_a = \bigcup_{q\in\mathbb{Z}} \Lambda_q$,
- $\Lambda_q \cap \Lambda_p = \emptyset, \quad p \neq q$.

We conclude that every Λ_q is a connected component of Λ_a.
We assume that the space dimension is odd then we conclude that for every $q \in \mathbb{Z}$ the component Λ_q is isomorphic to the component Λ_{-q}.

So for every $u \in \Lambda_a$ we can define the charge $ch(u) \in \mathbb{Z}$. Now, we consider the set of a minimizer of f_a in the open set

$$\Lambda^*{}_q = \{u \in \Lambda_a : ch(u) \neq 0\}.$$

Remark 4.3.1. *We can easily see that $ch(u) \neq 0$ implies $\|u\|_{L^\infty} > 1$.*

4.4 Properties of the energy functional

Lemma 4.4.1. *The functional f_a takes real values and it is continuous on Λ_a.*

Proof. We have

$$f_a(u) = \underbrace{\int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^2) + \frac{b}{2}|\nabla u|^p\right) dx}_{} + \underbrace{\int_{\mathbb{R}^n} V(u)dx}_{}.$$

First we show that

$$f_a(u) < \infty.$$

The first term on the left-hand side of energy f_a is finite and continuous. Let us prove that the second term is finite and continuous.

We have

$$V(\xi) = V(0) + V'(0)\xi + V''(0)\xi \cdot \xi + o(\xi^2),$$

by (V_1) and (V_2) then

$$V(\xi) = V''(0)\xi \cdot \xi + o(\xi^2).$$

By (V_3) there exist a small neighborhood of $0 \in \mathbb{R}^{n+1}$ and $M > 0$ such that, for every $\xi \in \mathbb{R}^{n+1}$,we have

$$V(\xi) \leq M|\xi|^2. \tag{4.4.1}$$

Since every $u \in E_a$ decays to zero at infinity (see (4.2.4)), there exists a ball B_u such that, for every $x \in \mathbb{R}^n/B_u$, $|u(x)| < \epsilon$,

by (6.4.1) and for ϵ sufficiently small

$$V(u(x)) \leq M|u(x)|^2. \tag{4.4.2}$$

From $u \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, we deduce

$$\int_{\mathbb{R}^n/B_u} V(u)dx < \infty.$$

On the other hand, since u is continuous (see (4.2.3)), we also have

$$\int_{B_u} V(u)dx < \infty.$$

Let$\{u_k\} \subset \Lambda_a$ be a sequence such that $f_a(u_k) < \infty$ and $u_k \to u \ \ in \ \ E_a$.
We show that

$$\int_{\mathbb{R}^n} V(u_k) \longrightarrow \int_{\mathbb{R}^n} V(u).$$

Since $f_a(u_k) < \infty$ and with Lemma 4.4.4, u belongs to Λ_a.

We have $u_k \to u \ on \ L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ see (4.2.2).

We deduce that $V(u_k) \to V(u) \ uniformly \ on \ \mathbb{R}$, then

$$\int_{B_u} V(u_k)dx \to \int_{B_u} V(u)dx. \tag{4.4.3}$$

On the other hand by (4.4.2) we have

$$\int_{\mathbb{R}^N \setminus B_u} V(u(x))dx \leq \int_{\mathbb{R}^N \setminus B_u} |u(x)|^2 dx,$$

and since $u_k \to u \in L^2(\mathbb{R}^N, \mathbb{R}^{N+1})$, and using the dominated convergence (see Theorem 2.5.3)

$$\int_{\mathbb{R}^N \setminus B_u} V(u_k)dx \to \int_{\mathbb{R}^N \setminus B_u} V(u)dx. \tag{4.4.4}$$

□

Lemma 4.4.2. *The map $f' : E_a \to E'_a$ defined by*

$$\begin{aligned} \langle f'_a(u), v \rangle &= \langle -a\Delta u - b\Delta_p u + V'(u), v \rangle \\ &= \int_{\mathbb{R}^n} \Big(a(\nabla u|\nabla v) + b\frac{p}{2}|\nabla u|^{p-2}(\nabla u|\nabla v) + V'(u).v \Big)\, dx \end{aligned}$$

is continuous .

Proof. We have

$$f'_a(u) = \underbrace{-a\Delta u - b\frac{p}{2}\Delta_p u}+\underbrace{V'(u)}.$$

The proof of the first term on the left-hand side of f'_a is given in the Appendix B. Let us prove that the second term is continuous.
Let $\{u_k\} \subset \Lambda_a$ be a sequence such that $f_a(u_k) < \infty$ and $u_k \longrightarrow u$.
We show that

$$V'(u_k) \longrightarrow V'(u_k) \ \ in \ \ E'_a.$$

Since $f_a(u_k) < \infty$ and with Lemma 4.4.4, u belongs to Λ_a. Recall that E_a is continuously embedded in L^∞ see (4.2.2). We have

$$\|V'(u_k) - V'(u)\|_{E'_a} = \sup_{\|h\|_{E_a} \le 1} < V'(u_k) - V'(u), h >_{E'_a \times E_a},$$

with

$$\langle V'(u_k) - V'(u), h \rangle_{E'_a \times E_a} = \textstyle\int_{\mathbb{R}^n} (V'(u_k) - V'(u))h \; dx$$

$$= \underbrace{\int_{B_u} (V'(u_k) - V'(u))h \; dx}_{1} + \underbrace{\int_{\mathbb{R}^N \setminus B_u} (V'(u_k) - V'(u))h \; dx}_{2}$$

in the term 1: since $\|h\|_{L^\infty} \le \|h\|_{E_a} \le 1$ with the same reasoning as in (4.4.3) we have

$$\int_{B_u} (V'(u_k) - V'(u))h \; dx < \frac{\epsilon}{2},$$

with the same choice of B_u as in proof of Lemma 4.4.1.
In the term 2: we have $V'(\xi) = (V''(0)\xi + o(\xi)$ then by $V3$

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_u}^{.} (V'(u_k))h \;\; dx &= M \int_{\mathbb{R}^N \setminus B_u}^{.} |u_k||h| \;\; dx \\ &\le \|u_k\|_{L^2}\|h\|_{L^2} \\ &\le \|u_k\|_{L^2}. \end{aligned} \tag{4.4.5}$$

From (4.4.5)with the same reasoning as in (4.4.4) we have

$$\int_{\mathbb{R}^n/B_u}(V'(u_k)-V'(u))h\ dx<\frac{\epsilon}{2}.$$

□

Lemma 4.4.3. *The functional f_a is coercive in Λ_a; that is, for every sequence $u_k \subset \Lambda_a$ such that $\|u_k\|_a \to \infty$, we have $f_a(u_k) \to \infty$.*

Proof. In the case $a > 0,\ n > 2$, we have

$$\|u\|_a = a\|\nabla u\|_{L^2} + \|\nabla u\|_{L^p} + \|u\|_{L^2}.$$

Let $u_k \in \Lambda_a$ such that

$$\|u_k\|_a \to \infty \ \text{ as } \ k \to \infty.$$

It is clear that, if

$$a\|\nabla u_k\|_{L^2} + \|\nabla u_k\|_{L^p} \to \infty \ \text{ as } \ k \to \infty, \tag{4.4.6}$$

we have

$$f_a(u_k) \to \infty \ \text{ as } \ k \to \infty.$$

Assume now that there exists $c_* > 0$ such that

$$a\|\nabla u_k\|_{L^2} + \|\nabla u_k\|_{L^p} < c_* \tag{4.4.7}$$

and

$$\|u_k\|_{L^2} \to \infty \ \text{ as } \ k \to \infty. \tag{4.4.8}$$

We shall prove that

$$\int_{\mathbb{R}^n} V(u_k)dx \ \text{ as } \ k \to \infty.$$

From (V_3), we have for every $r > 0$ there exists $\omega_r > 0$ such that

$$|\xi| \le r \Rightarrow V(\xi) \ge \omega_r|\xi|^2. \tag{4.4.9}$$

For every $k \in \mathbb{N}$, we set

$$A_k = \{x \in \mathbb{R}^n :\ |u_k(x)| \le r\},$$

where $u_k \in W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1})$. By the Sobolev inequality (see Theorem 2.1.8)

$$\|u_k\|_{L^{2^*}} \le c\|\nabla u_k\|_{L^2},\ 2^* = \frac{2n}{n-2},\ n > 2. \tag{4.4.10}$$

From (4.4.7), we obtain

$$\|u_k\|_{L^{2^*}} < c_*. \tag{4.4.11}$$

Moreover, from (4.2.1), there exists $M \geq 0$ independent of u_k, such that

$$\begin{aligned} |u_k(x)| &\leq \left| \int_{Q_k} u dy \right| + M \|\nabla u_k\|_{L^p(Q_k)}, \ mes(Q_k) = 1 \\ &\leq \|u\|_{L^{2^*}(Q_k)} + M \|\nabla u_k\|_{L^p(Q_k)}. \end{aligned}$$

By (4.4.6) and (4.4.11), for any $x \in \mathbb{R}^n$, we have

$$|u_k(x)| < c_* + Mc_*. \tag{4.4.12}$$

Then, there exists $c > 0$ such that

$$mes(\mathbb{R}^n \backslash A_k) < c. \tag{4.4.13}$$

From (4.4.12) and (4.4.13), we deduce that there exists $c_1 > 0$ such that

$$\int_{\mathbb{R}^n \backslash A_k} |u_k|^2 dx < c_1. \tag{4.4.14}$$

By (4.4.11), we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} V(u_k) dx &\geq \int_{A_k} V(u_k) dx \\ &\geq \omega_r \int_{A_k} \|u_k\|^2 dx \\ &\geq \omega_r \left(\|u_k\|_{L^2}^2 - \int_{\mathbb{R}^n \backslash A_k} |u_k|^2 dx \right). \end{aligned}$$

From (4.4.14) and (4.4.10), we have

$$\int_{\mathbb{R}^n} V(u_k) dx \geq \omega_r (\|u_k\|_{L^2}^2 - c_1) \to \infty \text{ as } k \to \infty.$$

In the case, $a = 0$ or $n = 2$, by (V_5), there exists $r_* > 0$ such that, for every $\xi \in \mathbb{R}^n$ with $|\xi| \geq r_*$, we have

$$V(\xi) \geq \frac{\nu}{2}. \tag{4.4.15}$$

Let $u_k \in \Lambda_a$ be a sequence such that

$$\|u_k\|_0 \to \infty \text{ as } k \to \infty.$$

Since the functional f_a is invariant with respect to translation in $\mathbb{R}^n$, we can assume

$$\|u_k\|_{L^\infty} = |u_k(0)|. \tag{4.4.16}$$

Now, we consider the case

$$\|\nabla u_k\|_{L^p} \leq M_* \ \text{ and } \ \|u_k\|_{L^2} \to \infty \text{ as } \ k \to \infty.$$

Here we have two subcases:

(a)

$$\|u_k\|_{L^\infty} \to \infty \text{ as } k \to \infty, \tag{4.4.17}$$

or

(b)

$$\|u_k\|_{L^\infty} \quad \text{is bounded.} \tag{4.4.18}$$

In the subcase (a), by (4.4.17), we can choose a sequence $(r_k) \subset (0, \infty)$ such that

$$r_* \leq \|u_k\|_{L^\infty} - K(r_k^{\frac{p-n}{p}}) \text{ and } r_k \to \infty, \tag{4.4.19}$$

where $K = cM_*$ and c is the same constant as in (4.2.3). For every $y \in \mathbb{R}^n$, we have

$$|u_k(0)| - |u_k(y)| \leq |u_k(0) - u_k(y)|.$$

Hence by (4.2.3), we obtain

$$|u_k(0)| - |u_k(y)| \leq K(|y|^{\frac{p-n}{p}}).$$

From (4.4.16), we get

$$|u_k(y)| \geq \|u_k\|_{L^\infty} - K(|y|^{\frac{p-n}{p}}).$$

For $|y| \leq r_k$ and (4.4.19), we have

$$|u_k(y)| \geq \|u_k\|_{L^\infty} - K(r_k^{\frac{p-n}{p}}) \geq r_*. \tag{4.4.20}$$

From (4.4.15) and (4.4.20), we get

$$\int_{\mathbb{R}^n} V(u_k)dx \geq \int_{B(0,r_k)} V(u_k)dx \geq \frac{\nu}{2} mes(B(0, r_k)).$$

This implies that

$$\int_{\mathbb{R}^n} V(u_k)dx \to \infty \text{ as } \ r_k \to \infty.$$

In the last subcase (b), we assume there exists $\bar{M} > 0$ such that

$$\|u_k\|_{L^\infty} \leq \bar{M}.$$

From (4.4.11), we obtain

$$\int_{\mathbb{R}^n} V(u_k)dx \geq \omega_{\bar{M}} \|u_k\|_{L^2} \to \infty \text{ as } \ k \to \infty.$$

$\square$

We are going to study the behaviour of energy f_a when u approaches the boundary of Λ_a; in the spirit of a well-known result of Gordon (see [85]), concerning strongly attractive potentials. We remark that $\partial\Lambda_a = E_a \setminus \Lambda_a$.

Lemma 4.4.4. *Let $(u_k) \subset \Lambda_a$ be a weakly converging sequence. If the weak limit belongs to $\partial\Lambda_a$, then*

$$f_a(u_k) \to \infty \text{ as } k \to \infty.$$

Proof. Let $(u_k) \subset \Lambda_a$ such that

$$u_k \rightharpoonup u \in \partial\Lambda_a \text{ as } k \to \infty.$$

Since $u \in \partial\Lambda_a$ then there exists $x_* \in \mathbb{R}^n$ such that $u(x_*) = \eta$.

Since V is nonnegative, it is sufficient to show that there exists a small ball centered at x_* such that

$$\lim_{k\to\infty} \int_{B(x_*,R)} V(u_k(x))dx = +\infty.$$

Using the fact that (u_k) is bounded in E_a, then by the uniform convergence on compact sets, we have

$$u_k(x_*) \to u(x_*) \quad \text{as } k \to \infty. \tag{4.4.21}$$

Since (u_k) is bounded in E_a, then ∇u_k is bounded in L^p.
Then from (4.2.3), we obtain

$$|u_k(x) - u_k(x_*)| \le c\sup(|x - x_*|^{\frac{p-n}{p}}). \tag{4.4.22}$$

Thus

$$|u_k(x) - u_k(x_*)| \le |u_k(x) - u_k(x_*)| + |u_k(x_*) - \eta| \tag{4.4.23}$$

By (4.4.21) and (4.4.22), there exists $\epsilon_k > 0$ such that

$$|u_k(x) - \eta| \le c\sup(|x - x_*|^{\frac{p-n}{p}}) + \epsilon_k$$

where

$$\epsilon_k \to 0 \text{ as } \ k \to \infty.$$

Let $x \in B(0,r)$. For r sufficiently small, there exists $\rho > 0$ such that

$$|u_k(x) - \eta| \le c\sup(r^{\frac{p-n}{p}}) + \epsilon < \rho. \tag{4.4.24}$$

From (4.4.24) and (V_4), we have

$$V(u_k(x)) \ge c(|u_k(x) - \eta|^{\frac{np}{p-n}}). \tag{4.4.25}$$

Then from (4.4.24) and (4.4.25), we obtain

$$V(u_k(x)) \geq \frac{c}{|x - x_*| + \epsilon_k}.$$

Restricting our attention to $B(x_*, r)$, we have

$$\lim_{k\to\infty} \int_{B(x_*,r)} V(u_k(x))dx \geq \int_{B(x_*,r)} \lim_{k\to\infty} V(u_k(x))dx \geq c \int_{B(x_*,r)} \frac{1}{|x - x_*|^n} dx = \infty.$$

□

Corollary 4.4.5. *For every $b > 0$, there exists $d = d(b)$ such that, for every $u \in \Lambda_a$ we have*

$$f_a(u) \leq b \Rightarrow \min_{x\in\mathbb{R}^n} |u(x) - \eta| \geq d.$$

Proof. Arguing by contradiction, assume that there exist $b > 0$ and a sequence $(u_k) \subset \Lambda_a$ such that

$$f_a(u_k) \leq b$$

and

$$\min_{x\in\mathbb{R}^n} |u_k(x) - \eta| < \frac{1}{k}. \tag{4.4.26}$$

For every $k \in \mathbb{N}$, by Remark 4.2.1 , there exists $x_k \in \mathbb{R}^n$ such that

$$|u(x_k) - \eta| = \min_{x\in\mathbb{R}^n} |u(x) - \eta|.$$

Then we can consider the sequence

$$\psi_k = u(\cdot + x_k).$$

Since

$$f_a(\psi_k) = f_a(u_k) \leq b \tag{4.4.27}$$

we have that $\{f_a\psi_k\}$ is bounded in E_a, then, up to a subsequence, it weakly converges to ψ .

Now, from the definition of ψ_k and (4.4.26), we obtain

$$\psi(0) = \lim_{k\to\infty} \psi_k(0) = \eta.$$

Therefore $\psi \in \partial\Lambda_a$. Taking into account (4.4.27) we have got a contradiction with Proposition 4.4.4.

□

Lemma 4.4.6. *The functional f_a is weakly lower semicontinuous in Λ_a.*

Proof. Let $u \in \Lambda_a$ and let a sequence $(u_k) \subset \Lambda_a$ weakly converge to u. We show that

$$\liminf_{k\to\infty} f_a(u_k) \geq f_a(u).$$

The result is obvious when

$$\liminf_{k\to\infty} f_a(u_k) = +\infty,$$

we have

$$f_a(u_k) = \underbrace{\frac{a}{2}\|\nabla u_k\|_{L^2}^2 + \frac{b}{2}\|\nabla u_k\|_{L^p}^p}_{A} + \underbrace{\int_{\mathbb{R}^n} V(u_k)dx,}_{B}$$

the part A is convex and strongly continuous then is weakly lower semicontinuous (see [28], Remark 6, page 61).
Now we have to study the part B

Since $\{u_k\}$ converges to u uniformly on every compact set, we fix a sphere $B_R(0)$ and we have

$$\lim_{k\to\infty} \int_{B_R(0)} V(u_k)dx = \int_{B_R(0)} V(u)dx.$$

On the other hand, since V is nonnegative, we have

$$\liminf_{k\to\infty} \int_{\mathbb{R}^n} V(u_k)dx \geq \liminf_{k\to\infty} \int_{B_R(0)} V(u_k)dx = \int_{B_R(0)} V(u)dx$$

and taking the limit for $R \to \infty$,we obtain

$$\liminf_{k\to\infty} \int_{\mathbb{R}^n} V(u_k)dx \geq \int_{\mathbb{R}^n} V(u)dx.$$

So, the proposition is completely proved.

□

Proposition 4.4.7. *There exists $\Delta_a > 0$ such that, for every $u \in \Lambda_a$ satisfying $\|u\|_{L^\infty} \geq 1$ we have*

$$f_a(u) \geq \Delta_a.$$

Proof. By continuous injection in Proposition 4.2.1,

$$\|u\|_a \geq \|u\|_{L^\infty} \geq 1,$$

and by the coercivity of f_a, we get

$$\|u\|_a \geq 1 \Rightarrow \exists\ \Delta_a > 0 \text{ such that } \ f_a(u) \geq \Delta_a.$$

□

4.5 Non trivial solution

We recover $\Lambda_a = \bigcup_{q\in\mathbb{Z}} \Lambda_a^q$ and the natural idea is to minimize E_a on each Λ_a^q. Unfortunately, in this approach the following problems arise:

Λ_a^q is not weakly closed, the operator Δ_p is not weakly continuous and the concentration-compactness methods cannot be applied directly These difficulties can be overcome if we are able to 'localize' the charge, so that every bump has its own charge and the charge of any configuration equals the sum of the charge of its bumps.

4.5.1 The splitting lemma

The proof of our main result is based on the following proposition, in the spirit of the Concentration-Compactness principle for unbounded domains (see [21, 65]).

Proposition 4.5.1. *(Splitting proposition) Let $(u_k) \in \Lambda_a^*$ be a sequence and M be a positive real number such that*

$$f_a(u_k) \leq M. \tag{4.5.1}$$

Then there exists $l \in \mathbb{N}$ such that

$$1 \leq l \leq M \setminus \Delta_a, \tag{4.5.2}$$

where Δ_a was introduced in Proposition 4.4.7, and there exist $\bar{u}_1, \ldots, \bar{u}_l \in \Lambda_a$, $(x_k^1), \ldots, (x_k^l) \subset \mathbb{R}^n$,$R_1, \ldots, R_l$ such that, up to a subsequence,

$$u_k(\cdot + x_k^i) \rightharpoonup \bar{u}_i; \tag{4.5.3}$$

$$\|\bar{u}_i\|_{L^\infty} \geq 1; \tag{4.5.4}$$

$$|x_k^i - x_k^j| \to \infty, \quad i \neq j; \tag{4.5.5}$$

$$\sum_{i=1}^{l} f_a(\bar{u}_i) \leq \liminf_{k\to\infty} f_a(u_k); \tag{4.5.6}$$

$$\forall x \in \mathbb{R}^n \setminus \bigcup_{i=1}^{l} B^i(x_n^i) : \qquad |u_n(x)| \leq 1. \tag{4.5.7}$$

Then we have also

$$ch(u_k) = \sum_{i=1}^{l} ch(\bar{u}_i), \tag{4.5.8}$$

$$\limsup_{k\to\infty} \|u_k - \sum_{i=1}^{l} \bar{u}_i(\cdot - x_k^i)\|_{L^\infty} \leq 1. \tag{4.5.9}$$

Proof. The proof is divided in two parts. Into the first part, with an iterative procedure, we prove the existence of $l \in \mathbb{N}$, $\bar{u}_1, \ldots, \bar{u}_l \in \Lambda_a$, $(x_k^1), \ldots, (x_k^l) \subset \mathbb{R}^n$, $R_1, \ldots, R_l$ such that (4.5.2)-(4.5.7) are satisfied;in the second part from these properties we shall easily deduce (4.5.8) and (4.5.9).
For the sake of simplicity, whenever it is necessary. We shall tacitly consider a subsequence of u_k. First of all we arbitrarily choose $\gamma \in]0\ 1[$. Let $x_k^1 \in \mathbb{R}^n$ be a maximum point for$|u_k|$; by Remark 4.3.1. We have $|u_k(x_k^1)| > 1$. We set

$$u_k^1 = u_k(\cdot + x_k^1)$$

and we obtain

$$\|u_k^1\|_\infty = |u_k^1(0)| > 1. \tag{4.5.10}$$

Since $f_a(u_k^1) = f_a(u_k)$ and the functional f_a is coercive, then the sequence $\{u_k^1\}$ is bounded in E_a and we have

$$u_k^1 \rightharpoonup \bar{u}_1 \in E_a. \tag{4.5.11}$$

From (4.5.10) and (4.2.2) it follows

$$\|\bar{u}_1\|_\infty \geq 1. \tag{4.5.12}$$

Since $u_k^1 \subset \Lambda_a$ and $f_a(u_k^1)$ is bounded, by (4.5.11) and Lemma 4.4.4, we get $\bar{u}_1 \in \Lambda_a$.

Since f_a is weakly lower semi-continuous, we have

$$f_a(\bar{u}_1) \leq \liminf_{k\to\infty} f_a(u_k^1) = \liminf_{k\to\infty} f_a(u_k). \tag{4.5.13}$$

We set $\bar{u}_1 = (\bar{u}_{01}, \tilde{\bar{u}}_1) \in \mathbb{R} \times \mathbb{R}^n$. Now, using (4.2.4), we consider $R_1 > 0$ such that

$$\forall x \in \mathbb{R}^n \setminus B(0, R_1), \qquad |\bar{u}_1(x)| \leq \gamma; \tag{4.5.14}$$

for simplicity we set

$$B_k^1 = B(x_k^1, R_1).$$

Now we distinguish two cases: either

$(A1)$ for k sufficiently large

$$\forall x \in \mathbb{R}^n \setminus B_k^1, \quad |u_k(x)| \leq 1,$$

or

$(B1)$ eventually passing to a subsequence,

$$\exists x \in \mathbb{R}^n \setminus B_k^1, \quad |u_k(x)| > 1.$$

In the case $(A1)$the first part of the Proposition is proved with $l = 1$; let us consider the case $(B1)$. Let x_k^2 be a maximum point for $|u_k|$ in $\mathbb{R}^n \setminus B_k^1$. We have that $u_k(x_k^2) > 1$. We set

$$u_k^2 = u_k(\cdot + x_k^2)$$

and we obtain

$$\|u_k^2\|_\infty = |u_k^2(0)| > 1.$$

As for $\{u_k^1\}$, we have that

$$u_k^2 \rightharpoonup \bar{u}_2 \in \Lambda_a, \tag{4.5.15}$$

with

$$\|\bar{u}_2\|_\infty \geq 1. \tag{4.5.16}$$

Now we have to show that

$$|x_k^1 - x_k^2| \to \infty. \tag{4.5.17}$$

We set

$$y_k = x_k^1 - x_k^2$$

and, arguing by contradiction, we assume that the sequence $\{y_k\}$ is bounded in $\mathbb{R}^n$; then, up to subsequence, we have that

$$y_k \to \tilde{y}.$$

Since $|y_k| = |x_k^1 - x_k^2| \geq R1$, we have $|\tilde{y}| \geq R1$; then, using (4.5.14),

$$|\bar{u}_1(\tilde{y})| \leq \gamma < 1. \tag{4.5.18}$$

On the other hand we have

$$1 \leq |u_k(x_k^2)| = |u_k(y_k + x_k^1)| = |u_k^1(y_k)|.$$

Then, by (4.5.18),

$$0 < 1 - |\bar{u}_1(\tilde{y})| \leq |u_k^1(y_k)| - |\bar{u}_1(\tilde{y})| \leq |u_k^1(y_k) - \bar{u}_1(\tilde{y})|$$

$$\begin{aligned} 0 < 1 - |\bar{u}_1(\tilde{y})| &\leq |u_k^1(y_k)| - |\bar{u}_1(\tilde{y})| \leq |u_k^1(y_k) - \bar{u}_1(\tilde{y})| \\ &\leq |u_k^1(y_k) - \bar{u}_1(y_k)| + |\bar{u}_1(y_k) - \bar{u}_1(\tilde{y})| \\ &\leq \left(\sup_{|y-\tilde{y}|\leq 1} |u_k^1(y) - \bar{u}_1(y)| \right) + |\bar{u}_1(y_k) - \bar{u}_1(\tilde{y})|, \end{aligned}$$

by (4.2.3) $\bar{u}_1$ is continuous and by (4.2.2) we have locally a compact injection $(E_a \subset L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}))$, then taking the limit for $k \to \infty$ we get a contradiction,

$$0 < 1 - |\bar{u}_1(\tilde{y})| \leq 0.$$

Now we show that

$$f_a(\bar{u}_1) + f_a(\bar{u}_2) \leq \lim_{k\to\infty} f_a(u_k). \tag{4.5.19}$$

Hereafter, for sake of simplicity, we set, for every $u \in \Lambda_a$ and $A \subset \mathbb{R}^n$

$$f_{a/A}(u) = \int_A \left(\frac{a}{2}|\nabla u|^2 + \frac{1}{2}|\nabla u|^p + V(u) \right) dx.$$

Since f_a is continuous, $f_a(0) = 0$ and by (4.2.4), then for a fixed $\eta > 0$, there exists $\rho > 0$ such that
$f_{a/C_{B_\rho(0)}}(\bar{u}_1) < \eta/2 \quad \text{and} \quad f_{a/C_{B_\rho(0)}}(\bar{u}_2) < \eta/2,$

$$C_{B_\rho(0)} = \mathbb{R}^{n+1} \setminus B(0,\rho).$$

From (4.5.17) it follows that the spheres $B_\rho(x_k^1)$ and $B_\rho(x_k^2)$ are disjoint for k sufficiently large, then we get:

$$\begin{aligned}
\liminf_{k\to\infty} f_a(u_k) &\geq \liminf_{k\to\infty} \left(f_{a/C_{B_\rho(x_k^1)}}(u_k) + f_{a/C_{B_\rho(x_k^2)}}(u_k) \right) \\
&\geq \liminf_{k\to\infty} f_{a/C_{B_\rho(x_k^1)}}(u_k) + \liminf_{k\to\infty} f_{a/C_{B_\rho(x_k^2)}}(u_k) \\
&= \liminf_{k\to\infty} f_{a/C_{B_\rho(0)}}(u_k^1) + \liminf_{k\to\infty} f_{a/C_{B_\rho(0)}}(u_k^2) \\
&\geq f_{a/C_{B_\rho(0)}}(\bar{u}_1) + f_{a/C_{B_\rho(0)}}(\bar{u}_2) \\
&> f_a(\bar{u}_1) + f_a(\bar{u}_2) - \eta.
\end{aligned}$$

From the arbitrariness of η, we get (4.5.19).

Finally, as well as for $\bar{u}_1$, from (4.2.4) we get $R_2 > 0$, such that

$$\forall x \in C_{B_{R_2}(0)}, \quad |\bar{u}_2(x)| \leq \gamma$$

and we set

$$B_k^2 = B_{R_2}(x_k^2).$$

Also in this second step we have an alternative: either

$(A2)$ for k sufficiently large

$$\forall x \in C_{(B_k^1 \cup B_k^2)}, \quad |u_k(x)| \leq 1,$$

or

$(B2)$ eventually passing to a subsequence,

$$\exists x \in C_{(B_k^1 \cup B_k^2)}, \quad |u_k(x)| > 1.$$

If case $(A2)$ holds true, the first part of the Proposition is proved with l =2; in the case $(B2)$ we consider a maximum point of $|u_k|$ in $C_{(B_k^1 \cup B_k^2)}$ and we repeat the same argument used in the case $(B1)$. This alternative process terminates in a finite number of steps. Indeed,now we prove (4.5.2).

From (4.5.12) and (4.5.16)

$$\|\bar{u}_i\|_\infty \geq 1 \qquad i = 1, \ldots, l;$$

and with Proposition 4.4.7 we get, $f_a(\bar{u}_i) \geq \Delta_a > 0$, then From (4.5.1) and (4.5.6)

$$l \cdot \Delta_a \leq \sum_{i=1}^{l} f_a(\bar{u}_i) \leq \liminf_{k \to \infty} f_a(u_k) \leq M.$$

So, we get (4.5.2); we notice that this estimate is independent of the sequence $\{u_k\}$.

Now we prove (4.5.8). We consider k sufficiently large so that (4.5.7) holds and

$$B_k^i \cap B_k^j = \emptyset \quad for \quad i \neq j. \tag{4.5.20}$$

By using the same arguments used in proposition4.3.1 $K_{u_k} \subset \bigcup_{i=1}^{l} B_k^i$ we have

$$\begin{aligned} ch(u_k) &= deg(\tilde{u_k}, \bigcup_{i=1}^{l} B_k^i, 0) \\ &= \sum_{i=1}^{l} deg(\tilde{u_k}, B_k^i, 0) \\ &= \sum_{i=1}^{l} deg(\tilde{u_k^i}, B_{R_i}(0), 0). \end{aligned} \tag{4.5.21}$$

On the other hand, for every $i \in \{1, \ldots, l\}$, since $\{u_k^i\}$ converges uniformly to$\{\bar{u}_i\}$ on $B_{R_i}(0)$ we obtain,

$$deg(\tilde{u_k^i}, B_{R_i}(0), 0) = deg(\tilde{\bar{u}}_i, B_{R_i}(0), 0), \tag{4.5.22}$$

recall that $u_k^i \rightharpoonup \bar{u}_i \in \Lambda_a$, $\bar{u}_i = (\bar{u}_{0i}, \tilde{\bar{u}}_i) \in \mathbb{R} \times \mathbb{R}^n$, $u_k^i = (u_{0k}^i, \tilde{u_k^i}) \in \mathbb{R} \times \mathbb{R}^n$.

And since we have $\forall x \in C_{(B_{R_i}(0))}, \quad |\bar{u}_i(x)| \leq \gamma < 1$. Then we have, by the excision property of the topological degree,

$$deg(\bar{\tilde{u}}_i, B_{R_i}(0), 0) = deg(\bar{\tilde{u}}_i, K_{u_i}, 0) + deg(\bar{\tilde{u}}_i, B_{R_i}(0) \setminus \bar{K}_{u_i}, 0).$$

Let $x \in B_{R_i}(0) \setminus \bar{K}_{\bar{u}_i}, 0)$. then $\gamma < u_{0i}(x) \leq 1$. From the arbitrariness of γ, we get $u_{0i}(x) = 1$ and since $\{\bar{u}_i\} \subset \Lambda_a$ then $\bar{\tilde{u}}_i \neq 0$ which implies by the solvability property of the topological degree,

$$deg(\bar{\tilde{u}}_i, B_{R_i}(0) \setminus \bar{K}_{u_i}, 0) = 0,$$

then

$$ch(\bar{u}_i) = deg(\bar{\tilde{u}}_i, B_{R_i}(0), 0). \tag{4.5.23}$$

From (4.5.23), (4.5.22) and (4.5.21)

$$ch(u_k) = \sum_{i=1}^{l} ch(\bar{u}_i).$$

Finally, in order to prove (4.5.9), since u_k^i converges uniformly to $\bar{u}_i$ in $B_{R_i}(0)$, for every $i \in \{1, \ldots, l\}$, we assume that,

$$\forall x \in B_i^k : \quad |u_k(x) - \bar{u}_i(x - x_i^k)| < \gamma. \tag{4.5.24}$$

We shall prove that, for k large enough,

$$\forall x \in \mathbb{R}^n : \quad |u_k(x) - \sum_{i=1}^{l} \bar{u}_i(x - x_i^k)| < 1 + l\gamma. \tag{4.5.25}$$

Indeed, if $x \in \bigcup_{i=1}^{l} B_i^k$, then, by (4.5.20), there exists a unique index $j \in \{1, \ldots, l\}$ such that $x \in B_j^k$ then

$$|u_k(x) - \sum_{i=1}^{l} \bar{u}_i(x - x_i^k)| \leq |u_k(x) - \bar{u}_j(x - x_j^k)| + \sum_{i \neq j} |\bar{u}_i(x - x_i^k)|$$

$$< \quad \gamma + (l-1)\gamma = l\gamma < 1 + l\gamma. \tag{4.5.26}$$

On the other hand, if $x \notin \bigcup_{i=1}^{l} B_i^k$, then, by (4.5.7),

$$|u_k(x) - \sum_{i=1}^{l} \bar{u}_i(x - x_i^k)| \leq |u_k(x)| + \sum_{i=1}^{l} |\bar{u}_i(x - x_i^k)|$$

$$\leq 1 + l\gamma.$$

Now fix $\eta > 1$; choosing γ sufficiently small we have

$$1 + l\gamma < \eta. \tag{4.5.27}$$

Substituting (4.5.27) in (4.5.25), we get

$$\forall x \in \mathbb{R}^n : \quad |u_k(x) - \sum_{i=1}^{l} \bar{u}_i(x - x_i^k)| < \eta,$$

and, by the arbitrariness of $\eta > 1$, we obtain (4.5.9).

□

4.5.2 Existence of minima in the connected components of Λ_a

The minimum is attained on the set Λ_a, and it is easy to see that $u \equiv 0$ is a trivial solution of the problem. But, of course, we are interested in nontrivial solutions. Now, we consider the following problem

$$I_* = \inf_{u \in \Lambda_a^*} f_a(u), \quad \Lambda_a^* = \{u \in E_a : ch(u) \neq 0\}.$$

The functional is bounded below and the set Λ_a is not empty. We consider fields u having the form

$$u(x) = (\frac{2}{1 + |x|^m}, \frac{1}{1 + |x|^m} x). \tag{4.5.28}$$

Lemma 4.5.2. *There exists a suitable $m \geq 1$, such that, the field u defined in (4.5.28) belongs to Λ_a^* .*

Proof. Clearly, if m is sufficiently large, then the field u defined in (4.5.28) belongs to E_a. For the sake of contradiction, suppose that there exists $\bar{x} \in \mathbb{R}^n$ such that $u(\bar{x}) = \eta = (1, 0)$. We deduce that

$$\frac{2}{1 + |\bar{x}|^m} = 1,$$

$$\frac{1}{1 + |\bar{x}|^m} \bar{x} = 0.$$

We get the contradiction: $|\bar{x}| = 1$ and $\bar{x} = 0$. So, $u \in \Lambda_a$.

We show that $ch(u) \neq 0$. Set $g(x) = \frac{1}{2}x$ then we have

$$K_u = \{x \in \mathbb{R}^n : \frac{2}{1 + |x|^m} > 1\} = B(0, 1),$$

$$if|x| = 1 \ \ then \ \ g(x) = \frac{1}{1 + |x|^m} x,$$

by the properties of the topological degree (see Theorem 2.8.1) we get,

$$deg(\frac{1}{1+|x|^m}x, B(0,1), 0) = deg(g(x), B(0,1), 0) \neq 0.$$

□

And moreover the set Λ_a^* is open in the space E_a; indeed, since the topological charge is continuous with respect to the uniform convergence, see Theorem 4.3.2, and the continuity of the embedding E_a in L^∞ (see Proposition 4.2.1) assure that the topological charge is continuous on Λ_a.

Theorem 4.5.3. *Let $a \geq 0$, $b > 0$, $p > n > 2$. If V satisfies $(V_1) - (V_5)$, then there exists a weak solution of (4.1.10) (i.e., a static solution of (4.1.8), which is a minimizer of the energy functional (4.1.12) in the class of maps whose topological charge is different from* 0.

Proof. By Lemma 4.4.1 and Proposition 4.4.7 we have,

$$\forall u \in \Lambda_a^*, 0 < \Delta_a \leq f_a(u) < \infty.$$

We consider a minimizing sequence$\{u_k\} \subset \Lambda_a^*$. It has obviously bounded energy; then we can apply Proposition 4.5.1. There exist $l \in \mathbb{N}$ and $\bar{u}_1, \ldots, \bar{u}_l \in \Lambda_a$ such that, up to a subsequence, (4.5.6), (4.5.8) hold true. Since $ch(u_k) \neq 0$, from (4.5.8) we deduce that there exists $\bar{i} \in \{1, \ldots, l\}$ such that $ch(\bar{u}_{\bar{i}}) \neq 0$. Then, from (4.5.6), we obtain

$$I_* \leq f_a(\bar{u}_{\bar{i}}) \leq \sum_{i=1}^{l} f_a(\bar{u}_i) \leq \liminf_{k\to\infty} f_a(u_k) = I_*.$$

So we conclude that

$$f_a(\bar{u}_{\bar{i}}) = I_*.$$

Moreover since Λ_a^* is an open set then there exists a weak solution of (4.1.10) (i.e., a static solution of (4.1.8)) then we deduce a solution of equation (4.1.8) having the form of equation(4.1.11).

□

Remark 4.5.1. *The functional exhibits an invariance for the symmetry group of rotations and translations; indeed, for every function u and $g \in O(n)$, if we set $ug(x) = u(gx)$, we have immediately*

$$f_a(gu) = f_a(u).$$

Then our theorem gives the existence of an orbit of minimum solutions. This orbit consists of two connected components, which are identified, respectively, by $\bar{u}$ and

$$\bar{u} \circ \mathcal{P}(x) = \bar{u}(-x).$$

Since typically $n = 3$ is odd, $\bar{u} \circ \mathcal{P}$ and $\bar{u}$ have opposite topological charge.

4.5.3 Resolution of static equation

In this subsection we prove the existence of a solution for the static equation.

Theorem 4.5.4. *The minimum points $u \in \Lambda_a$ for the functional f_a are weak solutions of the system (4.1.10).*

Proof. Let u be a minimum point of f_a and $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Let e_j denote the j^{th}-vector of the canonical basis in $\mathbb{R}^n$. If ϵ is sufficiently small, then $u + \epsilon e_j h \in \Lambda_a$ and $f_a(u + \epsilon e_j h) < \infty$. Since u is a minimum point of f_a, then

$$0 \quad = \quad \left.\frac{df(u+\epsilon e_j h)}{d\epsilon}\right|_{\epsilon=0} = \int_{\mathbb{R}^n} \left(a\nabla u_j \nabla h + b\frac{p}{2}(|\nabla u|^{p-2}\nabla u_j \nabla h) + \frac{\partial V(\xi)}{\partial \xi_j} h \right) dx.$$

By Green's formula,

$$\int_{\mathbb{R}^n} b\frac{p}{2}(|\nabla u|^{p-2}\nabla u_j \nabla h)dx = -\int_{\mathbb{R}^n} b\frac{p}{2} div(|\nabla \cdot u|^{p-2}\nabla u_j) h dx.$$

So

$$\int_{\mathbb{R}^n} \left(-a\Delta u_j - b\frac{p}{2} div(|\nabla \cdot u|^{p-2}\nabla u_j) + \frac{\partial V(\xi)}{\partial \xi_j} \right) h dx = 0,$$

for $1 \le j \le n+1$, and for any $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Then

$$\int_{\mathbb{R}^n} \left[-a\Delta u - b\frac{p}{2}\Delta_p u + V'(u) \right] \phi dx = 0, \text{ for every } \phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

This implies by density

$$-a\Delta u - \frac{b}{2}\Delta_p u + V'(u) = 0.$$

□

4.6 Compactness properties related to symmetry

We fix $a \in \mathbb{R}_+ \setminus \{0\}$; for sake of simplicity we assume $a = 1$; so hereafter in the notation we omit the index a. We consider the Banach space E, the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|_E = \|\nabla u\|_{L^2} + \|\nabla u\|_{L^p} + \|u\|_{L^2}.$$

By Corollary 4.2.3, the space E coincides with

$$W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n+1}) \cap W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

In the space E we can consider the following $O(n)$ action: for every $u \in E$, $g \in O(n)$

$$T_g u(x) = (u_0(gx), g^{-1}\tilde{u}(gx)) \tag{4.6.1}$$

We see below if the potential V satisfies a suitable symmetry property, it is possible to prove the existence of infinitely many finite energy solutions. More precisely we assume:

$(V6)$ There exist ρ_1 and $r > 1$ such that

$$|V'(\xi) - V''(0)\xi| \leq c_0|\xi|^{\ r}$$

whenever $|\xi| \leq \rho_1$;

$(V7)$ for every $\xi = (\xi_0, \tilde{\xi})$ and for every g in the orthogonal group $O(n)$,

$$V(\xi_0, g\tilde{\xi}) = V(\xi_0, \tilde{\xi}).$$

An easy calculation and assumption $(V7)$ give the following lemma.

Lemma 4.6.1. *The open set Λ and the functional f are invariant under the action (4.6.1), that is, for every $g \in O(n)$ and $u \in \Lambda$, we have*

$$T_g(u) \in \Lambda,$$

$$f(T_g(u)) = f(u).$$

Now, let F denote the subspace of fixed points

$$F = \{u \in E \mid T_g u = u, \forall g \in O(n)\}.$$

We shall show that

$$\Lambda_F = \Lambda \cap F$$

is a natural constraint to finding the critical points of f. This means that any $u \in \Lambda_F$ such that, for any $v \in F$,

$$\langle f'(u), v\rangle = 0$$

gives us $f'(u) = 0$ (see Lemma 4.6.2).This fact is usual in Hilbert spaces and unfortunately E is only a Banach space. Moreover we shall need a continuous projection $\mathbf{P} : E \to F$.We can define $\mathbf{P}$ by using the $O(n)$-continuous action. For every $u \in E$, we set

$$\mathbf{P}u = \int_{O(n)} T_g u \ dg, \tag{4.6.2}$$

dg being the Haar measure on the group $O(n)$. This map $\mathbf{P}$ is continuous and takes its values in F; moreover we have $\mathbf{P}\text{o}\mathbf{P} = \mathbf{P}$. So we conclude that $\mathbf{P}$ is a projection on F and F is a closed subspace.

Lemma 4.6.2. *For every $u \in \Lambda_F$ and $v \in E$ we have*

$$\langle f'(u), v\rangle = \langle f'(u), \mathbf{P}v\rangle,$$

$\mathbf{P}$ *being the projection of E onto F.*

Proof. Since the functional f is invariant, the map $f' : E \to E'$ is "equivariant", that is

$$\langle f'(u), T_g v\rangle = \langle f'(T_{g^{-1}}u), v\rangle. \tag{4.6.3}$$

Now we recall that the integral commutes with continuous linear forms, so we have

$$\begin{aligned}
\langle f'(u), \mathbf{P}v\rangle &= \langle f'(u), \int_{O(n)} T_g v \, dg\rangle = \int_{O(n)} \langle f'(u), T_g v\rangle dg \\
\textit{by } (4.6.3) & \\
&= \int_{O(n)} \langle f'(T_{g^{-1}}u), v\rangle dg \\
\textit{since } \ u \in F & \\
&= \int_{O(n)} \langle f'(u), v\rangle dg \\
&= \langle f'(u), v\rangle \int_{O(n)} dg = \langle f'(u), v\rangle,
\end{aligned}$$

where the last equality follows from the fact that $\int_{O(n)} dg = 1$.

From this lemma we deduce that every local minimum of f restricted to Λ_F is a critical point of f.

□

Proposition 4.6.3. *The space F, equipped with the norm on E, $\| \cdot \|_E$, is compactly embedded in $L^{\mathbf{s}}(\mathbb{R}^n, \mathbb{R}^{n+1})$ for every $\mathbf{s} \in]2, 2^*[$, where*

$$2^* = \begin{cases} +\infty & \textit{if } \ n = 2, \\ 2n/(n-2) & \textit{if } \ n > 2. \end{cases} \tag{4.6.4}$$

This proposition is an easy consequence of the following theorem, which is proved in Appendix A.

Theorem 4.6.4. *If $\mathcal{W}$ is a bounded subset of $W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1})$, then*

$$\mathcal{W}_{\mathcal{R}} = \{u \in \mathcal{W} \mid |u| \textit{is radial function}\}$$

is relatively compact in $L^{\mathbf{s}}(\mathbb{R}^n, \mathbb{R}^{n+1})$ for every $\mathbf{s} \in]2, 2^*[$.

Proof. (Proposition 4.6.3.)
We have to prove that every bounded set in F is relatively compact in $L^{\mathbf{s}}(\mathbb{R}^n, \mathbb{R}^{n+1})$. To prove this, we can employ Theorem 4.6.4; we have only to notice that, for every $u \in F$, $|u|$ is a radial function. If $u = (u_0, \tilde{u}) \in F$, the function u_0 is $O(n)$ invariant, as well as the field u is $O(n)$ equivariant, that is, for every $x \in \mathbb{R}^n, g \in O(n)$

$$u_0(gx) = u_0(x),$$

$$\tilde{u}(gx) = g\tilde{u}(x).$$

So we have

$$\begin{aligned} |u(gx)|^2 &= |u_0(gx)|^2 + |\tilde{u}(gx)|^2 \\ &= |u_0(x)|^2 + |g\tilde{u}(x)|^2 \\ &= |u_0(x)|^2 + |\tilde{u}(x)|^2 = |u(x)|^2. \end{aligned}$$

This means that $|u(x)|^2$ depends only on $|x|$. □

Proposition 4.6.5. *The functional $f/_{\Lambda_F}$ satisfies the Palais-Smale condition, i.e., for every sequence $\{u_k\} \in \Lambda_F$ such that*

(a) $f(u_k)$ is bounded,

(b) $f'/_{\Lambda_F}(u_k)$ converges to 0 in F', contains a convergent subsequence.

We remark that (b) means that, for every $v \in F$,

$$\langle f'(u_k), v\rangle \leq \varepsilon_k \|v\|,$$

where $\varepsilon_k \to 0$

In the proof we need the following lemmas.

Lemma 4.6.6. *The map*

$$\mathcal{A} \;\; E \longrightarrow E'$$

defined by

$$\begin{aligned} \langle \mathcal{A}u, v\rangle &= \langle -\Delta u - \Delta_p u + V''(0)u, v\rangle \\ &= \int_{\mathbb{R}^n} \big((\nabla u|\nabla v) + |\nabla u|^{p-2}(\nabla u|\nabla v) + V''(0)u.v\big)\, dx \end{aligned}$$

is invertible with continuous inverse.

The proof is given in the Appendix B.

Lemma 4.6.7. *For every $u \in F$, $v \in E$,*

$$\langle \mathcal{A}u, v\rangle = \langle \mathcal{A}u, \mathbf{P}v\rangle. \tag{4.6.5}$$

Proof. First we notice that, if $u \in F$, then for every $g \in O(n)$

$$\langle \mathcal{A}u, T_g v\rangle = \langle \mathcal{A}u, v\rangle.$$

Now, since the integral commutes with continuous linear forms, we have

$$\begin{aligned}
\langle \mathcal{A}u, \mathbf{P}v\rangle &= \langle \mathcal{A}u, \int_{O(n)} T_g v \; dg\rangle \\
&= \int_{O(n)} \langle \mathcal{A}u, v\rangle \; dg \\
&= \langle \mathcal{A}u, v\rangle.
\end{aligned}$$

□

Proof. Let $\{u_k\}$ be a sequence in Λ_F such that

$$f(u_k) \;\; is \;\; bounded, \tag{4.6.6}$$

$$f'/(u_k) = -\Delta u_k - \Delta_p u_k + V'(u_k) \to 0 \;\; in \;\; F'. \tag{4.6.7}$$

By (4.6.6), since the functional f is coercive, the sequence $\{u_k\}$ is bounded in E.

First we shall prove that, up to a subsequence, it is strongly convergent to $u \in E$. Using the operator $\mathcal{A}$ we can write

$$\mathcal{A}(u_k) = f'(u_k) - U'(u_k), \tag{4.6.8}$$

where

$$U(\xi) = V(\xi) - V''(0)\xi \cdot \xi$$

.

Using (4.6.6) and Corollary 4.4.5, we have that there exists $d > 0$ such that, for every $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$

$$|u_k(x) - \eta| \geq d. \tag{4.6.9}$$

From (4.6.9), since $\{u_k\}$ is bounded in L^∞, we deduce that, for a suitable $M > 0$,

$$|U'(u_k)| \leq M. \tag{4.6.10}$$

Now we set

$$A_k = \{x \in \mathbb{R}^n : |u_k| \geq \rho_1\},$$

where ρ_1 is introduced in $(V6)$. Since $\{u_k\}$ is bounded in L^2, we have that the measure of A_k is uniformly bounded.

Since $r > 1$ (see $(V6)$), we can find s such that

$$(2^*)' < s' < 2 \leq rs',$$

where

$$(2^*)' = \begin{cases} 1 & \text{if } 2^* = +\infty, \\ 2^*/(2^*-1) & \text{if } 2^* < +\infty. \end{cases}$$

Now, by $(V6)$, we have, for every $x \in \mathbb{R}^n/A_k$,
$U'(u_k(x)) \leq C_0|u_k(x)|^{\ r}$
and using (4.6.10), we get

$$\begin{aligned} \int_{\mathbb{R}^n} |U'(u_k)|^{\ s'} &\leq \int_{\mathbb{R}^n/A_k} |U'(u_k)|^{\ s'} dx + \int_{A_k} |U'(u_k)|^{\ s'} dx \\ &\leq \int_{\mathbb{R}^n/A_k} C_0^{s'} |u_k|^{\ rs'} + M^{s'} \text{meas}(A_k) \\ &\leq M_1 \|u_k\|^{\ rs'}_{L^{rs'}} + M_2. \end{aligned} \tag{4.6.11}$$

We know that $\{u_k\}$ is bounded in E; then Corollary 4.2.3 implies that $\{u_k\}$ is bounded in $L^{rs'}(R^n, R^{n+1})$. So, from (4.6.11), we deduce the boundedness of $U'(u_k)$ in $L^{s'}(R^n, R^{n+1})$.
Since

$$(2^*)' < s' < 2,$$

we have

$$2 < s = \frac{s'}{s'-1} < 2^*.$$

and, from Proposition 4.6.3, is compactly embedded into L^s. Then, since $U'(u_k)$ is bounded in $L^{s'}$ and with the Theorem 7.3.2 (Schauder theorem) $\{U'(u_k)\}$ is strongly convergent in F' (up to a subsequence).

$$f'(u_k) - U'(u_k) \longrightarrow \chi \in F'. \tag{4.6.12}$$

Now we can define $\mathcal{P}\chi \in E'$ by setting, for every $\omega \in E$,

$$\langle \mathcal{P}\chi, \omega \rangle = \langle \chi, \mathbf{P}\omega \rangle, \tag{4.6.13}$$

where $\mathbf{P}$ is defined in (4.6.2). Now we want to prove that

$$\mathcal{A}u_k \longrightarrow \mathcal{P}\chi$$

in E'; indeed, using (4.6.5), (4.6.13) and (4.6.8), for every $\omega \in E$ we have

$$\begin{aligned}\langle \mathcal{A}u_k - \mathcal{P}\chi, \omega\rangle &= \langle \mathcal{A}u_k - \chi, \mathbf{P}\omega\rangle \\ &= \langle f'(u_k) - U'(u_k) - \chi, \mathbf{P}\omega\rangle \longrightarrow 0.\end{aligned}$$

Using Lemma 4.6.6, we deduce

$$u_k = \mathcal{A}^{-1}\mathcal{A}u_k \longrightarrow \mathcal{A}^{-1}\mathcal{P}\chi = u.$$

Finally we have $u \in \Lambda_F$; in fact $u \in F$ since F is closed and $u \in \Lambda$ by Lemma 4.4.4 and (4.6.6). □

4.7 Infinitely many solutions

In this Section, we prove under some symmetry assumptions the existence of infinitely many solutions, which are constrained minima of the energy. More precisely, for every $N \in \mathbb{N}$ there exists a solution of charge N.

Theorem 4.7.1. *Assume $p > n \geq 2$ and $a \geq 0$. Assume that V satisfies $(V_1) - (V_5)$, moreover assume that*

$(V6)$ *There exist ρ_1 and $r > 1$ such that*

$$|V'(\xi) - V''(0)\xi| \leq c_0|\xi|^{\ r}$$

whenever $|\xi| \leq \rho_1$;

$(V7)$ *for every $\xi = (\xi_0, \tilde{\xi})$ and for every g in the orthogonal group $O(n)$,*

$$V(\xi_0, g\tilde{\xi}) = V(\xi_0, \tilde{\xi}).$$

Then, for any $N \in \mathbb{N}$, there exists u_N solution of (4.1.10) such that $ch(u_N) = N$. Moreover we have

$$\lim_N f(u_N) = +\infty.$$

For the proof of Theorem 4.7.1 we shall prove the following statements:

(A) for every $N \geq 1$, the connected component

$$\Lambda_F^N = \{u \in \Lambda_F \mid ch(u) = N\}$$

is not empty;

(B) for every $N \geq 1$, the energy functional attains its minimum in Λ_F^N;

(C) if we denote by u_N a minimizer of the energy in Λ_F^N then

$$\lim_N f(u_N) = +\infty.$$

4.7.1 Symmetric fields with arbitrary charge

This subsection is devoted to the proof of statement (A). We shall give a complete proof of

$$\Lambda_F^N \neq \emptyset \tag{4.7.1}$$

in the case of N odd; the case of N even is analogous.
To this end, we shall study suitable fields in F having the form

$$u(x) = (A(|x|), B(|x|)x), \tag{4.7.2}$$

A and B being two scalar fields such that $u \in \Lambda$. Indeed, an easy calculation shows that fields having the form (4.7.2), are fixed points for the action (4.6.1), so they belong to Λ_F. More precisely, we consider fields u having the form (4.7.2), with

$$A(|x|) = \frac{a}{1 + (|x|/2\pi)^m} \cos |x|, \tag{4.7.3}$$

$$B(|x|) = \frac{1}{1 + (|x|/2\pi)^m} \sin |x|. \tag{4.7.4}$$

We show that $u \in \Lambda_F$.

Lemma 4.7.2. *There exists a suitable $m \geq 1$, such that, for every $a \in \mathbb{R}/\mathbb{Q}$, the field u defined by (4.7.2)-(4.7.4)) belongs to Λ (by the above remark, it belongs to Λ_F).*

Proof. An easy calculation with polar coordinates and for m sufficiently large, we have that the field u defined by (4.7.2)-(4.7.4)) belongs to E, now we assume $a \in \mathbb{R} \setminus \mathbb{Q}$ and we prove that $u \in \Lambda$.
We have to prove that, for every $x \in \mathbb{R}^n$,

$$u(x) \neq \eta = (1, 0)$$

For the sake of contradiction, suppose that there exists $\bar{x} \in \mathbb{R}^n$ such that

$$u(\bar{x}) = (1, 0).$$

Using the definition of u, we deduce that

$$A(|\bar{x}|) = \frac{a}{1 + (|\bar{x}| \setminus 2\pi)^m} \cos |\bar{x}| = 1, \tag{4.7.5}$$

$$B(|\bar{x}|) = \frac{a}{1 + (|\bar{x}| \setminus 2\pi)^m} \sin |\bar{x}| = 0. \tag{4.7.6}$$

From (4.7.6) we deduce that $|\bar{x}| = k\pi$, $k \in \mathbb{N}$; then from (4.7.5) we get

$$\pm \frac{a}{1 + (k/2)^m} = 1,$$

and this contradicts $a \in \mathbb{R}/\mathbb{Q}$, so $\Lambda \neq \emptyset$. $\square$

Proposition 4.7.3. *Let $N = 2L + 1$ ($L \in N$) an odd number. Then any field u of type (4.7.2)-(4.7.4) with m as in Lemma 4.7.2 and $a \in \mathbb{R}/\mathbb{Q}$ such that*

$$1 + L^m < a < 1 + (L + 1)^m \tag{4.7.7}$$

has charge equal to N.

Proof. By the definition of charge (see Definition 4.3.1), we have to prove that

$$deg(\tilde{u}, K_u, 0) = 2L + 1,$$

where

$$\tilde{u}(x) = \frac{\sin |x|}{1 + (|x| \setminus 2\pi)^m} x$$

and

$$K_u = \{x \in \mathbb{R}^n : \frac{a \cos |x|}{1 + (|x| \setminus 2\pi)^m} > 1\}.$$

We fix $\epsilon \in]0, \pi/2[$ set

$$K^0 = \{|x| < \epsilon\};$$

moreover, for every $j = 1, \dots, L$, we set

$$K^j = \{2j\pi - \epsilon < |x| < 2j\pi + \epsilon\}.$$

The open subsets $\{K^j\}_{1 \leq j \leq L}$ are disjoint.
We show that

$$K^0 \cup K^1 \cup \ldots \cup K^L \subset K_u.$$

Let $x \in K^j$, then $2j\pi - \epsilon < |x| < 2j\pi + \epsilon$. We develop $\cos |x|$ on the ball $B(0, \epsilon)$ for ϵ small enough, we set $t = |x| \geq 0$

$$\cos t = 1 - \frac{t^2}{2} + \frac{t^3}{9} R(t),$$

such that $R(t) = \sin\theta t \quad \text{with} \quad 0 < \theta < 1.$
We deduce that

$$\frac{a\cos t}{1+(t\setminus 2\pi)^m} > \frac{a(1-\frac{t^2}{2})}{1+(j+t\setminus 2\pi)^m}.$$

So it is enough to prove that

$$\frac{a(1-\frac{t^2}{2})}{1+(j+t\setminus 2\pi)^m} > 1.$$

We consider the function

$$s(t) = \frac{(1+t^m)(1-\frac{t^2}{2})}{1+(j+t\setminus 2\pi)^m}, \tag{4.7.8}$$

where $0 \leq j \leq L$. The function s is continuous and strictly decreasing then s is locally bijective such that $s(0) = 1$, and for t small enough we have,

$$1 - s(t) < \epsilon_0. \tag{4.7.9}$$

From (4.7.7) we can choose $\epsilon_0 = 1 - \frac{1+t^m}{a}$ then from (4.7.9) and (4.7.8)

$$\frac{a(1-\frac{t^2}{2})}{1+(j+t\setminus 2\pi)^m} > 1.$$

So we have

$$K^0 \cup K^1 \ldots \cup K^L \subset K_u.$$

Moreover, using the right-hand side inequality of (4.7.7), we can prove

$$K_u = \{x \in K_u \mid \tilde{u}(x) = 0\} \subset K^0 \cup K^1 \cup \ldots \cup K^L.$$

So, by the excision and the additive properties of the topological degree, we conclude

$$\begin{aligned} deg(\tilde{u}, K_u, 0) &= deg(\tilde{u}, \bigcup_{j=1}^{L} K^j, 0) + \underbrace{deg(\tilde{u}, K_u \setminus \bigcup_{j=1}^{L} K^j, 0)}_{=0} \\ &= \sum_{j=1}^{L} deg(\tilde{u}, K^j, 0) \end{aligned}$$

Clearly the conclusion will follow if we prove that

$$deg(\tilde{u}, K^0, 0) = 1 \tag{4.7.10}$$

and, for $j = 1, \dots, L$

$$deg(\tilde{u}, K^j, 0) = 2. \tag{4.7.11}$$

First we prove (4.7.10). Consider the function

$$v_0 = \frac{\sin \epsilon}{1 + (\epsilon \setminus 2\pi)^m} x.$$

We notice that for every $x \in \partial K^0$ (that is such that $|x| = \epsilon$), we have

$$\tilde{u} = v_0 \neq 0.$$

We have

$$deg(v_0, K^0, 0) = \sum_{x \in v_0{}^{-1}(0)} sgn J_{v_0(x)} = 1,$$

where the equality follows from the fact that v_0 is the identity up to a multiplicative constant.

Since the degree depends only on the values on the boundary, we conclude that

$$deg(\tilde{u}, K^0, 0) = deg(v_0, K^0, 0) = 1.$$

Now, for every $j \in \{1, \dots, L\}$, we set

$$B_+^j = \{|x| < j\pi + \epsilon\}$$

$$B_-^j = \{|x| < j\pi - \epsilon\}.$$

Since $\overline{K^j} = \overline{B_+^j \setminus B_-^j}$, by the additive property of the degree, we have

$$deg(\tilde{u}, K^j, 0) = deg(\tilde{u}, B_+^j, 0) - deg(\tilde{u}, B_-^j, 0). \tag{4.7.12}$$

Then we consider the function

$$v_j^+(x) = \frac{\sin \epsilon}{1 + (|2j\pi + \epsilon|/2\pi)^m} x;$$

for every $x \in \partial B_j^+$,we have

$$\tilde{u} = v_j^+ \neq 0.$$

So, for the boundary dependence of the degree, we conclude, as before, that

$$deg(\tilde{u}, B_+^j, 0) = deg(v_j^+, B_+^j, 0) = 1. \tag{4.7.13}$$

Analogously we have

$$deg(\tilde{u}, B_-^j, 0) = deg(v_j^-, B_-^j, 0) = 1 \tag{4.7.14}$$

with

$$v_j^-(x) = \frac{\sin \epsilon}{1 + (|2j\pi - \epsilon| \setminus 2\pi)^m} x.$$

Substituting (4.7.13) and (4.7.14) into (4.7.12), we get (4.7.11). So the Proposition is completely proved.

□

By the preceding proposition, for every $N \geq 1$, N odd, we can construct a field $u \in \Lambda_F$ having the form (4.7.2) such that $ch(u) = N$.
The case of N even is analogous: We can consider again a field $u \in \Lambda_F$ having the form (4.7.2) with coefficients

$$A(|\bar{x}|) = \frac{a}{1 + (|\bar{x}| \setminus 2\pi)^m} \sin |\bar{x}| = 1, \tag{4.7.15}$$

$$B(|\bar{x}|) = -\frac{a}{1 + (|\bar{x}| \setminus 2\pi)^m} \cos |\bar{x}| = 0. \tag{4.7.16}$$

With the same choice of m as in Lemma 4.7.2, for every $L \geq 1$, we can find $a \in \mathbb{R} \setminus \mathbb{Q}$, such that the field defined by (4.7.2), (4.7.15), (4.7.16) has charge $2L$.

4.7.2 Minimizers in Λ_F^N

We recall that

$$\Lambda_F^N = \{u \in \Lambda_F \mid ch(u) = N\} = \Lambda^N \cap F \neq \emptyset,$$

Λ_F^N is a connected components of Λ_F.

Fix $N \geq 1$ and consider

$$c_N = \inf_{\Lambda_F^N} f.$$

The proof of our main result is based on lemma 2.7.2 (Eckland's lemma), Proposition 4.6.5, Lemma 4.6.2 and Proposition4.5.1 (Splitting lemma).

We recall that, for every $u \in \Lambda$ with $ch(u) \neq 0$, we have $\|u\|_{L^\infty} \geq 1$ (see Remark 4.3.1), which from Proposition 4.4.7 implies

$$f(u) \geq \Delta^* > 0. \tag{4.7.17}$$

So we conclude that

$$c_N \geq \Delta^* > 0.$$

We want to prove that the value c_N is attained in Λ_F^N .

For every $c \in \mathbb{R}$, the sublevels of f are given by

$$f^c = \{u \in \Lambda_F \mid \quad f(u) \leq c\}.$$

Taking into account Lemma 4.4.4 and since F is a closed subspace, it is easy to prove that f^c are complete in E, as well as in F. By (4.7.17) f is lower bounded on Λ_F^N, then from Lemma 2.7.2 (Eckland's Lemma) there exists a sequence $\{u_k^N\} \subset \Lambda_F^N$ such that

$$c_N \leq f(u_k^N) \leq c_N + \frac{1}{k}, \tag{4.7.18}$$

and

$$\forall v \in \Lambda_F^N, \quad f(v) + \frac{1}{k}\|v - u_k^N\|_E \geq f(u_k^N). \tag{4.7.19}$$

Since f is C^1 we have

$$f(v) = f(u_k^N) + < f'(u_k^N), v - u_k^N > + \circ (v - u_k^N).$$

Then by (4.7.19) we have

$$\forall v \in \Lambda_F^N, \quad < f'(u_k^N), u_k^N - v > \leq \frac{1}{k}\|v - u_k^N\|_E + \circ(v - u_k^N); \tag{4.7.20}$$

we take $v = u_k^N - \epsilon h$ such that $h \in F$, (ϵ small enough) then $v \in \Lambda_F^N = \Lambda^N \cap F$, indeed F is a subspace and Λ^N is open in Λ.
So for all $h \in F$ we have

$$\frac{\langle f'(u_k^N), h\rangle}{\|h\|_E} \leq \frac{1}{k} + \frac{\circ(\epsilon h)}{\|\epsilon h\|_E}.$$

Then

$$f'(u_k^N) \longrightarrow 0 \ \ inF', \tag{4.7.21}$$

and from (4.7.18)

$$f(u_k^N) \longrightarrow c_N. \tag{4.7.22}$$

Moreover the functional f restricted to Λ_F satisfies the Palais-Smale condition (see Proposition 4.6.5). Then by (4.7.21) and (4.7.22), up to a subsequence, such that

$$u_k^N \longrightarrow u^N \quad in\ \Lambda_F^N$$

and since f is C^1, we have

$$f(u^N) = c_N,$$

and

$$f'(u^N) = 0 \ \ in\ F'. \tag{4.7.23}$$

From (4.7.23) and Lemma 4.6.2 we deduce that

$$f'(u^N) = 0 \ \ in\ E',$$

u^N is a critical point of f. Then u^N is a weak solution of (4.1.10) (i.e., a static solution of (4.1.8)).

We want to show that $f(u^N) \to \infty$, for $N \to \infty$. For the sake of contradiction, assume that, up to a subsequence,

$$f(u^N) \leq M.$$

Then, by Proposition 4.5.1 (see (4.5.8)), there exists $Q \in \mathbb{N}$ such that (up to a subsequence)

$$ch(u^N) = Q,$$

and this contradicts

$$ch(u^N) = N \to \infty.$$

Chapter 5

Soliton in Generalized Sobolev Space

The aim of this chapter is to carry out an existence analysis of the finite-energy static solutions in more than one space dimension for a class of Lagrangian densities L which include (4.1.1) with variable exponents.We study a class of Lorentz invariant nonlinear field equations in several space dimensions. The main purpose is to obtain soliton-like solutions with variable exponent. The fields are characterized by a topological invariant, which we call the charge. We prove the existence of a static solution which minimizes the energy among the configurations with nontrivial charge. The study of partial differential equations with $p(x)-$growth condition has received more and more attention in recent years. The specific attention accorded to such kinds problems is due to applications in mathematical physics. More precisely, such an equation is used in electrorheological fluid [71] and in elastic mechanics [87]. They also have wide applications in different research fields; see [6, 34, 54] and the reference therein.

5.1 Statement of the Problem

The class of Lagrangian densities we consider generalizes the problem studied in [10], Lagrangian density with variable exponent, in such a way as to include the Lorentz invariant Lagrangian density proposed in [10]. First we introduce some notation. For n, m positive integers, we will denote, respectively, the physical space-time (typically $n = 3$) and the internal parameters space. We are interested in the multi-dimensional case, so we assume that $n \geq 2$. A point in $\mathbb{R}^{n+1}$ will be denoted by $X = (x, t)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The fields we are interested in are maps $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^m,\ \psi = (\psi_1, \dots, \psi_m)$. We set

$$\rho = |\nabla\psi|^2 - |\psi_t|^2,$$

$\nabla\psi$ and ψ_t denoting, respectively, the Jacobian with respect to x and the derivative with respect to t. Let

$$s : \mathbb{R}^{n+1} \to \mathbb{R}.$$

We shall consider Lagrangian densities of the form

$$\mathcal{L}(\psi, \rho) = -\frac{1}{2}\alpha(\rho, s) - V(\psi), \tag{5.1.1}$$

where the function V is a real function defined in an open subset $\Omega \subset \mathbb{R}^m$ and α is a real function defined by

$$\alpha(\rho, s) = a\rho + b|\rho|^{\frac{s(\cdot)}{2}},\ a \geq 0,\ b > 0,\ s(0) > n. \tag{5.1.2}$$

The results of Chapter 4 were concerned with the case: $s(\cdot) \equiv p$, (we fix the variable exponent). The action functional related to (5.1.1) is

$$\begin{aligned} S(\psi) &= \int_{\mathbb{R}^{n+1}} \mathcal{L}(\psi, \rho) dx dt \\ &= \int_{\mathbb{R}^{n+1}} -\frac{1}{2}\alpha(\rho, s) - V(\psi) dx dt. \end{aligned}$$

So the Euler-Lagrange equations are

$$\frac{\partial}{\partial t}(\alpha' \psi_t) - \nabla\big(\alpha' \nabla\psi\big) + V'(\psi) = 0, \tag{5.1.3}$$

where $\nabla\big(\alpha'\nabla\psi\big)$ denotes the vector whose $j-th$ component is given by $div\big(\alpha'\nabla\psi^j\big)$, and V' denotes the gradient of V. The equation (5.1.3) is Lorentz invariant. Static solutions $\psi(x, t) = u(x)$ of (5.1.3) solve the equation

$$-\nabla\big(\alpha'\nabla u\big) + V'(u) = 0. \tag{5.1.4}$$

Set $s(x, t) = p(x)$ on $\mathbb{R}^n$ (the restrictions of s on $\mathbb{R}^n$). Using (5.1.2) and (5.1.4) we obtain

$$-a\Delta u - \frac{b}{2}\Delta_{p(.)} + V'(u) = 0, \tag{5.1.5}$$

where

$$\Delta_{p(\cdot)} u = \nabla\big(p(\cdot)|\nabla u|^{p(\cdot)-2}\nabla u\big).$$

We introduce the following notations and functional spaces:

$$C_+(\mathbb{R}^n) = \{p \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) :\ p(x) > 1 \text{ for all } x \in \mathbb{R}^n\}$$

and

$$p^+ = \sup_{x\in\mathbb{R}^n} p(x), \quad p^- = \inf_{x\in\mathbb{R}^n} p(x).$$

We assume that

(p_1) $S(x,t) = p(\frac{x_1 - t\nu}{\sqrt{1-\nu^2}}, \ldots, x_n)$, where ν is a parameter used in the Lorentz transformation.

(p_2) $\lim\limits_{x\to\infty} p(x) = p_\infty = p^- > n$.

Recall that the results of Chapter 4 were concerned with the case

$$p(\cdot) \equiv p^- > n.$$

Under (p_1), it is easy to verify that, if $u = u(x)$ is a solution of the (5.1.3) and $v = (\nu, 0, \ldots, 0)$ with $|\nu| < 1$, the field

$$\psi_\nu(x,t) = u\left(\frac{x_1 - \nu t}{\sqrt{1-\nu^2}}, x_2, \ldots, x_n\right) \tag{5.1.6}$$

is solution of (5.1.3). Notice that the function undergoes a contraction by a factor,

$$\gamma = \frac{1}{\sqrt{1-\nu^2}},$$

in the direction of the motion; this is a consequence of the fact that (5.1.3) is Lorentz invariant. Clearly (5.1.5) are the Euler-Lagrange equations with respect to the energy functional

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^2 + \frac{b}{2}|\nabla u|^{p(x)} + V(u)\right) dx, \tag{5.1.7}$$

where $m = n+1$, so the time independent fields u are maps

$$u : \mathbb{R}^n \to \mathbb{R}^m.$$

For every $\xi \in \mathbb{R}^{n+1}$, we write $\xi = (\xi_0, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^n$. $V : \Omega \to \mathbb{R}$ where $\Omega = \mathbb{R}^{n+1} \setminus \{\eta\}$, $\eta = (1, 0)$, and V is positive and singular in η. More precisely we assume:

(V_1) $V \in C^1(\Omega, \mathbb{R})$.

(V_2) $V(\xi) \geq V(0) = 0$.

(V_3) V is twice differentiable in 0 and the Hessian matrix $V''(0)$ is nondegenerate.

(V_4) There exist $c, \rho > 0$ such that if $|\xi| < \rho$ then

$$V(\eta + \xi) \geq c\left(|\xi|^{-q^+} + |\xi|^{-q^-}\right)$$

where

$$\frac{1}{q^-} = \frac{1}{n} - \frac{1}{p^-}, \quad \frac{1}{q^+} = \frac{1}{n} - \frac{1}{p^+}.$$

(V_5) For every $\xi \in \Omega \backslash \{0\}$ we have

$$V(\xi) > 0, \text{ and } \lim_{|\xi| \to \infty} \inf V(\xi) = v > 0.$$

Example 5.1.1. *A potential satisfying the assumptions $(V_1) - (V_5)$ is*

$$V(\xi) = \omega_0^2 \left(|\xi|^2 + \frac{|\xi|^4}{|\xi - \eta|^{q^+} + |\xi - \eta|^{q^-}} \right).$$

Definition 5.1.1. *We call $p(x)$-*soliton *a solution of equation (5.1.3) having the form of equation (5.1.6), where u is a local minimum of the energy functional.*

5.2 Solution space

Let $p^- > n \geq 2$ and, with no loss of generality, we can consider the functional (5.1.7) with $b = 1$. It will be convenient to introduce the following notation:

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2} |\nabla u|^2 + \frac{1}{2} |\nabla u|^{p(x)} + V(u) \right) dx,$$

and we define the space E_a to be the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|_a = a \|\nabla u\|_{L^2} + \|\nabla u\|_{L^{p(\cdot)}} + \|u\|_{L^2}, \quad p^- > n \geq 2, \ a \geq 0,$$

i.e.,

$$E_a = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_a},$$

$$\|u\|_{L^2} = \left(\sum_{j=1}^{n+1} \|u_j\|_{L^2}^2 \right)^{\frac{1}{2}},$$

$$\|\nabla u\|_{L^2} = \left(\sum_{j=1}^{n+1} \|\nabla u_j\|_{L^2}^2 \right)^{\frac{1}{2}},$$

and

$$|\nabla u|_{p(x)} = \inf \left\{ \sigma > 0 : \mathbb{R}^n \to \mathbb{R} : \int_{\mathbb{R}^n} \left| \frac{\nabla u(x)}{\sigma} \right|^{p(x)} dx \leq 1 \right\}.$$

For every $a > 0$, the norms $\| \cdot \|_a$ are equivalent, so we have to study only two cases: $a = 0, \ a > 0$.

Proposition 5.2.1. *The Banach space E_0 is continuously embedded in $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$, for every $s \in [2, \infty]$.*

Proof. The space E_0 is continuously embedded in $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, therefore it is sufficient to show that E_0 is embedded also in $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Since $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in E_0, so it is sufficient to prove that there exists $c > 0$ such that, for every $u \in C_0(\mathbb{R}^n, \mathbb{R}^{n+1})$, we have

$$\|u\|_{L^\infty} \leq c\|u\|_0.$$

We fix $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ and consider a family of cubes $Q_k \subset \mathbb{R}^n$ such that

$$mes(Q_k) = 1, \ \cup_{k\in\mathbb{N}} Q_k = \mathbb{R}^n.$$

Then, by a well-known inequality (see below equation (5.2.4), in Proposition 5.4.6),

for every $k \in \mathbb{N}$ and $Q_k \subset \mathbb{R}^n$,

$$|u(x)| \leq \left| \int_{Q_k} u dy \right| + M\|\nabla u\|_{L^{p(\cdot)}(Q_k)}, \tag{5.2.1}$$

where $M \geq 0$ is independent of u. Thus

$$\begin{aligned} |u(x)| &\leq mes(Q_k)\|u\|_{L^2} + M\|\nabla u\|_{L^{p(\cdot)}(Q_k)} \\ &\leq \|u\|_{L^2(\mathbb{R}^n)} + M\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq (1+M)\|u\|_0. \end{aligned}$$

Hence

$$\|u\|_{L^\infty} \leq c\|u\|_0, \quad c = 1 + M.$$

□

Corollary 5.2.2. *The Banach space E_0 is continuously embedded in $L^{p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$.*

Proof. Since $2 \leq n < p_0 \leq p^- < p^+ < \infty$, $E_0 = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_0}$, so $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in E_0 and also in $L^{p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see [41, Theorem 3.4.12]), it is sufficient to prove that there exists $c > 0$ such that

$$\|u\|_{L^{p(\cdot)}} \leq c\|u\|_0, \text{for all} u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

Let B the support of u, then

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|u\|_{L^{p(\cdot)}(B)}.$$

From [41, Theorem 3.3.1, p. 82], we have

$$\|u\|_{L^{p(\cdot)}(B)} \leq \|u\|_{L^{p^+}(B)}.$$

It is clear that

$$\|u\|_{L^{p^+}(B)} \leq \|u\|_{L^{p^+}(\mathbb{R}^n)}.$$

From Proposition 5.2.1, we deduce that there exists $c > 0$ such that

$$\|u\|_{L^{p^+}(\mathbb{R}^n)} \leq c\|u\|_0.$$

This implies that

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|u\|_{L^{p(\cdot)}(B)} \leq \|u\|_{L^{p^+}(B)} \leq \|u\|_{L^{p^+}(\mathbb{R}^n)} \leq c\|u\|_0.$$

□

Corollary 5.2.3. *The Banach space E_0 is continuously embedded in $H_0^{1,p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$.*

Proof. By definition of the space E_0, we have for every $u \in E_0$

$$\|u\|_0 > \|\nabla u\|_{L^{p(\cdot)}}.$$

From Corollary 5.2.2 there exists $c_1 > 0$ such that

$$c_1\|u\|_0 > \|u\|_{L^{p(\cdot)}},$$

and so

$$\|u\|_0 > c\|u\|_{H_0^{1,p(\cdot)}}.$$

□

Corollary 5.2.4. *For every $a > 0$, the space E_a can be identified with the Banach space*

$$W = H_0^{1,p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1}) \cap W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1}),$$

equipped with the usual norm

$$\|u\|_W = \|u\|_{W^{1,2}} + \|u\|_{W^{1,p(\cdot)}}.$$

Proof. $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in $H_0^{1,p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see Definition 2.4.1), and $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in $W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1})$; see [28]. For any $u \in E_a$ we have

$$\|u\|_a \leq \sup(1, a)\|u\|_W.$$

From Corollary 5.2.2, there exists $c > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$, we have

$$\|u\|_a \geq c(\|u\|_{W^{1,2}} + \|u\|_{W^{1,p(\cdot)}})$$

□

Proposition 5.2.5. *Since $p > n$, for every value $a \geq 0$, the functions in E_a are bounded, continuous, and decay to zero at infinity. Furthermore, the following inequality holds:*

$$|u(x) - u(y)| \leq c \sup \left(|x-y|^{1-\frac{n}{p^-}}, |x-y|^{1-\frac{n}{p^+}} \right) \|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \text{ for all } x, y \in \mathbb{R}. \tag{5.2.2}$$

Proof. By Proposition 5.2.1 we have

$$E_a \subset E_0 \subset L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}) \tag{5.2.3}$$

and since $E_a = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_a}$, then it is easy to get that all functions in E_a are bounded, and decay to zero at infinity. Now we show the inequality .
Fix $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ and consider a family of cubes $Q_k \subset \mathbb{R}^n$ such that

$$mesQ_k = 1, \bigcup_{k\in\mathbb{N}} Q_k = \mathbb{R}^n,$$

with each Q_k an open cube, containing 0, whose sides-of length r-are parallel to the coordinate axes. For $x \in Q_k$ we have

$$u(x) - u(0) = \int_0^1 du(tx) = \int_0^1 \frac{du(tx)}{dt} dt,$$

where

$$\frac{du(tx)}{dt} = \sum_{i=1}^n \frac{\partial u}{\partial x_i}(tx) \cdot \frac{\partial(tx_i)}{\partial t} = \sum_{i=1}^n \frac{\partial u}{\partial x_i}(tx) \cdot x_i.$$

Then

$$|u(x) - u(0)| \leq \int_0^1 \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i}(tx) \right| |x_i| dt, \quad x \in Q_k, \ |x_i| < r.$$

Hence

$$|u(x) - u(0)| \leq r \int_0^1 \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i}(tx) \right| dt.$$

Let

$$\overline{u} = \frac{1}{|Q_k|} \int_{Q_k} u dx.$$

Integrating the last inequality on Q_k , we obtain for every $k \in \mathbb{N}$ and $Q_k \subset \mathbb{R}^n$ we have

$$\int_{Q_k} |u(x) - u(0)| dx \geq \left| \int_{Q_k} (u(x) - u(0)) dx \right| = |Q_k| |\overline{u} - u(0)|$$

and

$$|Q_k| |\overline{u} - u(0)| \leq r \int_{Q_k} dx \int_0^1 \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i}(tx) \right| dt, \ |Q_k| = r^n.$$

Then

$$\begin{aligned}
|\overline{u}-u(0)| &\leq \frac{1}{r^{n-1}}\int_0^1 dt\int_{Q_k}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}(tx)\right|dx \\
&\leq r^{n-1}\int_0^1 dt\int_{tQ_k}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}(y)\right|\frac{dy}{t^n}, \quad tQ_k\subset Q_k, \quad t\in(0,1),
\end{aligned}$$

From Hölder's inequality and Lemma 2.3.4, we have

$$\begin{aligned}
\int_{tQ_k}\left|\frac{\partial u}{\partial x_i}(y)\right|dy &\leq 2\|\frac{\partial u}{\partial x_i}\|_{L^{p(\cdot)}(Q_k)}+\|1\|_{L^{q(\cdot)}(tQ_k)} \\
&\leq 2\|\frac{\partial u}{\partial x_i}\|_{L^{p(\cdot)}(Q_k)}\left(|tQ_k|^{\frac{n}{q^-}}+|tQ_k|^{\frac{n}{q^+}}\right),
\end{aligned}$$

where $\frac{1}{p(x)}+\frac{1}{q(x)}=1$.

Then we have

$$|\overline{u}-u(0)|\leq\frac{2}{r^{n-1}}\int_0^1 dt\|\nabla u\|_{L^{p(\cdot)}(Q_k)}\frac{1}{t^n}\left((tr)^{\frac{n}{q^-}}+(tr)^{\frac{n}{q^+}}\right).$$

We can easily show that

$$2\frac{(r)^{\frac{n}{q^-}}}{r^{n-1}}\|\nabla u\|_{L^{p(\cdot)}(Q_k)}\int_0^1\frac{(t)^{\frac{n}{q^-}}}{t^n}dt=2\frac{r^{1-\frac{n}{p^-}}}{1-\frac{n}{p^-}}\|\nabla u\|_{L^{p(\cdot)}(Q_k)}$$

and

$$2\frac{(r)^{\frac{n}{q^+}}}{r^{n-1}}\|\nabla u\|_{L^{p(\cdot)}(Q_k)}\int_0^1\frac{(t)^{\frac{n}{q^+}}}{t^n}dt=2\frac{r^{1-\frac{n}{p^+}}}{1-\frac{n}{p^+}}\|\nabla u\|_{L^{p(\cdot)}(Q_k)}.$$

We deduce from this that

$$|\overline{u}-u(0)|\leq c\sup\left(r^{1-\frac{n}{p^-}},r^{1-\frac{n}{p^+}}\right)\|\nabla u\|_{L^{p(\cdot)}(Q_k)}.$$

By translation, this inequality remains true for all cubes Q_k whose sides-of length r-are parallel to the coordinate axes. Thus we have

$$|\overline{u}-u(x)|\leq c\sup\left(r^{1-\frac{n}{p^-}},r^{1-\frac{n}{p^+}}\right)\|\nabla u\|_{L^{p(\cdot)}(Q_k)}. \tag{5.2.4}$$

By adding these (and using the triangle inequality) we obtain

$$|u(x)-u(y)|\leq c'\sup\left(r^{1-\frac{n}{p^-}},r^{1-\frac{n}{p^+}}\right)\|\nabla u\|_{L^{p(\cdot)}(Q_k)}.$$

Given any two points $x,y\in\mathbb{R}^n$, there exists such a cube with side $r=2|x-y|$

$$\begin{aligned}
|u(x)-u(y)| &\leq c\sup\left(|x-y|^{1-\frac{n}{p^-}},|x-y|^{1-\frac{n}{p^+}}\right)\|\nabla u\|_{L^{p(\cdot)}(Q_k)} \\
&\leq c\sup\left(|x-y|^{1-\frac{n}{p^-}},|x-y|^{1-\frac{n}{p^+}}\right)\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

□

Remark 5.2.1. *We deduce from Proposition 5.4.6 that if $u \in E_a$ with $n < p^- < \infty$, then u is bounded and $\lim_{|x|\to\infty} u(x) = 0$.*

Recall that η is a singular point of the potential V, so it is reasonable to consider in space E_a, the open subset

$$\Gamma_a = \{u \in E_a : u(x) \neq \eta, \text{ for all } x \in \mathbb{R}^n\}.$$

The subset Γ_a is open in E_a. Indeed, by Remark 5.2.1, we have

$$\inf_{x\in\mathbb{R}^n} |u(x) - \eta| = d > 0.$$

Then, by (5.2.3) ($E_a$is continuously embedded in L^∞), we deduce that for all $u \in \Gamma_a$there exists a small neighborhood of u contained in Γ_a.
The boundary of Γ_a is given by

$$\begin{aligned} \partial\Gamma_a &= \{u \in E_a : \text{there exist } x \in \mathbb{R}^n \text{ such that } u(x) = \eta\} \\ &= E_a \setminus \Gamma_a. \end{aligned}$$

5.3 Topological charge and connected components of Γ_a

For the sake of simplicity, we consider the function space

$$C = \left\{ u : \mathbb{R}^n \to \mathbb{R}^{n+1} \backslash \{\eta\} \text{ is continuous and } \lim_{|x|\to\infty} u(x) = 0 \right\}$$

where $\eta = (1, 0)$. Every function $u \in C$ we write in the form $u(x) = (u_0(x), \tilde{u}(x)) \in \mathbb{R}^{n+1}$ where $u_0 : \mathbb{R}^n \to \mathbb{R}$ and $\tilde{u} : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 5.3.1. *For every function $u \in C$ we define the support of u*

$$K_u = \{x \in \mathbb{R}^n : u_0(x) > 1\}.$$

Then we define the topological charge of u

$$ch(u) := \begin{cases} deg(\tilde{u}, K_u, 0) & \text{if } K_u \neq \emptyset, \\ \\ 0 & \text{if } K_u = \emptyset, \end{cases}$$

such that

$$deg(\tilde{u}, K_u, 0) = \sum_{x \in \tilde{u}^{-1}(0)} sgn J_{\tilde{u}}(x),$$

(Brouwer degree) see Section 2.8;
where $J_{\tilde{u}}$ denotes the determinante of the Jacobian matrix.
For more information about this subject (Topological Degree), see [69].

As a consequence of the fact that u is continuous and $\lim_{|x|\to\infty} u(x) = 0$, we have that K_u is an open, bounded set in $\mathbb{R}^n$. Since $u \in \Gamma$, if $x \in \partial K_u$, then, together with $u(x) \neq \eta$, we have $\tilde{u}(x) \neq 0$. Therefore the previous definition is well posed. Moreover, the topological charge is continuous with respect to the uniform convergence [12]. We notice that this definition of charge is the same as in [10].

Now, for every $q \in \mathbb{Z}$ we set

$$\Gamma_a^q = \{u \in \Gamma_a : ch(u) = q\}.$$

Since the topological charge is continuous with respect to the uniform convergence and the continuity of the embeddings E_a in L^∞ assure that the topological charge is continuous on Γ_a, it follows that Γ_a^q is open in E_a, and we have also

- $\Gamma_a = \bigcup_{q\in\mathbb{Z}} \Gamma_a^q$,
- $\Gamma_a^q \cap \Gamma_a^p = \emptyset, \quad p \neq q.$

We conclude that every Γ_a^q is a connected component of Γ_a. We observe that for every $q \in \mathbb{Z}$ the component Γ_a^q is isomorphic to the component Γ_a^{-q}. So for every $u \in \Gamma_a$ we can define the charge $ch(u) \in \mathbb{Z}$. Now, we consider the set of a minimizer of f_a in the open set

$$\Gamma_a^* = \{u \in \Gamma_a : ch(u) \neq 0\}.$$

Remark 5.3.1. *We can easily see that $ch(u) \neq 0$ implies $\|u\|_{L^\infty} > 1$.*

5.4 Properties of the energy functional

Lemma 5.4.1. *The functional f_a takes real values and it is continuous on Γ_a.*

Proof. We have

$$\begin{aligned} f_a(u) &= \int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^2) + \frac{b}{2}|\nabla u|^{p(x)}\right) dx + \int_{\mathbb{R}^n} V(u)dx, \\ &= \underbrace{\frac{a}{2}\|\nabla u\|_{L^2}^2 + \frac{b}{2}\rho_{p(x)}(u)} + \underbrace{\int_{\mathbb{R}^n} V(u)dx}. \end{aligned}$$

The first term on the left-hand side of energy f_a is finite and continuous. Let us prove that the second term is finite and continuous.

We have $V(\xi) = (V''(0)\xi \cdot \xi + o(\xi^2)$. By (V_3) there exist a small neighborhood of $0 \in \mathbb{R}^{n+1}$ and $M > 0$ such that, for every $\xi \in \mathbb{R}^{n+1}$, we have

$$V(\xi) \leq M|\xi|^2. \tag{5.4.1}$$

Since every $u \in E_a$ decays to zero at infinity (see Proposition 5.4.6), there exists a ball B_u such that, for every $x \in \mathbb{R}^n \backslash B_u$, $|u(x)| < \epsilon$, by (5.4.1) and for ϵ sufficiently small

$$V(u(x)) \leq M|u(x)|^2, \tag{5.4.2}$$

From $u \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, we deduce

$$\int_{\mathbb{R}^n \backslash B_u} V(u)dx < \infty.$$

On the other hand, since u is continuous (see Proposition 5.4.6), we also have

$$\int_{B_u} V(u)dx < \infty.$$

Let$\{u_k\} \subset \Lambda_a$ be a sequence such that $f_a(u_k) < \infty$ and $u_k \to u \;\; in \;\; E_a$.
We show that

$$\int_{\mathbb{R}^n} V(u_k) \longrightarrow \int_{\mathbb{R}^n} V(u).$$

Since $f_a(u_k) < \infty$ and with Lemma 5.4.3, u belongs to Λ_a.

By (5.2.3) we have $u_k \to u \; on \; L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$, then $V(u_k) \to V(u)$ uniformly on $\mathbb{R}$. We deduce that

$$\int_{B_u}^{\cdot} V(u_k)dx \to \int_{B_u}^{\cdot} V(u)dx. \tag{5.4.3}$$

By (5.4.2)

$$\int_{\mathbb{R}^n \backslash B_u}^{\cdot} V(u(x))dx \leq \int_{\mathbb{R}^n \backslash B_u}^{\cdot} |u(x)|^2 dx,$$

and since $u_k \to u \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, and using the dominated convergence theorem

$$\int_{\mathbb{R}^n / B_u} V(u_k)dx \to \int_{\mathbb{R}^n / B_u} V(u)dx. \tag{5.4.4}$$

$\square$

Lemma 5.4.2. *The functional f_a is coercive in Γ_a; that is, for every sequence $u_k \subset \Gamma_a$ such that $\|u_k\|_a \to \infty$, we have $f_a(u_k) \to \infty$.*

Proof. In the case $a > 0, \; n > 2$, we have

$$\|u\|_a = a\|\nabla u\|_{L^2} + \|\nabla u\|_{L^{p(\cdot)}} + \|u\|_{L^2}.$$

Let $u_k \in \Gamma_a$ such that

$$\|u_k\|_a \to \infty \;\text{ as }\; k \to \infty.$$

It is clear that, if

$$a\|\nabla u_k\|_{L^2} + \|\nabla u_k\|_{L^{p(\cdot)}} \to \infty \quad \text{as} \quad k \to \infty, \tag{5.4.5}$$

we have

$$f_a(u_k) \to \infty \quad \text{as} \quad k \to \infty.$$

Assume now that there exists $c_* > 0$ such that

$$a\|\nabla u_k\|_{L^2} + \|\nabla u_k\|_{L^{p(\cdot)}} < c_* \tag{5.4.6}$$

and

$$\|u_k\|_{L^2} \to \infty \quad \text{as} \quad k \to \infty. \tag{5.4.7}$$

We shall prove that

$$\int_{\mathbb{R}^n} V(u_k)dx \quad \text{as} \quad k \to \infty.$$

From (V_3), we have for every $r > 0$ there exists $\omega_r > 0$ such that

$$|\xi| \leq r \Rightarrow V(\xi) \geq \omega_r |\xi|^2. \tag{5.4.8}$$

For every $k \in \mathbb{N}$, we set

$$A_k = \{x \in \mathbb{R}^n : \ |u_k(x)| \leq r\},$$

where $u_k \in W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1})$. By the Sobolev inequality

$$\|u_k\|_{L^{2^*}} \leq c\|\nabla u_k\|_{L^2}, \ 2^* = \frac{2n}{n-2}, \ n > 2. \tag{5.4.9}$$

From (5.4.6), we obtain

$$\|u_k\|_{L^{2^*}} < c_*. \tag{5.4.10}$$

Moreover, from (5.2.4), there exists $M \geq 0$ independent of u_k, such that

$$\begin{aligned} |u_k(x)| &\leq \left| \int_{Q_k} u_k dy \right| + M\|\nabla u_k\|_{L^{p(\cdot)}(Q_k)}, \ mes(Q_k) = 1 \\ &\leq \|u_k\|_{L^{2^*}(Q_k)} + M\|\nabla u_k\|_{L^{p(\cdot)}(Q_k)}. \end{aligned}$$

By (5.4.6) and (5.4.10), for any $x \in \mathbb{R}^n$, we have

$$|u_k(x)| < c_* + Mc_*. \tag{5.4.11}$$

Then, there exists $c > 0$ such that

$$mes(\mathbb{R}^n \backslash A_k) < c. \tag{5.4.12}$$

From (5.4.11) and (5.4.12), we deduce that there exists $c_1 > 0$ such that

$$\int_{\mathbb{R}^n \setminus A_k} |u_k|^2 dx < c_1. \tag{5.4.13}$$

By (5.4.8), we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} V(u_k)dx &\geq \int_{A_k} V(u_k)dx \\ &\geq \omega_r \int_{A_k} \|u_k\|^2 dx \\ &\geq \omega_r \left(\|u_k\|_{L^2}^2 - \int_{\mathbb{R}^n \setminus A_k} |u_k|^2 dx \right). \end{aligned}$$

From (5.4.13) and (5.4.7), we have

$$\lim_k \int_{\mathbb{R}^n} V(u_k)dx \geq \omega_r(\|u_k\|_{L^2}^2 - c_1) \to \infty \text{ as } k \to \infty.$$

In the case, $a = 0$ or $n = 2$, by (V_5), there exists $r_* > 0$ such that, for every $\xi \in \mathbb{R}^n$ with $|\xi| \geq r_*$, we have

$$V(\xi) \geq \frac{\nu}{2}. \tag{5.4.14}$$

Let $u_k \in \Gamma_a$ be a sequence such that

$$\|u_k\|_0 \to \infty \text{ as } k \to \infty.$$

Since the functional f_a is invariant with respect to translation in $\mathbb{R}^n$, we can assume

$$\|u_k\|_{L^\infty} = |u_k(0)|. \tag{5.4.15}$$

Now, we consider the case

$$\|\nabla u_k\|_{L^{p(\cdot)}} \leq M_* \text{ and } \|u_k\|_{L^2} \to \infty \text{ as } k \to \infty.$$

Here we have two subcases:

(a)

$$\|u_k\|_{L^\infty} \to \infty \text{ as } k \to \infty. \tag{5.4.16}$$

(b)

$$\|u_k\|_{L^\infty} \quad \text{is bounded.} \tag{5.4.17}$$

In the subcase (a), by (5.4.16), we can choose a sequence $(r_k) \subset (0, \infty)$ such that

$$r_* \leq \|u_k\|_{L^\infty} - K(r_k^{\frac{p^+ - n}{p^+}} + r_k^{\frac{p^- - n}{p^-}}) \text{ and } r_k \to \infty, \tag{5.4.18}$$

where $K = cM_*$ and c is the same constant as in (5.2.2). For every $y \in \mathbb{R}^n$, we have

$$|u_k(0)| - |u_k(y)| \leq |u_k(0) - u_k(y)|.$$

Hence by (5.2.2), we obtain

$$|u_k(0)| - |u_k(y)| \leq K(|y|^{\frac{p^+-n}{p^+}} + |y|^{\frac{p^--n}{p^-}}).$$

From (5.4.15), we get

$$|u_k(y)| \geq \|u_k\|_{L^\infty} - K(|y|^{\frac{p^+-n}{p^+}} + |y|^{\frac{p^--n}{p^-}}).$$

For $|y| \leq r_k$ and (5.4.18), we have

$$|u_k(y)| \geq \|u_k\|_{L^\infty} - K(r_k^{\frac{p^+-n}{p^+}} + r_k^{\frac{p^--n}{p^-}}) \geq r_*. \tag{5.4.19}$$

From (5.4.14) and (5.4.19), we get

$$\int_{\mathbb{R}^n} V(u_k)dx \geq \int_{B(0,r_k)} V(u_k)dx \geq \frac{\nu}{2} mes(B(0,r_k)).$$

This implies that

$$\int_{\mathbb{R}^n} V(u_k)dx \to \infty \text{ as } \ r_k \to \infty.$$

In the last subcase (b), we assume there exists $\bar{M} > 0$ such that

$$\|u_k\|_{L^\infty} \leq \bar{M}.$$

From (5.4.8), we obtain

$$\int_{\mathbb{R}^n} V(u_k)dx \geq \omega_{\bar{M}} \|u_k\|_{L^2} \to \infty \text{ as } \ k \to \infty.$$

□

We are going to study the behaviour of energie f_a when u approaches the boundary of Γ_a; we remark that $\partial\Gamma_a = E_a \setminus \Gamma_a$.

Lemma 5.4.3. *Let $(u_k) \subset \Gamma_a$ be a weakly converging sequence. If the weak limit belongs to $\partial\Gamma_a$, then*

$$f_a(u_k) \to \infty \text{ as } k \to \infty.$$

Proof. Let $(u_k) \subset \Gamma_a$ such that

$$u_k \rightharpoonup u \in \partial\Gamma_a \text{ as } k \to \infty.$$

Since $u \in \partial\Gamma_a$ then there exists $x_* \in \mathbb{R}^n$ such that $u(x_*) = \eta$. Using the fact that (u_k) is bounded in E_a, then by the uniform convergence on compact sets, we have

$$u_k(x_*) \to u(x_*) \quad \text{as } k \to \infty. \tag{5.4.20}$$

Since (u_k) is bounded in E_a, then ∇u_k is bounded in $L^{p(\cdot)}$. Then from (5.2.2), we obtain

$$|u_k(x) - u_k(x_*)| \leq c \sup(|x - x_*|^{\frac{p^+-n}{p^+}}, |x - x_*|^{\frac{p^--n}{p^-}}). \tag{5.4.21}$$

Thus

$$|u_k(x) - \eta| \leq |u_k(x) - u_k(x_*)| + |u_k(x_*) - \eta|. \tag{5.4.22}$$

By (5.4.20) and (5.4.21), there exists $\epsilon_k > 0$ such that

$$|u_k(x) - \eta| \leq c \sup(|x - x_*|^{\frac{p^+-n}{p^+}}, |x - x_*|^{\frac{p^--n}{p^-}}) + \epsilon_k \tag{5.4.23}$$

where

$$\epsilon_k \to 0 \text{ as } \ k \to \infty.$$

Let $x \in B(0, r)$, for r sufficiently small, there exists $\rho > 0$ such that

$$|u_k(x) - \eta| \leq c \sup(r^{\frac{p^+-n}{p^+}}, r^{\frac{p^--n}{p^-}}) + \epsilon_k < \rho. \tag{5.4.24}$$

From (5.4.24) and (V_4), we have

$$V(u_k(x)) \geq c(|u_k(x) - \eta|^{-(\frac{np^+}{p^+-n})} + |u_k(x) - \eta|^{-(\frac{np^-}{p^--n})}). \tag{5.4.25}$$

Then from (5.4.23) and (5.4.25), we obtain

$$V(u_k(x)) \geq \frac{c}{|x - x_*|^n + \epsilon_k}.$$

Restricting our attention to $B(x_*, r)$, we have

$$\lim_{k\to\infty} \int_{B(x_*,r)} V(u_k(x))dx \geq \int_{B(x_*,r)} \lim_{k\to\infty} V(u_k(x))dx \geq c \int_{B(x_*,r)} \frac{1}{|x - x_*|^n} dx = \infty.$$

□

Corollary 5.4.4. *For every $b > 0$, there exist $d* = d(b)$ such that, for every $u \in \Gamma_a$ we have*

$$f_a(u) \leq b \Rightarrow \min_x |u(x) - \eta| \geq d_*.$$

Proof. The proof is the same as in [Corollary4.4.5-Chapter 4] □

Lemma 5.4.5. *The functional f_a is weakly lower semicontinuous in Γ_a.*

Proof. The proof is the same as in [Lemma 4.4.6-Chapter 4] □

Proposition 5.4.6. *There exists $\Delta_a > 0$ such that, for every $u \in \Gamma_a$ such that for every $u \in \Gamma_a$ satisfied $\|u\|_{L^\infty} \geq 1$ we have*

$$f_a(u) \geq \Delta_a.$$

It is easy to see that $ch(u) \neq 0$ implies $\|u\|_{L^\infty} \geq 1$

Proof. By continuous injection in Proposition 5.2.1

$$\|u\|_a \geq \|u\|_0 \geq \|u\|_{L^\infty} \geq 1,$$

and by the coercivity of f_a, we get

$$\|u\|_a \geq 1 \Rightarrow \exists\ \Delta_a > 0 \text{ such that } \ f_a(u) \geq \Delta_a.$$

□

5.5 Existence result

Theorem 5.5.1. *The minimum points $u \in \Gamma_a$ for the functional f_a are weak solutions of the system (5.1.5).*

Proof. Let u be a minimum point of f_a and $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Let e_j denote the j^{th}-vector of the canonical basis in $\mathbb{R}^n$. If ϵ is sufficiently small, then $u + \epsilon e_j h \in \Gamma_a$ and $f_a(u + \epsilon e_j h) < \infty$. Since u is a minimum point of f_a, then

$$0 = \left.\frac{df(u+\epsilon e_j h)}{d\epsilon}\right|_{\epsilon=0} = \int_{\mathbb{R}^n} \left(a\nabla u_j \nabla h + \frac{b}{2}(p(\cdot)|\nabla u|^{p-2}\nabla u_j \nabla h) + \frac{\partial V(\xi)}{\partial \xi_j} h \right) dx.$$

By Green's formula,

$$\int_{\mathbb{R}^n} \frac{b}{2}(p(.)|\nabla u|^{p-2}\nabla u_j \nabla h)dx = \int_{\mathbb{R}^n} -\frac{b}{2} div(p(.)|\nabla .u|^{p-2}\nabla u_j) h dx,$$

$$\int_{\mathbb{R}^n} a\nabla u_j \nabla h = \int_{\mathbb{R}^n} -a\Delta u_j h.$$

So

$$\int_{\mathbb{R}^n} \left(-a\Delta u_j - \frac{b}{2} div(p(.)|\nabla .u|^{p-2}\nabla u_j) + \frac{\partial V(\xi)}{\partial \xi_j} \right) h dx = 0,$$

for $1 \leq j \leq n+1$, and for any $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Then

$$\int_{\mathbb{R}^n} \left[-a\Delta u - \frac{b}{2}\Delta_{p(\cdot)} u + V'(u) \right] \phi dx = 0, \text{ for every } \phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

This implies by density

$$-a\Delta u - \frac{b}{2}\Delta_{p(\cdot)} u + V'(u) = 0.$$

□

Proposition 5.5.2. *(Splitting proposition) Let $(u_k) \in \Gamma_a^*$ be a sequence and M be a positive real number such that*

$$f_a(u_k) \leq M.$$

Then there exists $l \in \mathbb{N}$ such that

$$1 \leq l \leq \frac{M}{\Delta_a}$$

where Δ_a was introduced in Proposition 5.4.6 and there exist $\bar{u}_1, \ldots, \bar{u}_l \in \Gamma_a$, $(x_k^1), \ldots, (x_k^l) \subset \mathbb{R}^n$ such that, up to a subsequence,

$$u_k(\cdot + x_k^i) \to \bar{u}_i,$$

$$|x_k^i - x_k^j| \to \infty, \quad i \neq j,$$

$$\sum_{i=1}^{l} f_a(\bar{u}_i) \leq \liminf_{k\to\infty} f_a(u_k),$$

and

$$ch(u_k) = \sum_{i=1}^{l} ch(\bar{u}_i).$$

Proof. From Lemmas 5.4.2, 5.4.3 and 5.4.5, and by the same method as used in [Proposition 4.5.1-Chapter 4], we can conclude the result of this proposition. □

The minimum is attained on the set Γ_a, and it is easy to see that $u \equiv 0$ is a trivial solution of the problem. But, of course, we are interested in nontrivial solutions, We consider the following problem

$$I_* = \inf_{u\in\Gamma_a^*} f_a(u), \quad \Gamma_a^* = \{u \in E_a : ch(u) \neq 0\}.$$

The functional is bounded below and the set E_a is not empty. We consider fields u having the form

$$u(x) = (\frac{2}{1+|x|^m}, \frac{1}{1+|x|^m}x). \tag{5.5.1}$$

Lemma 5.5.3. *There exists a suitable $m \geq 1$, such that, the field u defined in (5.5.1) belongs to Γ_a^* .*

Proof. Clearly, if m is sufficiently large, then the field u defined in (5.5.1) belongs to E_a. For the sake of contradiction, suppose that there exists $\bar{x} \in \mathbb{R}^n$ such that $u(\bar{x}) = \eta = (1, 0)$. We deduce that

$$\frac{2}{1+|\bar{x}|^m} = 1,$$

$$\frac{1}{1+|\bar{x}|^m}\bar{x} = 0.$$

We get the contradiction : $|\bar{x}| = 1$ and $\bar{x} = 0$. So, $u \in \Gamma_a$.
We show that $ch(u) \neq 0$. Set $g(x) = \frac{1}{2}x$, then we have

$$K_u = \{x \in \mathbb{R}^n : \frac{2}{1+|x|^m} > 1\} = B(0,1),$$

$$if |x| = 1 \quad then \quad g(x) = \frac{1}{1+|x|^m}x,$$

by the properties of the topological degree (see Section 2.8) we get,

$$deg(\frac{1}{1+|x|^m}x, B(0,1), 0) = deg(g(x), B(0,1), 0) \neq 0.$$

□

And moreover the set Γ_a^* is open in the space E_a; indeed,

- $\Gamma_a^* = \bigcup_{q \in \mathbb{N}^*} \Gamma_a^q$,
- $\Gamma_a^q \cap \Gamma_a^p = \emptyset, \quad p \neq q$.

where Γ_q is a connected component

Theorem 5.5.4. *Let $a \geq 0$, $b > 0$, $p^- > n > 2$. Then, if V satisfies $(V_1) - (V_6)$, and if p satisfies $(p_1) - (p_2)$, then there exists a weak solution of (5.1.5) (i.e., a static solution of (5.1.3)), which is a minimizer of the energy functional (5.1.7) in the class of maps whose topological charge is different from 0.*

Proof. By the Splitting Proposition and the same technique used in Chapter 4 - Theorem 4.5.3, we can conclude that there exists weak solution to (5.1.5).

And with suitable change of variable in (5.1.6) we deduce a solution of equation (5.1.3).

□

Remark 5.5.1. *The functional exibits an invariance for the symmetry group of rotations and translations; indeed, for every function u and $g \in O(n)$, if we set $u_g(x) = u(gx)$, we have immediately*

$$f_a(u_g) = f_a(u).$$

Then our theorem gives the existence of an orbit of minimum solutions. This orbit consists of two connected components, which are identified, respectively, by $\bar{u}$ and

$$\bar{u} \circ \mathcal{P}(x) = \bar{u}(-x).$$

Since typically $n = 3$ is odd, $\bar{u} \circ \mathcal{P}$ and $\bar{u}$ have opposite topological charge.

Chapter 6

Derrick's problem with twice variable exponent

In the mathematical models (soliton) studied in chapter 4 the space of the finite energy configurations (solution space) splits into infinitely many connected components according to the topological charge. In that Chapter was proved the existence of infinitely many solutions, which are constrained minima of the energy. More precisely, on every one connected component characterized by a topological charge equal to $N \in \mathbb{N}$ there exists a solution of charge N. Since p is arbitrary in the static equation, it is natural to consider $p = p(x)$ as a variable that depends on the connected component. The aim of this chapter is to carry out an existence analysis of the finite-energy static solutions in more than one space dimension ($n \geq 2$) for a class of Lagrangian densities L which include (5.1.1) in Chapter 5 and generalizing the results of the Chapter 4. More precisely we are concerned with Generalized Sobolev Spaces with twice variable exponent $r(\cdot) \leq n$ and $p(\cdot) > n$.

6.1 Statement of the Problem

The class of Lagrangian densities we consider generalizes the problem studied in Chapter 5, in such a way as to include the Derrick proposal.

First we introduce some notation. If n, m are positive integers, and they will denote, respectively, the physical space-time (typically $n = 3$) and the internal parameters space. We are interested in the multi-dimensional case, so we assume that $n \geq 2$. A point in $\mathbb{R}^{n+1}$ will be denoted by $X = (x, t)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The fields we are interested in are maps $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^m$, $\psi = (\psi_1, \ldots, \psi_m)$. We set

$$\rho = |\nabla\psi|^2 - |\psi_t|^2,$$

$\nabla\psi$ and ψ_t denoting, respectively, the Jacobian with respect to x and the derivative with respect to t. Let

$$s : \mathbb{R}^{n+1} \to \mathbb{R}.$$

We shall consider Lagrangian densities of the form

$$\mathcal{L}(\psi, \rho) = -\frac{1}{2}\alpha(\rho, k, s) - V(\psi), \tag{6.1.1}$$

where the function V is a real function defined in an open subset $\Omega \subset \mathbb{R}^m$ and α is a real function defined by

$$\alpha(\rho, s, k) = a\rho|\rho|^{\frac{k(\cdot)}{2}-1} + b|\rho|^{\frac{s(\cdot)}{2}},\ a \geq 0,\ b > 0,\ s(0) > n,\ 2 \leq k(\cdot) < n. \tag{6.1.2}$$

The results of [10] were concerned with the case: $s(\cdot) \equiv p$ and $k(\cdot) \equiv 2$, (we fix the variable exponent). The action functional related to (6.1.1) is

$$\begin{aligned} S(\psi) &= \int_{\mathbb{R}^{n+1}} \mathcal{L}(\psi, \rho)dxdt \\ &= \int_{\mathbb{R}^{n+1}} -\frac{1}{2}\alpha(\rho, k, s) - V(\psi)dxdt. \end{aligned}$$

So the Euler-Lagrange equations are

$$\frac{\partial}{\partial t}(\alpha'\psi_t) - \nabla\big(\alpha'\nabla\psi\big) + V'(\psi) = 0, \tag{6.1.3}$$

where $\nabla\big(\alpha'\nabla\psi\big)$ denotes the vector whose $j-th$ component is given by $div\big(\alpha'\nabla\psi^j\big)$, and V' denotes the gradient of V. The equation (6.1.3) is Lorentz invariant. Static solutions $\psi(x,t) = u(x)$ of (6.1.3) solve the equation

$$-\nabla\big(\alpha'\nabla u\big) + V'(u) = 0. \tag{6.1.4}$$

Set $k(x,t) = r(x), s(x,t) = p(x)$ on $\mathbb{R}^n$ (the restrictions of s on $\mathbb{R}^n$). Using (6.1.2) and (6.1.4) we obtain

$$-\frac{a}{2}\Delta_{r(\cdot)} - \frac{b}{2}\Delta_{p(\cdot)} + V'(u) = 0, \tag{6.1.5}$$

where

$$\Delta_{r(\cdot)}u = \nabla\big(r(\cdot)|\nabla u|^{r(\cdot)-2}\nabla u\big),\quad \Delta_{p(\cdot)}u = \nabla\big(p(\cdot)|\nabla u|^{p(\cdot)-2}\nabla u\big).$$

We introduce the following notations and functional spaces:

$$C_+(\mathbb{R}^n) = \{p \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) :\ p(x) > 1\ \text{ for all } x \in \mathbb{R}^n\}$$

and

$$p^+ = \sup_{x\in\mathbb{R}^n} p(x),\quad p^- = \inf_{x\in\mathbb{R}^n} p(x).$$

We assume that

(p_1) $S(x,t) = p(\frac{x_1 - t\nu}{\sqrt{1-\nu^2}}, \ldots, x_n)$, where ν is a parameter used in the Lorentz transformation.

(p_2) $\lim\limits_{x\to\infty} p(x) = p_\infty = p^- > n$.

(r_1) $k(x,t) = r(\frac{x_1 - t\nu}{\sqrt{1-\nu^2}}, \ldots, x_n)$, where ν is a parameter used in the Lorentz transformation.

(r_2) $\lim\limits_{x\to\infty} r(x) = r_\infty = 2 = r^- \leq r^+ \leq n$.

(r_3) There exists $c > 0$ such that for all balls, $\forall x \in B$ we have $|B|^{r_B^- - r(x)} < c$.

(r_4) For all $x \in \mathbb{R}^n, |r(x) - 2| < \frac{1}{log|e+|x||}$.

Recall that the results of [10] were concerned with the case

$$r(\cdot) \equiv 2 \textit{ and } p(\cdot) \equiv p^- > n.$$

Under (p_1), it is easy to verify that, if $u = u(x)$ is a solution of the (6.1.3) and $v = (\nu, 0, \ldots, 0)$ with $|\nu| < 1$, the field

$$\psi_\nu(x,t) = u\left(\frac{x_1 - \nu t}{\sqrt{1-\nu^2}}, x_2, \ldots, x_n\right) \tag{6.1.6}$$

is solution of (6.1.3). Notice that the function undergoes a contraction by a factor,

$$\gamma = \frac{1}{\sqrt{1-\nu^2}},$$

in the direction of the motion; this is a consequence of the fact that (6.1.3) is Lorentz invariant. Clearly (6.1.5) are the Euler-Lagrange equations with respect to the energy functional

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^{r(x)} + \frac{b}{2}|\nabla u|^{p(x)} + V(u)\right) dx, \tag{6.1.7}$$

where $m = n + 1$, so the time independent fields u are maps

$$u : \mathbb{R}^n \to \mathbb{R}^m.$$

For every $\xi \in \mathbb{R}^{n+1}$, we write $\xi = (\xi_0, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^n$. Let $V : \Omega \to \mathbb{R}$, where $\Omega = \mathbb{R}^{n+1} \setminus \{\eta\},\ \eta = (1, 0)$, and V is positive and singular in η. More precisely we assume:

(V_1) $V \in C^1(\Omega, \mathbb{R})$.

(V_2) $V(\xi) \geq V(0) = 0$.

(V_3) V is twice differentiable at 0 and the Hessian matrix $V''(0)$ is nondegenerate.

(V_4) There exist $c, \rho > 0$ such that if $|\xi| < \rho$ then

$$V(\eta + \xi) \geq c\left(|\xi|^{-q^+} + |\xi|^{-q^-}\right)$$

where

$$\frac{1}{q^-} = \frac{1}{n} - \frac{1}{p^-}, \quad \frac{1}{q^+} = \frac{1}{n} - \frac{1}{p^+}.$$

(V_5) For every $\xi \in \Omega \backslash \{0\}$ we have

$$V(\xi) > 0, \text{ and } \lim_{|\xi| \to \infty} \inf V(\xi) = v > 0.$$

(V_6) There exist $R > 0$, $|\xi| < R \Longrightarrow V(\xi) \geq \omega_R |\xi|^{r^+},\ \omega_R > 0.$

Example 6.1.1. *A potential satisfying the assumptions* $(V_1) - (V_6)$ *is*

$$V(\xi) = \omega_0^2\left(|\xi|^{r^+} + \frac{|\xi|^4}{|\xi - \eta|^{q^+} + |\xi - \eta|^{q^-}}\right).$$

6.2 Solution space

Let $p^- > n \geq 2$, $2 = r^- \leq r^+ \leq n$, and with no loss of generality, we can consider the functional (6.1.7) with $b = 1$. It will be convenient to introduce the following notation:

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^{r(x)} + \frac{1}{2}|\nabla u|^{p(x)} + V(u)\right) dx,$$

and we define the space E_a to be the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|_a = a\|\nabla u\|_{L^{r(\cdot)}} + \|\nabla u\|_{L^{p(\cdot)}} + \|u\|_{L^{r(\cdot)}}, \quad a \geq 0,$$

i.e.,

$$E_a = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_a};$$

$$\|u\|_{r(x)} = \inf\left\{\sigma > 0 : \mathbb{R}^n \to \mathbb{R} : \int_{\mathbb{R}^n} \left|\frac{u(x)}{\sigma}\right|^{r(x)} dx \leq 1\right\},$$

$$\|\nabla u\|_{r(x)} = \inf\left\{\sigma > 0 : \mathbb{R}^n \to \mathbb{R} : \int_{\mathbb{R}^n} \left|\frac{\nabla u(x)}{\sigma}\right|^{r(x)} dx \leq 1\right\},$$

and

$$\|\nabla u\|_{p(x)} = \inf\left\{\sigma > 0 : \mathbb{R}^n \to \mathbb{R} : \int_{\mathbb{R}^n} \left|\frac{\nabla u(x)}{\sigma}\right|^{p(x)} dx \leq 1\right\}.$$

For every $a > 0$, the norms $\|\cdot\|_a$ are equivalent, so we have to study only two cases: $a = 0,\ a > 0$.

Remark 6.2.1.
We have $\lim_{x\to\infty} r(x) = r_\infty = 2 = r^- \leq r^+ \leq n$.

- *From (r3), (r4), Lemma 2.4.2 and Definition 2.4.2 it's easy to see that* $r(\cdot)$ *is globally Hölder continuous.*
- *From Definition 2.4.3 and Remark 2.4.1 it's easy to see that* $r \in \mathcal{P}^{log}(\mathbb{R}^n)$.
- *Then* $L^{r(.)}(\mathbb{R}^n, \mathbb{R}^{n+1})$ *is continuously embedded in* $L^{r^-}(\mathbb{R}^n, \mathbb{R}^{n+1})$*; see ([41, Proposition 4.1.8 (page 103)]).*

Proposition 6.2.1. *The Banach space* E_0 *is continuously embedded in* $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$, *for every* $s \in [r^-, \infty], r^- = 2$.

Proof. From Remark 6.2.1, the space E_0 is continuously embedded in $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, therefore it is sufficient to show that E_0 is embedded also in $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$. Since $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in E_0, it is sufficient to prove that there exists $c > 0$ such that, for every $u \in C_0(\mathbb{R}^n, \mathbb{R}^{n+1})$, we have

$$\|u\|_{L^\infty} \leq c\|u\|_0.$$

We fix $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ and consider a family of cubes $Q_k \subset \mathbb{R}^n$ such that

$$mes(Q_k) = 1, \ \cup_{k\in\mathbb{N}} Q_k = \mathbb{R}^n.$$

Then, by a well-known inequality, see equation (5.2.4) in [Proposition 5.4.6 - Chapter 5], $Q_k \subset \mathbb{R}^n$, then

$$|u(x)| \leq \left| \int_{Q_k} u dy \right| + M\|\nabla u\|_{L^{p(\cdot)}(Q_k)}, \tag{6.2.1}$$

where $M \geq 0$ is independent of u. Thus

$$\begin{aligned} |u(x)| &\leq mes(Q_k)\|u\|_{L^{r(.)}} + M\|\nabla u\|_{L^{p(\cdot)}(Q_k)} \\ &\leq \|u\|_{L^{r(.)}(\mathbb{R}^n)} + M\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq (1 + M)\|u\|_0. \end{aligned}$$

Hence

$$\|u\|_{L^\infty} \leq c\|u\|_0, \quad c = 1 + M.$$

□

Corollary 6.2.2. *The Banach space* E_0 *is continuously embedded in* $L^{p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$.

Proof. Since $2 \leq n < p_0 \leq p^- < p^+ < \infty$, $E_0 = \overline{C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})}^{\|\cdot\|_0}$, so $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in E_0 and it is also dense in $L^{p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see [41, Theorem 3.4.12]). So it is sufficient to prove that there exists $c > 0$ such that

$$\|u\|_{L^{p(\cdot)}} \leq c\|u\|_0, \text{for all} u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

Let B be the support of u, then

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|u\|_{L^{p(\cdot)}(B)}.$$

From [41, Theorem 3.3.1, p. 82], we have

$$\|u\|_{L^{p(\cdot)}(B)} \leq \|u\|_{L^{p^+}(B)}.$$

It is clear that

$$\|u\|_{L^{p^+}(B)} \leq \|u\|_{L^{p^+}(\mathbb{R}^n)}.$$

From Proposition 6.2.1, we deduce that there exists $c > 0$ such that

$$\|u\|_{L^{p^+}(\mathbb{R}^n)} \leq c\|u\|_0.$$

This implies that

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|u\|_{L^{p(\cdot)}(B)} \leq \|u\|_{L^{p^+}(B)} \leq \|u\|_{L^{p^+}(\mathbb{R}^n)} \leq c\|u\|_0.$$

□

Corollary 6.2.3. *The Banach space E_0 is continuously embedded in $H_0^{1,p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$.*

Proof. By definition of the space E_0, we have for every $u \in E_0$

$$\|u\|_0 > \|\nabla u\|_{L^{p(\cdot)}}.$$

From Corollary 6.2.2 there exists $c_1 > 0$ such that

$$c_1\|u\|_0 > \|u\|_{L^{p(\cdot)}},$$

and so

$$\|u\|_0 > c\|u\|_{H_0^{1,p(\cdot)}}.$$

□

Remark 6.2.2.
We have $r_\infty = 2 = r^- \leq r^+ \leq n$.

- *From Remark 6.2.1 we have $r \in \mathcal{P}^{log}(\mathbb{R}^n)$, then $H_0^{1,r(.)}(\mathbb{R}^n) = W_0^{1,r(.)}(\mathbb{R}^n)$; see ([41, Corollary 11.2.4 (page 347)]).*

- *Since* $r \in \mathcal{P}(\mathbb{R}^n)$ *is bounded, then* $W_0^{1,r(.)}(\mathbb{R}^n) = W^{1,r(.)}(\mathbb{R}^n)$*; see ([41, Corollary 9.1.3 (page 291)]).*

Corollary 6.2.4. *For every* $a > 0$*, the space* E_a *can be identified with the Banach space*

$$W = H_0^{1,p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1}) \cap W^{1,r(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1}),$$

equipped with the usual norm

$$\|u\|_W = \|u\|_{W^{1,r(\cdot)}} + \|u\|_{W^{1,p(\cdot)}}.$$

Proof. $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in $H_0^{1,p(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$ (see Definition 2.4.1), and $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ is dense in $W^{1,r(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$; (see [41, Theorem 9.1.6 (page 291)]). For any $u \in E_a$ we have

$$\|u\|_a \leq \sup(1, a)\|u\|_W.$$

From Corollary 6.2.2, there exists $c > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$, we have

$$\|u\|_a \geq c(\|u\|_{W^{1,2}} + \|u\|_{W^{1,p(\cdot)}}).$$

□

Proposition 6.2.5. *Since* $p > n$*, for every value* $a \geq 0$*, the functions in* E_a *are bounded, continuous, and decay to zero at infinity. Furthermore, the following inequality holds:*

$$|u(x) - u(y)| \leq c \sup\left(|x-y|^{1-\frac{n}{p^-}}, |x-y|^{1-\frac{n}{p^+}}\right) \|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \text{ for all } x, y \in \mathbb{R}. \tag{6.2.2}$$

Proof. The proof is the same as in [Proposition 5.4.6 - Chapter 5]. □

Remark 6.2.3. *By Proposition 6.2.1 we have*

$$E_a \subset E_0 \subset L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}). \tag{6.2.3}$$

We deduce from Proposition 6.2.5 that if $u \in E_a$ *with* $n < p^- < \infty$*, then* u *is bounded and*

$$\lim_{|x|\to\infty} u(x) = 0.$$

Recall that η is a singular point of the potential V, so it is reasonable to consider in space E_a, the open subset

$$\Gamma_a = \{u \in E_a : u(x) \neq \eta, \text{ for all } x \in \mathbb{R}^n\}.$$

The subset Γ_a is open in E_a. Indeed, since by Remark 6.2.3, we have

$$\inf_{x\in\mathbb{R}^n} |u(x) - \eta| = d > 0.$$

Then, by (6.2.3) (E_a is continuously embedded in L^∞), we deduce that for all $u \in \Gamma_a$, there exists a small neighborhood of u contained in Γ_a.

The boundary of Γ_a is given by

$$\begin{aligned} \partial\Gamma_a &= \{u \in E_a : \text{there exist } x \in \mathbb{R}^n \text{ such that } u(x) = \eta\} \\ &= E_a \setminus \Gamma_a. \end{aligned}$$

6.3 Topological charge and connected components of Γ_a

For the sake of simplicity, we consider the function space

$$C = \left\{ u : \mathbb{R}^n \to \mathbb{R}^{n+1} \setminus \{\eta\} \text{ is continuous and } \lim_{|x|\to\infty} u(x) = 0 \right\}$$

where $\eta = (1, 0)$. Every function $u \in C$ we write in the form $u(x) = (u_0(x), \tilde{u}(x)) \in \mathbb{R}^{n+1}$ where $u_0 : \mathbb{R}^n \to \mathbb{R}$ and $\tilde{u} : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 6.3.1. *For every function $u \in C$ we define the support of u*

$$K_u = \{x \in \mathbb{R}^n : u_0(x) > 1\}.$$

Then we define the topological charge of u

$$ch(u) := \begin{cases} deg(\tilde{u}, K_u, 0) & \text{if } K_u \neq \emptyset, \\ \\ 0 & \text{if } K_u = \emptyset, \end{cases}$$

such that the Brouwer degree [69],

$$deg(\tilde{u}, K_u, 0) = \sum_{x \in \tilde{u}^{-1}(0)} sgn J_{\tilde{u}}(x),$$

where $J_{\tilde{u}}$ denotes the determinant of the Jacobian matrix.

From $\lim_{|x|\to\infty} u(x) = 0$, we have that K_u is an open, bounded set in $\mathbb{R}^n$. Since $u \in \Gamma$, if $x \in \partial K_u$, we have, together with $u(x) \neq \eta$, that $\tilde{u}(x) \neq 0$. Therefore the previous definition is well posed. Moreover, the topological charge is continuous with respect to the uniform convergence; see [12]. We notice that this definition of charge is the same as in [10]. Now, for every $q \in \mathbb{Z}$ we set

$$\Gamma_a^q = \{u \in \Gamma_a : ch(u) = q\}.$$

Since the topological charge is continuous with respect to the uniform convergence and the continuity of the embeddings E_a in L^∞ assure that the topological charge is continuous on Γ_a, it follows that Γ_a^q is open in E_a, since we have also

- $\Gamma_a = \bigcup_{q\in\mathbb{Z}} \Gamma_a^q$,
- $\Gamma_a^q \cap \Gamma_a^p = \emptyset, \quad p \neq q.$

We conclude that every Γ_a^q is a connected component of Γ_a. We observe that for every $q \in \mathbb{Z}$ the component Γ_a^q is isomorphic to the component Γ_a^{-q}. containedin the space C, which we have considered in the preceding Section. So for every $u \in \Gamma_a$ we can define the charge $ch(u) \in \mathbb{Z}$. Now, we consider the set of a minimizer of f_a in the open set

$$\Gamma_a^* = \{u \in \Gamma_a : ch(u) \neq 0\}.$$

Remark 6.3.1. *We can easily see that $ch(u) \neq 0$ implies $\|u\|_{L^\infty} > 1$.*

For more information about this subject, see [69].

6.4 Properties of the energy functional

Lemma 6.4.1. *The functional f_a takes real values and it is continuous on Γ_a.*

Proof. We have

$$f_a(u) = \int_{\mathbb{R}^n} \left(\frac{a}{2}|\nabla u|^{r(x)} + \frac{b}{2}|\nabla u|^{p(x)}\right) dx + \int_{\mathbb{R}^n} V(u)dx,$$

$$= \underbrace{\frac{a}{2}\rho_{r(x)}(u) + \frac{b}{2}\rho_{p(x)}(u)} + \underbrace{\int_{\mathbb{R}^n} V(u)dx}.$$

The first term on the left-hand side of energy f_a is finite and continuous. Let us prove that the second term is finite and continuous.

We have $V(\xi) = (V''(0)\xi \cdot \xi + o(\xi^2)$. By (V_3) there exist a small neighborhood of $0 \in \mathbb{R}^{n+1}$ and $M > 0$ such that, for every $\xi \in \mathbb{R}^{n+1}$, we have

$$V(\xi) \leq M|\xi|^2. \tag{6.4.1}$$

Since every $u \in E_a$ decays to zero at infinity (see Proposition 6.2.5), there exists a ball B_u such that,for every $x \in \mathbb{R}^n \backslash B_u$, $|u(x)| < \epsilon$, by (6.4.1), and for ϵ sufficiently small

$$V(u(x)) \leq M|u(x)|^2. \tag{6.4.2}$$

From $u \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, we deduce

$$\int_{\mathbb{R}^n \backslash B_u} V(u)dx < \infty.$$

On the other hand, since u is continuous (see Proposition 6.2.5), we also have

$$\int_{B_u} V(u)dx < \infty.$$

$$\int_{\mathbb{R}^n} V(u_k) \longrightarrow \int_{\mathbb{R}^n} V(u).$$

Since $f_a(u_k) < \infty$ and with Lemma 6.4.3, u belongs to Λ_a. By (6.2.3) we have $u_k \to u$ on $L^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$.

$$\int_{B_u} V(u_k)dx \to \int_{B_u} V(u)dx. \tag{6.4.3}$$

By (6.4.2)

$$\int_{\mathbb{R}^n \setminus B_u} V(u(x))dx \leq \int_{\mathbb{R}^n \setminus B_u} |u(x)|^2 dx,$$

and since $u_k \to u \in L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, the dominated convergence theorem gives

$$\int_{\mathbb{R}^n \setminus B_u} V(u_k)dx \to \int_{\mathbb{R}^n \setminus B_u} V(u)dx. \tag{6.4.4}$$

□

Lemma 6.4.2. *The functional f_a is coercive in Γ_a; that is, for every sequence $u_k \subset \Gamma_a$ such that $\|u_k\|_a \to \infty$, we have $f_a(u_k) \to \infty$.*

Proof. In the case $a > 0,\ n > r(x) \geq 2$, we have

$$\|u\|_a = a\|\nabla u\|_{L^{r(\cdot)}} + \|\nabla u\|_{L^{p(\cdot)}} + \|u\|_{L^{r(\cdot)}}.$$

Let $u_k \in \Gamma_a$ such that

$$\|u_k\|_a \to \infty \ \text{ as } \ k \to \infty.$$

It is clear that, if

$$a\|\nabla u_k\|_{L^{r(\cdot)}} + \|\nabla u_k\|_{L^{p(\cdot)}} \to \infty \ \text{ as } \ k \to \infty, \tag{6.4.5}$$

we have

$$f_a(u_k) \to \infty \ \text{ as } \ k \to \infty.$$

Assume now that there exists $c_* > 0$ such that

$$a\|\nabla u_k\|_{L^{r(\cdot)}} + \|\nabla u_k\|_{L^{p(\cdot)}} < c_* \tag{6.4.6}$$

and

$$\|u_k\|_{L^{r(\cdot)}} \to \infty \ \text{ as } \ k \to \infty. \tag{6.4.7}$$

We shall prove that

$$\int_{\mathbb{R}^n} V(u_k)dx \quad \text{as} \quad k \to \infty.$$

From (V_3), we have for every $R > 0$ there exists $\omega_R > 0$ such that

$$|\xi| \leq R \Rightarrow V(\xi) \geq \omega_R |\xi|^2, \ r^- = 2. \tag{6.4.8}$$

From (V_6), and (6.4.8), we have there exists $R > 0$ such that

$$|\xi| \leq R \Rightarrow V(\xi) \geq \omega_R |\xi|^{r(\cdot)}, \ \omega_R > 0. \tag{6.4.9}$$

For every $k \in \mathbb{N}$, we set

$$A_k = \{x \in \mathbb{R}^n : \ |u_k(x)| \leq R\},$$

where $u_k \in W^{1,r(\cdot)}(\mathbb{R}^n, \mathbb{R}^{n+1})$. By the inequality (see [41, Theorem 8.3.1 (page 265)])

$$\|u_k\|_{L^{r^*(\cdot)}} \leq c\|\nabla u_k\|_{L^{r(\cdot)}}, \ r^*(x) = \frac{r(x).n}{n - r(x)}, \ n > r(x) \geq 2. \tag{6.4.10}$$

From (6.4.6), we obtain

$$\|u_k\|_{L^{r^*(\cdot)}} < c_*. \tag{6.4.11}$$

Moreover, from (5.2.4), there exists $M \geq 0$ independent of u_k, such that

$$\begin{aligned} |u_k(x)| &\leq \left| \int_{Q_k} u_k dy \right| + M\|\nabla u_k\|_{L^{p(\cdot)}(Q_k)}, \ mes(Q_k) = 1, \\ &\leq \|u_k\|_{L^{r^*(\cdot)}(Q_k)} + M\|\nabla u_k\|_{L^{p(\cdot)}(Q_k)}. \end{aligned}$$

By (6.4.5) and (6.4.11), for any $x \in \mathbb{R}^n$, we have

$$|u_k(x)| < c_* + Mc_*. \tag{6.4.12}$$

Then, there exists $c > 0$ such that

$$mes(\mathbb{R}^n \backslash A_k) < c. \tag{6.4.13}$$

From (6.4.12) and (6.4.13), we deduce that there exists $c_1 > 0$ such that

$$\int_{\mathbb{R}^n \backslash A_k} |u_k|^{r(\cdot)} dx < c_1. \tag{6.4.14}$$

By (6.4.11), we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} V(u_k)dx &\geq \int_{A_k} V(u_k)dx \\ &\geq \omega_R \int_{A_k} \|u_k\|^{r(\cdot)} dx \\ &\geq \omega_R \left(\rho_{r(x)}(u) - \int_{\mathbb{R}^n \backslash A_k} |u_k|^{r(\cdot)} dx \right). \end{aligned}$$

From (6.4.14) and (6.4.7), we have

$$\lim_{k} \int_{\mathbb{R}^n} V(u_k)dx \geq \omega_R(\rho_{r(x)}(u) - c_1) \to \infty \text{ as } k \to \infty.$$

In the case, $a = 0$ or $n = 2 \equiv r(\cdot)$, by (V_5), there exists $r_* > 0$ such that, for every $\xi \in \mathbb{R}^n$ with $|\xi| \geq r_*$, we have

$$V(\xi) \geq \frac{\nu}{2}. \tag{6.4.15}$$

Let $u_k \in \Gamma_a$ be a sequence such that

$$\|u_k\|_0 \to \infty \text{ as } k \to \infty.$$

Since the functional f_a is invariant with respect to translation in $\mathbb{R}^n$, we can assume

$$\|u_k\|_{L^\infty} = |u_k(0)|. \tag{6.4.16}$$

$$\|\nabla u_k\|_{L^{p(\cdot)}} \leq M_* \quad \text{and} \quad \|u_k\|_{L^2} \to \infty \quad \text{as} \quad k \to \infty.$$

Here we have two subcases:

(a)

$$\|u_k\|_{L^\infty} \to \infty \text{ as } k \to \infty. \tag{6.4.17}$$

(b)

$$\|u_k\|_{L^\infty} \quad \text{is bounded.} \tag{6.4.18}$$

In the subcase (a), by (6.4.17), we can choose a sequence $(R_k) \subset (0, \infty)$ such that

$$R_* \leq \|u_k\|_{L^\infty} - K(R_k^{\frac{p^+-n}{p^+}} + R_k^{\frac{p^--n}{p^-}}) \text{ and } R_k \to \infty, \tag{6.4.19}$$

where $K = cM_*$ and c is the same constant as in (6.2.2). For every $y \in \mathbb{R}^n$, we have

$$|u_k(0)| - |u_k(y)| \leq |u_k(0) - u_k(y)|.$$

Hence by (6.2.2), we obtain

$$|u_k(0)| - |u_k(y)| \leq K(|y|^{\frac{p^+-n}{p^+}} + |y|^{\frac{p^--n}{p^-}}).$$

From (6.4.16), we get

$$|u_k(y)| \geq \|u_k\|_{L^\infty} - K(|y|^{\frac{p^+-n}{p^+}} + |y|^{\frac{p^--n}{p^-}}).$$

For $|y| \leq R_k$ and (6.4.19), we have

$$|u_k(y)| \geq \|u_k\|_{L^\infty} - K(R_k^{\frac{p^+-n}{p^+}} + R_k^{\frac{p^--n}{p^-}}) \geq R_*. \tag{6.4.20}$$

From (6.4.15) and (6.4.20), we get

$$\int_{\mathbb{R}^n} V(u_k)dx \geq \int_{B(0,r_k)} V(u_k)dx \geq \frac{\nu}{2} mes(B(0,R_k)).$$

This implies that

$$\int_{\mathbb{R}^n} V(u_k)dx \to \infty \text{ as } \ R_k \to \infty.$$

In the subcase (b), we assume there exists $\bar{M} > 0$ such that

$$\|u_k\|_{L^\infty} \leq \bar{M}.$$

From (6.4.8), we obtain

$$\int_{\mathbb{R}^n} V(u_k)dx \geq \omega_{\bar{M}} \|u_k\|_{L^2} \to \infty \text{ as } \ k \to \infty.$$

□

We are going to study the behaviour of energy f_a when u approaches the boundary of Γ_a. We remark that $\partial\Gamma_a = E_a \setminus \Gamma_a$.

Lemma 6.4.3. *Let $(u_k) \subset \Gamma_a$ be a weakly converging sequence. If the weak limit belongs to $\partial\Gamma_a$, then*

$$f_a(u_k) \to \infty \text{ as } k \to \infty.$$

Proof. The proof is the same as in [Lemma 5.4.3 - Chapter 5]. □

Corollary 6.4.4. *For every $b > 0$, there exists $d_* = d(b)$ such that, for every $u \in \Gamma_a$ we have*

$$f_a(u) \leq b \Rightarrow \min_x |u(x) - \eta| \geq d_*.$$

Proof. The proof is the same as in [Corollary 4.4.5 - Chapter 4] □

Lemma 6.4.5. *The functional f_a is weakly lower semicontinuous in Γ_a.*

Proof. The proof is the same as in [Lemma 4.4.6 - Chapter 4] □

Proposition 6.4.6. *There exists $\Delta_a > 0$ such that, for every $u \in \Gamma_a$ satisfying $\|u\|_{L^\infty} \geq 1$, we have*

$$f_a(u) \geq \Delta_a.$$

It is easy to see that $ch(u) \neq 0$ implies $\|u\|_{L^\infty} \geq 1$.

Proof. The proof is the same as in [Proposition 5.4.6 - Chapter 5] □

6.5 Existence result

Theorem 6.5.1. *The minimum points $u \in \Gamma_a$ for the functional f_a are weak solutions of the system (6.1.5).*

Proof. Let u be a minimum point of f_a and $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Let e_j denote the j^{th}-vector of the canonical basis in $\mathbb{R}^n$. If ϵ is sufficiently small, then $u + \epsilon e_j h \in \Gamma_a$ and $f_a(u + \epsilon e_j h) < \infty$. Since u is a minimum point of f_a, then for $1 \leq j \leq n+1$,

$$\begin{aligned} 0 &= \left. \frac{df(u+\epsilon e_j h)}{d\epsilon} \right|_{\epsilon=0} \\ &= \int_{\mathbb{R}^n} \left(\frac{a}{2}(r(\cdot)|\nabla u|^{r(\cdot)-2} \nabla u_j \nabla h) + \frac{b}{2}(p(\cdot)|\nabla u|^{p(\cdot)-2} \nabla u_j \nabla h) + \frac{\partial V(\xi)}{\partial \xi_j} h \right) dx. \end{aligned}$$

By Green's formula,

$$\int_{\mathbb{R}^n} \frac{b}{2}(p(\cdot)|\nabla u|^{p-2} \nabla u_j \nabla h) dx = \int_{\mathbb{R}^n} -\frac{b}{2} div(p(\cdot)|\nabla \cdot u|^{p-2} \nabla u_j) h dx.$$

So

$$\int_{\mathbb{R}^n} \left(-\frac{a}{2} div(r(\cdot)|\nabla \cdot u|^{r-2} \nabla u_j) - \frac{b}{2} div(p(\cdot)|\nabla \cdot u|^{p-2} \nabla u_j) + \frac{\partial V(\xi)}{\partial \xi_j} \right) \cdot h dx = 0,$$

for $1 \leq j \leq n+1$, and for any $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Then

$$\int_{\mathbb{R}^n} \left[-\frac{a}{2} \Delta_{r(\cdot)} u - \frac{b}{2} \Delta_{p(\cdot)} u + V'(u) \right] \phi \, dx = 0, \text{ for every } \phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

This implies by density

$$-\frac{a}{2} \Delta_{r(\cdot)} u - \frac{b}{2} \Delta_{p(\cdot)} u + V'(u) = 0.$$

□

Proposition 6.5.2. *(Splitting lemma) Let $(u_k) \in \Gamma_a^*$ be a sequence and M be a positive real number such that*

$$f_a(u_k) \leq M.$$

Then there exists $l \in \mathbb{N}$ such that

$$1 \leq l \leq \frac{M}{\Delta_a},$$

where Δ_a was introduced in Proposition 6.4.6, and there exist $\bar{u}_1, \ldots, \bar{u}_l \in \Gamma_a$, $(x_k^1), \ldots, (x_k^l) \subset \mathbb{R}^n$ such that, up to a subsequence,

$$u_k(\cdot + x_k^i) \to \bar{u}_i,$$

$$|x_k^i - x_k^j| \to \infty, \quad i \neq j,$$

$$\sum_{i=1}^{l} f_a(\bar{u}_i) \leq \liminf_{k\to\infty} f_a(u_k),$$

and

$$ch(u_k) = \sum_{i=1}^{l} ch(\bar{u}_i).$$

Proof. From Lemmas 6.4.2, 6.4.3 and 6.4.5, and by the same method as used in [Proposition 4.5.1 - Chapter 4], we can conclude the result of this proposition. □

The minimum is attained on the set Γ_a, and it is easy to see that $u \equiv 0$ is a trivial solution of the problem. But, of course, we are interested in nontrivial solutions, We consider the following problem

$$I_* = \inf_{u\in\Gamma_a^*} f_a(u), \quad \Gamma_a^* = \{u \in E_a : ch(u) \neq 0\}.$$

The functional is bounded below and the set E_a is not empty. We consider fields u having the form

$$u(x) = \left(\frac{2}{1+|x|^m}, \frac{1}{1+|x|^m}x\right). \tag{6.5.1}$$

Lemma 6.5.3. *There exists a suitable $m \geq 1$ such that, the field u defined in (6.5.1) belongs to Γ_a^*.*

Proof. Clearly, if m is sufficiently large, then the field u defined in (6.5.1) belongs to E_a. For the sake of contradiction, suppose that there exists $\bar{x} \in \mathbb{R}^n$ such that $u(\bar{x}) = \eta = (1, 0)$. We deduce that

$$\frac{2}{1+|\bar{x}|^m} = 1,$$

$$\frac{1}{1+|\bar{x}|^m}\bar{x} = 0.$$

We get the contradiction : $|\bar{x}| = 1$ and $\bar{x} = 0$. So, $u \in \Gamma_a$.

We show that $ch(u) \neq 0$. Set $g(x) = \frac{1}{2}x$, so we have

$$K_u = \{x \in \mathbb{R}^n : \frac{2}{1+|x|^m} > 1\} = B(0,1).$$

If $|x| = 1$, then

$$g(x) = \frac{1}{1+|x|^m}x,$$

by the properties of the topological degree (see [69]), we get

$$deg(\frac{1}{1+|x|^m}x, B(0,1), 0) = deg(g(x), B(0,1), 0) \neq 0.$$

□

And moreover the set Γ_a^* is open in the space E_a; indeed,

- $\Gamma_a^* = \bigcup_{q\in\mathbb{N}^*} \Gamma_a^q$,
- $\Gamma_a^q \cap \Gamma_a^p = \emptyset, \quad p \neq q$,

where Γ_q is a connected component

Theorem 6.5.4. *Let $a \geq 0,\ b > 0,\ p^- > n > 2$ and $2 = r^- \leq r^+ \leq n$. If V satisfies $(V_1) - (V_6)$, if p satisfies $(p_1) - (p_2)$, and if r satisfies $(r_1) - (r_4)$,then there exists a weak solution of (6.1.5) (i.e., a static solution of (6.1.3)), which is a minimizer of the energy functional (6.1.7) in the class of maps whose topological charge is different from* 0.

Proof. By Splitting Proposition and the same technique used in [Theorem 4.5.3-Chapter 4], we can conclude that there exists a weak solution (static) of equation (6.1.5). And with suitable change of variable (6.1.6), we deduce a solution of equation (6.1.3). □

Remark 6.5.1. *The functional exibits an invariance for the symmetry group of rotations and translations; indeed, for every function u and $g \in O(n)$, if we set $u_g(x) = u(gx)$, we have immediately*

$$f_a(u_g) = f_a(u).$$

Then our theorem gives the existence of an orbit of minimum solutions. This orbit consists of two connected components, which are identified, respectively, by $\bar{u}$ and

$$\bar{u} \circ \mathcal{P}(x) = \bar{u}(-x).$$

Since typically $n = 3$ is odd, $\bar{u} \circ \mathcal{P}$ and $\bar{u}$ have opposite topological charge.

Chapter 7

Appendix

7.1 Appendix A: Compact embedding

In this Appendix we first prove a result which slightly extends a compactness theorem in [23, 79]. We set

$$W^{1,2}_R(\mathbb{R}^n,\mathbb{R}) = \{u \in W^{1,2}(\mathbb{R}^n,\mathbb{R}) \mid u \ \text{ radial}\}.$$

Proposition 7.1.1. *Consider $n \geq 2$. The space $W^{1,2}_R(\mathbb{R}^n,\mathbb{R})$ is compactly embedded in $L^{\mathbf{s}}(\mathbb{R}^n,\mathbb{R})$ for every $\mathbf{s} \in]2, 2^*[$, where*

$$2^* = \begin{cases} +\infty & \text{if } \ n = 2, \\ 2n/(n-2) & \text{if } \ n > 2. \end{cases}$$

Proof. For $n > 2$, the proof is given in [23] (see Theorem A.I'). We give the proof for $n = 2$.
First we recall that, for every $m \in [2, +\infty[$,

$$W^{1,2}(\mathbb{R}^2,\mathbb{R}) \subset L^m(\mathbb{R}^2,\mathbb{R}) \tag{7.1.1}$$

Fix $s \in]2, +\infty[$ and consider a bounded sequence $\{u_k\} \subset W^{1,2}_R(\mathbb{R}^2,\mathbb{R})$; by (7.1.1) we have that $\{u_k\}$ is bounded $L^s(\mathbb{R}^2,\mathbb{R})$, so, up to a subsequence

$$u_k \rightharpoonup u \ in \ L^s(\mathbb{R}^2,\mathbb{R}). \tag{7.1.2}$$

Then we have to prove that the convergence is strong. Let $m \in]s, +\infty[$; clearly $\{u_k\}$ is bounded in $L^2(\mathbb{R}^2,\mathbb{R}) \cap L^m(\mathbb{R}^2,\mathbb{R})$. Now we apply the compactness Lemma A.I in [23] with

$$P(t) = t^s, \quad Q(t) = t^2 + t^m.$$

We conclude that

$$\|u_k\|_{L^s} \longrightarrow \|u\|_{L^s}. \tag{7.1.3}$$

From (7.1.2) and (7.1.3) we have

$$u_k \longrightarrow u \ in \ L^s(\mathbb{R}^2, \mathbb{R}).$$

□

Theorem 7.1.2. *If $\mathcal{W}$ is a bounded subset of $W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1})$, then*

$$\mathcal{W}_R = \{u \in \mathcal{W} \mid u \ \ radial\}$$

is relatively compact in $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$, for every $s \in]2, 2^*[$.

Proof. Fix $s \in]2, 2^*[$ and consider a sequence $\{u_k\} \in \mathcal{W}_R$. We have to show that there exists a subsequence that is strongly convergent in $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$. Since $\{u_k\}$ is bounded in $W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1})$, there exists $u \in W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1})$ such that, up to subtracting a subsequence

$$u_k \rightharpoonup u \ in \ W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1}). \tag{7.1.4}$$

By the continuous imbedding

$$W^{1,2}(\mathbb{R}^n, \mathbb{R}^{n+1}) \hookrightarrow L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$$

we deduce that

$$u_k \rightharpoonup u \ in \ L^s(\mathbb{R}^n, \mathbb{R}^{n+1}). \tag{7.1.5}$$

On the other hand, $\{|u_k|\}$ is bounded in $W^{1,2}(\mathbb{R}^n, \mathbb{R})$. Indeed

$$\int |\nabla |u_k||^2 \ \ dx \leq \int |\nabla u_k|^2 \ \ dx.$$

Then, by Proposition 7.1.1, we get

$$|u_k| \longrightarrow \chi \ in \ L^s(\mathbb{R}^n, \mathbb{R}), \tag{7.1.6}$$

and, up to a subsequence,

$$|u_k| \longrightarrow \chi \ \text{ a.e. in } \mathbb{R}^n.$$

Moreover, from (7.1.4) we deduce

$$u_k \rightharpoonup u \ in \ L^s_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^{n+1}),$$

and therefore, by a Cantor diagonal process, we can select a subsequence such that

$$u_k \longrightarrow u \ \text{ a.e. in } \mathbb{R}^n.$$

So we conclude that

$$\chi = |u|. \tag{7.1.7}$$

From (7.1.6) and (7.1.7) we deduce

$$\|u_k\|_{L^s} \longrightarrow \|u\|_{L^s}; \tag{7.1.8}$$

then (7.1.5) and (7.1.8) allow us to conclude that

$$u_k \longrightarrow u \ in \ L^s(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

□

7.2 Appendix B: Continuity & invertibility of Δ_p

Lemma 7.2.1. *The map $\Delta_p : E \to E'$ defined by*

$$\langle -\Delta_p u, v\rangle_{E'_a \times E_a} = \int_{\mathbb{R}^n} |\nabla u|^{p-2}(\nabla u|\nabla v)dx, \quad p > 2$$

is continuous.

Proof. Recall that E is the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$.
Let $h \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$

$$\begin{array}{rcl} \langle \Delta_p u - \Delta_p v, h\rangle & = & \displaystyle\int_{\mathbb{R}^n} \left(|\nabla v|^{p-2}(\nabla u|\nabla h) - |\nabla u|^{p-2}(\nabla v|\nabla h)\right) dx \\ & = & \displaystyle\int_{\mathbb{R}^n} \left(|\nabla v|^{p-2}\nabla u - |\nabla u|^{p-2}\nabla v \quad | \quad \nabla h\right) dx \\ & \leq & \displaystyle\int_{\mathbb{R}^n} \left||\nabla v|^{p-2}\nabla u - |\nabla u|^{p-2}\nabla v\right| \cdot |\nabla h|\, dx \\ \text{(from Lemma 7.6.2)} & \leq & \beta \displaystyle\int_{\mathbb{R}^n} \left||\nabla v|^{p-2} + |\nabla u|^{p-2}\right| \cdot |\nabla u - \nabla v| \cdot |\nabla h|\, dx \\ \text{(from Hölder's inequality)} & \leq & \beta \left(\|\nabla v\|_{L^p}^{p-2} + \|\nabla u\|_{L^p}^{p-2}\right) \cdot \|\nabla u - \nabla v\|_{L^p} \cdot \|\nabla h\|_{L^p}. \end{array}$$

□

Lemma 4.6.6 follows from the following result.

Theorem 7.2.2. *If H is a positive definite matrix of order $N+1$, then the map*

$$\mathcal{A} \quad E \longrightarrow E'$$

defined by

$$\begin{array}{rcl} \langle \mathcal{A}u, v\rangle & = & \langle -\Delta u - \Delta_p u + Hu, v\rangle \\ & = & \displaystyle\int_{\mathbb{R}^n} \left((\nabla u|\nabla v) + |\nabla u|^{p-2}(\nabla u|\nabla v) + Hu \cdot v\right) dx \end{array}$$

is invertible with continuous inverse.

For the proof we need some preliminary results. The first is concerned with the monotonicity of $-\Delta_p u$; for the utility of the reader we give a simple proof (see also [77] and [20] for the scalar case).

Lemma 7.2.3. *There exists a constant $c > 0$ such that, for every $u, v \in E$,*

$$\langle \Delta_p u - \Delta_p v, u - v\rangle \geq c\|\nabla u - \nabla v\|_{L^p}^p, \quad p > 2. \tag{7.2.1}$$

Proof. We prove (7.2.1) for $u, v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$; then our statement follows by density.
For every $u, v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$, we have

$$\langle \Delta_p u - \Delta_p v, u - v \rangle$$
$$= \int_{\mathbb{R}^n} \left[|\nabla u|^p + |\nabla v|^p - (\nabla u|\nabla v)\left(|\nabla u|^{p-2} + |\nabla v|^{p-2}\right)\right] dx$$

and

$$\|\nabla u - \nabla v\|_{L^p}^p = \int_{\mathbb{R}^n} |\nabla u - \nabla v|^p dx$$

where we mean

$$|\nabla u| = \sqrt{\sum_{j,\ i} \left(\frac{\partial u^j}{\partial x_i}\right)^2},$$
$$(\nabla u|\nabla v) = \sum_{j,\ i} \frac{\partial u^j}{\partial x_i}\frac{\partial v^j}{\partial x_i}.$$

So it is enough to prove that there exists $c > 0$ such that, for every $X, Y \in \mathbb{R}^{n(n+1)}$

$$|X|^p + |Y|^p - (X|Y)\left(|X|^{p-2} + |Y|^{p-2}\right) \geq c|X - Y|^p. \tag{7.2.2}$$

Substituting

$$-(X|Y) = \frac{1}{2}\left(|X - Y|^2 - |X|^2 - |Y|^2\right)$$

into (7.2.2), we get

$$\frac{1}{2}\left(|X|^p + |Y|^p\right) + \frac{1}{2}|X - Y|^2\left(|X|^{p-2} + |Y|^{p-2}\right)$$
$$\geq \ c|X - Y|^p + \frac{1}{2}\left(|Y|^2|X|^{p-2} + |X|^2|Y|^{p-2}\right). \tag{7.2.3}$$

We notice that (7.2.3) can be obtained by the inequalities

$$\frac{1}{2}\left(|X|^p + |Y|^p\right) \geq \frac{1}{2}\left(|Y|^2|X|^{p-2} + |X|^2|Y|^{p-2}\right),$$

$$\frac{1}{2}|X - Y|^2\left(|X|^{p-2} + |Y|^{p-2}\right) \geq c|X - Y|^p.$$

The first holds true for every pair of vectors X, Y see Lemma 7.6.5; the second is also true see Lemma 7.6.4 in Section 7.6.

□

Now we come back to Theorem 7.2.2.

Proof. (Theorem 7.2.2.)
First we prove that $\mathcal{A}$ is invertible. For every $h \in E'$, the solution of

$$\mathcal{A}u = h$$

can be obtained as the critical points of the functional

$$\mathcal{J}(u) = \frac{1}{p}\int |\nabla u|^p dx + \frac{1}{2}\int |\nabla u|^2 dx + \frac{1}{2}\int Hu \cdot u dx - \langle h, u\rangle.$$

Since the matrix H is positive definite, we have,

$$\frac{1}{p}\int |\nabla u|^p dx + \frac{1}{2}\int |\nabla u|^2 dx + \frac{1}{2}\int Hu \cdot u dx \geq \frac{1}{p}\|\nabla u\|^p_{L^p} + m\| \ u\|^2_{W^{1,2}}. \quad (7.2.4)$$

On the other hand, for every $\lambda > 0$

$$\langle h, u\rangle \leq \|h\|_{E'}\|u\|_E \leq \frac{1}{2\lambda}\|h\|_{E'} + \frac{\lambda}{2}\|u\|_E, \quad (7.2.5)$$

where

$$\|u\|_E = \|\nabla u\|_{L^p} + \| \ u\|_{W^{1,2}}.$$

Taking into account (7.2.4), (7.2.5), we conclude that the functional $\mathcal{J}$is lower bounded. Moreover it is strictly convex, so it has a unique critical point. Now, let $\{h_k\}$ be a sequence of elements of E' and $h \in E'$ such that $\{h_k\} \to h$ in E'. Then, we can consider $\{u_k\}$ and u in E such that

$$\mathcal{A}(u_k) = h_k, \qquad \forall k \in \mathbb{N},$$

$$\mathcal{A}(u) = h.$$

We want to prove that $u_k \to u$ in E. By (7.2.1), using again the fact that H is positive definite, we get $c_1 > 0$ such that

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle \geq c_1\left(\|\nabla u - \nabla v\|^p_{L^p} + \| \ u - v\|^2_{W^{1,2}}\right).$$

Then we have

$$\begin{array}{rcl} c_1\left(\|\nabla u_k - \nabla u\|^p_{L^p} + \| \ u_k - u\|^2_{W^{1,2}}\right) & \leq & \langle \mathcal{A}(u_k) - \mathcal{A}(u), u_k - u\rangle \\ & = & \langle h_k - h, u_k - u\rangle \\ & \leq & \| \ h_k - h\|_{E'}\| \ u_k - u\|_E \end{array}$$

that is,

$$\frac{1}{c_1}\| \ h_k - h\|_{E'} \geq \frac{\|\nabla u_k - \nabla u\|^p_{L^p} + \| \ u_k - u\|^2_{W^{1,2}}}{\|\nabla u_k - \nabla u\|_{L^p} + \| \ u_k - u\|_{W^{1,2}}}.$$

By applying Lemma 7.6.3, to

$$a_k = \|\nabla u_k - \nabla u\|_{L^p}$$

and using

$$b_k = \| \ u_k - u\|_{W^{1,2}},$$

we deduce that

$$\lim_k \|\nabla u_k - \nabla u\|_{L^p} = 0,$$

$$\lim_k \|\nabla u_k - \nabla u\|_{W^{1,2}} = 0,$$

so $u_k \to u$ in E.

$\square$

7.3 Appendix C: Linear operator

Proposition 7.3.1. *([28],Proposition 3.5, page 58)*
Let (x_n) be a sequence in E. Then

(i) $[x_k \rightharpoonup x \text{ weakly in } \sigma(E,E^*)] \Leftrightarrow [\langle f, x_k\rangle \to \langle f, x\rangle \forall \ \ f \in E^*]$.

(ii) *If* $x_k \to x$ *strongly, then* $x_k \rightharpoonup x$ *weakly in* $\sigma(E,E^*)$.

(iii) *If* $x_k \rightharpoonup x$ *weakly in* $\sigma(E,E^*)$, *then*$(\|x_k\|)$ *is bounded and* $\|x\| \leq \liminf \|x_k\|$.

(iv) *If* $x_k \rightharpoonup x$ *weakly in* $\sigma(E,E^*)$ *and* $f_k \to f$ *strongly* E^* *(i.e.,* $\|f_k - f\|_{E^*} \to 0$*), then* $\langle f_k, x_k\rangle \to \langle f, x\rangle$.

Theorem 7.3.2. *([28],Theorem 6.4 (Schauder),page 159)*
If $T \in \mathcal{K}(E,F)$, then $T^* \in \mathcal{K}(E^*,F^*)$. And conversely.

$\mathcal{K}(E,F)$: The set of all compact operators from E into F,

E^* : The dual space of E,

T^* : The adjoint of T.

Definition 7.3.1. *let E be a reflexive and separable Banach space , $\mathcal{A}$ application from E to E'*

$\mathcal{A}$ is monotone if:

$$\forall u, v \in E, \quad \langle \mathcal{A}u - \mathcal{A}v, u - v\rangle \geq 0.$$

$\mathcal{A}$ is coercive if:

$$\lim_{\|u\|_E \to +\infty} \frac{\langle \mathcal{A}u, v\rangle}{\|u\|_E} = +\infty.$$

7.4 Appendix D: Orthogonal group

Definition 7.4.1. *$g \in \mathcal{L}(\mathbb{R}^n)$ is said orthogonal when it preserves the scalar product $\forall x, y \in \mathbb{R}^n$,*

$$\langle g(x) \mid g(y) \rangle = \langle x \mid y \rangle.$$

We denote by $O(n)$ the set of orthogonal applications.

Properties:

1) g is orthogonal if and only if g preserves the norm.

2) If g is orthogonal, then it is bijective.

3) If $g \in O(n)$ then its inverse bijection is also in $O(n)$. Furthermore, if $f \in O(n)$ then $g \circ f \in O(n)$.

We conclude that $O(n)$ is a group. Furthermore it is a compact group.

Proposition 7.4.1. *The following are equivalent:*

i *g is orthogonal ;*

ii *there exists a b.o.n. in which the matrix of g is orthogonal;*

iii *in all b.o.n., the matrix of g is orthogonal.*

Here b.o.n: base orthonormal.

Definition 7.4.2. *The orthogonal matrices are matrices $M \in \mathcal{M}_n(\mathbb{R})$satisfying*

$${}^tM \cdot M = I_n;$$

We denote by $O_n(\mathbb{R})$ the set of orthogonal matrices.

Remark 7.4.1. *For all $g \in O(n)$ there exist $M \in O_n(\mathbb{R})$ such that*

$$g(x) = M \cdot x,$$

$$|g(x)| = |M \cdot x| = |x|;$$

with

$$detM = J_g = 1;$$

J being determining Jacobienne matrix.

Proposition 7.4.2. *For all $M \in O_n(\mathbb{R})$, $A \in \mathcal{M}_n(\mathbb{R})$*

$$|A \cdot M| = |M \cdot A| = |A|.$$

7.5 Appendix E: Measure of Haar

Let G a locally compact group.

Theorem 7.5.1. *(Weil1) - Definitions:*

- *Existence: There exists on G a measure of Radon (positive, nonworthless) invariant by translations on the left. Such a measure is called invariant measure of Haar on the left on G.*
- *Unicity: All measures of Haar on the left on G are proportional.*
- *Convention: If G is compact, there is a canonical choice of measure of Haar on G, namely the Haar measure left invariant that is a probability measure on G (i.e for which the measure of G is equal to 1). In general, we choose an invariant measure of Haar on the left on G, which one calls (wrongly) the measure of Haar to G and which one notes λG or more simply λ. Other notations: $d\lambda(x) = dx$.*

We notice that λ is translation invariant left, meaning: for any part Borel B of G, and for any $g \in G$, we have:

$$\lambda(gB) = \lambda(B).$$

7.6 Appendix F: Elementary calculus

Lemma 7.6.1. *([77],Lemma A.0.5, page 80)*
Let $x; y \in R^n$ and $\langle \cdot, \cdot \rangle$ be the standard scalar product in R^n. Then

$$\langle x|x|^{p-2} - y|y|^{p-2}, x - y\rangle \geq \begin{cases} c_p|x-y|^p & \text{if } \ p \geq 2, \\ c_p \frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } \ 1 < p < 2. \end{cases}$$

Lemma 7.6.2. *(see R. Glowinski and A. Marroco [62].)*

(i) $If p \in [2; \infty)$ then it holds that

$$\left|z|z|^{p-2} - y|y|^{p-2}\right| \leq \beta|z-y|(|z|+|y|)^{p-2} \quad for\ all \quad z, y \in \mathbb{R}^n$$

with β independent of y and z;

(ii) $If p \in (1; 2]$, then it holds that

$$\left|z|z|^{p-2} - y|y|^{p-2}\right| \leq \beta(|z|+|y|)^{p-1} \quad for\ all \quad z, y \in \mathbb{R}^n$$

with β independent of y and z.

Lemma 7.6.3. *Let a_k and b_k be two sequences of nonnegative numbers such that*

$$\lim_k \frac{a_k^p + b_k^2}{a_k + b_k} = 0. \tag{7.6.1}$$

Then

$$\lim_k a_k = \lim_k b_k = 0.$$

Proof. Since $a_k \geq 0$, $b_k \geq 0$, from (7.6.1), we immediately deduce

$$\lim_k \frac{a_k^p}{a_k + b_k} = 0, \tag{7.6.2}$$

$$\lim_k \frac{b_k^2}{a_k + b_k} = 0. \tag{7.6.3}$$

For contradiction, assume that, up to a subsequence,

$$a_k \geq \delta > 0. \tag{7.6.4}$$

From (7.6.2) and (7.6.4) we deduce

$$\lim_k (a_k + b_k) = +\infty.$$

Then, up to a subsequence, either

$$\lim_k a_k = +\infty, \tag{7.6.5}$$

or

$$\lim_k b_k = +\infty.$$

Suppose that (7.6.5) holds true. Then we write (7.6.2) in the following way,

$$\lim_k \frac{a_k^{p-1}}{1 + (b_k/a_k)} = 0,$$

from which we deduce

$$\lim_k \frac{b_k}{a_k} = +\infty.$$

So, for k sufficiently large,

$$a_k \leq b_k. \tag{7.6.6}$$

Then it is easy to deduce

$$\frac{1}{2}\, b_k \leq \frac{b_k^2}{a_k + b_k}. \tag{7.6.7}$$

Then, from (7.6.7) and (7.6.3) we deduce

$$\lim_k b_k = 0.$$

On the other hand, (7.6.5) and (7.6.6) imply

$$\lim_k b_k = +\infty.$$

So we get a contradiction. The proof for the other cases is analogous.

$\square$

Lemma 7.6.4. *Let* $a \geq 0$, $b \geq 0$ *and* $1 \leq p < \infty$. *we have*

$$(a+b)^p \leq 2^{p-1}(a^p + b^p).$$

Proof. If $a = 0$ we have the trivial case
If $a > 0$ the inequality is equivalent to proving that

$$(1+X)^p \leq 2^{p-1}(1+x^p).$$

We set

$$f(x) = \frac{(1+x)^p}{(1+x^p)}$$

which satisfies

$$f(0) = 1 = \lim_{x \to +\infty} f(x)$$

and
$f(x) > 0$ for all $0 < x < +\infty$.

So, for $x \geq 0$ f attains their a maximum only at the point $x = 1, f'(1) = 0$. With

$$f(1) = 2^{p-1},$$

which immediately gives the result.

$\square$

Lemma 7.6.5. *Let* $a \geq 0$, $b \geq 0$ *and* $p \geq 2$. *we have*

$$a^2 b^{p-2} + b^2 a^{p-2} \leq a^p + b^p.$$

Proof. By homogeneity we can assume that $a = 1$ and $b < 1$ The inequality is equivalent to proving that

$$b^p - b^{p-2} - b^2 + 1 \geq 0,$$

we have

$$b^p - b^{p-2} - b^2 + 1 = (b^2 - 1)(b^{p-2} - 1) \geq 0.$$

$\square$

Remark 7.6.1. *From Lemma 7.6.4, it is easy to see that under the assumption* $1 \leq p^- \leq p^+ < \infty$, *we have*

$$(a+b)^{p(x)} \leq 2^{p^+-1}(a^{p(x)} + b^{p(x)}).$$

From Lemma 7.6.5, and under the assumption $2 \leq p^- \leq p^+$, *we have*

$$a^2 b^{p(x)-2} + b^2 a^{p(x)-2} \leq a^{p(x)} + b^{p(x)}.$$

Conclusion and Perspectives

In this monograph we study a class of Lorentz invariant nonlinear field equations in several space dimensions with classical Sobolev space and general space Sobolev (variable exponent variable) as a functional setting. The main purpose is to obtain soliton-like solutions. The fields are characterized by a topological invariant, we call the charge. We prove the existence of a static solution which minimizes the energy among the configurations with nontrivial charge. And with the suitable change of variable we deduce the solution to the dynamic equation (soliton solution). Moreover, under some symmetry assumptions, we prove the existence of infinitely many solutions, which are constrained minima of the energy. More precisely, for every $N \in \mathbb{N}$ there exists a solution of charge N. We notice that the nature of the convergence of energy is the same one as that of the topological charge; when the charge explodes the energy explodes too. What gives us important information characterizing the solution.

We plan to look for generalization of the problem in general space Sobolev with variable exponent variable as a functional setting.

As anther perspective, we propose some possible ways:

- Generalization of the problem with an inclusion approach.
- Numerical treatment of soliton solution with finite element methods.
- The version of Chapter 4 and Chapter 5 Riemannian manifold.
- The stochastic version of the results of this thesis.

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